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ABSTRACT: A general method is developed with which various limit theorems are proven for a branching random walk under virtually minimal assumptions. Analogous results are obtained in continuous time.

1. INTRODUCTION

Consider a branching random walk, i.e. a Galton-Watson process $Z_0, Z_1, \ldots$ with offspring distribution $\{p_j\}$ on which we superimpose the additional structure of random walk on the line. A particle whose parent is at x, moves to $x+y$ and the y's of different particles are i.i.d. with common distribution function G. Let $\mu = \int_{-\infty}^{\infty} x \, dG(x)$, $\sigma^2 = \int_{-\infty}^{\infty} x^2 \, dG(x)$ and define for any Borel set B $Z_n(B)$ = the number of particles in the n^{th} generation located in B. It was first conjectured by Harris ([9], pg. 75), that in the supercritical case $m = \Sigma j p_j > 1$, $m^{-n} Z_n(]-\infty, n\mu + y\sigma\sqrt{n}])$ should converge in probability for any y to $\Phi(y)W$, where as usual $\Phi(y) = \int_{-\infty}^{y} 1/\sqrt{2\pi} \, e^{-x^2/2} \, dx$, $W = \lim_n m^{-n} Z_n$. Since then a number of papers ([10], [14], [15], [19]) have appeared resolving this conjecture and generalizing it considerably.

In an earlier paper ([1]) the above problem and some of its generalizations and related problems was studied from the L^2 point of view. In this work we present a different technique, which is useful in attacking Harris' conjecture and related problems. In particular we are able to settle Harris' conjecture assuming only that the underlying Galton-Watson process satisfies the well-known 'j log j'-condition, i.e. $\Sigma j \log j \, p_j < \infty$, and even to get a.s. convergence under only slightly stronger conditions. Also, our technique works equally well in various related situations. To demonstrate this, we give a local limit theorem for branching random walks and a limit result in continuous time for a branching diffusion, i.e. a Bellman-Harris process, where the particles move independently according to standard Brownian motion. The results will be fully stated in Section 2 and the set-up more fully explained in Sections 2 and 3. Section 3 also outlines the method of proof, while Section 4 gives the details in discrete time and Section 5 in continuous time.

2. STATEMENT OF RESULTS

For the branching random walk, we make the following assumptions throughout:

(1) $p_0 = 0,\ 1 < m = \Sigma\, j\, p_j < \infty,\ \Sigma\, j \log j\, p_j < \infty.$

(2) $\int_{-\infty}^{\infty} x\, dG(x) = 0, \qquad \int_{-\infty}^{\infty} x^2\, dG(x) = 1.$

(3) $Z_0 = 1$, and its position is at 0.

The assumption $p_0 = 0$ is not crucial but convenient since otherwise one has to keep qualifying 'on the set of explosion'. Also, (2) is just an appropriate scaling and (3) is no restriction in view of the additivity and translation properties of the process.

Theorem 1. Let $y_n = \sqrt{n}\, y + o(\sqrt{n})$. Then under the assumptions (1)-(3)

(4) $$m^{-n} Z_n(]-\infty, y_n]) \xrightarrow{P} \Phi(y) W.$$

If furthermore $\sum_{j=2}^{\infty} j(\log j)^{1+\delta} p_j < \infty$ for some $\delta > 0$, then the convergence in (4) holds a.s. .

Theorem 2. Assume in addition to (1)-(3) that G is non-lattice, $\int_{-\infty}^{\infty} |x|^3\, dG(x) < \infty$ and that $\sum_{j=2}^{\infty} j(\log j)^{3/2+\delta} p_j < \infty$ for some $\delta > 0$. Then for $-\infty < a < b < \infty$.

(5) $$\sqrt{2\pi n}\, m^{-n} Z_n([a,b]) \xrightarrow{a.s.} (b-a)W.$$

The assumptions on G are needed only so that an appropriate form of the central limit theorem can be applied.

Our assumptions on $\{p_j\}$ are much weaker than any we know of and not far from being the weakest one could hope for. E.g. in Theorem 1 the case $\Sigma j \log j\, p_j = +\infty$ is not really interesting since then a different norming is needed.

We consider now the continuous-time case. Let $\{Z_t\}_{t\geq 0}$ denote a Bellman-Harris process evolving from one particle at age 0 at $t = 0$. A particle whose parent was at x at its time of birth moves until it dies according to a standard Brownian motion starting at x. The motions of different particles are assumed independent and as above, $Z_t(B)$ denotes the number of particles at time t in B (a Borel set).

