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Introduction.

In Andersson [1] we introduced the general canonical hypothesis and proved that all hypotheses defined by symmetries were general canonical. In this paper we shall show that every general canonical hypothesis (with mean-value 0) can be defined by symmetries. We suppose that the reader is familiar with [1].

1. Notation.

$\acute{R}$ is the real field.

$\acute{C}$ is the complex field.

$\acute{K}$ is the quaternion field.

D is a field isomorphic with $\acute{R}$, $\acute{C}$ or $\acute{K}$.

A left [right] D-space means a left [right] vectorspace over D with finite dimension.

Let E be a left [right] D-space. E_0 is the deduced $\acute{R}$-space.

$P_D(E)$ is the semivectorspace over $\acute{R}_+$ of right [left] positive symmetrical sesquilinear definite functionals on E. If $D \simeq \acute{R}$, we set $P_{\acute{R}}(E) = P(E)$.

p.d.s.f. (p.d.f.) is an abbreviation of positive definite symmetrical sesquilinear functional (positive definite form). [1], § 3.

GL(E) is the locally compact group of bijective D-linear mappings from E to E. 1_E is the neutral element in GL(E).

If f is a D-linear mapping, f_0 denotes the $\acute{R}$-linear mapping deduced from f.

For $B \in P_D(E)$, $\mathring{O}(B)$ denotes the orthogonal group with respect to B.

$O(B)$ is a maximal compact subgroup in $GL(E)$. $B_0 = Re(B)$ is a p.d.f. on E_0. For definition and properties of the real tensorproduct of two p.d.s.f. see [1] § 12.

2. Reflexive subsemivectorspaces in $P(E)$.

2.1. Let E be an $\acute{R}$-space. For $\Sigma \subseteq P(E)$ we define

$$\acute{O}(\Sigma) = \{f \in GL(E) \mid \forall B \in \Sigma, \forall x,y \in E: B(f(x),f(y)) = B(x,y)\}.$$

Note that

$$\acute{O}(\Sigma) = \bigcap_{B \in \Sigma} O(B).$$

$\acute{O}(\Sigma)$ is a compact group.

$$\Sigma_1 \subseteq \Sigma_2 (\subseteq P(E)) \Rightarrow \acute{O}(\Sigma_1) \supseteq \acute{O}(\Sigma_2)$$

$$\acute{O}(P(E)) = \{1_E, -1_E\}.$$

$$O(\{\lambda B \mid \lambda \in \acute{R}_+\}) = O(B), \; B \in P(E).$$

2.2. For $S \subseteq GL(E)$ a relatively compact subgroup we define

$$\acute{P}(S) = \{B \in P(E) \mid \forall f \in S, \forall x,y \in E: B(f(x),f(y)) = B(x,y)\}.$$

Note that

$\acute{P}(S)$ is a subsemivectorspace in $P(E)$.

$S_1 \subseteq S_2 (\subseteq GL(E)) \Rightarrow \acute{P}(S_1) \supseteq \acute{P}(S_2)$.

$\acute{P}(\{1_E, -1_E\}) = P(E)$.

2.3. <u>Definition</u>: A subset $\Sigma \subseteq P(E)$ is <u>reflexive</u> if $\acute{P}(\acute{O}(\Sigma)) = \Sigma$.

A relatively compact subgroup $S \subseteq GL(E)$ is <u>reflexive</u> if $\acute{O}(\acute{P}(S)) = S$.

2.4. Reflexive subsets in $P(E)$ are subsemivectorspaces. Reflexive subgroups in $GL(E)$ are compact.

$\{1_E, -1_E\}$ and $P(E)$ are both reflexive. $O(B)$ and $\{\lambda B | \lambda \in \acute{R}_+\}$ are both reflexive for $B \in P(E)$.

2.5. <u>Proposition</u>: Let $\Sigma \subseteq P(E)$ and let S be a relatively compact subgroup in $GL(E)$. Then $\acute{O}(\Sigma)$ and $\acute{P}(S)$ are both reflexive.

Proof: Trivial.

2.6. <u>Proposition</u>: Let E be a left [right] D-space and let $B \in P_D(E)$. Then $O(B)_0$ is a compact irreducible subgroup of $O(B_0)$ of type D in E_0. $\acute{P}(O(B)_0) = \acute{P}(O(B_0))$ and $O(B)_0$ is a maximal element in the class of relatively compact subgroups of $GL(E_0)$ of type D.

Proof: Suppose that $O(B)_0$ is reducible and $D \simeq \acute{K}$. Let $F \subset E_0$ be a nontrivial $\acute{R}$-subspace invariant under $O(B)_0$. If 1,i,j, and k is a natural basis for $\acute{K}$, we have $E_0 = F \oplus_{\acute{R}} iF \oplus_{\acute{R}} jF \oplus_{\acute{R}} kF$, since $O(B)$ is irreducible in E. From this it follows that we can choose a $\acute{K}$-basis, ($\acute{R}$-basis for F) such that $O(B)_0$ can be described by real matrices in E. This is impossible, and we therefore have $O(B)_0$ irreducible in E_0. Since the elements in $O(B)_0$ are $\acute{K}$-linear, $O(B)_0$ is of type $\acute{K}$. The case $D \simeq \acute{C}$ is analogous and $D \simeq \acute{R}$ is trivial $\acute{P}(O(B)_0) = \acute{P}(O(B_0))$ follows from [1] prop. 13.6. Let $O(B)_0 \subseteq O_1 \subsetneq GL(E_0)$, where O_1 is a relatively compact subgroup (trivially irreducible) of type D. Then the elements in O_1 are also D-linear. The set of p.d.s.f. on E invariant under O_1 is a non-empty subsemivectorspace of the semivectorspace $\{\lambda B | \lambda \in \acute{R}_+\}$. Therefore $O_1 = O(B)$.

