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# BIRTH, DEATH, AND CONDITIONING OF MARKOV CHAINS†

by

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## 1. INTRODUCTION

Given a Markov chain $X_0, X_1, \ldots$ with stationary transition probabilities, we investigate random times $\tau$ with the property that the joint distribution of the pre-$\tau$ fragment $(X_0, \ldots, X_{\tau-1})$ and the post-$\tau$ fragment $(X_\tau, X_{\tau+1}, \ldots)$ can be described by saying that one or other of these fragments is Markovian with stationary transition probabilities, and that the two fragments are conditionally independent given the position of the inner endpoint of the Markovian fragment at $\tau-1$ or $\tau$. Such a description of the joint law of the pre-$\tau$ and post-$\tau$ processes for a random time $\tau$ will be called a path decomposition. For some examples of more sophisticated path decompositions which provided motivation for the present study see Williams [8], [9], Jacobsen [2], Pitman [4], [5], Pittenger and Shih [6]. Following Meyer, Smythe and Walsh [3] we refer to those random times $\tau$ for which the post-$\tau$ fragment is Markov as birth times, and to those for which the pre-$\tau$ fragment is Markov as death times. We show that for discrete time Markov chains with countable state space the analogues of the types of birth times and death times considered by Meyer, Smythe and Walsh for continuous time processes, namely optional, cooptional, terminal and coterminal times, all admit the additional conditional independence property described above, and that from these special types of random times it

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is possible to construct the most general random times determined by the evolution of the Markov chain which allow this kind of path decomposition.

To make things precise we assume that $(X_n, n \in \mathbb{N})$ is the coordinate process defined on the space $\Omega$ of all sequences in a countable set $J$ indexed by the non-negative integers $\mathbb{N}$, equipped with the usual product $\sigma$-field $F$. We say that a probability $P$ on $(\Omega, F)$ is Markov, or Markov(p), if $P$ is the distribution of a Markov chain with stationary transition probabilities $p$, i.e. if under $P$ the sequence $(X_n)$ is itself such a Markov chain. Background on this framework may be found in Freedman [1], and fuller definitions follow at the end of this introduction.

A random time $\tau = \tau(\omega)$ is now an $F$-measurable function of sequences $\omega \in \Omega$ with values in the extended time set $\mathbb{N} \cup \{\infty\}$. Given a Markov(p) probability $P$ and a random time $\tau$, we say that $\tau$ is a <u>birth time for</u> $P$ if the $P$-distribution of the post-$\tau$ process is Markov(q) for some transition matrix $q$, and say that $\tau$ is a <u>regular birth time for</u> $P$ if in addition the pre-$\tau$ and post-$\tau$ processes are conditionally independent given $X_\tau$ on $(\tau < \infty)$. Put another way, $\tau$ is a regular birth time for $P$ if there is a transition matrix $q$ such that conditional on the pre-$\tau$ process, $\tau < \infty$, and $X_\tau = x$ (for each state $x$) the $P$ distribution of the post-$\tau$ process is Markov(q) with starting state $x$. According to the strong Markov property each optional (stopping) time $\tau$ is a regular birth time for every Markov probability $P$, and in this case $q = p$. If all states are recurrent it will be seen that every regular birth time is a.s. equal to an optional time, but if there are transient states there will usually be many regular birth times $\tau$ for which $q$ differs from $p$, e.g. the last time $\tau$ that a certain set of states $H$ is visited, when the post-$\tau$ process is like the original process conditioned never to hit H.

It turns out quite generally that the Markov chain which emerges at a regular birth time $\tau$ can be described by conditioning a Markov chain with the same transition probabilities as the original. With this in mind we determine in Section 2 the collection of all events $C \in F$ with the property that when a Markov probability P on $(\Omega, F)$ is conditioned on C, another Markov probability results. The result of Section 2 is then applied in Section 3 to give a complete description of all regular birth times for a Markov probability P. It is shown that there is a class of random times $\mathcal{B}$ with the property that for each Markov probability P

(i) every $\tau \in \mathcal{B}$ is a regular birth time for P

(ii) every regular birth time for P is P a.s. equal to a random time in $\mathcal{B}$.

This canonical collection of regular birth times may be roughly described as comprising 'optional times after coterminal times'. It is interesting that the conditional independence hypothesis involved in regularity is quite essential for this type of result. We show that there exists no such canonical collection of plain birth times by exhibiting two Markov probabilities P and Q with the same null sets together with a random time $\tau$ which is a birth time for P but not for Q.

In Sections 4 and 5 we consider death times. We say that $\tau$ is a <u>death time for</u> P if the P distribution of the pre-$\tau$ process $(X_0, \ldots, X_{\tau-1})$ is Markov(r) for some sub-stochastic transition matrix r, and say that $\tau$ is a <u>regular death time for</u> P if in addition the pre-$\tau$ and post-$\tau$ processes are conditionally independent given $X_{\tau-1}$ on $(0 < \tau < \infty)$. Since the time reversal of a Markov fragment with finite lifetime and stationary transition probabilities is again Markov with stationary transition probabilities, the regularity condition for a death time is equivalent to demanding that there is some sub-stochastic transition matrix $\hat{r}$ such that conditional on the post-$\tau$ process, $0 < \tau < \infty$ and $X_{\tau-1} = x$ (for each state x), the reversed pre-$\tau$ fragment $(X_{\tau-1}, X_{\tau-2}, \ldots, X_0)$ is Markov($\hat{r}$) with starting state $x$. Thus the notion of a regular death time may be viewed as the dual under time reversal to the notion of a regular birth time.

In Section 5 we prove the existence of a canonical class $\mathcal{D}$ of regular death times which roughly speaking comprises 'co-optional times prior to terminal times'. This result is like a dual to the existence of the class $\mathcal{B}$ of regular birth times, but owing to the impossibility of reversing on $(\tau = \infty)$ we are unable to bridge between the two results by any direct use of time reversal. For the death time theorem we instead make use of a new method developed in Section 4, exploiting a functional equation satisfied by certain conditional probabilities associated with any death time, regular or not. Once again we show that there is no canonical collection of plain death times.

Section 6 is devoted to random times which are both regular birth times and regular death times. We show that for nice transition matrices $p$ these times are essentially either terminal times or coterminal times, and give a detailed description of the associated path decompositions. Finally, in Section 7 we discuss possible extensions of our results to Markov processes with more general time set or state space.

We set out now the basic notation and conventions which will be used throughout. We take it right from the start that our countable state space $J$ contains a conventional coffin state $\Delta$, but $\Delta$ is for use only after killing operations. Except where otherwise specified we assume that we are given a fixed Markov probability $P$ on the sequence space $(\Omega, F)$, with initial distribution $\lambda = (\lambda(x),\ x \in J)$ and transition probabilities $p = (p(x,y),\ x,y \in J)$ which are arbitrary subject to the coffin state conventions that $\lambda(\Delta) = 0$, $p(\Delta,\Delta) = 1$, $p(x,\Delta) = 0$, $x \neq \Delta$. Thus $P$ can be any Markov probability concentrating on the space $\Omega_0 \subset \Omega$ of all sequences in the state space $J \setminus \{\Delta\}$. For $x \in J$ we denote by $P^x$ the probability on $(\Omega, F)$ which is Markov(p) with starting state $x$: thus $P = \sum_{x \in J} \lambda(x) P^x$.

For $n \in \bar{N} = N \cup \{\infty\}$ we define the coordinate maps $X_n: \Omega \to J$, killing operators $K_n: \Omega \to \Omega$ and shift operators $\theta_n: \Omega \to \Omega$ as follows: for $\omega = (\omega_0, \omega_1, \ldots) \in \Omega$, $n \in N$,

$$
\begin{aligned}
X_n(\omega) &= \omega_n , \\
K_n(\omega) &= (\omega_0, \ldots, \omega_{n-1}, \Delta, \Delta, \ldots) , \\
\theta_n(\omega) &= (\omega_n, \omega_{n+1}, \ldots) ,
\end{aligned}
$$

while for $n = \infty$

$$X_\infty(\omega) = \Delta , \quad K_\infty(\omega) = \omega , \quad \theta_\infty(\omega) = \omega_\Delta ,$$

where $\omega_\Delta = (\Delta, \Delta, \ldots)$ is the dead sequence. For a random time $\tau: \Omega \to \bar{N}$ we define $F$-measurable mappings $X_\tau$, $K_\tau$ and $\theta_\tau$ in the obvious way: e.g. $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$, $\omega \in \Omega$. Thus $X_\tau$ gives the position of the process $(X_n)$ at time $\tau$, $K_\tau$ describes the strictly pre-$\tau$ fragment $(X_0, \ldots, X_{\tau-1})$ by identifying it with the more manageable process $(X_0, \ldots, X_{\tau-1}, \Delta, \Delta, \ldots)$, while $\theta_\tau$ describes the post-$\tau$ fragment $(X_\tau, X_{\tau+1}, \ldots)$. For $n \in N$ we define $F_n$ to be the sub-$\sigma$-field of $F$ generated by $X_0, \ldots, X_n$, and denote by $A_n$ the countable collection of all atoms of $F_n$, i.e. all events $A$ of the form $A = (X_k = x_k,\ 0 \le k \le n)$ for some $x_0, \ldots, x_n \in J$. For a random time $\tau$ define $F_\tau$, the $\sigma$-field of events up to and including time $\tau$, to be the $\sigma$-field generated by $K_{\tau+1}$. This agrees with the usual definition for an optional time $\tau$, and especially $F_\tau = F_n$ for the constant time $\tau = n$. We are only ever interested in the trace of $F_\tau$ on the space $\Omega_0$ of sequences avoiding the coffin state $\Delta$, and we find that for each $n \in \bar{\mathbb{N}}$ the trace of $F_\tau$ on the event $\Omega_0(\tau = n)$ is identical to the trace of $F_n$ on $\Omega_0(\tau = n)$, and that the event $\Omega_0(\tau < \infty)$ is the union of the countable collection $\{\Omega_0 A(\tau = n),$ $A \in A_n, n \in \mathbb{N}\}$ of atoms of $F_\tau$, where here and throughout, $AB$ stands for the intersection of two events $A$ and $B$ in $F$.

