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Section 1. Introduction:

In this paper, a probability inequality for sums of bounded symmetric random variables is obtained. This inequality sharpens and extends a previous result due to Eaton (1970) and it is assumed the reader is familiar with that paper.

Let $Y_1, \ldots, Y_n$ be independent symmetric random variables such that $|Y_i| \leqq 1$.

For real numbers $\theta_1, \ldots, \theta_n$ with $\sum_1^n \theta_i^2 = 1$, let $T_n(\theta) = \sum_1^n \theta_i Y_i$.

Theorem 1.1:

Let $\varphi(x) = 1/\sqrt{2\pi}\, e^{-\frac{1}{2}x^2}$ and suppose $\alpha > \sqrt{3}$. Then

$$P\{T_n(\theta) \geqq \alpha\} \leqq \frac{2e^{3/2}}{9} \frac{\varphi(\alpha)}{\alpha} W\left(1 - \frac{3}{\alpha^2}\right) \qquad (1.1)$$

where

$$W(x) = \frac{e^{\frac{3}{2}x}}{x^4}\left(1 + 5/2\,\frac{(1-x)}{x^2}\right), \quad 0 < x \leqq 1. \qquad (1.2).$$

Further, W is a strictly decreasing function on (0,1).

The proof of this result is given in Section 2. The method of proof is use the inequality

$$P(T_n(\theta) \geqq \alpha) \leqq \tfrac{1}{2} \int_{-\infty}^{\infty} f(x)\,\varphi(x)\,dx \qquad (1.3)$$

established in Corollary 2 of Eaton (1970).

The function f ranges over a class of functions described in Eaton (1970). Basically, the inequality (1.1) is derived by choosing f in (1.3) to be the function

$$f_\alpha^*(x) = \frac{(|x| - (\alpha - 3/\alpha))_+^3}{(3/\alpha)^3}, \quad \alpha > \sqrt{3}. \qquad (1.4).$$

Here $(y)_+ = y$ if $y \geqq 0$ and $(y)_+ = 0$ if $y < 0$.
However, the method of arriving at f_α^* is of interest and we will indicate how f_α^* arises.
To derive a "good" probability inequality from (1.3), one would like to minimize the right hand side of (1.3) over all f for which (1.3) is valid. But the class of functions in (1.3) is rather unwieldy and one is lead (for reasons given in the remark below) to consider the class F_α of functions defined as follows. For $v \in R$, let $(v)_+ = v$ if $v \geqq 0$ and $(v)_+ = 0$ if $v < 0$. Consider $\alpha > 0$ and let F_α be those functions $f : R \to (0,\infty)$ given by

$$f(x) = \int_0^\infty (|x| - u)_+^3 \, dF(u) \qquad (1.5)$$

where F is a non-decreasing function on $(0,\infty)$ such that

$$\int_0^\infty (\alpha - u)_+^3 \, dF(u) = 1.$$

For $f \in F_\alpha$, let

$$H(f) = \tfrac{1}{2} \int_{-\infty}^\infty f(x)\, \varphi(x)\, dx \qquad (1.6).$$

Proposition 2.1 shows that

$$\inf_{f \in F'_\alpha} H(f) = \inf_{0 \leqq u < \alpha} \int_u^\infty f_u(x)\, \varphi(x)\, dx \qquad (1.7)$$

where, for $0 \leqq u < \alpha$,

$$f_u(x) = \frac{(|x| - u)_+^3}{(\alpha - u)^3} \qquad (1.8).$$

Verifying the conditions of Corollary (2) in Eaton (1970) for f_u, we have

$$P\{T_n(\theta) \geqq \alpha\} \leqq \inf_{0 \leqq u < \alpha} \int_u^\infty f_u(x)\varphi(x)\, dx \qquad (1.9).$$

Minimizing (approximately) the right hand side of (1.9), we find that $u = \alpha - 3/\alpha$ for $\alpha > \sqrt{3}$ yields the approximate minimum and this gives the function f_α^* in (1.4).

An immediate corollary of Theorem 1.1 is

<u>Corollary 1.1</u>: If $\alpha \geqq \alpha_0 > \sqrt{3}$, then there exists a constant $K = K(\alpha_0)$ $= \frac{2\, e^{3/2}}{9} W\left(1 - \frac{3}{\alpha_0^2}\right)$ such that

$$P\{T_n(\theta) \geqq \alpha\} \leqq K \frac{\varphi(\alpha)}{\alpha} \qquad \text{for all } \alpha \geqq \alpha_0 \qquad (1.10)$$

<u>Remark</u>: Suppose $f : R \to (0,\infty)$ is a symmetric function with a derivative f'

which satisfies

$$\frac{1}{t}\ [f'(t+\Delta) - f'(-t+\Delta)] \qquad (1.11)$$

is non-decreasing in $t > 0$ for each $\Delta \geqq 0$. Then $f \in \acute{F}$ where $\acute{F}$ is defined in Eaton (1970).

