Steffen L. Lauritzen

Random Orthogonal Set Functions and Stochastic Models for the Gravity Potential of the Earth

Preprint
April
1974
2

Institute of Mathematical Statistics
University of Copenhagen

Steffen L. Lauritzen

RANDOM ORTHOGONAL SET FUNCTIONS AND STOCHASTIC MODELS FOR THE GRAVITY POTENTIAL OF THE EARTH

Preprint 1974 No. 2

INSTITUTE OF MATHEMATICAL STATISTICS

UNIVERSITY OF COPENHAGEN

April 1974

This is a reprint of Technical Report No. 14, 1973 from Department of Statistics, Stanford University, prepared under the auspices of Office of Navel Research Contract # N00014-67-A-0112 0030.

RANDOM ORTHOGONAL SET FUNCTIONS AND STOCHASTIC MODELS FOR THE GRAVITY POTENTIAL OF THE EARTH

by

Steffen L. Lauritzen*

Institute of Mathematical Statistics
University of Copenhagen

Abstract

The covariance function of the Newtonian potential of a random orthogonal set function on the unit sphere in three dimensions is derived, and it is shown that the coefficients to the series expansion of this are simply related to the moments of the covariance measure of the random set function.

Furthermore, as an application, it is shown that available gravity data indicates a mass distribution inside the Earth which becomes more and more irregular as one approaches the center of the Earth.

Key words: classical potential theory, geodesy, Hausdorff moment problem

* At present visiting Stanford University.

1. Introduction

In Lauritzen (1973) a derivation of stochastic models for the gravity potential of the Earth was given. It was shown how measurements of gravity anomalies and deflections of the vertical can be treated as linear operations on the random potential function, thus giving opportunity to estimate the covariance function under the assumption that it be invariant under the orthogonal group, and to predict linear functionals of the potential where these are unknown. A specific expression of the covariance function was shown to fit data very well.

However, the derivation was based entirely on analytical properties of harmonic functions and did not utilize the relation between the mass density in the interior of the Earth and the corresponding potential, thus making it difficult to give any physical interpretation of the results.

The present paper gives a derivation of stochastic models for the disturbing potential based on the variations of the mass and it is shown that the available data indicates the mass distribution becoming more and more irregular as one approaches the center of the Earth.

The disturbing mass is described by a random orthogonal set function and it is shown that the "degree-variances" of the potential covariance function are simply related to the moments of the covariance measure of the random orthogonal set function.

As a consequence, it is shown that the problem of determining the possible covariance functions for the disturbing potential arising from models with the above mentioned structure is equivalent to the classical Hausdorff moment problem.

Finally, the covariance measure corresponding to the model proposed in Lauritzen (1973) is derived.

2. Potentials of random orthogonal set functions on the unit sphere in 3-dimensional Euclidean space.

Let Z be a random orthogonal set function on the Borel sets of S, the closed unit sphere in $\underset{\sim}{R}^3$, 3-dimensional Euclidean space, with covariance measure μ satisfying $\mu(S) < +\infty$, i.e. Z is a stochastic process on the Borel sets of S with

i) $EZ(A) = 0$

ii) $EZ(A)Z(B) = \mu(A \cap B)$

iii) $Z(A) + Z(B) = Z(A \cup B)$ a.s. for $A \cap B = \emptyset$, A and B being arbitrary Borel sets of S.

For $t \notin S$, define the stochastic integral in the L_2-sense by

$$\xi(t) = \int_{u \in S} \frac{1}{||t-u||} Z(du) .$$

This defines a stochastic process on $\underset{\sim}{R}^3 \setminus S$ with mean value

$$E\xi(t) = \int_{u \in S} \frac{1}{||t-u||} EZ(du) = 0$$

and covariance function

$$E\xi(s)\xi(t) = \int_{u \in S} \int_{v \in S} \frac{1}{||t-u|| \; ||s-v||} E\{Z(du)Z(dv)\}$$

$$= \int_{u \in S} \frac{1}{||t-u|| \; ||s-v||} \mu(du) ,$$

see e.g. Grenander and Rosenblatt (1957) p. 25 ff.

The stochastic process ξ shall be called the Newtonian potential of Z, in accordance with the classical potential theory, where the Newtonian potential of a Radon measure ν on a compact set $K \subseteq \mathbb{R}^3$ is defined as the function $\phi: \mathbb{R}^3 \setminus K \to \mathbb{R}$ given by

$$\phi(x) = \int_{y \in K} \frac{1}{||x-y||} \nu(dy) .$$

As we have the expansion (e.g. Hobson, 1955)

$$\frac{1}{||t-u||} = \frac{1}{||t||} \sum_{n=0}^{\infty} \left(\frac{||u||}{||t||}\right)^n P_n (\cos \psi_{tu}) ,$$

where ψ_{tu} is the angle between the t and the u vectors and P_n are the Legendre polynomials, we get

$$E\xi(s)\xi(t) = \frac{1}{||t|| \; ||s||} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{u \in S} \frac{||u||^{n+m} P_n(\cos \psi_{tu}) P_m(\cos \psi_{su})}{||t||^n \; ||s||^m} \mu(du) .$$

