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## SUMMARY

Three OC equivalence concepts: fractile, slope and moment equivalence, and two ASN optimality concepts: minimax and Bayes ASN optimality, are discussed in relation to double sampling tests. As measure of the inverse efficiency, IE, is used the ASN for the double sampling test divided by the sample size for the equivalent single sampling test.

The OC and ASN functions for double sampling from a normal population with unknown mean and known variance are studied and optimum double sampling tests of given strength are determined corresponding to the various equivalence and optimality concepts. Rather complete tables of these tests are given.

The OC and ASN functions for double sampling from a Poisson population with unknown mean are studied. No simple mathematical method for determining optimum double sampling tests exists in this case, but a method of tabulating optimum and nearly optimum tests has been devised and some examples of tables are given. Finally approximation formulas are discussed.

Keywords: DOUBLE SAMPLING TESTS; OC EQUIVALENCE; OC FRACTILE; OC MOMENT; ASN OPTIMALITY; INVERSE EFFICIENCY; MINIMAX ASN DOUBLE SAMPLING TESTS; BAYES ASN DOUBLE SAMPLING TESTS; TABLES OF OPTIMUM DOUBLE SAMPLING TESTS FOR NORMAL AND POISSON POPULATIONS; APPROXIMATIONS TO OPTIMUM DOUBLE SAMPLING TESTS; MINIMAX ASN SPRT.

## 1. Introduction.

The purpose of the present paper is to determine optimum double sampling tests within the framework of the usual theory of testing statistical hypotheses. Taking into consideration the large body of theory existing for testing hypotheses by sequential sampling, see for instance the book by Ghosh (1970), it is a peculiar fact that similar results do not exist for double sampling. Of course, double sampling may be considered as a special case of sequential sampling and from that point of view no new general theory is required. It takes, however, a considerable amount of work to arrive at the results for double sampling from the general theory.

We shall begin with some general considerations on the operating characteristic (OC) and the efficiency of a test expressed by means of the average sample number (ASN). Next we shall derive the OC and ASN functions for double sampling from normal and Poisson distributed populations. Finally we shall determine the optimum double sampling tests for various restrictions on the OC and various ASN optimality criteria. The mathematics used will be the usual methods for optimization under constraints and the resulting equations will be solved iteratively by numerical methods.

The tests considered have the form: Take a first sample of size $n_1$, compute the average $\bar{x}_1$, accept if $\bar{x}_1 \leqq h_a$, reject if $\bar{x}_1 \geqq h_r$ and take a second sample of size $n_2$ if $h_a < \bar{x}_1 < h_r$. Use the combined sample of size $n = n_1 + n_2$ to compute the average $\bar{x}$, accept if $\bar{x} \leqq h$ and reject otherwise. For single sampling we shall write $n_0$ for the sample size, $\bar{x}_0$ for the average, accept if $\bar{x}_0 \leqq h_0$ and reject otherwise.

A more effective test may be obtained by letting the second sample size depend on $\bar{x}_1$. Even if this test is slightly more effective and the theory perhaps simpler than the one considered we shall not discuss it further because we regard the test with fixed second sample size as the more practical.

The first paper on double sampling is the one by Dodge and Romig (1941). They did not, however, use the ASN as measure of efficiency but used instead the average total inspection since they assumed that rejected lots should be totally inspected.

There exists an old rule of thumb saying that if there is not sufficient evidence in a sample to reach a conclusion then one should take a second sample twice as large as the first and draw the conclusion from the combined sample. This rule has been used in Military Standard 105A(1950) and the subsequent editions B and C. Let $(h_0, n_0)$ be a suitably chosen single sampling attribute plan. The five parameters of the corresponding double sampling plan are then disposed of as follows: $n_1 = 2n_0/3, n_2 = 4n_0/3$, the rejection numbers for the first sample and for the combined sample are set equal and the two remaining parameters are determined such that the OC's for the single and the double sampling plans match closely. In Military Standard 105D(1963) the relations were changed to $n_1 = n_2 \simeq 0.63\ n_0$ and the restriction on the two rejection numbers was removed.

Bowker and Goode (1952), Zeigler and Tietjen (1968) and Zeigler and Goldmann (1972) have used similar rules, viz. $n_1 \simeq 0.4\ n_0$ and $n_2 = 2n_1$ or $n_1 \simeq 0.6\ n_0$ and $n_2 = n_1$, the remaining three parameters being found such that the double sampling OC is closely fitted to the given single sampling OC in particular for the acceptance probabilities 0.95 and 0.10.

Hamaker and van Strik (1955) have given a thorough discussion of the efficiency characteristic $ASN/n_0$ for double and single sampling attribute plans with matching OC's.

Owen (1953) has determined double sample tests for the normal distribution. His first procedure has $n_1 = pn_0$, $n_2 = n_0$, the same test size as for the single sample test, but no use of the first sample is made at the second stage. His second procedure has $n_1 = pn_0$, $n_2 = n_0 - n_1$ and the same test size for single and double sampling. In both cases p is determined by minimizing the ASN under the null hypothesis leading to $p \simeq 0.50$ and $p \simeq 0.45$ respectively for a test size between 0.01 and 0.10.

## 2. Terminology and definitions.

We shall only discuss the case of one unknown parameter $\theta$ and one-sided tests with a differentiable OC, $P(\theta)$ say, decreasing from 1 to 0. The power function, $1 - P(\theta)$, may therefore be considered as a distribution function with a density $- P'(\theta)$, moments etc. This is the same consideration which led R.A. Fisher to introduce his fiducial probability. However, we do not consider $P(\theta)$ as a probability distribution and $\theta$ as a random variable, and we do not want to get involved in the discussion on the meaning of fiducial probability. We just want to use the concepts and techniques developed in probability theory as simple and familiar modes of characterizing the function $P(\theta)$.

We shall therefore speak of $- P'(\theta)$ as the OC density, $\dot{E}\{\theta^r\} = -\int\theta^r dP(\theta)$ as the OC moment of order r, the solution of the equation $P(\theta) = \beta$, $0 < \beta < 1$, which will be denoted by $\theta_\beta$, as the $\beta$ OC fractile etc. If it is clear from the context that we are speaking of the OC distribution we shall leave out the OC.

This terminology is very convenient. For a normally distributed random variable we may say that the single sampling OC distribution is normal and the OC distribution for the symmetric SPRT is logistic. For a Poisson distributed random variable the single sampling OC distribution is a gamma distribution. All these results are of course well-known.

Using this terminology we shall state three definitions of OC equivalence and strength.

Tests having the same $1 - \alpha$ and $\beta$ OC fractiles, $0 < \beta < 1 - \alpha < 1$, will be called fractile equivalent. Such tests are said to be of strength $(\theta_1, \alpha, \theta_2, \beta)$, $\theta_1 < \theta_2$, if $P(\theta_1) = 1 - \alpha$ and $P(\theta_2) = \beta$. This is the usual definition of strength within the Neyman-Pearson theory.

Tests having the same OC median and the same OC slope (or OC density) at that point will be called (median) slope equivalent. Such tests are said to be of strength $(\theta_0, s_0)$, $s_0 > 0$, if $P(\theta_0) = 1/2$ and $-P'(\theta_0) = s_0$. This is the definition of strength proposed by Hamaker (1950). It may be regarded as a limiting case of the fractile definition.

Tests having the same OC mean and variance will be called moment equivalent. Such tests are said to be of strength $(E\{\theta\}, V\{\theta\})$.

For given location of the OC, i.e. for given $(\theta_1, \theta_2)$, $\theta_0$ or $E\{\theta\}$, a test $S_1$ is said to be stronger than another test $S_2$ if $S_1$ has the steeper OC curve, i.e. if $S_1$ has smaller $\alpha$ and/or $\beta$, larger $s_0$ or smaller $V\{\theta\}$ than $S_2$.

Since a single sampling test is defined by means of two parameters $(h_0, n_0)$ it is fully determined by specifying two "properties" of the OC distribution. Any one of the three definitions of strength may therefore be used to determine a single sampling test and there exists a one-to-one correspondence between them. A double sampling test is defined by means of five parameters $(h_a, h_r, h, n_1, n_2)$ and specification of the strength of a test as above therefore only introduces two relations between the five parameters. It is usual practice to disregard the differences between OC's within an equivalence class and select the test within the class which is best with respect to the ASN.

Consider a class C of equivalent tests and let $\bar{n}(\theta, S)$, $S \in C$, denote the ASN. As usual $S^* \in C$ will be called ASN optimum at $\theta = \theta_0$ if $\bar{n}(\theta_0, S^*) = \inf_{S \in C} \bar{n}(\theta_0, S)$. In particular we are interested in determining $S^*$ such that $\sup_{\theta} \bar{n}(\theta, S^*) = \inf_{S \in C} \sup_{\theta} \bar{n}(\theta, S)$. Such a test will be called minimax ASN. Consider now a weight function, $w(\theta)$ say, and the average

$$\bar{n}_w(S) = \int \bar{n}(\theta, S)\, dw(\theta).$$

A test $S^* \in C$ is said to be Bayes ASN with respect to w if $\bar{n}_w(S^*) = \inf \bar{n}_w(S)$ for $S \in C$.

In particular we may restrict attention to two values of $\theta$ such that the minimax ASN test is defined by

$$\sup\{\bar{n}(\theta_1,S^*),\bar{n}(\theta_2,S^*)\} = \inf_{S\in C} \sup\{\bar{n}(\theta_1,S),\ \bar{n}(\theta_2,S)\}$$

and the Bayes ASN test with respect to w is defined by

$$\bar{n}_w(S^*) = \inf_{S\in C}\{w\bar{n}(\theta_1,S) + (1-w)\bar{n}(\theta_2,S)\},\ 0 \leq w \leq 1.$$

A test $S_1 \in C$ is said to be <u>better than</u> a test $S_2 \in C$ if $\bar{n}(\theta_1,S_1) \leq \bar{n}(\theta_1,S_2)$ and $\bar{n}(\theta_2,S_1) \leq \bar{n}(\theta_2,S_2)$ with strict inequality in at least one place, and a test $S^* \in C$ is said to be <u>admissible</u> if there exists no test in C better than $S^*$.

For a <u>given OC equivalence class of double sampling tests</u> we shall determine <u>the ASN optimum test</u> according to the above criteria.

## Part 1. Double sampling from a normal population.

### 3. The OC function, its derivatives and moments.

Let x be normally distributed with unknown mean $\theta$ and known variance $\sigma^2$. The standardized normal density and distribution functions will be denoted by $\phi$ and $\Phi$, and we shall write $\tau^2 = \sigma^2/n$ and $\tau_i^2 = \sigma^2/n_i$ for $i = 0, 1, 2$. From the five parameters $(h_a, h_r, h, n_1, n_2)$ we define $n = n_1 + n_2$, $\rho = n_1/n$, $y_a = (h - h_a)\sqrt{n_1}/\sigma$, $y_r = (h_r - h)\sqrt{n_1}/\sigma$ and the standardized variable $v = (h - \theta)\sqrt{n}/\sigma$.

Theorem 1. The double sampling OC for the normal distribution equals

$$P(\theta|h_a,h_r,h,n_1,n_2) = D(v,y_a,y_r,\rho)$$

$$= \int_v^\infty \phi(u)\Phi\left(\frac{a_1-u\sqrt{\rho}}{\sqrt{1-\rho}}\right)du + \int_{-\infty}^v \phi(u)\Phi\left(\frac{r_1-u\sqrt{\rho}}{\sqrt{1-\rho}}\right)du, \qquad (1)$$

where $a_1 = v\sqrt{\rho} - y_a$ and $r_1 = v\sqrt{\rho} + y_r$.

Proof. The probability of acceptance equals

$$Pr\{\bar{x}_1 \leq h_a\} + Pr\{(h_a < \bar{x}_1 < h_r) \cap (\bar{x} \leq h)\}$$

or equivalently

$$Pr\{\bar{x} \leq h\} + Pr\{(\bar{x} > h) \cap (\bar{x}_1 \leq h_a)\} - Pr\{(\bar{x} \leq h) \cap (\bar{x}_1 \geq h_r)\}.$$

From each of these expressions $P(\theta)$ may be found using the fact that the distribution of $(\bar{x}_1, \bar{x})$ is bivariate normal with coefficient of correlation equal to $\sqrt{\rho}$. Hence, $E\{\bar{x}_1|\bar{x}\} = \bar{x}$ and $V\{\bar{x}_1|\bar{x}\} = \tau^2(1-\rho)/\rho$ such that

$$Pr\{\bar{x}_1 \leq h_a|\bar{x}\} = \Phi\left(\frac{h_a - \bar{x}}{\tau}\sqrt{\frac{\rho}{1-\rho}}\right).$$

Using the second of the two expressions for $P(\theta)$ we get

$$P(\theta) = \Phi\left(\frac{h-\theta}{\tau}\right) + \int_h^\infty \Phi\left(\frac{h_a - x}{\tau} \Big/ \sqrt{\frac{\rho}{1-\rho}}\right)\phi\left(\frac{x-\theta}{\tau}\right)\frac{dx}{\tau}$$

$$- \int_{-\infty}^{h} \Phi\left(\frac{x-h_r}{\tau} \Big/ \sqrt{\frac{\rho}{1-\rho}}\right)\phi\left(\frac{x-\theta}{\tau}\right)\frac{dx}{\tau}$$

which is identical to (1).

Remark. For computational purposes it should be noted that

$$P(\theta) = \Phi(r_1) - L(v, a_1, \sqrt{\rho}) + L(v, r_1, \sqrt{\rho}),$$

where

$$L(h,k,r) = \int_h^\infty \phi(u)\Phi\left(\frac{ru-k}{\sqrt{1-r^2}}\right)du$$

has been tabulated in National Bureau of Standards (1959).

Derivatives of the OC.

The four partial derivatives of $D(v, y_a, y_r, \rho)$ will be denoted $D'_v$, $D'_a$, $D'_r$, $D'_\rho$, respectively, and further we introduce $Z_a = y_a/\sqrt{1-\rho}$, $Z_r = y_r/\sqrt{1-\rho}$, $a_2 = v\sqrt{1-\rho} + Z_a\sqrt{\rho}$ and $r_2 = v\sqrt{1-\rho} - Z_r\sqrt{\rho}$. Differentiating (1) and using Lemma 1 in the Appendix we get

$$D'_a = -\phi(a_1)\Phi(-a_2)$$

and

$$D'_r = \phi(r_1)\Phi(r_2).$$

Noting that

$$\frac{\partial}{\partial\rho}\left(\frac{a_1 - u\sqrt{\rho}}{\sqrt{1-\rho}}\right) = \frac{a_1 - u\sqrt{\rho}}{\sqrt{1-\rho}}\,\frac{1}{2\rho(1-\rho)} + \frac{Z_a}{2\rho}$$

and using Lemma 1 we find

$$D'_\rho = \frac{1}{2}\rho^{-1/2}[(1-\rho)^{-1/2}\{\phi(r_1)\phi(r_2) - \phi(a_1)\phi(a_2)\}$$

$$+ v\{\phi(r_1)\Phi(r_2) + \phi(a_1)\Phi(-a_2)\}]$$

$$= \frac{1}{2}\rho^{-1/2}[(1-\rho)^{-1/2}\phi(v)\{\phi(Z_r)-\phi(Z_a)\}$$

$$+ v\{\phi(r_1)\Phi(r_2) + \phi(a_1)\Phi(-a_2)\}].$$

Similarly we obtain

$$D'_v = \phi(v)\{\Phi(Z_r)-\Phi(-Z_a)\}+\sqrt{\rho}\{\phi(r_1)\Phi(r_2)+\phi(a_1)\Phi(-a_2)\}. \qquad (2)$$

Hence, the OC density becomes $-P'(\theta) = D'_v/\tau$ and $\partial P/\partial n = D'_v v/2n$. We note that $D'_a < 0$, $D'_r > 0$, $D'_v > 0$ and $\partial P/\partial n \gtreqless 0$ for $v \gtreqless 0$.

All the derivatives are simple to compute by means of a table of the normal distribution.

For a symmetric double sampling test, i.e. for $h - h_a = h_r - h$, we shall write $y_a = y_r = y$ and $Z_a = Z_r = Z$, which means that $D$ is a function of $(v,y,\rho)$ only. All the formulas above may be used directly apart from $D'_a$ and $D'_r$. Instead we have

$$D'_y = \phi(r_1)\Phi(r_2) - \phi(a_1)\Phi(-a_2). \qquad (3)$$

The OC moments.

Let $\mu_k$ denote the moment of order $k$ for the standardized normal distribution, i.e. $\mu_{2k+1} = 0$ and $\mu_{2k} = 1.3\ldots(2k-1)$, and let $m_k(Z)$ denote the incomplete moment $m_k(Z) = \int_Z^\infty (t-Z)^k\phi(t)dt$, $k = 0,1,\ldots$ .

The properties of $m_k(Z)$ have been discussed by Hald (1967a).

Theorem 2. The OC moment of order $k$ about $h$ for the normal double sampling test equals $\dot{E}\{(\theta-h)^k\} = (-\tau)^k\dot{E}\{v^k\}$, where

$$\dot{E}\{v^k\}= \mu_k+\sum_{i=0}^{k-1}\binom{k}{i}\mu_i\left(\frac{1-\rho}{\rho}\right)^{(k-i)/2}\{m_{k-i}(Z_r)+(-1)^k m_{k-i}(Z_a)\}. \qquad (4)$$

Proof. Before writing $D'_v$ in the form (2) we have as an intermediate expression

$$D'_v = \phi(v)\{\Phi(Z_r)-\Phi(-Z_a)\}+\int_v^\infty \phi(u)\phi\left(\frac{a_1-u\sqrt{\rho}}{\sqrt{1-\rho}}\right)du\Big/\sqrt{\frac{\rho}{1-\rho}}$$

$$+\int_{-\infty}^v \phi(u)\phi\left(\frac{r_1-u\sqrt{\rho}}{\sqrt{1-\rho}}\right)du\Big/\sqrt{\frac{\rho}{1-\rho}}.$$

The first term of $E\{v^k\} = \int v^k D_v' dv$ therefore equals $\mu_k'\{\Phi(Z_r) - \Phi(-Z_a)\} = \mu_k'\{1-m_0(Z_r) - m_0(Z_a)\}$. In the second term of $D_v'$ we first make the transformation $u = t + v$ which gives

$$\int_0^\infty \phi(t+v)\phi\left(t\sqrt{\frac{\rho}{1-\rho}} + Z_a\right)dt\sqrt{\frac{\rho}{1-\rho}} \ .$$

As

$$\int_{-\infty}^{\infty} v^k\phi(t+v)dv = \int_{-\infty}^{\infty}(u-t)^k\phi(u)du = \sum_{i=0}^{k}\binom{k}{i}\mu_i(-t)^{k-i}$$

the contribution to $E\{v^k\}$ becomes

$$\sum_{i=0}^{k}\binom{k}{i}\mu_i(-1)^k\int_0^\infty t^{k-i}\phi\left(t\sqrt{\frac{\rho}{1-\rho}}+Z_a\right)dt\sqrt{\frac{\rho}{1-\rho}}$$

$$= \sum_{i=0}^{k}\binom{k}{i}\mu_i(-1)^k\left(\frac{1-\rho}{\rho}\right)^{(k-i)/2} m_{k-i}(Z_a),$$

where we have used that $(-1)^i\mu_i = \mu_i$. A similar evaluation of the third term and a simple reduction leads to (4).

