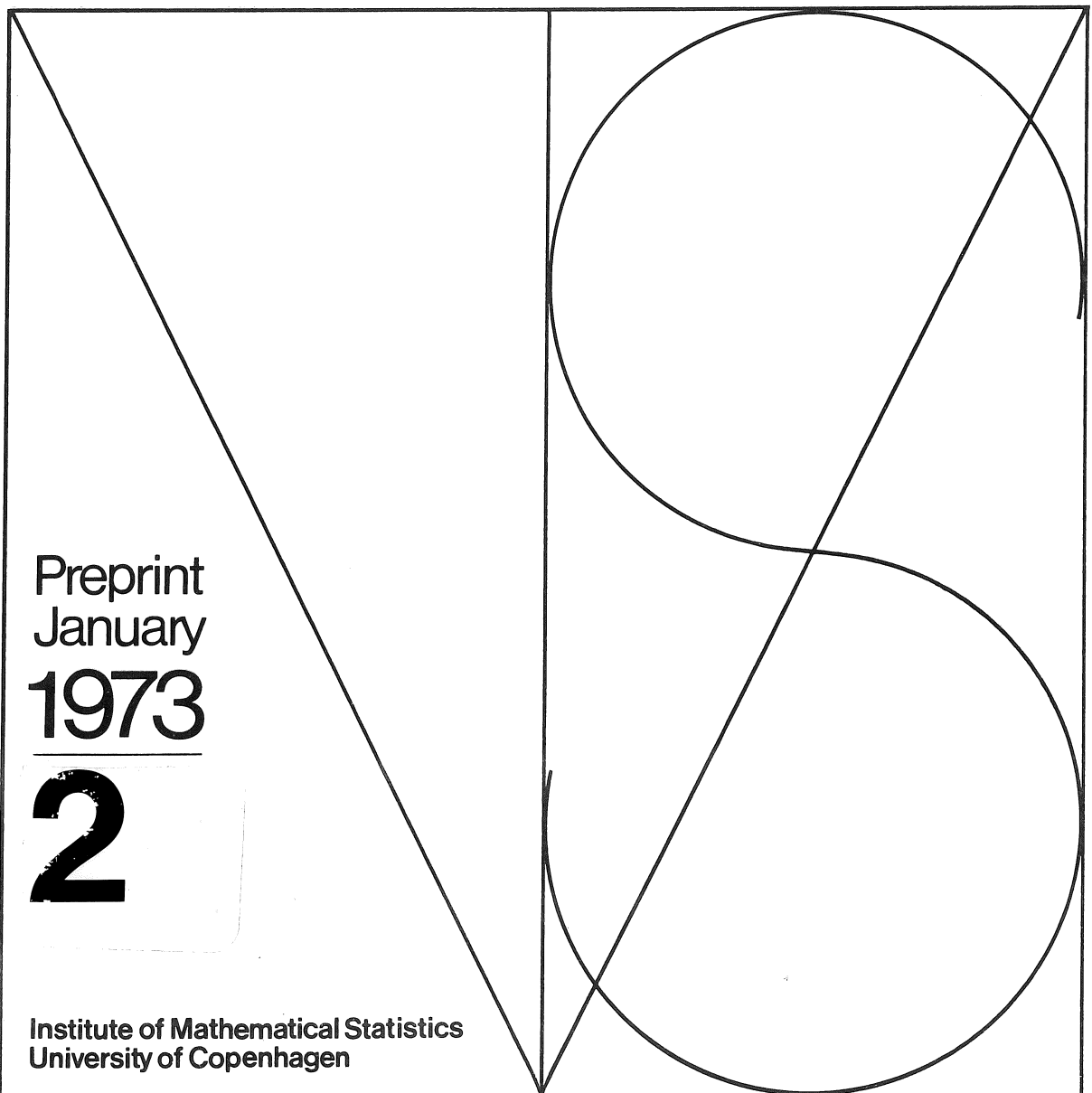


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The Extremal  
Convex Functions



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Preprint 1973 No. 2

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January 1973.