If we assume that μ is uniform on spheres of the form $\{||u|| = \rho\}$, and introduce spherical coordinates

$$s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} r_s \sin \theta_s \cos \lambda_s \\ r_s \sin \theta_s \sin \lambda_s \\ r_s \cos \theta_s \end{pmatrix} \sim \begin{pmatrix} r_s \\ \theta_s \\ \lambda_s \end{pmatrix}.$$

we get

$$E\xi(s)\xi(t) = \frac{1}{r_s \, r_t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\int_0^1 \frac{x^{n+m}}{r_s^n \, r_t^m} \eta(dx) \right) \left(\int_0^{2\pi} \int_0^{\pi} a_{nm}(s,t,\theta,\lambda) \sin \theta d\theta d\lambda \right) ,$$

where

$$\mu(dr, d\theta, d\lambda) = \eta(dr) \sin\theta \, d\theta \, d\lambda,$$

and

$$a_{nm}(s,t,\theta,\lambda) = P_n(\cos \psi_{su}) \, P_m(\cos \psi_{tu}) \; , \; u \sim (1,\theta,\lambda) \; .$$

Using the well-known formula

$$P_n(\cos \psi_{su}) = P_n(\cos \theta_s) \, P_n(\cos \theta) + 2 \sum_{i=1}^{n} \frac{(n-i)!}{(n+i)!} P_{ni}(\cos \theta_s) \, P_{ni}(\cos\theta) \cos i(\lambda_s - \lambda)$$

where P_{ni} are the Legendre functions, we get

$$a_{nm}(s,t,\theta,\lambda) = \sum_{i=0}^{n} \sum_{j=0}^{m} b_{ijnm}(s,t,\theta,\lambda) \; ,$$

where

$$\begin{aligned} b_{ijnm}(s,t,\theta,\lambda) = c_{ijnm}[& R_{ni}(\theta_s,\lambda_s) R_{mj}(\theta_t,\lambda_t) R_{ni}(\theta,\lambda) R_{mj}(\theta,\lambda) \\ & + R_{ni}(\theta_s,\lambda_s) S_{mj}(\theta_t,\lambda_t) R_{ni}(\theta,\lambda) S_{mj}(\theta,\lambda) \\ & + S_{ni}(\theta_s,\lambda_s) R_{mj}(\theta_t,\lambda_t) S_{ni}(\theta,\lambda) R_{mj}(\theta,\lambda) \\ & + S_{ni}(\theta_s,\lambda_s) S_{mj}(\theta_t,\lambda_t) S_{ni}(\theta,\lambda) S_{mj}(\theta,\lambda)] \; . \end{aligned}$$

Here

$$R_{ni}(\theta,\lambda) = P_{ni}(\cos \theta) \cos i\lambda$$

$$S_{ni}(\theta,\lambda) = P_{ni}(\cos \theta) \sin i\lambda$$

are the surface spherical harmonics. Using the orthogonality property of these, the only terms in the sums contributing anything to the integral will be those of the form $R_{ni}R_{ni}R_{ni}R_{ni}$ or $S_{ni}S_{ni}S_{ni}S_{ni}$. As for $i \neq 0$

$$\int_0^{2\pi}\int_0^{\pi} R_{ni}^2(\theta,\lambda)\ \sin\theta\ d\theta\ d\lambda = \int_0^{2\pi}\int_0^{\pi} S_{ni}^2(\theta,\lambda)\ \sin\theta\ d\theta d\lambda = \frac{2\pi(n+i)!}{(2n+1)(n-i)!} ,$$

$$\int_0^{2\pi}\int_0^{\pi} R_{n0}^2(\theta,\lambda)\ \sin\theta\ d\theta\ d\lambda = \frac{4\pi}{2n+1} , \quad S_{ni} \equiv 0 ,$$

and

$$C_{iinn} = \begin{cases} 4\left(\frac{(n-i)!}{(n+i)!}\right)^2 & \text{for } i \neq 0 \\ 1 & \text{otherwise,} \end{cases}$$

we obtain by using (*) again, that

$$\int_0^{2\pi}\int_0^{\pi} a_{nm}(s,t,\theta,\lambda)\ \sin\theta\ d\theta\ d\lambda = \delta_{nm}\frac{4\pi}{2n+1} P_n\ (\cos\psi_{st}) .$$

Hence

$$E\xi(s)\xi(t) = 4\pi \sum_{n=0}^{\infty}\left(\frac{1}{r_s\ r_t}\right)^{n+1} \frac{P_n(\cos\psi_{s,t})}{2n+1} \cdot \int_0^1 x^{2n}\eta(dx) .$$

Defining $\sigma_n^2 = \int_0^1 x^{2n}\ \eta(dx)$, we obtain that

$$\sigma_n^2 = a \cdot \gamma_n ,$$

where γ_n is the n'th moment of the distribution of the random variable X^2 , where X follows the distribution

$$P\{X \leq x\} = \frac{\int_0^x \eta(du)}{\int_0^1 \eta(du)} .$$

In terms of the covariance measure, we get

$$\sigma_n^2 = \frac{1}{4\pi} \int_{u \in S} ||u||^{2n} \mu(du) .$$

The problem of when a prescribed sequence γ_n can be a sequence of moments of a distribution on the unit interval is the classical Hausdorff moment problem, having the solution that $(\gamma_n, n=0,1,2, \ldots)$ is a moment sequence iff it is completely monotone, i.e. iff $(-1)^\nu \Delta^\nu \gamma_n \geqq 0$ for all ν, n, where $\Delta\gamma_n = \gamma_{n+1} - \gamma_n$, see Feller (1966).

