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Summary

The ASN/n for curtailed and semicurtailed single sampling by attributes is found as function of the acceptance probability (OC) for the binomial and the Poisson distribution. The two binomial ASN/n functions are approximated by means of the corresponding Poisson function. A table of ASN/n for the Poisson case is given and it is shown that the relative error of the approximations is small. An approximation based on the normal distribution is also derived.

Key Words

Curtailed sampling. Semicurtailed sampling. Binomial distribution. Poisson distribution. Table of ASN/n. Approximation to ASN/n.

Let n denote the sample size and c the acceptance number for a binomially distributed random variable with parameter p. It is well-known, see for example Statistical Research Group, Columbia University [6], that the ASN divided by n for fully curtailed sampling equals

$$e(p) = B(c,n+1,p)(n-c)/nq + \{1 - B(c+1,n+1,p)\}(c+1)/np,$$

where B(c,n,p) denotes the binomial distribution function. If curtailment takes place only in connection with rejection the first term in the formula above should be replaced simply by B(c,n,p) and we shall denote this semicurtailed ASN/n by e_b. A table of e and/or e_b will be rather voluminous because e depends on three variables (c,n,p), see Blyth and Hutchinson [1].

It seems natural to investigate the corresponding Poisson formula, e_g say, which depends on two parameters only, and try to approximate e and e_b using e_g as the main term in the approximation.

Deducing the Poisson formula directly or from the binomial formula by passing to the limit in the usual way ($p \to 0$, $n \to \infty$ and fixed $np = m$) we get

$$e_g(m) = G(c,m) + \{1 - G(c+1,m)\} (c+1)/m,$$

where G(c,m) denotes the Poisson distribution function. Note that e_g corresponds to e_b because we cannot have curtailment by acceptance under Poisson conditions.

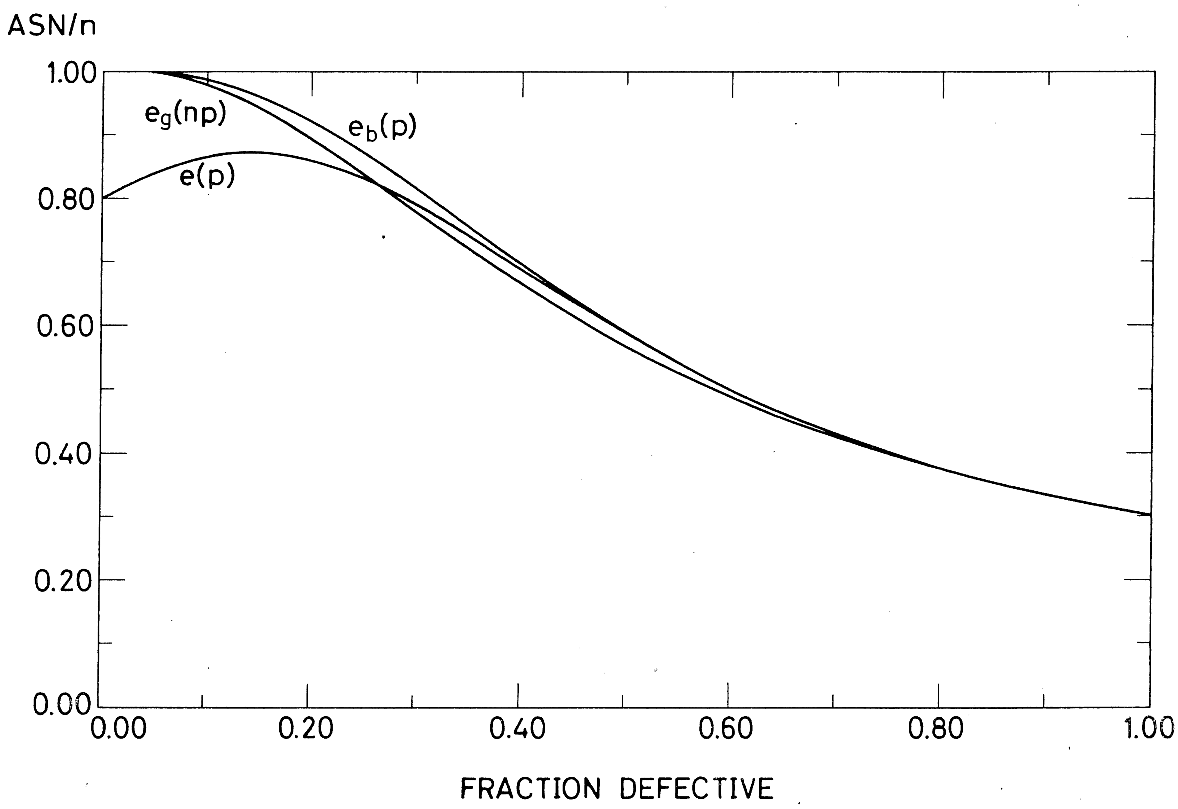
Since the derivative of $e_g(m)$ is negative it follows that $e_g(np)$ decreases from 1 to

$$e_g(n) = G(c,n) + \{1 - G(c+1,n)\} (c+1)/n$$

as p increases from 0 to 1. Similarly e_b (p) decreases from 1 to (c+1)/n.