1. Introduction and Summary. We shall consider the convex cone  $\bar{K}$  of finite continuous convex functions defined on a convex set  $K$  in  $\mathbb{R}^2$ . A large class of extremal functions is identified and it is proved that the extremal functions are dense in  $\bar{K}$ .

Thus the results are very different from the results obtained for convex functions on intervals, where all the extremal functions are of the form  $a \vee b$  for some affine functions  $a$  and  $b$ , see Blaschke and Pick [1].

It is easily seen that

$$\bar{K} \cap (-\bar{K}) = \bar{A},$$

where  $\bar{A}$  denote the affine functions. Hence the cone is not pointed and the usual definition of an extremepoint has to be modified as follows:

Definition: Let  $f$ ,  $g$ , and  $h$  be elements of  $\bar{K}$ , then  $f$  is called extremal if for all  $g$  and  $h$  such that

$$f = \frac{1}{2}(g+h)$$

there exist a constant  $\alpha > 0$  and an affine function  $a$ , such that

$$f = \alpha g + a.$$

We shall apply the following concepts from the theory of convex sets and functions, see Rockafellar [2].

A polyhedral set is a closed convex set which is the intersection of a finite number of halfspaces.

A polytope is a compact polyhedral set.

A face of a polyhedral set  $P$  is a subset  $F \subset P$  with the property that

$$x \in F, y \in P, z \in P, x = \frac{1}{2}(y+z) \Rightarrow y \in F, z \in F.$$

The 0 dimensional faces are the extreme points or the vertices. The 1-dimensional faces are the edges. A polyhedral set is bounded by a finite number of edges and has a finite number of vertices. A polytope is the convex hull of its vertices.

The family of polyhedral sets  $P_1, \dots, P_m$  is called a covering of the convex set  $K$  if

$$K \subset P_1 \cup \dots \cup P_m$$

and

$$\text{ri}P_i \cap \text{ri}P_j = \emptyset, \quad i \neq j.$$

Here  $\text{ri}$  denotes the relative interior. The order of a vertex of a polyhedral set in the covering is the number of polyhedral sets which contains it.

Let  $a_1, \dots, a_n$  denote a family of affine functions, then

$$f = \max_{1 \leq i \leq n} a_i$$

is called a polyhedral function and it is seen that it is convex and continuous, and that the sets

$$P_i = \{f = a_i\}, \quad \dim P_i = 2,$$

give rise to a covering of  $\mathbb{R}^2$ , and therefore of  $K$ , by polyhedral sets.

The class of extremal functions on  $\mathbb{R}^2$  which is found here can be described as polyhedral functions which induce a covering where the vertices are of order 3, see Theorem 1.

An alternative way of describing them is as follows: consider the cylinder

$$C = \{(x, \mu)\}: \mu \geq 0, x \in \mathbb{R}^2\}.$$

Any affine function  $a \in A$  can be thought of as cutting away from  $C$  the set

$$\{(x, \mu) : 0 \leq \mu \leq a(x), x \in \mathbb{R}^2\}$$

After having cut  $C$  by means of the affine functions  $a_1, \dots, a_n$  we are left with the epigraph of  $f = 0 \vee a_1 \vee \dots \vee a_n$

$$C_n = \{(x, \mu) : \mu \geq f(x), x \in \mathbb{R}^2\}.$$

If each function  $a_m$ ,  $m = 1, 2, \dots, n$  is chosen such that it does not cut through an extremepoint of  $C_{m-1}$  then  $f$  will be extreme, since then the vertices of the covering induced by  $f$  will have order 3.

The result that the functions thus constructed can approximate any continuous convex function  $f$  uniformly on a compact set is now rather obvious since the epigraph of  $f$  can be cut out of  $C$  by continuing the above procedure, each time avoiding the vertices already created, see Theorem 2.

The actual proofs for convex sets in  $\mathbb{R}^2$  are more complicated since we need extra conditions to ensure that there are enough vertices inside  $K$ , see the lemma.

