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1. Introduction.

The purpose of this paper is to exhibit a path decomposition for conditional Brownian motion in several dimensions (as defined by Doob [1]) which is in a similar vein to the path decomposition for one-dimensional diffusions described by Williams [6]. Fix an integer $k \geqq 3$ and consider Brownian motion in $\mathbb{R}^k$ ($BM(\mathbb{R}^k)$). Given two distinct points b and c in $\mathbb{R}^k$ let us say that a stochastic process

$$X = \{X(t),\ 0 \leqq t < \zeta_X\}$$

with continuous paths in $\mathbb{R}^k$ and lifetime ζ_X is BM($\mathbb{R}^k$) started at b and conditioned to converge to c ($BM(\mathbb{R}^k)_{b\to c}$) if X is a time homogeneous Markov process with starting state $X(0) = b$ and transition density $\vec{p}^{\,c}$ given by

$$\vec{p}^{\,c}(t,x,y) = p(t,x,y)\ g(c,y)/\ g(c,x), \qquad t \geqq 0, \quad x,y \in \mathbb{R}^k,$$

where p denotes the transition density of $BM(\mathbb{R}^k)$ and $g(x,y)$ is the free space Green function in $\mathbb{R}^k$,

$$g(x,y) = \int_0^\infty p(t,x,y)dt = \frac{1}{4}\,\Pi^{-\frac{k}{2}}\,\Gamma(\tfrac{k}{2} - 1)\ |x - y|^{2-k},$$

where $|\ \ |$ denotes the Euclidean distance.

Such a process X always exists, and with probability one ζ_X is finite, $X(\zeta_X^-) = c$, and $X(t)$ avoids both b and c for $0 < t < \zeta_X$.

Let us now take c to be the origin 0 in $\mathbb{R}^k$, and b a point distinct from 0. A $BM(\mathbb{R}^k)_b$ process (i.e. a $BM(\mathbb{R}^k)$ started at b) has probability zero of ever hitting the origin, but $BM(\mathbb{R}^k)_{b\to 0}$ can be described as the limit as $\varepsilon \to 0$ of $BM(\mathbb{R}^k)$ conditioned to hit the sphere $S(\varepsilon)$ of radius ε about 0 and killed at first contact with $S(\varepsilon)$. Indeed, if for $\varepsilon > 0$ and an $\mathbb{R}^k$-valued process $X = \{X(t),\ 0 \leqq t < \zeta_X\}$

we set

$$\tau_X^\varepsilon = \inf\{t : 0 < t < \zeta_X, \ |X(t)| \leqq \varepsilon\},$$

where $\inf \phi = \infty$, then it is easy to show that an $\mathbb{R}^k$-valued process X with continuous paths and lifetime ζ_X is a $BM(\mathbb{R}^k)_{b\to 0}$ process if and only if

$$\zeta_X = \lim_{\varepsilon\to 0} \tau_X^\varepsilon$$

and for each $\varepsilon > 0$ the process

$$\{X(t), \ 0 \leqq t \leqq \tau_X^\varepsilon\}$$

is identical in law to

$$\{Y(t), \ 0 \leqq t \leqq \tau_Y^\varepsilon\} \text{ conditional on } \{\tau_Y^\varepsilon < \infty\}$$

for a $BM(\mathbb{R}^k)_b$ process Y.

It is an important fact that the time reversal of a $BM(\mathbb{R}^k)_{b\to c}$ process is a $BM(\mathbb{R}^k)_{c\to b}$ process. This has the consequence that if K is a compact set containing b and σ_X^K is the last time that a $BM(\mathbb{R}^k)_b$ process X is in K (which is a.s. finite by the transience of $BM(\mathbb{R}^k)$ for $k \geqq 3$), then X can be defined so that the conditional distribution of the process

$$\{X(t), \ 0 \leqq t \leqq \sigma_X^K\}$$

given $X(\sigma_X^K) = x$ is the distribution of a $BM(\mathbb{R}^k)_{b\to x}$ process.