## 2. CONDITIONED MARKOV CHAINS

In this section we solve the following problem: given a Markov chain with stationary transition probabilities, on what events determined by the evolution of the Markov chain can one condition obtain a new Markov chain with stationary transition probabilities? For any probability $P$ on $(\Omega, F)$ and $C \in F$ with $P(C) > 0$ let $P_C$ or $P(\cdot|C)$ denote the probability on $(\Omega, F)$ obtained by conditioning on $C$:

$$P_C(F) = P(F|C) = P(FC)/P(C) \ , \quad F \in F \ .$$

Thus the problem becomes: given that $P$ is Markov, for which $C \in F$ is $P_C$ again Markov? We start by defining various collections of events contained in $F$:

(2.1) <u>Definition</u>. Let

$$C_0 = \{C\colon C \in F,\ C = (X_0 \in H) \text{ for some } H \subset J\} \ ,$$
$$C_* = \{C\colon C \in F,\ C = [(X_n, X_{n+1}) \in V,\ n \in \mathbb{N}] \text{ for some } V \subset J \times J\} \ ,$$
$$C_\infty = \{C\colon C \in F,\ C = (\theta_1 \in C)\} \ .$$

Thus $\mathcal{C}_0 = \mathcal{F}_0$ is the $\sigma$-field of <u>initial events</u> generated by $X_0$, $\mathcal{C}_\infty$ is the $\sigma$-field of <u>invariant events</u>, but the collection $\mathcal{C}_*$ of events which constrain all the transitions to be of a certain type is not a $\sigma$-field at all.

(2.2) <u>Definition</u>. Let

$$\mathcal{C}_+ = \{C\colon C \in \mathcal{F},\ C = C_* C_\infty \text{ for some } C_* \in \mathcal{C}_*,\ C_\infty \in \mathcal{C}_\infty\}\ ,$$
$$\mathcal{C} = \{C\colon C \in \mathcal{F},\ C = C_0 C_* C_\infty \text{ for some } C_0 \in \mathcal{C}_0,\ C_* \in \mathcal{C}_*,\ C_\infty \in \mathcal{C}_\infty\}\ .$$

Events in $\mathcal{C}_+$ will be called <u>coterminal events</u>, anticipating the connection between these events and coterminal times which is described in the next section. Events in $\mathcal{C}$ are intersections of initial events and coterminal events.

Now each of the collections $\mathcal{C}_\beta$, $\beta = 0, *, \infty$, is readily seen to have the property that if P is Markov then so is $P_C$ whenever $C \in \mathcal{C}_\beta$ and $P(C) > 0$, and it follows by repeated conditioning that the class $\mathcal{C}$ of all intersections of events from these collections must again have this property. The central result of this section is that no matter what Markov probability P we consider, the events in $\mathcal{C}$ are up to P-equivalence the only events for which $P_C$ is Markov:

(2.3) <u>Theorem</u>. Suppose P is Markov and C is an event with $P(C) > 0$. Then $P_C$ is Markov if and only if C is P-equivalent to an event in $\mathcal{C}$.

The theorem is an immediate consequence of Lemma (2.5) and Proposition (2.10) below. Proofs of these results take up the remainder of the section, but we mention first a simple corollary:

(2.4) Corollary. Suppose the Markov probability $P$ makes all states recurrent. Then $P_C$ is Markov if and only if $C$ is $P$-equivalent to an initial event.

Proof. If $P$ makes all states recurrent then it is found that every coterminal event is $P$-equivalent to an initial event (see Freedman [1], 1.120).

(2.5) Lemma. If $C$ is a coterminal event, then there are events $F_n \in \mathcal{F}_n$ such that

(i) $$C = F_n(\theta_n \in C) \ , \quad n \in \mathbb{N} \ .$$

Conversely, if $C$ is an event such that

(ii) $$C = F_1(\theta_1 \in C)$$

for some $F_1 \in \mathcal{F}_1$, then $C$ is a coterminal event. Furthermore, if (ii) holds only $P^x$ a.s. for all $x \in J$, where $P^x$ is Markov(p) starting at $x$, then there is a coterminal event which is $P^x$-equivalent to $C$ for all $x$.

Proof. The first assertion is obvious. For the converse suppose that $C \in \mathcal{F}$ satisfies (ii). Since $F_1 = ((X_0, X_1) \in V)$ for some $V \subset J \times J$ we get

$$C = ((X_0, X_1) \in V)(\theta_1 \in C) \quad ,$$

whence
$$C = ((X_{k-1}, X_k) \in V, \ 1 \le k \le n)(\theta_n \in C) \quad , \quad n \ge 1 \ ,$$

by iteration. But intersecting this identity over all $n \ge m$ gives us

$$C = C_V(\theta_n \in C, \ n \ge m)$$

where $C_V = ((X_{k-1}, X_k) \in V, \ 1 \le k < \infty) \in \mathcal{C}_*$, and taking the union of this identity over all $m$ gives $C = C_V C_\infty$ where $C_\infty = \liminf_{n \to \infty} (\theta_n \in C)$ is invariant, so that $C$ is indeed a coterminal event. For the final assertion the same sequence of identities is justified $P^x$ a.s. for all $x$ by using the fact that if two events $F_1$ and $F_2$ agree $P^x$ a.s. for all $x$ then so too do the events $(\theta_n \in F_1)$ and $(\theta_n \in F_2)$ for each $n \in \mathbb{N}$.

Our efforts now are directed toward establishing the final result (2.10) required for the proof of Theorem (2.3).

(2.6) Notation. Recall that $\mathcal{A}_n$ denotes the countable collection of all atoms of $\mathcal{F}_n$. Now for $y \in J$ let $\mathcal{A}_{ny}$ denote the subcollection of $\mathcal{A}_n$ comprising those atoms contained in the event $(X_n = y)$.

We observe that a probability $P$ on $(\Omega, \mathcal{F})$ is Markov if and only if for each $y \in J$ the $P_A$ distribution of the post-n process $\theta_n$ remains constant as $A$ varies over all events in $\mathcal{A}_{ny}$ with $P(A) > 0$ and $n$ varies over $\mathbb{N}$. When $P$ is Markov(p) this constant distribution is of course the probability $P^y$.

(2.7) Definition. Let $n \in \mathbb{N}$. For each event $A$ in $\mathcal{A}_n$, and each event $F \in \mathcal{F}$, define a set $F_A \subset \Omega$, the section of $F$ beyond $A$ as follows: For $A = (X_k = x_k,\ 0 \leq k \leq n) \in \mathcal{A}_n$, $F_A$ comprises those sequences $\omega = (\omega_0, \omega_1, \ldots)$ such that $\omega_0 = x_n$ and the sequence $(x_0, \ldots, x_n, \omega_1, \omega_2, \ldots)$ is in $F$.

Then $F_A$ is an event in $\mathcal{F}$ and we shall make repeated use of the identity

$$AF = A(\theta_n \in F_A), \quad A \in \mathcal{A}_n, \quad F \in \mathcal{F}. \tag{2.8}$$

Notice that if $A \in \mathcal{A}_{ny}$ then $F_A \subset (X_0 = y)$, $F \in \mathcal{F}$.

(2.9) Lemma. Suppose $P$ is Markov(p), $C \in \mathcal{F}$ with $P(C) > 0$. Then for $A \in \mathcal{A}_{ny}$ with $P(AC) > 0$, the $P_{AC}$ distribution of $\theta_n$ is $P^y_{C_A} = P^y(\cdot \mid C_A)$.

Proof. For $B \in F$, $A \in A_{ny}$ with $P(AC) > 0$, we have

$$P_{AC}(\theta_n \in B) = \frac{P[AC(\theta_n \in B)]}{P(AC)} = \frac{P[A(\theta_n \in C_A B)]}{P[A(\theta_n \in C_A)]}$$
$$= \frac{P_A(\theta_n \in C_A B)}{P_A(\theta_n \in C_A)} = \frac{P^y(C_A B)}{P^y(C_A)} = P^y(B|C_A) \; .$$

(2.10) Proposition. Suppose $P$ is Markov(p), $C \in F$ with $P(C) > 0$. Then $P_C$ is Markov if and only if there exists an event $D \in F$ such that

(i) $$C = C_0 D \qquad P \text{ a.s.}$$

for some initial event $C_0 \in C_0$, and

(ii) for each $n \in N$ there is an event $F_n \in F_n$ with

$$D = F_n(\theta_n \in D) \qquad P^x \text{ a.s.}$$

for all $x \in J$.

