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Sufficiency and Time Series Analysis

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1. Introduction

A standard problem of time series analysis is from observations of a part of the series, say $x_1, \ldots, x_n$, to infer about the structure and parameters of the probability mechanism generating the series and at the same time to predict future outcomes of the series.

The classical approach to time series analysis is to treat these problems of inference separately, i.e. the "structural" inference is performed using ordinary statistical methods, and the prediction is done under the assumption of complete knowledge of the probability structure of the series. This seems to be unsatisfactory and suggests a development of a theoretical framework allowing a simultaneous treatment of different aspects of inference in time series analysis.

Sufficiency is one of the basic concepts in classical statistical theory. This concept is clearly made up for the purpose of structural, i.e. parametric inference only as an answer to the question: how much can we reduce our data and still keep all available information about the parameters of our model? In time series analysis, the corresponding question is: how much can we reduce the data and still keep all available information about parameters and about unobserved values of the series?

It is the purpose of the present paper to define a concept of sufficiency corresponding to the last question and investigate
some of its basic properties.

But first we shall consider an example, which will play a fundamental role throughout the paper showing some of the difficulties and results.

**Example 1.1:** Let \((X_t, t = 0, \pm 1, \pm 2, \ldots)\) be a normal stationary autoregressive process of order 1 and mean value 0, i.e. \(X_t - \beta X_{t-1} = \xi_t\), where \((\xi_t, t = 0, \pm 1, \pm 2, \ldots)\) are independent and normally distributed with mean zero and variance \(\sigma^2\), \(|\beta| < 1\), \(\beta\) and \(\sigma^2\) are unknown parameters.

Suppose that \(x = (x_1, \ldots, x_n)\) are observed and we want to estimate \(\beta\) and \(\sigma^2\) and predict \(X_{n+1}, \ldots, X_{n+k}\). The likelihood function

\[
L(\theta, \sigma^2, x) = |\mathcal{C}(\theta, \sigma^2)| \cdot e^{-\frac{1}{2\sigma^2}(x_1^2 + x_n^2 + (1 + \beta^2) \sum_{i=2}^{n-1} x_i^2 - 2\beta \sum_{i=2}^{n} x_i x_i-1)}
\]

shows that the minimal sufficient statistic is

\[
(t_1(x), t_2(x), t_3(x)) = (x_1^2 + x_n^2, \Sigma x_i^2, \Sigma x_i x_i-1)
\]

The procedure would now be to find estimates \(\hat{\theta}, \hat{\sigma}^2\) of \((\theta, \sigma^2)\) based on the sufficient statistics and then use \((x_n, \hat{\beta})\) to predict \(X_{n+i}\) as

\[
\hat{X}_{n+i} = \hat{\beta}^i \cdot x_n \quad i = 1, \ldots, k.
\]

So obviously, if you throw away \(x_n\) and keep only the sufficient statistic \((t_1, t_2, t_3)\) you are not as well off as you would
like to be. Similarly, if it was interesting to say something about the past, you would like to keep $x_1$.

Suppose also, that you have two statisticians, one of them observing $(x_1, \ldots, x_k)$ and the other $(x_{k+1}, \ldots, x_n)$. If both statisticians reduce their data by sufficiency to respectively

\[ (x_1^2 + x_k^2, \sum_{i=2}^{k-1} x_i, \sum_{i=2}^{k-1} x_i x_{i-1}) \]

and

\[ (x_{k+1}^2 + x_n^2, \sum_{i=k+2}^{n-1} x_i^2, \sum_{i=k+2}^{n-1} x_i x_{i-1}) \]

you cannot from their reduced data get the sufficient reduction of $(x_1, \ldots, x_n)$; this, however, is possible if both statisticians also keep the first and last observation from their data.

We shall later see, that the statistic

\[ (x_1^2 + \sum_{i=2}^{n-1} x_i^2, \sum_{i=2}^{n-1} x_i x_{i-1}, x_n) \]

is what we call "totally sufficient".