<u>Remark</u>. The first two moments are

$$E\{\theta-h\} = \tau\left(\frac{1-\rho}{\rho}\right)^{1/2}\{m_1(Z_a) - m_1(Z_r)\} \qquad (5)$$

and

$$E\{\theta-h)^2\} = \tau^2\left[1+\frac{1-\rho}{\rho}\{m_2(Z_a)+m_2(Z_r)\}\right]. \qquad (6)$$

The moments are easily found from a table of $m_k(Z)$ or by means of the relation

$$m_k(Z) = Hh_k(Z)\Gamma(k+1)/\sqrt{2\pi},$$

since $Hh_k(Z)$ has been tabulated in British Association (1951).

## 4. Equivalent single sampling tests and efficiency.

Let $u_\beta$ denote the solution of the equation $\Phi(u) = \beta$, $0 < \beta < 1$. Similarly we shall write $v_\beta = v_\beta(y_a, y_r, \rho)$ for the solution of $D(v, y_a, y_r, \rho) = \beta$. Hence, $\theta_\beta = h - \tau v_\beta$.

For the single sampling test $(h_0, n_0)$ the OC is $P_1(\theta) = \Phi((h_0-\theta)/\tau_0)$, i.e. the OC distribution is normal with location parameter $h_0$ and scale parameter $\tau_0$. For the double sampling test the OC is $P_2(\theta) = D((h-\theta)/\tau, y_a, y_r, \rho)$, i.e. the OC distribution has h and $\tau$ as location and scale parameters and $(y_a, y_r, \rho)$ as shape parameters.

For a given single sampling test the three definitions of strength in Section 2 lead to the following relationships

$$h_0 = \frac{\theta_2 u_{1-\alpha} - \theta_1 u_\beta}{u_{1-\alpha} - u_\beta} = \theta_0 = E\{\theta\}$$

and

$$\sqrt{n_0} = \sigma \frac{u_{1-\alpha} - u_\beta}{\theta_2 - \theta_1} = \sigma s_0 \sqrt{2\pi} = \sigma / \sqrt{V\{\theta\}}.$$

For a given double sampling test we find

$$h = \frac{\theta_2 v_{1-\alpha} - \theta_1 v_\beta}{v_{1-\alpha} - v_\beta} = \theta_0 + \tau v_{.5}$$

$$= E\{\theta\} - \tau\left(\frac{1-\rho}{\rho}\right)^{1/2} \{m_1(Z_a) - m_1(Z_r)\}$$

and

$$\sqrt{n} = \sigma \frac{v_{1-\alpha} - v_\beta}{\theta_2 - \theta_1} = \sigma s_0 / D_v'(v_{.5}) = \sigma\lambda / \sqrt{V\{\theta\}},$$

where $D_v'(v_{.5}) = D_v'(v_{.5}, y_a, y_r, \rho)$ and

$$\lambda^2 = 1 + \frac{1-\rho}{\rho}\left[m_2(Z_a) + m_2(Z_r) - \{m_1(Z_a) - m_1(Z_r)\}^2\right]. \quad (7)$$

By means of the formulas for $(h_0, n_0)$ we may find three single sampling tests for any given double sampling test, each single sampling test corresponding to the definition of strength chosen.

Of course this is nothing else than fitting a normal distribution to the given double sampling OC distribution, using three different (rather primitive) methods of fitting. Other methods may be used as well.

The approximation $D((h-\theta)/\tau) \simeq \Phi((h_0-\theta)/\tau_0)$ may also be expressed in terms of the fractiles, i.e. $h - v_P\tau \simeq h_0 - u_P\tau_0$ or

$$v_P \simeq u_P\sqrt{n/n_0} + (h-h_0)\sqrt{n}/\sigma.$$

For fractile equivalence we get

$$v_P \simeq \frac{v_{1-\alpha}+v_\beta}{2} + \frac{v_{1-\alpha}-v_\beta}{u_{1-\alpha}-u_\beta}\left(u_P - \frac{u_{1-\alpha}+u_\beta}{2}\right).$$

Slope equivalence gives

$$v_P \simeq v_{.5} + u_P/\{\sqrt{2\pi}D_v'(v_{.5})\}$$

and moment equivalence leads to

$$v_P \simeq E\{v\} + \lambda u_P,$$

where $\lambda^2 = V\{v\}$ has been defined in (7).

Of the three approximations the last one is the simplest to compute because tables of $m_k(Z)$ are readily available whereas tables of the fractiles of v do not exist.

Note that the coefficient of $u_P$ gives $\sqrt{n/n_0}$ which is an important quantity for the evaluation of $ASN/n_0$.

By means of higher moments it is easy to obtain better approximations to $v_P$, for example by a Cornish-Fisher expansion. The double sampling OC distribution has considerably longer tails than the equivalent normal distributions. In Table 1 a comparison has been given of the exact values of $v_P$, the Cornish-Fisher approximation based on the first four moments and the three equivalent normal distributions defined above.

Table 1. Values of $v_P$ and four approximations. $y_a = 0.599$. $y_r = 0.899$. $\rho = 0.436$.

| 100P | Exact | C-F | Moment | Fractile | Slope |
|---|---|---|---|---|---|
| 0.1 | -3.79 | -3.79 | -3.47 | -3.43 | -3.38 |
| 0.5 | -3.07 | -3.08 | -2.90 | -2.87 | -2.83 |
| 1.0 | -2.74 | -2.75 | -2.63 | -2.60 | -2.56 |
| 2.5 | -2.27 | -2.28 | -2.23 | -2.21 | -2.17 |
| 5.0 | -1.89 | -1.90 | -1.88 | -1.86 | -1.83 |
| 10.0 | -1.47 | -1.47 | -1.48 | -1.47 | -1.44 |
| 20.0 | -0.98 | -0.97 | -1.00 | -0.99 | -0.97 |
| 50.0 | -0.06 | -0.06 | -0.07 | -0.08 | -0.06 |
| 80.0 | 0.84 | 0.83 | 0.85 | 0.84 | 0.84 |
| 90.0 | 1.31 | 1.31 | 1.33 | 1.31 | 1.31 |
| 95.0 | 1.71 | 1.71 | 1.73 | 1.71 | 1.70 |
| 97.5 | 2.06 | 2.06 | 2.08 | 2.05 | 2.04 |
| 99.0 | 2.48 | 2.49 | 2.48 | 2.45 | 2.43 |
| 99.5 | 2.78 | 2.79 | 2.76 | 2.72 | 2.70 |
| 99.9 | 3.42 | 3.43 | 3.32 | 3.28 | 3.25 |

The Cornish-Fisher approximation is based on $E\{v\} = -0.073$, $V\{v\} = 1.207$, $\gamma_1 = -0.069$ and $\gamma_2 = 0.242$. The moment equivalent single sampling test gives $v \simeq -0.073 + 1.099\, u$. The fractile approximation is based on $v_{.95} = 1.708$ and $v_{.10} = -1.469$ which gives $v \simeq -0.078 + 1.086\, u$. The (median) slope approximation uses $v_{.5} = -0.062$ and $D'_v\,(v_{.5}) = 0.372$ which lead to $v \simeq -0.062 + 1.073\, u$. As could be expected the moment approximation gives the best over-all fit among the three normal approximations. However, for $0.025 < P < 0.975$ the three approximations do not differ much.

Consider now

$$ASN = \bar{n}(\theta) = n_1 + n_2 \Pr\{h_a < \bar{x}_1 < h_r \mid \theta\}$$

and the <u>inverse efficiency</u>, $IE = \bar{n}(\theta)/n_0$, <u>of double sampling relative to the equivalent single sampling test</u>. We find

$$IE(\theta) = \frac{\bar{n}(\theta)}{n_0} = \frac{n}{n_0}\left[\rho + (1-\rho)\{\Phi(v\sqrt{\rho}+y_r) - \Phi(v\sqrt{\rho}-y_a)\}\right]. \tag{8}$$

By means of this expression and the approximations given above it is easy to investigate the efficiency of double sampling. As a simple example suppose we want to investigate symmetric double sampling tests. We then have $y_a = y_r = y$ and $v_P \simeq u_P\sqrt{n/n_0}$ such that

$$\frac{\bar{n}(\theta_P)}{n_0} \simeq \frac{n}{n_0}\left[\rho+(1-\rho)\{\Phi(y+u_P\sqrt{\rho n/n_0})-\Phi(-y+u_P\sqrt{\rho n/n_0})\}\right].$$

Setting P = 0.95, say, $\bar{n}(\theta_{.95})/n_0$ is easily tabulated as function of $(y,\rho)$ and a good double sampling plan may then be selected.

Another example (involving no approximation) is the case of moment equivalent tests with

$$\sup_\theta \bar{n}(\theta)/n_0 = \bar{n}(\theta_{.5})/n_0 = \frac{n}{n_0}\left[\rho+(1-\rho)\{2\Phi(y)-1\}\right]$$

as measure of efficiency, where $n/n_0 = 1 + 2(1-\rho)\rho^{-1}m_2(Z)$, see (7). Table 2 gives some values of this function. It will be seen that the test minimizing $\sup \bar{n}(\theta)/n_0$ is found for $\rho \simeq 0.6$ and $Z \simeq 1.0$, the exact values being $\rho = 0.586$ and $Z = 0.967$ as will be shown in the following section. The minimum of $\sup \bar{n}(\theta)/n_0$ equals 0.87.

Table 2. Values of $\sup \bar{n}(\theta)/n_0$ for moment equivalent, symmetric double sampling tests.

$Z = y/\sqrt{1-\rho}$

| $\rho$ | 0.0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 |
|---|---|---|---|---|---|---|
| 1/4 | 1.00 | 1.13 | 1.03 | 0.97 | 0.97 | 0.98 |
| 1/3 | 1.00 | 1.00 | 0.94 | 0.93 | 0.95 | 0.98 |
| 2/5 | 1.00 | 0.95 | 0.90 | 0.91 | 0.94 | 0.97 |
| 1/2 | 1.00 | 0.91 | 0.87 | 0.89 | 0.93 | 0.96 |
| 2/3 | 1.00 | 0.90 | 0.87 | 0.89 | 0.92 | 0.95 |
| 4/5 | 1.00 | 0.92 | 0.90 | 0.91 | 0.93 | 0.95 |
| 10/11 | 1.00 | 0.96 | 0.94 | 0.95 | 0.95 | 0.96 |

## 5. Double sampling tests minimizing max $\bar{n}(\theta)$.

From

$$\bar{n}'(\theta) = \frac{n_2}{\tau_1}\{\phi\left(\frac{h_a-\theta}{\tau_1}\right) - \phi\left(\frac{h_r-\theta}{\tau_1}\right)\}$$

it follows that max $\bar{n}(\theta) = \bar{n}(\frac{1}{2}(h_a+h_r))$. Further, $\bar{n}(\theta)$ is increasing for $\theta < (h_a+h_r)/2$, decreasing for $\theta > (h_a+h_r)/2$, $\bar{n}(\theta) \to n_1$ for $|\theta| \to \infty$ and

$$\max_{\theta} \bar{n}(\theta) = n\left[\rho+(1-\rho)\{2\Phi(\frac{1}{2}(y_a+y_r))-1\}\right].$$

We shall write $\bar{n}$ for max $\bar{n}(\theta)$ in the present section, and tests minimizing max $\bar{n}(\theta)$ will be called minimax $\bar{n}(\theta)$ tests. We shall determine such tests for the three equivalence definitions.

Theorem 3. Let there be given a single sampling test $(h_0, n_0)$. The moment equivalent, minimax $\bar{n}(\theta)$, double sampling test is given by $h = h_0$, $y_a = y_r = 0.622$, $\rho = 0.586$ and $n = 1.114\ n_0$. For this test max $\bar{n}(\theta) = 0.868\ n_0$.

Proof. Moment equivalence means that the double sampling tests considered satisfy the two equations

$$g_1 = h + \tau(1-\rho)^{1/2}\rho^{-1/2}\{m_1(Z_a)-m_1(Z_r)\} - h_0 = 0 \qquad (9)$$

and

$$g_2 = \tau^2\left[1+(1-\rho)\rho^{-1}\{m_2(Z_a)+m_2(Z_r)\}\right] - \tau_0^2 - (h_0-h)^2 = 0, \qquad (10)$$

see (5) and (6). Hence, the problem consists in minimizing max $\bar{n}(\theta)$ under the two restrictions. Defining $G = \bar{n} + \lambda_1 g_1 + \lambda_2 g_2$ the solution may be found by setting the derivatives of G equal to zero and solving for the five parameters $(h, y_a y_r, \rho, n)$. We shall use that $m'_{k+1}(Z) = -(k+1)m_k(Z)$, $k = 0,1,\ldots,$ and $m_0(Z) = \Phi(-Z)$.

From $\partial G/\partial h = \partial G/\partial y_a = \partial G/\partial y_r = 0$ we get

$$\lambda_1 + 2\lambda_2(h_0-h) = 0, \qquad (11)$$

$$\lambda_1 m_0(Z_a) + 2\lambda_2\tau(1-\rho)^{1/2}\rho^{-1/2}m_1(Z_a) = n\rho^{1/2}(1-\rho)\phi(\bar{y})/\tau, \quad (12)$$

$$-\lambda_1 m_0(Z_r) + 2\lambda_2\tau(1-\rho)^{1/2}\rho^{-1/2}m_1(Z_r) = n\rho^{1/2}(1-\rho)\phi(\bar{y})/\tau, \quad (13)$$

where $\bar{y} = (y_a+y_r)/2$. Subtracting (13) from (12), using (9) to introduce $h_0 - h$ and (11) to eliminate $\lambda_2(h_0-h)$ we obtain $\lambda_1(1-m_0(Z_a)-m_0(Z_r)) = 0$. As $h_a < h_r$ we have $Z_a + Z_r > 0$ which means that $1 - m_0(Z_a) - m_0(Z_r) \neq 0$ and hence $\lambda_1 = 0$. From (12) and (13) we then get $\lambda_2 \neq 0$ and $m_1(Z_a) = m_1(Z_r)$, which gives $Z_a = Z_r = Z$, say, and from (11) $h = h_0$. We have thus proved that the solution is to be found among the symmetric tests which is also intuitively clear since the specification is symmetric.

Rewriting (10) and (12) we have

$$g_2 = \tau^2\{1+2(1-\rho)\rho^{-1}m_2(Z)\} - \tau_0^2 = 0 \quad (14)$$

and

$$(\lambda_2\tau^2/n)m_1(Z) = \tfrac{1}{2}\rho(1-\rho)^{1/2}\phi(y). \quad (15)$$

From $\partial G/\partial\rho = \partial G/\partial n = 0$ we get

$$(\lambda_2\tau^2/n)\{Zm_1(Z)+\rho^{-1}m_2(Z)\} = \rho m_0(y)$$

and

$$(\lambda_2\tau^2/n)\{1+2(1-\rho)\rho^{-1}m_2(Z)\} = 1 - 2(1-\rho)m_0(y).$$

Solving (15) for $\lambda_2\tau^2/n$, inserting into the two equations above and using the relationship $m_2(Z) + Zm_1(Z) = m_0(Z)$ we find after some reduction

$$\frac{m_0(Z)}{m_1(Z)} - (1-\rho)Z = \frac{2\rho m_0(y)}{\sqrt{1-\rho}\,\phi(y)}$$

and

$$\frac{1-2m_0(Z)}{2m_1(Z)} = \frac{1-2m_0(y)}{\rho\sqrt{1-\rho}\,\phi(y)}.$$

Solving the two equations by iterative methods we find Z =

0.9672 and $\rho = 0.5864$. From (14) we get $n/n_0 = 1 + 2(1-\rho)\rho^{-1} m_2(Z) = 1.114$.

Corollary 1. For given $(\rho, n)$ the optimum test is determined by $h = h_0$ and $y = Z\sqrt{1-\rho}$, where Z is found from the equation

$$m_2(Z) = \frac{1}{2}\left(\frac{n}{n_0} - 1\right)\frac{\rho}{1-\rho}.$$

Corollary 2. For given $\rho$ the optimum test is determined by $h = h_0$, $y = Z\sqrt{1-\rho}$, where Z is found from the equation

$$\frac{1+2(1-\rho)\rho^{-1}m_2(Z)}{2m_1(Z)} = \frac{1-2(1-\rho)m_0(y)}{\rho\sqrt{1-\rho}\,\phi(y)},$$

and n is found from

$$n/n_0 = 1 + 2(1-\rho)\rho^{-1}m_2(Z),$$

see Table 3.

Table 3. Moment equivalent, minimax $\bar{n}(\theta)$, double sampling tests for given $\rho$.

| $n_2/n_1$ | $\rho$ | y | $n/n_0$ | $\bar{n}/n_0$ |
|---|---|---|---|---|
| 0.25 | 0.8000 | 0.469 | 1.034 | 0.902 |
| 0.50 | 0.6667 | 0.562 | 1.080 | 0.873 |
| 0.71 | 0.5864 | 0.622 | 1.114 | 0.868 |
| 1.00 | 0.5000 | 0.709 | 1.150 | 0.875 |
| 1.50 | 0.4000 | 0.890 | 1.162 | 0.901 |
| 2.00 | 0.3333 | 1.103 | 1.133 | 0.929 |
| 3.00 | 0.2500 | 1.511 | 1.072 | 0.966 |
| 4.00 | 0.2000 | 1.851 | 1.038 | 0.984 |

Remark on the equivalent SPRT. To illustrate the application of the above method to a sequential test we consider a normal process $x_n$, say, n being continuous, with unknown mean $\theta n$ and known variance $\sigma^2 n$. The continuation region for the symmetric SPRT may be written as $-k\sigma + n\theta_0 < x_n < k\sigma + n\theta_0$ and the corresponding OC is

$$P(\theta) = (1+e^w)^{-1} = F(w) \text{ for } w = 2k(\theta-\theta_0)/\sigma.$$

Hence, the OC distribution is logistic with mean $\theta_0$ and variance $\sigma^2\pi^2/(12k^2)$. The expected sample size is $\bar{n}(\theta) = 2k^2(1-2F(w))/w$ and max $\bar{n}(\theta) = \bar{n}(\theta_0) = k^2$.

