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An Intrinsic Time Scale for Non-Homogeneous Markov Branching Processes

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1. Summary.

Let $\{X_t,\ t \geqq 0\}$ be a non-homogeneous Markov branching process with $X_0 = 1$. Under the assumption that $P\{X_t = 1 | X_s = 1\}$ and $E(X_t | X_s = 1)$ are continuous in (s,t) we prove an inequality which implies that $P\{N(s,t) = 0\}$ and $EN(s,t)$ are finite, positive and continuous, where $N(s,t)$ is the number of jumps of the process in $[s,t[$. This result is then applied to prove that we can change the time scale of the process by either

$$\phi(t) = -\ln P\{N(0,t) = 0\}$$

or

$$\psi(t) = EN(0,t)$$

such that the transition probabilities satisfy a Lipschitz condition. This then again implies the existence of intensities for the process which can be found as the unique solution to the backward Kolmogorov equation.

2. General background.

The problem of finding an intrinsic time scale for a Markov process was first treated by Goodman [5] who proved that for a finite state space and under the assumption of continuity of the transition probabilities, one can change the time scale by means of

$$\alpha(t) = -\ln \operatorname{Det} P(0,t)$$

and obtain a Lipschitz condition for the transition probabilities. This is then used to obtain the existence of intensities almost everywhere and to prove that the process can be recovered as the unique solution to the backward Kolmogorov equations. Such a time scale is called an intrinsic time scale.

For the case of a countable state space it was proved in Goodman and Johansen [6] that under the assumption of uniform continuity of the transition probabilities one can apply a result of Doeblin [2] to obtain that

$$\phi(t) = -\ln P\{N(0,t) = 0\}$$

and

$$\psi(t) = EN(0,t)$$

are admissible intrinsic time scales.

Goodman [3] considered the semigroup of schlicht mappings of the unit disc into itself which have 0 as fixed point, and found that one could use

$$\eta(t) = -\ln D f_{0,t}(0)$$

as an intrinsic time scale, see also [4].

This has immediate applications to Markov branching processes with no deaths, since the generating function is a schlicht mapping of the unit disc into itself, such that $f(0) = 0$. An account of the application of these ideas to the imbedding problem for branching processes with no deaths is given in Johansen [7].

The purpose of the present note is to prove that for the general non-homogeneous Markov branching process one can apply Doob's Martingale inequation to obtain an inequality which proves that again ϕ and ψ are admissible intrinsic timescales.

The result is purely analytical but at the crucial point of the proof, a version of the underlying stochastic process is used. Thus one can either consider the result as a probabilistic proof of an analytic result or as an analytic formulation of the existence of a well behaved version of the process, i.e. a process given by its intensities and with a finite number of jumps on any finite interval.

The application to the imbedding problem will appear elsewhere.

3. A combinatorial inequality.

Let $\{X_t, t \geq 0\}$ be a non-homogeneous Markov branching process with $X_0 = 1$. It is well known that if $E X_t < \infty$ then $Y_t = X_t / E X_t$ is a martingale. It is however a special martingale in that it jumps with jumps of size almost equal to 1 for small values of t if $E X_t$ is close to 1. Let us introduce

$$h(t) = \sup_{0 \leq u \leq t} |1 - (E X_u)^{-1}|$$

$$k(t) = \sup_{0 \leq u \leq t} |1 - P\{X_u = 1\}|$$

and let D denote the partition

$$0 = u_0 < u_1 < \ldots . < u_N = t.$$

Then we can prove the following

<u>3.1. Lemma</u>. Let t be such that $h(t) \leq \frac{1}{4}$.

Then

$$1 - P\{X_u = 1, u \in D\} \leq 10(h(t) + k(t)) .$$

<u>Proof</u>. Let $\gamma^+(a,b)$ denote the number of upcrossings of the process $\{Y_u, u \in D\}$ over the interval $[a,b]$, see Neveu [9]. Similarly let $\gamma^-(a,b)$ denote the number of downcrossings.