It may be proved that $e_b(p) > e_g(np)$ for $0 < p \leqq 1$. It follows from the definitions that $e(p) < e_b(p)$ for $0 < p < 1$. Furthermore, for $0 < c < n - 1$ $e(p)$ first increases from $e(0) = (n-c)/n$ to a maximum and then decreases to $e(1) = (c+1)/n$. Consequently $e(p)$ and $e_g(np)$ intersect. An example has been shown in Fig.1. (To demonstrate the characteristic features of the curves and the differences between them we have chosen rather small values of n and c.)

Fig. 1. Comparison of the three ASN/n curves as functions of the fraction defective for n = 10 and c = 2.



Normally we are interested in the ASN corresponding to a known value of the OC. We shall therefore transform the three functions above so that they become functions of the acceptance probability. Let us define p_α and m_α as solutions to the equations $B(c,n,p_\alpha) = \alpha$ and $G(c,m_\alpha) = \alpha$, respectively, $0 \leqq \alpha \leqq 1$. Note that this definition of m_α means that m_α is different from np_α. As shown by Hald [3] it follows from a result by Wise [7] that

$$m_\alpha \,/\, np_\alpha = 1 + (m_\alpha - c)/2n + 0\ (n^{-2}), \tag{1}$$

which may be used to find p_α from m_α with sufficient accuracy for most applications in sampling inspection. Tables of m_α have been given by Hald and Kousgaard [4] and Burstein [2]; $m_\alpha = m_\alpha(c)$ may also be found as $\frac{1}{2}\,\chi^2_{1-\alpha}\ (2c+2)$.

Setting $e_g(m_\alpha) = E_g(\alpha)$ and $e(p_\alpha) = E(\alpha)$ it is straightforward to show that

$$E_g(\alpha) = \alpha + (1-\alpha)(c+1)/m_\alpha - g(c,m_\alpha)\ , \tag{2}$$

$$E_b(\alpha) = \alpha + (1-\alpha)(c+1)/np_\alpha - b(c,n,p_\alpha)(n-c)/n \tag{3}$$

