Søren Asmussen

# Some Convergence Rates for Supercritical Branching Processes

Preprint
Juli
1974
5

Institute of Mathematical Statistics
University of Copenhagen

Søren Asmussen

# Some Convergence Rates for Supercritical Branching Processes.

Preprint 1974 No. 5.

INSTITUTE OF MATHEMATICAL STATISTICS
University of Copenhagen.

July 1974.

## 1. INTRODUCTION

Let $Z_0, Z_1, \ldots$ be a Galton-Watson process with mean $m = E(Z_1 | Z_0 = 1)$; we shall consider the supercritical case, where $1 < m < \infty$. A classical result on the process is, that if $W_n = Z_n / m^n$, then $W = \lim_n W_n$ exists and is finite a.s.. In the recent years, some finer limit theorems (e.g. certain results on asymptotic normality ([2]) and a law of the iterated logarithm ([3] and [4])) has been proved under the assumption $\sigma^2 = \mathrm{Var}(Z_1 | Z_0 = 1) < \infty$; these results, which may be seen as parallels to classical results for sequences of sums of i.i.d. random variables, completely solves the problem of determining the order of magnitude of the deviations $W - W_n$ under the assumption cited. In the present paper we shall be concerned with giving some rates of the convergence $W_n \to W$ for the Galton-Watson process and some more general branching processes under weaker assumption than $\sigma^2 < \infty$; here again, the results have wellknown analogues in some refinements of the law of large numbers (compare [9], prop. IV. 7.1 and the remarks at the beginning of section 2).

Our main result will be given in the multitype case and before stating the theorem, we introduce the basic notation and set-up, which is essentially that of [6] . Let k be the number of types and $F_{i,j}$, $i,j = 1,\ldots,k$, the offspring distributions. Then $Z_n$ is a k-vector, $Z_n = (Z_n^1 \ldots Z_n^k)$, and given $Z_n$, $Z_{n+1}^j$ is distributed as

$$\sum_{i=1}^{k} \sum_{\nu=1}^{Z_n^i} X_{n,\nu}^{i,j}$$

where the X's are independent and $X_{n,\nu}^{i,j}$ is distributed according to $F_{i,j}$. M denotes the matrix with elements $m_{i,j} = \int_0^\infty x dF_{i,j}(x)$ such that $E(Z_{n+1} | Z_n) = Z_n M$. We assume M to be positively regular, that is, all elements of $M^t$ are strictly positive for some integer $t > 0$; it is then well-known (see appendix 2 of [5]), that M has a largest positive eigenvalue $\rho$ with associated right and left eigenvectors u and v with strictly positive coordinates. We assume $\rho > 1$ and normalize u and v by $vu' = 1$. By a result of Kesten & Stigum ([6]), there exists a one-dimensional random variable $W$ such that $\lim_n Z_n/\rho^n = W \cdot v$ a.s.. We shall prove:

<u>Theorem 1</u>

(i) If

$$\int_0^{\infty} x \cdot (\log x)^{\alpha+1} \, dF_{i,j}(x) < \infty, \quad i,j = 1,\ldots,k \tag{1.1}$$

for some $\alpha \geq 0$, then a.s.

$$n^{\alpha}\left(W \cdot v - \frac{Z_n}{\rho^n}\right) \to 0 \tag{1.2}$$

Furthermore, the series

$$\sum_{n=1}^{\infty} \left(W \cdot v - \frac{Z_n}{\rho^n}\right) \tag{1.3}$$

converges a.s. for $\alpha \geq 1$

(ii) If

$$\int_0^{\infty} x^p \, dF_{i,j}(x) < \infty, \quad i,j = 1,\ldots,k \tag{1.4}$$

for some $p$ with $1 < p < 2$, then a.s.

$$\rho^{n/q}\left(W - \frac{Z_n u'}{\rho^n}\right) \to 0 \tag{1.5}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The author has some rather complete results on the converse of the theorem in the one-dimensional case. Thus it may be proved, that (1.5) implies (1.4), while on the contrary a counterexample shows, that (1.1) is not necessary for (1.2) to hold. However, if (1.1) fails to hold, then for any $\varepsilon > 0$ $n^{\alpha+\varepsilon} \cdot (W-W_n) \to -\infty$ on the set $\{W > 0\}$. Also, it should be noted that (1.5) could not be replaced by $\rho^{n/q}(W \cdot v - Z_n/\rho^n) \to 0$; this may be seen from the results in [7] on the magnitude of $Z_n a'$ for vectors with $va' = 0$. The proofs of (i) and (ii) are entirely analogous and only the proof of (i) will be given. The proof of theorem 1 carries with some modifications over to continuous-time Markov branching processes; see further the end of section 2, where a proof of this fact is outlined.