Proof. Fix $P$ and $C \in F$ with $P(C) > 0$, and define $I \subset J$ to be the essential range of $(X_n)$ under $P_C$:

$$I = \{y \in J\colon\, P_C(X_n = y) > 0 \text{ for some } n\} \; .$$

For $y \in J$ define $A^+_{ny}$ to be the collection of all atoms $A$ in $A_{ny}$ with $P_C(A) > 0$, and set $A^+_y = \bigcup_n A^+_{ny}$. Thus $A^+_y$ is non-empty if and only if $y \in I$. Now $P_C$ is Markov if and only if for each $y \in I$ the $P_{AC}$ distribution of $\theta_n$ is constant as $A$ varies over $A^+_{ny}$ and $n$ varies over $N$. Thus by (2.9) $P_C$ is Markov if and only if for each $y \in I$ the probabilities $P^y(\cdot|C_A)$ are identical, $A \in A^+_y$, i.e. if and only if the events $C_A$ are $P^y$ a.s. identical, $A \in A^+_y$. But if there is a $D$ satisfying (i) and (ii), then we clearly have for all $A \in A_y^+$

$$C_A = (X_0=y)D \qquad P^y \text{ a.s.},$$

hence $P_C$ is Markov. Conversely, if $P_C$ is Markov, say Markov(q), then for each $y \in I$ we can select a representative event $C_A$ with $A \in A_y^+$, call it $D_y$, set $D = \bigcup_{y \in I} D_y$, and then have for each $y \in I$ the identity

$$(2.11) \qquad P^y(\cdot|D) = P^y(\cdot|D_y) = P^y(\cdot|C_A) = Q^y(\cdot) \;, \qquad A \in A_y^+ \;,$$

where $Q^y$ is Markov(q) starting at $y$. Obviously this $D$ satisfies (i) with $C_0 = (X_0 \in H)$ for $H = \{y: P_C(X_0=y) > 0\}$, and this $D$ also satisfies (ii), as can be seen by the following argument. For any $y \in I$, $P_D^y$ is Markov(q) so that $D = D_A$ $P^z$ a.s. for any $A \in A_{nz}$ with $P^y(AD) > 0$. Consequently $AD = A(\theta_n \in D)$ $P^y$ a.s. for any $A \in A_n$ with $P^y(AD) > 0$ and taking the union over all such atoms $A$ we arrive at a representation $D = F_{ny}(\theta_n \in D)$ $P^y$ a.s. with $F_{ny} \in F_n$. Defining $F_n = \bigcup_{y \in I} (X_0=y)F_{ny}$ we now see that $D = F_n(\theta_n \in D)$ $P^x$ a.s. for arbitrary $x \in I$, and since $P^x(D) = P^x(F_n) = 0$ for $x \notin I$, $D$ satisfies (ii), and the proof is complete.

## 3. REGULAR BIRTH TIMES

The main result of this section is Theorem (3.9) which describes all regular birth times for a Markov probability $P$ in terms of certain fundamental birth times associated with the coterminal events of the previous section, i.e. the coterminal times of Meyer, Smythe and Walsh [3].

Using the notation defined at the end of the introduction, a random time $\tau$ is a regular birth time for $P$ if and only if a $P$ conditional distribution of $\theta_\tau$ given $F_\tau$ is equal to $Q^x$ on $(\tau < \infty, X_\tau = x)$, where $Q^x$ is Markov(q) with starting state $x$ for some transition matrix $q$. Put another way, $\tau$ is a regular birth time for $P$ if and only if under $P$ the post-$\tau$ sequence $(X_{\tau+n}, n \in \mathbb{N})$ is Markov($q$) with respect to the increasing sequence of $\sigma$-fields $(F_{\tau+n}, n \in \mathbb{N})$.

Suppose now that $C$ is a coterminal event as defined in (2.2), i.e.

(3.1) $$C = C_V C_\infty$$

where

(3.2) $$C_V = [(X_{n-1}, X_n) \in V, \ 1 \le n < \infty]$$

for some $V \subset J \times J$, and $C_\infty$ is invariant.

(3.3) Definition. The coterminal time associated with $C$ is the random time $\tau_C$ defined by

$$\tau_C = \inf\{n \in \mathbb{N}: \theta_n \in C\} .$$

(Here and elsewhere we use the convention $\inf \emptyset = \infty$, $\sup \emptyset = 0$).

Since for coterminal events $C$

(3.4) $$(\theta_k \in C) \subset (\theta_m \in C) , \qquad 0 \le k \le m < \infty ,$$

we have the identity

(3.5) $$(\tau_C \le n) = (\theta_n \in C) , \qquad n \in \mathbb{N} .$$

In particular, if $C = C_V$, then $\tau_C$ is the time that the last transition in $V^c$ is completed:

$$\tau_{C_V} = \sup\{n \ge 1: (X_{n-1}, X_n) \in V^c\} ,$$

while if $C = C_\infty$ is invariant, then

$$\tau_{C_\infty} = 0 \text{ on } C_\infty , \quad \infty \text{ on } C_\infty^c ,$$

and in general for $C = C_V C_\infty$ we have that $\tau_C$ is simply the maximum of $\tau_{C_V}$ and $\tau_{C_\infty}$.

It is easy to check that $\tau_C$ is indeed a coterminal time as defined by Meyer, Smythe and Walsh in [3], i.e. that the random time $\tau = \tau_C$ has the properties

$$(3.6) \qquad \begin{array}{ll} \text{(i)} & \tau \circ \theta_n = (\tau - n)^+ , \quad n \in \mathbb{N} , \\ \text{(ii)} & \tau \circ K_n = \tau \text{ on } (\tau < n) , \quad n \in \mathbb{N} . \end{array}$$

Conversely, if $\tau$ is a coterminal time, then $C = (\tau = 0)$ is a coterminal event and $\tau = \tau_C$. To see this observe that (3.6)(i) implies

$$(3.7) \qquad (\tau \leq n) = (\tau \circ \theta_n = 0) = (\theta_n \in C) , \quad n \in \mathbb{N} ,$$

so that by (3.5) it suffices to show that $C$ is a coterminal event. But by (3.6)(ii) for $n = 2$ and (3.7) for $n = 1$ we have

$$C = (\tau = 0, \tau < 2) = (\tau \circ K_2 = 0, \tau < 2) = (\tau \circ K_2 = 0, \theta_1 \in C) ,$$

and since $(\tau \circ K_2 = 0) \in F_1$, we conclude from (2.5)(ii) that $C$ is a coterminal event.

We shall see shortly that each coterminal time $\tau = \tau_C$ is a birth time for each Markov probability $P$: indeed a $P$ conditional distribution for $\theta_\tau$ given $F_\tau$ equals $P_C^x$ on $(X_\tau = x)$, where $P_C^x = Q^x$ is Markov(q) for some q by (2.3). This is just the analogue in the present context of Theorem 5.1 of [3]. For a more detailed description of the path decomposition at $\tau_C$ giving the transition probabilities $q$ of the post-$\tau_C$ process, see Section 6.

(3.8) Definition. Let $\mathcal{B}$ denote the class of all random times $\tau$ of the form

$$\tau = \tau_C + \rho$$

where $\tau_C$ is the coterminal time associated with a coterminal event $C$, and $\rho$ is an optional time for the increasing sequence of $\sigma$-fields $(F_{\tau_C+n}, n \in \mathbb{N})$, i.e. $(\rho=n) \in F_{\tau_C+n}$, $n \in \mathbb{N}$.

Once we know that each $\tau_C$ is a regular birth time for $P$, it follows at once from the strong Markov property of the sequence $(X_{\tau_C+n}, n \in \mathbb{N})$ adapted to $(F_{\tau_C+n}, n \in \mathbb{N})$ that each $\tau \in \mathcal{B}$ is again a regular birth time for $P$. Our principal result is that the random times in $\mathcal{B}$ form a complete, canonical collection of birth times in that no matter what Markov probability $P$ we start off with, every regular birth time for $P$ is $P$-equivalent to a random time in $\mathcal{B}$.

(3.9) Theorem. A random time $\tau$ is a regular birth time for a Markov probability $P$ if and only if $\tau$ is $P$-equivalent to a random time in $\mathcal{B}$.

The proof of this theorem takes up the rest of the section, but we mention first the following corollary:

(3.10) Corollary. If $P$ makes all states recurrent then every regular birth time for $P$ is $P$-equivalent to a stopping time.

Proof. Just as for (2.4).

(3.11) Definition. A random time $\tau$ is a conditional independence time for a Markov probability $P$ if under $P$ the pre-$\tau$ and post-$\tau$ processes are conditionally independent given $X_\tau$, i.e. if there is a conditional distribution of $\theta_\tau$ given $F_\tau$ within $(\tau < \infty)$ which is a function of $X_\tau$ alone.

(3.12) Lemma. A random time $\tau$ is a conditional independence time for $P$ if and only if there are events $F_n \in F_n$, $n \in N$, $G \in F$ with

$$(\tau = n) = F_n(\theta_n \in G) \qquad P \text{ a.s.}, \quad n \in N, \tag{3.13}$$

and there is then a conditional distribution of $\theta_\tau$ given $F_\tau$ which equals $P_G^x$ on $(\tau < \infty, X_\tau = x)$.

