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STRONG LIMIT THEOREMS FOR SUPERCRITICAL IMMIGRATION-BRANCHING PROCESSES WITH A GENERAL SET OF TYPES

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Consider an increasing sequence of random times and a corresponding sequence of random populations. Let these be the starting times and initial values of otherwise equivalent Markov branching processes. The resulting superposition is an immigration-branching process. Admitting a general set of types, let the underlying branching process be supercritical and positively regular in the sense of [1], where the limiting behavior of such processes has been established. It is readily conjectured that, if only the immigration is dominated in some appropriate sense by the branching, the immigration-branching process, averaged and normalized as the branching process, converges almost surely to a superposition of the limits for the composing branching processes. We verify this conjecture, thereby satisfactorily sharpening a result of [2]. The known theory for a finite set of types ([3], [5]) is not only considerably generalized and simultaneously extended to the continuous time case, it is also sharpened. Moreover, it becomes clear that the finite, discrete time case reduces to a triviality, once the limiting behavior of processes without immigration is known. This seems to have been overlooked by some authors.

1. Let X be an arbitrary set, $X^{(n)}$ the symmetrized n-fold direct product of X, θ some extra point, and $X^{(0)} = \{\theta\}$. Define

$$\hat{X} := \bigoplus_{n=0}^{\infty} X^{(n)}$$

$$\hat{x} + \hat{y} := \hat{x}; \qquad \hat{x} \in \hat{X},\ \hat{y} = \theta,$$

$$:= \langle x_1, \dots, x_n,\ y_1, \dots, y_m \rangle,$$

$$\hat{x} = \langle x_1, \dots, x_n \rangle,\ \hat{y} = \langle y_1, \dots, y_m \rangle \in \hat{X},$$

$$\hat{x}[A] := 0; \quad \hat{x} = \theta,$$

$$:= \sum_{\nu=1}^{n} 1_A(x_\nu); \ \hat{x} = \langle x_1, \dots, x_n \rangle \in \hat{X},$$

$$\hat{x}[\xi] := \int_X \xi(x)\ \hat{x}[dx],$$

for $A \subset X$ and every real valued function ξ on X.

Let A be a σ-algebra on X and $\hat{A}$ the σ-algebra induced on $\hat{X}$ by A. Let either $T = N = \{0,1,2,\dots\}$, or $T = R_+ = [0,\infty[$, and suppose to be given

(a) the <u>immigration process</u> $\{\tau_\nu, \hat{y}_\nu, P\}$, where $0 \leq \tau_\nu \uparrow \infty$ is a sequence of (not necessarily finite) random times and $\{\hat{y}_\nu\}_{\nu \in N}$ is a random sequence in $(\hat{X}, \hat{A})$, both defined on the same space with probability measure P,

(b) the <u>Markov branching process</u> $\{\hat{x}_t, P^{\hat{x}}\}$ that is an $(\hat{X}, \hat{A})$-valued Markov process with parameter set T and stationary transition probabilities satisfying the branching condition as in [1].

For the case that $T = R_+$ we shall need some topological structure, in order to get beyond the consideration of discrete skeletons:

(C) <u>If</u> $T = R_+$ <u>the set</u> X <u>is a separable metric space,</u> A <u>is the topological Borel algebra, and</u> $\{\hat{x}_t, P^{\hat{x}}\}$ <u>is right continuous.</u>

Standard examples for the immigration process are the following.

(i) The τ_ν are the epochs of a renewal process, and the $\hat{y}_\nu$ are independent and identically distributed.

(ii) The τ_ν form a Poisson process determined by a density p_t, and the $\hat{y}_\nu$ are independent with the distribution of $\hat{y}_\nu$ depending only on τ_ν, [2].

(iii) Given a decomposable branching process consider the immigration into one component from all other components, [4].