Finally, in section 3, we shall give an application of theorem 1 to show a refinement of a limit theorem of Kesten & Stigum ([8]) for decomposable multitype Galton-Watson processes; the announcement of the result is postponed until section 3.

## 2. SOME LEMMAS; PROOF OF THEOREM 1

As mentioned in the introduction, theorem 1 has a counterpart in some sharp versions of the law of large numbers and also its proof is based on an idea similar to the one employed here. If $Y_1, Y_2, \ldots$ are i.i.d. with mean zero, the law of large numbers, $(Y_1+\ldots+Y_n)/n \to 0$, may be shown by first proving the convergence of the series $\sum_{n=1}^{\infty} Y_n/n$ and next combine this fact with summation by parts (in form of the Kronecker lemma) to obtain the desired result. What we shall do, is (in the one-dimensional case) first to prove the convergence of the series $\Sigma m^{\alpha}(W_{m+1}-W_m)$ by some arguments inspired by those of Kesten and Stigum ([6] and [7]), and next use summation by parts to estimate the tail sums $\sum_{n}^{\infty}(W_{m+1}-W_m) = W - W_n$ of the series $\Sigma(W_{m+1}-W_m)$.

Lemma 1. Let $Y_{jm}$; $j = 1,2,\ldots$; $m = 1,2,\ldots$ be i.i.d. random variables with $EY_{jm} = 0$ and let $\mathcal{B}_0, \mathcal{B}_1, \ldots$ be an increasing sequence of $\sigma$-algebras such that the $Y_{jm}$'s are $\mathcal{B}_n$-measurable for $m \le n$ and independent of $\mathcal{B}_n$ for $m > n$. Further, let for each m $Z_m$ be a $\mathcal{B}_{m-1}$-measurable random variable with values in $\{0,1,2,\ldots\}$ such that $M = \sup_m Z_m/\rho^m < \infty$ a.s.. If

$$\int_0^{\infty} x.(\log x)^{\alpha+1}\, dF(x) < \infty,$$

where $F$ is the common distribution function of the $|Y_{jm}|$'s, then the series

$$\sum_{m=1}^{\infty} \frac{m^{\alpha}}{\rho^{m}} \sum_{j=1}^{Z_m} Y_{jm}$$

converges a.s. for $\rho > 1$.

Proof. Let $Y'_{jm} = Y_{jm} 1_{\{|Y_{jm}| \leq \rho^m/m^{\alpha}\}}$, $S_m = \frac{m^{\alpha}}{\rho^m} \sum_{j=1}^{Z_m} Y_{jm}$ and $S'_m = \frac{m^{\alpha}}{\rho^m} \sum_{j=1}^{Z_m} Y'_{jm}$.

Since obviously $S_m$ and $S'_m$ are $\mathcal{B}_m$-measurable, it is easy to see by referring to [9], prop. V. 6.2 and the corollary of prop. V.6.3, that it suffices to show the convergence of each of the series

$$\sum_{m=1}^{\infty} P(S_m \neq S'_m | \mathcal{B}_{m-1}), \quad \sum_{m=1}^{\infty} E(S'_m|\mathcal{B}_{m-1}) \text{ and } \sum_{m=1}^{\infty} \mathrm{Var}(S'_m|\mathcal{B}_{m-1}).$$

However, let for any $x \geq 0$ $p = p(x) = \sup\{m | \rho^m/m^{\alpha} < x\}$. Then $p = O(\log x)$, $\rho^p = O(x(\log x)^{\alpha})$ and thus

$$\sum_{m=1}^{\infty} P(S_m \neq S'_m | \mathcal{B}_{m-1}) \leq \sum_{m=1}^{\infty} Z_m \int_{\rho^m/m^{\alpha}}^{\infty} dF(x) \leq$$

$$M \int_0^{\infty} \sum_{m=1}^{\infty} \rho^m 1_{\{\rho^m/m^{\alpha}<x\}} dF(x) = M \int_0^{\infty} O(\rho^p) dF(x) =$$

$$M \int_0^{\infty} O(x(\log x)^{\alpha}) dF(x) < \infty,$$

$$\sum_{m=1}^{\infty} |E(S'_m|\mathcal{B}_m)| = \sum_{m=1}^{\infty} \frac{m^{\alpha}}{\rho^m} Z_m |EY_{1m} 1_{\{|Y_{1m}|>\rho^m/m^{\alpha}\}}| \leq$$