## 2. A Combinatorial Lemma.

Lemma. Let  $C$  be an open convex set of dimension 2. Let  $P_1, \dots, P_m$  be a covering of  $C$  with convex closed polyhedral sets of dimension 2 such that

- 1) Each polyhedral set has a vertex in  $C$ .
- 2) Any two vertices in  $C$  can be connected by a path of edges in  $C$ .
- 3) No vertex is in the relative interior of an edge.
- 4) Each vertex is of order 3.

Let finally  $f$  be a continuous function which is affine on each  $P_i$ ,  $i = 1, \dots, m$ . Then if  $f \equiv 0$  on two polyhedral sets with a common edge then  $f \equiv 0$  on  $C$ .

Proof: Let  $P_1$  and  $P_2$  have a common edge  $L = P_1 \cap P_2$  and let  $f = 0$  on  $P_1 \cup P_2$ . We want to prove that there is a third polyhedral set  $P_3$  which has an edge  $L_1$  in common with  $P_1$  and an

edge  $L_2$  in common with  $P_2$ , then since  $f$  is affine on  $P_3$  and 0 on  $L_1$  and  $L_2$ , the condition 3) will ensure that  $f = 0$  on  $P_3$ .

If  $L \cap C$  contains no vertex of  $P_1$  (or  $P_2$ ) then  $L$  would bisect  $C$ , such that  $P_1$  and  $P_2$  would be on different sides of  $L$ . Now  $P_1$  and  $P_2$  each have a vertex in  $C$ , by condition 1), and by 2) they can be connected by a path of edges inside  $C$ . This path must meet  $L \cap C$  and hence  $L \cap C$  does contain a vertex. Let therefore  $V_1$  be a vertex in  $L \cap C$ . Since  $V_1$  is of order 3 there is a third polyhedral set  $P_3$  which meets  $P_1$  and  $P_2$  at  $V_1$ . This set clearly has a common edge with  $P_1$  and with  $P_2$  and by the above argument  $f = 0$  on  $P_3$ .

This was the start of the induction and we can now prove that the conditions 1) through 4) are sufficient that the presented argument spreads to all of  $C$ .

Let us assume that  $f = 0$  on

$$C_k = P_1 \cup \dots \cup P_k, \quad (k \leq m),$$

which is connected and contains a vertex in the interior, namely  $V_1$ .

If  $C \setminus C_k = \emptyset$  we have proved that  $f = 0$  on  $C$ . Otherwise let  $x \in C \setminus C_k$ , and let  $P$  be a polyhedral set which contains  $x$ . Let  $V_2$  be a vertex in  $C \cap P$ .

Now consider the connected path that leads from  $V_1$  to  $V_2$  inside  $C$ . Let  $V_3$  be the first vertex on this path which lies on the boundary of  $C_k$ .

Notice that since  $V_3$  is reach from inside  $C_k$ , the three edges that meet at  $V_3$  all lie in  $C_k$ , but since  $V_3$  is on the boundary of  $C_k$  and in  $C$  there must be a polyhedral set  $P_{k+1}$  say, which is outside  $C_k$  and which meets the boundary of  $C_k$  at  $V_3$  and at two edges. Hence since  $f$  is affine on  $P_{k+1}$  and 0 on  $C_k$  we find that  $f$  is 0 on  $P_{k+1}$  as well. This completes the induction and the proof of the lemma.

Let us remark, that in case  $C = \mathbb{R}^2$  there are two cases possible. Either there are no vertices at all in which case the covering  $(P_1, \dots, P_m)$  consists of parallel strips covering the plane, or there is at least one vertex, in which case 1) and 2) are automatically satisfied.

Notice also that if the covering is induced by a convex function then 3) is automatically satisfied and hence we see that the most important condition is the fourth condition that each vertex should have order 3.

3. The Extremal Functions. We can not identify all the extremal convex functions, but we can find so many that there are enough to prove the main result that they are dense in  $\hat{K}$ .