Our main object is to prove the following theorem, which provides a straightforward construction of the $BM(\mathbb{R}^k)_{b\to 0}$ process that does not involve any conditioning.

<u>Theorem</u> Let $k \geqq 3$ be an integer and let $b \in \mathbb{R}^k$. On a suitable probability triple $(\Omega,\mathfrak{F},P)$ set up three independent random elements:

(i) a random variable V taking values in the interval $[|b|,\infty)$ with

$$P\{V > v\} = |b|^{k-2}/v^{k-2}, \qquad v \geqq |b|;$$

(ii) a $BM(\mathbb{R}^k)_b$ process $X_b = \{X_b(t),\ t \geqq 0\}$;

(iii) a $BM(\mathbb{R}^k)_0$ process $X_0 = \{X_0(t),\ t \geqq 0\}$;

For $a = b$ or 0 define

$$\tau_a^V = \inf\{t: |X_a(t)| = V\},$$

set $\xi_a = X_a(\tau_a^V)$ and define

$$\begin{aligned} X(t) &= X_b(t), & 0 \leqq t \leqq \tau_b^V \\ &= T_{\xi_0\xi_b} X_0(\tau_b^V + \tau_0^V - t), & \tau_b^V \leqq t < \tau_b^V + \tau_0^V, \end{aligned}$$

where for $x, y \in \mathbb{R}^k$ with $|x| = |y| > 0$, T_{xy} is the unique orthogonal transformation of $\mathbb{R}^k$ which takes the point x to the point y by a rotation with axis through 0 perpendicular to the plane of the points $\{0,x,y\}$. Then

$$\{X(t),\ 0 \leqq t < \tau_b^V + \tau_0^V\}$$

is a $BM(\mathbb{R}^k)_{b\to 0}$ process.

Notice that the random variable V is equal to the furthest distance that the process X ever gets from the point 0 to which it converges, and that the time τ_b^V is the unique time at which the distance is achieved. Thus the theorem provides a decomposition of the path of this $BM(\mathbb{R}^k)_{b\to 0}$ process X into the two path fragments before and after the time $\rho_X = \tau_b^V$

at which the path is furthest away from the point 0, in the sense that the joint distribution of the two fragments is specified as follows: $|X(\rho_X)| = V$ has the stated distribution, conditional on $V = v$ the pre-ρ_X process is $BM(\mathbb{R}^k)_b$ run until it first hits the sphere $S(v)$ centred on 0 with radius v, and conditional on the whole pre-ρ_X process the post-ρ_X process reversed is $BM(\mathbb{R}^k)_0$ run until it first hits $S(v)$ rotated so that it meets up with the pre-ρ_X path at $X(\rho_X) = \xi_b$ (this rotation having the same effect as conditioning the $BM(\mathbb{R}^k)_0$ to hit $S(v)$ at ξ_b). It should be observed that the random time ρ_X is neither a stopping time of X nor of its reverse, but ρ_X is a splitting time of X in the sense of Jacobsen [3], and there is a Markovian split at ρ_X in the sense that the pre-ρ_X and the post-ρ_X processes are conditionally independent given $X(\rho_X)$.

A reversal of time shows that there is a similar decomposition of $BM(\mathbb{R}^k)_{b\to c}$ at the time when it is furthest from its starting point b. Put together with the earlier remark about last exit times it is easy to see how this implies the following decomposition for the path of a $BM(\mathbb{R}^k)_0$ process prior to the last time it leaves the sphere $S(r)$ of radius r about 0:

Corollary. Let $r > 0$ and let b be a point in $\mathbb{R}^k$ on the sphere $S(r)$. Let V, X_b and X_0 be just as in the theorem, and define τ_a^V and ξ_a as before. Set

$$X^*(t) = X_0(t), \qquad 0 \leq t \leq \tau_0^V$$

$$= T_{\xi_b\xi_0} X_b(\tau_0^V + \tau_b^V - t), \qquad \tau_0^V \leq t < \tau_0^V + \tau_b^V.$$

Then the process

Then the process

$$\{X^*(t),\ 0 \leq t < \tau_0^V + \tau_b^V\}$$

is identical in law to a $BM(\mathbb{R}^k)_0$ process prior to the last time it leaves the sphere $S(r)$.

