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Introduction.

This paper gives a construction of branching processes with discrete time and a continuous number of types. The approach differs from Moyal [3] or Harris [2] by leaning on the theory of integration in locally compact spaces, as treated in Bourbaki [1].

The types of the process will be assumed to be the points in a locally compact space X. On basis of the topology on X, a locally compact topology is constructed on Ω_X, the states of the process or the point-distributions (section 1). Section 2 deals with integration and harmonic analysis (i.e. generating functions) on Ω_X. In section 3, the transition operator is constructed; and finally, in section 4, a few of the classical results, concerning the form of the generating functions and the probabilities of extinction, are established in the given set-up.

1. The topology on the state space.

We start by recalling some algebraic and topological concepts. A monoid is a set with an associative composition $(x,y) \to xy$ with a neutral element. If E,F are monoids with neutral elements i_E, i_F, a homomorphism from E to F is a map $f : E \to F$ satisfying $f(xy) = f(x)f(y)$ for all $x,y \in E$ and $f(i_E) = i_F$. A topological monoid is a monoid equipped with a topology in which the composition is continuous. We shall deal only with commutative monoids; for convenience we use the abbreviation CTM for commutative topological monoid. Important examples of CTM's are N_0, the additive monoid of non-negative integers in the discrete topology, and Δ, the multiplicative monoid of complex numbers with modulus $\leqq 1$ in the usual topology.

Let X be a set; X will later be the types of the process. We let Ω_X denote the set of all maps $\omega : X \to N_0$ with finite support supp ω (the "point distributions"); Ω_X is a monoid with pointwise addition. For $\omega \in \Omega_X$ $\omega(X)$ denotes the total mass of ω, $\omega(X) = \sum_{x \in X} \omega(x)$. The imbedding $\varepsilon : X \to \Omega_X$ is defined the obvious way, that is, $\varepsilon(x)$ is the point distribution, which is 1 in x and 0 otherwise. For $n \geqq 1$, $r_n : X^n \to \Omega_X$ is defined by $r_n(x_1,\ldots,x_n) = \varepsilon(x_1) + \ldots + \varepsilon(x_n)$ and r_0 is defined as the map from an (arbitrary) one-point set $\{a\}$ to Ω_X with $r_0(a) = 0$.

Suppose, now, that X is a (Hausdorff) topological space. We shall then equip Ω_X with the final topology defined by $(r_n)_{n \in N_0}$.

1.1. Lemma. The topology on Ω_X has the following properties:

(1) For each $n \in N_0$, $\{\omega \in \Omega_X | \omega(X) = n\}$ is closed and open; in particular, 0 is isolated point.

(2) Let $\omega \in \Omega_X \setminus \{0\}$, $\omega = \varepsilon(x_1) + \ldots + \varepsilon(x_n)$. The sets $\varepsilon(V_1) + \ldots + \varepsilon(V_n)$, where V_i is a neighbourhood of x_i in X, form a basis for the system of neighbourhoods of ω in Ω_X.

(3) The addition $(\omega_1, \omega_2) \to \omega_1 + \omega_2$ is continuous, i.e. Ω_X is a CTM.

(4) $\varepsilon : X \to \Omega_X$ is a homeomorphism $X \to \varepsilon(X)$.

If X is locally compact, furthermore:

(5) The sets

$$\{\omega \in \Omega_X | \omega(X) \leqq N, \text{ supp } \omega \subseteq K\},$$

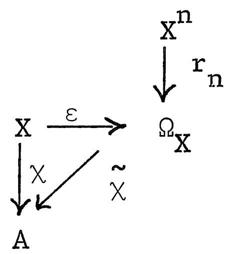
where $N \in N_0$ and $K \subseteq X$ is compact, form a basis for the compacts in Ω_X in the sense, that each such set is compact and that each compact in Ω_X is contained in such a set.

(6) Ω_X is locally compact and σ-compact if X is so.