Remark. The proof can easily be sharpened to show that $\tau$ is a conditional independence time for $P$ if and only if $\tau$ is P a.s. equal to a $\tau^*$ with $(\tau^* = n) = F_n(\theta_n \in G)$ exactly for some $F_n \in F_n$, $G \in F$. Every such time is thus a.s. equal to a splitting time, defined in Jacobsen [2] as a random time $\tau$ for which $(\tau = n) = F_n(\theta_n \in G_n)$ for some $F_n \in F_n$, $G_n \in F$. The present argument will also show that splitting times are characterized by conditional independence of the pre-$\tau$ and post-$\tau$ processes given both $X_\tau$ and $\tau$.

Proof. Working on atoms as in the proof of (2.9), let $A_{nx}^+$ be the collection of all atoms $A$ of $F_n$ contained in $(X_n = x)$ with $P(AG_n) > 0$, where $G_n = (\tau = n)$, so that $(X_n = x)G_n$ is $P$ a.s. equal to the union of the sets $AG_n$ over all $A$ in $A_{nx}^+$. Defining $G_{nA}$ as in (2.7) to be the section of $G_n$ beyond $A$, we have from (2.9) that the $P$ conditional distribution of $\theta_n$ given $AG_n$ is $P^x(\cdot|G_{nA})$, $A \in A_{nx}^+$. But $\tau$ is a conditional independence time if and only if this conditional distribution is a function of $x$ alone for all $A \in A_{nx}^+$, $n \in N$, i.e. if and only if for some $G_x$ in $F$ with

$G_x \subset (X_0 = x)$ we have for each $x \in J$

(3.14) $$G_{nA} = G_x \qquad P^x \text{ a.s.}, \quad A \in A^+_{nx} .$$

But if (3.13) holds we have (3.14) with $G_x = G(X_0 = x)$, while if (3.14) holds we get (3.13) with $G = \bigcup_{x \in J} G_x$, and $F_n$ the union of all $A \in A_n$ with $P(AG_n) > 0$.

Proof of Theorem (3.9). We have that $\tau$ is a regular birth time for $P$ if and only if $\tau$ is a conditional independence time for $P$ and the $P$ distribution of $\theta_\tau$ is Markov. Thus the 'if' part follows at once from (3.12) and (2.3). For the converse, suppose $\tau$ is a regular birth time for $P$. Then (3.12) shows that there are events $F_n \in F_n$, $G \in F$ with

(3.15) $$(\tau = n) = F_n(\theta_n \in G) \qquad P \text{ a.s.},$$

and also that the conditional distribution of $\theta_\tau$ given $F_\tau$ is $P^x_G$ on $(\tau < \infty, X_\tau = x)$. Since the regular birth time property tells us that this probability is Markov(q) for some $q$ not depending on $x$, we deduce from Theorem (2.3) that for every $x$ with $P(\tau < \infty, X_\tau = x) > 0$, $G$ is $P^x$ a.s. equal to a coterminal event $C$ which because of (2.10)(ii) and (2.5)(ii) may be chosen so as not to depend on $x$. It now follows that we can replace the set $G_x = G(X_0 = x)$ which appears in (3.14) by the set $C_x = C(X_0 = x)$ to deduce that in fact

(3.16) $$(\tau = n) = F_n(\theta_n \in C) \qquad P \text{ a.s.}, \quad n \in N .$$

But now define

(3.17) $$C_n = F_n(\theta_n \in C)$$

and define a random time $\tau'$ by setting

$$\tau' = n \quad \text{on} \quad C_n \setminus \bigcup_{k=0}^{n-1} C_k \;, \quad n \in \mathbb{N}$$

$$= \infty \quad \text{on} \quad \Big(\bigcup_{k=0}^{\infty} C_k\Big)^c \;.$$

(Note that we could not define $\tau'$ to be $n$ on $C_n$ because the sets $C_n$ might not be disjoint). We clearly have $\tau' = \tau$ $P$ a.s., and we now conclude by showing that $\tau' \geq \tau_C$ and that $\rho = \tau' - \tau_C$ is an optional time for $(F_{\tau_C+m}, m \in \mathbb{N})$. But from (3.17), (3.4) and (3.5) we have

$$(\tau' \leq n) = \bigcup_{k=0}^{n} C_k \subset \bigcup_{k=0}^{n} (\theta_k \in C) = (\tau_C \leq n) \;, \quad n \in \mathbb{N} \;,$$

which implies $\tau' \geq \tau_C$. Furthermore, setting $\rho = \tau' - \tau_C$ we have for $m \in \mathbb{N}$ the identity

$$(3.18) \qquad (\rho \leq m) = \bigcup_{n=m}^{\infty} (\tau_C = n-m, \; \tau' \leq n) \;.$$

But

$$(3.19) \qquad (\tau_C = n-m, \; \tau' \leq n) = (\tau_C = n-m)\Big[\bigcup_{k=0}^{n} C_k\Big] = \bigcup_{k=0}^{n} B_k$$

where $B_k = (\tau_C = n-m)C_k$. Now if $k < n-m$ we have by (3.17) and (3.5) that $C_k \subset (\theta_k \in C) = (\tau_C \leq k)$ so that $B_k = \emptyset$, while if $n-m \leq k \leq n$ we have from (3.5) and (3.4) that $(\tau_C = n-m) \subset (\theta_{n-m} \in C) \subset (\theta_k \in C)$ so that using (3.17) we get $B_k = (\tau_C + m = n)F_k$ where $F_k \in F_k \subset F_n$. In either case we see that $B_k \in F_{\tau_C+m}$, and thus working back through (3.19) to (3.18) we get also that $(\rho \leq m) \in F_{\tau_C+m}$, which is to say that $\rho$ is an optional time of $(F_{\tau_C+m}, m \in \mathbb{N})$. The proof is complete.

We conclude this section with some remarks about plain birth times. For an interesting example fix $m \geq 1$ and define

$$\tau = \inf\{n \geq 1 \colon (X_n, \ldots, X_{n+m}) = (X_0, \ldots, X_m)\} .$$

If $P$ makes all states recurrent then $\tau$ is $P$ a.s. finite and since $\tau+m$ is a stopping time and $(X_\tau, \ldots, X_{\tau+m}) = (X_0, \ldots, X_m)$ on $(\tau < \infty)$ it is easy to see that the $P$ distribution of $\theta_\tau$ is $P$. Thus $\tau$ is a birth time for $P$, but certainly not a regular birth time, since knowledge of the pre-$\tau$ process completely determines the first $m$-moves of the post-$\tau$ process. Examples show that $\tau$ can fail to be a birth time if there are transient states. But it is of greater interest to use the idea behind this example to construct two Markov probabilities $P$ and $R$ with the same null sets and a $\tau$ which is a birth time for $P$ but not for $R$, since this shows that there exists no canonical class $\mathcal{B}^*$ of plain birth times with the property that $\tau$ is a birth time for a Markov probability $P$ if and only if $\tau$ is $P$-equivalent to a time in $\mathcal{B}^*$.

(3.20) Example. Ignore the coffin state $\Delta$ and let $J = \{1,2,3\}$. Define transition matrices $p$ and $r$ on $J$ by

$$p = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \qquad r = \begin{bmatrix} 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} .$$

Let $P$ and $R$ be Markov($p$) and Markov($r$) respectively, both starting at 1. Then $P$ and $R$ have the same null sets but the random time $\tau$ defined by

$$\tau = \begin{cases} \inf\{n \colon X_n = 2, X_{n+1} = 2\} & \text{if } X_1 = 2 \\ \inf\{n \colon X_n = 2, X_{n+1} = 3\} & \text{if } X_1 = 3 \end{cases}$$

is a birth time for $P$ only.

## 4. DEATH TIMES

Given a Markov(p) probability $P$ on $(\Omega,F)$ we now investigate death times for $P$, i.e. random times $\tau$ such that under $P$ the distribution of the pre-$\tau$ fragment $(X_0,\ldots,X_{\tau-1})$ is Markov with stationary sub-stochastic transition probabilities, or, what is the same, that the killed process $K_\tau = (X_0,\ldots,X_{\tau-1},\Delta,\Delta,\ldots)$ is Markov with stationary transition probabilities.

Let $\tau$ be a death time for $P$. If we let $J_+$ denote the essential range of the pre-$\tau$ path,

$$J_+ = \{x \in J\setminus\{\Delta\}\colon P(\tau > n,\ X_n = x) > 0 \text{ for some } n \in \mathbb{N}\} ,$$

then the transition probabilities $q(x,y)$ of the killed chain are well defined for all $x$, $y \in J_+ \cup \{\Delta\}$ and induce a family of probabilities $\{Q^x,\ x \in J_+ \cup \{\Delta\}\}$ on $\Omega$ which concentrate on paths $\omega$ which remain forever within this restricted state space $J_+ \cup \{\Delta\}$. Clearly if $P(\tau > 0,\ X_0 = x) > 0$ then $Q^x$ is identical to the $P^x$ distribution of $K_\tau$ given $(\tau > 0)$. This conclusion may be either false or meaningless if $P(\tau > 0,\ X_0 = x) = 0$, but we shall see that matters can be rectified by redefining $\tau(\omega)$ properly for paths $\omega$ starting at such points $x$.

