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Steffen L. Lauritzen

Testing the Order of Autoregression

UNIVERSITY OF COPENHAGEN
INSTITUTE OF
MATHEMATICAL STATISTICS

Steffen L. Lauritzen

AN EXACT SIMILAR TEST OF THE ORDER OF AN AUTOREGRESSIVE SCHEME

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1. Introduction.

The order of an autoregressive scheme is commonly tested by two different methods (see [1]): (a) changing the model a little bit (e.g. making the model circular) and deriving the exact distribution of a test; (b) deriving approximate distributions of a test in the exact model. The present note shows a test in the exact model with an F-distribution not depending on the other parameters of the model. The derivation of the test is very easily performed using a conditioning technique. A similar technique has been used by J.E. Besag [2] in connection with nearest-neighbour systems.

2. Testing Independence in the Autoregressive Scheme of Order 1.

Let $(X_t,\ t = \ldots,-1,0,1,\ldots)$ be a stationary, normal autoregressive scheme of order 1, i.e. $(X_t - \xi) - \beta(X_{t-1} - \xi) = \varepsilon_t$, where the ε_t- s are independent, identically, normally distributed with $E\varepsilon_t = 0$, $E\varepsilon_t^2 = \sigma^2 > 0$, $-\infty < \xi < \infty$ and $-1 < \beta < 1$. Let $X_1,\ldots,X_{2n+1}$, where $n > 2$ is an integer be observations from the above process. The conditional distribution of $(X_{2i}, i=1,\ldots,n)$ given $(X_{2i+1},\ i = 0,\ldots,n)$ is then multivariate normal, X_{2i} independent of X_{2j} when $i \neq j$,

$$E(X_{2i}|X_{2j+1}, j = 0,\ldots,n) = \alpha + \gamma(X_{2i-1} + X_{2i+1}) \tag{1}$$

$$\mathrm{Var}(X_{2i}|X_{2j+1},\ j=0,\ldots,n) = \tau^2$$

where (α,γ,τ^2) are related to $(\xi,\beta,\ \sigma^2)$ by the formulas

$$\left.\begin{aligned} \tau^2 &= \sigma^2/(1+\beta^2) \\ \gamma &= \beta\tau^2/(1-\beta^2) \\ \alpha &= \xi(1+2\gamma) \end{aligned}\right\} \quad . \tag{2}$$

The hypothesis $\beta = 0$ is seen to be equivalent to $\gamma = 0$, and it is well-known from regression analysis that the test for $\gamma = 0$ is of the form $\Phi > f$, where

$$\Phi = \frac{SPD^2(n-2)}{SSD'SSD'' - SPD^2} \tag{3}$$

$$SPD = \sum_{i=1}^{n} X_{2i}(X_{2i-1} + X_{2i+1}) - n\bar{X}'\bar{X}''$$

$$= \sum_{i=1}^{2n} X_i X_{i+1} - n\bar{X}'\bar{X}'' \tag{4}$$

$$SSD' = \sum_{i=1}^{n} X_{2i}^2 - n\bar{X}'^2 \tag{5}$$

$$SSD'' = \sum_{i=1}^{n} (X_{2i-1} + X_{2i+1})^2 - n\bar{X}''^2$$

$$= X_1^2 + 2(\sum_{i=1}^{n-1} X_{2i+1}^2 + \sum_{i=1}^{n} X_{2i-1} X_{2i+1}) + X_{2n+1}^2 - n\bar{X}''^2 \tag{6}$$

$$\bar{X}' = \sum_{i=1}^{n} X_{2i}/n \tag{7}$$

$$\bar{X}'' = \sum_{i=1}^{n} (X_{2i-1} + X_{2i+1})/n$$

$$= (X_1 + 2\sum_{i=1}^{p-1} X_{2i+1} + X_{2n+1})/n. \tag{8}$$

Under the hypothesis $\gamma = 0$, Φ has an F-distribution with D.F.(1,n-2). The distribution is conditional, but as it does not depend on $(X_{2i+1}, i=0,\ldots,n)$, the distribution is also marginal.

If the number of observations is not odd, the above technique can only be used if you "throw away" one observation.

3. Testing the Order q Against Order p, where p > q.

Conditioning upon $X_1,\ldots,X_p,X_{p+2},\ldots,X_{2p+1},\ldots X_{mp+m+p}$, where $n = mp+m+p$ is the number of observations, the techniques of ordinary regression analysis again can be used to get an F-test with D.F.(p-q,m-p-1).

4. Comment.

The test criterion Φ is in a way analogous to the difference test in the normal two sample problem. Suppose $(X_1,\ldots,X_n)$ and $(Y_1,\ldots,Y_n)$ are independent random variables from two normal distributions with $EX_i = \xi_X$, $EY_i = \xi_Y$, $Var(X_i) = \sigma_X^2$ and $Var(Y_i) = \sigma_Y^2$. If σ_X^2 and σ_Y^2 are not equal, you can avoid the Fisher-Behrens problem using

$$T = \sqrt{n}(\bar{X}-\bar{Y})/s, \qquad (9)$$

where

$$(n-1)s^2 = \sum_{i=1}^{n} (X_i - Y_i)^2 - n(\bar{X}-\bar{Y})^2, \qquad (10)$$

as a similar test criterion for the hypothesis $\xi_X = \xi_Y$. The "price" is loss of degrees of freedom.

The same thing is happening above.

References.

[1] Andersson, T.W. (1971): The Statistical Analysis of Time Series. Wiley, New York.

[2] Besag, J.E. (1972): On the Statistical Analysis of Nearest-Neighbour Systems. Paper presented at the European Meeting of Statisticians, Budapest.

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