Let $\{p_j\}$ be the offspring distribution and F the lifetime distribution. We assume that (1) holds for $\{p_j\}$ and as usual

(6) $$F \text{ is non-lattice and } F(0) = 0.$$

The Malthusian parameter α is defined as the unique solution at the equation $m\int_0^\infty e^{-\alpha y}\, dF(y) = 1$. It has recently been proved ([3]), that under (1) and (6) $W = \lim_t e^{-\alpha t} Z_t$ exists a.s. and $P(W > 0) > 0$.

<u>Theorem 3. Assume in addition to (1) and (6) that</u> $\sum_{j=2}^{\infty} j^2 p_j < \infty$. <u>Then for any Borel set B with $|\partial B| = 0$</u>

(7) $$e^{-\alpha t} Z_t(\sqrt{t}\, B) \xrightarrow{a.s.} \Phi(B)W$$

and for any bounded Borel set B with $|\partial B| = 0$

(8) $$\sqrt{2\pi t}\; e^{-\alpha t}\; Z_t(B) \xrightarrow{a.s.} |B|\, W$$

(Here $\sqrt{t}\, B = \{\sqrt{t}\, b \mid b \in B\}$, $\Phi(B) = \int_B 1/\sqrt{2\pi}\; e^{-x^2/2} dx$, $|B| = \int_B dx =$ the Lebesgue measure of B, $\partial B =$ the boundary of B).

Though the assumptions on B are slightly different formulated, (7) and (8) are close analogs to Theorems 1 and 2. This follows by standard facts on weak convergence. For a branching diffusion, it is possible to some extent to dispense with the hypothesis $|\partial B| = 0$ and also the second moment hypothesis on $\{p_j\}$ may be weakened to conditions corresponding to those of the discrete case for a large class of lifetime distributions containing e.g. the exponential distribution. For partial results in these directions, see Lemmas 5 and 10 of Section 5. Several generalizations of Theorem 3 suggest themselves. Thus one would expect the proportion of particles in B, which are of age at most x, to tend to $A(x)$, where A is the stable age-distribution ([3]). We have formulated our lemmas in sufficient generality to deal with this case as well, while we feel that the details of the argument is a straightforward combination of the methods of the present paper and of [3]. Also, there is no difficulty in modifying the brownian motion, say by considering a more general stable process or by allowing an absorbing barrier. Processes of this last type were studied in [5], [16], [17], [20] with methods which required a Markovian structure of $\{Z_t\}_{t \geq 0}$.

3. Preliminaries and outline of proofs

We now return to discrete time. Following the notation in [9], Ch. 5, we denote any particle in the n^{th} generation by $\underset{\sim}{i}_n = \langle i_1 i_2 \ldots i_n \rangle$. Let $Z_n(\underset{\sim}{i}_k)$ denote the number of descendants of $\underset{\sim}{i}_k$ at time $n \geq k$ and $X_{\underset{\sim}{i}_n}$ the position of $\underset{\sim}{i}_n$. Then

$$X_{\underset{\sim}{i}_n} = X_{\langle i_1 i_2 \ldots i_{n-1} \rangle} + Y_{\underset{\sim}{i}_n}$$

where $Y_{\underset{\sim}{i}_n}$ is the displacement of $\underset{\sim}{i}_n$. Thus the $Y_{\underset{\sim}{i}_n}$'s are i.i.d. with law G. For $n \geq 1$, let $\mathcal{F}_n = \sigma(Z_\ell(\underset{\sim}{i}_k); k \leq \ell \leq n)$ and $\mathcal{D}_n = \sigma(Z_\ell(\underset{\sim}{i}_k), Y_{\underset{\sim}{i}_k}; k \leq \ell \leq n)$. A key observation is, that under (3) the law of $X_{\underset{\sim}{i}_n}$ conditioned upon $\mathcal{F}_n$ is G_n, the n^{th} convolution of G.

We introduce some more notation . $Z_n(\underset{\sim}{i}_k;B)$ $(k \leq n)$ denotes the number of descendants of $\underset{\sim}{i}_k$ at time n in B and we let $W_n(\underset{\sim}{i}_k) = m^{-(n-k)} Z_n(\underset{\sim}{i}_k)$, and $W_n(\underset{\sim}{i}_k;B) = m^{-(n-k)} Z_n(\underset{\sim}{i}_k;B)$.