We consider now the whole family of probabilities $\{P^x\}_{x\in J}$ where $P^x$ on $(\Omega,F)$ is Markov(p) with starting state $x$. For each random time $\tau$ we define $J_\tau \subset J$ by

$$J_\tau = \{x \in J\setminus\{\Delta\}\colon P^x(\tau > 0) > 0\} ,$$

and say that $\tau$ is a <u>death time for the family</u> $\{P^x\}_{x\in J}$ if there is a transition matrix $q$ on $J_\tau \cup \{\Delta\}$ such that for each $x \in J_\tau$ the $P^x$ distribution of $K_\tau$ given $(\tau > 0)$ is $Q^x$, where $Q^x$ on $(\Omega, F)$ is Markov($q$) with starting state $x$. In particular $Q^x$ concentrates on the set of sequences in $J_\tau \cup \{\Delta\}$.

Obviously every death time for the family $\{P^x\}_{x \in J}$ is a death time for each Markov($p$) probability $P$. Conversely we have

(4.1) Proposition. Let $\tau$ be a death time for a Markov($p$) probability $P$. Then there is a death time $\tau^*$ for the family $\{P^x\}_{x \in J}$ such that $\tau^*(\omega) = \tau(\omega)$ for all paths $\omega$ starting at points $x \in J$ with $P(\tau > 0, X_0 = x) > 0$.

Remark. If $P = P^y$ for a fixed state $y$ then $\tau^*$ and $\tau$ agree $P$ a.s.

Proof. As before let $J_+$ be the essential range of the pre-$\tau$ path under $P$, and suppose $K_\tau$ is Markov($q$) under $P$. For those paths $\omega$ starting at an $x \in J \backslash J_+$ we set $\tau^*(\omega) = 0$, while for those starting at an $x \in J_+$ with $P(\tau > 0, X_0 = x) > 0$ we put $\tau^*(\omega) = \tau(\omega)$. Finally, if $x \in J_+$ and $P(\tau > 0, X_0 = x) = 0$ we find an $m \geq 1$ and $x_0, \ldots, x_m \in J_+$ with $x_m = x$ such that $P[A(\tau > m)] > 0$ where $A = (X_0 = x_0, \ldots, X_m = x_m)$, and then for $\omega = (\omega_0, \omega_1, \ldots)$ with $\omega_0 = x$ define

$$\tau^*(\omega) = (\tau(x_0, \ldots, x_{m-1}, \omega_0, \omega_1, \ldots) - m)^+ .$$

Obviously $J_{\tau^*} = J_+$ and it remains only to check that the $P^x$ distribution of $K_{\tau^*}$ given $(\tau^* > 0)$ is $Q^x$ for $x \in J_+$ with $P(\tau > 0, X_0 = x) = 0$, where $Q^x$ is Markov($q$) starting at $x$. For this it suffices to show that for $n \in \mathbb{N}$, $B = (X_0 = y_0, \ldots, X_n = y_n)$, where $y_0 = x$, $y_1, \ldots, y_n \in J_+$, we have

$$P^x[B(\tau^* > n) \mid \tau^* > 0] = Q^x(B) ,$$

where $Q^x(B) = q(y_0,y_1)\cdots q(y_{n-1},y_n)$ by definition.

But let $m$ and $n$ be as above. Then by first using the Markov property of $P$ and then the fact that $\tau^*\circ\theta_m = (\tau-m)^+$ on $A$ we have

$$P^x[B(\tau^* > n)] = P[A(\theta_m \in B,\ \tau^*\circ\theta_m > n)]/P(A)$$
$$= P[A(\theta_m \in B,\ \tau > m+n)]/P(A)\ .$$

Similarly $P^x(\tau^* > 0) = P[A(\tau > n)]/P(A)$, whence

$$P^x[B(\tau^* > n)|\ \tau^* > 0] = P[A(\theta_m \in B,\ \tau > m+n)]/P[A(\tau > n)]$$

which equals $Q^x(B)$ because the $P$ distribution of $K_\tau$ is Markov(q).

This result reduces the problem of describing the death times for a given Markov probability $P$ to that of characterizing the death times for a family $\{P^x\}_{x\in J}$. Such a characterization is provided by the following proposition. Write $P(\circ|F_n)$ for any of the identical conditional probabilities $P^x(\circ|F_n)$ and given a random time $\tau$ introduce

$$f(x) = P^x(\tau > 0)\ ,\quad Z_n = P(\tau > n|\ F_n)\ ,\quad n \in N\ .$$

Note that $Z_0 = f(X_0)$ and recall that $J_\tau = \{x \in J\backslash\{\Delta\}:\ f(x) > 0\}$.

(4.2) Proposition. A random time $\tau$ is a death time for the family $\{P^x\}_{x\in J}$ if and only if for each $x \in J\backslash\{\Delta\}$, $m,n \in N$ the identity

$$Z_{m+n} = \begin{cases} Z_m Z_n\circ\theta_m/f(X_m) & \text{on } (X_m \in J_\tau) \\ 0 & \text{on } (X_m \notin J_\tau) \end{cases}$$

holds $P^x$ a.s.

Proof. With $Q^x$ the $P^x$ distribution of $K_\tau$ given $(\tau > 0)$, the condition that $\tau$ be a death time for $\{P^x\}$ is equivalent to requiring that for all $m, n, k \in N$, $x_0,\ldots,x_{m+n}$, $x \in J_\tau$,

$$(4.3)\quad Q^{x_0}(X_0=x_0,\ldots,X_{m+n}=x_{m+n}) = Q^{x_0}(X_0=x_0,\ldots,X_m=x_m)Q^{x_m}(X_0=x_m,\ldots,X_n=x_{m+n}) ,$$

$$(4.4)\qquad Q^{x_0}(X_k \notin J_\tau \cup \{\Delta\}) = 0 .$$

Introducing the atoms $A = (X_0=x_0,\ldots,X_m=x_m)$, $B = (X_0=x_m,\ldots,X_n=x_{m+n})$, it is seen that (4.3) and (4.4) are equivalent to

$$(4.5)\qquad P^{x_0}[A(\theta_m \in B,\ \tau > m+n)] = P^{x_0}[A(\tau > m)]P^{x_m}[B(\tau > n)]/f(x_m)$$

$$(4.6)\qquad P^{x_0}(X_k \notin J_\tau,\ \tau > k) = 0 .$$

But the left side of (4.5) equals

$$P^{x_0}[Z_{m+n};\ A(\theta_m \in B)]$$

while the right side becomes

$$P^{x_0}(Z_m;A)P^{x_m}(Z_n;B)/f(x_m) = P^{x_0}[Z_m Z_n \circ \theta_m/f(X_m);\ A(\theta_n \in B)] .$$

Since $A$ and $B$ are arbitrary atoms for paths within $J_\tau$ and since (4.6) is equivalent to demanding that $Z_k, Z_{k+1},\ldots$ vanish $P^x$ a.s. on $(X_k \notin J_\tau)$ the result follows.

We finish this section with an example of two Markov probabilities $P$ and $R$, with the same null sets, and a random time $\tau$ which is a death time for $P$ but not for $R$, thus proving that there exists no collection of random times $\mathcal{D}^*$ such that $\tau$ is a death time for a Markov probability $P$ if and only if

$\tau$ is $P$ a.s. equal to a random time in $\mathcal{D}^*$ (cf. Example (3.20)).

(4.7) Example. Ignoring the coffin state $\Delta$ let $J = \{1,2,3,4\}$. Define transition matrices $p$ and $r$ on $J$ by

$$p = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad r = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $P$ and $R$ be Markov($p$) and Markov($r$) respectively, both starting at 1. Clearly $P$ and $R$ have the same null sets. Define $\rho$ as the time of entry into $\{2,3\}$, $\sigma$ as the time to absorption in state 4, put $\delta = \sigma - \rho$ and finally let $\tau$ be the minimum of $\rho$ and $\delta$.

Under both $P$ and $R$ the random times $\rho$ and $\delta$ are independent; under $P$ they both have a geometric distribution, whence so too does $\tau$, but under $R$ the random time $\rho$ is geometric while $\delta$ is not, and thus $\tau$ is not geometric either. Since the pre-$\tau$ process never leaves state 1 it is therefore Markov under $P$ but not under $R$.

## 5. REGULAR DEATH TIMES

In this section we consider death times $\tau$ for a Markov probability $P$ on $(\Omega, F)$ with the additional property, discussed in the introduction, that conditional on the final position $X_{\tau-1}$ of the pre-$\tau$ process on $(0 < \tau < \infty)$ the pre-$\tau$ and post-$\tau$ processes are independent. Such a death time for $P$ will be referred to as regular.