The theorem above is intimately related to the path decomposition for one-dimensional diffusions given by Williams [6], and indeed the proof of the theorem which is provided in the next section relies on an application of Williams' one-dimensional result to the radial parts of the processes considered. The theorem can also be viewed as a specially neat case of a decomposition which applies in several situations to the path of a Brownian motion in a domain D which has been conditioned to converge to a point on the Martin Boundary of D. This more general decomposition is discussed in the final section.

2. Proof of the theorem.

Suppose throughout that k is an integer with $k \geq 3$. Let S^{k-1} denote the sphere $\{x: |x| = 1\}$ in $\mathbb{R}^k$. For $x \in \mathbb{R}^k$ write $x = [r,\theta]$ where $r = |x| \in [0,\infty)$ and $\theta = x/r \in S^{k-1}$. A k-dimensional Bessel process (BES(k)) is a continuous process with state space $[0,\infty)$ which is identical in law to the radial part R of a $BM(\mathbb{R}^k)$ process $X = [R,\Theta]$.
The invariance of $BM(\mathbb{R}^k)$ under the orthogonal group shows that BES(k) is a diffusion process with generator which is the radial part

$$\frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{k-1}{r}\frac{d}{dr}\right)$$

of the k-dimensional Laplace operator. For further information about Bessel processes see Ito-Mckean [2] and Williams [6].

Let us describe a continuous process with state space $[0,\infty)$ which is identical in law to the radial part of a $BM(\mathbb{R}^k)_{x\to 0}$ process as a $BES(k)_{r\to 0}$ process, for $r = |x| > 0$. Clearly a $BES(k)_{r\to 0}$ process can be interpreted as a $BES(k)_r$ process conditioned to hit zero: up to the first passage time to ε the $BES(k)_{r\to 0}$ process is identical in law to a $BES(k)_r$ process conditioned to hit the level ε, $0 < \varepsilon < r$ (cf. Jacobsen [3], Williams [6]). Using the fact that BES(k) has scale function s given by

$$s(r) = -1/r^k, \qquad 0 < r < \infty$$

it can be shown that BES(k) conditioned to hit the zero is the diffusion on $(0,\infty)$ with generator

$$\frac{1}{2}\left(\frac{d^2}{dr^2} - \frac{k-3}{r}\frac{d}{dr}\right),$$

the process being killed when it gets to zero (see Williams [6]). Note the interesting fact that for $k = 3$ this process is just Brownian motion on $(0,\infty)$ killed at the first contact with zero. For further information concerning this case see Pitman [5].

The key to the proof of the path decomposition for $BM(\mathbb{R}^k)_{b\to 0}$ is Williams path decomposition result for a one-dimensional diffusion, which shows that the radial motion splits up as it should:

<u>Lemma 1.</u> Let $k \geqq 3$. On a suitable probability triple $(\Omega,\mathcal{F},P)$ define three independent random elements, a $BES(k)_r$ process R_r, a $BES(k)_0$ process R_0, and a random variable V with $P(V > v) = (r/v)^{k-2}$, $r \leqq v < \infty$.

For $q = 0$ or r let $\tau_q^V = \inf\{t: t > 0, R_q(t) = V\}$, and define

$$R(t) = R_r(t), \qquad 0 \leqq t \leqq \tau_r^V$$

$$= R_0(\tau_r^V + \tau_0^V - t) \qquad \tau_r^V \leqq t < \tau_r^V + \tau_0^V.$$

Then the process

$$\{R(t),\ 0 \leqq t < \tau_r^V + \tau_0^V\}$$

is a $BES(k)_{r\to 0}$ process.