2. Totally Sufficient Subfields

Let $(\Omega, \mathcal{A})$ be a measurable space, $\bar{\pi}$ a family of probability measures on $(\Omega, \mathcal{A})$, $T$ an arbitrary set and $(\xi_t, t \in T)$ a real valued stochastic process on $(\Omega, \mathcal{A})$. Let $\mathcal{A}_{T_0}$ denote the $\sigma$-field generated by $(\xi_t, t \in T_0)$ for $T_0 \subseteq T$ and suppose $\mathcal{A}_T = \mathcal{A}$. For $B \subseteq \Omega$, $1_B$ is indicator of $B$, i.e.

\[ 1_B(\omega) = \begin{cases} 1 & \text{for } \omega \in B \\ 0 & \text{for } \omega \notin B \end{cases} \]
For $S, B$ subfields of $A$ with $S \subseteq B$ we write $S \text{suf}_{\pi} (\pi, B)$ if $S$ is sufficient for $\pi$ on $B$, i.e. if $\forall B \in B: \exists \phi_B - \text{measurable } \forall p \in \pi$:

$$\phi_B = E_p \mathcal{I}_{B \cdot S} (p)$$

For $S, B, C$ subfields of $A$ we write $S \perp C$ if $B$ and $C$ are conditionally independent given $S$ according to $p$, i.e. if $\forall B \in B: \forall C \in C$:

$$E_p^{S \perp B \cdot C} = E_p^{S \perp B} E_p^{S \perp C} \text{ a.s. } (p)$$

In the following, we shall not always explicitly write "a.s. (p)", hoping that the reader will not be confused.

**Definition 2.1**: A subfield $S \subseteq A_{T_0}$ is said to be totally sufficient for $\pi$ relative to $T_0$, and we write $S_{tsuf}(\pi, T_0)$, if

(a) $S_{tsuf}(\pi, A_{T_0})$  
(b) $\forall p \in \pi: A_{T_0} \bot_p S_{0} A_{T \cap T_0}$

Condition (a) and (b) ensures, that all information concerning $\pi$ and $A_{T \cap T_0}$ available from $A_{T_0}$ is summarized in $S_0$. You might call (a) "parameter sufficiency" and (b) "prediction sufficiency".

**Proposition 2.1**: $A_{T_0} \text{tsuf} (\pi, T_0)$

**Proof**: (a) is true as we can use $l_A$ as $\phi_A$ in the definition of sufficiency; (b) is true as for $A \in A_{T_0}, A^* \in A_{T \cap T_0}$:

$$E_p^{A_{T_0}} 1_{A \cap A^*} = E_p^{A_{T_0}} 1_{A^*}$$
Before we proceed to more interesting propositions, we shall prove the following lemmas:

Lemma 2.1: Let \((\Omega, \mathcal{A}, p)\) be a probability space, \(\mathcal{A}_0, \mathcal{A}_1, \mathcal{S}_0, \mathcal{S}_1\) subfields of \(\mathcal{A}\) so that \(\mathcal{S}_0 \subseteq \mathcal{A}_0, \mathcal{S}_1 \subseteq \mathcal{A}_1\). Then

\[
\mathcal{S}_0 \subseteq \mathcal{A}_0 \wedge \mathcal{S}_1 \subseteq \mathcal{A}_1 \Rightarrow \forall \mathcal{S}_0 \in \mathcal{A}_0 \forall \mathcal{S}_1 \in \mathcal{A}_1:
\]

(3) 

\[
P (\mathcal{S}_0 \cup \mathcal{S}_1) = P (\mathcal{S}_0 \cap \mathcal{S}_1)
\]

Proof: To prove "\(\Rightarrow\)" it is enough to show, that for

\(\mathcal{S}_0 \in \mathcal{S}_0, \mathcal{S}_1 \in \mathcal{S}_1, \mathcal{A}_0 \in \mathcal{A}_0, \mathcal{A}_1 \in \mathcal{A}_1\):

(4) 

\[
P (\mathcal{S}_0 \cap \mathcal{A}_0 \cap \mathcal{S}_1) = P (\mathcal{S}_0 \cap \mathcal{A}_0 \cap \mathcal{S}_1)
\]

as the sets \((\mathcal{S}_0 \cap \mathcal{S}_1, \mathcal{S}_0, \mathcal{S}_1)\) form a semialgebra generating \(\mathcal{S}_0 \cup \mathcal{S}_1\).