$$M \sum_{m=1}^{\infty} m^{\alpha} \int_{\rho^m/m^{\alpha}}^{\infty} x\, dF(x) = M \int_0^{\infty} x \sum_1^p m^{\alpha} dF(x) =$$

$$M \int_0^{\infty} x \cdot O((\log x)^{\alpha+1}) dF(x) < \infty$$

and finally,

$$\sum_{m=1}^{\infty} \mathrm{Var}(S'_m|\mathcal{B}_{m-1}) = \sum_{m=1}^{\infty} \frac{m^{2\alpha}}{\rho^{2m}} Z_m \mathrm{Var}\, Y'_{1m} \leq$$

$$M\sum_{m=1}^{\infty}\frac{m^{2\alpha}}{\rho^{m}}\int_{0}^{\rho^{m}/m^{\alpha}} x^{2}\,dF(x) \le M\int_{0}^{\infty} x^{2}\sum_{m=p}^{\infty}\frac{m^{2\alpha}}{\rho^{m}}\,dF(x) =$$

$$M\int_{0}^{\infty} x^{2}\,O\Big(\int_{p}^{\infty}\frac{t^{2\alpha}}{\rho^{t}}\,dt\Big)dF(x) =$$

$$M\int_{0}^{\infty} x^{2}O\Big(O\Big(-\Big[\frac{t^{2\alpha}}{\rho^{t}}\Big]_{p}^{\infty}\Big)\Big)dF(x) = M\int_{0}^{\infty} x^{2}O\Big(\frac{p^{\alpha}}{x}\Big)dF(x) =$$

$$M\int_{0}^{\infty} O(x(\log x)^{\alpha})\,dF(x) < \infty,$$

completing the proof.

Now consider a positively regular Galton-Watson process $Z_0$, $Z_1,\ldots$ with $M,\rho,u,v$ etc. as defined in the introduction.

Lemma 2. Under assumption (1.1), the series

$$\sum_{n=1}^{\infty}\frac{n^{\alpha}}{\rho^{n}}(Z_n-Z_{n-1}M) \tag{2.1}$$

converges a.s..

Proof. Write $Z_n^q = \sum_{p=1}^{k} Z_n^{p,q}$, where $Z_n^{p,q}$ denotes the number of type q-individuals in the n'th generation with parents of type p. Now

$$(Z_n-Z_{n-1}M)^q = \sum_{p=1}^{k}(Z_n^{p,q} - Z_{n-1}^{p}m_{p,q}) =$$

$$\sum_{p=1}^{k}\sum_{j=1}^{Z_{n-1}^{p}}(X_{j,n}^{p,q} - m_{p,q}),$$

where for each p and q the $X_{j,n}^{p,q}$'s are i.i.d. with common distribution $F_{p,q}$ and independent of the first n-1 generations. Since lemma 1 immediately gives the convergence of each of the sums

$$\sum_{n=1}^{\infty}\frac{n^{\alpha}}{\rho^{n}}\sum_{j=1}^{Z_{n-1}^{p}}(X_{j,n}^{p,q} - m_{p,q}),$$

the convergence of (2.1) is clear.

It is obvious from the proofs of lemma 1 and 2, that lemma 2 is still valid if (2.1) is replaced by

$$\sum_{n=1}^{\infty} \frac{n^{\alpha}}{\rho^{n}} (Z_n - Z_{n-1} M) A^{(n)},$$

where the sequence of matrices $A^{(n)} = (a_{i,j}^{(n)})$ satisfies $\sup_n |a_{i,j}^{(n)}| < \infty$ for all i and j. We shall make use of this remark in the proof of

<u>Lemma 3</u>. Under assumption (1.1) the series

$$\sum_{n=1}^{\infty} \frac{n^{\alpha}}{\rho^{n}} Z_n a'$$

converges a.s. for any a with $va' = 0$.