It follows from the additivity property of the branching process that for $k \leq n$

$$Z_n(B) = \sum_{\underset{\sim}{i}_k} Z_n(\underset{\sim}{i}_k;B) \tag{9}$$

where $\sum_{\underset{\sim}{i}_k}$ extends over all Z_k particles in the k^{th} generation. Conditioned upon $\mathcal{D}_k$, the random variables on the right-hand side of (9) are independent and the law of $Z_n(\underset{\sim}{i}_k;B)$ is that of $Z_{n-k}(B-\omega)$, where $B-\omega = \{b-\omega \mid b \in B\}$ and $\omega = X_{\underset{\sim}{i}_k}$. For each n, we choose an integer $k_n < n$. We then obtain the following representation from (9), which is basic in the proof of Theorem 1:

$$(10) \qquad W_n(]-\infty, y_n]) = A_n + B_n + W_{k_n}\Phi(y)$$

where

$$A_n = m^{-k_n} \sum_{\underset{\sim}{i}_{k_n}} \{W_n(\underset{\sim}{i}_{k_n};]-\infty, y_n]) - EW_n(\underset{\sim}{i}_{k_n};]-\infty, y_n])\}$$

$$B_n = m^{-k_n} \sum_{\underset{\sim}{i}_{k_n}} \{ EW_n(\underset{\sim}{i}_{k_n};]-\infty, y_n]) - \Phi(y)\}$$

and

$$W_{k_n} = m^{-k_n} Z_{k_n} .$$

The idea is now to choose k_n with $k_n \to \infty$ such that both A_n and B_n becomes small. For A_n, we use an inequality of Kurtz ([12]), while the analysis of B_n essentially reduces to a study of the mean value function of the process.

For Theorem 2, a similar argument works since we are able to obtain a representation analogous to (10) for $\sqrt{2\pi n}\, m^{-n} Z_n([a,b])$. Also, for the branching diffusion, a similar representation will yield convergence in probability. The a.s. convergence proof is, however, more complicated since we are dealing with a continuum of random variables. However, the method of proof outlined will give the a.s. convergence for t restricted to lattices of the form $\{n\delta\}$, $n = 0,1,\ldots$, $\delta > 0$ a rational. Some technical arguments are then used to push the convergence to the whole line. An idea similar to this was used in [3].

4: Proofs: Discrete time

We start by giving two preliminary technical lemmas.

Lemma 1. *Let $X_1,\dots,X_n$ be independent random variables with mean 0 such that $P(|X_i| > t) \leq \int_t^\infty dQ(x)$ for a distribution Q on $[0,\infty[$ with finite mean. Then for $\delta > 0$*

$$(11) \qquad P(|\bar{X}_n| > \delta) \leq c(n\int_n^\infty dQ(x) + \frac{1}{n}\int_0^n x^2 dQ(x))$$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and c may depend on δ and Q but not on n. In particular, $\bar{X}_n \xrightarrow{P} 0$ as $n \to \infty$.

Proof: Let $\eta = \frac{1}{n}\int_n^\infty (x-n)dQ(x)$, $e = \int_n^\infty (x-n)dQ(x)$. For n large, $\eta < e < \delta$ and by taking $\varphi(u) = u^2$ in part (a) of Theorem (3.1) of [12] it then follows, that

$$(12) \qquad P(|\bar{X}_n| > 2\delta) \leq P(|\bar{X}_n| > \delta + e)$$

$$\leq (\frac{4}{(2\delta-\eta)^2} + 1)n\int_0^1 2u\int_{nu}^\infty dQ(x)du$$

$$= c(n\int_n^\infty dQ(x) + \frac{1}{n}\int_0^n x^2 dQ(x)) \quad .$$

To obtain (11), we need only to modify c such that (12) holds for all n and replace δ by $\delta/2$. Finally, the last part of the lemma follows from

$$\overline{\lim_n}\, n\int_n^\infty dQ(x) \leq \overline{\lim_n}\int_n^\infty x dQ(x) = 0 \,,$$

$$\overline{\lim_n}\,\frac{1}{n}\int_0^n x^2 dQ(x) = \overline{\lim_n}\int_0^\infty x(\frac{x}{n}1_{\{x\leq n\}})dQ(x) = 0,$$

where we have used the dominated convergence theorem.

Lemma 2. Let $Z_0, Z_1, \ldots$ be a supercritical Galton-Watson process with offspring distribution $\{p_j\}$ and mean $m = \sum_{j=0}^{\infty} j\, p_j$. Define $M = \sup_n m^{-n} Z_n$. Then for $\alpha \geq 0$ $EM(\log^+ M)^\alpha < \infty$ if and only if $\sum_{j=2}^{\infty} j(\log j)^{1+\alpha} p_j < \infty$.