But

\[
P (\mathcal{S}_0 \cap \mathcal{A}_0 \cap \mathcal{S}_1) = P (\mathcal{A}_0 \cap \mathcal{S}_0 \cap \mathcal{A}_1 \cap \mathcal{S}_1)
\]

\[
P (\mathcal{E}_0 (\mathcal{A}_0 \cap \mathcal{S}_0 \cap \mathcal{A}_1 \cap \mathcal{S}_1)) = P (\mathcal{E}_0 (\mathcal{A}_0 \cap \mathcal{S}_0 \cap \mathcal{A}_1 \cap \mathcal{S}_1))
\]

\[
P (\mathcal{E}_0 (\mathcal{A}_1 \cap \mathcal{S}_1 \cap \mathcal{A}_0 \cap \mathcal{S}_0)) = P (\mathcal{E}_0 (\mathcal{A}_1 \cap \mathcal{S}_1 \cap \mathcal{A}_0 \cap \mathcal{S}_0))
\]

\[
P (\mathcal{E}_1 (\mathcal{A}_1 \cap \mathcal{S}_1 \cap \mathcal{A}_0 \cap \mathcal{S}_0)) = P (\mathcal{E}_1 (\mathcal{A}_1 \cap \mathcal{S}_1 \cap \mathcal{A}_0 \cap \mathcal{S}_0))
\]

\[
P (\mathcal{E}_1 (\mathcal{A}_0 \cap \mathcal{S}_0 \cap \mathcal{A}_1 \cap \mathcal{S}_1)) = P (\mathcal{E}_1 (\mathcal{A}_0 \cap \mathcal{S}_0 \cap \mathcal{A}_1 \cap \mathcal{S}_1))
\]

\[
P (\mathcal{E}_1 (\mathcal{A}_0 \cap \mathcal{S}_0 \cap \mathcal{A}_1 \cap \mathcal{S}_1)) = P (\mathcal{E}_1 (\mathcal{A}_0 \cap \mathcal{S}_0 \cap \mathcal{A}_1 \cap \mathcal{S}_1))
\]

\[
P (\mathcal{E}_1 (\mathcal{A}_0 \cap \mathcal{S}_0 \cap \mathcal{A}_1 \cap \mathcal{S}_1)) = P (\mathcal{E}_1 (\mathcal{A}_0 \cap \mathcal{S}_0 \cap \mathcal{A}_1 \cap \mathcal{S}_1))
\]
and (4) is proved. The conditional independence is used only to establish the equalities marked \( \dagger \), otherwise only standard properties of conditional expectation are used.

If, on the other side, we have

\[
E_p^{A_0 \cap A_1} = E_p^{A_0} E_p^{A_1}
\]

we can put \( A_0 = \Omega \) and get

\[
E_p^{A_1} = E_p^{A_0 (E_p^{A_0} A_1)} = E_p^{A_0} E_p^{A_1}
\]

and therefore, using \( E_p^{A_0} \) on both sides of (5)

\[
E_p^{A_0 \cap A_1} = E_p^{A_0} E_p^{A_0 \cap A_1} = E_p^{A_0} E_p^{A_1}
\]

The conditional independence is established and the proof is complete.

**Lemma 2.2:** Let \((\Omega, A, P)\) be a probability space, \(A_0, A_1, S_0, S_1\) subfields of \(A\) so that \(S_0 \subseteq A_0, S_1 \subseteq A_1\) and \(A_0 \perp_{P} A_1\). Then if \(A_0 \subseteq A_0, A_1 \subseteq A_1\)

\[
E_p^{A_0 \cap A_1} = E_p^{A_0} E_p^{A_1}
\]

**Proof:** We have

\[
E_p^{A_0 \cap A_1} = E_p^{A_0} (E_p^{A_1} A_0 \cap A_1)
\]

Using lemma 2.1 with \( S_1 = A_1 \), (9) is equal to
(10) \[ \mathcal{S}_0 \vee \mathcal{S}_1 \mathcal{E}_p (E_p^0 \mathcal{A}_0 E_p^1 \mathcal{A}_1) = E_p^0 \mathcal{A}_0 E_p^1 \mathcal{A}_1 \]

and (8) is proved.

Now we are able to show a proposition on combination of totally sufficient subfields.

