PREPRINT

JUNE 1972

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INSTITUTE OF
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POLYTEKNISK FORLAG OG POLYTEKNISK TRYKKERI

1. Introduction and summary.

In the statistical theory for the multidimensional normal distribution symmetry hypotheses in the mean and the variance play a fundamental role. Andersson [1], Consul [7], James [9], Wotaw [14], and Olkin and Press [13] have investigated some special cases, but a general theory seems not to exist. (Compare however Maclaren [12]). In this paper we define a class of hypotheses, the general canonical hypotheses, which includes all (not necessary finite) symmetry hypotheses in the mean and the variance. It also includes the class of canonical hypotheses defined in Brøns, Henningsen, and Jensen [6].

In the first part (section 2-6) we define what we shall call a real tensor-product of positive sesquelineary symmetric functionals and give some of its properties.

In the second part (section 7-11)we define a general canonical hypothesis and derive the maximum likelihood estimator of the mean and the variance and its distribution under this hypothesis.

In the third part (section 12-14) we show that the class of general canonical hypotheses include all symmetry hypotheses. The exposition in the second part leans heavily on the ideas and methods of [6]. The class of general canonical hypotheses is a generalization of the canonical hypotheses in the sense of [6] and the derivation of the maximum likelihood estimator and its distribution is in principle the same.

The idea to the present paper arose from the wrong conjecture in [6] that the class of canonical hypotheses defined therein contains all hypotheses given by finite symmetries.

I am much indebted to one of the authors of [6], S.T. Jensen. Together we constructed a symmetry hypothesis not canonical in the sense of [6] and

I have also otherwise profited greatly from discussions with him.

2. Notation.

$\mathbb{R}$ is the real field.

$\mathbb{C}$ is the complex field.

$\mathbb{K}$ is the quaternion field.

D is a field isomorphic with $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{K}$.

There exists a unique conjugation in D ([2], chap. III, § 2, No. 4, prop. 4).

This will be written $d \mapsto \overline{d}$, $d \in D$.

We have $\mathrm{Re}(d) = \frac{1}{2}(d+\overline{d})$. Note that $\mathrm{Re}(d_1 d_2) = \mathrm{Re}(d_2 d_1)$.

E, F, H will denote left [right] vectorspaces over D.

We use the abbreviation D-space for a finite dimensional vectorspace over D.

The scalar multiplication in a left [right] D-space E will be denoted

$(d,x) \to dx$ $[(d,x) \to xd]$, $d \in D$, $x \in E$.

$L_D(E,F)$ denotes the linear operators from the left [right] D-space E to the left [right] D-space F.

For D commutative, $L_D(E,F)$ can in a natural way be organized as a D-space.

$L_D(E,E)$ is an algebra over D. ([2])

$GL_D(E)$ are those elements in $L_D(E,E)$ which have an invers.

$GL_D(E)$ is a group. ([2])

1_E is the neutral element in $GL_D(E)$.

G will denote a given fixed group.

3. Positive forms.

Let E be a left [right] D-space.

3.1. Definition: ([4], § 1, No. 2.) A right [left] sesquilinear functional on E is a mapping $B:E \times E \to D$ with the properties:

$$B(x+x',y) = B(x,y) + B(x',y). \qquad x,x',y \in E. \qquad (1)$$

$$B(x,y+y') = B(x,y) + B(x,y'). \qquad x,y,y' \in E. \qquad (2)$$

$$B(dx,y) = dB(x,y). \qquad d \in D, \quad x,y \in E. \qquad (3)$$

$$[B(xd,y) = \overline{d}B(x,y). \qquad d \in D, \quad x,y \in E.] \qquad (3')$$

$$B(x,dy) = B(x,y)\overline{d}. \qquad d \in D. \quad x,y \in E. \qquad (4)$$

$$[B(x,yd) = B(x,y)d. \qquad d \in D. \quad x,y \in E.] \qquad (4')$$

We call a sesquilinear functional B symmetrical if

$$B(x,y) = \overline{B(y,x)}. \qquad x,y \in E. \qquad (5)$$

and positive if

$$B(x,x) \geqq 0. \qquad \forall\, x \in E. \qquad (6)$$

and positive definite if

$$\forall\, x \in E: x \neq 0 \Rightarrow B(x,x) > 0. \qquad (7)$$

In the case $D \simeq \mathbb{R}$ we call a sesquilinear, symmetrical, positive (definite) functional for a positive (definite) form.

3.2. We shall use the following abbreviations:

p.f. for positive form.

p.d.f. for positive definite form.

s.f. for sesquilinear functional.

s.s.f. for symmetrical sesquilinear functional.

p.s.f. for positive symmetrical sesquilinear functional.

p.d.s.f. for positive definite symmetrical sesquilinear functional.

3.3. For a s.f. B we set $Q(x) = B(x,x)$, $x \in E$, and we have the identity

$$2(B(x,y) + B(y,x)) = Q(x+y) - Q(x-y). \qquad x,y \in E,$$

which shows that a s.s.f. B on E is determined by its values on the diagonal in $E \times E$. We have

$$\text{for } D \simeq \mathbb{R}\text{: } B(x,y) = \frac{1}{4}(Q(x+y) - Q(x-y)). \qquad x,y \in E. \tag{8}$$

for $D \simeq \mathbb{C}$: (right): $(d-\bar{d})\, B(x,y)$

$$= \frac{1}{2}(dQ(x+y) - dQ(x-y) - Q(x+dy) + Q(x-dy)). \quad d \in D.\ x,y \in E. \tag{9}$$

(left): $(d-\bar{d})\, B(x,y)$

$$= \frac{1}{2}(dQ(x+y) - dQ(x-y) - Q(x+y\bar{d}) + Q(x-y\bar{d})). \quad d \in D.\ x,y \in E. \tag{10}$$

for $D \simeq K$ (right): $2B(x,y)(\bar{d}\bar{e} - \bar{e}\bar{d}) = Q(x-dey) - Q(x+dey) + dQ(x+ey)$

$- dQ(x-ey) + Q(x+dy)\bar{e} - Q(x-dy)\bar{e} + dQ(x-y)\bar{e}$

$- dQ(x+y)\bar{e}.$ $\qquad d,e \in D.\ x,y \in E.$ (11)

(left): $2(\bar{d}\bar{e}-\bar{e}\bar{d})\, B(y,x) = Q(x-yde) - Q(x+yde) + \bar{d}Q(x+ye)$

$- \bar{d}Q(x-ye) + Q(x+yd)e - Q(x-yd)e + \bar{d}Q(x-y)e$

$- \bar{d}Q(x+y)e.$ $\qquad d,e \in D.\ x,y \in E.$ (12)

The first three formulas are well-known and the last two are easily verified. Note that in cases $D \simeq C$ and $D \simeq K$ it is not necessary to assume symmetry.

3.4. Let E be a left [right] D-space. If $\overline{E}$ denotes the conjugated space, $\overline{E}$ is a right [left] D-space. ([4], § 1, No. 2, def. 5). If B is a right [left] s.f. on E, then $\overline{B}: \overline{E} \times \overline{E} \to D$ defined by $\overline{B(x,y)} = B(y,x)$ is a left [right] s.f. on $\overline{E}$. $B \to \overline{B}$ is a one to one correspondance between the right [left] s.f. on E and the left [right] s.f. on $\overline{E}$.