Let $\{\hat{x}_{\nu,t};\ t \geq \tau_\nu\}$ be the branching process initiated at τ_ν by $\hat{y}_\nu$ and developing according to $P^{\hat{y}_\nu}$. Set

$$N_t = \max\{\nu : \tau_\nu \leq t\}.$$

The immigration-branching process $\{\hat{z}_t, \tilde{P}\}$ is then given by

$$\hat{z}_t = \sum_{\nu \leq N_t} \hat{x}_{\nu,t}$$

and the corresponding probability measure $\tilde{P}$ defined on the appropriate product space. The formal construction is the same as in [2] except for the trivial adaptation to the more general, not necessarily Poissonian case considered here.

Let B be the Banach algebra of all bounded, A-measurable functions ξ with supremum-norm, B_+ the non-negative cone in B.

We assume that $\{\hat{x}_t, P^{\hat{x}}\}$ is <u>positively regular</u>, i.e. satisfies the following condition:

(M) <u>The first moment semigroup</u> $\{E^{\langle\cdot\rangle}\hat{x}_t[\cdot]\}_{t\in T}$ <u>exists and can be represented in the form</u>

$$E^{\langle x\rangle}\hat{x}_t[\eta] = \rho^t\varphi^*[\eta]\varphi(x) + Q_t^{\langle x\rangle}[\eta], \quad x \in X, \ t \in T, \ \eta \in B,$$

<u>with</u> $\rho \in]0,\infty[$, $\varphi \in B_+$, <u>and</u> φ^* <u>a non-negative bounded linear functional on</u> B <u>such that</u>

$$\varphi^*[\varphi] = 1,$$

$$\varphi^*[Q_t^{\langle\cdot\rangle}[\cdot]] \equiv 0, \qquad Q_t^{\langle\cdot\rangle}[\varphi] \equiv 0,$$

$$|Q_t^{\langle x\rangle}[\eta]| \leq \alpha_t\varphi^*[\eta]\varphi(x), \quad x \in X, \ \eta \in B_+, t > 0,$$

<u>for some</u> $\alpha\colon T \to [0,\infty[$ <u>satisfying</u>

$$\rho^{-t}\alpha_t \to 0, \qquad t > \infty.$$

The condition (M) implies that $\varphi^*[1_A]$ is a measure on A. We set $\varphi^*[1] = 1$, where $1(\cdot) = 1$.

For a finite set of types our definition of positive regularity is the usual one, in the infinite case its motivation derives from branching diffusions, [1].

As is well known,

$$W_t := \rho^{-t}\hat{x}_t[\varphi], \qquad t \in T,$$

is a martingale with respect to $\sigma(\hat{x}_s;\ 0 \leq s \leq t)$, $t \in T$, and it follows by the martingale convergence theorem that

$$W := \lim_{t\to\infty} W_t \quad \text{a.s.}$$

exists. Assuming in addition that $\{\hat{x}_t, P^{\hat{x}}\}$ is <u>supercritical</u>, i.e. $\rho > 1$, there exists a necessary and sufficient moment condition for the nondegeneracy of W, [1].

2. We assume throughout that (M) is satisfied with $\rho > 1$.

Define

$$W_{\nu,t} := \rho^{-(t-\tau_\nu)} \hat{x}_{\nu,t}[\varphi] \, (t \geq \tau_\nu), \quad W_\nu := \lim_{t\to\infty} W_{\nu,t} \, .$$

THEOREM 1. _Let_ $T = N$, _or_ $T = R_+$, _and suppose_

$$\sum_{\nu=1}^{\infty} \rho^{-\tau_\nu} \hat{y}_\nu[\varphi] < \infty \quad \text{a.s.} \, . \tag{2.1}$$

Then

$$\tilde{W} := \lim_{t\to\infty} \rho^{-t} \hat{z}_t[\varphi]$$

exists and is finite a.s., _and_

$$\tilde{W} = \sum_{\nu=1}^{\infty} \rho^{-\tau_\nu} W_\nu \quad \text{a.s.} \, .$$

PROOF: Let $I = \sigma(\tau_1, \tau_2, \ldots, \hat{y}_1, \hat{y}_2, \ldots)$. Then, conditioned upon I,

$$\tilde{W}_t := \rho^{-t} \hat{z}_t[\varphi] = \sum_{\tau_\nu \leq t} \rho^{-\tau_\nu} W_{\nu,t} \tag{2.2}$$

is a submartingale and

$$\sup_t \tilde{E}(\tilde{W}_t | I) = \sup_t \sum_{\tau_\nu \leq t} \rho^{-\tau_\nu} \hat{y}_\nu[\varphi] < \infty \quad \text{a.s.} \, .$$

Thus the first part of the theorem follows by the convergence theorem for non-negative submartingales.