<u>Proof</u>. It is well-known ([5]), that

$$M^n = \rho^n u'v + O(\rho_1^n) \tag{2.2}$$

for some $\rho_1$ with $0 < \rho_1 < \rho$; thus

$$\frac{M^n a'}{\rho^n} = O((\frac{\rho_1}{\rho})^n). \tag{2.3}$$

Now

$$\sum_{n=1}^{N} \frac{n^{\alpha}}{\rho^{n}} Z_n a' = \sum_{n=1}^{N} \frac{n^{\alpha}}{\rho^{n}} (Z_n - Z_0 M^n) a' + Z_0 \sum_{n=1}^{N} \frac{n^{\alpha}}{\rho^{n}} M^n a'$$

and here by (2.3) the second term has a limit as $N \to \infty$. For each r

$$A^{(r)} = \sum_{n=0}^{\infty} (\frac{n+r}{r})^{\alpha} \frac{M^n a'}{\rho^n}$$

exists and for each i,j $\sup_r |a_{i,j}^{(r)}| < \infty$.

Since

$$\sum_{n=1}^{N} \frac{n^\alpha}{\rho^n}(Z_n - Z_0 M^n)a' = \sum_{n=1}^{N} \frac{n^\alpha}{\rho^n} \sum_{r=1}^{n} (Z_r - Z_{r-1}M)M^{n-r}a' =$$

$$\sum_{r=1}^{N} \frac{r^\alpha}{\rho^r}(Z_r - Z_{r-1}M) \sum_{n=r}^{N} \binom{n}{r}^{\alpha} \frac{M^{n-r}a'}{\rho^{n-r}} =$$

$$\sum_{r=1}^{N} \frac{r^\alpha}{\rho^r}(Z_r - Z_{r-1}M)(A^{(r)} - \sum_{n=N-r+1}^{\infty} \binom{n+r}{r}^{\alpha} \frac{M^n a'}{\rho^n})$$

it suffices by the remark following the proof of lemma 2 to show that

$$\sum_{r=1}^{N} \frac{r^\alpha}{\rho^r}(Z_r - Z_{r-1}M) \sum_{n=N-r+1}^{\infty} \binom{n+r}{r}^{\alpha} \frac{M^n a'}{\rho^n} =$$

$$\sum_{n=1}^{\infty} \Big( \sum_{r=N-n+1}^{N} \frac{r^\alpha}{\rho^r}(Z_r - Z_{r-1}M)\binom{n+r}{r}^{\alpha} \Big) \frac{M^n a'}{\rho^n}$$

tends to zero; but since

$$\Big|\Big| \sum_{r=N-n+1}^{N} \frac{r^\alpha}{\rho^r}(Z_r - Z_{r-1}M)\binom{n+r}{r}^{\alpha} \Big|\Big| \le$$

$$n(n+1)^\alpha \sup_{r \ge N-n+1} \Big|\Big| \frac{r^\alpha}{\rho^r}(Z_r - Z_{r-1}M) \Big|\Big|,$$

where the sup tends to zero by lemma 2, and the series $\sum_{n=1}^{\infty} n^{\alpha+1} M^n a'/\rho^n$ is absolute convergent, this follows by the dominated convergence theorem.

<u>Lemma 4</u>. (i) Let $\alpha_m$ be an increasing sequence of non-negative numbers tending to $+\infty$. If the series $\sum_{m=1}^{\infty} \alpha_m \beta_m$ converges, so does $\sum_{m=1}^{\infty} \beta_m$ and furthermore

$$\alpha_N \sum_{m=N+1}^{\infty} \beta_m \to 0, \; N \to \infty \tag{2.4}$$

(ii) The existence of $\sum_{m=1}^{\infty} m\beta_m$ implies that of $\sum_{m=1}^{\infty} \sum_{n=m}^{\infty} \beta_n$.

Proof. Let $S_N = \sum_{m=1}^{N} \alpha_m \beta_m$ and $K_N = \sup_{m,n \geq N} |s_m - s_n|$; then $K_N \to 0$, $N \to \infty$. For $p, q \geq N$

$$\left| \sum_{m=p+1}^{q} \beta_m \right| = \left| \sum_{m=p+1}^{q} \frac{1}{\alpha_m}(s_m - s_{m-1}) \right| =$$

$$\left| \sum_{m=p+1}^{q} \left( \frac{1}{\alpha_{q+1}} + \sum_{r=m}^{q} \left( \frac{1}{\alpha_r} - \frac{1}{\alpha_{r+1}} \right) \right) (s_m - s_{m-1}) \right| \leq$$

$$\frac{1}{\alpha_{q+1}} |s_q - s_p| + \left| \sum_{r=p+1}^{q} \left( \frac{1}{\alpha_r} - \frac{1}{\alpha_{r+1}} \right) (s_r - s_p) \right| \leq$$

$$\frac{1}{\alpha_{q+1}} K_N + \left( \frac{1}{\alpha_{p+1}} - \frac{1}{\alpha_{q+1}} \right) K_N = \frac{K_N}{\alpha_{p+1}} \leq \frac{K_N}{\alpha_N}$$