4. The one to one correspondence between positive forms and positive symmetrical, sesquilinear functionals.

Since D is an algebra over R, every left [right] D-space E is also in a natural way a R-space, E_0. If F_0 is an R-space and also has a structure as a D-space, such that the restriction to the reals in D precisely is the original structure on F_0, we denote this D-space F. Remark: $(\overline{E})_0 = E_0$.

If B is a right [left] p.s.f. on the D-space E, the mapping $B_0: (x,y) \to Re(B(x,y))$ is a p.f. on E_0. Note that $(\overline{B})_0 = B_0$.

4.1. <u>Proposition</u>: The mapping from the set of right [left] p.s.f. on E to the set of p.f. on E_0 with the property

$$(*) \qquad \Phi(dx,y) = \Phi(x,\bar{d}y), \qquad d \in D,\ x,y \in E$$

$$[(*) \qquad \Phi(xd,y) = \Phi(x,y\bar{d}), \qquad d \in D,\ x,y \in E]$$

where Φ is a p.f. on E_0 defined by $B \to B_0$, is a positive, homogeneous, semilinear one to one correspondence.

B is definite if and only if B_0 is definite.

<u>Proof</u>: The case $D \simeq R$ is trivial. The case $D \simeq C$ follows from [4], § 3, No. 3 prop. 1. To verify (*) in the case $D \simeq K$ use the fact $\mathrm{Re}(d_1 d_2) = \mathrm{Re}(d_2 d_1)$, $d_1, d_2 \in D$. (Note that for $D \simeq K$ we do not have the usual identity $B(dx,y) = B(x,\bar{d}y)$.)

From formula 11 (12) in § 3 it follows that there exists one and only one p.s.f. B on E with the property $B_0 = \Phi$. The last assertion is trivial.

4.2. Let E and F be two left [right] D-spaces. The two R-spaces $(E \oplus_D F)_0$ and $E_0 \oplus_R F_0$ are canonical isomorphic. Let A and B be two right (left) p.s.f. on E and F, respectively. Then

$$(A \oplus_D B)_0 = A_0 \oplus_R B_0 .$$

4.3. When E is a left [right] D-space the dual space, E* is a right [left] D-space. The mapping $\theta : x^* \to \mathrm{Re}(x^*)$ gives a natural isomorphism between the R- spaces $(E^*)_0$ and $(E_0)^*$.

Let now B be a right [left] p.s.f. on E. The mapping $f_B: E \to E^*$ defined by $y \to (x \to B(x,y))$ [$(x \to B(y,x))$] is "conjugating linear". The mapping is a bijection if and only if B is definiet ([4], § 7, No. 1, prop. 2). In the same way we can define the $\acute{R}$-linear mapping $f_{B_0}: E_0 \to (E_0)^*$. It is easy to see that $f_{B_0} = \theta \circ f_B$.

4.4. Definition: Let B be definite. The left [right] p.d.s.f. B^{-1} on the right [left] D-space E^* defined by

$$B^{-1}(x^*,y^*) = B(f_B^{-1}(x^*), f_B^{-1}(y^*)). \qquad x^*,y^* \in E^*$$

is called the inverse to B.

4.5. Proposition: Let B be definite. Then $(B^{-1})_0 = (B_0)^{-1}$ if we identify $(E^*)_0$ and $(E_0)^*$ through θ.

Proof: $(B_0)^{-1}(\theta(x^*),\theta(y^*)) = B_0(f_{B_0}^{-1}(\theta(x^*)), f_{B_0}^{-1}(\theta(y^*))) =$

$B_0(f_B^{-1}(x^*), f_B^{-1}(y^*)) = \mathrm{Re}(B(f_B^{-1}(x^*), f_B^{-1}(y^*))) = \mathrm{Re}(B^{-1}(x^*,y^*)) =$

$(B^{-1})_0(x^*,y^*)$.

5. A generalization of the tensorproduct of positive forms.

Let E be a right and F a left D-space. $E \otimes_D F$ is a $\acute{Z}$-modul ([2], chap.2, § 3). Because of the structure of D as an algebra over $\acute{R}$, $E \otimes_D F$ can also be considered as a $\acute{R}$-space, $(E \otimes_D F)_0$.

We shall say that a mapping $w: E \times F \to H$, where H is a $\acute{R}$-space, has the property (**) if w is $\acute{R}$-bilinear and

$$w(xd,y) = w(x,dy) \qquad d \in D,\ x \in E,\ y \in F.$$

This determinates one and only one $\acute{R}$-linear mapping w' from $(E \otimes_D F)_0$ into H with the property $w'(x \otimes_D y) = w(x,y)$ ([2], chap. 2, § 3, No. 1, prop. 1).

5.1. Let now A be a left and B a right p.s.f. on E and F, respectively. The mapping $\delta: (E \times F) \times (E \times F) \to \acute{R}$ defined by

$$(x,y,u,v) \to \operatorname{Re}(A(x,u)\,\overline{B(y,v)})$$

has the property

$$\delta(xd,y,ue,v) = \delta(x,dy,u,ev), \qquad d,e \in D,\ x,u \in E,\ y,v \in F.$$

From ([2], chap. 2, § 3, No. 9, remarque 2) it follows that δ determines one and only one $\acute{R}$-bilinear functional Φ on $(E \otimes_D F)_0$ with the property $\Phi(x \otimes_D y,\ u \otimes_D v) = \delta(x,y,u,v)$.

It is easily seen that Φ is a p.f. on $(E \otimes_D F)_0$.

5.2. <u>Definition</u>: The p.f. on the $\acute{R}$-space $(E \otimes_D F)_0$ defined above is called the <u>real tensorproduct</u> of A and B. We donote it $(A \mathbin{\tilde{\otimes}_D} B)_0$.

5.3. Remark: For $D \simeq \mathbb{R}$ the above real tensorproduct is the usually tensorproduct of positive forms.

Remark: Let A be a left and B a right p.s.f. on the $\mathbb{C}$-spaces E and F. If we define $\tilde{A}$, resp. $\tilde{B}$, by $\tilde{A}(x,y) = A(y,x)$, resp. $\tilde{B}(x,y) = B(y,x)$, we have $(\tilde{A} \otimes_{\mathbb{C}} B)_0 = (A \otimes_{\mathbb{C}} \tilde{B})_0 = (A \tilde{\otimes}_{\mathbb{C}} B)_0$. ([4], § 1, No. 9).

5.4. Let E and F be as above. For $x^* \in E^*$ and $y^* \in F^*$ we define the mapping $\delta_1 : E \times F \to \mathbb{R}$ as $\delta_1(x,y) = \mathrm{Re}(y^*(y)\, x^*(x))$. δ_1 has the property (**) and therefore defines an $\mathbb{R}$-linear mapping $y^* \otimes x^* : (E \otimes_D F)_0 \to \mathbb{R}$. $y^* \otimes x^*$ is an element in $((E \otimes_D F)_0)^*$. The mapping $\delta_2 : F^* \times E^* \to ((E \otimes_D F)_0)^*$ defined by $\delta_2(y^*,x^*) = y^* \otimes x^*$ also have the property (**) and defines another $\mathbb{R}$-linear mapping $\psi : (F^* \otimes_D E^*)_0 \to ((E \otimes_D F)_0)^*$. It is easy to see that ψ is an isomoprhism of $\mathbb{R}$-spaces.

5.5. Proposition: Let E,F,A,B be defined as in 5.1. Then $(A \tilde{\otimes}_D B)_0$ is definite if and only if A_0 $[(A)]$ and B_0 $[(B)]$ are definite. In that case $(A \tilde{\otimes}_D B)_0^{-1} = (B^{-1} \tilde{\otimes}_D A^{-1})_0$, if we identify $((E \otimes_D F)_0)^*$ and $(F^* \otimes_D E^*)_0$ through the ψ defined above.