Lemma 1.1: If $f : R \to (0,\infty)$ is symmetric, f''' exists, and if $f'''(x)$ is non-decreasing for $x \geqq 0$, then f satisfies (1.11).

Proof: For $t > 0$ and $\Delta \geqq 0$

$$f'''(t+\Delta) - f'''(-t+\Delta) \geqq 0 \qquad (1.12)$$

so

$$t[f'''(t+\Delta) - f'''(-t+\Delta)] + f''(t+\Delta) + f''(-t+\Delta)$$

$$\geqq f''(t+\Delta) + f''(-t+\Delta). \qquad (1.13).$$

Hence

$$\frac{d}{dt}\,[t(f''(t+\Delta) + f''(-t+\Delta)]$$

$$\geqq \frac{d}{dt}\,[f'(t+\Delta) - f'(-t+\Delta)] \qquad (1.14)\ .$$

Therefore

$$t[f''(t+\Delta) + f''(-t+\Delta)]$$

$$\geqq f'(t+\Delta) - f'(-t+\Delta) \qquad (1.15).$$

But

$$\frac{d}{dt}\left[\frac{f'(t+\Delta) - f'(-t+\Delta)}{t}\right] =$$

$$\frac{t\,[f''(t+\Delta) + f''(-t+\Delta)] - [f'(t+\Delta) - f'(-t+\Delta)]}{t^2} \geqq 0 \qquad (1.16)$$

by (1.15).//

Now, assume f satisfies the conditions of Lemma 1.1 and $f^{(j)}(0) = 0$ for $j = 0,1,2,3$. From the integral form of Taylor's Theorem,

$$f(x) = \int_0^x \frac{(x-u)_+^3}{3!}\, dF(u) \qquad , x \geqq 0 \qquad (1.17)$$

where F is a non-decreasing function. The condition $\int_0^\infty \frac{(\alpha - u)_+^3}{3!}\, dF(u) = 1$ is simply a normalization $(f(\alpha) = 1)$ so Corollary 2 in Eaton (1970) can be applied. It is the above considerations which lead to $\bar{F}_\alpha$.

Section 2: Proof of Theorem 1.1:

For $f \in \mathcal{F}_\alpha$, let

$$H(f) = \int_0^\infty f(x)\, d\Phi(x) = \tfrac{1}{2} \int_{-\infty}^\infty f(x)\varphi(x)\, dx \qquad (2.1)$$

where $\Phi(x)$ is the cumulative distribution function of a $N(0,1)$.

Proposition 2.1: For $\alpha > 0$,

$$\inf_{f \in \mathcal{F}_\alpha} H(f) = \inf_{0 \leq u < \alpha} \int_0^\infty f_u(x)\, d\Phi(x) \qquad (2.2)$$

where $0 \leq u < \alpha$ and f_u is given by (1.8).

Proof: For $f \in \mathcal{F}_\alpha$,

$$H(f) = \int_0^\infty f(x)\, d\Phi(x) = \int_0^\infty \int_0^x (x-u)_+^3\, dF(u) d\Phi(x)$$

$$= \int_0^\infty \int_u^\infty (x-u)_+^3\, d\Phi(x) dF(u) = \int_0^\infty w(u)\, dF(u) \qquad (2.3)$$

where $w(u) = \int_u^\infty (x-u)_+^3\, d\Phi(x)$

and F satisfies the side condition

$$\int_0^\infty (\alpha - u)_+^3 \, dF(u) = 1 \qquad (2.4).$$

Clearly, the infimum over F will occur for F with all its increase in the interval $(0,\alpha)$.

Let $dG(u) = (\alpha - u)_+^3 \; dF(u)$ so (2.4) becomes

$$\int_0^\alpha dG(u) = 1 \qquad (2.5).$$

Thus

$$\inf_{f \in \not{F}_\alpha} H(f) = \inf_G \int_0^\alpha \frac{w(u)}{(\alpha - u)^3} \, dG(u) \leqq \inf_{0 \leqq u < \alpha} \frac{w(u)}{(\alpha - u)^3} \qquad (2.6)$$

with equality for the G which puts mass 1 at a minimum of $w(u)/(\alpha - u)^3$.