Proof. Each of the assertions follows rather easily from the preceding ones or the definition and we omit the details.□

1.2. Proposition. Let $\chi : X \to A$, where A is a commutative monoid.

(1) χ has an unique extension to a homomorphism $\tilde{\chi} : \Omega_X \to A$, given by $\tilde{\chi}(0) = 0_A$ and $\tilde{\chi}(\varepsilon(x_1) + \ldots + \varepsilon(x_n)) = \chi(x_1) + \ldots + \chi(x_n)$ (*)



(2) If A is a CTM and χ continuous, $\tilde{\chi}$ is continuous.

Proof. It is obvious that $\tilde{\chi}$ is uniquely determined and well-defined by (*) and a homomorphism. Under the assumptions of (2), the continuity of $\tilde{\chi}$ follows from the fact that $\tilde{\chi} \circ r_n$, being the map $(x_1, \ldots, x_n) \to \chi(x_1) + \ldots + \chi(x_n)$, is continuous for each n.□

Expressed in another way:

Ω_X is the free CTM over X.

2. Integration and harmonic analysis on Ω_X.

In the following, X (and therefore Ω_X) is assumed to be locally compact. Let Z be a locally compact space (here X or Ω_X). Following mainly Bourbaki [1], we use the following notation for spaces of continuous, complex-valued functions on Z:

$f \in K(Z) \iff$ f has compact support

$f \in C_0(Z) \iff$ f vanishes at infinity

$f \in C^b(Z) \iff$ f is bounded

$f \in C(Z,\Delta) \iff f(z) \in \Delta$ for all $z \in Z$.

We shall consider K(Z), $C_0(Z)$ and $C^b(Z)$ in the norm topology and $C(Z,\Delta)$ in the compact open topology (the topology for uniform convergence on compacts).

$M^1(Z)$ denotes the set of bounded measures on Z, i.e. the dual of K(Z), $M^1_+(Z)$ the set of positive $\mu \in M^1(Z)$ and $P(Z) = \{\mu \in M^1_+(Z) \mid \mu(1) = 1\}$ the set of probability measures on Z. We shall consider $M^1(Z)$, $M^1_+(Z)$ and P(Z) in the C^b-topology, defined by the linear functionals $\mu \to \mu(f)$, $f \in C^b(Z)$.

For $z \in Z$, δ_z denotes Dirac-measure in z; one shows easily that δ is a homeomorphism $Z \to \delta(Z)$. A discrete measure is a measure of the form $\sum_{i=1}^{n} \alpha_i \delta_{z_i}$.

2.1. Lemma. The positive discrete measures are dense in $M^1_+(Z)$ and the discrete measures in $M^1(Z)$.

Proof. By the bipolar theorem, using that $C^b(Z)$ is the dual of $M^1(Z)$. □

Occassionally we shall use the norm topology on $M^1(Z)$ in which $M^1(Z)$ is a Banach space. For $Z = \Omega_X$, $M^1(\Omega_X)$ becomes a commutative Banach algebra with convolution

$$\mu * \nu(f) = \int f(\omega_1 + \omega_2)\, d\mu(\omega_1)\, d\nu(\omega_2),$$

as multiplication; Dirac-measure in 0 is neutral element by convolution.

Important properties of convolution are

(a) $\delta_{\omega_1} * \delta_{\omega_2} = \delta_{\omega_1 + \omega_2}$

(b) $f(\omega_1 + \omega_2) = f(\omega_1)f(\omega_2) \Rightarrow \mu * \nu(f) = \mu(f)\nu(f)$.

Let E be a CTM. The <u>dual</u> CTM $\hat{E}$ of E is defined as the set of continuous homomorphisms $E \to \Delta$ (<u>semicharacters</u>), with pointwise multiplication and the compact open topology. If E is a group, then $|\chi(x)| = 1$ for all $\chi \in \hat{E}$ and $x \in E$, i.e., the dual CTM of E is the dual group of E.

Proposition 1.2 sets up a one-to-one correspondance between $\chi \in C(X,\Delta)$ and $\tilde{\chi} \in \widehat{\Omega_X}$. Furthermore:

2.2. <u>Proposition</u>. The identification between $\widehat{\Omega_X}$ and $C(X,\Delta)$ is a homeomorphism in the compact open topologies.