Since the symmetric SPRT depends on two parameters only, viz. $\theta_0$ and k, it is fully determined by the first two moments such that $\acute{E}\{\theta\} = \theta_0 = h_0$ and $\acute{V}\{\theta\} = \sigma^2\pi^2/(12k^2) = \sigma^2/n_0$ or $k^2 = n_0\pi^2/12 = 0.8225\ n_0$. The inverse efficiency then becomes 0.8225 (as compared to 0.863 for double sampling) and the continuation region is

$$-0.907\sigma\sqrt{n_0} + nh_0 < x_n < 0.907\sigma\sqrt{n_0} + nh_0.$$

Theorem 4. Let there be given a single sampling test $(h_0, n_0)$. The slope equivalent, minimax $\bar{n}(\theta)$, double sampling test is given by $h = h_0$, $y_a = y_r = 0.560$, $\rho = 0.465$ and $n = 1.216\ n_0$. For this test max $\bar{n}(\theta) = 0.842\ n_0$.

Proof. It is clear that the optimum test must be symmetric because the specification is so. Slope equivalence means that $P(\theta_0) = \frac{1}{2}$ and $-P'(\theta_0) = s_0$, which immediately give $h = h_0 = \theta_0$ and

$$D_v'(0)/\tau = \phi(0)/\tau_0 = s_0, \qquad (16)$$

where

$$D_v'(0) = 2\sqrt{\rho}\phi(y)\Phi(-Z\sqrt{\rho}) + \phi(0)\{2\Phi(Z)-1\}.$$

Introducing the auxiliary function $G = \bar{n} - \lambda D_v'(0)/\tau$ the unknown parameters $(y,\rho,n)$ may be found by solving the three equations $\partial G/\partial y = \partial G/\partial\rho = \partial G/\partial n = 0$ together with the restriction (16). Noting that $\phi(y)\phi(Z\sqrt{\rho}) = \phi(0)\phi(Z)$ and eliminating $\lambda$ from the three equations we get

$$\rho y + (1-\rho)\frac{\phi(y)}{2\Phi(-y)} = \sqrt{\rho(1-\rho)}\ \frac{\phi(Z\sqrt{\rho})}{\Phi(-Z\sqrt{\rho})}$$

and

$$\frac{\phi(y)}{\phi(0)}\ \frac{1-2\Phi(-y)}{2\Phi(-y)} = \sqrt{\rho}\ \frac{1-2\Phi(-Z)}{2\Phi(-Z\sqrt{\rho})}.$$

The solution is $y = 0.5596$ and $\rho = 0.4652$. Inserting into (16) we find $n/n_0 = \{\phi(0)/D'_v(0)\}^2 = 1.216$ and finally $\bar{n}/n_0 = 0.842$.

Theorem 5. Let there be given a single sampling test $(h_0, n_0)$. For $\alpha = \beta$ the $1-\alpha$ fractile equivalent, minimax $\bar{n}(\theta)$, symmetric double sampling test is given by $h = h_0$, $(y,\rho)$ are determined by (17) and (18), and $n = (v_{1-\alpha}/u_{1-\alpha})^2 n_0$, see Table 4.

Proof. Considering symmetric tests only the fractile specification $P(\theta_1) = 1-P(\theta_2) = 1-\alpha$ gives $h = h_0 = (\theta_1+\theta_2)/2$. The problem is to minimize $\bar{n} = n[\rho+(1-\rho)\{2\Phi(y)-1\}]$ under the restriction $D(v_1,y,y,\rho) = 1 - \alpha$, where $v_1 = (h-\theta_1)/\tau$. Introducing $G = \bar{n} - \lambda D(v_1,y,y,\rho)$ and eliminating $\lambda$ from the three equations $\partial G/\partial y = \partial G/\partial \rho = \partial G/\partial n = 0$ we get

$$\frac{(1-\rho)\phi(y)}{\Phi(-y)} = \frac{D'_y}{D'_\rho} \tag{17}$$

and

$$\frac{1-2(1-\rho)\Phi(-y)}{\Phi(-y)} = \frac{vD'_v}{D'_\rho}, \tag{18}$$

where $D'_y$, $D'_\rho$ and $D'_v$ have been defined in Section 3. The two equations together with the restriction have been solved by iteration starting from suitably chosen values of $(y,\rho)$, the only difficulty in the procedure being the solution of the equation $D(v,y,y,\rho) = 1 - \alpha$ with respect to $v$. A special algorithm has been developed for this purpose.

Remarks on Table 4. A summary of numerical results from Theorems 3,4 and 5 has been given in Table 4. Besides the parameters characterizing the optimum tests the table contains the most important OC fractiles and corresponding values of $\bar{n}(\theta)/n_0$ such that the OC and ASN functions may be easily found.

It will be seen that the moment equivalent test and the 0.96 fractile equivalent test are nearly equal. For the commonly used values of $\alpha$, $0.01 \leq \alpha \leq 0.05$ say, we have $n \simeq 1.10n_0$, $\rho \simeq 0.6$, that is $n_2/n_1 \simeq 2/3$, $y \simeq 0.63$, max $\bar{n}(\theta) \simeq 0.87n_0$

Table 4. Table of symmetric double sampling tests minimizing max $\bar{n}(\theta)$.

| | Mom. equiv. | OC-Fractile equivalence | | | | | | | Slope Equiv. |
|---|---|---|---|---|---|---|---|---|---|
| $100(1-\alpha)$ | - | 99.9 | 99.5 | 99.0 | 97.5 | 95.0 | 90.0 | 80.0 | - |
| $u^2_{1-\alpha}$ | - | 9.550 | 6.635 | 5.412 | 3.841 | 2.706 | 1.642 | 0.7083 | - |
| $n/n_0$ | 1.114 | 1.064 | 1.079 | 1.088 | 1.104 | 1.121 | 1.144 | 1.175 | 1.216 |
| $\rho$ | .5864 | .6764 | .6441 | .6266 | .5988 | .5730 | .5417 | .5046 | .4652 |
| $y$ | .6220 | .6725 | .6580 | .6498 | .6362 | .6230 | .6060 | .5845 | .5596 |
| $\bar{n}/n_0$ | .8682 | .8914 | .8830 | .8785 | .8716 | .8654 | .8582 | .8499 | .8417 |
| 100P | Table of v for $D(v,y,y,\rho) = P$. | | | | | | | | |
| 99.9 | 3.340 | 3.188 | 3.229 | 3.256 | 3.308 | 3.365 | 3.449 | 3.572 | 3.733 |
| 99.5 | 2.752 | 2.647 | 2.676 | 2.694 | 2.730 | 2.769 | 2.826 | 2.912 | 3.027 |
| 99.0 | 2.474 | 2.387 | 2.411 | 2.427 | 2.456 | 2.488 | 2.535 | 2.605 | 2.700 |
| 97.5 | 2.073 | 2.008 | 2.026 | 2.038 | 2.059 | 2.083 | 2.118 | 2.169 | 2.238 |
| 95.0 | 1.734 | 1.683 | 1.697 | 1.706 | 1.723 | 1.741 | 1.768 | 1.806 | 1.858 |
| 90.0 | 1.346 | 1.310 | 1.320 | 1.327 | 1.339 | 1.352 | 1.370 | 1.397 | 1.433 |
| 80.0 | 0.882 | 0.859 | 0.866 | 0.870 | 0.877 | 0.885 | 0.896 | 0.912 | 0.934 |
| 50.0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 100P | Table of $\bar{n}(\theta)/n_0$ for $D(v,y,y,\rho) = P$. | | | | | | | | |
| 99.9 | .665 | .728 | .705 | .693 | .673 | .655 | .633 | .607 | .580 |
| 99.5 | .684 | .742 | .720 | .709 | .691 | .674 | .654 | .631 | .606 |
| 99.0 | .697 | .752 | .732 | .721 | .704 | .689 | .670 | .649 | .625 |
| 97.5 | .724 | .773 | .755 | .745 | .730 | .717 | .701 | .682 | .662 |
| 95.0 | .752 | .795 | .779 | .771 | .758 | .746 | .732 | .716 | .700 |
| 90.0 | .788 | .824 | .811 | .804 | .793 | .783 | .772 | .759 | .745 |
| 80.0 | .829 | .858 | .848 | .842 | .833 | .826 | .816 | .806 | .795 |
| 50.0 | .868 | .891 | .883 | .879 | .872 | .865 | .858 | .850 | .842 |
| $n_1/n_0$ | .653 | .720 | .695 | .682 | .661 | .642 | .620 | .593 | .566 |

and $\bar{n}(\theta_{1-\alpha}) = \bar{n}(\theta_\beta) \simeq 0.73n_0$.

Example 1. Suppose that we want to test the hypothesis $\theta \leq 0$ against $\theta > 0$ for $\sigma = 10$ and that we require $P(\theta_1) = 1-P(\theta_2) = 0.95$ for $\theta_1 = 0$ and $\theta_2 = 3$, say, for the single sampling test. Hence $h_0 = 1.5$ and $n_0 = 100 \cdot 2.706 \cdot (2/3)^2 = 120.3$.

By means of Table 4 we obtain the following equivalent double sampling tests.

Equivalence.

| | Moment | Slope | 0.95 Fractile |
|---|---|---|---|
| $n$ | 134 | 146 | 135 |
| $n_1$ | 79 | 68 | 77 |
| $y$ | 0.622 | 0.560 | 0.623 |
| $h_a$ | 0.80 | 0.82 | 0.79 |
| $h_r$ | 2.20 | 2.18 | 2.21 |
| max $\bar{n}(\theta)$ | 104 | 101 | 104 |
| $\bar{n}(\theta_1)=\bar{n}(\theta_2)$ | 90 | 85 | 90 |

It will be seen that the three specifications of strength lead to slightly different double sampling tests.

Non-Symmetric double sampling tests. We shall consider tests of strength $(\theta_1,\alpha,\theta_2,\beta)$, $\alpha \neq \beta$, minimizing max $\bar{n}(\theta)$. The equations for determining the parameters will be given in Section 6. Some results for $\beta = 2\alpha$ are presented in Table 5. Comparing with Table 4 it will be seen that $\rho$, $n/n_0$ and max $\bar{n}(\theta)/n_0$ can be found with great accuracy as the average of the values in Table 4 for $1-\alpha$ and $1-\beta$.

Table 5. Table of double sampling tests of strength $(\theta_1,\alpha,\theta_2,\beta)$ minimizing max $\bar{n}(\theta)$.

| $1-\alpha$ | $\beta$ | $y_a$ | $y_r$ | $\rho$ | $n/n_0$ | $\frac{\max\bar{n}(\theta)}{n_0}$ | $v_{1-\alpha}$ | $-v_\beta$ |
|---|---|---|---|---|---|---|---|---|
| 0.999 | 0.002 | 0.6500 | 0.6894 | 0.6702 | 1.067 | 0.8897 | 3.185 | 2.980 |
| 0.995 | 0.01 | 0.6278 | 0.6801 | 0.6355 | 1.083 | 0.8808 | 2.671 | 2.431 |
| 0.990 | 0.02 | 0.6145 | 0.6754 | 0.6166 | 1.094 | 0.8760 | 2.421 | 2.160 |
| 0.975 | 0.05 | 0.5910 | 0.6684 | 0.5861 | 1.112 | 0.8686 | 2.050 | 1.751 |
| 0.950 | 0.10 | 0.5660 | 0.6632 | 0.5575 | 1.132 | 0.8618 | 1.727 | 1.386 |

## 6. Optimum double sampling tests of strength $(\theta_1, \alpha, \theta_2, \beta)$.

Consider the class of all double sampling tests satisfying $P(\theta_1) = 1 - \alpha$ and $P(\theta_2) = \beta$, $\theta_1 < \theta_2$ and $0 < \beta < 1 - \alpha < 1$, and the corresponding point set $\{\bar{n}(\theta_1), \bar{n}(\theta_2)\}$. Introducing randomized tests the point set of ASN's will obviously be convex and bounded below, and the admissible tests correspond to the part of the lower boundary lying between the horizontal and the vertical tangent to the convex set. The admissible tests may therefore be found as the Bayes ASN tests by minimizing $\bar{n}_w = w\,\bar{n}(\theta_1) + (1-w)\bar{n}(\theta_2)$, since for each $0 \leq w \leq 1$ we get a supporting tangent and thus a point on the lower boundary of the convex set. The procedure has been illustrated in Fig. 1 for $(\alpha, \beta) = (0.05, 0.10)$ by plotting $(\bar{n}(\theta_1)/n_0, \bar{n}(\theta_2)/n_0)$ for the admissible tests and showing the supporting tangent for $w = 2/3$.

Setting $v_i = (h-\theta_i)/\tau$, $a_{1i} = v_i\sqrt{\rho} - y_a$ and $r_{1i} = v_i\sqrt{\rho} + y_r$ for $i = 1,2$ we get $\bar{n}_w = n\{\rho + (1-\rho)c\}$, where

$$c = w\{\Phi(r_{11}) - \Phi(a_{11})\} + (1-w)\{\Phi(r_{12}) - \Phi(a_{12})\}.$$

The problem is to minimize $\bar{n}_w$ under the two constraints

$$D(v_1, y_a, y_r, \rho) = 1 - \alpha \text{ and } D(v_2, y_a, y_r, \rho) = \beta. \qquad (19)$$

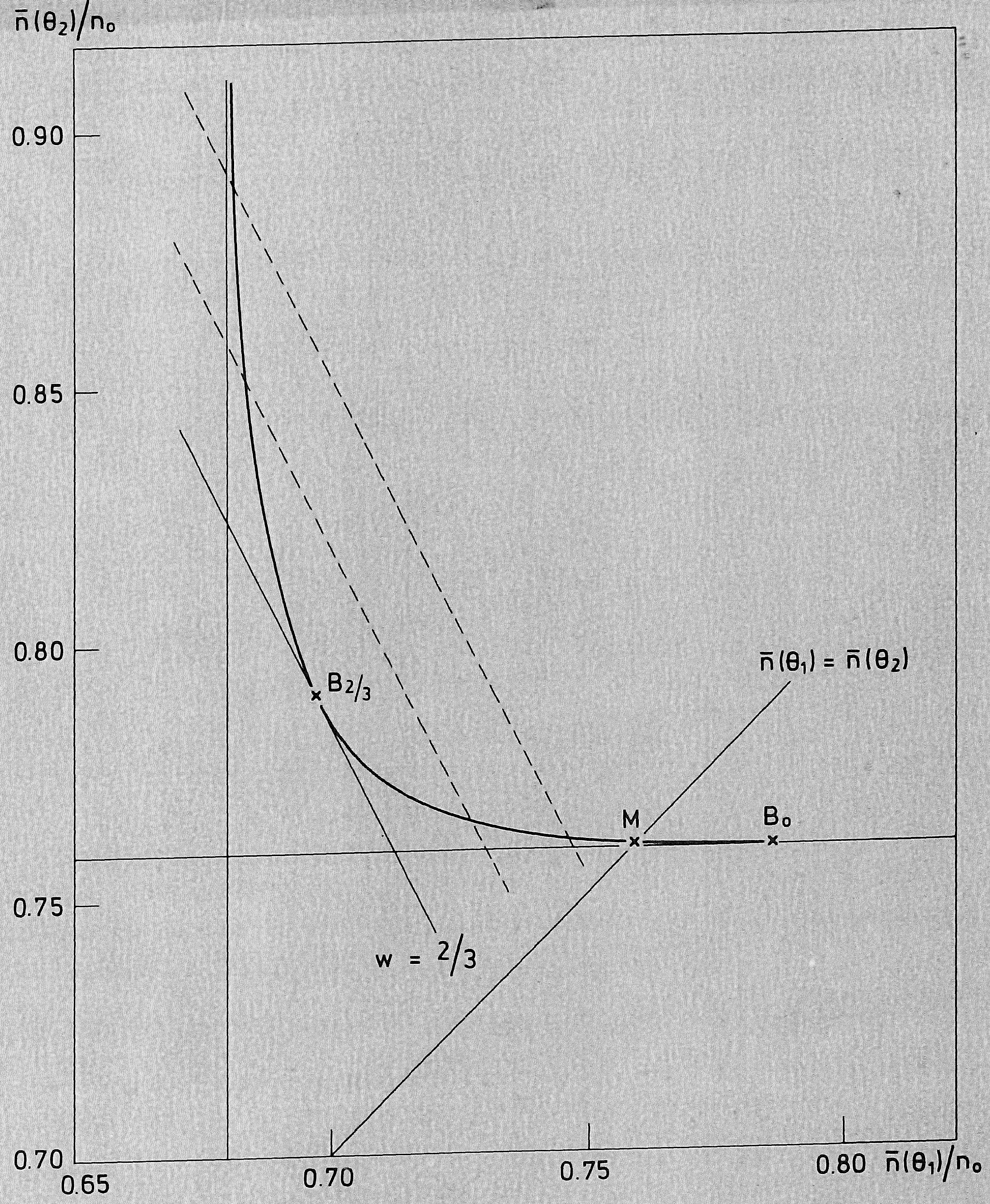
Since there exists a one-to-one relation between $(h,n)$ and $(v_1, v_2)$ we shall treat $(v_1, v_2, y_a, y_r, \rho)$ as the unknown parameters and determine $(h,n)$ from $h = (\theta_2 v_1 - \theta_1 v_2)/(v_1 - v_2)$ and $\sqrt{n} = \sigma(v_1 - v_2)/(\theta_2 - \theta_1)$.

Writing $D(v)$ for $D(v, y_a, y_r, \rho)$ and $G = \bar{n}_w - \lambda_1 D(v_1) - \lambda_2 D(v_2)$ we get five equations of the form $\bar{n}'_w = \lambda_1 D'(v_1) + \lambda_2 D'(v_2)$. The derivatives of D have been given in Section 3 and it is straightforward to find $\bar{n}'_w$ noting that $\partial n/\partial v_1 = -\partial n/\partial v_2 = 2n/(v_1 - v_2)$. Denoting the five derivatives of $\bar{n}_w$ with the subscripts 1, 2, a, r and ρ, respectively, we get

$$\lambda_1 D'_v(v_1) = \bar{n}'_{w1} \text{ and } \lambda_2 D'_v(v_2) = \bar{n}'_{w2}, \qquad (20)$$

and

Fig. 1. Inverse efficiency, $\bar{n}(\theta_1)/n_0$ and $\bar{n}(\theta_2)/n_0$, for the admissible double sampling tests of strength ($\theta_1$, 0.05, $\theta_2$, 0.10). $B_{2/3}$ corresponds to the Bayes ASN test for w = 2/3 and M corresponds to the minimax$\{\bar{n}(\theta_1),\bar{n}(\theta_2)\}$ test.



$$\lambda_1 D'_a(v_1) + \lambda_2 D'_a(v_2) = \bar{n}'_{wa}, \qquad (21)$$

$$\lambda_1 D'_r(v_1) + \lambda_2 D'_r(v_2) = \bar{n}'_{wr}, \qquad (22)$$

$$\lambda_1 D'_\rho(v_1) + \lambda_2 D'_\rho(v_2) = \bar{n}'_{w\rho}. \qquad (23)$$

The seven equations may be solved by iteration. Starting from a suitably chosen value of $(y_a, y_r, \rho)$ we find $(v_1, v_2)$ from (19) and $(\lambda_1, \lambda_2)$ from (20). Inserting into (21) - (23) we may find an improvement of the starting value and then start a new cycle.