Since $\frac{K_N}{\alpha_N} \to 0$, $N \to \infty$, the convergence of $\sum_{m=1}^{\infty} \beta_m$ is clear and by letting $p = N$ and $q \to \infty$ in the inequality just derived, we see that

$$\left| \sum_{m=N+1}^{\infty} \beta_m \right| \leq \frac{K_N}{\alpha_N},$$

from which (2.4) follows. Finally, (ii) follows immediately from (i) by taking $\alpha_m = m$ and observing that the last term in the identity

$$\sum_{m=1}^{N} \sum_{n=m}^{\infty} \beta_n = s_N + N \cdot \sum_{n=N+1}^{\infty} \beta_n$$

tends to zero.

Proof of theorem 1. Under assumption (1.1),

$$n^{\alpha}(Z_n/\rho^n - W \cdot v)a' = n^{\alpha} Z_n a'/\rho^n \to 0$$

by lemma 3 for any a with $va' = 0$. Furthermore, since

$$\left(W \cdot v - \frac{Z_n}{\rho^n}\right)u' = \lim_{m\to\infty} \frac{Z_m u'}{\rho^m} - \frac{Z_n u'}{\rho^n} = \qquad (2.5)$$

$$\sum_{m=n}^{\infty} \frac{Z_{m+1}u'}{\rho^{m+1}} - \frac{Z_m u'}{\rho^m} = \sum_{m=n}^{\infty} \frac{1}{\rho^{m+1}}(Z_{m+1} - Z_m M)u',$$

$n^{\alpha}(Z_n/\rho^n - W \cdot v)u' \to 0$ by lemma 2 and 4. Now (1.2) follows from the fact that the vectors orthogonal to v together with u spans $R^k$. For $\alpha \geq 1$, the convergence of

$$\sum_{m=1}^{\infty} \frac{m}{\rho^m}(Z_m - Z_{m-1}M)$$

(lemma 2) implies that of

$W - Z_n/\rho^n$

$$\sum_{n=0}^{\infty} \boxed{(W \cdot v - Z_n/\rho^n)}u' = \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} \frac{1}{\rho^{m+1}}(Z_{m+1} - Z_m M)u'$$

by (2.5) and part (ii) of lemma 4. Since also for any a orthogonal to v

$$\sum_{n=0}^{\infty}\left(\frac{Z_n}{\rho^n} - Wv\right)a' = \sum_{n=0}^{\infty} \frac{Z_n a'}{\rho^n}$$

exists, the desired convergence follows as before.

In the remainder of this section, we briefly indicate the main modifications of the proof of theorem 1 needed in the case of a continuous-time Markov branching process $\{Z_t;\ t \geq 0\}$. We use here the minimal process and the split times (compare [1], sec.III.9). That is, the jumps $\xi_1, \xi_2, \ldots$ of the process are i.i.d. such that $\xi_i + 1$ is distributed according to the offspring distribution F(we shall only consider the case of a single type); as pointed out in [1], it is no restriction to take $F(\{1\}) = 0$ and we shall also for simplicity assume $F(\{0\}) = 0$. Further, the time $T_n$ between the n-1'th and the n'th jump has the form $T_n = X_n/S_{n-1}$, where $S_{n-1} = 1 + \xi_1 + \ldots + \xi_{n-1}$ (we assume $Z_0=1$) and the X's are independent of the $\xi$'s and i.i.d. with exponential distribution with mean $1/\beta$. Then, if

$\tau_n = T_1 + \ldots + T_n$ denotes the time of the n'th jump and $\lambda = \beta\int_0^\infty (x-1)dF(x)$ $(\lambda>0)$, it is well-known ([1]) that $\lim_n(\tau_n - \log n/\lambda)$ exists a.s. under assumption (2.6) below, that hence $\tau_n \uparrow \infty$ a.s. and that $Z_t = S_n$ for any t with $\tau_n \leq t < \tau_{n+1}$. Let $W_t = e^{-\lambda t}Z_t$ and $W = \lim_t W_t$. In order to show that $t^\alpha(W-W_t) \to 0$ a.s. under the assumption