<u>Proof</u>. Clearly $\tilde{\chi}_i \to \tilde{\chi}$ in $\widehat{\Omega_x}$ implies $\chi_i \to \chi$ in $C(X,\Delta)$. To show the converse, it suffices by lemma 1.1, (5), to show that $\tilde{\chi}_i \to \tilde{\chi}$, uniformly on each of the sets $\{\omega \in \Omega_X | \omega(X) \leq N,\ \mathrm{supp}\ \omega \subseteq K\}$, where $K \subseteq X$ is compact and $N \in N_0$, and to show this it suffices to show that the convergence is uniform on each of the sets

$$A_n = \{\omega \in \Omega_X | \omega(X) = n,\ \mathrm{supp}\ \omega \subseteq K\}, \quad n = 0, 1, \ldots, N.$$

This is easily obtained by induction in n, using that each $\omega \in A_n$ can be written $\omega = \omega_1 + \varepsilon(x)$, $\omega_1 \in A_{n-1}$, $x \in K$, giving

$$|\tilde{\chi}_i(\omega) - \tilde{\chi}(\omega)| = |\tilde{\chi}_i(\omega_1)\chi_i(x) - \tilde{\chi}(\omega_1)\chi(x)| \leq$$

$$|\chi_i(\omega_1)(\chi_i(x) - \chi(x))| + |(\chi_i(\omega_1) - \tilde{\chi}(\omega_1))\chi(x)| \leq$$

$$|\chi_i(x) - \chi(x)| + |\tilde{\chi}_i(\omega_1) - \tilde{\chi}(\omega_1)|. \square$$

$\widehat{\Omega_X} = C(X,\Delta)$ is thus not in general locally compact (cf the situation for a group).

2.3. <u>Definition</u>. For $\mu \in M^1(\Omega_X)$, the <u>generating function</u> $F\mu : C(X,\Delta) \to C$ is defined by

$$F\mu(\chi) = \tilde{\mu}(\chi) = \int \tilde{\chi}(\omega)\, d\mu(\omega), \quad \chi \in C(X,\Delta).$$

The definition is analogous with that of Moyal [3] . From the remarks above it is seen that the generating function is a generalization of the Fourier transform on a group. Also, for $X = \{1,\dots,k\}$, where $C(X,\Delta) = \Delta^k$, $F\mu$ for $\mu \in P(\Omega_X)$ is the classical probability generating function.

Some of the main properties of the generating function will be given in proposition 2.5. For the proof, we prepare the following

2.4. <u>Lemma</u>. Let B be the set of real $\chi \in C(X,\Delta)$ with $||\chi|| < 1$, which vanishes at infinity, and A the real vectorspace spanned by $\tilde{B}$. A is contained in $C_0(\Omega_X)$ and is dense in the space of real $f \in C_0(\Omega_X)$.

<u>Proof</u>. Let $\chi \in B$ and $\varepsilon > 0$. We choose a compact $K \subseteq X$ with $|\chi(x)| \leqq \varepsilon$ $x \notin K$, and $N \in N_0$ with $||\chi||^N \leqq \varepsilon$. If $\omega(X) > N$, then obviously $|\tilde{\chi}(\omega)| \leqq \varepsilon$ If supp $\omega \not\subseteq K$, we write $\omega = \omega_1 + \varepsilon(x)$, $x \notin K$, and then $|\tilde{\chi}(\omega)| = |\tilde{\chi}(\omega_1)\chi(x)| \leqq \varepsilon$. This shows that $|\tilde{\chi}(\omega)| \leqq \varepsilon$ for all ω outside the compact $\{\omega \in \Omega_X | \omega(X) \leqq N,\ \text{supp}\ \omega \subseteq K\}$ and we conclude that $\tilde{\chi} \in C_0(\Omega_X)$ and that $A \subseteq C_0(\Omega_X)$.