Proof: The mapping $\delta_3 : (E \times F) \to (F^* \otimes_D E^*)_0$ defined by $\delta_3(x,y) = f_B(y) \otimes_D f_A(x)$ has the property (**) and defines an $\mathbb{R}$-linear mapping $f_B \otimes f_A$ from the $\mathbb{R}$-space $(E \otimes_D F)_0$ to the $\mathbb{R}$-space $(E^* \otimes_D E^*)_0$.

Let $u \otimes v \in E \otimes_D F$, then

$$f_{(A \tilde{\otimes}_D B)_0}(u \otimes v) = (x \otimes y \to (A \tilde{\otimes}_D B)_0(u \otimes v, x \otimes y))$$

$$= (x \otimes y \to \mathrm{Re}(A(u,x)\, \overline{B(v,y)}))$$

and

$$\psi \circ (f_B(v) \otimes f_A(u)) = (x \otimes y \to \mathrm{Re}(f_A(u)(x)\ f_B(v)(y))$$

$$= (x \otimes y \to \mathrm{Re}(A(u,x)\ \overline{B(v,y)})).$$

From this follows the identity

$$f_{(A\ \tilde{\otimes}_D B)_0} = \psi \circ (f_B \otimes f_A).$$

Now

$$(B^{-1} \tilde{\otimes}_D A^{-1})_0 (y^* \otimes x^*, v^* \otimes u^*) = \mathrm{Re}(B^{-1}(y^*,v^*)\overline{A^{-1}(x^*,u^*)})$$

$$= \mathrm{Re}(B(f_B^{-1}(y^*),\ f_B^{-1}(v^*))\ \overline{A(f_A^{-1}(x^*),\ f_A^{-1}(u^*))})$$

$$= (A \tilde{\otimes}_D B)_0((f_B \otimes f_A)^{-1}(y^* \otimes x^*),\ (f_B \otimes f_A)^{-1}(v^* \otimes u^*))$$

$$= (A \tilde{\otimes}_D B)_0(f_{(A \tilde{\otimes} B)_0}^{-1}\ (\psi(y^* \otimes x^*),\ f_{(A \tilde{\otimes} B)_0}^{-1}\ (\psi(v^* \otimes u^*)))$$

$$= (A \otimes_D B)_0^{-1}(\psi(y^* \otimes x^*),\ \psi(v^* \otimes u^*)). \qquad y^*,v^* \in F^*.\ x^*,u^* \in E^*.$$

It is trivial to see the following

5.6. Proposition: Let E be a right and F_1, F_2 left D-spaces, and let A be a p.s.f. on E, B_1 and B_2 right p.s.f. on F_1 and F_2, respectively.

Then

$$(A \tilde{\otimes}_D (B^1 \oplus_D B^2))_0 = (A \otimes_D B^1)_0 \oplus_{\acute{R}} (A \otimes_D B^2)_0,$$

if we identify $(E \otimes_D (F_1 \oplus_D F_2))_0$ and $(E \otimes_D F_1)_0 \oplus_{\acute{R}} (E \otimes_D F_2)_0$ through the natural $\acute{R}$-space isomorphism.

6. Positive tensors.

Let E be a right [left] D-space and B a right [left] p.s.f. on the left [right] D-space E*. The bilinear mapping $B_0 : \overline{E^*} \times E^* \to \acute{R}$ [$B_0 : E^* \times \overline{E^*} \to \acute{R}$] has the property (**) and defines a unique $\acute{R}$-linear form on $(\overline{E^*} \otimes_D E^*)_0$[$(E^* \otimes_D \overline{E^*})_0$]. The D and $\acute{R}$-isomorphisms $\overline{E^*} \simeq (\overline{E})^*$, $E^{**} \simeq E$ and $((\overline{E^*} \otimes_D E^*)_0)^* \simeq (E \otimes_D \overline{E})_0$[$((E^* \otimes_D \overline{E^*})_0)^* \simeq (\overline{E} \otimes_D E)_0$] shows that the set of p.f. on E_0^* (see 4.3.) can be identified with $P_D(E)$, the semivectorspace over $\acute{R}_+$ of positive tensors in $(E \otimes_D \overline{E})_0$[$(\overline{E} \otimes_D E)_0$], i.e. the elements in $(E \otimes_D \overline{E})_0$[$(\overline{E} \otimes_D E)_0$], which can be written in the form $\sum_\nu (x_\nu \otimes_D x_\nu)$, $x_\nu \in E$. Note that $P_{\acute{R}}(E) = P(E)$ where P(E) is defined in [6] for $D \simeq \acute{R}$. Let E and E' be two left (right) D-spaces. Each $f \in L_D(E, E')$ gives a semilinear mapping $P_D(f) : P_D(E) \to P_D(E')$ defined by $P_D(f)(\sum_\nu (x_\nu \otimes_D x_\nu)) = \sum_\nu (f(x_\nu) \otimes_D f(x_\nu))$ (compare [6]). In the case $E = E'$, $GL_D(E)$ will define a transitive, left action on $P_D(E)_r$ (= the definite elements in $P_D(E)$). The existence of a $GL_D(E)$-invariant measure ν on $P_D(E)_r$, unique up to a multiplicative positive constant follows from [5], § 3, No. 3, ex. 8.

6.2. Let H be a right and F a left D-space, $A_0 \in P_D(H)$ and $B_0 \in P_D(F)$. If A_0 resp. B_0 can be written on the form

$$A_0 = \sum_k (h_k \otimes_D h_k), \quad h_k \in H$$

resp.

$$B_0 = \sum_p (f_p \otimes_D f_p), \quad f_p \in F,$$

then $(B \tilde{\otimes}_D A)_0$ is determinated by the following equations:

$D \simeq \acute{R}$:

$$(B \tilde{\otimes}_D A)_0 = \sum_{k,p} (h_k \otimes_D f_p) \otimes_{\acute{R}} (h_k \otimes_D f_p)$$

$D \simeq \acute{C}$:

$$\|d\|^2 (B \tilde{\otimes}_D A)_0 = \sum_{k,p} (\|d\|^2 (h_k \otimes_D f_p) \otimes_{\acute{R}} (h_k \otimes_D f_p) + (h_k \otimes_D df_p) \otimes_{\acute{R}} (h_k \otimes_D df_p)),$$

$\forall\, d \in \acute{C}$, with the property $\mathrm{Re}(d) = 0$.

$D \simeq \acute{K}$:

$$\|de\| (B \tilde{\otimes}_D A)_0 = \sum_{p,k} (\|de\|^2 (h_k \otimes_D f_p) \otimes_{\acute{R}} (h_k \otimes_D f_p) + \|d\|^2 (h_k \otimes_D ef_p)) \otimes_{\acute{R}}$$

$$(h_k \otimes_D ef_p) + \|e\|^2 (h_k \otimes_D df_p) \otimes_{\acute{R}} (h_k \otimes_D df_p) + (h_k \otimes_D def_p) \otimes_{\acute{R}} (h_k \otimes_D def_p))$$

$\forall$ d, e $\in \acute{K}$ with the properties Re(e) = Re(d) = Re(de) = 0.