<u>Proof</u>. This result follows from Theorem 2.4 of Williams [6].

To derive from this result the full path decomposition for $BM(\mathbb{R}^k)_{b\to 0}$ we make use of the description of $BM(\mathbb{R}^k)$ as a skew product of its radial motion and an independent spherical Brownian motion $BM(S^{k-1})$ run with a clock depending on the progress of the radial motion:

<u>Lemma 2.</u> Let $x = [r,\theta] \in \mathbb{R}^k$, with $r > 0$. Let R be a $BES(k)_r$ process and let Φ be an independent $BM(S^{k-1})_\theta$ process. Then the process Θ defined by

$$(*) \qquad \Theta(t) = \Phi\left(\int_0^t 1/\ R^2(s)ds \right), \qquad t \geqq 0$$

is such that $X = [R,\Theta]$ is a $BM(\mathbb{R}^k)_x$. Conversely, if $X = [R,\Theta]$ is a $BM(\mathbb{R}^k)_x$ process, then the process Φ defined by (*) is a $BM(S^{k-1})_\theta$ independent of the $BES(k)_r$ process R.

<u>Remark</u>. In the last statement we mean of course that Φ is to be defined by

$$\Phi(s) = \Theta(\alpha^{-1}(s)), \qquad s \geqq 0$$

where $\alpha(t) = \int_0^t 1/R^2(s)ds.$

Proof. See Ito-Mckean [2], P. 270.

Lemma 2 does not apply for $x = 0$ since for a $BES(k)_0$ process we have

$$\int_0^t 1/R^2(s)ds = \infty \qquad \text{a.s. for all } t > 0,$$

Corrresponding to the fact that the spherical part of a $BM(R^k)_0$ process spins more and more violently as time $t \to 0$. However by making a time reversal we obtain the following result, where only the part corresponding to the second half of Lemma 2 is stated:

Lemma 3. Let $X = [R,\Theta]$ be a $BM(\mathbb{R}^k)_0$, let V be a positive random variable independent of X, and let

$$\tau = \inf\{t: R(t) = V\}$$

Then the process $\Phi = \{\Phi(s), s \geqq 0\}$ defined through

$$\Theta(t) = \Phi\left(\int_t^\tau 1/R^2(s)ds\right), \qquad 0 < t \leqq \tau$$

is a $BM(S^{k-1})$ which is independent of R and V and which starts uniformly distributed on S^{k-1}.

Proof. It is evidently sufficient to consider constant random variables V, and the result for this case can be deduced from Lemma 2 by considering the $BM(\mathbb{R}^k)_0$ after it first hits a sphere of small radius about the origin and using the fact that $BM(S^{k-1})$ started with the uniform distribution on S^{k-1} is invariant under a reversal of time. (cf. Ito-Mckean [2], P. 276).

Finally, we require the representation of $BM(\mathbb{R}^k)_{x\to 0}$ as a skew product:

Lemma 4. Let $x = [r,\theta] \in \mathbb{R}^k$, with $r > 0$. Let $R = \{R(t),\ 0 \leqq t \leqq \tau_R^0\}$ be a $BES(k)_{r\to 0}$ and let Φ be an independent $BM(S^{k-1})_\theta$. Set

$$\Theta(t) = \Phi\left(\int_0^t 1/R^2(s)ds \right), \qquad 0 \leqq t < \tau_R^0$$

Then the process $X = [R,\Theta]$ with lifetime τ_R^0 is a $BM(R^k)_{x\to 0}$.

Proof. This follows immediately from Lemma 1 and the description of $BM(\mathbb{R}^k)_{x\to 0}$ given in the Introduction as a process which up to the first passage time to the sphere $S(\varepsilon)$ of radius ε about the origin is identical in law to $BM(\mathbb{R}^k)_x$ conditioned to hit $S(\varepsilon)$.