Table 6 shows the results for $(\alpha,\beta) = (0.05, 0.10)$ and various values of w, see also Fig. 1, $n/n_0$ being found as $(v_1-v_2)^2/(u_{1-\alpha}-u_\beta)^2$. It will be seen that the minimax ASN test corresponds to $w \simeq 0.05$.

We remark that the problem of determining the double sampling test of strength $(\theta_1, \alpha, \theta_2, \beta)$ minimizing max $\bar{n}(\theta)$ may be solved by the same method since we only have to replace $\bar{n}_w$ by max $\bar{n}(\theta)$. Numerical results for this case have already been presented in Table 5.

In the <u>symmetric</u> case, that is, for $y_a = y_r = y$ and $\alpha = \beta$, we have $\bar{n}(\theta_1) = \bar{n}(\theta_2)$, which means that the equations become much simpler. As $h = h_0$ we only have to determine $(y, \rho, n)$ by minimizing $\bar{n}(\theta_1)$ under the restriction $D(v_1, y, y, \rho) = 1 - \alpha$.

We shall now present some numerical results and also compare the efficiency of the optimum double sampling test with the efficiency of the SPRT which has been given by Ghosh (1970, p.138).

For the symmetric case results are given in Table 7 which is analogous to Table 4 for the minimax $\bar{n}(\theta)$ tests. For the commonly used values of $\alpha$ we have $n \simeq 1.18n_0$, $\rho \simeq 0.43$, that is $n_2/n_1 \simeq 4/3$.

A comparison with the SPRT is given in Table 8.

Table 6. Bayes ASN double sampling tests for $\alpha = 0.05$ and $\beta = 0.10$.

| $w$ | $y_a$ | $y_r$ | $\rho$ | $n/n_0$ | $\bar{n}(\theta_1)/n_0$ | $\bar{n}(\theta_2)/n_0$ | $v_{1-\alpha}$ | $-v_\beta$ |
|---|---|---|---|---|---|---|---|---|
| 0.000 | 1.155 | 0.606 | 0.463 | 1.138 | 0.786 | 0.759 | 1.869 | 1.253 |
| 0.050 | 1.001 | 0.618 | 0.464 | 1.149 | 0.758 | 0.760 | 1.855 | 1.281 |
| 0.125 | 0.886 | 0.638 | 0.462 | 1.159 | 0.740 | 0.761 | 1.839 | 1.313 |
| 0.200 | 0.813 | 0.662 | 0.459 | 1.168 | 0.730 | 0.763 | 1.823 | 1.339 |
| 0.250 | 0.776 | 0.678 | 0.456 | 1.172 | 0.725 | 0.765 | 1.813 | 1.355 |
| 0.333 | 0.726 | 0.710 | 0.451 | 1.179 | 0.718 | 0.768 | 1.796 | 1.381 |
| 0.400 | 0.693 | 0.737 | 0.447 | 1.183 | 0.713 | 0.770 | 1.783 | 1.400 |
| 0.500 | 0.652 | 0.786 | 0.439 | 1.188 | 0.707 | 0.776 | 1.760 | 1.430 |
| 0.600 | 0.616 | 0.845 | 0.431 | 1.191 | 0.701 | 0.783 | 1.735 | 1.459 |
| 0.667 | 0.595 | 0.893 | 0.425 | 1.192 | 0.697 | 0.789 | 1.716 | 1.480 |
| 0.750 | 0.570 | 0.968 | 0.417 | 1.192 | 0.693 | 0.801 | 1.689 | 1.506 |
| 0.800 | 0.556 | 1.026 | 0.412 | 1.191 | 0.690 | 0.810 | 1.671 | 1.523 |
| 0.875 | 0.536 | 1.146 | 0.403 | 1.187 | 0.686 | 0.833 | 1.638 | 1.550 |
| 0.950 | 0.516 | 1.366 | 0.393 | 1.179 | 0.682 | 0.879 | 1.596 | 1.581 |
| 1.000 | 0.501 | 1.900 | 0.384 | 1.170 | 0.680 | 0.990 | 1.557 | 1.609 |

Table 7. Table of symmetric double sampling tests of strength $(\theta_1, \alpha, \theta_2, \alpha)$ minimizing $\bar{n}(\theta_1)$.

| | OC-Fractile equivalence | | | | | | | Slope equiv. |
|---|---|---|---|---|---|---|---|---|
| $100(1-\alpha)$ | 99.9 | 99.5 | 99.0 | 97.5 | 95.0 | 90.0 | 80.0 | - |
| $u^2_{1-\alpha}$ | 9.550 | 6.635 | 5.412 | 3.841 | 2.706 | 1.642 | 0.7083 | |
| $n/n_0$ | 1.159 | 1.170 | 1.174 | 1.180 | 1.186 | 1.193 | 1.203 | 1.216 |
| $\rho$ | .3948 | .4075 | .4147 | .4260 | .4359 | .4466 | .4569 | .4652 |
| $y$ | 1.133 | .9750 | .9066 | .8165 | .7488 | .6814 | .6162 | .5596 |
| 100P | Table of $v$ for $D(v,y,y,\rho) = P$. | | | | | | | |
| 99.9 | 3.327 | 3.434 | 3.484 | 3.549 | 3.597 | 3.643 | 3.689 | 3.733 |
| 99.5 | 2.717 | 2.786 | 2.821 | 2.870 | 2.908 | 2.947 | 2.988 | 3.027 |
| 99.0 | 2.437 | 2.493 | 2.521 | 2.562 | 2.594 | 2.628 | 2.665 | 2.700 |
| 97.5 | 2.038 | 2.078 | 2.009 | 2.129 | 2.154 | 2.181 | 2.210 | 2.238 |
| 95.0 | 1.703 | 1.733 | 1.748 | 1.772 | 1.791 | 1.812 | 1.835 | 1.858 |
| 90.0 | 1.322 | 1.343 | 1.354 | 1.370 | 1.384 | 1.400 | 1.416 | 1.433 |
| 80.0 | 0.866 | 0.878 | 0.885 | 0.895 | 0.903 | 0.913 | 0.923 | 0.934 |
| 50.0 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 100P | Table of $\bar{n}(\theta)/n_0$ for $D(v,y,y,\rho) = P$. | | | | | | | |
| 99.9 | .576 | .554 | .549 | .547 | .551 | .558 | .569 | .580 |
| 99.5 | .655 | .621 | .610 | .600 | .595 | .595 | .599 | .606 |
| 99.0 | .697 | .660 | .646 | .631 | .624 | .621 | .622 | .625 |
| 97.5 | .762 | .721 | .704 | .685 | .674 | .666 | .662 | .662 |
| 95.0 | .817 | .774 | .757 | .735 | .721 | .710 | .703 | .700 |
| 90.0 | .874 | .833 | .814 | .791 | .775 | .762 | .751 | .745 |
| 80.0 | .932 | .892 | .873 | .849 | .832 | .816 | .804 | .795 |
| 50.0 | .979 | .942 | .924 | .900 | .882 | .866 | .852 | .842 |
| $n_1/n_0$ | .458 | .477 | .487 | .503 | .517 | .533 | .550 | .566 |

Table 8. Inverse efficiency, $\bar{n}(\theta_{1-\alpha})/n_0$, for fractile equivalent, symmetric double sampling tests and SPRT's.

| $100\alpha$ | 0.5 | 1 | 5 | 10 |
|---|---|---|---|---|
| Double | 0.62 | 0.65 | 0.72 | 0.76 |
| SPRT | 0.40 | 0.42 | 0.49 | 0.53 |

Results for the non-symmetric case are given in Table 6 and 9, see also Fig. 1 and 2. For $w = 1/2$ a good approximation to the solution may be read from Table 7.

Since errors of "the first kind" are considered more serious than errors of "the second kind" we choose $\alpha \leqq \beta$. Accordingly it is desirable to have $\bar{n}(\theta_1) \leqq \bar{n}(\theta_2)$, which means that the minimax$\{\bar{n}(\theta_1), \bar{n}(\theta_2)\}$ test is a limiting case. Among the admissible tests it seems reasonable to choose the Bayes ASN test with $w = \beta/(\alpha+\beta)$, see Fig. 2. For $\beta = 2\alpha$ there is no essential difference between tests with $w = 2/3$ and $w = 1/2$.

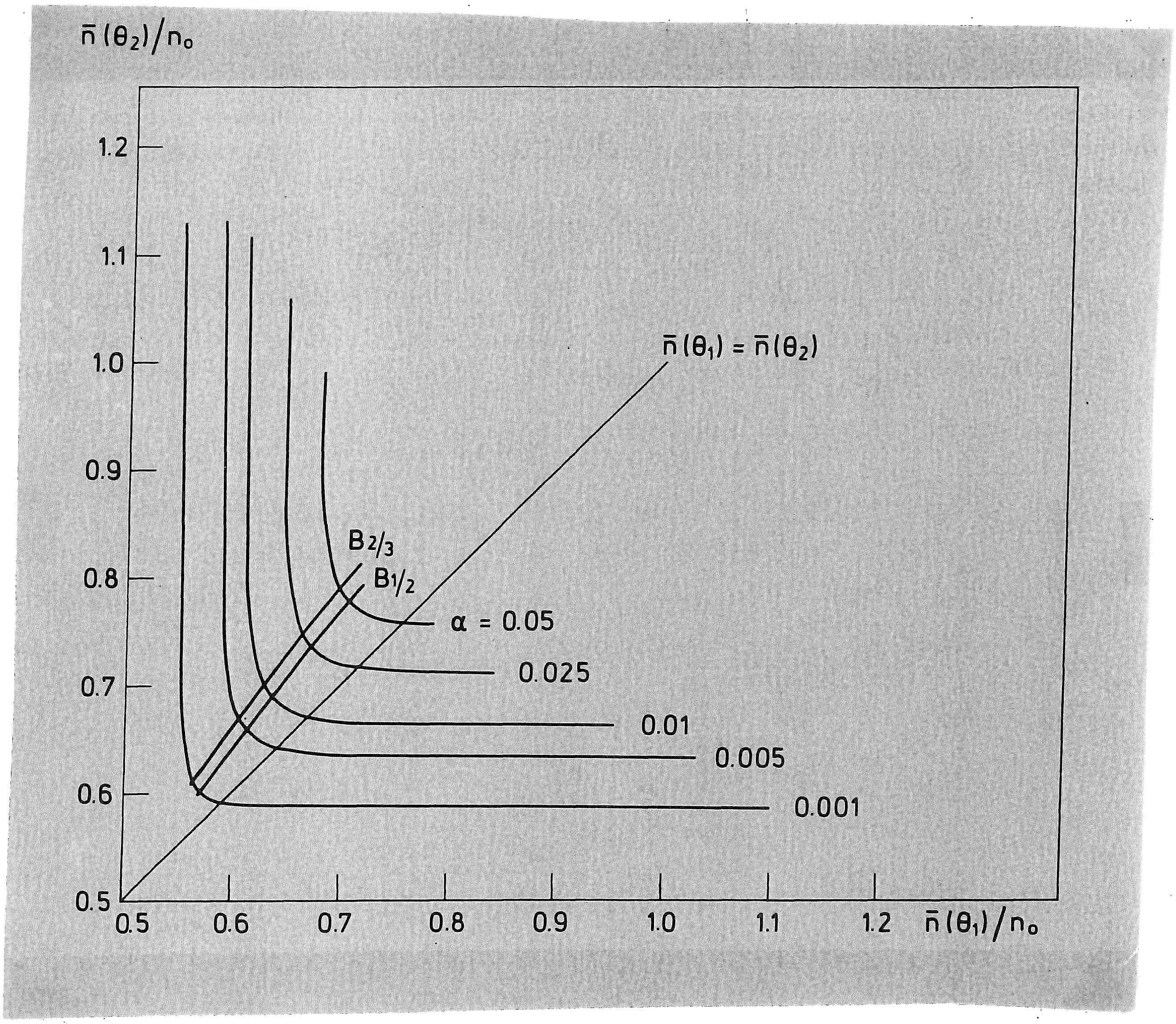
Table 9 contains sufficient information for finding the Bayes ASN tests for other values of $w$ and $\alpha$ by interpolation.

Example 2. For the problem considered in Example 1 we may find the 0.95 fractile equivalent double sampling test minimizing $\bar{n}(\theta_1) = \bar{n}(\theta_2)$ by means of Table 7. From $n_0 = 120.3$ we get immediately $n = 143$, $n_1 = 62$, $\bar{n}(\theta_1) = \bar{n}(\theta_2) = 87$ (as compared to 90 in Example 1) and max $\bar{n}(\theta) = 106$ (as compared to 104).

Example 3. Considering the problem in Example 1 for $P(\theta_1) = 0.95$ and $P(\theta_2) = 0.10$ we get $h_0 = 1.69$ and $n_0 = 95.2$. The fractile equivalent minimax $\bar{n}(\theta)$ test and the fractile equivalent Bayes ASN test for $w = 2/3$ may be found from Table 5 and Table 9, respectively, by straightforward computations. The results are

| | $n$ | $n_1$ | $h_a$ | $h_r$ | $h$ | $\bar{n}(\theta_1)$ | $\bar{n}(\theta_2)$ | $\bar{n}_w$ | max $\bar{n}$ |
|---|---|---|---|---|---|---|---|---|---|
| Minimax | 108 | 60 | 0.93 | 2.52 | 1.66 | 70 | 74 | 71 | 82 |
| Bayes | 113 | 48 | 0.75 | 2.90 | 1.61 | 66 | 75 | 69 | 84 |

Fig. 2. Inverse efficiency, $\bar{n}(\theta_1)/n_0$ and $\bar{n}(\theta_2)/n_0$, for the admissible double sampling tests of strength $(\theta_1, \alpha, \theta_2, \beta)$ for $\beta = 2\alpha$. $B_{2/3}$ and $B_{1/2}$ correspond to the Bayes ASN tests for $w = 2/3$ and $w = 1/2$, respectively.



Even if there are considerable differences between the parameters of the two tests the differences expressed in terms of the ASN's are rather small. Of course, more pronounced differences will be found for smaller values of $\alpha$.

## 7. Moment equivalent, Bayes ASN, double sampling tests with normal weight function.

Consider the class of moment equivalent, double sampling tests, that is, tests satisfying equations (9) and (10), and suppose that we average $\bar{n}(\theta)$ with respect to a weight function which is chosen as a normal distribution function with parameters $(\mu,\omega^2)$. This gives

$$\bar{n}_w = \int_{-\infty}^{\infty} \bar{n}(\theta)\omega^{-1}\phi\{(\theta-\mu)/\omega\}d\theta$$

$$= n_1 + n_2\left\{\Phi\left(\frac{h_r-\mu}{\sqrt{\omega^2+\tau_1^2}}\right) - \Phi\left(\frac{h_a-\mu}{\sqrt{\omega^2+\tau_1^2}}\right)\right\}.$$

To characterize the weight function in relation to the given OC distribution with parameters $(h_0,\tau_0^2)$ we introduce $\mu_0 = (h_0-\mu)/\tau_0$ and $\omega_0 = \omega/\tau_0$.

Further, we introduce

$$k = (h-\mu)/\tau_0,$$

$$c = 1 + \frac{\omega^2}{\tau_1^2} = 1 + \omega_0^2\,\rho\,\frac{n}{n_0},$$

$$a = \left(-y_a + k\sqrt{\frac{\rho n}{n_0}}\right)c^{-1/2}$$

and

$$r = \left(y_r + k\sqrt{\frac{\rho n}{n_0}}\right)c^{-1/2}$$

such that

$$\bar{n}_w = n[\rho + (1-\rho)\{\Phi(r) - \Phi(a)\}].$$

The Bayes ASN test may then be found by minimizing $\bar{n}_w$ with respect to k, $n/n_0$, $y_a$, $y_r$ and $\rho$ under the constraints (9) and (10) using the usual procedure.

To tabulate the solution we have to choose $\mu$ and $\omega$ in a reasonable way. Suppose that we want to put the greatest weight at a certain value of $\theta$, $\theta_{1-\alpha}$ say. This means that $\mu = \theta_{1-\alpha}$ or $\mu_0 = u_{1-\alpha}$. Consider next a larger value of $\theta$, $\theta_\beta$ say, $0 < \beta < 1 - \alpha < 1$, and suppose that we want the weight at that point to be a certain fraction $\lambda$, $0 < \lambda < 1$, of the maximum weight. This leads to the equation $\theta_\beta = h_0 + u_{1-\beta}\tau_0$ and

$$-\frac{1}{2\omega^2}\left(\theta_\beta - \theta_{1-\alpha}\right)^2 = \ln \lambda$$

with the solution

$$\omega_0 = (u_{1-\beta} - u_\alpha)(-2 \ln \lambda)^{-1/2}.$$

Some examples of Bayes ASN tests have been given in Table 10. The interpretation of the results is rather obvious. Note that $h_0 - h = (\mu_0 - k)\tau_0$. Similar computations for $\beta = \alpha$ show that the parameters of the double sampling tests are nearly the same as for $\beta = 2\alpha$.

Table 10. Moment equivalent, Bayes ASN, double sampling tests. $\beta = 2\alpha$.

| $100(1-\alpha)$ | $\mu_0$ | $\lambda$ | $\omega_0$ | $\mu_0-k$ | $y_a$ | $y_r$ | $\rho$ | $n/n_0$ | $\bar{n}_w/n_0$ |
|---|---|---|---|---|---|---|---|---|---|
| 99.9 | 3.090 | 1/2 | 5.069 | .028 | 1.108 | 1.297 | .261 | 1.172 | .535 |
| | | 1/10 | 2.781 | .071 | 0.930 | 1.426 | .293 | 1.170 | .577 |
| 99.5 | 2.576 | 1/2 | 4.164 | .032 | 1.036 | 1.241 | .289 | 1.169 | .573 |
| | | 1/10 | 2.284 | .076 | 0.864 | 1.376 | .323 | 1.166 | .615 |
| 99.0 | 2.326 | 1/2 | 3.720 | .033 | 0.997 | 1.210 | .305 | 1.168 | .595 |
| | | 1/10 | 2.041 | .077 | 0.828 | 1.346 | .340 | 1.163 | .637 |
| 97.5 | 1.960 | 1/2 | 3.062 | .037 | 0.930 | 1.159 | .334 | 1.164 | .632 |
| | | 1/10 | 1.680 | .080 | 0.772 | 1.292 | .369 | 1.158 | .672 |
| 95.0 | 1.645 | 1/2 | 2.486 | .041 | 0.864 | 1.108 | .365 | 1.160 | .669 |
| | | 1/10 | 1.364 | .081 | 0.720 | 1.232 | .400 | 1.151 | .706 |
| 90.0 | 1.282 | 1/2 | 1.803 | .047 | 0.774 | 1.038 | .411 | 1.151 | .719 |
| | | 1/10 | 0.989 | .080 | 0.659 | 1.136 | .443 | 1.142 | .751 |

Comparing the results in Table 10 and Table 9 it will be seen that $\rho$ does not vary much with $\lambda$ and w, respectively, whereas $\rho$ depends strongly on $\alpha$ in Table 10 but not in Table 9. For $\alpha = 0.05$, small values of $\lambda$ and large values of w we have in both cases values of $\rho$ about 0.4.