6.3. Let H'be a right, F'a left D-space, $t \in L_D(H,H')$, and $s \in L_D(F,F')$. $(t \otimes_D s)_0 \in L_{\acute{R}}((H \otimes_D F)_0, (H' \otimes_D F')_0)$ and 6.2 gives

$$P_{\acute{R}}((t \otimes_D s)_0)((B \tilde{\otimes}_D A)_0) = (P_D(s)(B) \tilde{\otimes}_D P_D(t)(A))_0 .$$

7. The regular normal distribution.

7.1. Let E be a $\acute{R}$-space, λ a lebesgue measure on E, B a p.d.f. on E^* and $\xi \in E$.

The function $\varphi_{B,\xi} : E \to \acute{R}$ defined by

$$\varphi_{B,\xi}(x) = e^{-\frac{1}{2}B^{-1}((x-\xi) \otimes_{\acute{R}} (x-\xi))}$$

is continuous and λ-integrable. We define

$$a(B) = \int_E \varphi_{B,\xi} d\lambda .$$

The measure

$$\mu_{B,\xi} = a(B)^{-1} \varphi_{B,\xi} \lambda$$

is the regular normal distribution with mean-value ξ and variance B. When $B = B_1 \oplus_{\acute{R}} B_2$ and $\xi = \xi_1 \oplus \xi_2$ then

$$\mu_{B_1 \oplus B_2, \xi_1 \oplus \xi_2} = \mu_{B_1,\xi_1} \otimes \mu_{B_2,\xi_2} ,$$

special will

$$a(B_1 \oplus B_2) = a(B_1)a(B_2).$$

For $f \in GL_{\acute{R}}(E)$ we have

$$f\mu_{B,\xi} = \mu_{P_{\grave{R}}(f)B,\, f(\xi)},$$

hence $a(P_{\grave{R}}(f)B) = |\det f|\ a(B)$ (compare [6]).

8. The general canonical hypothesis.

8.1. A general canonical hypothesis in the mean and the variance in a multidimensional normal distribution is a decomposition $E = \bigoplus_{i \in I}(H_i \otimes_{D_i} (F_i \oplus_{D_i} F_i'))_0$ og E together with a parametrization $\bigoplus_{i \in I}(H_i \otimes_{D_i} F_i') \to E$ of the mean and the variance determinated by the injections $(H_i \otimes_{D_i} F_i')_0 \to (H_i \otimes_{D_i} (F_i \oplus_{D_i} F_i'))_0$ and $((A_i)_0)_{i \in I} \to \bigoplus_{i \in I}((B_i \oplus_{D_i} B_i') \tilde{\otimes}_{D_i} A_i)_0$, where $D_i \simeq \acute{R}, \acute{C}$ or $\acute{K}$, $(A_i)_0 \in P_{D_i}(H_i)_r$, $(B_i)_0 \in P_{D_i}(F_i)_r$ and $(B_i')_0 \in P_{D_i}(F_i')_r$. Here A_i is unknown and B_i and B_i' is known p.d.s.f. ($i \in I$) (cf. [6] section 4).

8.2. In the case $D_i \simeq \acute{R}$ for all $i \in I$, we have the canonical hypothesis from [6].

9. Estimation.

9.1. In this section we shall find the maximum likelihood estimator for the mean and the variance in the problem defined in 8. We give a generalization of [6] section 5. Since the problem spilts into a product of independent estimation problems, we can restrict ourselves to the simpel hypothesis

$$E = (H \otimes_D (F \oplus_D F'))_0 = (H \otimes_D F)_0 \oplus_{\tilde{R}} (H \otimes_D F')_0$$

$$A_0 \to ((B \oplus_D B') \tilde{\otimes}_D A)_0 \text{ where } D \simeq \tilde{R},\ \tilde{C} \text{ or } \tilde{K}.$$

The measure

$$\mu_{((B \oplus_D B') \tilde{\otimes}_D A)_0, \xi} \qquad (A_0 \in P_D(H),\ \xi \in (H \otimes_D F')_0)$$

has the density

$$(x,y) \to a((B \tilde{\otimes}_D A)_0)^{-1}\, e^{-\frac{1}{2}(A^{-1} \tilde{\otimes}_D B^{-1})_0 (x \otimes_{\tilde{R}} x)} .$$

$$a((B' \tilde{\otimes}_D A)_0)^{-1}\, e^{-\frac{1}{2}(A^{-1} \otimes_D B'^{-1})_0 ((y-\xi) \otimes_{\tilde{R}} (y-\xi))}$$

$$x \in (H \otimes_D F)_0, \quad y \in (H \otimes_D F')_0$$

with respect to lebesgue measure $\lambda_1 \otimes \lambda_2$ on $(H \otimes_D F)_0 \oplus_{\tilde{R}} (H \otimes_D F')_0$.

The mapping $\delta:(H \times F) \times (H \times F) \to (H \otimes_D \overline{H})_0$ defined by $\delta(h_1,f_1,h_2,f_2) = h_1 \otimes_D B^{-1}(f_1,f_2)h_2$ determinates one and only one $\vec{R}$-linear mapping

$$\theta_B:(H \otimes_D F)_0 \otimes_{\vec{R}} (H \otimes_D F)_0 \to (H \otimes_D \overline{H})_0$$

with the property $\theta_B((h_1 \otimes_D f_1) \otimes_{\vec{R}} (h_2 \otimes_D f_2)) = \delta(h_1,f_1,h_2,f_2)$.

Define $s(x) = \theta_B(x \otimes_{\vec{R}} x)$.

It is obvious that $(s(x),y)$ is a sufficient statistic for (A_0,ξ). $((A^{-1} \tilde{\otimes}_D B^{-1})_0(x \otimes_{\vec{R}} x) = A_0^{-1}(s(x)))$.

Note for $D \simeq \vec{R}$ $s(x)$ is the unnormed empirical variance defined in [6].

$s(x) \in P_D(H)$ since $(H \otimes_D F)_0$ is $\vec{R}$-isomorphic with $(L_D(F^*,H))_0$ ([2], chap. 2, § 4, No. 2) and a direct calculation shows that $s(x) = P_D(x)(B_0^{-1})$.

If $\dim_D(H) \leqq \dim_D(F)$ then the elements of $(H \otimes_D F)_0 \simeq (L_D(F^*,H))_0$ of full rank constitutes an open dense subset $((H \otimes_D F)_0)_r$. So $s(x) \in P_D(H)_r$ with probability 1 and s is a surjection from $((H \otimes_D F)_0)_r$ to $P_D(H)_r$.

A direct calculation shows that s commutes with the action of $GL_D(H)$ on $((H \otimes_D F)_0)_r$ and $P_D(H)_r$.

For $\theta \in P_D(H)_r$, θ' denotes the p.d.s.f with the property $\theta = \mathrm{Re}(\theta')$ (see 4.1).

Let the measure η on $P_D(H)_r$ be the image by s of λ_1. The facts that s commutes with the actions of $GL_D(H)$ gives that the measure ν on $P_D(H)_r$ defined by

$$d\nu(\theta) = \frac{1}{\left|a((B \tilde{\otimes}_D \theta')_0)\right|} d\eta(\theta), \qquad \theta \in P_D(H)_r$$

is the $GL_D(H)$-invariant measure on $P_D(H)_r$ (see [5] 6.2 amd 6.3).