Now A is formed of all finite linear combinations $\Sigma\alpha_i\tilde{\chi}_i$, $\alpha_i \in R$, $\chi_i \in B$. Since for $\chi_1,\chi_2 \in B$, we have $\chi_1\chi_2 \in B$ and $\widetilde{\chi_1\chi_2} = \tilde{\chi}_1\tilde{\chi}_2$, A is closed under multiplication, that is, an algebra. That A is dense is now an easy consequense of the Stone-Weierstrass theorem in its locally compact form.□

2.5. <u>Proposition</u>.

(1) F is an isomorphism of the algebra $M^1(\Omega_X)$ onto an algebra of continuous functions on $C(X,\Delta)$.

For positive measures, furthermore:

(2) $(\mu,\chi) \to F\mu(\chi)$ is continuous.

(3) $\mu_i \to \mu$ iff $F\mu_i(\chi) \to F\mu(\chi)$ for all $\chi \in C(X,\Delta)$.

Proof. F, obviously being linear, is a homomorphism by (b) p. 4. Suppose $F\mu = F\nu$ and let $f \in K(\Omega_X)$ be real. By lemma 2.4, there exists a sequence $f_n \in A$ with $||f - f_n|| \to 0$. Since $\mu(g) = \nu(g)$ for all $g \in A$, $\mu(f) = \nu(f)$; hence F is one-to-one. The continuity of $F\mu$ will follow from (2), since μ without loss of generality can be assumed to be positive.

To prove (2), suppose $(\mu_i, X_i) \to (\mu, X)$, where μ and the μ_i's are positive. Now

$$|F\mu_i(X_i) - F\mu(X)| \leq |F\mu_i(X_i) - F\mu_i(X)| + |F\mu_i(X) - F\mu(X)|$$

and since $\mu_i \to \mu$, the second term approaches 0. Let $\varepsilon > 0$. We choose $f : \Omega_X \to [0,1]$ with compact support K and $\mu(f) > \mu(1) - \varepsilon$. For sufficiently large i $|\mu_i(1) - \mu(1)| < \varepsilon$, $|\mu_i(f) - \mu(f)| < \varepsilon$ and $|\tilde{X}_i(\omega) - \tilde{X}(\omega)| < \varepsilon/(\mu(1) + \varepsilon)$ for $\omega \in K$. Then $\mu_i(1-f) < 3\varepsilon$ and thus

$$\mu_i(K) \leq \mu_i(1) < \mu(1) + \varepsilon$$

and

$$\mu_i(\Omega_X \setminus K) = \mu_i(1 - 1_K) \leq \mu_i(1-f) \leq 3\varepsilon$$

such that

$$|F\mu_i(X_i) - F\mu_i(X)| \leq \int |\tilde{X}_i(\omega) - \tilde{X}(\omega)| \, d\mu_i(\omega)$$

$$= \int_K + \int_{\Omega_X \setminus K} |\tilde{X}_i(\omega) - \tilde{X}(\omega)| \, d\mu_i(\omega) \leq \frac{\varepsilon}{\mu(1) + \varepsilon} \mu_i(K) + 2\mu_i(\Omega_X \setminus K) < \varepsilon + 6\varepsilon$$

and we conclude that $F\mu_i(X_i) \to F\mu(X)$.

To prove (3), suppose $F\mu_i(X) \to F\mu(X)$ for all $X \in C(X, \Delta)$, where μ and the μ_i's are positive. Then $\mu_i(1) \to \mu(1)$ and, in the notation of lemma 2.4 $\mu_i(h) \to \mu(h)$ for all $h \in A$. Let $g \in C^b(\Omega_X)$ be real and $\varepsilon > 0$. There exists $X \in B$ such that X is positive and $\mu(\tilde{X}) > \mu(1) - \varepsilon$. Now $\tilde{X}g$ is real and vanishes at infinity and by lemma 2.4 we can find a $h \in A$ with $||\tilde{X}g - h|| < \varepsilon$. For sufficiently large i

$$|\mu_i(1) - \mu(1)| < \varepsilon, \quad |\mu_i(\tilde{X}) - \mu(\tilde{X})| < \varepsilon \text{ and } |\mu_i(h) - \mu(h)| < \varepsilon .$$

