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BRANCHING PROCESSES WITH VARYING AND RANDOM GEOMETRIC OFFSPRING DISTRIBUTIONS

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Abstract

The class of fractional linear generating functions is used to illustrate various aspects of the theory of branching processes in varying and random environments. In particular, it is shown that Church's theorem on convergence of the varying environments process admits of an elementary proof in this particular case. For random environments, examples are given on the asymptotic behavior of extinction probabilities in the supercritical case and conditional expectation given nonextinction in the subcritical case.

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1. Introduction.

This note gives some examples of branching processes (Z_n) in varying and random environments where all offspring distributions have fractional linear generating functions. The iterates of such generating functions being again fractional linear, the distribution of Z_n, given the environment, may be stated explicitly and this class of distributions may thus be used to illustrate the general theory. Agresti (1973) has recently used this class to derive bounds on the extinction times of (Z_n).

We state the basic composition result for convenience in Section 2, comment in Section 3 on the theorem on convergence of (Z_n) (given by Church (1971) and sharpened by Lindvall (1973)), and in Section 4 we give examples on the distribution of the extinction probability for supercritical branching processes in random environments (BPRE) and the conditional expectation, given nonextinction, for subcritical BPRE, illustrating the basic work of Smith and Wilkinson (1969), Athreya and Karlin (1971a,b) and Kaplan (1972).

A concise introduction to branching processes in varying environments is given by Jagers (1974).

2. Galton-Watson processes with varying two-parameter geometric offspring distributions.

Consider a branching process $Z_0, Z_1, Z_2, \ldots$ where $Z_0 = 1$ and the distribution of Z_n given that $Z_{n-1} = z$ is the convolution of z two-parameter geometric distributions given by $p_n(k) = P\{Z_n = k | Z_{n-1} = 1\}$ where $p_n(0) = r_n$, $p_n(k) = (1-r_n)(1-c_n)c_n^{k-1}$, $k = 1,2,\ldots$, $0 \leqq r_n < 1$

and $0 < c_n < 1$. The generating function $f_n(s) = \sum_k p_n(k)s^k = r_n + (1-r_n)\cdot$ $(1-c_n)s/(1-c_n s)$. We shall prefer the parametrization by $r_n = f_n(0)$ and $\ell_n = Df_n(1-) = E(Z_n|Z_{n-1} = 1) = (1-r_n)/(1-c_n)$, in which

$$f_n(s) = \frac{r_n\ell_n+(1-r_n-r_n\ell_n)s}{\ell_n+(1-r_n-\ell_n)s} .$$

Let $g_n(s)$ be the generating function of Z_n, then $g_n = g_{n-1} \circ f_n$, $n = 1,2,\ldots$.

Theorem 2.1. The distribution of Z_n is two-parameter geometric with parameters $q_n = g_n(0) = P\{Z_n = 0\}$ and $m_n = Dg_n(1-) = EZ_n$ given by

$$\frac{1}{1-q_n} = 1 + \sum_{j=1}^{n} \frac{r_j}{(1-r_j)m_{j-1}} \tag{2.1}$$

and

$$m_n = \prod_{j=1}^{n} \ell_j . \tag{2.2}$$

The distribution of Z_n, given that $Z_n > 0$, is geometric with generating function $h_n(s) = s/\{\mu_n + (1-\mu_n)s\}$, the parameter $\mu_n = E(Z_n|Z_n > 0)$ being given by

$$\mu_n = \frac{m_n}{1-q_n} = m_n + \sum_{j=1}^{n} \frac{r_j}{1-r_j} \frac{m_n}{m_{j-1}} . \tag{2.3}$$

Proof. The results are all obtained by simple algebra, cf. also Agresti (1973).

Theorem 2.2. Assume that $m_n \to \infty$ and $q_n \to q$ where $0 \leqq q < 1$. Then there exists a random variable W such that $Z_n/m_n \to W$ a.s. and the distribution of W is given by $P\{W = 0\} = q$ and $P\{W \leqq w | W > 0\} = 1-e^{-w}$.

Proof. The a.s. convergence follows by the usual martingale argument, cf. Athreya and Karlin (1971b) or Jagers (1974). The asymptotic distribution is easily derived from the generating functions $g_n(s)$ defined above.