Investigation of the symmetric case $\mu = h_0 (\mu_0 = 0)$ shows that the parameters vary rather much with $\omega_0$. For example we get $\rho = 0.586, 0.436, 0.200$ for $\omega_0 = 0.1, 2.0, 10.0$ respectively. For $\omega_0 \to 0$ we get the minimax test described in Theorem 3.

Example 4. As in Example 3 we start from a single sampling test with $h_0 = 1.69$ and $n_0 = 95.2$ such that $\theta_1 = 0$ and $\theta_2 = 3$ have acceptance probabilities equal to 0.95 and 0.10, respectively. From Table 10 we may determine the moment equivalent, Bayes ASN test with a normal weight function. Suppose the largest weight is put on $\theta = 0$ and that 1/10 of that weight is put on $\theta = 3$. The parameters may then be found in Table 10 for $1 - \alpha = 0.95$ and $\lambda = 1/10$. For comparison with Example 3 the results are tabulated below:

| $n$ | $n_1$ | $h_a$ | $h_r$ | $h$ | $\bar{n}(\theta_1)$ | $\bar{n}(\theta_2)$ | $\bar{n}_w$ | $\max \bar{n}$ |
|---|---|---|---|---|---|---|---|---|
| 110 | 44 | 0.52 | 3.46 | 1.60 | 67 | 81 | 67 | 88 |

It will be seen that $\bar{n}(\theta_2)$ and $\max \bar{n}(\theta)$ here are larger than in Example 3 because more weight has been put on the smaller values of $\theta$. This has also resulted in a smaller value of $\bar{n}_w$.

## 8. Discussion

The purpose of double sampling as compared to single sampling is to obtain a test having the same OC and on the average fewer observations. This formulation immediately raises two questions: (1) What does it mean that the OC's are the same? (2) For what values of $\theta$ should $\bar{n}(\theta)$ be small?

The first question has been answered by defining equivalent tests as tests having two properties of the OC in common, disregarding all other features of the OC distribution. One of the reasons for choosing two properties only is that the single sampling test and the SPRT both are fully determined by means of two parameters. It should be noted, however, that as a consequence of this equivalence definition we are led to consider the normal and the logistic distributions as OC equivalent. This is common usage in controlling the risks of error in testing hypotheses but a similar point of view would not be considered good statistical practice in analysing data.

Usually two fractiles are employed for specifying the OC. As another possibility we have considered the first two moments. If one wants precise control of the risks of error at two specified values of the parameter fractile equivalence should be used, but if one wants good over-all agreement of the two OC's then moment equivalence is preferable. Furthermore, moment equivalence has the advantage of being much easier to work with in mathematical and computational respects.

The second question above has no unambiguous answer in the case of double sampling. For fractile equivalence we have the surprising result that the SPRT minimizes the ASN at both the specified values of the parameter but if we limit ourselves to double sampling tests a similar result does not exist. In accordance with general decision-theoretic concepts we have therefore introduced minimax and Bayes ASN double sampling tests and tabulated a reasonable selection of such tests. The choice of a good test among the admissible ones is then as usual left to the "client" who has to provide the necessary background knowledge for the choice. However, looking at the tables it is gratifying (and somewhat surprising) to find that for the values of $\alpha$ and $\beta$ usually employed there is not much difference between the ASN functions for the minimax and the Bayes tests for values of the parameter between the .975 and .025 fractiles, see Fig. 3 for an example. Naturally, for smaller values of $\alpha$ and $\beta$ the difference becomes considerably larger and in such cases the choice of optimality criterion becomes crucial.

The effect of using different optimality criteria may be studied in detail by comparison of the results in Tables 4 and 7. Some of the main results have been exhibited in Fig. 4. In the one case we minimize $\bar{n}(\theta_{.5})$, in the other $\bar{n}(\theta_{\alpha})$. For $\alpha = 0.5$ the two optimality criteria are identical. For larger values of $1-\alpha$ the differences between the parameters become more pronounced, but differences between the corresponding ASN's are still rather small for $1 - \alpha < 0.95$. Looking at the results for the commonly used values of $\alpha$, $0.01 \leqq \alpha \leqq 0.05$ say, it will be seen that $n_2/n_1 \simeq 2/3$ if min $\bar{n}(\theta_{.5})$ is used as optimality criterion, whereas $n_2/n_1 \simeq 4/3$ if min $\bar{n}(\theta_{\alpha})$ is used. The old rule with $n_2 = 2n_1$ cannot be recommended in these cases, whereas one is tempted to say that $n_2 = n_1$ which has been used in Mil-Std 105 D is a happy compromise between the two results above. Also in the non-symmetric case discussed in Table 9 a value of $n_2/n_1$ of about 4/3 will give nearly optimum tests. For the Bayes ASN tests considered in Table 10, however, a somewhat larger value between 1.5 and 2, say, should be used.

In acceptance sampling one is often interested in keeping the amount of sampling inspection down for lots of process average quality and it is therefore reasonable to use a Bayes ASN test with a large weight on the process average quality. In other fields of application a less extreme distribution of weights may be more reasonable.

In Section 6 it was recommended as a rule of thumb to

<u>Fig. 3</u>. Inverse efficiency for 0.95 fractile equivalent, symmetric double sampling tests minimizing max $\bar{n}(\theta)$ and minimizing $\bar{n}(\theta_{.95})$, respectively.

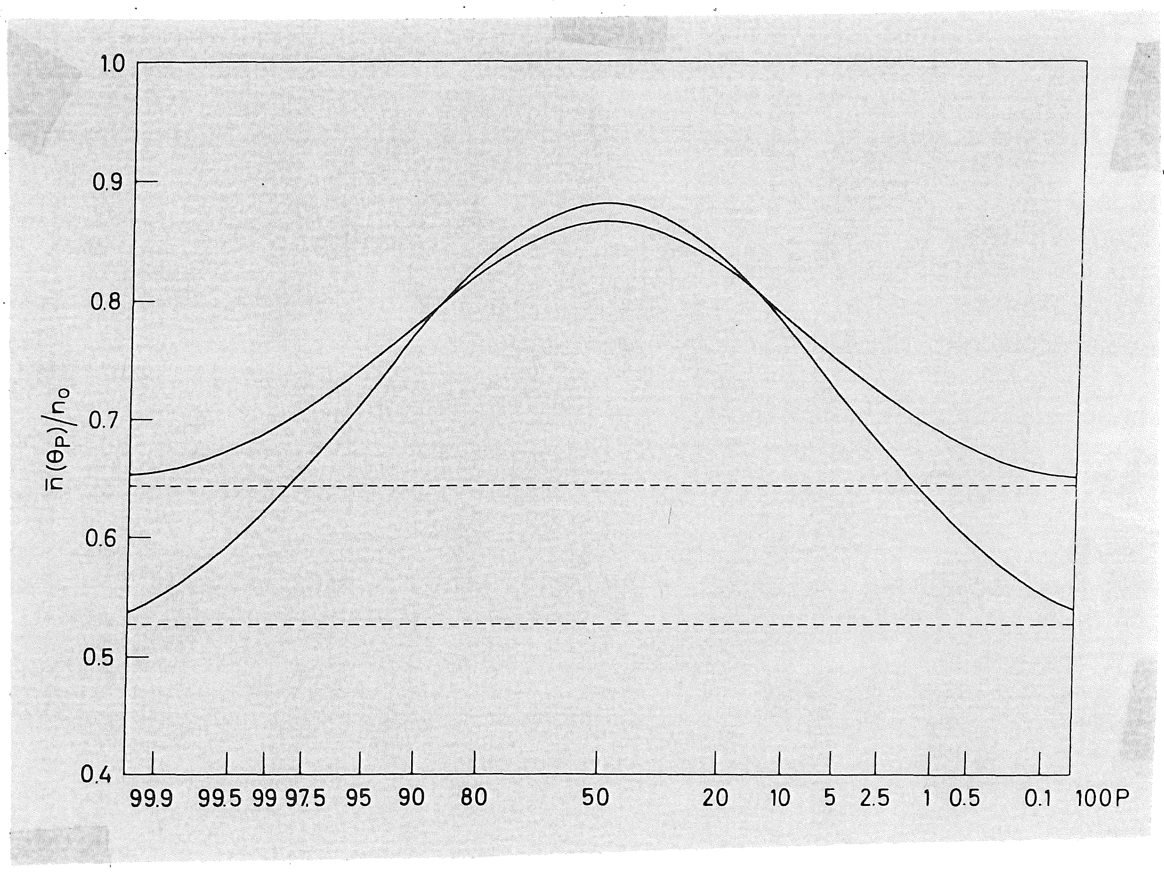
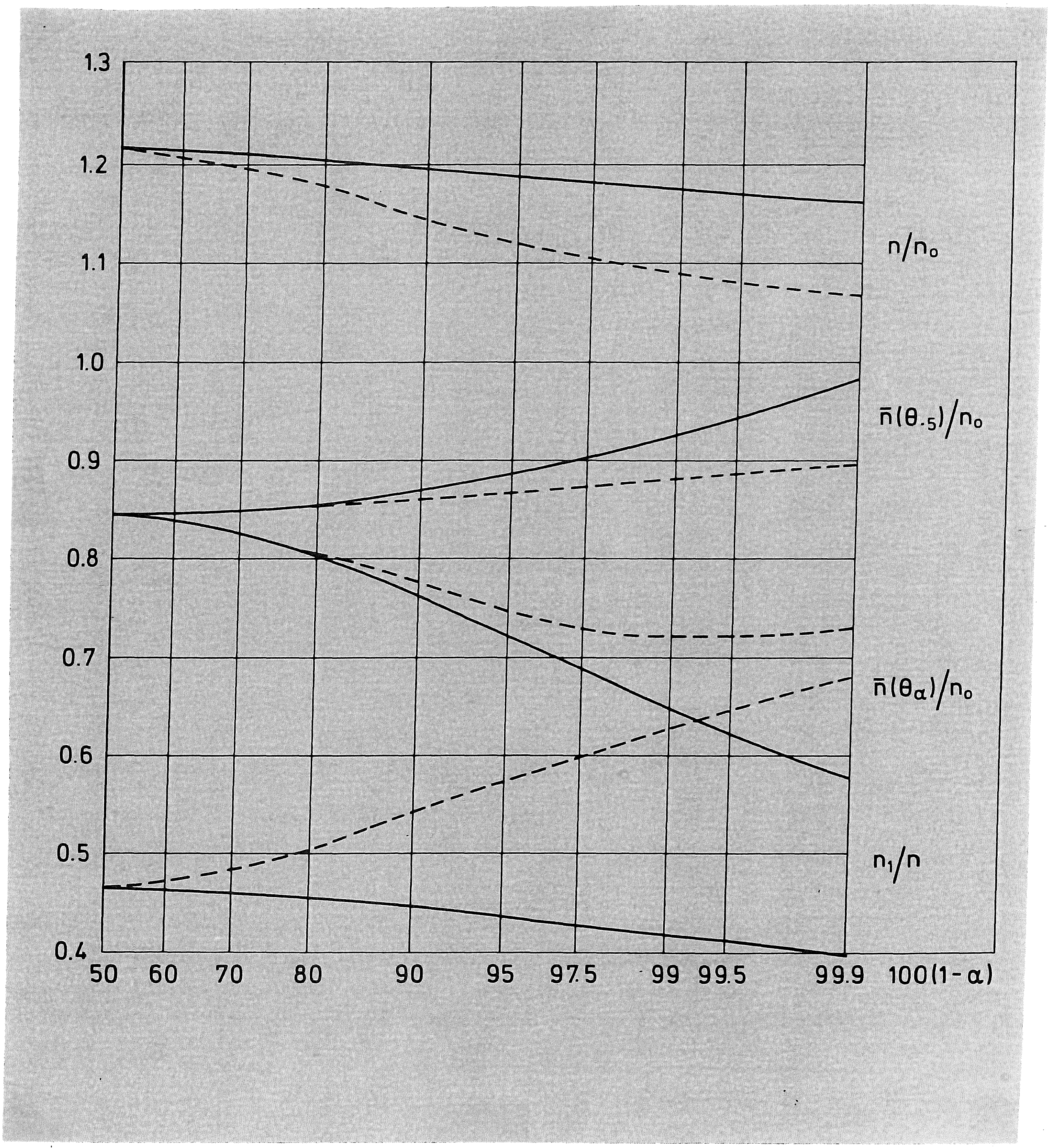


Fig. 4. Graph of main results for fractile equivalent, symmetric double sampling tests minimizing $\overline{n}(\theta_{.5})$, see Table 4 and minimizing $\overline{n}(\theta_\alpha)$, see Table 7.



choose the Bayes ASN test with $w = \beta/(\alpha+\beta)$ among the admissible tests of strength $(\theta_1, \alpha, \theta_2, \beta)$ because this leads to a test with small values of the ASN for <u>both</u> $\theta_1$ and $\theta_2$ and with $\bar{n}(\theta_1) \leqq \bar{n}(\theta_2)$. It may be of some interest to compare the ASN's for the corresponding SPRT. It is well-known that for a normal process the ratio of the two ASN's for the SPRT equals

$$\frac{\bar{n}(\theta_2)}{\bar{n}(\theta_1)} = \frac{(1-\beta)\ \ln \frac{1-\beta}{\alpha} + \beta\ \ln \frac{\beta}{1-\alpha}}{(1-\alpha)\ \ln \frac{1-\alpha}{\beta} + \alpha\ \ln \frac{\alpha}{1-\beta}} .$$

This ratio and the corresponding ratio for the Bayes ASN double sampling test has been tabulated in Table 11 for $\beta = 2\alpha$. It will be seen that to obtain the same ratio of the ASN's as for the SPRT the weight for $\theta_1$ has to be chosen to approximately 0.8.

<u>Table 11</u>. Values of $\bar{n}(\theta_2)/\bar{n}(\theta_1)$ for $\beta = 2\alpha$.

| | Bayes ASN double sampling test | | | |
|---|---|---|---|---|
| $100\alpha$ | w=1/2 | w=2/3 | w=4/5 | SPRT |
| 0.1 | 1.06 | 1.09 | 1.12 | 1.11 |
| 0.5 | 1.07 | 1.11 | 1.15 | 1.14 |
| 1.0 | 1.08 | 1.12 | 1.16 | 1.16 |
| 2.5 | 1.09 | 1.13 | 1.17 | 1.18 |
| 5.0 | 1.10 | 1.13 | 1.17 | 1.19 |

From the tables (and graphs) of $\bar{n}(\theta)/n_0$ it follows that a considerable saving in the number of observations may be obtained by using (optimum) double sampling instead of single sampling. Table 8 also shows that the step from single to double sampling gives more than half of the saving obtainable from single to sequential sampling.

Besides being of direct value in constructing optimum double sampling tests for a normally distributed random variable with known standard deviation the results obtained are also valuable considered as large sample results for non-normal random variables. The importance of this point of view is enhanced by the fact that the minimum considered is very flat such that there is room for considerable variation in the parameters without changing the inverse efficiency very much. However, the values of the parameters must be properly balanced.

It is of course desirable to find the optimum double sampling tests for many other cases than the one considered here, for example for a normal distribution with unknown standard deviation, for Poisson and binomial distributions etc. However, so long as the exact solution is lacking it is important to notice that a good approximation may be obtained from the results given here. One obvious procedure would be to start from the optimum value of $\rho$ for the normal case and then determine the other parameters by minimization. Besides $\rho$ other parameters could be taken from the large sample solution. The procedure has been illustrated in Corollary 1 and 2 in Section 5. It will also be used for the Poisson distribution in the following sections.

A theory of double sampling based on prior distributions and linear costs has recently been developed. For a continuous prior $n_2/n_1$ tends slowly to infinity for $n_1 \to \infty$, see for example Hald and Keiding (1969, 1972), in contradistinction to the results found here where $n_2/n_1$ is constant. It is a well-known fact that Bayesian decision rules have quite different (asymptotic) properties than rules corresponding to a specified strength. For discrete priors, however, we have that $n_2/n_1$ tends slowly to 4 (from below), see Hald (1973).

For simplicity and for practical reasons we have used the ASN for a double sampling test divided by the sample size for the equivalent single sampling test as (inverse) efficiency measure. In accordance with general statistical theory it would have been more natural to measure efficiency by the ratio of the ASN for the optimum test (a sequential test) within the equivalence class and the ASN for the double sampling test in question, see Moriguti (1956). We have not, however, pursued this line of thought.

## Part 2. Double sampling from a Poisson population.

### 9. The OC function and its moments.

Consider a Poisson process with intensity $\lambda$ per observational unit. For a sample of size m we define $g(x,m\lambda) = e^{-m\lambda}(m\lambda)^x/x!$ and

$$G(c,m\lambda) = \sum_{x=0}^{c} g(x,m\lambda) = \int_{m\lambda}^{\infty} e^{-t}t^c dt/\Gamma(c+1)$$

for $\lambda \geq 0$, $m \geq 0$ and $c = 0,1,\ldots$ . However, the last expression for $G(c,m\lambda)$ makes it possible to extend the definition to non-integer values of c.

Let a single sampling test be given by $(a_0,m_0)$. The OC then becomes $P(\lambda) = G(a_0,m_0\lambda)$ with density $-P'(\lambda) = m_0 g(a_0,m_0\lambda)$ and moments $E\{(m_0\lambda)^k\} = (a_0+k)^{(k)}$, where $a^{(k)} = a(a-1)\ldots(a-k+1)$. It follows that $E\{m_0\lambda\} = V\{m_0\lambda\} = a_0+1$.