The following argument gives the distribution of s(x)

$$d(s\mu_{(B \tilde{\otimes}_D A)_0, 0})(\theta) = \frac{1}{a((B \tilde{\otimes}_D A)_0)} e^{-\frac{1}{2}A_0^{-1}(\theta)} d(s\lambda_1)(\theta)$$

$$= \frac{1}{a((B \tilde{\otimes}_D A)_0)} e^{-\frac{1}{2}A_0^{-1}(\theta)} d\eta(\theta)$$

$$= \frac{a((B \tilde{\otimes}_D \theta')_0)}{a((B \tilde{\otimes}_D A)_0)} e^{-\frac{1}{2}A_0^{-1}(\theta)} d\nu(\theta), \qquad \theta \in P_D(H)_r.$$

The measure

$$dw(\theta) = \frac{a((B \tilde{\otimes}_D \theta')_0)}{a((B \tilde{\otimes}_D A)_0)} e^{-\frac{1}{2}A_0^{-1}(\theta)} d\nu(\theta)$$

is called the D-Wishart distribution on $P_D(H)_r$ with parameters (B,A) (compare [6]).

9.2. <u>Theorem</u>: The maximum likelihood estimator for (A_0,ξ) is

$$\left(\frac{s(x)}{\dim_{\bar{R}}(D)\ \dim_D(F \oplus_D F')}\ ,\ y\right).$$

<u>Proof</u>: The natural $\bar{R}$-linear surjection from $H_0 \otimes_{\bar{R}} H_0$ into $(H \otimes_D \bar{H})_0$ defined in an obvious way an $\bar{R}$-linear surjection from $L_{\bar{R}}((H \otimes_D (F \oplus_D F'))_0, H_0)$ $\otimes_{\bar{R}} L_{\bar{R}}((H \otimes_D (F \oplus_D F')), H_0)$ into $L_{\bar{R}}((H \otimes_D (F \oplus_D F'))_0 \otimes_{\bar{R}} (H \otimes_D (F \oplus F'))_0, (H \otimes_D \bar{H})_0)$.

This shows that we can extend the linearity argument in the proof from [6] section 5. Now a direct calculation, using the formulas in 6.2, gives that

$$\theta_{B \oplus B'}((B \oplus_D B') \tilde{\otimes}_D A) = \dim_{\acute{R}}(D) \dim_D(F \oplus_D F').$$

This finishes the proof.

10. Matrixformulation.

10.1. The case $D \simeq \acute{K}$.

Let 1,i,j,k be a basis for the $\acute{R}$-space D with the properties $i^2 = j^2 = k^2 = -1$, $ij=-ji=k$, $jk=-kj=i$, and $ki=-ik=j$. Every element $d \in D$ can be written uniquely as $d = p+qi+rj+sk$, $r,s,p,q \in \acute{R}$. $\bar{d} = p-qi-rj-sk$. We have $Re(d) = p$, $I(d) = q$, $J(d) = r$, and $K(d) = s$.

Let H,F,A, and B be as defined in 6.2 (and assume A and B definite). Formulas 11 and 12 in § 3 give ($d = i$ and $e = j$)

$$A(x,u) = A_0(x,u) + A_0(x,iu)i + A_0(x,ju)j + A_0(x,ku)k, \qquad x,u \in H^*$$

$$B(y,v) = B_0(y,v) + B_0(yi,v)\, i + B_0(yj,v)j + B_0(yk,v)k, \qquad y,v \in F^*.$$

I(A), J(A), K(A), I(B), J(B), and K(B) are all antisymmetric $\acute{R}$-bilineary forms.

$$(B \tilde{\otimes}_D A)_0(y \otimes_D x, v \otimes_D u) = B_0(y,v)A_0(x,u) + B_0(yi,v)A_0(x,iu)$$

$$+ B_0(yj,v)A_0(x,ju) + B_0(yk,v)A_0(x,ku), \quad y \otimes_D x, v \otimes_D u \in (F^* \otimes_D H^*)_0.$$

Let $f_1,\dots,f_n$ resp. $h_1,\dots h_m$ be a basis for F resp. H, such that $B = I_n$ (identity matrix of dimension n) with respect to $f_1^*,\dots,f_n^*$ (the dual basis to $f_1,\dots,f_n$). Since the $\mathbb{R}$-dual basis to $\mathbb{R}$-basis $h_1,\dots,h_m$, $h_1 i,\dots,h_m i$, $h_1 j,\dots,h_m j$, $h_1 k,\dots,h_m k$ for H_0 is $h_1^*,\dots,h_m^*$, $-ih_1^*,\dots,-ih_m^*$, $-jh_1^*,\dots,-jh_1^*$, $-kh_1^*,\dots,-kh_m^*$, and analog for F_0 only, we have $B_0 = I_{4n}$ and

$$A_0 = \begin{Bmatrix} Re(A) & -I(A) & -J(A) & -K(A) \\ +I(A) & Re(A) & -K(A) & +J(A) \\ +J(A) & +K(A) & Re(A) & -I(A) \\ +K(A) & -J(A) & +I(A) & Re(A) \end{Bmatrix} \quad \text{(A is the matrix of A with respect to } h_1^*,\dots,h_m^*)$$

with respect to the $\mathbb{R}$-basis above.

$$(B \tilde{\otimes}_D A)_0 = \begin{Bmatrix} \overbrace{A_0 & & & & 0 \\ & A_0 & & & \\ & & \ddots & & \\ & & & \ddots & \\ 0 & & & & A_0}^{\text{n-times}} \end{Bmatrix} \text{4nm columns and rows}$$

with respect to the $\mathbb{R}$-basis

$$
\begin{array}{cccccccc}
f_1^*\otimes_D h_1^*,\dots, & f_1^*\otimes_D h_m^*, & f_1^*\otimes_D -ih_1^*,\dots, & f_1^*\otimes_D -ih_m^*, & f_1^*\otimes_D -jh_1^*,\dots, & f_1^*\otimes_D -jh_m^*, & f_1^*\otimes_D -kh_1^*,\dots, & f_1^*\otimes_D -kh_m^* \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
f_n^*\otimes_D h_1^*,\dots, & f_n^*\otimes_D h_m^*, & f_n^*\otimes_D -ih_1^*,\dots, & f_n^*\otimes_D -ih_m^*, & f_n^*\otimes_D -jh_1^*,\dots, & f_n^*\otimes_D -jh_m^*, & f_n^*\otimes_D -kh_1^*,\dots, & f_n^*\otimes_D -kh_m^*.
\end{array}
$$

In the case $D \simeq C$ we get with an analogously notation $B_0 = I_{2n}$ and

$$
A_0 = \begin{Bmatrix} Re(A) & -I(A) \\ I(A) & Re(A) \end{Bmatrix} \qquad (B \tilde{\otimes}_D A)_0 = \underbrace{\begin{Bmatrix} A_0 & & & 0 \\ & A_0 & & \\ & & \ddots & \\ 0 & & & A_0 \end{Bmatrix}}_{n\text{-times}} \text{ 2 nm columns and rows}
$$

and for $D \simeq R$

$$
B_0 = B = I_n \qquad (B \tilde{\otimes}_D A) = \underbrace{\begin{Bmatrix} A & & & 0 \\ & A & & \\ & & \ddots & \\ 0 & & & A \end{Bmatrix}}_{n\text{-times}} \text{ nm columns and rows}
$$

$A_0 = A.$

11. Estimation in the case of a fixed basis.

We first take the case with meanvalue 0. ($n \leq m$).

$D \simeq \acute{K}$.