**Proposition 2.2:** Let \( T_1, T_2 \subseteq T \), \( T_1 \cap T_2 = \emptyset \). If \( \mathcal{S}_1 \text{tsuf}(\pi, T_1) \) \( \mathcal{S}_2 \text{tsuf}(\pi, T_2) \) and \( E_p^1 \mathcal{A}_1 = \phi^1_{A_1} \), \( E_p^2 = \phi^2_{A_2} \), where \( \phi^1 \) and \( \phi^2 \) can be chosen as regular conditional probabilities, on \( A_1 \) resp. \( A_2 \), then \( \mathcal{S}_1 \vee \mathcal{S}_2 \text{tsuf}(\pi, T_1 \cup T_2) \).

**Proof:** For \( A_1 \in \mathcal{A}_{T_1}, A_2 \in \mathcal{A}_{T_2} \), lemma 2.1 gives:

\[
(11) \quad \mathcal{E}_p^1 \mathcal{A}_1 \mathcal{A}_2 = \phi^1_{A_1} \phi^2_{A_2}
\]

The sets \( (A_1 \cap A_2, A_1 \in \mathcal{A}_{T_1}, A_2 \in \mathcal{A}_{T_2}) \) form a semialgebra generating \( \mathcal{A}_{T_1 \cup T_2} \). As \( \phi^1(\omega) \) and \( \phi^2(\omega) \) are probability measures for all \( \omega \), the function

\[
(12) \quad \phi^{12}_{A_1 \cap A_2} = \phi^1_{A_1} \phi^2_{A_2}
\]

uniquely extends to \( \mathcal{A}_{T_1 \cup T_2} \). It is trivial to verify, that for \( S_{12} \in \mathcal{S}_{12} \)

\[
(13) \quad \int_{S_{12}} \phi^{12}_{A_1 \cap A_2} \, dp = p(A_1 \cap S_{12}),
\]

as the finite measure on \( \mathcal{A}_{T_1 \cup T_2} \)

\[
(14) \quad p_{12}(A_{12}) = p(A_1 \cap S_{12})
\]
is determined on sets of the form

\[(A_1 \cap A_2 | A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2)\].

We have now proved the "parameter sufficiency" of \(S_1 \vee S_2\).

Let \(A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2, A \in \mathcal{A}_{T_1 \cap (T_1 \cup T_2)}\). Lemma 2.2 gives

\[(15) \quad S_1 \vee S_2 = S_1 \wedge S_2, \quad E P A_1 \cap A_2 \cap A^* = E P A_1 E P A_2 \cap A^*\]

but, again from lemma 2.2

\[(16) \quad E P A_2 \cap A^* = E P A_2 E P A^*\]

Combining (11), (15) and (16), we have

\[(15) \quad S_1 \vee S_2 = S_1 \wedge S_2, \quad E P A_1 \cap A_2 \cap A^* = E P A_2 \cap A_1 \cap A_2 \cap A^*\]

Again the function \(p(\cdot, A^*, S_{12})\), defined as

\[(16) \quad p(A_{12}, A^*, S_{12}) = \int S_{12} \phi_{A_{12}} E P A_1 \cap A^* dP\]

is a finite measure on \(\mathcal{A}_{T_1 \cup T_2}\) determined on sets of the form

\[(A_1 \cap A_2 | A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2), \text{ so that}\]

\[(17) \quad S_1 \vee S_2 = \phi_{A_{12}} E P A_1 \cap A_2 \cap A^*,\]

and the prediction sufficiency is proved.

The above proposition ensures the possibility of combining totally sufficient reductions of different observations of data concerning the same problem, even if these observations
happen to be dependent. The next proposition shows, that the condition (b) in the definition of total sufficiency with mild restrictions on \( \pi \) is necessary to ensure a "calculus of sufficient reduction".