$$\int_0^\infty x\cdot(\log x)^{\alpha+1}dF(x) < \infty \tag{2.6}$$

it suffices by integrating by parts as in lemma 4 to show the existence of

$$\int_0^\infty t^\alpha dW_t = \lim_T \int_0^T t^\alpha dW_t$$

Now $dW_t$ is the measure with atoms of weight $e^{-\lambda\tau_n}\xi_n$ in each $\tau_n$ and with density $-\lambda e^{-\lambda t}S_n$ with respect to Lebesgue measure dt on each of the intervals $[\tau_n, \tau_{n+1}[$ and thus, if $\tau_n \leq T < \tau_{n+1}$,

$$\int_0^T t^\alpha dW_t = U_n - \lambda S_n \int_{\tau_n}^T t^\alpha e^{-\lambda t}dt \tag{2.7}$$

where

$$U_n = \int_0^{\tau_n} t^\alpha dW_t = \sum_{i=1}^n (\tau_i^\alpha e^{-\lambda\tau_i}\xi_i - \lambda S_{i-1}\int_{\tau_{i-1}}^{\tau_i} t^\alpha e^{-\lambda t}dt).$$

The sequence $U_1, U_2, \ldots$ is easily seen to form a martingale with respect to $\mathcal{B}_1, \mathcal{B}_2, \ldots$, where $\mathcal{B}_n = \sigma(\xi_1, T_1, \ldots, \xi_n, T_n)$, and the existence of $\lim_n U_n$ may be established in the spirit of the proof of lemma 1 by truncating $\xi_n$ at $n/(\log n)^\alpha$; we leave the calculations to the reader. We need thus only show, that the last term of (2.7) tends to zero. But let c be some number such that $t^\alpha e^{-\lambda t}$ is decreasing for $t \geq c$. Then

$$S_n \int_{\tau_n}^T t^\alpha e^{-\lambda t}dt \leq$$

$$X_{n+1}(\tau_n^{\alpha} e^{-\lambda\tau_n} 1_{\{\tau_n > c\}} + \sup_t t^{\alpha} e^{-\lambda t} \circ 1_{\{\tau_n \leq c\}})$$

Since $\tau_n \uparrow \infty$, the last term tends to zero, and we conclude the proof by remarking that $\tau_n^{\alpha} e^{-\lambda\tau_n} = O((\log n)^{\alpha}/n)$ and that $X_{n+1}(\log n)^{\alpha}/n$ tends to zero by an simple application of the Borel-Cantelli lemma.

## 3. REFINEMENT OF A LIMIT THEOREM FOR DECOMPOSABLE MULTITYPE GALTON-WATSON PROCESSES.

In the present section we consider a multitype Galton-Watson process with mean matrix M: $k \times k$, which can be partitioned

$$M = \begin{pmatrix} M(1,1) & 0 \\ M(2,1) & M(2,2) \end{pmatrix}$$

where $M(1,1)$: $k_1 \times k_1$ and $M(2,2)$: $k_2 \times k_2$ are positively regular (for further discussion of the set-up, see [8]), and where the largest positive eigenvalues $\rho_1$ and $\rho_2$ associated with $M(1,1)$ and $M(2,2)$ by the Frobenius theorem are equal to some common number $\rho > 1$. Let for $i = 1,2$ $u(i)$ and $v(i)$ be the right and left eigenvectors of $M(i,i)$ corresponding to $\rho$ and write corresponding to the partition of M $Z_n = (Z_n(1)\ Z_n(2))$. Since the individuals of the first $k_1$ types cannot produce offspring of the last $k_2$ types, the $Z_n(2)$-process is positively regular and by [6], there exists a (one-dimensional) random variable W such that $Z_n(2)/\rho^n \to Wv(2)$ a.s.. The main result in [8] for the situation considered is, that if

$$\int_0^{\infty} x \cdot \log x \, dF_{i,j}(x) < \infty, \quad i,j = 1,\ldots,k,$$

then

$$\frac{Z_n(1)}{n\rho^n} \overset{a.s.}{\to} \frac{W}{\rho} v(2)M(2,1)u'(1)v(1).$$

We shall prove the following refinement of this result:

Theorem 2. Under the assumption

$$\int_0^{\infty} x \cdot (\log x)^2 \, dF_{i,j}(x) < \infty, \quad i,j = 1,\ldots,k, \tag{3.1}$$

there is a one-dimensional random variable U such that

$$U\cdot v(1) + W\cdot b = \lim_{n\to\infty}\left(\frac{Z_n(1)}{\rho^n} - n\frac{W}{\rho}v(2)M(2,1)u'(1)v(1)\right)$$

$$= \lim_{n\to\infty}\left(\frac{Z_n(1)}{\rho^n} - n\frac{Z_n(2)}{\rho^{n+1}}M(2,1)u'(1)v(1)\right)$$

where b is the $k_1$-vector

$$\frac{1}{\rho}v(2)M(2,1)\sum_{m=0}^{\infty}\left(\frac{M(1,1)^m}{\rho^m} - u'(1)v(1)\right).$$