Theorem 1. A polyhedral convex function  $f$  is extremal in  $\hat{K}$  if the covering of  $K$  given by  $f$  satisfies the conditions 1), 2), and 4) for  $C = \text{int } K$ .

Proof: Let  $g$  and  $h$  be elements of  $K$  and let

$$f = \frac{1}{2}(g + h).$$

Let  $P_1, \dots, P_m$  denote the covering induced by  $f$  which satisfies the conditions 1), 2) and 4). The condition 3) will then automatically be satisfied.

Clearly  $g$  and  $h$  must be affine on each of the polyhedral sets  $P_1, \dots, P_m$  and we shall assume that  $g$  and  $h$  are polyhedral functions.

Now let  $a, b,$  and  $c$  denote affine functions, such that

$$f_1 = f - a, \quad g_1 = g - b, \quad \text{and} \quad h_1 = h - c$$

all vanish on  $P_1$ , then

$$f_1 = \frac{1}{2}(g_1 + h_1).$$

Let then  $x_0 \in \text{int } P_2 \cap \text{int } K$ , where  $P_2$  has an edge in common with  $P_1$ .

If  $g_1(x_0) = 0$  then  $g_1$  is 0 on  $P_2$  and  $P_1$  and by the lemma  $g_1 \equiv 0$  on  $K$  which proves that  $g$  is affine and hence that  $f$  is extremal.

Let us therefore assume that  $g_1(x_0) > 0$  and by a similar argument that  $h_1(x_0) > 0$ .

Let us then define

$$f_2 = f_1/f_1(x_0), \quad g_2 = g_1/g_1(x_0), \quad h_2 = h_1/h_1(x_0)$$

then

$$f_2 = \alpha g_2 + (1-\alpha)h_2$$

where

$$0 < \alpha = g_1(x_0)/2f_1(x_0) < 1.$$

Now we have that

$$f_2 = g_2 = h_2 = 0 \text{ on } P_1$$

and

$$f_2(x_0) = f_2(x_0) = h_2(x_0) = 1$$

but then

$$f_2 = g_2 = h_2 \text{ on } P_2.$$

If we apply the lemma to the piecewise affine function  $f_2 - g_2$  we get that

$$f_2 = g_2 = h_2 \text{ on } K$$

which implies that  $f$  is extremal. This completes the proof of Theorem 1.

Proposition: Let  $a$  and  $b$  be affine functions then  $a$  and  $a \vee b$  are extremal functions. If  $c$  is an affine function such that the equations



$$a(x) = b(x) = c(x)$$

have only one solution in  $\text{int } K$  then  $a \vee b \vee c$  is extremal.

Proof: It is easily seen that  $a$  is extremal, and that the construction in the proof of Theorem 1 will give that  $a \vee b$  is extremal. The above condition on  $c$  ensures the existence of a vertex in  $\text{int } K$  and the covering induced by  $a \vee b \vee c$  satisfies the conditions 1), 2), and 4) of the lemma.

Corollary. In the convex cone of finite continuous convex functions defined on  $R^2$ , the polyhedral functions which induce coverings with vertices of order 3 are extremal.

In particular the functions  $a$ ,  $a \vee b$ , and  $a \vee b \vee c$  are extremal if the equations

$$a(x) = b(x) = c(x)$$

have only one solution.

Proof: This follows from the remarks after the proof of the lemma, together with Theorem 1 and the proposition.

We shall now prove the main result.

Theorem 2. Any finite continuous convex function on the convex set  $K$  can be approximated uniformly on any convex compact subset of  $K$  by an extremal convex function.

Proof: Let  $f \in K$  be given as a finite continuous convex function on  $K$ . Let  $K_1 \subset K$  be a compact convex set of dimension 2.

We shall prove that  $f$  can be approximated by modifying the function on  $K_1$  a finite number of times in such a way that the final modification is an extremal function in  $K$ , which differs less than  $\epsilon$  from  $f$  on  $K_1$ .