(5.1) Definition. Let $\mathcal{D}$ denote the class of all random times $\tau$ of the form

$$\tau = \tau_{V,F} = \sup\{n\colon 1 \le n \le \tau_V,\ \theta_{n-1} \in F\}$$

for some $V \subset J \times J$, $F \in \mathcal{F}$, where $\tau_V$ is the terminal time associated with $V$:

$$\tau_V = \inf\{n\colon n \ge 1,\ (X_{n-1}, X_n) \in V\} .$$

Our main result is that the random times in $\mathcal{D}$ form a complete canonical collection of regular death times for each Markov probability $P^x$ on $(\Omega, \mathcal{F})$ with a fixed starting state $x$. For random starts $\mathcal{D}$ must be enlarged to include all times of the form $\tau 1(X_0 \in H)$ for $H \subset J$, but we shall ignore this trivial complication.

(5.2) Theorem. A random time $\tau$ is a regular death time for a Markov probability $P^x$ with starting state $x$ in $J \backslash \{\Delta\}$ if and only if $\tau$ is $P^x$-equivalent to a random time in $\mathcal{D}$.

The proof of this theorem will be taken up later in the section. We mention first some facts about the collection $\mathcal{D}$ which derive from the following result.

(5.3) Proposition. The random times in $\mathcal{D}$ are characterized by the following two structural properties: for every $n \in \mathbb{N}$

(i) $\qquad (\tau > n) = F_n(\tau \circ \theta_n > 0)$ for some $F_n \in \mathcal{F}_n$ ,

(ii) $\qquad \tau \circ \theta_n = \tau - n$ on $(\tau > n)$ .

Indeed, if $\tau$ satisfies (i) and (ii), then $\tau = \tau_{V,F}$ with $F = (\tau > 0)$ and $V$ such that $F_1 = ((X_0, X_1) \in V^c)$, and then

$$F_n = (\tau_V > n) = ((X_k, X_{k+1}) \in V^c,\ 0 \le k \le n-1) .$$

Proof. See Lemma (5.4) below. The present criterion is just a restatement of the criterion (a) of that lemma.

We note that for an optional time $\tau$ we have $(\tau > n) \in \mathcal{F}_n$ so that the two conditions (i) and (ii) above collapse to (ii) alone. Since an optional time satisfying (ii) is by definition a terminal time, we find that an optional time is in $\mathcal{D}$ if and only if it is a terminal time. Furthermore it is plain from the original definition of $\mathcal{D}$, that $\tau$ is a terminal time if and only if $\tau$ is the first time that the path either enters $H$ or completes a jump in $V$ for some $H \subset J$, $V \subset J \times J$ (cf. Walsh and Weil [7]).

For a cooptional time we have by definition that $\tau \circ \theta_n = (\tau - n)^+$, and since this obviously implies both (i) and (ii) we see that $\mathcal{D}$ includes all cooptional times. We find also that every cooptional time $\tau$ can be represented as $\tau = \sup\{n\colon n \ge 1,\ \theta_{n-1} \in F\}$ where $F = (\tau > 0)$, and since for cooptional $\tau$ we must have $(\tau > 0) = (\theta \in G)$ for some $G \in \mathcal{F}$ we deduce that $\tau$ is a cooptional time if and only if for some $G \in \mathcal{F}$

$$\tau = \sup\{n\colon n \in \mathbb{N},\ \theta_n \in G\} .$$

Finally we mention two important closure properties of $\mathcal{D}$ which are readily checked using Proposition (5.3): if $\sigma$ and $\tau$ are two random times in $\mathcal{D}$, then so are (i) their minimum $\sigma \wedge \tau$, and (ii) the random time $\sigma \circ K_\tau$ obtained by applying $\sigma$ after first killing at $\tau$, provided $\sigma \le \inf\{n \in \mathbb{N}\colon X_n = \Delta\}$.

(5.4) Lemma. Each of the following conditions (a) and (b) is necessary and sufficient for a random time $\tau$ to belong to $\mathcal{D}$:

(a) For each $m \in \mathbb{N}$ there is an $F_m \in \mathcal{F}_m$ such that

$$(\tau > m+n) = F_m(\tau\circ\theta_m > n) , \qquad n \in \mathbb{N} .$$

(b) There is an $F_1 \in \mathcal{F}_1$ such that for $\bar{F}_n = F_1(\theta_1 \in F_1)\cdots(\theta_{n-1} \in F_1)$ we have

$$\begin{array}{lll} \text{(i)} & (\tau = n+1) = \bar{F}_n(\tau\circ\theta_n = 1) , & \\ & & n \geq 1 . \\ \text{(ii)} & (\tau = \infty) = \bar{F}_n(\tau\circ\theta_n = \infty) , & \end{array}$$

Proof. If $\tau = \tau_{V,F} \in \mathcal{D}$ then (a) is immediate with $F_m = (\tau_V > m)$. Conversely, if $\tau$ satisfies (a) we will show that $\tau = \tau_{V,F}$ with $F = (\tau > 0)$ and the $V$ defined by $F_1 = ((X_0,X_1) \in V^c)$. As a preliminary we claim that for $m \geq 1$ the $F_m$ appearing in (a) can be replaced by the $\bar{F}_m$ determined from $F_1$ as in (b). This is trivial for $m = 1$, while if true for $m$ and all $n$ we find that

$$\begin{aligned}(\tau > (m+1) + n) = (\tau > m + (n+1)) &= \bar{F}_m(\tau\circ\theta_m > n+1) \\ &= \bar{F}_m[\theta_m \in F_1(\tau\circ\theta > n)] = \bar{F}_{m+1}(\tau\circ\theta_{m+1} > n) ,\end{aligned}$$

establishing the fact for $m+1$, and the assertion is verified. To show now that $\tau = \tau_{V,F}$ for the given $V$ and $F$ it suffices to check that $(\tau_{V,F} > m) = (\tau > m)$ for $m \in \mathbb{N}$, i.e. that

$$\bigcup_{n \geq m} (\tau_V > n,\ \theta_n \in F) = \bar{F}_m(\tau\circ\theta_m > 0) .$$

But since $\bar{F}_m = (\tau_V > m)$ and $(\tau\circ\theta_m > 0) = (\theta_m \in F)$, the right hand event is identical to $(\tau_V > m,\ \theta_m \in F)$, and we must therefore show that for each $n \geq m$ this event contains the event

(5.5) $$(\tau_V > n,\ \theta_n \in F) = (\tau_V > m,\ \tau_V \circ \theta_m > n-m,\ \theta_n \in F)\ .$$

But because $(\tau > 0) \supset (\tau > 1) = F_1(\tau \circ \theta > 0)$, we find by induction that $(\tau > 0) \supset \bar{F}_k(\tau \circ \theta_k > 0) = (\tau_V > k,\ \theta_k \in F)$ and using this with $k = n-m$ in (5.5) yields the desired conclusion.

For the second characterization (b) it is obvious that if $\tau = \tau_{V,F}$ is in $\mathcal{D}$, then both (i) and (ii) hold with $F_1 = ((X_0,X_1) \in V^c)$, while if these identities obtain it is easy to check that (a) holds with $F_m = \bar{F}_m$.

The same proofs show that if $\Omega^* \subset \Omega$ is invariant under $\theta$, i.e. if $\Omega^* \subset (\theta \in \Omega^*)$, then knowing that either (5.4)(a) or (5.4)(b) holds just on $\Omega^*$, e.g. that

$$(\tau = n+1)\Omega^* = \bar{F}_n(\tau \circ \theta_n = 1)\Omega^*\ , \qquad (\tau = \infty)\Omega^* = \bar{F}_n(\tau \circ \theta_n = \infty)\Omega^*\ ,$$

we can deduce that $\tau = \tau_{V,F}$ on $\Omega^*$ for some $V \subset J \times J$, $F \in \mathcal{F}$.

We also note that an obvious modification of Lemma (3.12) tells us that for a random time $\tau$ and a Markov probability $P$ the pre-$\tau$ and post-$\tau$ processes are conditionally independent given $X_{\tau-1}$ and the event $(0 < \tau < \infty)$ if and only if for every $n \in \mathbb{N}$ there exists $F_n \in \mathcal{F}_n$, $G \in \mathcal{F}$ such that

(5.6) $$(\tau = n+1) = F_n(\theta_n \in G) \qquad P \text{ a.s.}\ .$$

<u>Proof of Theorem (5.2)</u>. Fix $x \in J\backslash\{\Delta\}$. Assuming first that $\tau = \tau_{V,F}$ $P^x$ a.s., it follows from (5.4)(a) that for $Z_n = P(\tau_{V,F} > n \mid \mathcal{F}_n)$ we have $Z_{m+n} = 1_{F_m} Z_n \circ \theta_m$ a.s. where here and throughout this proof 'a.s.' on its own means '$P^y$ a.s. for all $y \in J\backslash\{\Delta\}$'. But Proposition (4.2) is then easily applied to show that $\tau_{V,F}$ is a death time for $P^x$, and hence so too is $\tau$. Since (5.4)(b) applied to $\tau_{V,F}$ shows that (5.6) holds with $G = (\tau_{V,F} = 1)$ it follows that $\tau$ is a regular death time for $P^x$.