**Proposition 2.3:** Let \((\Omega, \mathcal{A})\) be a measurable space, \(\pi\) a family of probability measures on \((\Omega, \mathcal{A})\), \(\mathcal{A}_1, \mathcal{A}_2\) subfields of \(\mathcal{A}\), \(\mathcal{S}_1 \subseteq \mathcal{A}_1\), \(\mathcal{S}_2 \subseteq \mathcal{A}_2\) so that \(\mathcal{S}_1 \text{ suf } (\pi, \mathcal{A}_1)\), \(\mathcal{S}_2 \text{ suf } (\pi, \mathcal{A}_2)\) and \(\mathcal{S}_1 \lor \mathcal{S}_2 \text{ suf } (\pi, \mathcal{A}_1 \lor \mathcal{A}_2)\). Then

\[
\begin{align*}
\mathcal{S}_1 \lor \mathcal{S}_2 &\quad \text{ and } \\
\mathcal{S}_1 \lor \mathcal{S}_2 &\quad \text{ and }
\end{align*}
\]

**Proof:** Suppose \(\mathcal{A}_1 \mathcal{S}_1 \mathcal{A}_2\) and \(\mathcal{A}_1 \mathcal{S}_2 \mathcal{A}_2\). Then, from lemma 2.1

\[
\begin{align*}
\mathcal{S}_1 \lor \mathcal{S}_2 &\quad \text{ and } \\
\mathcal{S}_1 \lor \mathcal{S}_2 &\quad \text{ and }
\end{align*}
\]

The sufficiency of \(\mathcal{S}_1, \mathcal{S}_2\) and \(\mathcal{S}_1 \lor \mathcal{S}_2\) gives, that the right and left sides of (19) are independent of \(p_0\), so that

\[
\begin{align*}
\mathcal{S}_1 \lor \mathcal{S}_2 &\quad \text{ and } \\
\mathcal{S}_1 \lor \mathcal{S}_2 &\quad \text{ and }
\end{align*}
\]

Using lemma 2.1, we have

\[
\begin{align*}
\mathcal{S}_1 \lor \mathcal{S}_2 &\quad \text{ and } \\
\mathcal{S}_1 \lor \mathcal{S}_2 &\quad \text{ and }
\end{align*}
\]

and the proof is complete.
If, for example, $A_1$ and $A_2$ are independent for just one $p \in \pi$, the theorem applies, i.e. you must have prediction sufficiency to be able to combine parameter sufficient subfields.

3. Relation between sufficiency and total sufficiency.

Spring 1972, Barndorff-Nielsen pointed out to me, that he together with Skibinsky worked out a concept of adequate subfields [1], which in a later paper by Skibinsky [3] was generalized to something seemingly equivalent to total sufficiency.

Apart from a slightly different motivation and framework, the two concepts can easily be proved to be identical, i.e.

$S_0 \text{ tsuf } (\pi, T_0) \Leftrightarrow S_0 \text{ adq } (\hat{A}_{T_0}; \hat{A}_{T \cap T_0}, \pi).$

Skibinsky proves the equivalence between adequacy and sufficiency with respect to a certain family of conditional distributions (Theorems 1 and 2 of [2]). In this formulation, roughly speaking, $S_0$ is totally sufficient for $\pi$ relative to $T_0$ iff $S_0$ is sufficient for the family of (regular) conditional probabilities $(p^{T \cap T_0}, p \in \pi)$ on $\hat{A}_{T_0}$. This equivalence gives rise to the "translation" of most standard theorems on sufficiency to adequacy. Results like proposition 2.2 and 2.3 are not available from this equivalence as the structure of the index set and the subfields of $\hat{A}$ are used.

We shall briefly mention some definitions analogous to the theory of sufficiency.
Definition 3.1: \( \mathcal{S}_0 \subseteq \mathcal{A}_{T_0} \) is said to be minimal totally sufficient for \( \pi \) relative to \( T_0 \), and we write \( \mathcal{S}_0 \text{min tsuf } (\pi, T_0) \) if \( \mathcal{S}_0 \text{tsuf } (\pi, T_0) \) and

\[
\mathcal{S}_0 \text{tsuf } (\pi, T_0) = \mathcal{S}_0 \subseteq \mathcal{S}_0[\mathcal{A}_{T_0}, \pi]
\]

Definition 3.2: An \( \mathcal{A}_{T_0} \)-measurable statistic \( t_0: (\Omega, \mathcal{A}) \rightarrow (\Omega^*_\mathcal{A}^*) \) where \( (\Omega^*_\mathcal{A}^*) \) is a measurable space is said to be totally sufficient relative to \( T_0 \) if \( t_0^{-1}(\mathcal{A}^*) \text{tsuf } (\pi, T_0) \).