2.7. Let E_1, E_2 be $\acute{R}$-spaces. For $\Sigma_1 \subseteq P(E_1)$, $\Sigma_2 \subseteq P(E_2)$, S_1, S_2 relatively compact subgroups in $GL(E_1)$ and $GL(E_2)$ respectively. We define

$$\Sigma_1 \oplus \Sigma_2 = \{B_1 \oplus B_2 \in P(E_1 \oplus E_2) \mid B_1 \in \Sigma_1, B_2 \in \Sigma_2\}.$$

$$S_1 \times S_2 = \{g_1 \oplus g_2 \in GL(E_1 \oplus E_2) | g_1 \in S_1, g_2 \in S_2\}.$$

We have

$$\acute{O}(\Sigma_1 \oplus \Sigma_2) \supseteq \acute{O}(\Sigma_1) \times \acute{O}(\Sigma_2) \qquad (*)$$

$$\acute{P}(S_1 \times S_2) \supseteq \acute{P}(S_1) \oplus \acute{P}(S_2) \qquad (**).$$

2.8. <u>Proposition</u>: If Σ_1, $\Sigma_2(S_1,S_2)$ are reflexive, then (*) ((**)) are equalities.

Proof: Trivial.

2.9. <u>Proposition</u>: Let H be a right, F a left D-space. For $B \in P_D(F)$ we define

$$P_D(H)\ \tilde{\otimes}_D B = \{A\ \tilde{\otimes}_D B \in P_{\acute{R}}((H \otimes_D F)_0)\,|\,A \in P_D(H)\}.$$

It follows that $P_D(H)\ \tilde{\otimes}_D B$ is reflexive and

$$\acute{O}(P_D(H)\ \tilde{\otimes}_D B) = \begin{cases} 1_H \otimes_D O(B), & \text{if } \dim_D(H) \geqq 2. \\ O(B_0), & \text{if } \dim_D(H) = 1. \end{cases}$$

Proof: The last equation is trivial. Since $\acute{P}(1_H \otimes_D O(B)) = P_D(H)\ \tilde{\otimes}_D B$, ([1], prop. 13.9) $P_D(H)\ \tilde{\otimes}_D B$ is reflexive after 2.5. $\acute{O}(P_D(H)\ \tilde{\otimes}_D B)$ must be of the form $1_H \otimes_D O$ ($\dim_D H \geqq 2$), where O_0 is irreducible and of type D ([1] prop. 13.[illegible] and 2.5). Since $1_H \otimes_D O(B) \subseteq \acute{O}(P_D(H)\ \tilde{\otimes}_D B)$ and $O(B)_0$ is maximal (2.6) the first equation follows.

2.10. <u>Theorem</u>: Let E be an $\acute{R}$-space and $\Sigma \subseteq P(E)$. Then Σ is reflexive if and only if there exist (unique) decompositions of E and S of the following form:

$$E = \bigoplus_{i \in I} (H_i \otimes_{D_i} F_i)_0$$

$$\Sigma = \bigoplus_{i \in I} (P_{D_i}(H_i)\ \tilde{\otimes}_{D_i} B_i),$$

where $D_i \simeq R, C$ or K, H_i is a right, F_i a left D_i-space and B_i a right p.d.s.f. on F_i.

In that case $O(\Sigma)$ can be obtained from 2.8 and 2.9.

Proof: Follows 2.8, 2.9, and [1], prop. 13.10.

3. The general canonical hypothesis.

3.1. Theorem 2.10 shows that a hypothesis in the variance in a multidimensional normal distribution with mean-value 0 given by a subset $\Sigma \subseteq P(E^*)$ (see [1] § 8) is reflexive if and only if the hypothesis is general canonical.

In that case the hypothesis can be given by a (group) symmetry.

Let $\bigoplus_{i \in I} (B_i \tilde{\otimes}_{D_i} P_{D_i}(H_i^*)) \subseteq P(E^*)$ ([1], § 8) be a general canonical hypothesis. It is natural to choose the symmetry as follows:

$G = \prod_{i \in I} G_i$, where

$$G_i = \begin{cases} O((B_i)_0), & \text{if } \dim_{D_i}(H_i) = 1. \\ O(B_i)_0, & \text{if } \dim_{D_i}(H_i) \geqq 2, \end{cases}$$

and $\Pi((g_i)_{i \in I}) = \bigoplus_{i \in I} (g_i \otimes_{D_i} 1_{H_i^*})$.

3.2. Note that the set of canonical hypothesis in the sense of [2] (mean-value 0) is precisely those hypothesis which can be obtained by a direct sum of symmetries of type R and multiplicityfree symmetries.

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References.

[1] Andersson, S.: Invariant Hypotheses I. Preprint No. 6, Institute of Mathematical Statistics, University of Copenhagen, Copenhagen 1972.

[2] Brøns, H., Henningsen, I., and Jensen, S.T.: A Canonical Hypothesis in the Multidimensional Normal Distribution. Preprint No. 7, Institute of Mathematical Statistics, University of Copenhagen, Copenhagen 1971.

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