Proof of the theorem. Let V, X_b, X_0 and X be as defined in the statement of the theorem. Setting $b = [r,\theta]$, $X = [R,\Theta]$, $X_a = [R_a,\Theta_a]$, $a = 0$ or b, it is clear from the construction of X that

$$\begin{aligned} R(t) &= R_b(t) & 0 \leqq t \leqq \tau_b^V \\ &= R_0(\tau_b^V + \tau_0^V - t), & \tau_b^V \leqq t < \tau_0^V + \tau_0^V \end{aligned}$$

where $\tau_a^V = \inf\{t\colon R_a(t) = V\}$. Lemma 1 now implies that R is a $BES(k)_{r\to 0}$ process, and thus from Lemma 4 we see that to establish the assertion of the theorem that X is a $BM(\mathbb{R}^k)_{b\to 0}$ we have only to show that the process $\Phi = \{\Phi(s),\ s \geqq 0\}$ defined by

$$\Theta(t) = \Phi\left(\int_0^t 1/R^2(s)ds \right)$$

is a $BM(S^{k-1})_\theta$ process independent of R. But let Φ_b be the $BM(S^{k-1})_\theta$ associated with X_b by Lemma 2, so that

$$\Theta_b(t) = \Phi_b\left(\int_0^t 1/R_b^{\,2}(s)ds \right),$$

and let Φ_0 be the $BM(S^{k-1})$ with uniform initial distribution which is associated with X_0 prior to $\overset{V}{\tau}_0$ by Lemma 3:

$$\Theta_0(t) = \Phi_0\left(\int_t^{\overset{V}{}} 1/R_0^{\,2}(s)ds\right), \qquad 0 < t \leq \overset{V}{\tau}_0.$$

Then since from the construction of X we have

$$\begin{aligned} \Theta(t) &= \Phi_b(t), & 0 \leq t \leq \overset{V}{\tau}_b \\ &= T_{\xi_0\xi_b}\,\Theta_0(\overset{V}{\tau}_b + \overset{V}{\tau}_0 - t), & \overset{V}{\tau}_b \leq t \leq \overset{V}{\tau}_b + \overset{V}{\tau}_0, \end{aligned}$$

it follows that if we set

$$\nu_b = \int_0^{\overset{V}{\tau}_b} 1/R_b^{\,2}(s)ds,$$

and define random variables ϕ_b and ϕ_0 taking values in S^{k-1} by

$$\begin{aligned} \phi_b &= \Phi_b(\nu_b) = \Theta_b(\overset{V}{\tau}_b) = \xi_b/|\xi_b|, \\ \phi_0 &= \Phi_0(0) \;\; = \Theta_0(\overset{V}{\tau}_0) = \xi_0/|\xi_0| \end{aligned}$$

then the process Φ defined above is given by

$$\begin{aligned} \Phi(s) &= \Phi_b(s), & 0 \leq s \leq \nu_b \\ &= T_{\xi_0\xi_b}\,\Phi_0(s - \nu_b), & \nu_b \leq s < \infty. \end{aligned}$$

Now conditional on the processes R_b and R_0 and the random variable V the random time ν_b is fixed and Φ_b and Φ_0 are independent $BM(S^{k-1})$ processes, so that the Markov property of $BM(S^{k-1})$ and the invariance of $BM(S^{k-1})$ under the action of the orthogonal group ensure that the process Φ defined above is a $BM(S^{k-1})$. Thus Φ must be a $BM(S^{k-1})_\theta$ independent of R_b, R_a and V together, hence also independent of R, and this is just what is required.