If for some $u \in D$, $X_u \neq 1$, then either $X_u = 0$, in which case $Y_u = 0$ and $\gamma^-(0, 1 - h(t)) \geq 1$, or $X_u \geq 2$ in which case $Y_u \geq 2/ E X_u \geq 2(1 - h(t))$ which implies that $\gamma^+(1 + h(t), 2(1 - h(t))) \geq 1$.

In terms of probability this becomes:

$$1 - P\{X_u = 1, u \in D\}$$

$$\leq P\{\gamma^+(1 + h(t), 2(1 - h(t))) \geq 1\} + P\{\gamma^-(0,1 - h(t)) \geq 1\}$$

$$\leq E\ \gamma^+(1 + h(t), 2(1 - h(t)) + E\ \gamma^-(0, 1 - h(t)).$$

We next evaluate this by Doob's inequality and obtain

$$(1 - 3h(t))\ E\ \gamma^{+}(1 + h(t),\ 2(1 - h(t)) \leqq \sup_{u\in D} E\left(1 + h(t) - \frac{X_u}{E\ X_u}\right)^{+}$$

and

$$(1 - h(t))\ E\ \gamma^{-}(0,\ 1 - h(t)) \leqq \sup_{u\in D} E\left(\frac{X_u}{E\ X_u} - (1 - h(t))\right)^{+}.$$

Continuing we obtain

$$E\left(1 + h(t) - \frac{X_u}{E\ X_u}\right)^{+} = (1 + h(t))\ p_u(0) + \left(1 + h(t) - \frac{1}{E\ X_u}\right) p_u(1)$$

$$\leqq (1 + h(t))\ (1 - p_u(1)) + 2h(t)\ p_u(1) \leqq 2(h(t) + k(t)),$$

since $k/\ E\ X_u \geqq k(1 - h(t)) > (1 + h(t))$ whenever $k \geqq 2$ and $h(t) \leqq \frac{1}{4}$. Here $p_u(k) = P\{X_u = k\}$. Similarly we get

$$E\left(\frac{X_u}{E\ X_u} - (1 - h(t))\right)^{+} = \sum_{k=1}^{\infty} \left(\frac{k}{E\ X_u} - (1 - h(t))\right) p_u(k)$$

$$= \frac{E\ X_u}{E\ X_u} - (1 - h(t))\ (1 - p_u(0)) \leqq h(t) + p_u(0) \leqq h(t) + k(t).$$

If we combine these inequalities and use that $1 - 3h(t) \geqq 1/4$ and $(1 - h(t)) \geqq 3/4$ then we get the result of the Lemma.

<u>3.2. Corollary</u>. If $E(X_t|\ X_s = 1)$ and $P\{X_t = 1|\ X_s = 1\}$ are continuous functions of (s,t), then

$$g(s,t) = P\{X_u = 1,\ s \leqq u \leqq t|\ X_s = 1\} = P\{N(s,t) = 0|\ X_s = 1\}$$

is positive and continuous.

Proof. Let $\{X_t\}$ be constructed as the second canonical process, then we can use the relation

$$P\{X_u = 1,\ 0 \leqq u \leqq t\} = \inf_D P\{X_u = 1,\ u \in D\},$$

see Meyer [8] or Goodman and Johansen [6].

From the inequality we obtain

$$1 - P\{X_u = 1,\ 0 \leqq u \leqq t\} \leqq 10(h(t) + k(t))$$

whenever t is so small that $h(t) \leqq 1/4$. This implies the continuity at $t = 0$. From the multiplicativity

$$g(0,t) = g(0,u)\ g(u,t), \qquad 0 \leqq u \leqq t,$$

the rest follows.

3.3. Corollary. If $E(X_t | X_s = 1)$ and $P\{X_t = 1 | X_s = 1\}$ are continuous functions of (s,t), then $E\ N(s,t)$ is finite and continuous.