Let

$$
\begin{array}{cccccccccccc}
X^r_{11}, & \dots, & X^r_{m1}, & X^i_{11}, & \dots, & X^i_{m1}, & X^j_{11}, & \dots, & X^j_{m1}, & X^k_{11}, & \dots, & X^k_{m1} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
X^r_{1n}, & \dots, & X^r_{mn}, & X^i_{1n}, & \dots, & X^i_{mn}, & X^j_{1n}, & \dots, & X^j_{mn}, & X^k_{1n}, & \dots, & X^k_{mn}
\end{array}
$$

be the coordinates with respect to the $\acute{R}$-basis

$$
\begin{array}{cccccccccccc}
h_1 \otimes_D f_1, & \dots, & h_m \otimes_D f_1, & h_1 i \otimes_D f_1, & \dots, & h_m i \otimes_D f_1, & h_1 j \otimes_D f_1, & \dots, & h_m j \otimes_D f_1, & h_1 k \otimes_D f_1, & \dots, & h_m k \otimes_D f_1 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
h_1 \otimes_D f_n, & \dots, & h_m \otimes_D f_n, & h_1 i \otimes_D f_n, & \dots, & h_m i \otimes_D f_n, & h_1 j \otimes_D f_n, & \dots, & h_m j \otimes_D f_n, & h_1 k \otimes_D f_n, & \dots, & h_m k \otimes_D f_n
\end{array}
$$

in $(H \otimes_D F)_0$.

We set

$$
X^r = (X^r_{\mu\nu})_{\substack{\nu=1,\dots,n.\\ \mu=1,\dots,m.}}, \quad X^i = (X^i_{\mu\nu})_{\substack{\nu=1,\dots,n.\\ \mu=1,\dots,m.}}, \quad X^j = (X^j_{\mu\nu})_{\substack{\nu=1,\dots,n.\\ \mu=1,\dots,m.}}, \quad X^k = (X^k_{\mu\nu})_{\substack{\nu=1,\dots,n\\ \mu=1,\dots,m}}
$$

and the maximum likelihood estimator for A_0 is given by

$$
\hat{A}_0 = \frac{1}{4n}\left\{
\begin{array}{c|c|c|c}
\begin{matrix} X^r(X^r)'+X^i(X^i)'+ \\ X^j(X^j)'+X^k(X^k)' \end{matrix} &
\begin{matrix} X^k(X^j)'-X^j(X^k)'+ \\ X^i(X^r)'-X^r(X^i)' \end{matrix} &
\begin{matrix} X^i(X^k)'-X^k(X^i)'+ \\ X^j(X^r)'-X^r(X^j)' \end{matrix} &
\begin{matrix} X^j(X^i)'-X^i(X^j)'+ \\ X^k(X^r)'-X^r(X^k)' \end{matrix} \\
\hline
\begin{matrix} X^j(X^k)'-X^k(X^j)'+ \\ X^r(X^i)'-X^i(X^r)' \end{matrix} &
\begin{matrix} X^r(X^r)'+X^i(X^i)'+ \\ X^j(X^j)'+X^k(X^k)' \end{matrix} &
\begin{matrix} X^j(X^i)'-X^i(X^j)'+ \\ X^k(X^r)'-X^r(X^k)' \end{matrix} &
\begin{matrix} X^k(X^i)'-X^i(X^k)'+ \\ X^r(X^j)'-X^j(X^r)' \end{matrix} \\
\hline
\begin{matrix} X^k(X^i)'-X^i(X^k)'+ \\ X^r(X^j)'-X^j(X^r)' \end{matrix} &
\begin{matrix} X^i(X^j)'-X^j(X^i)'+ \\ X^r(X^k)'-X^k(X^r)' \end{matrix} &
\begin{matrix} X^r(X^r)'+X^i(X^i)'+ \\ X^j(X^j)'+X^k(X^k)' \end{matrix} &
\begin{matrix} X^k(X^j)'-X^j(X^k)'+ \\ X^i(X^r)'-X^r(X^i)' \end{matrix} \\
\hline
\begin{matrix} X^i(X^j)'-X^j(X^i)'+ \\ X^r(X^k)'-X^k(X^r)' \end{matrix} &
\begin{matrix} X^i(X^k)'-X^k(X^i)'+ \\ X^j(X^r)'-X^r(X^j)' \end{matrix} &
\begin{matrix} X^j(X^k)'-X^k(X^j)'+ \\ X^r(X^i)'-X^i(X^r)' \end{matrix} &
\begin{matrix} X^r(X^r)'+X^i(X^i)'+ \\ X^j(X^j)'+X^k(X^k)' \end{matrix}
\end{array}
\right\}
$$

In the case $D \simeq \check{C}$ we get with analog notation

$$\hat{A}_0 = \frac{1}{2n} \left\{ \begin{array}{c|c} X^r(X^r)' + X^i(X^i)' & X^i(X^r)' - X^r(X^i)' \\ \hline X^r(X^i)' - X^i(X^r)' & X^r(X^r)' + X^i(X^i)' \end{array} \right\}$$

and for $D \simeq \check{R}$

$$\hat{A}_0 = \frac{1}{n} X(X)'$$

The second estimation problem is well known (see [8], [10] and [11]) and the last one is a simpel canonical hypothesis in the sense of [6] (compare also [1]).

12. Group representations.

12.1. Definition: A representation of a group G on $\check{R}$-space E is a homomorphism π of the group G into the group GL(E).
E will be called the representations space for π and we write $E = E_\pi$.
The dimension of π is the dimension of E.
If π is a representation of G, E is in a natural way a left module over the algebra $\check{R}^{(G)}$ over R. ([3] § 13 № 1 Remarque)

12.2. Let π and ρ be two representations of G.

Definition: A representation-morphism from π to ρ is a $\check{R}$-linear operator f from E_π to E_ρ with the property

$$\forall\, g \in G : f \circ \pi(g) = \rho(g) \circ f.$$

$L(\pi,\rho)$ denotes the representations-morphism from π to ρ .
$L(\pi,\rho)$ is a $\check{R}$-space. In the case $\pi=\rho$ $L(\pi,\pi)$ will be an algebra over $\check{R}$.

12.3. $L(\pi,\rho)$ is the same as the $\acute{R}^{(G)}$-modul homomorphism from the $\acute{R}^{(G)}$-modul E_π to the $\acute{R}^{(G)}$-modul E_ρ.

This ensures the existence of the following ([3]):

Isomorphic representation. Sub- and quotient representation.

Kernel, cokernel, image, and coimage for a representation morphism.

Direct sum and product. Irreducible (simple), reducible (semi-simple), isotropic and disjoint representations.

Note that:

π and ρ disjoint is equivalent to $L(\pi,\rho) = 0$.

ρ isotrop is equivalent to ρ isomorphic with a direct sum of isomorphic irreducible representations. (Not necessarily in a unique way).

12.4. <u>Proposition</u>: Let π and ρ be two representations (of G). For $f \in L(\pi,\rho)$ and $f \neq 0$ we have:

π irreducible $\Rightarrow$ f injectiv.

ρ irreducible $\Rightarrow$ f surjectiv.

π and ρ irreducible $\Rightarrow$ f is a isomorphism.

The proof is trivial, see [3] § 4, No. 3, lemma 2.

12.5. From 12.4 it follows that two irreducible representations are either disjoint or isomorphic. Further if π is irreducible, $L(\pi,\pi)$ is a field (or a division algebra over $\acute{R}$). The field $L(\pi,\pi)$ is finite dimensional. From this it follows that only the three cases below can occur.

1) $L(\pi,\pi)$ is isomorphic with $\mathbb{R}$. $L(\pi,\pi) = \{\lambda 1_E \mid \lambda \in \mathbb{R}\}$.