For the second part note that by (2.1)

$$\tilde{W}^* := \sum_{\nu=1}^{\infty} \rho^{-\tau_\nu} W_\nu < \infty \quad \text{a.s.} \, .$$

For $0 < s < t < \infty$ write

$$\tilde{W}-\tilde{W}^* = (\tilde{W}-\tilde{W}_t) + \sum_{\tau_\nu \leq s} \rho^{-\tau_\nu}(W_{\nu,t}-W_\nu) + \sum_{s<\tau_\nu \leq t} \rho^{-\tau_\nu} W_{\nu,t} - \sum_{\tau_\nu > s} \rho^{-\tau_\nu} W_\nu .$$

First let $t \to \infty$. Then almost surely the first and the second term on the right tend to zero and the third to a finite limit $U_s \geq 0$. Now let $s \to \infty$. Since U_s is nonincreasing in s, $U = \lim_{s\to\infty} U_s \geq 0$ a.s. exists, and $U = 0$ a.s., since

$$\tilde{E}(U|I) \leq \liminf_{s\to\infty} \liminf_{t\to\infty} \tilde{E}\Big(\sum_{s<\tau_\nu \leq t} \rho^{-\tau_\nu} W_{\nu,t} \Big| I\Big)$$

$$= \liminf_{s\to\infty} \sum_{s<\tau_\nu} \rho^{-\tau_\nu} \hat{y}_\nu[\varphi] = 0 .$$

The remaining term tends to zero a.s. by (2.1). $\square$

PROPOSITION 1. _Let_ $\tau_1, \tau_2, \ldots$ _be the epochs of a renewal process and the_ $\hat{y}_\nu$ _be i.i.d. Then for any A-measurable_ $\eta \geq 0$ _the condition_

(2.3) $$E \log^+ \hat{y}_1[\eta] < \infty$$

is sufficient for

$$\sum_{\nu=1}^{\infty} \rho^{-\tau_\nu} \hat{y}_\nu[\eta] < \infty \quad \text{a.s.}.$$

It is necessary if the mean interarrival time m _is finite._

PROOF: Let $\gamma > 1$. Kolmogorov's three series criterion implies that

$$\sum_{\nu=1}^{\infty} \gamma^{-\nu}\hat{y}_\nu[\eta] < \infty \quad \text{a.s.}$$

if and only if (2.3) holds. Since $\tau_\nu/\nu \to m$ a.s., there is a (random) $\gamma > 1$ such that $\rho^{-\tau_\nu} \leq \gamma^{-\nu}$ for ν sufficiently large and, if $m < \infty$, a γ_1 such that $\rho^{-\tau_\nu} \geq \gamma_1^{-\nu}$. □

In general, (2.1) holds at least if

$$E \sum_{\nu=1}^{\infty} \rho^{-\tau_\nu}\hat{y}_\nu[\varphi] < \infty. \tag{2.4}$$

In the Poisson case $(T = R_+)$ this reduces to

$$\int_0^\infty \rho^{-s} p_s M^s[\varphi]ds < \infty,$$

where $M^s[\cdot] = E(\hat{y}_\nu[\cdot] \mid \tau_\nu = s)$, cf. [2]. For a decomposable branching process with two components (2.4) is automatic if $\rho = \rho_1 > \rho_2$ in the notation of [4].

We continue with the general theory.