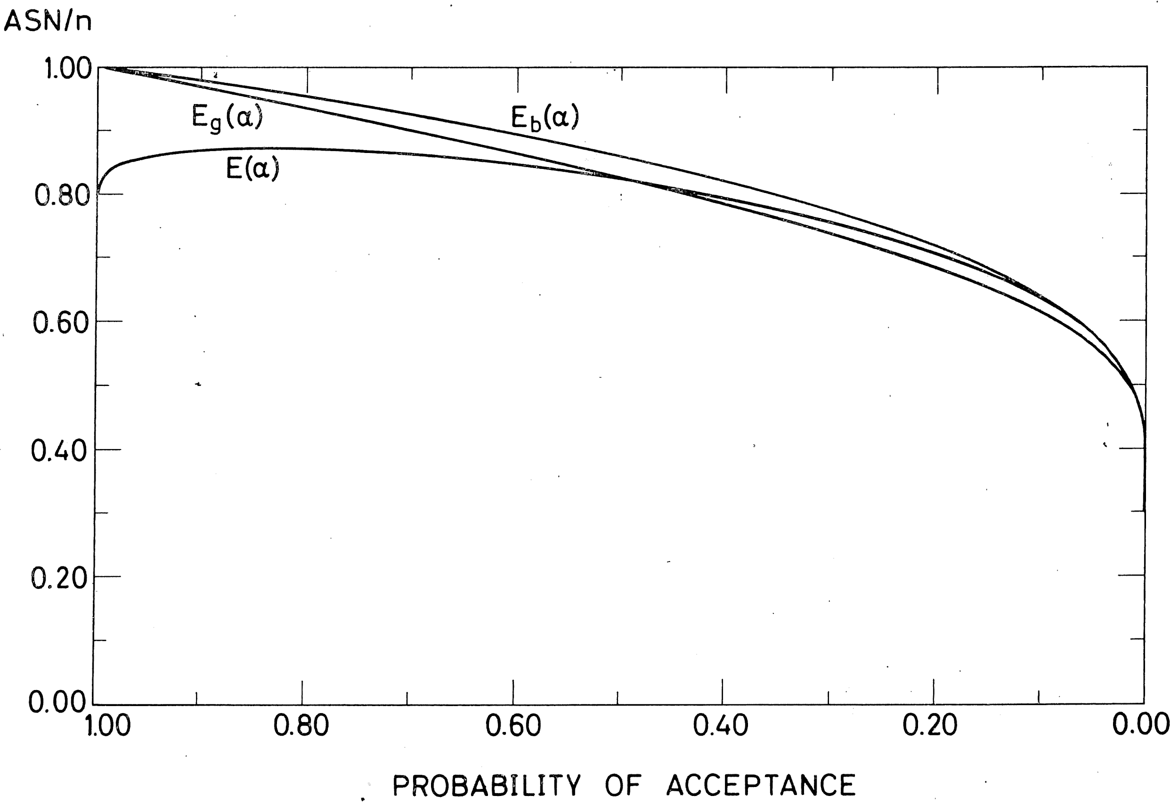
and

$$E(\alpha) = E_b(\alpha) + \alpha(np_\alpha - c)/nq_\alpha - b(c,n,p_\alpha)\ p_\alpha\ (n-c)/nq_\alpha, \tag{4}$$

where $g(c,m)$ and $b(c,n,p)$ denote the frequency functions for the Poisson and binomial distributions, respectively. Closely related results for the binomial have been given by Shah and Phatak [5].

Graphs of the three functions corresponding to Fig. 1 are shown in Fig. 2.

Fig.2. Comparison of the three ASN/n curves as functions of the probability of acceptance for n = 10 and c = 2.



From (2) and (3) we have

$$E_b(\alpha) = E_g(\alpha) + (1-\alpha)\,\frac{c+1}{m_\alpha}\left[\frac{m_\alpha}{np_\alpha} - 1\right] + g(c,m_\alpha) - b(c,n,p_\alpha)\,\frac{n-c}{n}.$$

Replacing $b(c,n,p_\alpha)$ by $g(c,m_\alpha)(1+(c/2n))$, see Lemma 2 in the Appendix, using (2) to eliminate $g(c,m_\alpha)$ and inserting (1) we get

$$E_b(\alpha) = E_g(\alpha) + \{(1-\alpha) + c(1-E_g(\alpha))\}/2n + O(n^{-2}). \qquad (5)$$

Noting that the correction to $E_b(\alpha)$ in (4) is $0(n^{-1})$ we get

$$E(\alpha) = E_b(\alpha) + \{\alpha(m_\alpha - c) - g(c,m_\alpha)m_\alpha\}/\ n + 0(n^{-2})$$
$$= E_b(\alpha) + \{m_\alpha E_g(\alpha) + \alpha - c - 1\}/\ n + 0(n^{-2}),$$

where we have used (2) to eliminate $g(c,m_\alpha)$. Inserting (5) we finally get

$$E(\alpha) = E_g(\alpha) + \{E_g(\alpha)(2m_\alpha - c) + \alpha - c - 1\}/\ 2n + 0(n^{-2}). \quad (6)$$

To compute the approximation (5) to $E_b(\alpha)$ we need only a table of $E_g(\alpha)$ whereas the approximation (6) to $E(\alpha)$ requires a table of m_α as well. Both approximations are simple to compute and rather accurate as will be shown in the following.

Table 1 contains values of $E_g(\alpha)$ for 9 commonly used values of α and $c = 1(1)20(2)50(5)70(10)100$.

Table 2 contains for $c = 5$ and $n = 20$ and 50 the values of $E_g(\alpha)$, $E_b(\alpha)$ and $E(\alpha)$ and the errors, Δ_b and Δ, i.e. the approximations computed form (5) and (6) minus the exact values. It will be seen that the error decreases with n and that the error even for $n = 20$ is rather small.

A survey of the relative error is given in Table 3 which shows that the absolute value of the relative error by using the approximations for $n \geqq 20$ and $(c+1)/(n+1) \leqq 0.25$ is at most 1.80 per cent for $0.01 \leqq \alpha \leqq 0.99$. The maximum of the relative error is normally found for rather small values of n, so that for large values of n the relative error will be considerably smaller than the maximum shown in Table 3.

Table 1. The ASN/n = $E_g(\alpha)$ as function of the acceptance probability for the Poisson distribution.