Then

$$|\mu_i(g-h)| \leqq |\mu_i(\tilde{X}g-h)| + |\mu_i((1-\tilde{X})g)| \leqq$$

$$\varepsilon\cdot\mu_i(1) + ||g|| \mu_i(1-\tilde{X}) \leqq \varepsilon(\mu(1) + \varepsilon) + ||g||\cdot 3\varepsilon.$$

Similarly, $|\mu(g-h)| \leqq \varepsilon\mu(1) + ||g||\varepsilon$ and thus

$$|\mu_i(g) - \mu(g)| \leqq |\mu_i(g-h)| + |\mu_i(h) - \mu(h)| + |\mu(h-g)|$$

$$\leqq \varepsilon(\mu(1) + \varepsilon + 3||g|| + 1 + \mu(1) + ||g||),$$

proving that $\mu_i(g) \to \mu(g)$. □

2.6. Corollary. Convolution of positive measures is continuous.

2.7. Corollary. $P(\Omega_X)$ is a CTM with convolution as composition.

3. The transition operator.

Let $T : X \to P(\Omega_X)$ be continuous; the interpretation of Tx is as the distribution of the progeny of an object of type x. We shall now show that T has an unique extension to a transition operator $P(\Omega_X) \to P(\Omega_X)$ for a branching process.

We shall allow ourselves some liberties concerning notation: We shall denote the extension, which we for convenience will construct on the whole of $M^1(\Omega_X)$, by T, also, and write $T\omega$ instead of $T\delta_\omega$, Tx instead of $T\varepsilon(x)$ or $T\delta_{\varepsilon(x)}$.

The branching property of T is equivalent with the formula $T(\mu * \nu) = T\mu * T\nu$.

3.1. Proposition. $T : X \to P(\Omega_X)$ has an unique extension to a continuous operator $T : M^1(\Omega_X) \to M^1(\Omega_X)$ satisfying $T(\mu * \nu) = T\mu * T\nu$. This extension is continuous in the norm, $||T\mu|| \leqq ||\mu||$, and maps probability measure into probability measure. For $\mu \in M^1(\Omega_X)$, $T\mu$ determined by

$$T\mu(f) = \int T\omega(f)d\mu(\omega), \quad f \in C^b(\Omega_X).$$

Proof. We start by showing the uniqueness. Let $\omega \in \Omega_X \backslash \{0\}$, $\omega = \varepsilon(x_1) + \ldots + \varepsilon(x_n)$. Since T should be a homomorphism of the algebra $M^1(\Omega_X)$ into itself, we must have $T\omega = Tx_1 * \ldots * Tx_n$. This implies $T\delta_0 = \delta_0$, and thus the linearity and continuity of T gives uniqueness by lemma 2.1.

To show existence, we start by extending T to Ω_X by the formulas just given. By proposition 1.2 and corollary 2.7 this gives a continuous homomorphism $\Omega_X \to P(\Omega_X)$. For an arbitrary μ we then define

$$T\mu(f) = \int T\omega(f)\, d\mu(\omega), \quad f \in K(\Omega_X).$$

This is well-defined since $\omega \to T\omega(f)$ is continuous and bounded (by $||f||$). An immediately check now gives that $f \to T\mu(f)$ is linear and continuous, i.e. $T\mu \in M^1(\Omega_X)$, and that $\mu \to T\mu$ is linear, continuous and extends $T : \Omega_X \to P(\Omega_X)$. By definition $T(\omega_1 + \omega_2) = T\omega_1 * T\omega_2$, and since convolution is bilinear and continuous when restricted to $M^1_+(\Omega_X)$ (corollary 2.6), we obtain $T(\mu * \nu) = T\mu * T\nu$ for positive μ, ν and then, by bilinearity, for all μ, ν.