Proof: The case $\alpha = 0$ is treated in [11], where also the existence of constants A, B such that

$$P(M > Ax) \leq B\, P(W > x) \tag{13}$$

is proven under the assumption $\Sigma\, j \log j\, p_j < \infty$. Thus for $\alpha > 0$ it follows upon integration by parts that $EM(\log^+ M)^\alpha < \infty$ if $EW(\log^+ W)^\alpha < \infty$ and also the converse is clear. Finally in [2] $EW(\log^+ W)^\alpha < \infty$ is proved to be equivalent to $\Sigma\, j\,(\log j)^{1+\alpha} p_j < \infty$.

We now start the proof of Theorem 1 by choosing β with $0 < \beta < 1$ such that $1/\beta \leq 1+\delta$ and next $\alpha > 0$ such that $\alpha/2.(1/\beta - 1) > 1$. For j integer and $j^{\alpha/\beta} \leq n < (j+1)^{\alpha/\beta}$ we then set $k_n = a_j = [j^\alpha]$

Lemma 3. $\lim_n B_n = 0$ a.s. without conditions on $\{p_j\}$.

Proof: By the standard central limit theorem $D_n = \sup_{-\infty < x < \infty} |\Phi(\frac{x}{\sqrt{n-k_n}}) - G_{n-k_n}(x)|$ tends to 0 as $n \to \infty$. Furthermore,

$$\begin{aligned} |B_n| &= |m^{-k_n} \sum_{\underset{\sim}{i}_{k_n}} (G_{n-k_n}(y_n - X_{\underset{\sim}{i}_{k_n}}) - \Phi(y))| \\ &\leq m^{-k_n} \sum_{\underset{\sim}{i}_{k_n}} |\Phi(\frac{y_n - X_{\underset{\sim}{i}_{k_n}}}{\sqrt{n-k_n}}) - \Phi(y)| + W_{k_n} D_n \\ &\leq C m^{-k_n} \sum_{\underset{\sim}{i}_{k_n}} \frac{|X_{\underset{\sim}{i}_{k_n}}|}{\sqrt{n-k_n}} + W_{k_n} D_n \end{aligned}$$

where C is a suitable constant. The last inequality follows from elementary calculus. Thus in view of the definition of k_n it suffices to show that

$$E_j = m^{-a_j} \sum_{\underset{\sim}{i}_{a_j}} \frac{|X_{\underset{\sim}{i}_{a_j}}|}{j^{\alpha/2\beta}}$$

tends to 0 as $j \to \infty$. But this follows by standard arguments, since

$$\sum_{j=1}^{\infty} E(E_j|\mathcal{F}_{a_j}) = \sum_{j=1}^{\infty} m^{-a_j} Z_{a_j} \frac{E(|X_{\underset{\sim}{i}_{a_j}}||\mathcal{F}_{a_j})}{j^{\alpha/2\beta}} \leq$$
$$\sum_{j=1}^{\infty} W_{a_j} \frac{\sqrt{E(X^2_{\underset{\sim}{i}_{a_j}}|\mathcal{F}_{a_j})}}{j^{\alpha/2\beta}} = \sum_{j=1}^{\infty} W_{a_j} \frac{a_j^{1/2}}{j^{\alpha/2\beta}} < \infty \text{ a.s.} .$$

The proof of Theorem 1 is completed by

<u>Lemma 4</u>. $\sum_{j=2}^{\infty} j \log j p_j < \infty \Longrightarrow A_n \xrightarrow{P} 0$, and $\sum_{j=2}^{\infty} j(\log j)^{1+\delta} p_j < \infty \Longrightarrow A_n \xrightarrow{a.s.} 0$.

<u>Proof</u>: Since $W > 0$ a.s., also $\inf Z_n/m^n > 0$ a.s. and thus we may replace A_n by

$$\overline{A_n} = \frac{1}{Z_{k_n}} \sum_{\underset{\sim}{i}_{k_n}} (W_n(\underset{\sim}{i}_{k_n}]-\infty, y_n]) - EW_n(\underset{\sim}{i}_{k_n};]-\infty, y_n])) \quad .$$

Obviously $W_n(\underset{\sim}{i}_{k_n};]-\infty,y_n])$ is stochastically smaller than W_{n-k_n} and thus than $M = \sup_n Z_n/m^n$. Consequently $\overline{A_n}$ is of the same form as $\overline{X}_n$ in Lemma 1, where we can take Q as the law of $M+EM$. Clearly, Q possesses the same moments as M . It then follows from