Probability of Acceptance

c	.990	.950	.900	.750	.500	.250	.100	.050	.010
0	.995	.975	.949	.869	.721	.541	.391	.317	.215
1	.997	.982	.964	.903	.783	.625	.483	.409	.300
2	.997	.986	.970	.919	.814	.672	.538	.466	.355
3	.998	.988	.975	.929	.835	.703	.576	.507	.397
4	.998	.989	.977	.936	.850	.726	.606	.538	.430
5	.998	.990	.979	.942	.861	.744	.629	.563	.457
6	.998	.991	.981	.946	.870	.759	.648	.584	.479
7	.999	.992	.982	.949	.877	.771	.664	.602	.499
8	.999	.992	.983	.952	.883	.781	.678	.617	.516
9	.999	.993	.984	.954	.888	.790	.690	.631	.531
10	.999	.993	.985	.956	.893	.798	.700	.643	.545
11	.999	.993	.986	.958	.897	.805	.710	.654	.558
12	.999	.994	.986	.960	.901	.811	.718	.663	.569
13	.999	.994	.987	.961	.904	.817	.726	.672	.579
14	.999	.994	.987	.962	.907	.822	.733	.680	.589
15	.999	.994	.988	.964	.910	.827	.740	.688	.598
16	.999	.995	.988	.965	.912	.831	.746	.695	.606
17	.999	.995	.988	.966	.914	.835	.752	.701	.613
18	.999	.995	.989	.967	.916	.839	.757	.707	.621
19	.999	.995	.989	.967	.918	.843	.762	.713	.627
20	.999	.995	.989	.968	.920	.846	.766	.718	.634
22	.999	.995	.990	.970	.923	.852	.775	.728	.645
24	.999	.996	.990	.971	.926	.857	.782	.737	.656
26	.999	.996	.991	.972	.929	.862	.789	.745	.666
28	.999	.996	.991	.973	.931	.866	.795	.752	.674
30	.999	.996	.991	.974	.933	.870	.801	.758	.682
32	.999	.996	.992	.975	.935	.873	.806	.764	.690
34	.999	.996	.992	.975	.937	.877	.811	.770	.697
36	.999	.996	.992	.976	.939	.880	.815	.775	.703
38	.999	.996	.992	.977	.940	.882	.819	.780	.709
40	.999	.997	.992	.977	.941	.885	.823	.784	.714
42	.999	.997	.993	.978	.943	.887	.826	.789	.720
44	.999	.997	.993	.978	.944	.890	.830	.792	.725
46	.999	.997	.993	.979	.945	.892	.833	.796	.729
48	.999	.997	.993	.979	.946	.894	.836	.800	.734
50	.999	.997	.993	.979	.947	.896	.839	.803	.738
55	1.000	.997	.994	.980	.949	.900	.845	.811	.748
60	1.000	.997	.994	.981	.952	.904	.851	.817	.756
65	1.000	.997	.994	.982	.953	.907	.856	.823	.764
70	1.000	.997	.994	.983	.955	.910	.860	.829	.771
80	1.000	.998	.995	.984	.958	.916	.868	.838	.783
90	1.000	.998	.995	.985	.960	.920	.875	.846	.793
100	1.000	.998	.995	.985	.962	.924	.881	.853	.802

Table 2. Comparison of the three ASN/n functions and evaluation of the errors by using the approximations (5) and (6).

α	c = 5	c = 5		n = 20		c = 5		n = 50	
	E_g	E_b	Δ_b	E	Δ	E_b	Δ_b	E	Δ
.990	.998	.999	.000	.829	.008	.998	.000	.933	.001
.950	.990	.993	.000	.860	.009	.991	.000	.941	.001
.900	.979	.985	-.001	.875	.009	.982	.000	.940	.001
.750	.942	.956	.000	.884	.008	.947	.000	.920	.001
.500	.861	.893	-.002	.856	.004	.873	.000	.860	.001
.250	.744	.799	-.004	.785	-.002	.765	-.001	.760	.000
.100	.629	.705	-.007	.700	-.006	.657	-.001	.656	.000
.050	.563	.651	-.009	.649	-.009	.596	-.001	.595	-.001
.010	.457	.563	-.013	.562	-.013	.496	-.002	.496	-.001

Table 3. Absolute value of maximum relative error expressed as percentage by using (5) (upper entry)and (6) (lower entry) to compute $E_b(\alpha)$ and $E(\alpha)$ for $n \geq 20$.

