K. B. Athreya

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Institute of Mathematical Statistics
University of Copenhagen

STOCHASTIC ITERATION OF STABLE PROCESSES

BY

K. B. ATHREYA.

Abstract. Let $\{X_n(t); -\infty<t<\infty\}$ n=1,2,... be a sequence of i.i.d. stable processes of order α so that for each n, w.p.1. $X_n(0) = 0$, $\{X_n(t)\}$ has independent increments, and $|t|^{-\alpha^{-1}} X_n(t)$ has the same distribution of $X_n(1)$. Define $Y_1(t) = X_1(t)$ and recursively $Y_{n+1}(t) = X_{n+1}(Y_n(t))$ for $n \geq 1$. It is shown here that

a) if $\alpha<1$ then $\alpha^n \log|Y_n(t)| - \log|Y_o(t)|$ converges w.p.1. to a realvalued random variable Y w.p.1. whose distribution is independent of t.

b) if $\alpha=1$ then $\sigma^{-1} n^{-\frac{1}{2}}(\log|Y_n(t)| - n\mu) \overset{d}{\to} N(0,1)$ where μ and σ^2 are the mean and variance of $\log|X_1(1)|$ and $\overset{d}{\to}$ means convergence in distribution.

c) if $\alpha>1$ then $\log|Y_n(t)|$ converges in distribution and the limit is independent of t.

The limiting behavior of $Y_n(t)$ and $|Y_n(t)|$ are also deduced from the above.

Keywords: Stochastic iteration, stable processes.

Footnotes.

1. The work was initiated at the University of Wisconsin, Madison, U.S.A. and completed at the Institut For Matematisk Statistik, University of København, Denmark.

2. The author is on leave from the Indian Institute of Science Bangalore 560012, India.

3. This is a revised version of an invited talk given by the author at the Summer Institute on Stochastic Processes at Bloomington, Indiana, U.S.A. in July - August 1974.

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K. B. ATHREYA [1,2,3,4.]

1. Introduction. In a recent paper [1] the concept of stochastic iteration was introduced by the author and was shown to be a generalization of the notion of branching processes. Roughly speaking the idea is this. Let $\{X_n(t,\omega);\ t\in T\}$ $n=1,2,...$ be a sequence of stochastic processes defined on a common probability space (Ω,B,P) where the index set T is also the state space of the processes. Define $Y_1(t,\omega) = X_1(t,\omega)$ and recursively $Y_{n+1}(t,\omega) = X_{n+1}(Y_n(t,\omega),\omega)$ for $n\geq 1$. Then the sequence $\{Y_n\}$ is called the stochastic iterate of $\{X_n\}$. If the X_n's are i.i.d. random walks on the nonnegative integer lattice, then for each t the sequence $\{Y_n(t);\ n=1,2,...\}$ is a Galton-Watson branching process with $Y_1(t)$ as the initial number and the distribution generating the random walk as the offspring distribution. If the X_n's are i.i.d. random walks on the whole integer lattice then for each t the sequence $\{Y_n(t)\}$ is a selfannihilating branching process (see [3]). If the X_n's are i.i.d. processes on $[0,\infty)$ with stationary and independent nonnegative increments then for each t the sequence $\{Y_n(t)\}$ is a continuous state space branching process with $Y_1(t)$ as the initial amount. One can go on like this and realize branching processes in random environments, branching processes with state dependent population growth and so on as stochastic iterates. It is wellknown (see [1]) that in most branching processes context the population either dies out or explodes and there is no stability. A natural question is

what happens to the stochastic iterate $\{Y_n\}$ of a sequence $\{X_n\}$ more general than the ones mentioned in the above examples. Specifically, what is the limiting behavior of $\{Y_n\}$. We answer this question fairly completely when the X_n's are i.i.d. stable processes on $(-\infty,\infty)$. There is a trichotomy depending on the order α of the process. There is stability if and only if $\alpha>1$. For $\alpha<1$, $|Y_n|^{\alpha^n}$ has a limit and for $\alpha=1$ $(|Y_n|e^{-n\mu})^{n^{-\frac{1}{2}}}$ is asymptotically normal for some appropriate μ. For $\alpha>1$, $Y_n(t)$ has a limit distribution independent of $t\neq 0$.

<u>2. Statement of the results</u>. Let $\{X_n(t,\omega);\ -\infty<t<\infty\}$ be a sequence of i.i.d. processes defined by the conditions

i) $\{X_1(t,\omega);\ t\geq 0\}$ has stationary independent increments, $X_1(0,\omega)=0$ w.p.1. and for any $t>0$, $t^{-\alpha^{-1}}X_1(t)$ has a distribution independent of t.

ii) $\{X_1(-t,\omega);\ t\geq 0\}$ is an independent copy of $\{X_1(t);\ t\geq 0\}$.