Lemmas 1 and 2, that $P(|\bar{A}_n|>\delta|\mathcal{B}_{k_n}) \to 0$ for $\Sigma j \log j\, p_j < \infty$, proving the first part of the lemma. To prove a.s. convergence, we note that from Lemma 1

$$\sum_{n=1}^{\infty} P(|A_n| > \delta | \mathcal{B}_{k_n}) \leq$$

$$c\int_0^\infty \{ \sum_{n=1}^{\infty} (Z_{k_n} 1_{\{Z_{k_n} < x\}} + \frac{x^2}{Z_{k_n}} 1_{\{Z_{k_n} \geq x\}})\} dQ(x) \leq$$

$$U \int_0^\infty x(\log^+ x)^{1/\beta - 1} dQ(x)$$

where $U < \infty$ a.s.. The last inequality follows by elementary but tedious calculus from the fact that $\sup Z_{k_n}/m^{n^\beta} < \infty$, $\inf Z_{k_n}/m^{n^\beta} > 0$ a.s. . Finally, the last expression is finite a.s. if and only if $\Sigma\, j(\log j)^{1/\beta} p_j < \infty$ (Lemma 2), so that the extended Borel-Cantelli lemma ([13], pg. 151) completes the proof

The proof of Theorem 2 follows along similar lines. We shall not carry out all the details but only indicate the main modifications needed. We use the following expansion similar to (10):

(13) $$\sqrt{2\pi n}\; m^{-n} Z_n([a,b]) = A_n + B_n + W_{k_n}(b-a)$$

where now

$$A_n = \sqrt{2\pi n}\; m^{-k_n} \sum_{\underset{\sim}{i}_{k_n}} (W_n(\underset{\sim}{i}_{k_n};[a,b]) - EW_n(\underset{\sim}{i}_{k_n};[a,b]))$$

$$B_n = m^{-k_n} \sum_{\underset{\sim}{i}_{k_n}} E[\sqrt{2\pi n}\; W_n(\underset{\sim}{i}_{k_n};[a,b]) - (b-a)]$$

To define k_n, we first choose β such that $0 < \beta < 1$ and that

$3/2\beta \le 3/2 + \delta$ and next $\alpha > 0$ such that $\alpha(1/\beta - 1) > 1$. We set $k_n = a_j = [j^{\alpha}]$ for $j^{\alpha/\beta} \le n < (j+1)^{\alpha/\beta}$. Thus k_n is of magnitude n^{β} and that A_n tends to 0 a.s., follows along the lines of the proofs of Lemmas 1 and 4 by appealing to part (a) of Theorem (3.1) of [12]. To deal with B_n, we proceed as follows:

$$|B_n| = m^{-k_n} \Big| \sum_{\underset{\sim}{i}_{k_n}} \left(\sqrt{2\pi n}\ \{G_{n-k_n}(b - X_{\underset{\sim}{i}_{k_n}}) - G_{n-k_n}(a - X_{\underset{\sim}{i}_{k_n}})\} - (b-a)\right)\Big|$$

$$\le m^{-k_n} \sum_{\underset{\sim}{i}_{k_n}} \left|\sqrt{2\pi n}\ \left\{\Phi\left(\frac{b - X_{\underset{\sim}{i}_{k_n}}}{\sqrt{n-k_n}}\right) - \Phi\left(\frac{a - X_{\underset{\sim}{i}_{k_n}}}{\sqrt{n-k_n}}\right)\right\} - (b-a)\right| + W_{k_n} D_n$$

$$\le m^{-k_n} \sum_{\underset{\sim}{i}_{k_n}} \frac{X_{\underset{\sim}{i}_{k_n}}^2}{n-k_n} + W_{k_n} D_n$$

where D_n is non-random and tends to 0. Here we have used the extended central limit theorem ([8], pg. 210) for the first inequality and elementary calculus for the second. Thus in view of the definition of k_n it suffices to show that

$$E_j = m^{-a_j} \sum_{\underset{\sim}{i}_{a_j}} \frac{X_{\underset{\sim}{i}_{a_j}}^2}{j^{\alpha/\beta}}$$

tends to 0 as $j \to \infty$. But this follows from

$$\sum_{j=1}^{\infty} E(E_j \mid \mathcal{F}_{a_j}) = \sum_{j=1}^{\infty} W_{a_j} \frac{a_j}{j^{\alpha/\beta}} < \infty \quad \text{a.s.} .$$

5: Proofs: Continuous time.

The proofs in continuous time resemble those in discrete time, except that we have the added complication of particles dying at different times. We prove only (8) since the proof of (7) is similar.