	Probability of Acceptance								
$\frac{c+1}{n+1}$	0.99	0.95	0.90	0.75	0.50	0.25	0.10	0.05	0.01
0.001	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00
0.01	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.00 0.00	0.01 0.01	0.02 0.02
0.05	0.00 0.03	0.00 0.03	0.00 0.02	0.00 0.02	0.01 0.01	0.03 0.02	0.06 0.06	0.10 0.10	0.20 0.20
0.10	0.00 0.12	0.00 0.11	0.00 0.10	0.01 0.08	0.03 0.04	0.09 0.05	0.19 0.17	0.28 0.27	0.54 0.53
0.20	0.00 0.56	0.00 0.51	0.01 0.46	0.03 0.36	0.11 0.17	0.28 0.12	0.54 0.47	0.77 0.73	1.34 1.33
0.25	0.00 0.93	0.00 0.84	0.01 0.77	0.05 0.61	0.16 0.29	0.40 0.16	0.76 0.66	1.06 1.01	1.80 1.79
0.33	0.00 1.86	0.01 1.69	0.02 1.55	0.09 1.19	0.28 0.56	0.68 0.25	1.27 1.09	1.74 1.64	2.86 2.84
0.50	0.00 5.61	0.02 5.03	0.06 4.48	0.19 3.20	0.56 1.46	1.29 0.35	2.31 1.94	3.07 2.89	4.78 4.74

To complete the investigation we shall derive an approximation to $E_g(\alpha)$ based on the normal distribution. From the Edgeworth expansion we have

$$g(c,m) = \{\phi(u)/\sqrt{c+1}\}\{1 + (u^3 - 3u)/(3\sqrt{c+1}) + O(c^{-1})\},$$

where $\phi(u)$ denotes the standardized normal density function and $u = (m-c-1)/\sqrt{c+1}$. (The expansion is found from the expansion of the gamma distribution using the relation between the Poisson and the gamma distributions.) Similarly, the Cornish- Fisher expansion gives

$$m_\alpha = (c+1)\{1 - u_\alpha/\sqrt{c+1} + (u_\alpha^2-1)/(3(c+1)) + O(c^{-3/2})\},$$

where $\Phi(u_\alpha) = \alpha$, $\Phi(u)$ denoting the standardized normal distribution function. Inserting these expansions into (2) we get

$$E_g(\alpha) = 1 + \{(1-\alpha)u_\alpha - \phi(u_\alpha)\}/\sqrt{c+1}$$
$$+ \{(1-\alpha)(2u_\alpha^2+1) - \phi(u_\alpha)(u_\alpha^3 - 3u_\alpha)\}/(3(c+1)) + O(c^{-3/2}). \quad (7)$$

The error of this approximation has been illustrated in Table 4. It will be seen that the error is a decreasing function of c and that the approximation is satisfactory for most practical purposes for $c > 50$.

Analogous results may be found for $E_b(\alpha)$ and $E(\alpha)$ by means of expansions of the beta distribution.

Table 4. Errors by using the approximation (7) to $E_g(\alpha)$.

α	c = 20		c = 50		c = 100	
	E_g	Δ	E_g	Δ	E_g	Δ
.990	.999	.000	.999	.000	1.000	.000
.950	.995	.006	.997	.003	.998	.001
.900	.989	.012	.993	.005	.995	.002
.750	.968	.016	.979	.006	.985	.003
.500	.920	.001	.947	.000	.962	.000
.250	.846	-.012	.896	-.005	.924	-.003
.100	.766	.000	.839	-.002	.881	-.001
.050	.718	.014	.803	.003	.853	.001
.001	.634	.046	.738	.013	.802	.005

Appendix

Lemma 1. For fixed c and m , $0 < m < \infty$, $p = m/n$ and $n \to \infty$ we have

$$b(c,n,p) = g(c,np)\{1 + p(c - \tfrac{1}{2}\, np) - (c^2-c)/\, 2n + 0(n^{-2})\}.$$

Proof. This result follows immediately from an expansion of $\ln\,\{b(c,n,p)/\, g(c,np)\}$ using Stirling's formula for $\ln n!$ and the power series expansion for $\ln(1-p)$.

Lemma 2. For fixed c and α , $0 < \alpha < 1$, and $n \to \infty$ we have

$$b(c,n,p_\alpha) = g(c,m_\alpha)\{1 + c/\, 2n + 0(n^{-2})\}.$$

Proof. From (1) we get

$$np_\alpha = m_\alpha - m_\alpha(m_\alpha-c)/\, 2n + 0(n^{-2}),$$

which by means of Taylor's expansion gives

$$g(c,np_\alpha) = g(c,m_\alpha)\{1 + (m_\alpha-c)^2/\, 2n + 0(n^{-2})\}.$$

Combining this result with Lemma 1 for $p = p_\alpha$ Lemma 2 follows.

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