These hypotheses imply that the characteristic function of $X_1(t)$ is stable and is of the form $\exp(t\psi(\theta))$ where

$$\psi(\theta) = \begin{cases} ia\theta - c|\theta|^{\alpha}\{1+i\beta\frac{\theta}{|\theta|}\tan\frac{\pi\alpha}{2}\} & \text{if } \alpha\neq 1 \\ ia\theta - c|\theta|^{\alpha}\{1+i\beta\frac{\theta}{|\theta|}\frac{2}{\pi}\log|\theta|\} & \text{if } \alpha=1 \end{cases}$$

with a and β real, $|\beta|\leq 1$, $c>0$, $0<\alpha\leq 2$.

Further since $t^{-\alpha^{-1}}X_1(t)$ has the same distribution as $X_1(1)$ the function $\psi(\theta)$ must satisfy $t\psi(\theta t^{-\alpha^{-1}}) = \psi(\theta)$ for all θ real and $t>0$.

This imposes the following constraints on a and β

i) If $\alpha \neq 1$ then $a = 0$ and hence if $\alpha=2$ then $\{\frac{X_1(t,\omega)}{c}$; $t \geq 0\}$ is a standard Brownian motion process. Note, however, that if $\alpha \neq 1$ and $\alpha < 2$ then the process need not even be symmetric about the origin.

ii) If $\alpha=1$ then $\beta=0$ but a need not vanish. Thus $\{X_1(t)$; $t \geq 0\}$ is a noncentered Cauchy process with drift at and a scale coeffecient c. Thus $\{\frac{X_1(t) - at}{c}$; $t \geq 0\}$ is a standard Cauchy process.

If $\alpha \geq 1$ then the support of the probability distribution of $X_1(1)$ is the whole real line. This is so since the density function is analytic at least on the strip $|\mathrm{Im}Z| < c$. If $\alpha \leq 1$ and $|\beta| = 1$ then the distribution of $X_1(1)$ is one-sided (bounded on the left if $\beta = -1$ and on the right if $\beta = +1$). For proofs of those observation see [4]. Notice that these are consistent with our results. If the distribution of $X_1(1)$ was one-sided then from the classical branching process theory we would expect instability for $\{Y_n\}$. This is indeed the case even if $X_1(1)$ were not onesided so long as $\alpha < 1$. We are now ready to state our results.

Theorem 1. Let $0 < \alpha < 1$. Then,

i) $\lim_{n \to \infty} |Y_n(t,\omega)|$ exists w.p.1., but assumes only two values namely 0 and ∞.

ii) $\lim_{n \to \infty} |Y_n(t,\omega)|^{\alpha^n}$ exists w.p.1. and equals $\exp\{|t| + Z(t,\omega)\}$ wher $Z(t,\omega)$ is a real valued random variable having on absolutely continuons distribution that is identical to that of

$\sum_1^\infty \alpha^j \mu_j$ where $\{\mu_j: j=1,2,...\}$ are i.i.d. as $\log|X_1(1)|$. Thus, the distribution of $Z(t,\omega)$ is independent of t.

iii) On the set $\{\omega: |t| + Z(t,\omega) > 0\}$ $\lim_n |Y_n(t,\omega)| = \infty$ but $\{Y_n(t,\omega)\}$ need not converge if $P\{X_1(1)<0\}P\{X_1(1)>0\} > 0$. On the set $\{\omega: |t| + Z(t,\omega) < 0\}$ $\lim_n |Y_n(t,\omega)| = 0$ and hence $Y_n(t,\omega) \to 0$ also.

Theorem 2. Let $\alpha=1$ and $\mu=E\log|X_1(1)|$. Then;

i)
$$\lim_{n\to\infty} |Y_n(t,\omega)| = \begin{cases} \infty & \text{w.p.1. if } \mu>0 \\ 0 & \text{w.p.1. if } \mu<0 \end{cases}$$

ii) If $\mu=0$ then w.p.1. both 0 and ∞ are limit points for the sequence $\{|Y_n(t,\omega)|\}$.

iii) There exist no normalising sequence c_n such that $c_n|Y_n|$ converges w.p.1. or in law to a nondegenerate proper limit distribution.

iv) $(|Y_n|e^{-n\mu})^{n^{-\frac{1}{2}}} \xrightarrow{d} e^{N\sigma}$ where N is a standard normal random variable, σ^2 the variance of $\log|X_1(1)|$ and $\xrightarrow{d}$ means convergence in distribution.

Theorem 3. Let $\alpha>1$. Then, $Y_n(t,\omega) \xrightarrow{d} Y$ where the random variable Y has a distribution function independent of t given by

$$P(Y \leqq y) = \int_0^\infty G(y|x|^{\alpha^{-1}})\, dH(x)$$

where $G(x) = P(X_1(1) \leqq x)$, $H(x) = P\{\exp(\sum_{j=1}^\infty \alpha^{-j}\mu_j) \leqq x\}$ the μ_j's being i.i.d. as $\log|X_1(1)|$. Further, the sequence $\{Y_n(t,\omega)\}$ cannot converge with probability one.