Remark. The asymptotic distribution derived in this theorem generalizes the result of Harris (1963, Sec. I.8.5) and furnishes for random environment an example regarding Athreya and Karlin's (1971b) Theorem 1.

Example 2.1. Discrete observations from a birth-and-death process.

Let $\{X_t: t \geqq 0\}$ be a linear birth-and-death process with birth and death intensities λ and μ and let $0 = t_0 < t_1 < t_2 < \cdots$. It is well known (Harris 1963, Sec. V.5.1) that the Markov chain $\{Z_n: n=0,1,2,...\}$ given by $Z_n = X_{t_n}$ is a Galton-Watson process with varying geometric offspring distributions. In this case

$$f_n(s) = \frac{\mu(s-1)-e^{(\mu-\lambda)(t_n-t_{n-1})}(\lambda s-\mu)}{\lambda(s-1)-e^{(\mu-\lambda)(t_n-t_{n-1})}(\lambda s-\mu)}$$

so that $\ell_n = \exp\{(\lambda-\mu)(t_n-t_{n-1})\}$ and

$$r_n = \frac{\mu(e^{(\mu-\lambda)(t_n-t_{n-1})}-1)}{\mu e^{(\mu-\lambda)(t_n-t_{n-1})}-\lambda} .$$

It is furthermore immediate that $Z_n = X_{t_n}$ has a modified geometric distribution with $m_n = \exp\{(\lambda-\mu)t_n\}$,

$$q_n = \frac{\mu(e^{(\mu-\lambda)t_n}-1)}{\mu e^{(\mu-\lambda)t_n}-\lambda}$$

and

$$\mu_n = (\mu-\lambda e^{(\lambda-\mu)t_n})/(\mu-\lambda) .$$

The relations in Theorem 2.1 are easily verified, and if $m_n \to \infty$ and $\lambda > \mu$, $t_n \to \infty$ and $q_n \to \mu/\lambda < 1$ so that the asymptotic distribution quoted in Theorem 2.2 is also well known for this particular case.

3. Convergence of Z_n. Church's Theorem.

Let (Z_n) be a Galton-Watson process in varying environments. The following theorem is due to Church (1971), cf. Athreya and Karlin (1971a).

Theorem 3.1. There exists a random variable Z_∞ such that $Z_n \overset{W}{\to} Z_\infty$ as $n \to \infty$. $P\{0 < Z_\infty < \infty\} > 0$ if and only if $\Sigma\{1-p_n(1)\} < \infty$.

The proof of this theorem is quite involved and we shall see that if all offspring distributions are two-parameter geometric, a slightly stronger result may be proved by elementary means.

Theorem 3.2. Let (Z_n) be a Galton-Watson process with varying two-parameter geometric offspring distributions. Then either

(a) $\Sigma\{1-p_n(1)\} < \infty$, $m_n \to m$, $0 < m < \infty$, $Z_n \overset{W}{\to} Z_\infty$ where $P\{Z_\infty < \infty\} = 1$ and $P\{Z_n > 0\} > 0$,

(b) $q_n \to 1$, that is, $Z_n \overset{P}{\to} 0$, or

(c) $m_n \to \infty$, $q_n \to q < 1$ and $P\{Z_n \to \infty\} = 1-q$.

In cases (b) and (c), $\Sigma\{1-p_n(1)\} = \infty$.

<u>Proof</u>. Assume first that $\Sigma\{1-p_n(1)\} < \infty$. Then $\Sigma r_n \leqq \Sigma\{1-p_n(1)\} < \infty$ and since $p_n(1) = (1-r_n)^2/\ell_n \to 1$, it follows that $\ell_n \to 1$. We may then in turn conclude the convergence of $\Sigma(\ell_n-(1-r_n)^2)$ and therefore of $\Sigma(\ell_n-1)$. Hence $m_n = \prod_1^n \ell_j$ converges towards a finite limit and thus $P\{Z_n \to \infty\} = 0$. The convergence of $(1-q_n)^{-1}$ (cf. (2.1)) is now seen to follow from the convergence of Σr_n, and therefore $q = \lim q_n < 1$, which proves (a).