We introduce first some notation. $\{(a_i, y_i);\ i = 1,\dots,Z_t\}$ is the chart of ages and positions of the particles alive at time t. Let $\mathcal{F}_t$, $\mathcal{D}_t$ be the σ-algebras containing the information up to time t on the Bellman-Harris process (including the a_i' s), respectively on the whole branching diffusion (that is, in addition also on the y_i' s). By the additivity properties of the process we may write for any Borel set B and $s_t < t$

$$(14)\qquad Z_t(B) = \sum_{i=1}^{Z_{s_t}} Z_{t-s_t}(a_i, y_i; B)$$

where $Z_{t-s_t}(a_i, y_i; B)$ is the number of descendants at time t in B of the i^{th} particle alive at time s_t. We let

$$W_{t-s_t}(a_i, y_i; B) = e^{-\alpha(t-s_t)} Z_{t-s_t}(a_i, y_i; B)\ ,$$

$$W_{t-s_t}(a_i) = W_{t-s_t}(a_i, y_i; \mathbb{R}),$$

$$M_{t-s_t}(a_i, y_i; B) = E(W_{t-s_t}(a_i, y_i; B) \mid \mathcal{D}_{s_t}),$$

$$M_{t-s_t}(a_i) = E(W_{t-s_t}(a_i) \mid \mathcal{D}_{s_t}) \quad \text{etc.}$$

Here e.g. $e^{\alpha(t-s_t)} M_{t-s_t}(a_i)$ is the expected number of particles alive at time $t-s_t$ in a Bellman-Harris process, where the original particle was of age a_i at $t = 0$. We get the following expansion similar to (10)

$$\sqrt{2\pi t}\quad e^{-\alpha t}\, Z_t(B) = A_t + B_t + C_t \tag{15}$$

where

$$A_t = \sqrt{2\pi t}\quad e^{-\alpha s_t} \sum_{i=1}^{Z_{s_t}} \{W_{t-s_t}(a_i, y_i; B) - M_{t-s_t}(a_i, y_i; B)\}$$

$$B_t = e^{-\alpha s_t} \sum_{i=1}^{Z_{s_t}} \{\sqrt{2\pi t}\; M_{t-s_t}(a_i, y_i; B) - |B|\; M_{t-s_t}(a_i)\}$$

$$C_t = |B|\, e^{-\alpha s_t} \sum_{i=1}^{Z_{s_t}} M_{t-s_t}(a_i)\ .$$

By taking $t = n\delta$ $(\delta > 0)$, it now follows that

<u>Lemma 5. For any $\delta > 0$, and any bounded Borel set B</u>

$$\sqrt{2\pi n\delta}\; e^{-\alpha n\delta}\; Z_{n\delta}(B) \xrightarrow{a.s.} |B|\, W$$

The proof is very similar to the ones of Section 4. Using the easily verified fact that

$$\mathrm{Var}(W_{t-s_t}(a_i, y_i; B) | \mathcal{D}_{s_t})) \leq C$$

for some constant $C < \infty$, it follows that $\sum_{n=1}^{\infty} \mathrm{Var}(A_{n\delta} | \mathcal{B}_{s_{n\delta}}) < \infty$

and thus that $\lim_n A_{n\delta} = 0$ a.s. . To handle $B_{n\delta}$ and $C_{n\delta}$, we need the following two lemmas, which are formulated in greater generality than needed at present:

Lemma 6. For any two bounded measurable functions g, h.

$$E\left(\sum_{i=1}^{Z_t(a,y;\mathbb{R})} g(a_i)h(y_i)\right) = E\left(\sum_{i=1}^{Z_t(a)} g(a_i)\right)E(h(\xi_y(t)))$$

where ξ_y is standard Brownian motion starting from y. Lemma 6 is clear once it is observed that the distribution of y_i conditioned upon $\mathcal{F}_t$ is that of $\xi_y(t)$. In particular, if $g \equiv 1$ and $h = 1_B$, then

$$M_{t-s_t}(a_i, y_i; B) = M_{t-s_t}(a_i)P(\xi_y(t-s_t) \in B)$$

and since $M_{t-s_t}(a_i) \leq C < \infty$, the argument in the proof of Theorem 2 can be repeated verbatum to yield $\lim_n B_n = 0$ a.s.

Before stating our next lemma, we introduce some notation. Let

$$A(x) = \frac{\int_0^x e^{-\alpha y}(1-F(y))dy}{\int_0^\infty e^{-\alpha y}(1-F(y))dy}, \qquad F_a(x) = \frac{F(a+x)-F(a)}{1-F(a)},$$

$$V(x) = \frac{me^{\alpha x}\int_x^\infty e^{-\alpha y}dF(y)}{1-F(x)} = m\int_0^\infty e^{-\alpha y}dF_a(y),$$

$$n_1 = EW = \frac{\int_0^\infty e^{-\alpha y}(1-F(y))dy}{m\int_0^\infty ye^{-\alpha y}dF(y)}.$$