3. A more general path decomposition.

Let $\mathbb{D}$ be a Greenian domain in $\mathbb{R}^k$, where k is now any positive integer. (see Ito-Mckean[2],P. 237). Denote Brownian motion in $\mathbb{D}$ by BM($\mathbb{D}$), and for a positive harmonic function h on $\mathbb{D}$ let us refer to a Doob h-process associated with BM($\mathbb{D}$) as a $\mathrm{BM}(\mathbb{D})^h$ process. A $\mathrm{BM}(\mathbb{D})^h$ process $X^h = \{X^h(t),\ 0 < t < \zeta\}$ is a time homogeneous Markov process with continuous paths in the state space $\mathbb{D}$ which proceeds with transition density p^h given by

$$p^h(t,x,y) = p(t,x,y)\, h(x)/\, h(y), \qquad t > 0,\ x,y \in \mathbb{D},$$

where p is the transition density of BM($\mathbb{D}$), and the right side is taken to be zero if either h(x) or h(y) is infinite.

We mention now some basic facts concerning h-processes which can be found in Doob [1]. If X^h is a $\mathrm{BM}(\mathbb{D})^h$ process then $X^h(0^+)$ and $X^h(\zeta^-)$ are well defined random variables provided $\mathbb{D}$ is compactified by the addition of its Martin boundary $\mathbb{D}'$. If $X^h(0^+) = b$ a.s. for a point b in $\mathbb{D}$ or $\mathbb{D}'$ let us say that X^h is a $\mathrm{BM}(\mathbb{D})^h_b$ process. There is a $\mathrm{BM}(\mathbb{D})^h_b$ process defined for each point b in $\mathbb{D}$ with $h(b) < \infty$, but for points b in $\mathbb{D}'$ it is difficult to say in general whether or not there exists a $\mathrm{BM}(\mathbb{D})^h_b$ process.

Suppose now that h is minimal, so that h is the harmonic function associated with a single point c on the Martin boundary $\mathbb{D}'$. Then a $\mathrm{BM}(\mathbb{D})^h_b$ process X^h exists for each b in $\mathbb{D}$, and for such a process X^h we have $X^h(\zeta^-) = c$ a.s.. In this case the $\mathrm{BM}(\mathbb{D})^h_b$ process can be described as '$\mathrm{BM}(\mathbb{D})_b$ conditioned to converge to c'.

One concrete interpretation of this statement can be made as follows: assuming for the sake of interest that h is unbounded, the sets $\{h > u\}$ for $u \in (0,\infty)$ converge to c as $u \to \infty$ in the sense that these sets eventually get inside any neighbourhood of c in the Martin topology, and if for a $\mathbb{D}$-valued process $X = \{X(t),\ 0 < t < \zeta_X\}$ with continuous paths we define

$$\tau_X^u = \inf\{t\colon 0 < t < \zeta_X,\ h(X(t)) = u\},$$

then X is a $BM(\mathbb{D})_b^h$ process if and only if

$$\zeta_X = \lim_{u\to\infty} \tau_X^u \quad \text{a.s.}$$

and for each $u < \infty$ the process

$$\{X(t),\ 0 < t < \tau_X^u\}$$

is identical in law to

$$\{Y(t),\ 0 < t < \tau_Y^u\} \quad \text{conditional on } \{\tau_Y^u < \infty\}$$

for a $BM(\mathbb{D})_b$ process Y.

We now consider a proposition which is the natural generalization to this situation of the theorem proved in the last section. I am only able to establish parts (i) and (ii) of this proposition in full generality, but examples will be given below of several situations where it is possible to prove the full decomposition asserted by (iii).

Proposition. Let b be a point in $\mathbb{D}$ and suppose that h is a minimal positive harmonic function in $\mathbb{D}$ which is unbounded. Let

$$X^h = \{X^h(t),\ 0 \leq t < \zeta\}$$

be a $BM(\mathbb{D})_b^h$ process, and define random variables M and ρ by

$$M = \inf_{0 \leq t \leq \zeta} h(X^h(t)),$$

$$\rho = \inf\{t: h(X^h(t)) = M\}$$

Then X^h can be defined in such a way that

(i) M is uniformly distributed on $[0, h(b)]$

(ii) Conditional on $M = m$ the pre-ρ process

$$\{X^h(t), 0 \leq t < \rho\}$$

is identical in law to a $BM(\mathbb{D})_b$ process run until it first hits the level surface $\{h = m\}$ of the harmonic function h.