Now assume conversely that $\tau$ is a regular death time for $P^x$. In particular by (5.6), $(\tau = n+1) = F_n(\theta_n \in G)$ $P^x$ a.s. for some $F_n \in \mathcal{F}_n$, $G \in \mathcal{F}$. Replacing $\tau$ by the $\tau^*$ constructed in Proposition (4.1) we get a random time which is $P^x$ a.s. equal to $\tau$ and which is a death time for the family $\{P^y\}_{y \in J}$. Furthermore, it is not difficult to verify that $(\tau^* = n+1) = F_n^*(\theta_n \in G)$ a.s., $n \in \mathbb{N}$, where $F_n^*$ is the union of the events $F_{n,y}$ over $y$ in $J_{\tau^*}$, with $F_{n,y}$ the section of $F_{m+n}$ beyond the atom $A_y = (X_0 = x_0, \ldots, X_m = y)$ used to define $\tau^*$ on paths starting at $y \in J_{\tau^*}$ (see (2.7) and the proof of (4.1)). Thus we may as well drop the stars and take it from the start that $\tau$ is such that there are events $F_n \in \mathcal{F}_n$, $G \in \mathcal{F}$ with

$$(5.7) \qquad (\tau = n+1) = F_n(\theta_n \in G) \quad \text{a.s.}$$

We now aim to show that such a $\tau$ satisfies (i) and (ii) of (5.4)(b) a.s. Both in this and later arguments there will be much tacit use of the fact that if two events $F_1$ and $F_2$ are a.s. equal then so too are $(\theta_n \in F_1)$ and $(\theta_n \in F_2)$ for each $n \in \mathbb{N}$. Writing

$$M = \{y \in J \setminus \{\Delta\} : P^y(\tau = 1) > 0\}$$

we first claim that

$$(5.8) \qquad (\tau = n+1,\ X_n \in M) = F_n(X_n \in M,\ \tau \circ \theta_n = 1) \quad \text{a.s.}$$

To see this observe that by (5.7), $(X_0 \in M) = F_0(P^{X(0)}G > 0)$ so that also $(X_n \in M) = (\theta_n \in F_0,\ P^{X(n)}G > 0)$. But then, again using (5.7)

$$(\tau = n+1,\ X_n \in M) = F_n(\theta_n \in F_0 G,\ P^{X(n)}G > 0) \quad \text{a.s.}$$

and (5.8) follows. Introducing now

$$f(y) = P^y(\tau > 0) \ , \quad Z_n = P(\tau > n | F_n) \ , \quad W_n = P(\tau = n+1 | F_n) \ ,$$

it follows from Proposition (4.2) that

$$Z_n = W_n + Z_n[f(X_n)]^{-1} 1(X_n \in J_\tau) P^{X(n)}(\tau > 1) \quad \text{a.s.}$$

so that because $(Z_n > 0) \subset (X_n \in J_\tau)$,

(5.9) $$(W_n > 0) = (Z_n > 0, \ X_n \in M) \quad \text{a.s.}$$

Because $(\tau = n+1) \subset (W_n > 0)$ we find in particular that $(\tau = n+1) \subset (X_n \in M)$ so that (5.8) may be written

$$(\tau = n+1) = F_n(X_n \in M, \ \tau \circ \theta_n = 1) \quad \text{a.s.}$$

whence $(W_n > 0) = F_n(X_n \in M)$, and comparing with (5.9) it develops that

$$(\tau = n+1) = (W_n > 0, \ \tau \circ \theta_n = 1) = (Z_n > 0, \ X_n \in M, \ \tau \circ \theta_n = 1) \quad \text{a.s.}$$

Since $(\tau \circ \theta_n = 1) \subset (X_n \in M)$ and $\tau$ is a death time for $\{P^x\}_{x \in J}$, Proposition (4.2) implies that (i) of (5.4)(b) holds a.s. with $F_1 = (Z_1 > 0)$. To establish (ii) of (5.4)(b) observe that by the martingale convergence theorem $1(\tau = \infty) = \lim_{m \uparrow \infty} Z_m$ a.s. But because of Proposition (4.2) that limit equals

$$Z_n[f(X_n)]^{-1} 1(X_n \in J_\tau) \lim_m Z_m \circ \theta_m = Z_n[f(X_n)]^{-1} 1(X_n \in J_\tau, \ \tau \circ \theta_n = \infty) \quad \text{a.s.}$$

for every $n \in N$, and the desired conclusion that $(\tau = \infty) = (Z_n > 0, \ \tau \circ \theta_n = \infty)$ a.s. is now immediate.

Thus, for each $y \in J \setminus \{\Delta\}$, ignoring a $P^y$-null set $L_y$, we have that $\tau$ satisfies the identities of (5.4)(b) for all $n$ with $F_1 = (Z_1 > 0)$. But the set $L = \bigcup_{y \in J \setminus \{\Delta\}} (X_0 = y) L_y$ is a null set for all $P^y$, $y \in J \setminus \{\Delta\}$ simultaneously and consequently, defining $\Omega^* = (\bigcup_{n \in \mathbb{N}} (\theta_n \in L))^c$ we see that for $n \in \mathbb{N}$ the identities

$$(\tau = n+1)\Omega^* = (Z_1 > 0, \ldots, Z_1 \circ \theta_{n-1} > 0, \ \tau \circ \theta_n = 1)\Omega^* ,$$
$$(\tau = \infty)\Omega^* = (Z_1 > 0, \ldots, Z_1 \circ \theta_{n-1} > 0, \ \tau \circ \theta_n = \infty)\Omega^* ,$$

hold exactly, and since $\Omega^*$ is invariant under $\theta$, the remark following Lemma (5.4) shows that on $\Omega^*$, $\tau$ is of the form $\tau_{V,F}$ for some $V \subset J \times J$, $F \in F$. Since also $P^y(\Omega^*) = 1$ for every $y$, in particular for $y = x$, we conclude that $\tau$ is $P^x$ a.s. equal to some $\tau_{V,F}$, and the proof is complete.

As a final comment on death times, it may be observed that there exist random times $\tau$ which are death times for all Markov probabilities simultaneously without being regular. A simple example is obtained by taking an integer $a \geq 2$ and defining

$$(\tau > 0) = \Omega , \quad (\tau > n) = (X_0 = \cdots = X_{an}) , \quad n \geq 1 .$$

## 6. BIRTH AND DEATH TIMES

We now consider random times which are both regular death times and regular birth times for each Markov(p) probability P. For such a random time $\tau$ it is seen that a path decomposition specifying the joint distribution of the pre-$\tau$ and post-$\tau$ processes can be given in terms of just four quantities determined by $\tau$ and $p$, namely the function $f: J \to [0,1]$ and the three transition matrices $q$, $r$ and $s$ on $J$ such that under the probability $P^x$ on $(\Omega, F)$ which makes $(X_n)$ Markov(p) starting at $x$ we have

(6.1) (i) $P^x(\tau > 0) = f(x)$, $x \in J$.

(ii) Conditional on $\tau > 0$ the $P^x$ distribution of the pre-$\tau$ process is Markov(q) starting at $x$,

(iii) Conditional on $0 < \tau < \infty$ and a pre-$\tau$ path with $X_{\tau-1} = y$ the $P^x$ distribution of $X_\tau$ is $r(y,\cdot)$,

(iv) Conditional on $\tau < \infty$, the pre-$\tau$ path and $X_\tau = z$, the $P^x$ distribution of the post-$\tau$ process is Markov(s) starting at $z$.

With $f$, $q$, $r$ and $s$ specified by (6.1) the path decomposition involved in (6.1) can be expressed more intuitively by saying that the following probabilistic motion describes a Markov chain with stationary transition probabilities $p$: Start at $x$, and then with probability $1 - f(x)$ move off according to a Markov chain with transition probabilities $q$; when (if ever) this chain dies, look back at the position $y$ where the chain was at the instant before it died and instead of dying make a single transition according to $r(y,\cdot)$; if this gets you to state $z$ (where $z = x$ if there was no motion according to $q$) complete the motion by moving forevermore according to a Markov chain with transition probabilities $s$ starting at $z$.

There are two basic kinds of random times which induce a path decomposition of this kind: terminal times and coterminal times. We first indicate how the parameters $f$, $q$, $r$, and $s$ are obtained for these times, and then show that for nice transition matrices $p$ these are essentially the only random times inducing such a path decomposition.

Suppose first that $\tau$ is a terminal time. Then as mentioned below (5.3) there is a subset $H$ of $J$ and a subset $V$ of $H \times H$ such that

$$(\tau > 0) = (X_0 \in H) \;, \quad (\tau > n) = ((X_k, X_{k+1}) \in V,\ 0 \le k \le n-1) \;, \quad n \ge 1 \;.$$

With $H$ and $V$ so defined it easily is checked that the parameters $f$, $q$, $r$ and $s$ are given by

$$f(x) = 1_H(x) \;,\quad x \in J \;,$$

$$q(x,y) = \begin{cases} p(x,y)1_V(x,y) & \text{if } y \neq \Delta \\ 1 - \sum\limits_{z \neq \Delta} q(x,z) & \text{if } y = \Delta \end{cases}$$

$$r(x,y) = \begin{cases} p(x,y)1_{V^c}(x,y)/\sum\limits_z p(x,z)1_{V^c}(x,z) & \text{if } x \in H \\ \text{arbitrary} & \text{if } x \in H^c \;, \end{cases}$$

$$s(x,y) = p(x,y) \;,$$

where $1_B$ stands for the indicator function of a subset $B$ of either $J$ or $J \times J$.