LEMMA 1. <u>Let</u> $T = N$ <u>or</u> $T = R_+$, <u>and assume in addition to</u> (2.1) <u>that</u>

$$\lim_{t\to\infty} \rho^{-t}\hat{x}_t[\xi] = W\varphi^*[\xi] \text{ a.s. } [P^{\hat{x}}] \quad \forall \hat{x} \in X \tag{2.5}$$

<u>for some non-negative</u> $\xi \in L^1_{\varphi^*}$. <u>Then</u>

$$\liminf_{t\to\infty} \rho^{-t}\hat{z}_t[\xi] \geq \widetilde{W}\varphi^*[\xi] \text{ a.s. } [\widetilde{P}]. \tag{2.6}$$

PROOF:

$$\liminf_{t\to\infty} \rho^{-t}\hat{z}_t[\xi] = \liminf_{t\to\infty} \sum_{\tau_\nu \leq t} \rho^{-\tau_\nu}\rho^{-(t-\tau_\nu)}\hat{x}_{\nu,t}[\xi]$$

$$\geq \sum_{\nu=1}^{\infty} \rho^{-\tau_\nu} W_\nu \varphi^*[\xi] \text{ a.s. } . \qquad \square$$

PROPOSITION 2. Let $T = N$ and suppose in addition to (2.1) that

(2.7) $$\lim_{n\to\infty} \rho^{-n}\hat{z}_n[\vartheta] = \tilde{W}\varphi^*[\vartheta] \quad \text{a.s.}$$

for some non-negative $\vartheta \in L^1_{\varphi^*}$. Then

(2.8) $$\lim_{n\to\infty} \rho^{-n}\hat{z}_n[\vartheta\eta] = \tilde{W}\varphi^*[\vartheta\eta] \text{ a.s. } .$$

for all $\eta \in B$.

PROPOSITION 2'. Let $T = R_+$ and assume in addition to (2.1) that

(2.9) $$\lim_{t\to\infty} \rho^{-t}\hat{x}_t[\vartheta] = W\varphi^*[\vartheta] \text{ a.s. } [P^{\hat{x}}] \quad \forall \hat{x} \in \hat{X}$$

(2.10) $$\lim_{t\to\infty} \rho^{-t}\hat{z}_t[\vartheta] = \tilde{W}\varphi^*[\vartheta] \text{ a.s. } [\tilde{P}]$$

for some $\vartheta \in B_+$ which is lower semi-continuous a.e. $[\varphi^*]$. Then for any φ^*-a.e. continuous $\eta \in B$,

(2.11) $$\lim_{t\to\infty} \rho^{-t}\hat{z}_t[\vartheta\eta] = \tilde{W}\varphi^*[\vartheta\eta] \text{ a.s. } .$$

PROOF OF PROPOSITIONS 2,2': We may assume $0 \leq \eta \leq 1$. By [1], (2.5) holds with $\xi = \vartheta\eta$ and $\xi = \vartheta(1-\eta)$. Hence by (2.6)

$$\limsup_{t\to\infty} \rho^{-t}\hat{z}_t[\vartheta\eta]$$

$$\leq \limsup_{t\to\infty} \rho^{-t}\hat{z}_t[\vartheta] - \liminf_{t\to\infty} \rho^{-t}\hat{z}_t[\vartheta(1-\eta)]$$

$$\leq \tilde{W}\varphi^*[\vartheta] - \tilde{W}\varphi^*[\vartheta(1-\eta)] = \tilde{W}\varphi^*[\vartheta\eta] \text{ a.s. } .$$

Combined with (2.6) for $\xi = \vartheta\eta$, this completes the proof. □

By Theorem 1 we certainly have (2.8) and (2.11) with $\vartheta = \varphi$. However, if $\inf \varphi = 0$ as for example in the case at branching diffusions with absorbing barriers, this is unsatisfactory, since it leaves, e.g., the limiting behaviour of the total population size, $\hat{z}_t[1]$, as an open question. In Section 3, we return to this problem, which is, in fact, the only one considered here, whose solution is not a corollary of results for processes without immigration. For a finite set of types, of course, φ is uniformly positive, and the problem does not arise, so that Propositions 2 and 2' settle this case. If for a finite set of types the usual $x \log x$ condition holds, (2.1) is also necessary for $\tilde{W} < \infty$ a.s., i.e. for the immigration to be dominated by the branching in the sense that ρ^{-t} remains the proper normalizing factor. So at least in the finite case our theory covers all models in which the branching dominates in the above sense, and specific assumptions on the immigration process, as can be found in the literature, serve only to ensure (2.1). For example, Propositions 1 and 2 combine to strengthen the convergence in probability in Theorem 4.1 of [3] to a.s. convergence.