For double sampling the two sample sizes will be denoted by $m_1$ and $m_2$, $m = m_1 + m_2$ and $\rho = m_1/m$. The outcome of the sampling is denoted by $x_1$ and $x_2$, respectively, and $x = x_1 + x_2$. The decision rule is as follows: Accept if $x_1 \leq a_1$ and reject if $x_1 \geq r_1$. If $a_1 < x_1 < r_1$ take a second sample, accept if $x \leq a$ and reject otherwise. The probability of acceptance may be found as

$$Pr\{x \leq a\} + Pr\{(x>a)\cap(x_1 \leq a_1)\} - Pr\{(x \leq a)\cap(r_1 \leq x_1 \leq a)\}$$

such that the OC becomes

$$P(\lambda | a_1,r_1,a,m_1,m_2) = H(a_1,r_1,a,\rho,m_1\lambda)$$

$$G(a,m\lambda) + \sum_{x_1=0}^{a_1} g(x_1,m_1\lambda) \sum_{x_2=a+1-x_1}^{\infty} g(x_2,m_2\lambda) - \sum_{x_1=r_1}^{a} g(x_1,m_1\lambda) \sum_{x_2=0}^{a-x_1} g(x_2,m_2\lambda).$$

Using the fact that

$$g(x_1,m_1\lambda)g(x_2,m_2\lambda) = g(x_1+x_2,m\lambda)b(x_1,x_1+x_2,\rho),$$

where $b(x,n,\rho) = \binom{n}{x}\rho^x(1-\rho)^{n-x}$, and setting $v = m_1\lambda$ we get

$$H(a_1, r_1, a, \rho, v) = G(a, v/\rho) + \sum_{x_1=0}^{a_1} \sum_{x=a+1}^{\infty} b(x_1, x, \rho) g(x, v/\rho)$$

$$- \sum_{x_1=r_1}^{a} \sum_{x=x_1}^{a} b(x_1, x, \rho) g(x, v/\rho). \qquad (25)$$

The OC density is now easily found since $\partial P/\partial\lambda = m_1 \partial H/\partial v$, $G_m'(x,m) = -g(c,m)$ and $g_m'(c,m) = g(c-1,m) - g(c,m)$ for $c = 0,1,.$ setting $g(-1,m) = 0$.

To evaluate the moments of v we need the repeated sums of the binomial distribution $B_{k+1}(c,n,\rho) = \sum_{x=0}^{c} B_k(x,n,\rho)$ for $k = 0,1,\dots,$ where $B_0(x,n,\rho) = b(x,n,\rho)$. ($B_{k+1}$ is analogous to the repeated integral of the normal distribution which in turn is proportional to the k'th incomplete normal moment).

Theorem 6. The OC moment of order k+1 for the Poisson double sampling test equals $E\{\lambda^{k+1}\} = E\{v^{k+1}\}/m_1^{k+1}$, where

$$E\{v^{k+1}\} = (r_1+k)^{(k+1)}$$

$$- \sum_{\mu=0}^{k} \rho^{k-\mu} (k+1)^{(\mu+1)} (a+k+1)^{(k-\mu)} \Big[ B_{\mu+2}(x, a+\mu+1, \rho) \Big]_{x=a_1}^{r_1-1}.$$

Proof. To evaluate $E\{v^{k+1}\} = -\int v^{k+1} dH(v)$ we first note that

$$-\int_0^{\infty} v^{k+1} \{g_v'(x, v/\rho)\}\, dv = (k+1)\rho^{k+1}(x+k)^{(k)}$$

such that

$$E\{(v/\rho)^{k+1}\} = (a+k+1)^{(k+1)} + (k+1) \sum_{x_1=0}^{a_1} \sum_{x=a+1}^{\infty} (x+k)^{(k)} b(x_1, x, \rho)$$

$$-(k+1) \sum_{x_1=r_1}^{a} \sum_{x=x_1}^{a} (x+k)^{(k)} b(x_1, x, \rho).$$

The two sums may be changed to

$$\sum_{0}^{a_1} \sum_{a+1}^{\infty} - \sum_{r_1}^{a} \left( \sum_{x_1}^{\infty} - \sum_{a+1}^{\infty} \right) = \sum_{0}^{a} \sum_{a+1}^{\infty} - \sum_{r_1}^{a} \sum_{x_1}^{\infty} - \sum_{a_1+1}^{r_1-1} \sum_{a+1}^{\infty}.$$

Hence

$$E\{v^{k+1}\} = \rho^{k+1}(a+k+1)^{(k+1)} + S,$$

where S denotes the three sums above with summand $(k+1)\rho^{k+1}$ $(x+k)^{(k)}b(x_1,x,\rho)$. By means of Lemma 2 in the Appendix we get for the first sum

$$\sum_{\mu=0}^{k} \rho^{k-\mu}(k+1)^{(\mu+1)}(a+k+1)^{(k-\mu)}B_{\mu+2}(a,a+\mu+1,\rho)$$

$$= \sum_{\mu=0}^{k} \rho^{k-\mu}(k+1)^{(\mu+1)}(a+k+1)^{(k+1)}(1-\rho)^{\mu+1}/(\mu+1)!$$

$$= (a+k+1)^{(k+1)}(1-\rho^{k+1}),$$

where we have used the general formula for repeated sums

$$B_{k+1}(c) = \sum_{x=0}^{c} B_k(x) = \sum_{x=0}^{c} \binom{c+k-x}{k} B_0(x)$$

to prove that

$$B_{\mu+2}(a,a+\mu+1,\rho) = \binom{a+\mu+1}{\mu+1}(1-\rho)^{\mu+1}.$$

The second sum becomes

$$\sum_{r_1}^{a} (k+1)(x_1+k)^{(k)} = (a+k+1)^{(k+1)} - (r_1+k)^{(k+1)}.$$

The terms evaluated so far are easily reduced to $(r_1+k)^{(k+1)}$.

Lemma 2 gives directly the third sum which is the last term of $E\{v^{k+1}\}$ in the theorem. This concludes the proof.

Remark. The first two moments are

$$E\{m_1\lambda\} = r_1 - \Big[B_2(x,a+1,\rho)\Big]_{a_1}^{r_1-1} \tag{27}$$

and

$$E\{(m_1\lambda)^2\} = (r_1+1)^{(2)} - 2\rho(a+2)\Big[B_2(x,a+1,\rho)\Big]_{a_1}^{r_1-1}$$

$$-2\Big[B_3(x,a+2,\rho)\Big]_{a_1}^{r_1-1} . \tag{28}$$

The moments may easily be found by means of a table of the binomial distribution since $B_2$ and $B_3$ are just repeated sums of B. Usually $(a_1, r_1, a)$ are small numbers. For large values of a the approximation by means of $m_k(z)$ given in Lemma 2 in the Appendix may be used.

The moments of the OC for double sampling in the binomial case may be found by the same technique using the fact that the last two terms of the OC may be written as sums of the product of a hypergeometric and a binomial probability, compare (25). The summations necessary to find the moments are not simple, however, unless the hypergeometric probability is approximated by a binomial.

## 10. Equivalent single sampling tests and efficiency.

Let $\lambda_\beta$ denote the solution of the equation $P(\lambda) = \beta$, $0 < \beta < 1$. Introducing the abbreviations $b = (a_1, r_1, a, \rho)$ and $H(a_1, r_1, a, \rho, v) = H(b, v)$ we define $v_\beta = v_\beta(b)$ as the solution of the equation $H(b, v) = \beta$.

For the single sampling test $(a_0, m_0)$ the OC is $P_1(\lambda) = G(a_0, m_0\lambda)$ and for the double sampling test $(b, m_1)$ the OC is $P_2(\lambda) = H(b, m_1\lambda)$.

Since $P_1(\lambda) = Pr\{\chi^2(2a_0+2) > 2m_0\lambda\}$ the single sampling test of strength $(\lambda_1, \alpha, \lambda_2, \beta)$ may be found by solving the equations $2m_0\lambda_1 = \chi^2_\alpha(2a_0+2)$ and $2m_0\lambda_2 = \chi^2_{1-\beta}(2a_0+2)$. Allowing non-integral values of $a_0$ we find $(a_0, m_0)$ from

$$\chi^2_{1-\beta}(2a_0+2)/\chi^2_\alpha(2a_0+2) = \lambda_2/\lambda_1$$

and $m_0 = \chi^2_\alpha(2a_0+2)/(2\lambda_1)$.

Hamaker (1950) has shown that the test of strength $(\lambda_0, s_0)$ may be found with good approximation from

$$a_0 = 2\pi(\lambda_0 s_0)^2 - 0.73 \text{ and } m_0 = (a_0 + \tfrac{2}{3})/\lambda_0.$$

Finally we have

$$a_0 + 1 = E^2\{\lambda\}/V\{\lambda\} \text{ and } m_0 = (a_0+1)/E\{\lambda\}.$$

The corresponding formulas for a double sampling test of strength $(\lambda_1, \alpha, \lambda_2, \beta)$ are $v_\beta(b)/v_{1-\alpha}(b) = \lambda_2/\lambda_1$ and $m_1 = v_{1-\alpha}(b)/\lambda_1$.

For a given double sampling test we therefore find the fractile equivalent single sampling test by solving the equation

$$\chi^2_{1-\beta}(2a_0+2)/\chi^2_\alpha(2a_0+2) = v_\beta(b)/v_{1-\alpha}(b)$$

with respect to $a_0$ and computing $m_0 = m_1\chi^2_\alpha(2a_0+2)/\{2v_{1-\alpha}(b)\}$.

Note that $a_0$ and $m_1/m_0$ are functions of b only.

Hamaker and van Strik (1955) have shown how to find $(\lambda_0, s_0)$ for a given double sampling test.

For the moment equivalent test we have that $E\{m_1\lambda\} = (a_0+1)m_1/m_0$ and $V\{m_1\lambda\} = (a_0+1)(m_1/m_0)^2$ such that

$$a_0 + 1 = E^2\{m_1\lambda\}/V\{m_1\lambda\} \text{ and } m_0 = m_1 E\{m_1\lambda\}/V\{m_1\lambda\}.$$

Since $(E\{m_1\lambda\}, V\{m_1\lambda\})$ according to Theorem 6 are functions of b only we have that the moment equivalent $a_0$ and $m_1/m_0$ are fully determined by b.

We have thus shown how to fit a gamma distribution to the given double sampling OC distribution using three different methods of fitting. Computationally, fitting by moments is much simpler than the other methods.

Expressed in terms of fractiles we have

$$v_P(b) \simeq \chi^2_Q(2a_0+2)(m_1/2m_0).$$

Naturally, better approximations may be obtained by using more fractiles or more moments for the fitting.

For small values of a fitting by moments tends to give smaller values of $a_0$ and larger values of $m_1/m_0$ than fitting by fractiles. However, the product $m_1\bar{\lambda} = (a_0+1)m_1/m_0$ is nearly the same. This has been illustrated in Table 12.

From the average sample size

$$ASN = \bar{m}(\lambda) = m_1 + m_2\Big[G(x, m_1\lambda)\Big]_{a_1}^{r_1-1} \tag{29}$$

we get the inverse efficiency of double sampling relative to the equivalent single sampling test

Table 12. Comparison of fractile and moment equivalent single sampling tests for double sampling tests with $\rho = 0.419$.

| | | | Fractile equiv. | | | Moment equiv. | | |
|---|---|---|---|---|---|---|---|---|
| $a_1$ | $r_1$ | $a$ | $a_0$ | $m_1/m_0$ | $m_1\bar{\lambda}$ | $a_0$ | $m_1/m_0$ | $m_1\bar{\lambda}$ |
| 0 | 2 | 1 | 0.80 | .689 | 1.24 | 0.52 | .776 | 1.18 |
| 0 | 2 | 2 | 1.30 | .609 | 1.40 | 1.11 | .655 | 1.38 |
| 0 | 3 | 2 | 1.75 | .545 | 1.50 | 1.40 | .606 | 1.45 |
| 0 | 3 | 3 | 2.60 | .495 | 1.78 | 2.35 | .525 | 1.76 |
| 0 | 3 | 4 | 3.01 | .505 | 2.03 | 3.05 | .505 | 2.05 |
| 0 | 4 | 4 | 3.77 | .453 | 2.16 | 3.58 | .469 | 2.15 |
| 1 | 4 | 5 | 3.97 | .553 | 2.75 | 3.58 | .590 | 2.70 |
| 1 | 5 | 5 | 4.37 | .525 | 2.82 | 3.85 | .567 | 2.75 |
| 1 | 5 | 6 | 5.34 | .489 | 3.10 | 4.96 | .514 | 3.06 |
| 1 | 5 | 7 | 6.02 | .481 | 3.38 | 5.88 | .491 | 3.38 |
| 2 | 6 | 8 | 6.76 | .523 | 4.05 | 6.22 | .553 | 3.99 |
| 2 | 6 | 9 | 7.46 | .508 | 4.30 | 7.18 | .523 | 4.28 |
| 2 | 7 | 9 | 8.14 | .480 | 4.39 | 7.66 | .502 | 4.35 |
| 3 | 7 | 11 | 8.81 | .534 | 5.24 | 8.39 | .553 | 5.19 |
| 3 | 7 | 12 | 9.19 | .534 | 5.44 | 9.13 | .539 | 5.45 |

$$IE(\lambda) = \frac{\overline{m}(\lambda)}{m_0} = \frac{m_1}{m_0}\left\{1 + \frac{1-\rho}{\rho}\left[G(x,m_1\lambda)\right]_{a_1}^{r_1-1}\right\}. \tag{30}$$

From

$$\overline{m}'(\lambda) = - m_1 m_2 [g(x,m_1\lambda)]_{a_1}^{r_1-1}$$

it follows that max $\overline{m}(\lambda) = \overline{m}(\lambda^*)$, where

$$(m_1\lambda^*)^{r_1-a_1-1} = (r_1-1)^{(r_1-a_1-1)} . \tag{31}$$

Hence, there is one and only one maximum and $\overline{m}(\lambda)$ is increasing for $0 < \lambda < \lambda^*$ and decreasing for $\lambda > \lambda^*$. A rather good approximation to $m_1\lambda^*$ is $(a_1+r_1)/2$. For $\lambda \to 0$ and $\lambda \to \infty$ we have $\overline{m}(\lambda) \to m_1$.

## 11. A method for tabulating optimum tests.

As an example consider moment equivalent tests minimizing max $\bar{m}(\lambda)$. As shown in Section 10 the problem is to minimize $\bar{m}(\lambda^*)/m_0 = F(a_1,r_1,\rho,m_1/m_0)$ under the two constraints $m_1/m_0 = f(b)$ and $a_0 = g(b)$, say, where $b = (a_1,r_1,a,\rho)$. Eliminating $m_1/m_0$ we get $\bar{m}(\lambda^*)/m_0 = F^*(b)$, say, which has to be minimized for $a_0 = g(b)$, that is, the minimum and the optimum value of b become functions of $a_0$. The solution is therefore somewhat more complicated than for the normal distribution where we had to minimize a function $F^*(y_a,y_r,\rho)$ without any constraints on the variables because the equation corresponding to $a_0 = g(b)$ contained the parameter h, which did not enter into $F^*$.

To find the minimum we could in principle use the same technique as in Part 1. The derivatives of F are, however, so complicated that we have not succeeded in obtaining a workable solution. Furthermore, we are only interested in integer values of $(a_1,r_1,a)$. It seems therefore natural to work with differences with respect to $a_1, r_1$ and <u>a</u> and derivatives with respect to $\rho$ and $m_1/m_0$, but even if the differences are rather simple the constraints complicate the problem so much that neither by this method have we been able to find a solution. We have, however, worked out a method for tabulating the optimum tests.

Consider first the problem for a <u>given value of</u> $\rho$. All double sampling tests may then be ordered according to increasing values of <u>a</u> and for each <u>a</u> ordered according to increasing values of $a_1$ and $r_1$. For each <u>b</u> we then find the equivalent $(a_0,m_1/m_0)$ and the IE $= \bar{m}(\lambda^*)/m_0$. Consider now the point set $(a_0, \text{IE})$. Extending this point set by randomization it will be seen that the optimum tests correspond to the lower boundary of the convex set. To find the non-randomized optimum tests we rearrange the tests in the table according to increasing values of $a_0$ for all $a_0 \leq a_0^*$, say. Starting from the test with the smallest $a_0$ we compute the slope of the IE with respect to $a_0$ for all the following tests and select the test with the smallest slope. The procedure is then repeated starting from the test selected. In this manner we get the tests on the convex lower boundary of the tabulated values of $(a_0, IE)$. Disregarding the end effect from using a finite $a_0^*$ it turns out that the IE for the optimum tests is a decreasing function of $a_0$ which was to be expected since the IE for $a_0 \to \infty$ must tend to the value for the normal distribution.

Repeating the whole procedure for various values of $\rho$ and comparing the boundaries we may determine the set of optimum tests, at least approximately.

Investigating several cases for $a_0 < 20$, as well for moment and fractile equivalence as for minimax and Bayes tests, it turns out that this procedure leads to rather few non-randomized optimum tests, typically 5-10. From a practical point of view it is, however, desirable to have a collection of optimum and nearly optimum double sampling tests such that the difference between successive values of $a_0$ is at most 1 and at least 0.2, say. Such a collection has been obtained from the optimum tests in the following way: If the difference between two successive values of $a_0$ is greater than 1 we consider all tests having values of $a_0$ between the lower endpoint of the interval plus 0.2 and the upper endpoint minus 0.2. Among these tests we select the one with the lowest IE. The procedure is then applied to the two sub-intervals and so on until it terminates. (For practical reasons the rule employed is slightly different and it may occasionally lead to tests with a difference between successive $a_0$'s less than 0.2.) An example has been shown in Fig. 5. Obviously, the numbers 1 and 0.2 are arbitrary and they may be changed such that the number of tests selected is increased or decreased. The rule used here leads usually to a number of double sampling tests about 30 for $a_0 < 20$. In the following we shall call such a collection of tests optimum (Bayes or minimax) even if it contains some tests which are only "nearly optimum". Further comments will be given in Section 14.

## 12. Double sampling tests minimizing max $\bar{m}(\lambda)$.

### Moment equivalent test.

The method of tabulation described above depends on $\rho$ being fixed. For moment equivalent tests we are, however, able to find the optimum value of $\rho$ as function of $(a_1, r_1, a)$ which leads to a more direct method of solution.

Consider the IE as function of $\rho$ for given $(a_1, r_1, a)$, that is,

$$IE = \frac{m_1}{m_0}\left\{1 + \frac{1-\rho}{\rho}G\right\} \text{ for } G = \left[G(x, m_1\lambda^*)\right]_{a_1}^{r_1-1}.$$

From $D_\rho IE = 0$ we get $G/\{\rho^2 + \rho(1-\rho)G\} = D_\rho \ln(m_1/m_0)$.

Hence, the value of $\rho$ minimizing IE may be found as a root of this equation.

Setting $E = E\{m_1\lambda\}$ and $V = V\{m_1\lambda\}$ we find from (27) and (28) that

$$D_\rho \ln(m_1/m_0) = D_\rho \ln V - D_\rho \ln E$$

$$= (a+1)\left[B(x,a,\rho)\right]_{a_1}^{r_1-1}\{2V^{-1}(\rho(a+2)-E) - E^{-1}\},$$

where we have used the fact that

$$D_\rho B_k(x,a,\rho) = -aB_{k-1}(x,a-1,\rho).$$

For each $(a_1, r_1, a)$ we may thus find the optimum value of $\rho$. We then order these "suboptimum" tests according to $a_0$ and determine the lower boundary as described in Section 11. Fig. 5 shows the IE's and the boundary for $a_0 \leq 5$.