Thus

$$\inf_{f \in \not{F}_\alpha} H(f) = \inf_{0 \leqq u < \alpha} \int_u^\infty \frac{(x - u)^3}{(\alpha - u)^3} \, d\Phi(u) \qquad (2.7).$$

This completes the proof. //

Proposition 2.2: For f_u defined by (1.8),

$$P\{T_n(\theta) \geqq \alpha\} \leqq \int_u^\infty f_u(x) \, d\Phi(x) \qquad (2.8).$$

Proof: It is straight forward to check that f_u satisfies the assumptions of Corollary 2 in Eaton (1970).//

Now, to minimize (approximately) $\int_u^\infty f_u(x)d\Phi(x)$, first note that

$$\frac{1}{3!}\int_u^\infty (x-u)^3 d\Phi(x) = -\left(\frac{u^3}{6}+\frac{u}{2}\right)\Phi(-u) + \left(\frac{u^2}{6}+\frac{1}{3}\right)\varphi(u). \qquad (2.9).$$

Equation (2.9) follows from Chernoff and Ray (1965, equation 4.9).

Also, for u > 0 (Feller (1950), problem 1,p.179)

$$\Phi(-u) \geqq \varphi(u)\left[\frac{1}{u}-\frac{1}{u^3}+\frac{3}{u^5}-\frac{15}{u^7}\right] \qquad (2.10).$$

Using (2.11) in (2.10), we have the inequality

$$\frac{1}{6}\int_u^\infty (x-u)^3\, d\Phi(x) \leqq$$

$$\varphi(u)\left[\frac{u^2}{6}+\frac{1}{3}+\left(\frac{u^3}{6}+\frac{u}{2}\right)\left(-\frac{1}{u}+\frac{1}{u^3}-\frac{3}{u^5}+\frac{15}{u^7}\right)\right]$$

$$= \varphi(u)\left[\frac{1}{u^4}+\frac{15}{2u^6}\right] \qquad (2.11).$$

Thus, we have

$$\int_u^\infty \frac{(x-u)^3}{(\alpha-u)^3}\, d\Phi(u) \leqq \frac{6\varphi(u)}{(\alpha-u)^3}\left[\frac{1}{u^4}+\frac{15}{2u^6}\right] \qquad (2.12),$$

and therefore

$$P\{T_n(\theta) \geqq \alpha\} \leqq \inf_{0 \leqq u \leqq \alpha} \frac{6\varphi(u)}{(\alpha - u)^3}\left[\frac{1}{u^4} + \frac{15}{2u^6}\right] \qquad (2.13).$$

The approximate infimum of the right hand side of (2.13) (for α large) is achieved by setting $u = \alpha - \frac{3}{\alpha}$ for $\alpha > \sqrt{3}$.

Thus

$$P\{T_n(\theta) \geqq \alpha\} \leqq \frac{6\varphi(\alpha - 3/\alpha)}{\frac{3^3}{\alpha^3}(\alpha - 3/\alpha)^4}\left[1 + \frac{15\alpha^2}{2(\alpha^2 - 3)^2}\right]$$

$$= \frac{6\varphi(\alpha)\, e^{\frac{3}{2}}\, e^{\frac{3}{2}(1 - \frac{3}{\alpha^2})}}{3^3\alpha(1 - 3/\alpha^2)^4}\left[1 + \frac{15/\alpha^2}{2(1 - 3/\alpha^2)^2}\right]$$

$$= \frac{e^{3/2}}{3^3}\ \frac{6\varphi(\alpha)}{\alpha}\ W(1 - 3/\alpha^2) \qquad (2.14)$$

where

$$W(x) = \frac{e^{\frac{3}{2}x}}{x^4}\left[1 + 5/2\,\frac{(1 - x)}{x^2}\right], 0 < x \leqq 1. \qquad (2.15).$$

It is not hard to show that $W'(x) < 0$ for $0 < x \leqq 1$ so W is strictly decreasing on (0,1). Clearly $W(1) = e^{3/2}$. This completes the proof of Theorem 1.1.

Remarks: Two questions concerning the above inequality remain open. First, can the function W be improved to yield a better inequality? It is the author's opinion that the techniques used in this paper will not yield a better function W.

Second, can the inequality (1.1) be extended to random variables which have mean 0 (rather than being symmetric)? The author has had no success in attempting to generalize the argument above to the non-symmetric case.

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