3. Application to the gravity potential of the Earth.

Lauritzen (1973) studied stochastic models for the disturbing gravity potential of the Earth, and showed that the general form of a rotational invariant covariance function for a random potential is

$$E\xi(s)\xi(t) = \sum_{n=0}^{\infty} a_n^2 \left(\frac{1}{r_s\ r_t}\right)^{n+1} P_n\ (\cos\ \theta) ,$$

where a_n^2 just have to be so that the series is convergent.

The random disturbing potential can be represented in a series expansion with random coefficients α_{nm} and β_{nm}

$$\xi(t) = \sum_{n=0}^{\infty} \left(\frac{1}{r_t}\right)^{n+1} \left(\sum_{m=0}^{n} \alpha_{nm} R_{nm}\ (\theta_t, \lambda_t) + \sum_{m=0}^{n} \beta_{nm} S_{nm}\ (\theta_t, \lambda_t) \right) .$$

As one has rather precise knowledge of the coefficients α_{nm}, β_{nm} for $n=0,1,2$, one could consider those as fixed, i.e. consider the covariance function of $\xi(t)$ conditional on the known values of α_{nm}, β_{nm} for $n = 0, 1, 2$, as also done in Lauritzen (1973); this can be shown (same paper) to affect the covariance function only by putting $a_0^2 = a_1^2 = a_2^2 = 0$.

From an empirical covariance function obtained from gravity anomalies, it was demonstrated that an expression of the form

$$a_n^2 = \begin{cases} 0 & \text{for } n = 0, 1, 2 \\ \dfrac{A}{(n-1)(n-2)} & \text{for } n \geqq 3 \end{cases}$$

gave an extremely good fit to the observations.

The investigations in section 2 of the present paper give us a possibility of an interpretation of these coefficients in terms of the variations in the mass distribution inside the Earth. We have the relation

$$a_n^2 = \sigma_n^2 \frac{4\pi}{2n+1} \Longleftrightarrow \sigma_n^2 = \frac{2n+1}{4\pi} a_n^2 \quad .$$

Now, suppose that the Earth is the unit sphere in $\underset{\sim}{R}^3$ and the deviations between a homogeneous and the actual mass distribution is given by a random orthogonal set function Z with covariance measure absolutely continuous w.r.t. Lebesgue measure on S and density f given by

$$f(u) = \begin{cases} 0 & \text{for } ||u|| < \varepsilon \\ \dfrac{A}{2\pi}\left(\dfrac{5}{||u||^7} - \dfrac{3}{||u||^5}\right) & \text{for } \varepsilon \leqq ||u|| \leqq 1 \ . \end{cases}$$

This corresponds to $\eta(dx) = g(x)dx$, where

$$g(x) = \begin{cases} 0 & \text{for } x < \varepsilon \\ \dfrac{A}{2\pi}\left(\dfrac{5}{x^5} - \dfrac{3}{x^3}\right) & \text{for } \varepsilon \leqq x \leqq 1 \ , \end{cases}$$

corresponding to for $n \geqq 3$

$$\sigma_n^2 = \frac{A}{4\pi} \frac{2n+1}{(n-1)(n-2)} + \frac{A}{4\pi} \varepsilon^{2n-4} \left(\frac{3\varepsilon^2}{n-1} - \frac{5}{n-2} \right) .$$

For ε being small we then have the approximation

$$a_n^2 = \frac{A}{(n-1)(n-2)} + 0(\varepsilon^2) \cdot 0(\frac{1}{n}) \quad \text{for } n \geqq 3 ,$$

the main term of which exactly is the one proposed by Lauritzen (1973).

The good fit of this form for a_n^2 should then indicate a mass distribution inside the Earth which gets more and more irregular in absolute variation as one approaches the center of the Earth.

References

Feller, W. (1966): An Introduction to Probability Theory and Its Applications, Vol. II, Wiley, N.Y.

Grenander, U. and Rosenblatt, M. (1957): Statistical Analysis of Stationary Time Series, Wiley, N.Y.

Hobson, E. W. (1955): The Theory of Spherical and Ellipsoidal Harmonics, Chelsea, N.Y.

Lauritzen, S. L. (1973): The Probabilistic Background of some Statistical Methods in Physical Geodesy. Medd. Geodetic Institute, No. 48, Copenhagen.