Table 13 gives the tests on the boundary for $a_0 < 20$. It will be seen that apart from the first test in the table ($a_0$ = 0.944) the values of $\rho$ and $m/m_0$ do not deviate much from the asymptotic values, see Theorem 3. Also the IE converges rapidly to 0.868.

Fig. 5. Moment equivalent, minimax ASN, double sampling tests. Each point represent the equivalent $a_0$ and the $\text{minimum}_{\rho}\text{max}_{\lambda}$ $\overline{m}(\lambda)/m_0$ = IE corresponding to a value of $(a_1, r_1, a)$.

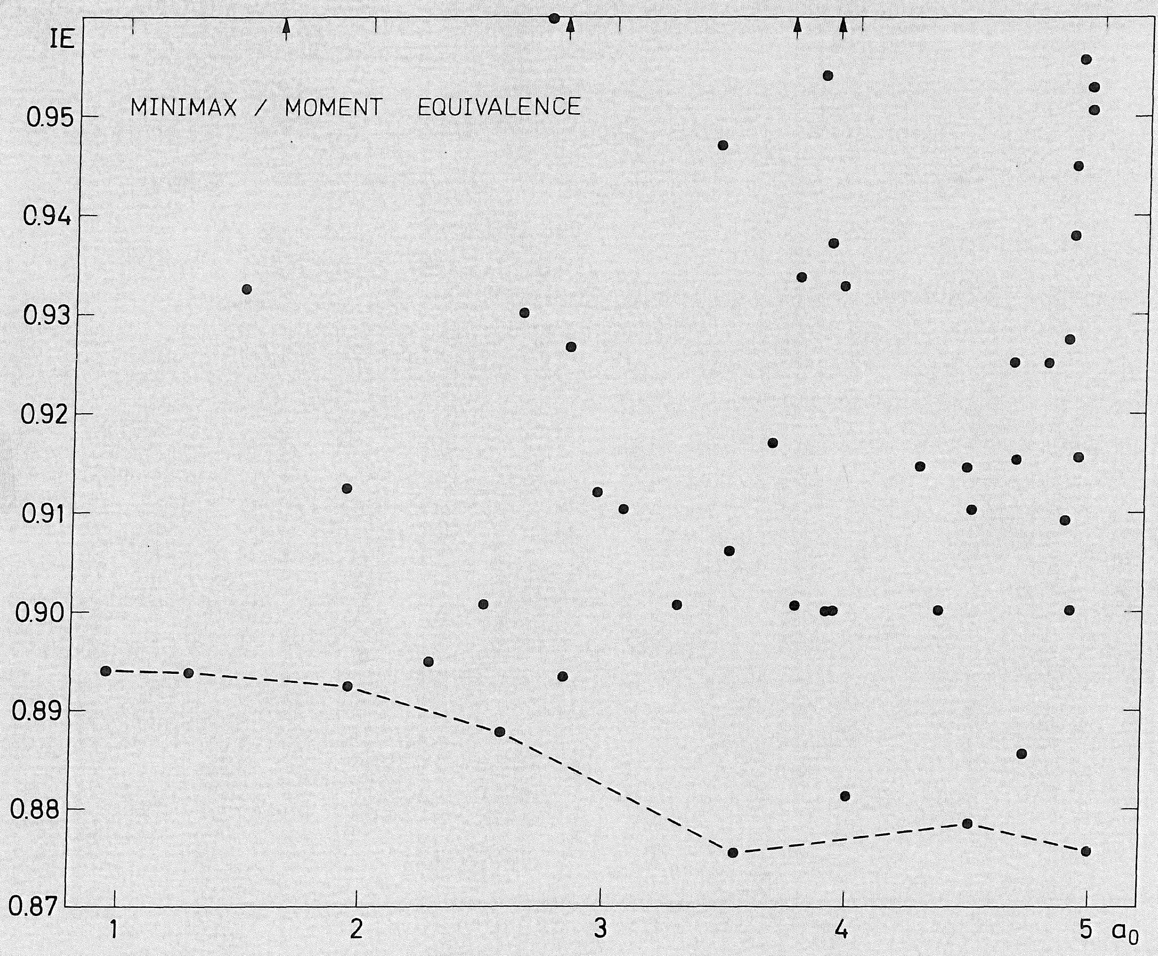


Table 13. Moment equivalent, minimax ASN, double sampling tests.

| $a_1$ | $r_1$ | $a$ | $\rho$ | $a_0$ | $m_1/m_0$ | $m/m_0$ | IE |
|---|---|---|---|---|---|---|---|
| 0 | 2 | 1 | .723 | .944 | .783 | 1.08 | .894 |
| 0 | 2 | 2 | .575 | 1.29 | .703 | 1.22 | .894 |
| 0 | 3 | 2 | .653 | 1.94 | .680 | 1.04 | .892 |
| 0 | 3 | 3 | .539 | 2.58 | .591 | 1.10 | .888 |
| 1 | 4 | 4 | .618 | 3.54 | .678 | 1.10 | .875 |
| 2 | 5 | 5 | .671 | 4.52 | .733 | 1.09 | .878 |
| 2 | 5 | 6 | .595 | 4.99 | .687 | 1.15 | .876 |
| 3 | 6 | 7 | .639 | 5.97 | .729 | 1.14 | .878 |
| 2 | 6 | 7 | .577 | 6.36 | .625 | 1.08 | .876 |
| 3 | 7 | 8 | .619 | 7.33 | .669 | 1.08 | .874 |
| 3 | 7 | 9 | .566 | 7.89 | .632 | 1.12 | .873 |
| 4 | 8 | 10 | .602 | 8.87 | .669 | 1.11 | .871 |
| 5 | 9 | 11 | .632 | 9.85 | .698 | 1.10 | .872 |
| 5 | 9 | 12 | .589 | 10.3 | .671 | 1.14 | .872 |
| 6 | 10 | 13 | .616 | 11.3 | .697 | 1.13 | .873 |
| 5 | 10 | 13 | .580 | 11.9 | .629 | 1.09 | .873 |
| 6 | 11 | 14 | .606 | 12.9 | .655 | 1.08 | .872 |
| 6 | 11 | 15 | .571 | 13.4 | .631 | 1.10 | .871 |
| 7 | 12 | 16 | .595 | 14.4 | .655 | 1.10 | .870 |
| 8 | 13 | 17 | .616 | 15.4 | .676 | 1.10 | .871 |
| 8 | 13 | 18 | .587 | 15.9 | .656 | 1.12 | .870 |
| 9 | 14 | 19 | .606 | 16.9 | .675 | 1.11 | .870 |
| 9 | 14 | 20 | .579 | 17.4 | .658 | 1.14 | .870 |
| 10 | 15 | 21 | .598 | 18.4 | .676 | 1.13 | .870 |
| 11 | 16 | 22 | .614 | 19.4 | .691 | 1.13 | .871 |
| Asymptote | | | .586 | | .653 | 1.11 | .868 |

Computations with fixed values of $\rho$, $\rho = 0.5$, 0.586, 0.6, 0.7, show that $\rho = 0.586$ and $\rho = 0.6$ lead to tests with practically the same IE as in Table 13 whereas $\rho = 0.5$ and $\rho = 0.7$ lead to considerably poorer tests, that is, tests with considerably higher IE's.

Fractile equivalent tests.

For fractile equivalence the two constraints are so complicated that a workable expression for $D_\rho \ln(m_1/m_0)$ does not exist. We have therefore used the method with fixed values of $\rho$ described in Section 11. For $\alpha = 0.05$ and $\beta = 0.10$ the boundaries corresponding to three values of $\rho$ have been shown in Fig. 6. It is obvious that for $a_0 > 5$ the best solution is obtained for $\rho = 0.5575$, the asymptotic value of $\rho$, see Table 5. Also for $a_0 \leqq 5$ rather good results are obtained for this value of $\rho$. Table 14 contains the corresponding tests for $a_0 < 20$. ($R_2$ is defined as $v_{.10}/v_{.95}$). It is of course possible for small values of $a_0$ to replace some of the tests tabulated by slightly better tests by finding the optimum value of $\rho$ by trial and error but we have not attempted this.

Fig. 6. Comparison of boundaries for fractile equivalent ($\alpha$ = 0.05 and $\beta$ = 0.10), minimax ASN, double sampling tests for fixed values of $\rho$.

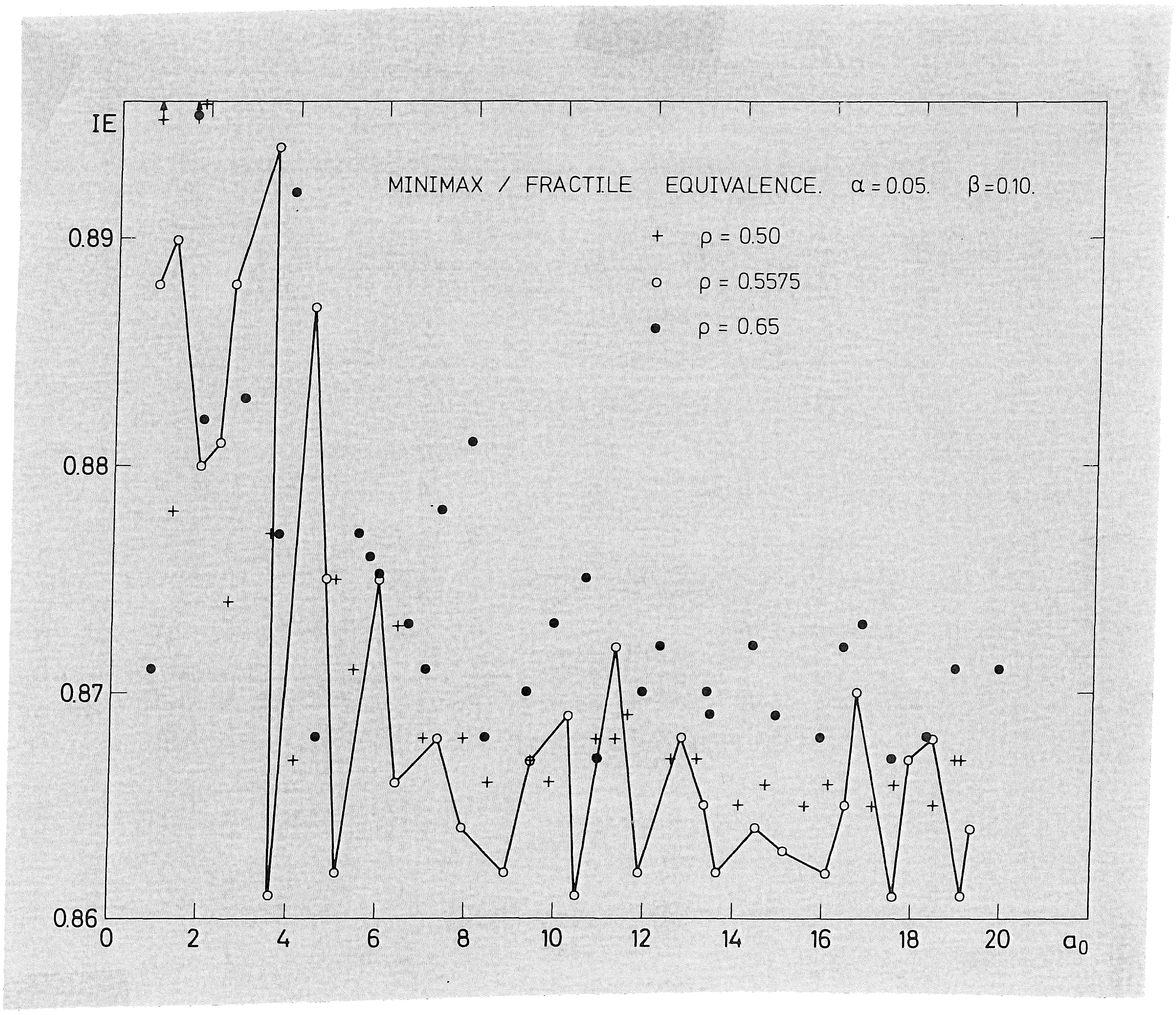


**Table 14.** Fractile equivalent, minimax ASN, double sampling tests. $1 - \alpha = 0.95$, $\beta = 0.10$, $\rho = 0.5575$.

| $a_1$ | $r_1$ | $a$ | $v_{.95}$ | $v_{.10}$ | $R_2$ | $a_0$ | $m_1/m_0$ | $m/m_0$ | IE |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 2 | 1 | .222 | 2.59 | 11.7 | .918 | .687 | 1.23 | .888 |
| 0 | 2 | 2 | .328 | 2.97 | 9.06 | 1.29 | .689 | 1.24 | .890 |
| 0 | 3 | 2 | .471 | 3.14 | 6.67 | 1.94 | .600 | 1.08 | .880 |
| 1 | 3 | 3 | .715 | 4.18 | 5.84 | 2.32 | .725 | 1.30 | .881 |
| 0 | 3 | 3 | .681 | 3.70 | 5.43 | 2.57 | .606 | 1.09 | .888 |
| 2 | 4 | 5 | 1.29 | 5.62 | 4.37 | 3.55 | .759 | 1.36 | .894 |
| 1 | 4 | 4 | 1.08 | 4.68 | 4.34 | 3.59 | .627 | 1.12 | .861 |
| 2 | 5 | 5 | 1.51 | 5.74 | 3.81 | 4.43 | .671 | 1.20 | .887 |
| 1 | 5 | 5 | 1.45 | 5.27 | 3.65 | 4.77 | .587 | 1.05 | .875 |
| 2 | 5 | 6 | 1.73 | 6.10 | 3.52 | 5.06 | .652 | 1.17 | .862 |
| 3 | 6 | 7 | 2.22 | 7.14 | 3.21 | 5.97 | .680 | 1.22 | .875 |
| 2 | 6 | 7 | 2.16 | 6.67 | 3.09 | 6.42 | .603 | 1.08 | .866 |
| 3 | 7 | 8 | 2.63 | 7.58 | 2.89 | 7.34 | .622 | 1.12 | .868 |
| 3 | 7 | 9 | 2.89 | 8.02 | 2.77 | 7.96 | .620 | 1.11 | .864 |
| 4 | 8 | 10 | 3.40 | 8.93 | 2.63 | 8.94 | .632 | 1.13 | .862 |
| 4 | 8 | 11 | 3.64 | 9.34 | 2.57 | 9.40 | .636 | 1.14 | .867 |
| 4 | 9 | 11 | 3.84 | 9.45 | 2.46 | 10.3 | .599 | 1.07 | .869 |
| 5 | 9 | 12 | 4.18 | 10.2 | 2.45 | 10.5 | .642 | 1.15 | .861 |
| 6 | 10 | 13 | 4.73 | 11.2 | 2.37 | 11.3 | .661 | 1.19 | .872 |
| 6 | 10 | 14 | 4.97 | 11.5 | 2.32 | 11.9 | .653 | 1.17 | .862 |
| 7 | 11 | 15 | 5.54 | 12.5 | 2.25 | 12.8 | .666 | 1.20 | .868 |
| 7 | 11 | 16 | 5.77 | 12.8 | 2.22 | 13.3 | .664 | 1.19 | .865 |
| 6 | 11 | 15 | 5.47 | 12.1 | 2.20 | 13.6 | .615 | 1.10 | .862 |
| 7 | 12 | 16 | 6.01 | 12.9 | 2.15 | 14.5 | .625 | 1.12 | .864 |
| 7 | 12 | 17 | 6.30 | 13.3 | 2.12 | 15.1 | .623 | 1.12 | .863 |
| 8 | 13 | 18 | 6.86 | 14.2 | 2.07 | 16.1 | .630 | 1.13 | .862 |
| 8 | 13 | 19 | 7.12 | 14.6 | 2.05 | 16.5 | .632 | 1.13 | .865 |
| 7 | 13 | 18 | 6.80 | 13.9 | 2.04 | 16.7 | .598 | 1.07 | .870 |
| 9 | 14 | 20 | 7.71 | 15.5 | 2.01 | 17.6 | .636 | 1.14 | .861 |
| 9 | 14 | 21 | 7.95 | 15.8 | 1.99 | 17.9 | .641 | 1.15 | .867 |
| 10 | 15 | 21 | 8.28 | 16.4 | 1.97 | 18.4 | .648 | 1.16 | .868 |
| 10 | 15 | 22 | 8.56 | 16.7 | 1.95 | 19.1 | .643 | 1.15 | .861 |
| 9 | 15 | 21 | 8.21 | 16.0 | 1.94 | 19.3 | .608 | 1.09 | .864 |
| | | | Asymptote | | | | .631 | 1.13 | .862 |

## 13. Optimum double sampling tests of strength $(\lambda_1, \alpha, \lambda_2, \beta)$.

The Bayes ASN test is defined by minimizing

$$\frac{\bar{m}_w}{m_0} = \frac{m_1}{m_0} \left\{ 1 + \frac{1-\rho}{\rho} c \right\}.$$

where

$$c = w\left[G(x, v_{1-\alpha})\right]_{a_1}^{r_1 - 1} + (1-w)\left[G(x, v_\beta)\right]_{a_1}^{r_1 - 1},$$

$0 \leqq w \leqq 1$, under the constraints $H(b, m_1 \lambda_1) = 1 - \alpha$ and $H(b, m_1 \lambda_2) = \beta$. An example is shown in Table 15. The IE for $\lambda = \lambda_1$ and $\lambda = \lambda_2$ is denoted by $IE_1$ and $IE_2$.