Also, for g measurable, bounded and positive,

$$K_t(a,g) = E\left(\sum_{i=1}^{Z_t(a)} g(a_i)\right), \quad M_t(a,y) = e^{-\alpha t} K_t(a,g) \quad .$$

Lemma 7. *Let* $g(x)(1-F(x))e^{-\alpha x}$ *be directly Riemann integrable. Then*

$$\lim_{t\to\infty,\, t-s_t\to\infty} e^{-\alpha s_t} \sum_{i=1}^{Z_{s_t}} M_{t-s_t}(a_i,g) = W \int_0^\infty g(x)\,dA(x) \quad \text{a.s.}$$

Proof: In the usual way one can show that $K_t(a,g)$ satisfies

(16) $$K_t(a,g) = g(a+t)(1-F_a(t)) + m\int_0^t K_{t-u}(0,g)\,dF_a(u)$$

and consequently by the renewal theorem ([4], pg. 147)

(17) $$\lim_{t\to\infty} M_t(0,g) = n_1 \int_0^\infty g(x)\,dA(x) \quad .$$

It follows from (16) and (17) that

(18) $$\overline{\lim_{t\to\infty}} \sup_a \left| M_t(a,g) - n_1 V(a) \int_0^\infty g(x)\,dA(x) \right| = 0.$$

Also it is well-known ([3]), that

(19) $$\lim_{t\to\infty} e^{-\alpha t} \sum_{i=1}^{Z_t} n_1 V(a) = W \quad \text{a.s.}$$

The lemma follows from (18) and (19).

By taking $g \equiv 1$, it follows that $\lim_n C_{n\delta} = W|B|$ a.s., completing the proof of Lemma 5.

Remark 1. The added generality of Lemmas 6 and 7 allows us to prove with no extra difficulty the following variants of Lemma 5:

Variant 1. *Let* g *satisfy the hypothesis of Lemma 7. Then for any* $\delta > 0$

$$\lim_n \sqrt{2\pi n\delta}\, e^{-\alpha n\delta} \sum_{i=1}^{Z_{n\delta}} g(a_i) 1_{\{y_i \in B\}} = W|B| \int_0^\infty g(x)\,dA(x) \quad \text{a.s.}$$

Variant 2. *Let* h *be any measurable positive bounded function such that* $\int_{-\infty}^{\infty} x^2 h(x)\,dx < \infty$. *Then for any* $\delta > 0$

$$\lim_n \sqrt{2\pi n\delta}\, e^{-\alpha n\delta} \sum_{i=1}^{Z_{n\delta}} h(y_i) = W \int_{-\infty}^{\infty} h(x)\,dx \qquad \text{a.s.}$$

The assumption $\int_{-\infty}^{\infty} x^2 h(x)\,dx < \infty$ is needed to show $\lim_n B_{n\delta} = 0$ a.s. We omit the details.

We are now ready to complete the proof of (8). Let $\epsilon > 0$ and define

$$B_\epsilon = \{x \mid x \in B,\ \rho(x,\partial B) > \epsilon\}, \quad B^\epsilon = \{x \mid \rho(x,B) > \epsilon\}$$

where ∂B is the boundary of B and $\rho(x,A) = \inf_{y\in A} |y-x|$ for any Borel set A. Also, we let $\bar{\xi}(\delta) = \sup_{0\le t\le \delta} |\xi_0(t)|$. Note that

$$\lim_{\epsilon\to 0} |B_\epsilon| = \lim_{\epsilon\to 0} |B^\epsilon| = |B| \quad \text{if} \quad |\partial B| = 0.$$

Lemma 8. $\varliminf_{t} \sqrt{2\pi t}\ e^{-\alpha t}\ Z_t(B) \geq W|B|$

Proof: Let $\delta > 0$ and define for $i = 1,\dots,Z_{n\delta}(B_\epsilon)$ $A_{n,i}$ to be the event, that for all $t \in [n\delta,(n+1)\delta]$ there is at least one particle in the line of descent initiated by i in B. Clearly,

$$Z_t(B) \geq \sum_{i=1}^{Z_{n\delta}(B_\epsilon)} 1_{A_{n,i}}$$

for all $t \in [n\delta,(n+1)\delta]$. As above, it follows that

$$\lim_{n} \sqrt{2\pi n\delta}\ e^{-\alpha n\delta} \sum_{i=1}^{Z_{n\delta}(B_\epsilon)} \{1_{A_{n,i}} - PA_{n,i}\} = 0 \quad \text{a.s.}$$

Furthermore,

$$PA_{n,i} \geq P(A_{n,i} | i \text{ does not split before } t) \geq P(\bar{\xi}(\delta) \leq \epsilon).$$

Thus

$$\varliminf_{t} \sqrt{2\pi t}\ Z_t(B) \geq e^{-\alpha\delta} \varliminf_{n} \sqrt{2\pi n\delta}\ e^{-\alpha n\delta} \sum_{i=1}^{Z_{n\delta}} PA_{n,i} \geq$$

$$e^{-\alpha\delta} |B_\epsilon| P(\bar{\xi}(\delta) \leq \epsilon)$$

and the conclusion follows by letting first δ and then ϵ tend to 0.