Proof. We first prove an inequality:

$$\begin{aligned}
P\{X_u \neq X_s\} &= \sum_m P\{X_u \neq m | X_s = m\}\ P\{X_s = m\} \\
&= \sum_m (1 - P\{X_u = m | X_s = m\})\ P\{X_s = m\} \\
&\leqq \sum_m (1 - P\{X_v = m,\ s \leqq v \leqq u | X_s = m\})\ P\{X_s = m\} \\
&= \sum_m (1 - g(s,u)^m)\ P\{X_s = m\} \\
&\leqq (1 - g(s,u)) \sum_m m\ P\{X_s = m\} \leqq (\phi(u) - \phi(s))\ E\ X_s,
\end{aligned}$$

where $\phi(u) = -\ln g(0,u)$.

Now we can find $E\,N(s,t)$ as follows

$$E\,N(s,t) = \sup_{S} \sum_{k=1}^{n} E\,I\{X_{s_k} \neq X_{s_{k-1}}\},$$

where S is the partition $s = s_0 < s_1 < \dots < s_n < t$ of the interval $[s,t[$.

Using the above inequality we get

$$E\,N(s,t) \leqq \sup_{s \leqq u \leqq t} E(X_u \mid X_s = 1) \sup_{S} \sum_{k=1}^{n} (\phi(s_k) - \phi(s_{k-1}))$$

$$\leqq (\phi(t) - \phi(s)) \sup_{s \leqq u \leqq t} E(X_u \mid X_s = 1).$$

This proves the finiteness of $E\,N(s,t)$ and the continuity for $s = t$. The additivity

$$E\,N(s,t) = E\,N(s,u) + E\,N(u,t)$$

then implies the rest.

4. The change of time scale.

Let the generating function $f_{s,t}$ be defined by

$$f_{s,t}(z) = \sum_{k=0}^{\infty} z^k P\{X_t = k \mid X_s = 1\}.$$

We then have the Chapman-Kolmogorov equation

(4.1) $$f_{s,u}(f_{u,t}(z)) = f_{s,t}(z), \qquad 0 \leqq z \leqq 1, \quad s \leqq u \leqq t,$$

(4.2) $$f_{s,t}(z) = z, \qquad s = t.$$

We also assume continuity:

(4.3) $f_{s,t}(z)$ is continuous in (s,t) uniformly in $z \in [0,1]$.

We shall call the process regular if

(4.4) $E\, X_t = D\, f_{0,t}(1)$ is finite and continuous.

The family $\{f_{s,t},\ 0 \leqq s \leqq t < \infty\}$ satisfying (4.1) - (4.4) will be called a regular continuous family. We shall deal with this family rather than the family $\{P(s,t),\ 0 \leqq s \leqq t < \infty\}$ of transition probabilities for the non-homogeneous Markov branching process.

<u>4.1 Proposition</u>. For a regular process there exists a change of time scale

$$\phi(t) = -\ln P\{N(0,t) = 0\}$$

or

$$\psi(t) = E\, N(0,t),$$

such that the functions $f_{\cdot,t}(z)$ satisfy a Lipschitz condition uniformly in $z \in [0,1]$.

<u>Proof</u>. From Corollary 3.2 we get that ϕ and ψ are well defined. We can then change the time scale as follows.

If $u = \phi(t)$, $v = \phi(s)$ then we define

$$_{*}f_{v,u}(z) = f_{s,t}(z).$$

This is a consistent definition, since if for instance $u = \phi(t_1) = \phi(t_2)$, $t_1 < t_2$, then $g(t_1,t_2) = 1$ and hence

$$P\{X_{t_2} = 1 \mid X_{t_1} = 1\} \geqq g(t_1,t_2) = 1$$

which implies that $f_{t_1,t_2}(z) = z$ and hence

$$f_{s,t_2}(z) = f_{s,t_1}(f_{t_1,t_2}(z)) = f_{s,t_1}(z).$$

Similarly one can see that ${}_*f_{s,t}$ satisfies the continuity conditions (4.3) and (4.4).

A similar consideration shows that we could use ψ as the change of time scale.

Let now $i(z) = z$ and for any function $g: [0,1] \to [-1,1]$ we define $|g| = \sup_z |g(z)|$.