3. Again it is assumed throughout that (M) is satisfied with $\rho > 1$. If $\inf \varphi = 0$, it is not difficult to construct examples showing that (2.1) is not sufficient to guarantee the well-behaviour of $\rho^{-t}\hat{z}_t[1]$. We therefore sharpen (2.1) to

$$\sum_{\nu=1}^{\infty} \rho^{-\tau_\nu} \hat{y}_\nu[1] < \infty \quad \text{a.s.} \; . \tag{3.1}$$

The verification of (3.1) in the examples follows the discussion on (2.1) verbally.

THEOREM 2. Let $T = N$. If (3.1) holds, then

$$\lim_{n\to\infty} \rho^{-n}\hat{z}_n[\eta] = \tilde{W}\varphi^*[\eta] \quad \text{a.s.}$$

for all $\eta \in B$.

To deal with $T = R_+$, we need - as in [1]-some additional structure, which is automatic, e.g., for branching diffusions. Define the split times $\sigma_1, \sigma_2, \ldots$ by

$$\sigma_1 := \inf\{t : \hat{x}_t[1] \neq \hat{x}_0[1]\},$$

$$\sigma_{n+1} := \inf\{t > \sigma_n : \hat{x}_t[1] \neq \hat{x}_{\sigma_n}[1]\}.$$

THEOREM 2'. Let $T = R_+$ and assume in addition to (3.1) that the σ_ν; $\nu = 1,2,\ldots$ are Markov times and that

$$P^{\langle\cdot\rangle}(\sigma_1 = t) \equiv 0 \quad \forall\, t > 0$$

$$\limsup_{t\downarrow 0} \| E^{\langle\cdot\rangle}(\hat{x}_t[1] + \sum_{n=1}^{\infty} n 1_{\{\sigma_n \leq t < \sigma_{n+1}\}}) \| = 1. \tag{3.2}$$

Then

$$\lim_{t\to\infty} \rho^{-t} \hat{z}_t[\eta] = \tilde{W}\varphi^*[\eta] \quad \text{a.s.}$$

for all φ^*- a.e. continuous $\eta \in B$.

LEMMA 2. Let $T = N$ or $T = R_+$ and let $\delta, m \in T \setminus \{0\}$. Let $Y_{n,i,\nu}$; $n\delta < \tau_\nu \leq (n+m)\delta$; $i = 1,\ldots,\hat{y}_\nu[1]$ be non-negative random variables with $\tilde{E}(Y_{n,i,\nu}|I) \leq \gamma < \infty$. Then

$$\lim_{n\to\infty} \rho^{-n\delta} \sum_{n\delta<\tau_\nu\leq(n+m)\delta} \; \sum_{i=1}^{\hat{y}_\nu[1]} Y_{n,i,\nu} = 0 \quad \text{a.s.} \;.$$

PROOF:

$$\sum_{n=0}^{\infty} \tilde{E}(\rho^{-n\delta} \sum_{n\delta<\tau_\nu\leq(n+m)\delta} \; \sum_{i=1}^{y_\nu[1]} Y_{n,i,\nu}|I)$$

$$\leq \gamma m \rho^{m\delta} \sum_{\nu=1}^{\infty} \rho^{-\tau_\nu} \hat{y}_\nu[1] < \infty \quad \text{a.s.} \;. \quad \Box$$