Assume secondly that $\Sigma(1-p_n(1)) = \infty$. Then the result is that

$$g_n(s) = q_n + \frac{(1-q_n)^2 s}{m_n+(1-q_n-m_n)s} \to \lim q_n = q$$

for all $0 \leqq s < 1$ as $n \to \infty$. In case (b), when $q_n \to 1$, this is trivial. Assume therefore $q < 1$. We will show that then necessarily $m_n \to \infty$ and the result is then obvious. We proceed indirectly.

If $m_n \not\to \infty$ then either (m_n) has a finite limit m or $\liminf m_n < \limsup m_n \leqq \infty$. If $m_n \to m$, the assumed finiteness of

$$\lim (1-q_n)^{-1} = 1 + \sum_1^\infty \frac{r_j}{1-r_j} m_{j-1}^{-1}$$

implies $\Sigma r_j < \infty$ and since $m_n = \prod_1^n \ell_j$, $\Sigma(\ell_j-1) < \infty$ and in particular $\ell_n \to 1$. Hence

$$\Sigma(1-p_n(1)) = \Sigma(\ell_n-(1-r_n)^2)/\ell_n$$
$$= \Sigma(\ell_n-1)/\ell_n - \Sigma r_n^2/\ell_n + 2\Sigma r_n/\ell_n < \infty$$

contrary to the assumption of $\Sigma(1-p_n(1)) = \infty$.

If $A = \liminf m_n < B = \limsup m_n \leqq \infty$, there will be infinitely many downcrossings by the sequence (m_n) through the interval (C,D), $A < C < D < B$. Let $(m_{t_i},\ldots,m_{u_i})$, $i = 1,2,\ldots$ be these downcrossings defined in the obvious way, that is, $m_{t_i} \geqq D$, $C < m_j < D$ for $j = m_{t_i+1},\ldots,m_{u_i-1}$ and $m_{u_i} \leqq C$. Obviously, $\prod_{t_i}^{u_i} \ell_j \leqq \frac{C}{D}$ for all i from which we infer

$$1 - \sum_{t_i}^{u_i} r_j \leqq \prod_{t_i}^{u_i} (1-r_j) \leqq \prod_{t_i}^{u_i} \ell_j \leqq C/D \ .$$

It follows that

$$\lim(1-q_n)^{-1} = 1 + \sum_1^\infty \frac{r_j}{1-r_j} \ m_{j-1}^{-1} \geqq \sum_1^\infty r_j m_j^{-1} \geqq \sum_J r_j m_j^{-1} \geqq D^{-1} \sum_J r_j = \infty$$

where $J = \bigcup_i \{t_i,\ldots,u_i\}$ is an infinite union of sets of indices, each contributing not less than $1 - C/D > 0$ to the sum. But this is in contrast to the assumption that $\lim q_n < 1$ and this concludes the proof.

<u>Example 3.1</u>. <u>Discrete observations from a birth-and-death process</u>.

Lindvall (1973) has recently shown that in Theorem 3.1, the convergence holds almost surely. In Example 2.1, $Z_n = X_{t_n}$ which certainly converges almost surely as $n \to \infty$. Thus, for this particular case, it is seen that the result amounts to asserting that $t_n \to \infty$ if and only if $\Sigma\{1-p_n(1)\} < \infty$. This is easily checked directly: Let $\tau_n = t_n - t_{n-1}$ and assume $\mu > \lambda$ (the cases $\mu = \lambda$ and $\mu < \lambda$ can be treated similarly). Then

$$1-p_n(1) = \frac{\mu^2(e^{(\mu-\lambda)\tau_n}-1)-\lambda^2(1-e^{(\lambda-\mu)\tau_n})}{(\mu e^{(\mu-\lambda)\tau_n/2}-\lambda e^{(\lambda-\mu)\tau_n/2})^2} .$$

(a) If τ_n does not converge to 0 there will exist $\epsilon > 0$ such that $\tau_n > \epsilon$ and hence $e^{(\mu-\lambda)\tau_n} > e^{(\mu-\lambda)\epsilon} > 1$ for infinitely many values of n. At such n,

$$1-p_n(1) > \left(\frac{\mu(1-e^{(\lambda-\mu)\tau_n})}{(\mu-\lambda e^{(\lambda-\mu)\tau_n})} \right)^2 > (1-e^{(\lambda-\mu)\epsilon})^2 > 0$$

so that $\Sigma(1-p_n(1)) = \infty$.