The basic inequality is now the following: For $s < u < t$,

$$|f_{u,t} - f_{s,t}| = |f_{u,t} - f_{s,u} \circ f_{u,t}|$$

$$= |i - f_{s,u}| \leqq 1 - P\{X_u = 1 \mid X_s = 1\}$$

$$\leqq 1 - g(s,u) \leqq \phi(u) - \phi(s).$$

If the time scale is changed to ϕ this inequality implies that the function $f_{\cdot,t}(z)$ satisfies a Lipschitz condition uniformly in z.

If ψ had been used we would evaluate as follows:

$$1 - g(s,u) = P\{N(s,u) \geq 1\} \leq E\, N(s,u) = \psi(u) - \psi(s).$$

Now we prove the main result which follows from the inequality.

4.2 Theorem. For a regular continuous process there exists a change of time scale such that the intensities

$$(4.5) \qquad h_s = \lim_{u\downarrow 0, v\downarrow 0} (f_{s-u,t} - f_{s+v,t})(u+v)^{-1}$$

exist for $s \notin N$, where N is a null set for Lebesgue measure. The intensities satisfy

$$(4.6) \qquad \int_s^t D\, h_u(1)du < \infty, \qquad s < t.$$

The derivatives

$$(4.7) \qquad \partial_s f_{s,t} = \lim_{u\downarrow 0, v\downarrow 0} (f_{s+v,t} - f_{s-u,t})(u+v)^{-1}$$

exist for $s \notin N$ and $t > s$.

When the intensities satisfy (4.6) the function $f_{\cdot,t}(z)$ is given as the unique solution to the backward Kolmogorov equation:

$$(4.8) \qquad \partial_s f_{s,t}(z) = -h_s(f_{s,t}(z)), \qquad s \notin N$$

or

$$(4.9) \qquad z - f_{s,t}(z) = -\int_s^t h_v(f_{v,t}(z))dv, \qquad s \leq t,$$

with initial condition (4.2).

Proof. Let us first change the time scale by ϕ. Using Proposition 4.1

we then immediately get that the derivatives

$$(4.10)\qquad \partial_s f_{s,t}(z) = \lim_{u\downarrow 0, v\downarrow 0} (f_{s+v,t}(z) - f_{s-u,t}(z))\ (u + v)^{-1}$$

exist for $s \notin N$ as long as t is rational and z is rational.

Now define for $s > u > 0$, $v > 0$,

$$(4.11)\qquad g_{s,u,v}(z) = (f_{s-u,s+v}(z) - z)\ (u + v)^{-1} + z, \qquad 0 \leqq z \leqq 1,$$

then it follows from

$$|z - f_{s-u,s+v}(z)| \leqq 1 - P\{X_{s+v} = 1 |\ X_{s-u} = 1\}$$

$$\leqq 1 - g(s - u, s + v) \leqq u + v,$$

that $g_{s,u,v}$ is a probability generating function.

Now consider

$$(4.12)\qquad g_{s,t,u,v}(z) = g_{s,u,v}(f_{s+v,t}(z))$$

$$= (f_{s-u,t}(z) - f_{s+v,t}(z))\ (u + v)^{-1} + f_{s+v,t}(z).$$

If we use (4.10) then we get that for t and z rational and $s \notin N$, the limit as $u \downarrow 0$, $v \downarrow 0$ exists and we define

$$(4.13)\qquad g_{s,t}(z) = \lim_{u\downarrow 0, v\downarrow 0} g_{s,t,u,v}(z) = -\partial_s f_{s,t}(z) + f_{s,t}(z).$$

We want to extend this convergence to all z. But $g_{s,t,u,v}(\cdot)$ is an increasing convex function and that easily implies that the limit exists for all $0 \leqq z < 1$ and that the limit function $g_{s,t}(\cdot)$ is continuous on $[0,1[$.

For $z = 1$ we have $g_{s,t,u,v}(1) = 1$, which means that (4.13) holds for $z \in [0,1]$, t rational and $s \notin N$.

We next want to prove that $g_{s,t}(\cdot)$ is continuous at 1, i.e. $\lim_{z \uparrow 1} g_{s,t}(z) = 1$.