To show the formula $T\mu(f) = \int T\omega(f)\, d\mu(\omega)$ to hold for all $f \in C^b(\Omega_X)$, we can without loss of generality assume μ and f to be positive. Then there exists an increasing net $f_i \in K(\Omega_X)$ of positive functions with $\lim_i f_i(\omega) = f(\omega), \forall\, \omega \in \Omega_X$, and repeated application of a well-known result on Radon measures then gives

$$T\mu(f) = \lim_i T\mu(f_i) = \lim_i \int T\omega(f_i)\, d\mu(\omega)$$

$$= \int (\lim_i T\omega(f_i))\, d\mu(\omega) = \int T\omega(f)\, d\mu(\omega).$$

If $\mu \in P(\Omega_X)$ it is obvious that $T\mu$ is positive and by the formula just proved, $T\mu(1) = 1$, i.e. $T\mu \in P(\Omega_X)$. Since for $f \in C^b(\Omega_X)$, $||f|| \leq 1$ $|T\mu(f)| \leq \int T\omega(|f|)\, d|\mu|\omega \leq ||\mu||$, we obtain $||T\mu|| \leq ||\mu||$. ◻

4. The probability generating functions and the probability of extinction.

For $\chi \in C(X,\Delta)$ we define

$$T^{*}\chi : X \to C$$

by

$$T^{*}\chi(x) = Tx(\tilde{\chi}) = FTx(\chi).$$

One easily checks that $T^{*}\chi \in C(X,\Delta)$.

The classical result on the generating functions has now the following form:

4.1. Proposition. $FT\mu(\chi) = F\mu(T^{*}\chi)$

Proof. Both sides are linear, multiplicative and continuous in μ, so it suffices to take Dirac-measure in $\varepsilon(x)$ for μ. But then the identity reduces to the definition of $T^{*}\chi$. ▯

4.2. Corollary. $FT^{n}\mu(\chi) = F\mu(T^{*n}\chi)$.

Let $\mu \in P(\Omega_X)$. The mass of μ in 0 is $\mu(\{0\}) = \mu(\tilde{0}) = F\mu(0)$, since $\tilde{0}$, the extension of the zero map $X \to \Delta$ to a continuous homomorphism $\Omega_X \to \Delta$, is given by $\tilde{0}(0) = 1$ and $\tilde{0}(\omega) = 0$, $\omega \neq 0$. Obviously $T\mu(\tilde{0}) =$
$\int T\omega(\tilde{0})d\mu(\omega) \geqq \int \tilde{0}(\omega)d\mu(\omega) = \mu(\tilde{0})$.

Given we start with one object of type x, we define $q_n(x)$ as the probability that the process becomes extinct at time n and $q(x)$ as the probability that it becomes extinct sooner or later. q_n is determined by $q_n(x) = T^{n}x(\tilde{0})$ and this formula easily gives that q_n is continuous, increasing and - by induction - that $q_n = T^{*n}0$. Since q_n is increasing, $q_n \uparrow q$.

4.3. Proposition. q and $\tilde{q}$ are lower semi-continuous and satisfies the functional equation $q(x) = Tx(\tilde{q})$.

If q is continuous, $q_n \uparrow q$, uniformly on compacts, and

$$q(x) = FTx(q),$$

i.e.

$$q = T^{*}q.$$

Proof. q and $\tilde{q}$ are l.s.c. as the upper bounds of the sequences q_n and $\tilde{q}_n$ of continuous functions. The equation $q(x) = Tx(\tilde{q})$ follows by monotone convergence from the identity

$$q_{n+1}(x) = T^{*n+1}\, 0(x) = T^{*} q_n(x) = Tx(\tilde{q}_n).$$

The last part of the proposition follows immediately from the theorem of Dini.□

An easy counter-example, showing that q is not in general continuous, is obtained by taking $X = [0,1]$ and Tx as the distribution with mass x in 0 and 1-x in $\varepsilon(x)$. Here $q(0) = 0$ and $q(x) = 1$, $x > 0$.

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