For $\tau$ a coterminal time there is a subset $V$ of $J \times J$ and an invariant event $C_\infty$ such that

$$(\tau \leq n) = (\theta_n \in C) \;, \quad n \in \mathbb{N} \;,$$

where $C$ is the coterminal event $((X_k,X_{k+1}) \in V,\ k \in \mathbb{N})C_\infty$. Define functions $f$, $\bar{f}$, $g$ and $h$ from $J$ to $[0,1]$ by setting for $x \in J$

$$f(x) = P^x(\tau > 0) = P^x(C^c)$$
$$\bar{f}(x) = P^x(\tau = 0) = P^x(C)$$
$$g(x) = P^x(\tau = 1) = P^x[(X_0,X_1) \in V^c,\ \theta_1 \in C]$$
$$h(x) = P^x(\tau = \infty) = P^x[((X_n,X_{n+1}) \in V^c \text{ infinitely often}) \cup C_\infty^c] \;.$$

Then $f$, $\bar{f}$ and $g$ are related by the identities $f + \bar{f} = 1$ and

$$g(x) = \sum_y p(x,y)1_{V^c}(x,y)f(y) \;, \quad x \in J \;,$$

and it may also be observed that $f$ is p-excessive, that $h$ is p-harmonic, and that the Riesz decomposition of $f$ is $f = Ug + h$ where $U = \sum_{n=0}^{\infty} p^n$ is the potential operator associated with $p$. The parameters $\bar{f}$, $q$, $r$ and $s$ for the path decomposition are now readily seen to be specified as follows: $\bar{f}$ has already been defined,

$$q(x,y) = \begin{cases} p(x,y)f(y)/f(x) & \text{if } f(y) > 0 \ , \quad y \neq \Delta \ , \\ 0 & \text{if } f(y) = 0 \ , \quad y \neq \Delta \ , \\ 1 - \sum_{z \neq \Delta} q(x,z) & \text{if } y = \Delta \ , \end{cases}$$

$$r(x,y) = \begin{cases} p(x,y)1_{V^c}(x,y)\bar{f}(y) & \text{if } g(y) > 0 \ , \\ \text{arbitrary} & \text{if } g(y) = 0 \ , \end{cases}$$

$$s(x,y) = \begin{cases} p(x,y)1_V(x,y)\bar{f}(y)/\bar{f}(x) & \text{if } \bar{f}(x) > 0 \ , \\ \text{arbitrary} & \text{if } \bar{f}(x) = 0 \ . \end{cases}$$

In this case parts (i), (ii) and (iv) of the path decomposition statement (6.1) are the discrete analogues of Theorems (2.1) and (5.1) of Meyer, Smythe and Walsh [3]. Part (iii) provides the inner link between the pre-$\tau$ and post-$\tau$ processes which is required for the full statement of the path decomposition.

We now establish a characterization of terminal and coterminal times by the path decomposition (6.1).

(6.2) <u>Theorem</u>. Let the Markov probabilities $\{P^x, x \in J\}$ be induced by a transition matrix $p$ on $J$ which makes all states in $J \setminus \{\Delta\}$ form a single closed communicating class. Then a random time $\tau$ is both a regular birth time and a regular death time for $P^x$ if and only if $\tau$ is $P^x$-equivalent to a random time which is either terminal or coterminal.

Remark. The characterization fails without some hypothesis on $p$. For instance if $p$ induces two closed communicating classes $A$ and $B$ and a transient state $x$ from which absorption into either $A$ or $B$ is possible, then $\tau$ could be equal to a terminal time on paths entering $A$ and to a coterminal time on paths entering $B$.

Proof. The 'if' part is contained in Theorems (3.9) and (5.2). For the 'only if' part observe first that if $\tau$ is a regular birth time and a regular death time for $P^x$, then by the same results

(6.3) $$\tau = \tau_C + \rho \qquad P^x \text{ a.s. },$$

(6.4) $$\tau = \tau_{V,F} \qquad P^x \text{ a.s.}$$

where $\tau_C$ is a coterminal time associated with some coterminal event $C$, $\rho$ is an optional time for $\{F_{\tau_C+n}, n \geq 0\}$ and $\tau_{V,F}$ is given as in (5.1) for some $V \subset J \times J$, $F \in \mathcal{F}$. In particular, if $\tau_V \geq 1$ is the terminal time associated with $V$, we have

(6.5) $$\tau_C \leq \tau \leq \tau_V \qquad P^x \text{ a.s.}$$

Now $C$ is the intersection of an invariant event $C_\infty$ with an event requiring that all transitions belong to some subset, $V'^c$ say, of $J \times J$. By (6.5) therefore $\tau_C = \tau = \tau_V = \infty$, $P^x$ a.s. on $C_\infty^c$ so that $C_\infty^c \subset ((X_n, X_{n+1}) \in V^c, n \in \mathbb{N})$, $P^x$ a.s., which because the states form one communicating class is possible only if either $P^x C_\infty^c = 0$ or transitions in $V$ are impossible, i.e. we must have that either $\tau_C$ equals $P^x$ a.s. the last time $\sigma_{V'}$ a transition in $V'$ is completed or else $\tau_V = \infty$, $P^x$ a.s. But in the first case the inequality $\sigma_{V'} \leq \tau_V$ shows that it is impossible to perform a transition in $V'$ after a transition in $V$, and again because we have one communicating class, this means that either $V'$ or $V$ ocnsists of transitions which are impossible under $p$. Thus either $\tau_C = 0$, $P^X$ a.s. or $\tau_V = \infty$, $P^X$ a.s. and by (6.3), (6.4) this means that either there is an optional time $\rho$ which equals $\tau$, $P^X$ a.s., or there is an event $F \in F$ such that $\tau = \tau_{\emptyset,F} = \sup\{n \geq 1: \theta_{n-1} \in F\}$, $P^X$ a.s.

We have already seen that if, e.g. $\tau$ is itself an optional time satisfying (6.4) exactly, then $\tau$ is a terminal time. If we only know that $\tau = \rho$, $P^X$ a.s. with $\rho$ optional it follows that $P^X(\tau > n | F_n) = 1_{(\tau > n)}$, $P^X$ a.s. while because of (6.4) the same conditional probability equals $1_{(\tau_V > n)} P^{X(n)}(\tau_{V,F} > 0)$. But then

$$(\tau > n) = (\tau_V > n,\ P^{X(m)}(\tau_{V,F} > 0) > 0,\ 0 \leq m \leq n) \quad P^X \text{ a.s.}$$

showing that $\tau$ is $P^X$-equivalent to a terminal time.

Similarly, if $\tau = \tau_{\emptyset,F}$ $P^X$ a.s. we have $P^X(\tau \leq n | F_n) = P^{X(n)}(\tau_{\emptyset,F} = 0)$ while because of (6.3), $(\tau \leq n) = F_n(\theta_n \in C)$ for some $F_n \in F_n$ so that the conditional probability equals $1_{F_n} P^{X(n)}C$ and consequently $F_n(P^{X(n)}C > 0) = (P^{X(n)}(\tau_{\emptyset,F} = 0) > 0)$ $P^X$ a.s. whence

$$(\tau \leq n) = F_n(\theta_n \in C,\ P^{X(n)}C > 0) = (\theta_n \in C,\ P^{X(m)}(\tau_{\emptyset,F} = 0) > 0,\ m \geq n) \quad P^X \text{ a.s.},$$

showing that $\tau$ is $P^X$-equivalent to a coterminal time. The theorem is proved.

## 7. POSSIBLE EXTENSIONS

Though we have restricted ourselves in this paper to Markov processes in discrete time with countable state space, the concepts of birth times, death times, and conditional independence times can all be formulated for Markov processes with more general time set or state space. We conclude in this section with some comments on the difficulties involved in extending our results to apply to these situations.

A few of the results do carry over to apply to Markov processes with abstract measurable state space and time set $T$ either $N$ or $[0,\infty)$. Copying the definition from Section 4 of a death time for a family of Markov probabilities, a generalization of Proposition (4.2) remains valid, and by adopting an analogous definition of a conditioning event for a family rather than a single Markov probability, it can be shown that $F$ is such an event if and only if there are events $C \in F$, $F_t \in F_t$, $t \in T$ such that $F = F_t(\theta_t \in C)$, $P^x$ a.s. for every $x$ and every $t \geq 0$ (cf. (2.10) and Jacobsen [2], Lemma 1).

As for our other results, the assumption of a countable state space is used chiefly to avoid measure theoretical problems in the proofs of the harder 'only if' assertions of Theorems (2.3), (3.9) and (5.2), while the restriction to discrete time is essential for our treatment of conditional independence across a random time. The basic criterion for deciding whether a random time possesses the conditional independence property is Lemma (3.12), the proof of which relies on the fact that within the set $A(\tau = n)$, where $A$ is an arbitrary atom in $F_n$, the conditional probability law of $\theta_\tau$ given the pre-$\tau$ field may be determined as a conditional $P^y$-probability given the event $G_{nA}$ which is the section of $(\tau = n)$ beyond $A$. A generalization of this to processes in continuous time fails, partly because the conditioning event $G_{nA}$ may now have measure $0$ for more than a negligible collection of atoms $A$, and partly because, even when this is not the case, it is no longer obvious that the desired conditional probability given $F_\tau$ results. For some criteria for conditional independence and some of the subtleties involved see Jacobsen [2], Pittenger and Shih [6].

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