(b) If $\tau_n \to 0$, $\{1-p_n(1)\}/\{(\lambda+\mu)\tau_n\} \to 1$ so that $\Sigma(1-p_n(1)) < \infty$ $\Longleftrightarrow$ $\Sigma\tau_n < \infty \Longleftrightarrow (t_n)$ has a finite limit.

4. Random environments.

For the definition of branching processes in random environments, see Athreya and Karlin (1971a). We shall assume in this section that the random environment is given by a sequence ζ of independent identically distributed random variables $(\ell_1,r_1),(\ell_2,r_2),\ldots,$ where $P\{0 \leqq r_n < 1\} = P\{0 < \ell_n < \infty\} = 1$ and ℓ_n and r_n are parameters of two-parameter geometric distributions, cf. Section 2. If the integral of $\log \ell_1$ exists (and is possibly infinite), and if $E\{\log(1-r_1)\} < \infty$, the BPRE is called <u>supercritical</u>, <u>critical</u>, or <u>subcritical</u> according as $E(\log \ell_1) > 1$, $= 0$, or < 0. Let $q(\zeta)$ be the random extinction probability, defined as the random variable on the environment by $q(\zeta) = P\{Z_n \to 0|\zeta\}$. Then

$P\{q(\zeta) < 1\} = 1$ in the supercritical case and $P\{q(\zeta) = 1\} = 1$ in the critical and subcritical cases. Calculations of expected extinction probabilities have caused heavy problems, cf. Wilkinson (1969), for a practical application see Mountford (1971).

In this section we shall show by Examples 4.1 to 4.3 how the extinction probability distribution (which is the same as Smith and Wilkinson's (1969) "stationary distribution of the dual process") looks in some supercritical cases.

Example 4.1. If $0 < E(\ell_1^{-1}) < 1$ and $0 \leqq E\{r_1/(1-r_1)\} < \infty$, clearly $\log \ell_1$ is integrable, $E(\log \ell_1) > 0$ and $E\{-\log(1-r_1)\} < \infty$, so that by the general theory, $P\{q(\zeta) < 1\} = 1$. In this case, we get from (2.1) that

$$E[\{1-q_n(\zeta)\}^{-1}] = 1 + E\{r_1/(1-r_1)\} \sum_{j=1}^{n} \{E(\ell_1^{-1})\}^{j-1}$$

and since the monotone convergence theorem is clearly applicable,

$$E[\{1-q(\zeta)\}^{-1}] = 1 + E\{r_1/(1-r_1)\}/\{1-E(\ell_1^{-1})\} ,$$

implying in particular that $q(\zeta) < 1$ a.s.

Example 4.2. Let $P\{\ell_1 = 2\} = 1$ and $P\{r_1/(1-r_1) = i\} = 1/2$ for $i = 0$ and 1. Then by applying (2.1) it is found that $P\{(1-q_n)^{-1} = 1 + i2^{-n-1}\} = 2^{-n}$ for $i = 0,1,\ldots,2^n-1$ and consequently $(1-q_n)^{-1} \xrightarrow{w}$ the uniform distribution on [1,3]. The distribution of the extinction probability q is therefore given by $P\{q(\zeta) \leqq u\} = u/(2-2u)$, $0 \leqq u \leqq 2/3$, the limits 0 and 2/3 of course being the extinction probabilities corresponding to the two possible offspring distributions. The marginal extinction probabilities $P\{Z_n \to 0 | Z_0 = k\}$ are the k'th moments of the distribution of $q(\zeta)$. In particular, $P\{Z_n \to 0 | Z_0 = 1\} = 1 - \frac{1}{2}\log 3$.