2) $L(\pi,\pi)$ is isomorphic with $\mathbb{C}$. There exists $I \in L(\pi,\pi)$ with the property $I^2 = -1_E$, so $L(\pi,\pi) = \{\lambda 1_E + \mu I \mid \lambda, \mu \in \mathbb{R}\}$.

3) $L(\pi,\pi)$ is isomorphic with $\mathbb{K}$. There ixists $I,J,K \in L(\pi,\pi)$ with the properties $I^2 = J^2 = K^2 = -1_E$, $IJ = -JI = K$, $JK = -KJ = I$, and $KI = -IK = J$, so $L(\pi,\pi) = \{\lambda 1_E + \mu I + \nu J + \gamma K \mid \lambda,\mu,\nu,\gamma \in \mathbb{R}\}$.

We shall say that a irreducible representation π is of type D, if $L(\pi,\pi)$ is isomorphic with D.

12.6. Let π be a finite dimensional irreducible representation on the $\mathbb{R}$-space F_0. The mapping $(f,x) \to f(x)$ from $L(\pi,\pi) \times F_0$ into F_0 gives F_0 a D-space structure, $(D = L(\pi,\pi))$. Note that the restriction to the reals in $L(\pi,\pi)$ gives the original $\mathbb{R}$-space structure on F_0. Let H be a right D-space. The homomorphism $1_H \otimes_D \pi : G \to GL_{\mathbb{R}}((H \otimes_D F)_0)$ defined by $g \to 1_H \otimes_D \pi(g)$ gives a new representation on the $\mathbb{R}$-space $(H \otimes_D F)_0$. <u>The representation</u> $1_H \otimes_D \pi$ <u>is π-isotropic</u>. The mapping $(d,f) \to f \circ d$ from $D \times L(\pi, 1_H \otimes_D \pi)$ into $L(\pi, 1_H \otimes_D \pi)$ gives $L(\pi, 1_H \otimes_D \pi)$ a right D-space structure. The mapping $\alpha : H \to L(\pi, 1_H \otimes_D \pi)$ defined by $h \to (x \to h \otimes x)$ is the canonical D-space isomorphism between H and $L(\pi, 1_H \otimes_D \pi)$. If both H and H' are right D-spaces, the mapping $S : L_D(H,H') \to L(1_H \otimes_D \pi, 1_{H'} \otimes_D \pi)$ defined by $f \to f \otimes 1_F$ is a $\mathbb{R}$-space isomorphism. In the case $H = H'$ it is also an algebra isomorphism. On the other hand, let ρ be a π-isotrop representation on the $\mathbb{R}$-space E. Again $L(\pi,\rho)$ is a right D-space. The mapping $\beta : L(\pi,\rho) \otimes_D F \to E$ defined by $f \otimes x \to f(x)$ is the canonical representations isomorphism between $1_{L(\pi,\rho)} \otimes_D \pi$ and ρ. If both ρ and ρ' are π-isotrop, the mapping $T : L(\rho,\rho') \to L_D(L(\pi,\rho), L(\pi,\rho'))$ defined by $u \to (f \to u \circ f)$ is an $\check{R}$-space isomorphism. In the case $\rho = \rho'$, it is also an algebra isomorphism. For further details [3] § 1.

From this and [3] § 3, No. 4, prop. 9 follows:

12.7. Proposition: If ρ is a reducible representation of G on the $\check{R}$-space E then ρ is canonically isomorphic with a direct sum of disjoint isotopic representation and each isotrop representation is canonically isomprphic with a representation of the form $1_H \otimes_D \pi$ on the $\check{R}$-space $(H \otimes_D F)_0$. Here π is an irreducible representation on the $\check{R}$-space F_0 of type D. If ρ' is another reducible representation then $L(\rho,\rho')$ is $\check{R}$-isomorphic with a direct sum of $\check{R}$-space of the form $L_D(H,H')$, where H and H' comming from the isotropic parts of ρ and ρ' respectively.

12.8. From this we conclude that it is enough to look at representations of the form

$$E = \bigoplus_q (H_q \otimes_{D_q} F_q)_0$$

$$\rho = \bigoplus_q (1_{H_q} \otimes_{D_q} \pi_q)$$

where H_q is a right D_q-space, D_q isomorphic with $\check{R}$, $\check{C}$ or $\check{K}$, and π_q an irreducible representation of type D_q on the $\check{R}$-space $(F_q)_0$.

12.9. Let $E = \bigoplus_q \left(\bigoplus_{t \in T_q} E_{qt} \right)$

$$\rho = \bigoplus_q \left(\bigoplus_{t \in T_q} \pi_{qt} \right)$$

be a decomposition of ρ in irreducible parts, such that $\pi_{qt} \simeq \pi_{q't'} \Leftrightarrow q=q'$. Hence $\dim_D(H_q)$ = the number of elements in T_q.

12.10. <u>Definition</u>: Let π be a representation of G on the $\acute{R}$-space E. The dual of π is a representation π^* of G on the $\acute{R}$-space E^* defined by $(\pi^*(g)(x^*))(x) = x^*(\pi(g^{-1})x)$. $x \in E$, $x^* \in E^*$, $g \in G$.

12.11. <u>Remark</u>: π irreducible of type D $\Leftrightarrow$ π^* irreducible of type D.

13. <u>Representation-invariant positive forms</u>.

13.1. <u>Definition</u>: Let π be a representation of G on the $\acute{R}$-space E. A p.f. B on E is said to be π-invariant if

$$\forall\, g \in G,\ \forall\, x,y \in E:\ B(\pi(g)x,\ \pi(g)y) = B(x,y).$$

13.2. <u>Remark</u>: To ensure B π-invariant it is enough that $B(\pi(g)x, \pi(g)x) = B(x,x)$. B is π-invariant if and only if $f_B \in L(\pi,\pi^*)$.

13.3. <u>Lemma</u>: The nullspace of a π-invariant p.f. is π-invariant, i.e. defines a subrepresentation.

The proof is trivial.

13.4. Note that the subrepresentation defined by the nullspace may not have a complement. Therefore we shall usually assume reducibility of π. Also note that the existence of a π-invariant p.d.f. automatically gives reducibility of π.

13.5. <u>Proposition</u>: Let π be a representation. There exists a π-invariant p.d.f. if and only if $\pi(G) \subseteq GL(E_\pi)$ is releatively compact.

For proof, see [5] § 3, No. 1, prop. 1.

13.6. <u>Proposition</u>: Let π be an irreducible representation. Then either

a) the nullform is the only π-invariant p.f.

or

b) all π-invariant p.f. different from the nullform are definite and proportional.

<u>Proof</u>: a) and definiteness in b) follow from lemma 13.3.
To prove all π-invariante p.d.f.'s proportional let B_1 and B_2 be π-invariant p.d.f.. Choose a basis for E such that B_1 is the identity matrix and B_2 is a diagonal matrix. It is easy to see that there exists a $\lambda \neq 0$, such that $\lambda B_1 - B_2$ is positive but not definite. The result then follows from Lemma 13.3.

13.7. Let π be an irreducible representation on the R-space F_0. Hence π is then of type $L(\pi,\pi)$. If B_0 is a π-invariant p.d.f. on F_0, the adjoint mapping with respect to B_0 in $L(\pi,\pi)$ has all properties of a conjugation ([2] chap. III, § 2, No. 4, prop. 4) and therefore is the conjugation in the field $L(\pi,\pi)$. B_0 has the property (*) with respect to $L(\pi,\pi)$.