The theorem contains as corollary some results on the asymptotic behaviour of $Z_n(1)a'$, where a is some $k_1$-vector. If $v(1)a' \neq 0$, obviously $Z_n(1)a'/n\rho^n \to W/\rho v(2)M(2,1)u'(1)v(1)a'$, while for $v(1)a' = 0$ $Z_n(1)a'/\rho^n \to Wba'$. We shall not here discuss the complications of the case $v(1)a' = ba' = 0$ in detail. However, some phenomena similar to those of the positively regular case (compare [7], section 2) seems to occur. E.g. it may be proved by a rather straightforward combination of the proof of theorem 1 and theorem 2.1 of [7], that if $\gamma$ and $\rho_a$ is defined as in [7] (that is, loosely speaking, such that $M(1,1)^n a'$ is of the magnitude $n^\gamma|\rho_a|^n$), if furthermore $|\rho_a^2| > \rho$ and (3.1) is replaced by the stronger assumption

$$\int_0^\infty x^2\, dF_{i,j}(x) < \infty,\ i,j = 1,\dots,k$$

then we may find real constants $\phi_\alpha,\dots,\phi_\beta$ and (complexvalued) random variables $X_\alpha,\dots,X_\beta$ such that

$$\frac{Z_n(1)a'}{n^\gamma|\rho_a|^n} - \sum_{\delta=\alpha}^{\beta} e^{in\phi_\delta}X_\delta \xrightarrow{a.s.} 0$$

In particular, the normalizing constant for $Z_n(1)a'$ is the same as in the positively regular case.

<u>Proof of theorem 2.</u> For any $m \geq 1$ and $q = 1,\dots,k_1$, let $T_m^q$ be the number of type q-individuals in the m'th generation, whose parents are of some of the last $k_2$ types, let $T_m =$

$(T_m^1 \ldots T_m^{k_1})$ and let $U_{m,n,j}^q$ (a $k_1$-vector) be the offspring at time n ($n \geq m$) of the j'th among the $T_m^q$'s. Now

$$Z_{n+1}(1) = \sum_{m=1}^{n+1} \sum_{q=1}^{k_1} \sum_{j=1}^{T_m^q} U_{m,n+1,j}^q =$$

$$\sum_{m=1}^{n+1} \sum_{q=1}^{k_1} \sum_{j=1}^{T_m^q} (U_{m,m,j}^q M(1,1)^{n+1-m} + \sum_{r=m}^{n} (U_{m,r+1,j}^q - U_{m,r,j}^q M(1,1))M(1,1)^{n-r}) =$$

$$\sum_{m=1}^{n+1} T_m M(1,1)^{n+1-m} + \sum_{q=1}^{k_1} \sum_{r=1}^{n} \sum_{m=1}^{r} \sum_{j=1}^{T_m^q} (U_{m,r+1,j}^q - U_{m,r,j}^q M(1,1))M(1,1)^{n-r}$$

and thus

$$Z_{n+1}(1)/\rho^{n+1} - (n+1)\frac{W}{\rho}v(2)M(2,1)u'(1)v(1) = \tag{3.2}$$

$$\sum_{m=1}^{n+1} \frac{1}{\rho^m} (T_m - Z_{m-1}(2)M(2,1))M(1,1)^{n+1-m}/\rho^{n+1-m}$$

$$+ \sum_{m=1}^{n+1} (Z_{m-1}(2)/\rho^m - \frac{W}{\rho}v(2))M(2,1)M(1,1)^{n+1-m}/\rho^{n+1-m}$$

$$+ \frac{W}{\rho} v(2)M(2,1) \sum_{m=1}^{n+1} (M(1,1)^{n+1-m}/\rho^{n+1-m} - u'(1)v(1))$$

$$+ \sum_{q=1}^{k_1} \sum_{r=1}^{n} \frac{1}{\rho^r} \sum_{m=1}^{r} \sum_{j=1}^{T_m^q} (U_{m,r+1,j}^q - U_{m,r,j}^q M(1,1))M(1,1)^{n-r}/\rho^{n+1-r}$$