LEMMA 3. Let $T = N$ and let $Y_{n,i}$; $n = 0,1,2,\ldots$; $i = 1,\ldots,\hat{z}_n[1]$ be non-negative random variables, independent conditioned upon $\tilde{F}_n = \sigma(\hat{z}_m;\, m \leq n)$, and such that the distribution function $G_{\langle x_i\rangle}$ of $Y_{n,i}$ depends only on the type x_i of particle i. Suppose

$$\mu(\cdot) := \int_0^\infty y \; dG_{\langle\cdot\rangle}(y) \in B.$$

Then

$$\limsup_{n\to\infty} \rho^{-n} \sum_{i=1}^{\hat{z}_n[1]} Y_{n,i} \leq \limsup_{n\to\infty} \rho^{-n} \hat{z}_n[\mu] \quad \text{a.s.} \;.$$

PROOF: Write $\hat{z}_n = \hat{z}_n^* + \hat{z}_n^{**}$, where

$$\hat{z}_n^* := \sum_{\nu=1}^{N_{n-1}} \hat{x}_{\nu,n} , \quad \hat{z}_n^{**} := \sum_{N_{n-1}<\nu\leq N_n} \hat{y}_\nu .$$

By (M) and (3.1) there is a c, $0 < c < \infty$, such that

$$\tilde{E}\, \hat{z}_n^*[\eta] \leq c\rho^n \varphi^*[\eta] \quad \forall \eta \geq 0.$$

Therefore the proof of Lemma 2 of [1] goes through verbatim to yield

$$\limsup_{n\to\infty} \rho^{-n} \sum_{i=1}^{\hat{z}_n^*[1]} Y_{n,i} = \limsup_{n\to\infty} \rho^{-n} \sum_{i=1}^{\hat{z}_n^*[1]} \tilde{E}(Y_{n,i} 1_{\{Y_{n,i}\leq\rho^n\}} | \tilde{F}_n)$$

$$\leq \limsup_{n\to\infty} \rho^{-n} \hat{z}_n^*[\mu] \leq \limsup_{n\to\infty} \rho^{-n} \hat{z}_n[\mu] \quad \text{a.s.} .$$

Finally by Lemma 2

$$\lim_{n\to\infty} \rho^{-n} \sum_{i=1}^{\hat{z}_n^{**}[1]} Y_{n,i} = 0 \quad \text{a.s.} . \quad \square$$

PROOF OF THEOREM 2: From Lemma 1 and [1],

$$\liminf_{n\to\infty} \rho^{-n} \hat{z}_n[1] \geq \tilde{W} .$$

Let m be fixed, $Y_{n,i}$ the number of descendants at time $n+m$ at the i^{th} particle alive at time n, and $Y_{n,i,\nu}$ the number of descendants at time $n+m$ at the i^{th} of the particles that immigrated at time τ_ν, $n < \tau_\nu \leq n+m$. From (M),

$$\mu(x) \leq \rho^m c_m^+ \varphi(x) \quad \text{with} \quad c_m^+ \to 1 \quad \text{as} \quad m \to \infty.$$

Using Lemmata 2,3 and Proposition 2

$$\limsup_{n\to\infty} \rho^{-(n+m)} \hat{z}_{n+m}[1]$$

$$\leq \limsup_{n\to\infty} \rho^{-(n+m)} \sum_{i=1}^{\hat{z}_n[1]} Y_{n,i} + \limsup_{n\to\infty} \rho^{-(n+m)} \sum_{n<\tau_\nu\leq n+m} \sum_{i=1}^{\hat{y}_\nu[1]} Y_{n,i,\nu}$$

$$\leq \limsup_{n\to\infty} \rho^{-(n+m)} \hat{z}_n[\mu] = \rho^{-m} \tilde{W}\varphi^*[\mu] \leq \tilde{W} c_m^+ .$$

Letting $m \to \infty$ and using Proposition 2 with $\vartheta = 1$ completes the proof. □

PROOF OF THEOREM 2': We shall show that

$$(3.3) \qquad \limsup_{t\to\infty} \rho^{-t}\hat{x}_t[1] \leq W \quad \text{a.s.} \quad [P^{\hat{x}}] \quad \forall \hat{x} \in \hat{X}$$

$$(3.4) \qquad \limsup_{t\to\infty} \rho^{-t}\hat{z}_t[1] \leq \tilde{W} \quad \text{a.s.} \quad [\tilde{P}].$$

Admitting this for the moment, the proof is easily completed. First (3.3) and Lemma 8 of [1] with $\vartheta = 1$, $U = X$ yields (2.9) with $\vartheta = 1$ and next (3.4) and (2.6) with $\xi = 1$ combine to give (2.10) with $\vartheta = 1$. Thus Proposition 2' applies.