13.8. <u>Proposition</u>: Let π_1 and π_2 be two reducible disjoint representations on the R-spaces E_1 and E_2, respectively. Let B be a p.f. on the R-space $E_1 \oplus E_2$. Then B is $\pi_1 \oplus \pi_2$-invariant if and only if there exists π_1-invariant p.f. B_1 on E_1 and π_2-invariant p.f. B_2 on E_2 such that $B = B_1 \oplus B_2$.

Proof: "if" is trivial. "only if": The subrepresentation defined by the nullspace of B is a direct sum of two subrepresentations of π_1 and π_2 ([2] § 3, No. 4, prop. 4). Therefore we can restrict ourselves to the case B definite. Then E_2 is the orthogonal complement of E_1 with respect to B ([3] § 3, No. 4, prop. 9).

13.9. Proposition: Let $\rho = 1_H \otimes_D \pi$ be a π-isotropic representation of G on the $\vec{R}$-space $(H \otimes_D F)_0$. Here H is a right D-space and π is an irreducible representation of type D on the $\vec{R}$-space F_0. Let B be a p.f. on $(H \otimes_D F)_0$. B is ρ-invariant if and only if there exists a left p.s.f. A on H, such that $B = (A \tilde{\otimes}_D B_\pi)_0$, where B_π is a right p.s.f. on the left D-space F constructed from a π-invariant p.f. $(B_\pi)_0$ on F_0.

Proof: For every $h \in H$ the mapping $x \to B(h \otimes x, h \otimes x)$ is the diagonal part of a π-invariant p.f. on F_0. If the only π-invariant form is the trivial one, it follows that $B = 0$ and $B = (A \tilde{\otimes}_D 0)_0$, where A is a p.s.f. on H. Therefore assume $(B_\pi)_0$ be a non-zero (and therefore definite) π-invariant p.f. on F_0. 13.6. b) and the above remark shows that $\forall\, h \in H$, $\exists! Q(h) \geqq 0 : B(h \otimes x, h \otimes x) = Q(h)(B_\pi)_0(x,x) \;\forall\, x \in F$. $h \to Q(h)$ is the diagonal part of the p.f. $A_0 : (h_1,h_2) \to B(h_1 \otimes x, h_2 \otimes x)((B_\pi)_0(x,x))^{-1}$ on H_0. A_0 has the property (*), for $A_0(h_1 d, h_2) = B(h_1 d \otimes x, h_2 \otimes x)((B_\pi)_0(x,x))^{-1} = \|d\|^{-2} B(h_1 \otimes dx, h_2 \bar{d} \otimes dx)((B_\pi)_0(x,x))^{-1} = B(h_1 \otimes dx, h_2 \bar{d} \otimes dx)((B_\pi)_0(dx,dx))^{-1}$ $A_0(h_1, h_2 \bar{d})$. $x \in F, h_1, h_2 \in H_0$, $d \in D$, $x \neq 0$, $d \neq 0$. This finishes the proof.

13.10. Theorem: Let ρ be a reducible representation of G on the $\vec{R}$-space E. If we identify ρ with its canonical decomposition $\rho \simeq \bigoplus_{q \in Q} (1_{H_q} \otimes_{D_q} \pi_q)$ (12.6), then a p.f. B on E is ρ-invariant if and only if there for every

$q \in Q$ exists a left p.s.f. A_q on H_q, such that $B = \bigoplus_{q \in Q} (A_q \tilde{\otimes}_{D_q} B_{\pi_q})_0$, where B_{π_q} is a p.s.f. constructed from a π_q-invariant p.f. $(B_{\pi_q})_0$ on $(F_q)_0$.

Proof: 12.6, 12.7, 13.6, 13.8, and 13.9.

13.11. Corollary: If ρ is multiplicity free ($\dim_{D_q}(H_q) = 1$ for every $q \in Q$), then B is ρ-invariant if and only if for every $q \in Q$ there exists $a_q \geq 0$, such that $B = \bigoplus_{q \in Q} (a_q B_{\pi_q})_0$.

13.12. Proposition: Let ρ be a representation of G on the $\bar{R}$-space E. If there exists a ρ-invariant p.d.f. on E, then ρ is reducible ($\rho \simeq \bigoplus_{q \in Q} (1_{H_q} \otimes_{D_q} \pi_q)$) and for all $q \in Q$ there exists π_q-invariant p.d.f. $(B_{\pi_q})_0$ on $(F_q)_0$. The mapping $(A_q)_{q \in Q} \to B = \bigoplus_{q \in Q} (A_q \tilde{\otimes}_{D_q} B_{\pi_q})_0$ from the families of left p.s.f. on H_q to the ρ-invariant p.f. on $E \simeq \bigoplus_{q \in Q} (H_q \otimes_{D_q} F_q)_0$ is one to one. The nullspace of B is $\bigoplus_{q \in Q} (N_q \otimes_{D_q} F_q)_0$, where N_q is the nullspace of A_q. Note that B is definite if and only if A_q is definite for every $q \in Q$.

The proof is trivial.

13.13. Corollary: Let the situation be as in 13.10. Define $P = \{q \in Q \mid \pi_q$ has only a trivial π_q-invariant p.f.$\}$. The nullspace of B is then $(\bigoplus_{q \in P} (H_q \otimes_{D_q} F_q)_0) \oplus (\bigoplus_{q \in Q/P} (N_q \otimes_{D_q} F_q)_0)$, where N_q is the nullspace of A_q, $q \in Q/P$.

The proof is trivial.

14. Symmetry hypothesis in the mean and the variance.

In this section we shall show that all symmetry hypotheses are general canonical (§ 8) and describe their form. Further we shall investigate the relationships between symmetry hypotheses, canonical hypotheses ([6]) and the general canonical hypotheses defined in § 5.

Let E be a $\vec{R}$-space and π a representation on E.

14.1. Theorem: Let a hypothesis in the mean-value $\xi \in E$ and the variance $B \in P(E)$ be given by $\pi(g)(\xi) = \xi$ and B regular and π^*-invariant. If there exists π^*(or π)-invariant p.d.f. (else the model is empty) we have two possibilities:

a) π^*(or π) do not have the identity representation as a subrepresentation. This gives the general canonical hypothesis with $F_i' = 0$ for all $i \in I$. (see 8.1).

b) π^*(or π) have the identity representation as a subrepresentation. This gives the canonical hypothesis with $F_i' = 0$ for all $i \in I$ except one $i_0 \in I$, where $F_{i_0} = 0$.

Proof: The remarks about the mean-value is trivial and the rest follows from 13.12.

14.2. In the case of regular estimation ($\dim F_i \leqq \dim H_i, \forall\, i \in I$) the multiplicity of the identity representation in b) above is 1. ($\dim_{\vec{R}} H_{i_0} = 1$). Since the identity representation has dimension 1 and is of type R, the mean-value must belong to a one dimensional subspace in E!

14.3. There exist canonical hypotheses not given by symmetry.

There exist symmetry hypotheses which are not canonical.

All canonical - and all symmetry hypotheses are in the class of general canonical hypotheses.

There exist general canonical hypotheses not canonical and not given by symmetry. In a general canonical hypothesis the maximum likelihood estimator for the mean and the variance is sufficient and its distribution can be obtained (§ 9). A symmetry hypothesis is canonical ([6]) if and only if all irreducible subrepresentations in the symmetry not of type $\acute{R}$ are multiplicity free. (13.11)

Could a general canonical hypothesis be described by a symmetry? This is easily seen to be a question of existence of a group with a finite family of inequivalent irreducible representations with prescribed dimensions and types.

The above seems to be the road to the development of a theory about general canonical subhypotheses.

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