1) The first modification is to approximate  $f$  by a polyhedral function

$$f_1 = \sup_{1 \leq i \leq n} a_i$$

as follows: For each  $x \in K_1$  we find a subgradient  $a_x$  and determine a neighbourhood  $N_x$  of  $x$ , such that  $f(y) < a_x(y) + \epsilon/4$ ,  $y \in N_x$ .

By compactness we can pick out a finite number of neighbourhoods which cover  $K_1$  and the corresponding subgradients provide us with the function  $f_1$ .

2) The next step consists in modifying  $f_1$  such that the polyhedral covering induced by  $f_1$  satisfies condition 1) of the lemma. Let therefore  $P$  be such a polyhedral set where  $f_1$  is affine and such that

$$\text{int } P \cap \text{int } K_1 \neq \emptyset.$$

The set  $P$  need not have any vertices in  $\text{int } K_1$  but let us choose

$$x_1 \in \text{int } P \cap \text{int } K_1.$$

Now choose three affine functions  $a$ ,  $b$ , and  $c$ , such that

$$a(x) = b(x) = c(x)$$

only have the solution  $x = x_1$ . The function

$$f_1 + \delta a \vee b \vee c$$

will be a convex polyhedral function with a vertex at  $x_1$ . This function clearly induces a covering of  $K_1$  with more polyhedral sets than before, but each new polyhedral set will have a vertex in  $\text{int } K_1$ .

We then repeat this construction for each polyhedral set from  $f_1$  which does not possess a vertex in  $\text{int } K_1$ . The final modification  $f_2$  will consist of  $f_1$  plus a sum of simple extremal convex functions and will have an induced covering satisfying the first condition of the lemma. For  $\delta$  sufficiently small  $|f_2 - f_1| < \epsilon/4$  on  $K_1$ .

3) The vertices of  $f_2$  need not be connected but let  $x_1$  and  $x_2$  be any two vertices in  $\text{int } K_1$ . Let  $a$  be an affine func-

tion such that

$$a(x_1) = a(x_2) = 0$$

and such that  $a \neq 0$ .

The function

$$f_2 + \delta(Ova)$$

is a convex polyhedral function whose covering of  $K_1$  will contain some new polyhedral sets. Each new set however, will have a vertex on the line determined by  $x_1$  and  $x_2$  inside  $\text{int } K_1$ , and  $x_1$  and  $x_2$  can be connected by a path of edges in  $\text{int } K_1$ , also lying on the line  $[x_1, x_2]$ . This procedure is repeated by replacing  $x_2$  by any of the vertices from  $f_2$  each time adding an a simple extremal function. We end up with a function  $f_3$  whose covering satisfies conditions 1) and 2) of the lemma. For  $\delta$  sufficiently small we get  $|f_2 - f_3| < \epsilon/4$  on  $K_1$ .

4) The final step consists in ensuring that all vertices have order 3.

Assume  $V$  to be a vertex of order  $s > 3$  and let  $a$  be a subgradient such that

$$f_3(x) > a(x), \quad x \neq V$$

$$f_3(x) = a(x), \quad x = V$$

Then

$$f_3 \vee (a + \delta)$$

is a convex polyhedral function with the property that for  $\delta$  sufficiently small the corresponding polyhedral covering will be changed only around  $V$  in such a way that  $V$  will be surrounded by a small polytope with  $s$  vertices each of order 3.

Clearly the new polyhedral sets constructed this way still have vertices in  $\text{int } K_1$  and these can still be connected in-

side int  $K_1$ .

This construction is repeated for each vertex of order  $> 3$  each time taking the maximum of the function so far obtained and a suitable affine function. The final function will induce a covering with all the desired properties listed in the lemma.

For  $\delta$  sufficiently small this function  $f_4$  will lie within  $\epsilon/4$  of  $f_3$  on  $K_1$ .

Thus we can apply Theorem 1 and we get that  $f_4$  is an extremal function which differs less than  $\epsilon$  from  $f$  on  $K_1$  as was to be proved.

4. References.

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