The proofs of (3.3) and (3.4) are similar and we treat only the more complicated case (3.4). We first remark that for any $\delta > 0$ we can consider $\hat{z}_n^* = \hat{z}_{n\delta}$ as a discrete time immigration-branching process, with immigration times $\tau_\nu^* = ([\tau_\nu/\delta] + 1)\delta$ and $\hat{y}_\nu^* = \hat{x}_{\nu,\tau_\nu^*}$. From (3.2),

$$\gamma = \sup_{0\leq t\leq\delta} \| E^{\langle\cdot\rangle} \hat{x}_t[1] \| < \infty$$

and therefore

$$\tilde{E}\Big(\sum_{\nu=1}^{\infty} \rho^{-\tau_\nu^*} \hat{y}_{\nu*}[1] \Big| I\Big) \le \gamma \sum_{\nu=1}^{\infty} \rho^{-\tau_\nu} \hat{y}_\nu[1] < \infty \quad \text{a.s.},$$

so that (3.1) holds for the skeleton process and Lemma 3 and Theorem 2 apply.

Consider now the i^{th} particle alive at time $n\delta$ and let $Y_{n,i}$ be the number of descendant of i at time $(n+1)\delta$ plus the number of splits in i's line of descent in $[n\delta, (n+1)\delta]$. Similarly, let for $n\delta < \tau_\nu \le (n+1)\delta$, $i = 1,\ldots,\hat{y}_\nu[1]$, $Y_{n,i,\nu}$ be the number of descendants of i at time $(n+1)\delta$ plus the number of splits in i's line of descent in $[\tau_\nu, (n+1)\delta]$. Then for any $t \in [n\delta, (n+1)\delta]$

$$(3.5) \quad \rho^{-t}\hat{z}_t[1] \le \rho^{-n\delta} \sum_{i=1}^{\hat{z}_{n\delta}[1]} Y_{n,i} + \rho^{-n\delta} \sum_{n\delta<\tau_\nu\le(n+1)\delta} \sum_{i=1}^{\hat{y}_\nu[1]} Y_{n,i,\nu}.$$

By (3.2),

$$\tilde{E}(Y_{n,i}|\tilde{F}_{n\delta}) \le h_\delta , \quad \tilde{E}(Y_{n,i,\nu}|I) \le h_\delta$$

with $h_\delta \to 1$ as $\delta \downarrow 0$. Thus the second term on the right-hand side of (3.5) tends to zero by Lemma 2 and using Lemma 3 and the skeleton convergence, we get from (3.5)

$$\limsup_{t\to\infty} \rho^{-t}\hat{z}_t[1] \le \limsup_{n\to\infty} \rho^{-n\delta} \sum_{i=1}^{\hat{z}_{n\delta}[1]} Y_{n,i} \le$$

$$\le \limsup_{n\to\infty} \rho^{-n\delta}\hat{z}_{n\delta}[1]h_\delta = \tilde{W}h_\delta .$$

As $\delta \downarrow 0$, (3.4) follows. $\square$

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REFERENCES:

[1] Asmussen, S. and Hering, H.: Strong limit theorems for general supercritical branching processes with applications to branching diffusuons. (1975). To appear.

[2] Hering, H.: Asymptotic behaviour of immigration-branching processes with general set of types. II.: Supercritical branching part. Adv. Appl. Prob. 7 (1975).

[3] Kaplan, N.: The supercritical p-dimensional Galton-Watson process with immigration. Math. Biosci. 22, 1-18 (1974).

[4] Kesten, H. and Stigum, B.P.: Limit theorems for decomposable multi-dimensional Galton-Watson Processes. J. Math. Anal. Appl. 17, 309-338 (1967).

[5] Mode, C.J.: Theory and Application of Multi-type Branching Processes. Elsevier, Amsterdam 1971.

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