(iii) (Conjecture) The pre-ρ and the post-ρ processes are conditionally independent given $X^h(\rho)$, and conditional on a pre-ρ fragment with $X^h(\rho) = d$ and $M = h(d) = m$ the post-ρ process

$$X^h(\rho + t), \rho \leq t < \zeta - \rho\}$$

is a Doob $(h - m)$-process in the domain $\{h > m\}$ which starts at the point d on the surface $\{h = m\}$.

Proof of (i) and (ii) Part (i) can be deduced from the description of $BM(\mathbb{D})_b^h$ given above and the fact that if Y is a $BM(\mathbb{D})_b$ then for each $u \in (h(b), \infty)$ the process A_u defined by

$$\begin{aligned} A_u(t) &= h(Y(t)), \qquad 0 \leq t < \sigma_Y^u \\ &= h(Y(\sigma_Y^{u-})), \quad \sigma_Y^u \leq t < \infty \end{aligned}$$

(where $\sigma_Y^u = \min(\zeta_Y, \tau_Y^u)$, ζ_Y being the lifetime of Y and τ_Y^u being defined as before) is a martingale with continuous paths such that $A_u(\infty) = 0$ or u a.s. (see Doob [1]).

Part (ii) can be established after first showing that for each fixed ℓ with $0 < \ell < h(b)$, on the set $\{M < \ell\}$ the conditional distribution given M of the X^h process prior to its first hit to $\{h = \ell\}$ is identical to the distribution of a $BM(\mathbb{D})_b$ prior to its first hit to $\{h = \ell\}$. However the details are left to the reader.

Part (iii) remains unproved. This is the kind of splitting result toward which Jacobsen's paper [3] is directed,but unfortunately it does not seem to be easy to apply Jacobsen's method to the present situation. The difficulty is that in general it is hard to say much about the existence or uniquenes of an $(h - m)$-process started at a point d on the boundary $\{h = m\}$, let alone the weak continuity at d of the marginal distributions of an $(h - m)$-process as functions of its starting position in $\{h > m\}$, and these are the sort of things one needs to know about before Jacobsen's method can be applied. However the following examples show that (iii) does in fact hold true in a variety of interesting situations which can be reduced to the one-dimensional case by using space transformations and time substitutions.

<u>Example 1.</u> $\mathbb{D} = \mathbb{R}^k \setminus \{0\}$ for $k \geq 3$, $h = g(0,.)$. In this case the post-ρ process described in (iii) is $BM(\mathbb{R}^k)$ started at the point $X(\rho)$ on the surface of a sphere about the origin and conditioned to 'hit the origin without ever touching the sphere again', and when reversed in time this process becomes $BM(\mathbb{R}^k)_0$ run till it first hits the sphere and either conditioned to hit the sphere at $X(\rho)$ or else rotated so that the endpoint of the path is at $X(\rho)$. (This identity can be derived from the one-dimensional results of Williams [6] applied to the radial motions.) It is now plain that in this case the proposition above reduces to the theorem in the introduction.

motions.) It is now plain that in this case the proposition above reduces to the theorem in the Introduction.