3. Proof of the theorems.

Fix $t\neq 0$. Then since the X_n's are stable processes we conclude that $Y_n(t,\omega)\neq 0$ for any n. Thus, we may take logarithm of $|Y_n|$ to yield

$$Z_{n+1}(t,\omega) = \eta_{n+1}(t,\omega) + \alpha^{-1} Z_n(t,\omega) \qquad (1)$$

where $Z_n(t,\omega) = \log|Y_n(t,\omega)|$,

$$\eta_{n+1}(t,\omega) = \log\left|\frac{X_{n+1}(Y_n(t,\omega),\omega)}{|Y_n(t,\omega)|^{\alpha^{-1}}}\right|$$

for $n=0,1,2,\ldots$ (By definition, $Y_o(t,\omega)=t$ w.p.1.) Suppressing t and ω for convenience and iterating (1) yields

$$Z_{n+1} = \sum_{j=0}^{n} \alpha^{-j}\eta_{n+1-j} + \alpha^{-(n+1)}\log|t|. \qquad (2)$$

Let $F_n \equiv \sigma(Y_j(t);\ j=1,2,\ldots,n)$ be the sub-σ-algebra generated by $Y_j(t)$ $j=1,2,\ldots,n$ of the basic σ-algebra $\acute{B}$ of the triplet $(\Omega,\acute{B},P)$ on which all the stochastic processes mentioned so far are defined. It is clear that the random variables $\{\eta_j\}$ are adapted to this family $\{F_j\}$ and that the conditional distribution of η_{j+1} given F_j is independent of the conditioning and the same as that of $\log|X_1(1)|$. This yields the following

Lemma 1. Fix $t \neq 0$. The random variables $\{\eta_j(t,\omega);\ j=1,2,\ldots\}$ are i.i.d. as $\log|X_1(1)|$ where the η_j's are as in (1).

With this background we now proceed to the proofs of the three theorems.

Proof of Theorem 1. Here $\alpha<1$. Multiply both sides of (2) by α^{n+1} to get

$$\alpha^{n+1}Z_{n+1} = \sum_{j=1}^{n} \alpha^{j}\eta_j + \log|t|. \qquad (3)$$

Since the η_j's are identically distributed and since $\log|X_1(1)|$ has a finite mean (infact, all its moments are finite) we get $E(\sum_{j=1}^{\infty} \alpha^j |\eta_j|) = (E|\eta_1|)(1-\alpha)^{-1} < \infty$ and hence that $\sum_{j=1}^{j} \alpha^j \eta_j$ converges w.p.1. This proves part (ii) of Theorem 1. Part (i) is a trivial consequence of part (ii). Part (iii) follows by noting that under the hypothesis $P(X_1(1)>0)P(X_1(1)<0) > 0$ the sets $\{\omega: Y_n(t,\omega)$ is ultimately positive$\}$ and $\{\omega: Y_n(t,\omega)$ is ultimately negative$\}$ have probability zero.

Proof of Theorem 2. Here $\alpha=1$. Equation (2) yields

$$Z_{n+1} = \sum_{j=1}^{n+1} \eta_j + \log|t|.$$

The random variable $\log|X_1(1)|$ has all moments and in particular the mean and the variance. Part (i) easily follows by the strong law of large numbers. When $\mu=0$, the law of the iterated logarithm asserts that w.p.1. the sequence

$\{ \frac{1}{\sqrt{2n\log\log n}} \sum_{j=1}^{n} \eta_j : n=1,2,\ldots\}$ has the entire interval $[-\sigma,+\sigma]$ as the set of its limit points. This shows (ii) and the fact that the sequence $\{Z_n\}$ does not converge at all. To prove (iii) we just note that since the η_j's are i.i.d. with finite mean and variance there cannot exist constant constants d_n such that $\sum_{j=1}^{n} \eta_j - d_n$ converges in law or w.p.1. Finally part (iv) follows from the central limit theorem.

Proof of Theorem 3. Here $\alpha>1$. Equation (2) and the fact that the X_n's are i.i.d. yields the conclusion that Z_{n+1} has the same distribution as

$$Z^*_{n+1} = \sum_{j=1}^{n} \alpha^{-j} \eta_j + \alpha^{-(n+1)} \log|t|.$$

As in the proof of Theorem 1, the series $\sum_{j=1}^{\infty} \alpha^{-j} \eta_j$ converges w.p.1. and the sequence Z^*_{n+1} converges w.p.1. to $\sum_{j=1}^{\infty} \alpha^{-j}\eta_j$. This shows that $|Y_n|$ converges in distribution to $H(\cdot)$. Now note that

$$P(Y_{n+1}\leq y) = P\left(\frac{X_{n+1}(Y_n)}{|Y_n|^{\frac{1}{\alpha}}} \leq y|Y_n|^{-\frac{1}{\alpha}}\right)$$
$$= {}_0\int^{\infty} G(y|x|^{-\frac{1}{\alpha}})\,dP(|Y_n|\leq x).$$

The function $G(y|x|^{-\frac{1}{\alpha}})$ is continuous for x in $(0,\infty)$ and bounded. Also $|Y_n| \xrightarrow{d} H$. Thus $Y_n \xrightarrow{d} F$ where F is defined in Theorem 3. The limiting distribution F is easily seen to be nondegenerate. Now $\{Y_n\}$ being a Markov Chain having a nondegenerate limiting distribution cannot converge with probability one.

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