Table 15. Fractile equivalent, Bayes ASN, double sampling tests. $1 - \alpha = 0.95$, $\beta = 0.10$, $w = 2/3$, $\rho = 0.425$.

| $a_1$ | $r_1$ | $a$ | $v_{.95}$ | $v_{.10}$ | $R_2$ | $a_0$ | $m_1/m_0$ | $m/m_0$ | $IE_1$ | $IE_2$ | $IE_w$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2 | 1 | .187 | 2.39 | 12.8 | .808 | .664 | 1.56 | .804 | .861 | .823 |
| 0 | 2 | 2 | .297 | 2.60 | 8.76 | 1.34 | .592 | 1.39 | .769 | .746 | .761 |
| 0 | 3 | 2 | .376 | 2.66 | 7.08 | 1.78 | .530 | 1.25 | .750 | .841 | .780 |
| 0 | 3 | 3 | .567 | 3.03 | 5.35 | 2.63 | .489 | 1.15 | .762 | .733 | .753 |
| 1 | 4 | 4 | .895 | 4.11 | 4.60 | 3.29 | .582 | 1.37 | .749 | .840 | .780 |
| 1 | 4 | 5 | 1.08 | 4.36 | 4.02 | 4.06 | .540 | 1.27 | .737 | .757 | .744 |
| 1 | 4 | 6 | 1.23 | 4.65 | 3.79 | 4.47 | .541 | 1.27 | .768 | .733 | .757 |
| 1 | 5 | 6 | 1.40 | 4.74 | 3.39 | 5.42 | .483 | 1.14 | .741 | .770 | .750 |
| 2 | 5 | 8 | 1.80 | 5.88 | 3.26 | 5.80 | .572 | 1.35 | .753 | .753 | .753 |
| 3 | 6 | 9 | 2.28 | 6.93 | 3.04 | 6.63 | .613 | 1.44 | .751 | .798 | .767 |
| 2 | 6 | 8 | 2.00 | 5.96 | 2.98 | 6.90 | .512 | 1.20 | .724 | .780 | .743 |
| 2 | 6 | 9 | 2.21 | 6.28 | 2.84 | 7.59 | .502 | 1.18 | .744 | .741 | .743 |
| 3 | 7 | 10 | 2.64 | 7.18 | 2.72 | 8.30 | .538 | 1.27 | .723 | .793 | .746 |
| 3 | 7 | 11 | 2.85 | 7.45 | 2.62 | 9.01 | .524 | 1.23 | .731 | .753 | .738 |
| 3 | 8 | 11 | 2.98 | 7.50 | 2.52 | 9.80 | .495 | 1.16 | .719 | .806 | .748 |
| 4 | 8 | 13 | 3.50 | 8.62 | 2.46 | 10.4 | .544 | 1.28 | .727 | .766 | .740 |
| 3 | 8 | 12 | 3.23 | 7.83 | 2.43 | 10.7 | .482 | 1.14 | .735 | .763 | .744 |
| 5 | 9 | 15 | 4.18 | 9.79 | 2.34 | 11.7 | .563 | 1.33 | .728 | .778 | .745 |
| 4 | 9 | 14 | 3.90 | 8.96 | 2.30 | 12.2 | .498 | 1.17 | .722 | .771 | .738 |
| 4 | 9 | 15 | 4.12 | 9.28 | 2.25 | 12.8 | .495 | 1.17 | .743 | .745 | .744 |
| 5 | 10 | 16 | 4.58 | 10.1 | 2.20 | 13.6 | .514 | 1.21 | .716 | .780 | .738 |
| 5 | 10 | 17 | 4.81 | 10.4 | 2.16 | 14.3 | .507 | 1.19 | .730 | .753 | .738 |
| 6 | 11 | 18 | 5.28 | 11.2 | 2.12 | 14.9 | .529 | 1.24 | .715 | .789 | .740 |
| 6 | 11 | 19 | 5.50 | 11.5 | 2.09 | 15.7 | .520 | 1.22 | .723 | .761 | .736 |
| 6 | 11 | 20 | 5.70 | 11.8 | 2.07 | 16.1 | .521 | 1.23 | .742 | .746 | .744 |
| 6 | 12 | 19 | 5.66 | 11.5 | 2.04 | 16.7 | .495 | 1.16 | .713 | .800 | .742 |
| 7 | 12 | 21 | 6.21 | 12.6 | 2.03 | 17.1 | .532 | 1.25 | .720 | .769 | .736 |
| 6 | 12 | 20 | 5.92 | 11.9 | 2.00 | 17.7 | .486 | 1.14 | .725 | .769 | .739 |
| 8 | 13 | 23 | 6.94 | 13.7 | 1.98 | 18.4 | .545 | 1.28 | .719 | .778 | .739 |
| 7 | 13 | 22 | 6.63 | 12.9 | 1.95 | 19.1 | .497 | 1.17 | .717 | .775 | .736 |
| 7 | 13 | 23 | 6.87 | 13.2 | 1.93 | 19.8 | .494 | 1.16 | .733 | .753 | .740 |
| Asymptote | | | | | | | .507 | 1.19 | .697 | .789 | .728 |

## 14. Applications of the tables.

Consider first a single sampling test of strength $(\lambda_1, \alpha, \lambda_2, \beta)$. In practice $a_0$ has to be an integer and we therefore have to reformulate the problem in the following way: Find $(a_0, m_0)$, $a_0$ being an integer as small as possible, such that $P_1(\lambda_1) \geqq 1 - \alpha$ and $P_1(\lambda_2) \leqq \beta$.

It is well-known, see for example Hald (1967b), that $a_0$ is determined from the inequality

$$R_1(a_0-1) > \lambda_2/\lambda_1 \geqq R_1(a_0),$$

where

$$R_1(c) = \chi^2_{1-\beta}(2c+2)/\chi^2_{\alpha}(2c+2), \quad c = 0,1,\ldots,$$

and that

$$\chi^2_{1-\beta}(2a_0+2)/2\lambda_2 \leqq m_0 \leqq \chi^2_{\alpha}(2a_0+2)/2\lambda_1.$$

For $m_0$ equal to the lower limit we get $P_1(\lambda_2) = \beta$ and $P_1(\lambda_1) > 1 - \alpha$ unless $\lambda_2/\lambda_1 = R_1(a_0)$ in which case $m_0$ is uniquely determined and we have equality in both places.

If the strength is specified by $(E\{\lambda\}, V\{\lambda\})$ we find $a_0 = [E^2/V]$. Keeping the location fixed we get $m_0 = (a_0+1)/E\{\lambda\}$ which means that $(a_0+1)/m_0^2 \leqq V\{\lambda\}$ so that the OC is "on the average" steeper than specified.

For double sampling we require that $a_1$, $r_1$ and $a$ are integers and that the test is optimum ASN. This leads to

$$P_2(\lambda_1) = H(b, m_1\lambda_1) \geqq 1 - \alpha \text{ and } P_2(\lambda_2) = H(b, m_1\lambda_2) \leqq \beta.$$

Introducing the decreasing function

$$R_2(b) = v_\beta(b)/v_{1-\alpha}(b) \text{ for } b = b_1, b_2, \ldots ,$$

where $b_1, b_2, \ldots$ represent the values of b for the optimum tests in question, the optimum value of b, $b = b_i$ say, is determined from the inequality

$$R_2(b_{i-1}) > \lambda_2/\lambda_1 \geqq R_2(b_i)$$

and

$$v_\beta(b_i)/\lambda_2 \leqq m_1 \leqq v_{1-\alpha}(b_i)/\lambda_1.$$

One of the reasons for including some non-optimum tests in the tables is the feeling that we should be able to meet the strength specifications as least as well by double as by single sampling. This requires, however, that the successive differences of $R_2$ are smaller than for $R_1$ (or that the differences between the successive values of the equivalent $a_0'$ s are smaller than 1). One may say that our solution is a compromise between strength and optimality requirements.

Denoting the moment equivalent $a_0$ by $a_0(b)$ we determine the optimum value of b from the inequality $a_0(b_{i-1}) < (E^2/V)-1 \leqq a_0(b_i)$. Corresponding to $b_i$ we find $m_1/m_0$ in the table and $m_0 = (a_0(b_i)+1)/E\{\lambda\}$.

Example 5. Testing $\lambda_1 = 1$ against $\lambda_2 = 6$ with $\alpha = 0.05$ and $\beta = 0.10$ leads to $a_0 = 2.24$, $m_0 = 0.942$ and $E\{\lambda\} = 3.44$, $V\{\lambda\} = 3.65$. In practice we use $a_0 = 3$ and $1.11 \leqq m_0 \leqq 1.37$.

The moment equivalent, minimax ASN, double sampling test is found from Table 13. As $(E^2/V) - 1 = 2.24$ we get $a_0(b_i) = 2.58$, such that $b_i = (0,3,3,0.539)$, $m_0 = 1.04$, $m_1 = 0.615$, $m = 1.14$ and max $\overline{m}(\lambda) = 0.924$.

The fractile equivalent, minimax ASN test is found from Table 14. As $\lambda_2/\lambda_1 = 6$ we get $R_2(b_i) = 5.84$, such that $b_i = (1,3,3,0.5575)$ and $0.697 \leqq m_1 \leqq 0.715$. Choosing $m_1 = 0.697$, say, we get $m = 1.25$ and max $\overline{m}(\lambda) = 0.847$.

The fractile equivalent, Bayes ASN test with $w = 2/3$ is found from Table 15. As $\lambda_2/\lambda_1 = 6$ we get $R_2(b_i) = 5.35$, such that $b_i = (0,3,3,0.425)$ and $0.505 \leqq m_1 \leqq 0.567$. Choosing $m_1 = 0.505$ we get $m = 1.19$.

A more detailed comparison of the OC and ASN functions has been given in Table 16. It will be seen that the single sampling test with $a_0 = 3$ is somewhat stronger than required and that the same is true for the double sampling tests, although to a lesser extent in the present case. Comparison of the ASN's is hampered because of the differences between the OC's but it is clear that a considerable saving may be obtained as compared to single sampling.

Table 16. Comparison of OC and ASN functions for testing $\lambda_1 = 1$ against $\lambda_2 = 6$.

| $a_0$ or $b$ | 2.24 | 3 | 0,3,3,0.539 | | 1,3,3,0.5575 | | 0,3,3,0.425 | |
|---|---|---|---|---|---|---|---|---|
| $m_0$ or $m_1$ | 0.942 | 1.11 | 0.615 | | 0.697 | | 0.505 | |
| $\lambda$ | $P_1(\lambda)$ | $P_1(\lambda)$ | $P_2(\lambda)$ | $\overline{m}(\lambda)$ | $P_2(\lambda)$ | $\overline{m}(\lambda)$ | $P_2(\lambda)$ | $\overline{m}(\lambda)$ |
| 0 | 1.000 | 1.000 | 1.000 | .615 | 1.000 | .697 | 1.000 | .505 |
| 1/2 | .993 | .997 | .995 | .752 | .993 | .721 | .996 | .656 |
| 1 | .950 | .973 | .960 | .844 | .953 | .764 | .964 | .766 |
| 3/2 | .868 | .912 | .885 | .897 | .872 | .803 | .892 | .840 |
| 2 | .759 | .815 | .778 | .920 | .762 | .830 | .786 | .883 |
| 3 | .521 | .574 | .531 | .910 | .519 | .846 | .540 | .905 |
| 4 | .322 | .352 | .320 | .862 | .317 | .829 | .328 | .873 |
| 6 | .100 | .101 | .091 | .753 | .100 | .771 | .100 | .757 |
| 8 | .026 | .023 | .022 | .680 | .029 | .730 | .028 | .652 |

## 15. Approximations

By means of the results in Part 1 we may construct large sample optimum tests in the usual manner. The problem is, however, to find suitable "continuity corrections" which will make the large sample results useful also for smaller samples.

For any double sampling test we define

$$m_1\bar{\lambda} = (a_0+1)m_1/m_0 \tag{32}$$

which is a function of b for any equivalence definition considered. Consider now the following formulas from which $a_1$, $r_1$ and a are found by rounding to the nearest integer:

$$\left.\begin{aligned} a_1 + 1 &= m_1\bar{\lambda} - y_a\sqrt{m_1\bar{\lambda}} \\ r_1 &= m_1\bar{\lambda} + y_r\sqrt{m_1\bar{\lambda}} \\ a + 1 &= m_1\bar{\lambda}/\rho . \end{aligned}\right\} \tag{33}$$

Tests found from these formulas by using the values of $a_0$, $m_1/m_0$ and $\rho$ for optimum tests, as for example in Tables 13, 14 and 15, are either optimum or very nearly so. It is a remarkable fact that this statement holds not only asymptotically but for all values of $a_0$ and not only in the middle part of the OC distribution but for all values of $\alpha$ (and $\beta$) $\geq 0.001$. It seems therefore natural to use (33) as starting point for the construction of approximations.

In practice we have to take not only $(y_a, y_r, \rho)$ but also $m_1/m_0$ from the theory in Part 1. For minimax ASN tests it turns out that (33) leads to optimum tests or very nearly so also when the asymptotic value of $m_1/m_0$ is used. For Bayes ASN tests, however, (33) is not quite satisfactory. We have therefore investigated the effect of using other continuity corrections and have found that the corrections depend on $\alpha$ and $\beta$. For $\alpha = 0.05$, $\beta = 0.10$ and $w = 2/3$ it turns out that $a_1 + 1$ should be replaced by $a_1 + 0.75$. It is easy to check by means of Table 15 that (33) with this modification gives excellent results. In a forthcoming paper more detailed information on continuity corrections will be given.

In practice (33), or (33) suitably modified, is used in the following way: From the strength requirements we find the exact single sampling test $(a_0, m_0)$, that is, $a_0$ should be determined to one or two decimal places. From the tables in Part 1 we find $(y_a, y_r, \rho, m_1/m_0)$. By means of (32) and (33) we get $(a_1, r_1, a)$. We then have to check that the test found satisfies the strength requirements and that the IE is reasonably near the asymptotic IE.

From $b = (a_1, r_1, a, \rho)$ we compute $a_0(b)$ and if $a_0(b) \geqq a_0$ the test is at least as strong as required and we may then determine the corresponding $m_1$. If $a_0(b) < a_0$ we have to increase the strength by changing $(a_1, r_1, a)$. For $a_0 < 20$ it is normally sufficient to change one of the components by 1.

In the binomial case we have analogously

$$a_1 + 1 = n_1\bar{p} - y_a\sqrt{n_1\overline{pq}}$$

$$r_1 = n_1\bar{p} + y_r\sqrt{n_1\overline{pq}}$$

$$a + 1 = n_1\bar{p}/\rho,$$

where
$$\bar{p} = (a_0+1)/(n_0+1).$$

Example 6. Consider the problem in Example 5. From Table 4 we get for moment equivalence $y_a = y_r = 0.622$, $\rho = 0.586$ and $m_1/m_0 = 0.653$ such that (33) gives $(a_1, r_1, a) = (0,3,3)$. Hence, $a_0(b) = 2.57$ which shows that the test is stronger than required. From the corresponding $m_1/m_0 = 0.632$ and $m_0 = 3.57/E\{\lambda\} = 1.04$ we get $m_1 = 0.656$. The approximation thus leads to a test which is nearly optimum.

For fractile equivalence we get from Table 5 $y_a = 0.566$, $y_r = 0.663$, $\rho = 0.5575$ and $m_1/m_0 = 0.631$ such that (33) gives $(a_1, r_1, a) = (0,3,3)$. Table 14 shows that $a_0(b) = 2.57$ and $0.617 \leqq m_1 \leqq 0.681$. The approximation has led us to a test which is slightly stronger than necessary since $a_0(b) = 2.32$

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## Appendix.

*Lemma 1.* Let $\sigma = (1+\beta^2)^{-1/2}$. Then

$$\phi(u)\phi(\alpha+\beta u) = \phi(\alpha\sigma)\phi(\alpha\beta\sigma+u\sigma^{-1}),$$

$$\int_v^\infty \phi(u)\phi(\alpha+\beta u)\,du = \sigma\phi(\alpha\sigma)\Phi(-\alpha\beta\sigma-v\sigma^{-1}),$$

and

$$\int_v^\infty \phi(u)\phi(\alpha+\beta u)(\alpha+\beta u)\,du$$

$$= \sigma^2\phi(\alpha\sigma)\{\beta\phi(\alpha\beta\sigma+v\sigma^{-1}) + \alpha\sigma\Phi(-\alpha\beta\sigma-v\sigma^{-1})\}.$$

*Proof.* The proof of the first two propositions is straightforward. The last formula may be found by differentiating the last but one with respect to $\alpha$.

*Lemma 2.* For $0 < \rho < 1$ and $a \geqq x_1$ we have

$$\rho^{k+1} \sum_{x=a}^{\infty} (x+k)^{(k)} b(x_1,x,\rho)$$

$$= \sum_{\mu=0}^{k} \rho^{k-\mu} k^{(\mu)} (a+k)^{(k-\mu)} B_{\mu+1}(x_1,a+\mu,\rho)$$

and

$$\rho^{k+1} \sum_{x=x_1}^{\infty} (x+k)^{(k)} b(x_1,x,\rho) = (x_1+k)^{(k)}.$$

*Proof.* To prove the last proposition we write

$$\rho^{k+1} \sum_{x=x_1}^{\infty} (x+k)^{(k)} b(x_1,x,\rho) =$$

$$= \rho^{x_1+k+1} \sum_{x=x_1}^{\infty} (x+k)^{(x_1+k)} (1-\rho)^{x-x_1}/x_1!$$

$$= (x_1+k)^{(k)} \rho^{x_1+k+1} \sum_{x=0}^{\infty} \binom{x+x_1+k}{x} (1-\rho)^x = (x_1+k)^{(k)}.$$

Setting $\sigma = \sqrt{x\rho(1-\rho)}$ it is easy to see that $B_{k+1}(x_1,x,\rho) \simeq m_k[\{\rho x - x_1 - \frac{1}{2}(k+1)\}\sigma^{-1}]\sigma^k/k!$.

Since $m_k(z) = O(e^{-z^2} z^{-k-1})$ for $z \to \infty$ it follows that $\lim\{x^\nu B_\mu(x_1,x,\rho)\} = 0$ for $x \to \infty$, $x_1$ and $\rho$ fixed, and $\nu = 0,1,\ldots$, $\mu = 1,2,\ldots$ . Using this result and the fact that $b(x_1,x,\rho) = -\rho^{-1}\Delta_x B_1(x_1,x,\rho)$ we shall prove the first proposition by induction.

$$\rho^{k+1} \sum_{x=a}^{\infty} (x+k)^{(k)} b(x_1,x,\rho) = -\rho^k \sum_{x=a}^{\infty} (x+k)^{(k)} \Delta_x B_1(x_1,x,\rho)$$

$$= -\rho^k \Big[ (x+k)^{(k)} B_1(x_1,x,\rho) \Big]_{x=a}^{\infty} + k\rho^k \sum_{x=a}^{\infty} (x+k)^{(k-1)} B_1(x_1,x+1,\rho).$$

The first term corresponds to the term for $\mu = 0$ in the lemma. The second term may be written as

$$k \sum_{y=0}^{x_1} \rho^k \sum_{x=a}^{\infty} (x+k)^{(k-1)} b(y,x+1,\rho)$$

$$= k \sum_{y=0}^{x_1} \rho^k \sum_{x=a+1}^{\infty} (x+k-1)^{(k-1)} b(y,x,\rho).$$

Using the lemma for k-1 we get

$$k \sum_{y=0}^{x_1} \sum_{\mu=0}^{k-1} \rho^{k-\mu-1} (k-1)^{(\mu)} (a+k)^{(k-\mu-1)} B_{\mu+1}(y,a+\mu+1,\rho)$$

$$= \sum_{\mu=0}^{k-1} \rho^{k-\mu-1} k^{(\mu+1)} (a+k)^{(k-\mu-1)} B_{\mu+2}(x_1,a+\mu+1,\rho)$$

which equals the last k terms in the formula in question.