The proof of (8) is now completed by

Lemma 9. $\varlimsup_{t} \sqrt{2\pi t}\ e^{-\alpha t} Z_t(B) \leq W|B|$.

Proof: Let $\delta > 0$ and define $\overline{Z}_\delta(a_i, y_i; B)$ as the total number of particles that ever enter B before time $t = (n+1)\delta$ in the line of descent initiated by a particle of age a_i and position y_i at time $t = n\delta$. Then for $t \in [n\delta, (n+1)\delta]$

$$Z_t(B) \leq \sum_{i=1}^{Z_{n\delta}} \overline{Z}_\delta(a_i, y_i; B)$$

and since $\mathrm{Var}(\overline{Z}_\delta(a_i, y_i; B)) \leq C < \infty$, it follows as above that

$$(20) \quad \overline{\lim_t} \sqrt{2\pi t}\, e^{-\alpha t} Z_t(B) \leq \overline{\lim_n} \sqrt{2\pi n\delta}\, e^{-\alpha n\delta} \sum_{i=1}^{Z_{n\delta}} \overline{Z}_\delta(a_i, y_i; B) =$$

$$\overline{\lim_n} \sqrt{2\pi n\delta}\, e^{-\alpha n\delta} \sum_{i=1}^{Z_{n\delta}} E\overline{Z}_\delta(a_i, y_i; B) .$$

To deal with this last expression, we note that

$$(21) \qquad E\overline{Z}_\delta(a_i, y_i; B) \leq 1 + c_1 F_{a_i}(\delta)$$

$$(22) \qquad E\overline{Z}_\delta(a_i, y_i; B) \leq c_2 P(\overline{\xi}(\delta) > \rho(y_i, B))$$

where c_1, c_2 are constants. Thus for $\epsilon > 0$

$$\sum_{i=1}^{Z_{n\delta}(B^\epsilon)} E\overline{Z}_\delta(a_i, y_i; B) \leq Z_{n\delta}(B^\epsilon) + c_1 \sum_{i=1}^{Z_{n\delta}(B^\epsilon)} F_{a_i}(\delta)$$

$$\sum_{i=1}^{Z_{n\delta}(\overline{B}^\epsilon)} E\overline{Z}_\delta(a_i, y_i; B) \leq c_2 \sum_{i=1}^{Z_{n\delta}(\overline{B}^\epsilon)} P(\overline{\xi}(\delta) > \rho(y_i, B)) \quad (\overline{B}^\epsilon = \mathbb{R} - B^\epsilon) .$$

It is not difficult to show that $F_x(\delta)(1-F(x))e^{-\alpha x} = (F(x+\delta) - F(x))e^{-\alpha x}$ is directly Riemann integrable and that

$$\int_{-\infty}^{\infty} y^2 P(\bar{\xi}(\delta) > \rho(y,B))dy < \infty .$$

E.g. it follows from Doob's martingale inequality ([13], pg. 133), that $P(\bar{\xi}(\delta) > \rho(y,B)) \leq c\delta^2/\rho(y,B)^4$. So appealing to Lemma 5 and Remark 1 we have from (20), (21) and (22)

$$\overline{\lim_t} \sqrt{2\pi t}\, e^{-\alpha t} Z_t(B) \leq$$

$$W(|B^\epsilon| + c_1 \int_0^\infty F_a(\delta)dA(a) + c_2 \int_{\overline{B}^\epsilon} P(\bar{\xi}(\delta) > \rho(y,B))dy).$$

It is easy to see, that the two last integrals tends to 0 as $\delta \to 0$. Thus the conclusion follows by letting first δ and then ϵ tend to 0.

Remark 2. The assumption $\Sigma j^2 p_j < \infty$ can be weakened to the same extent as in the discrete case providing we could prove the continuous time analog of (13). We state without proof a partial result.

Lemma 10. *Suppose* $\inf_x V(x) > 0$. *Then there exists constants* A,B *such that*

$$P(\sup_{t \geq 0} W_t > Ax) \leq BP(W > x) .$$

The condition of the lemma is satisfied for F exponential or F with bounded support. It would seem reasonable that the lemma should hold with no assumptions on F.

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