Example 2. $\mathbb{D} = (0,\infty)$. The Martin boundary consists of two points 0 and ∞ corresponding to the minimal harmonic functions h_0 and h_∞ given by

$$h_0(x) = 1, \qquad h_\infty(x) = x, \qquad x \in (0,\infty).$$

Taking $h = h_\infty$ it turns out that $BM(0,\infty)_b^h$, i.e. '$BM(0,\infty)_b$ conditioned to hit ∞ before 0' is defined for all b in $[0,\infty)$ and is identical to $BES(3)_b$ (this is the dual of an observation made earlier, namely that for $b \in (0,\infty)$ '$BES(3)_b$ conditioned to hit 0 before ∞' is $BM(0,\infty)_b$: see Williams [6] and Pitman [5]). In this case the random variable M is just the overall minimum value of the process X^h itself, and the statement (iii) is equivalent to the statement that the process

$$\{X^h(\rho + t) - M, \qquad 0 \leqq t \leq \infty\}$$

is a $BES(3)_0$ independent of the pre-ρ process. This is the path decomposition for $BES(3)_b$ given by Williams as Theorem 3.1 of [6] (see also Theorem 4.1 of Pitman [5]).

Example 3. $\mathbb{D}$ is the open half plane $\mathbb{H} = (0,\infty)\times(-\infty,\infty)$. The Martin boundary can be identified with the line $\{0\}\times(-\infty,\infty)$ together with a single point Δ at infinity. Taking h to be the minimal harmonic function h_Δ associated with Δ:

$$h_\Delta(u,v) = u\,, \qquad (u,v) \in \mathbb{H}\,,$$

it is obvious from the previous example that a $BM(\mathbb{H})^h$ process can be started at any point $b = (u,v)$ in $[0,\infty)\times(-\infty,\infty)$ and if X^h is a $BM(\mathbb{H})_b^h$ process then

$$X^h = (U,V)$$

where U is a $BES(3)_u$ process and V is an independent $BM(\mathbb{R}^1)_v$ process. In this case M is the closest that X^h gets to the boundary line $\{u = 0\}$ and (iii) states that the process

$$\{X^h(\rho + t) - X^h(\rho),\ 0 \leq t < \infty\}$$

is independent of the pre-ρ process and identical in law to the process (U_0,V_0) obtained from a $BES(3)_0$ process U_0 and an independent $BM(\mathbb{R}^1)_0$ process V_0. It is obvious how this can be derived from the previous example.

Example 4. $\mathbb{D}$ is a simply connected open subset of the plane which is not the whole plane, h any minimal positive harmonic function (this extends the previous example). The Riemann mapping theorem and conformal invariance of the Green function show that $\mathbb{D}$ together with its Martin boundary are homeomorphic to the interior of the unit disc together with the unit circle in the usual topology. From similar considerations and the familiar situation for the interior of the disc we see that h is necessarily unbounded, and that if g denotes the harmonic function conjugate to h then the conformal mapping $f: \mathbb{D} \to \mathbb{H}$ defined by

$$f(z) = h(z) + ig(z), \qquad z \in \mathbb{D}$$

is a bijective mapping onto the open half plane $\mathbb{H} = \{w: \operatorname{Re} w > 0\}$.

Now let $b \in \mathbb{D}$ and suppose that $X = \{X(t), 0 \leq t < \zeta_X\}$ is a $BM(\mathbb{D})_b$. Then defining a process $Y = \{Y(s), 0 \leq s < \zeta_Y\}$ with values in $\mathbb{H}$ by

$$Y(s) = f(X(\alpha^{-1}(s))), \qquad 0 \leq s < \zeta_y = \alpha(\zeta_x),$$

where

$$\alpha(t) = \int_0^t |f'(X(u))|^2 du,$$

we have that Y is a $BM(\mathbb{H})_{f(b)}$ (see Ito-Mckean [2], § 7.18, or Mckean [4], § 4.6). It follows easily that if Y^h is the same transformation of a $BM(\mathbb{D})_b^h$ process X^h, then Y^h is a $BM(\mathbb{H})_{f(b)}^{h_\Delta}$ process as described in the example above, and it is easy to check that the path decomposition given above for that process transforms as it should to yield the desired decomposition of $BM(\mathbb{D})_b^h$.

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