To prove this consider

$$\int_{s_1}^{s_2} g_{s,t}(z)ds = \int_{s_1}^{s_2} f_{s,t}(z)ds + f_{s_1,t}(z) - f_{s_2,t}(z)$$

The right hand side converges to $(s_2 - s_1)$ as $z \uparrow 1$ and the integrand on the left hand is monotone in z. By the theorem of monotone convergence we get that

$$\int_{s_1}^{s_2} g_{s,t}(1)ds = (s_2 - s_1),$$

which implies that $g_{s,t}(1) = 1$, $s \notin N$, t rational.

Hence $g_{s,t}(\cdot)$ is continuous on $[0,1]$ and this means that the convergence (4.13) is uniform in z and that $g_{s,t}$ is a probability generating function.

Next consider

$$|g_{s,u,v}(f_{s+v,t}(z)) - g_{s,u,v}(f_{s,t}(z))|$$

where t is rational, $z < 1$ and $s \notin N$. Take ε such that $f_{s,t}(z) < 1 - \varepsilon$ and find v_0 such that also $f_{s+v,t}(z) < 1 - \varepsilon$ for $v \leqq v_0$. Then this difference can be evaluated by

$$D\, g_{s,u,v}(1 - \varepsilon)|\, f_{s+v,t}(z) - f_{s,t}(z)| \leqq \varepsilon^{-2}\, v,$$

since for any probability generating function $g(z) = \Sigma_{k=0}^{\infty} z^k p_k$ we have

$$D\, g(z) = \sum_{k=0}^{\infty} k\, z^{k-1}\, p_k \leqq \sum_{k=0}^{\infty} k\, z^{k-1} = (1 - z)^{-2}.$$

This proves that

$$g_{s,t}(z) = \lim_{u\downarrow 0, v\downarrow 0} g_{s,u,v}(f_{s,t}(z)),$$

for t rational, $s \notin N$, $z < 1$. It is easily seen to hold for $z = 1$ as well.

This implies the existence of the limit

$$g_s(z) = \lim_{u\downarrow 0, v\downarrow 0} g_{s,u,v}(z)$$

for any $z \in \,]0,1]$, $s \notin N$.

Because for any such z and s, we can find t rational such that $f_{s,t}(0) < z$, but then $z = f_{s,t}(w)$ for some $w \in [0,1]$.

Again $g_{s,u,v}$ is increasing and convex and converges on $]0,1]$. This easily implies that it converges on $[0,1]$ and that the limit g_s is continuous. This implies that g_s is a probability generating function and that the convergence is uniform in $z \in [0,1]$.

Let us now define $h_s(z) = g_s(z) - z$, then (4.5) is proved.

Going back to (4.12) we let $u \downarrow 0$ and $v \downarrow 0$ and we obtain for $z \in [0,1]$, $s \notin N$ and any t,

(4.14) $$g_s(f_{s,t}(z)) = -\partial_s f_{s,t}(z) + f_{s,t}(z),$$

i.e. the derivatives exist and the convergence is uniform. This proves (4.7) and (4.8) and after integration (4.9).

In order to prove (4.6) we use (4.14) and obtain

$$\int_{s_1}^{s_2} \frac{g_s(f_{s,t}(z)) - 1}{z - 1} ds = \frac{f_{s_1,t}(z) - f_{s_2,t}(z)}{z - 1} + \int_{s_1}^{s_2} \frac{f_{s,t}(z) - 1}{z - 1} ds.$$

Now let $z \uparrow 1$. By the monotone convergence theorem and using the convexity of $f_{s,t}$ and $g_s \circ f_{s,t}$ we prove that

$$\int_{s_1}^{s_2} D\, g_s(1)\, D\, f_{s,t}(1) ds = D\, f_{s_1,t}(1) - D\, f_{s_2,t}(1) + \int_{s_1}^{s_2} D\, f_{s,t}(1) ds,$$

which means that $D\, g_s(1)$ is locally integrable. This proves (4.6). Finally we only have to prove that $f_{\cdot,t}(z)$ is uniquely given as a solution to (4.8). This follows however from Carathéodory [1] p. 674, since the function h_s satisfies the Lipschitz condition

$$|h_s(z) - h_s(z')| \leqq D\, h_s(1)\, |z - z'|$$

where $D\, h_s(1)$ is locally integrable.

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