As $n \to \infty$, the third term tends to Wb (the existence of b is clear from (2.3)). Furthermore, each of the series

$$\sum_{r=1}^{\infty} \frac{1}{\rho^r} \sum_{m=1}^{r} \sum_{j=1}^{T_m^q} (U_{m,r+1,j}^q - U_{m,r,j}^q M(1,1)) \quad (q=1,\ldots,k_1) \tag{3.3}$$

$$\sum_{m=1}^{\infty} \frac{1}{\rho^m} (T_m - Z_{m-1}(2)M(2,1)) \tag{3.4}$$

$$\sum_{m=1}^{\infty} \left(\frac{Z_{m-1}(2)}{\rho^m} - \frac{W}{\rho} v(2)\right) \tag{3.5}$$

converges a.s.; this follows for (3.5) by theorem 1 and is for (3.3) and (3.4) seen by arguments similar to those employed in the proofs of lemma 1 and lemma 2. The reader should note here, that the number of terms of i.i.d. random variables in (3.3) for each r is of the magnitude $r\rho^r$ (by referring to the theorem of Kesten & Stigum or alternatively by a rather direct application of the law of large numbers), that thus a modification of lemma 1 is needed and that actually assumption (3.1) is essential for the convergence of (3.3); we leave it to the reader to check the details.

Since $M(1,1)u'(1) = \rho u'(1)$, it is now clear that

$$\left(\frac{Z_n(1)}{\rho^n} - n\,\frac{W}{\rho}\,v(2)M(2,1)u'(1)v(1)\right)u'(1) \tag{3.6}$$

has a limit. Furthermore, if the (vector) series $\Sigma\alpha_m$ converges and $v(1)a' = 0$,

$$\sum_{m=0}^{n} \alpha_m \frac{M_{11}^{n-m}a'}{\rho^{n-m}} = \sum_{m=0}^{n} \alpha_{n-m} \frac{M_{11}^{m}a'}{\rho^{m}} \to 0,\ n \to \infty$$

by the dominated convergence theorem and (2.3). Thus for any such a

$$\left(\frac{Z_n(1)}{\rho^n} - n\,\frac{W}{\rho}\,v(2)M(2,1)u'(1)v(1)\right)a' \to Wba'. \tag{3.7}$$

and arguing as in the proof of theorem 1,

$$\lim_{n\to\infty}\left(\frac{Z_n(1)}{\rho^n} - n\,\frac{W}{\rho}\,v(2)M(2,1)u'(1)v(1)\right)$$

exists and may by (3.7) be written on the form $Uv(1) + Wb$. Finally,

$$\lim_{n\to\infty}\left(\frac{Z_n(1)}{\rho^n} - n\,\frac{Z_n(2)}{\rho^{n+1}}\,M(2,1)u'(1)v(1)\right) =$$

$$\lim_{n\to\infty}\left(\frac{Z_n(1)}{\rho^n} - n\,\frac{W}{\rho}\,v(2)M(2,1)u'(1)v(1)\right)$$

by (1.2).

## REFERENCES

[1] Athreya, K.B. & Ney, P.E.: Branching Processes. Springer Verlag (1972).

[2] Heyde, C.C.: A Rate of Convergence Result for the Supercritical Galton-Watson Process. J. Appl.Prob. 7, 451-454 (1970).

[3] Heyde, C.C.: Some Almost Sure Convergence Theorems for Branching Processes. Z. Wahrscheinlichkeitstheorie verw. Geb. 20, 189-192 (1971).

[4] Heyde, C.C. & Leslie, J.R.: Improved Classical Limit Analogues for Galton-Watson Processes with or without Immigration. Bull.Austr.Math.Soc. 5, 145-155(1971)

[5] Karlin, S.: A First Course in Stochastic Processes. Academic Press, New York (1966).

[6] Kesten, H. & Stigum, B.P.: A Limit Theorem for Multidimensional Galton-Watson Processes. Ann.Math.Stat. 37, 1211-1223 (1966).

[7] Kesten, H. & Stigum, B.P.: Additional Limit Theorems for Indecomposable Multidimensional Galton-Watson Processes. Ann.Math.Stat. 37, 1463-1481 (1966).

[8] Kesten, H. & Stigum, B.P.: Limit Theorems for Decomposable Multi-Dimensional Galton-Watson Processes. Journal of Mathematical Analysis and Applications 17, 309-338 (1967).

[9] Neveu, J.: Mathematical Foundations of the Calculus of Probability. Holden-Day, San Fransisco (1965).