Example 4.3. Let the environment consist of the two offspring distributions given by the generating functions $g(s) = s/(4-3s)$ and $h(s) = (3-2s)/(5-4s)$ and assumed with probability 1/2 each. The distribution of $q(\zeta)$ is the invariant measure for the Markov process given by $X_0 = s_0 \in [0,1)$, $X_n = f_n(X_{n-1})$ where f_n is g or h with probability 1/2 each and f_n are all independent (Smith and Wilkinson (1969)). To compute this invariant measure, assume first $X_0 = 0$. Then $X_1 = g(0) = 0$ or $h(0) = \frac{3}{5}$ with probability 1/2 each, $X_2 = g(g(0)) = 0$, $g(h(0)) = \frac{3}{11}$, $h(g(0)) = h(0) = \frac{3}{5}$ or $h(h(0)) = \frac{9}{13}$, each with probability 1/4. Since $g(h(0)) < h(g(0))$, it is seen that with this initial condition, the Markov process is stochastically increasing. Correspondingly, choosing as initial condition $X_0' = \frac{3}{4}$ (the fixed point of h), we get $X_1' = \frac{3}{7}$ or $\frac{3}{4}$ and $X_2' = \frac{3}{19}, \frac{3}{7}, \frac{15}{23}, \frac{3}{4}$, the latter values being attained each with probability 1/4. This process is stochastically decreasing. It follows from the first process that for the stationary measure π, $\pi[0,\frac{3}{5}] = \frac{1}{2}$ and from the second process that $\pi[\frac{3}{7},\frac{3}{4}] = \frac{1}{2}$ so that $\pi(\frac{3}{7},\frac{3}{5}) = 0$. Similarly, it follows that $\pi(\frac{3}{19},\frac{3}{11}) = \pi(\frac{15}{23},\frac{9}{13}) = 0$ and in general it is seen that every $i \cdot 2^{-n}$-fractile of π is a nondegenerate interval. (The crucial properties of the example being that $g(s) < h(s)$ for $0 \leqq s < 1$ and $g(\frac{3}{4}) = \frac{3}{7} < \frac{3}{5} = h(0)$). It follows from this that π or equivalently, the distribution of $q(\zeta)$, has a continuous distribution function which is singular with respect to Lebesgue measure.

In conclusion we shall give two examples of subcritical processes where the conditional expectation $\mu_n = E(Z_n | Z_n > 0)$ may be calculated explicitly.

Example 4.4. Let $P\{\ell_1 = \frac{1}{2}\} = 1$ and $P\{r_1/(1-r_1) = i\} = \frac{1}{2}$ for $i = 0$ and 1. Then (see (2.3)) $P\{\mu_n = i \cdot 2^{-n}\} = 2^{-n}$ for $i = 1,2,\ldots,2^n$. It follows that $\mu_n \overset{w}{\to}$ the uniform distribution on $[0,1]$. Notice that the weak convergence cannot be sharpened to convergence in probability. In fact, it is seen from (2.3) that in general

$$\mu_{n+1} = \ell_{n+1}\mu_n + \frac{r_{n+1}}{1-r_{n+1}} \ell_{n+1} .$$

Therefore, $P\{2\mu_{n+1} = \mu_n\} = P\{2\mu_{n+1} = \mu_n + 1\} = \frac{1}{2}$ or

$$P\{|\mu_{n+1}-\mu_n| \geqq \frac{1}{4}\}$$
$$= P\{|\mu_{n+1}-\mu_n| \geqq \frac{1}{4}, 0 \leqq \mu_n \leqq \frac{1}{2}\} + P\{|\mu_{n+1}-\mu_n| \geqq \frac{1}{4}, \mu_n > \frac{1}{2}\} = \frac{1}{2} .$$

This illustrates Kaplan's (1972) discussion of possible strengthening of weak convergence. Notice that this example is not covered by Kaplan who requires $P\{\ell_1 > 1\} > 0$ for the failure of convergence in probability.

Example 4.5. Assume that $P\{\frac{r_1}{1-r_1} \frac{\ell_1}{1-\ell_1} = c\} = P\{1-r_1 < \ell_1 \leqq a < 1\} = 1$, where $0 < c < 1$ and $0 < a < 1$. The process is then clearly subcritical. A recursion formula for μ_n is given by $\mu_n = r_n\ell_n/(1-r_n) + \ell_n\mu_{n-1}$; call the linear function on the right hand side $f_n(\mu_{n-1})$. Then by assumption $f_n(c) = c$ a.s. and since the slope $\ell_n \leqq a < 1$ a.s., a fixed point argument shows that $\mu_n \to c$ a.s. as $n \to \infty$. The generating function of the conditional distribution of Z_n, given the environment ζ, is by Theorem 2.1 equal to $h_n(s,\zeta) = s/\{\mu_n + (1-\mu_n)s\}$, $\mu_n = \mu_n(\zeta)$. It is seen that in this case $h_n(s,\zeta) \to s/\{c+(1-c)s\}$ a.s. as $n \to \infty$, cf. Kaplan (1972).

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