A minimal totally sufficient statistic can be defined analogously to a minimal sufficient statistic, and in regular cases, where a minimal totally sufficient statistic \( t_{T_0} \) for any \( T_0 \) exists and generates a min tsuf subfield, proposition 2.2 gives for \( T_0 \cap T_1 = \emptyset \).

\[
t_{T_0UT_1}(\xi_s, s \in T_0UT_1) = \phi(t_{T_0}(\xi_s, s \in T_0), t_{T_1}(\xi_s, s \in T_1))
\]

or in words, that the min tsuf statistic for the union of two disjoint subsets of \( T \) is a function of the min tsuf statistics for the subsets.

This property seems to make the theory applicable in classical statistical theory, where a lot of interesting models should be seen included in a "pattern of repetition" more complicated than independent repetitions of the same experiment.

4. Some examples.

Example 4.1: Put \( T = \mathbb{Z} \) and let \((X_t, t \in \mathbb{Z})\) be a stationary autoregressive process of order 1 (see Ex. 1.1) with \(|\beta| < 1\) and
For $T_0 = \{1, \ldots, n\}$ a minimal sufficient statistic is easily seen to be

$$t_{T_0}(X_1, \ldots, X_n) = (X_1, \sum_{i=1}^{n} X_i^2, \sum_{i=2}^{n} X_i X_{i-1}, X_n) =$$

$$(t_{T_0}^{(1)}, t_{T_0}^{(2)}, t_{T_0}^{(3)}, t_{T_0}^{(4)})$$

Put $T_1 = \{n+1, \ldots, m\}$ and notice, that

$$t_{T_0 \cup T_1}^{(1)}(X_1, \ldots, X_m) = t_{T_0}^{(1)}(X_1, \ldots, X_n)$$

$$t_{T_0 \cup T_1}^{(2)}(X_1, \ldots, X_m) = t_{T_0}^{(2)}(X_1, \ldots, X_n) + t_{T_1}^{(2)}(X_{n+1}, \ldots, X_m);$$

and further with a short notation

$$t_{T_0 \cup T_1}^{(3)} = t_{T_0}^{(3)} + t_{T_1}^{(3)}$$

$$t_{T_0 \cup T_1}^{(4)} = t_{T_1}^{(4)}$$

which illustrates the effect of proposition 2.2 (compare with example 1.1). It is easy to see, that all autoregressive processes admit proper totally sufficient reductions for $T_0$ being an interval of $\mathbb{Z}$.

The following example shows, that the existence of a $\pi$ giving conditional independence is not a superfluous condition in proposition 2.3.

Example 4.2: Let $(X_1, \ldots, X_n), (Y_1, \ldots, Y_n)$ be normally distributed random variables with $EX_i = \xi, EY_i = \eta, Var(X_i) = Var(Y_i) = 1$. 

$$\sigma^2 > 0 \text{ unknown.}$$
\[ \text{Cov}(X_i, Y_i) = \rho \delta_{ij} \text{ where } \xi, \eta \in \mathbb{R} \text{ are unknown, and } \rho \text{ is a fixed,} \]
known correlation between \( X_i \) and \( Y_i \). Let \( S_1 = \sigma(\sum_{i=1}^{n} X_i) \),
\[ S_2 = \sigma(\sum_{i=1}^{n} Y_i), \ A_1 = \sigma(X_1, \ldots, X_n), \ A_2 = \sigma(Y_1, \ldots, Y_n), \ \pi \text{ the family of probability measures on } \mathbb{R}^{2n} \text{ defined above. It is clear, that } S_1 \text{ sur } (\pi, A_1), \ S_2 \text{ sur } (\pi, A_2) \text{ and } S_1 \lor S_2 \text{ sur } (\pi, A_1 \lor A_2). \text{ If however, } \rho \neq 0 \text{ it is not true, that } A_1 \perp A_2 \text{ for any value of } \xi, \eta. \]

5. Questions to be discussed.

The present paper is to be seen as a preliminary report on some ideas about total sufficiency. A lot of open questions is still left to investigation, some of them being of a probabilistic and some of statistical nature.

For \( T = Z \) or \( T = Z^n \), what kind of families \( \pi \) on \( \mathbb{R}^T \) admit proper totally sufficient reductions of dimension \( p \), whenever \( T_0 \) is something nice, say an interval ? Exactly when will total sufficiency and sufficiency coincide ?

How shall we use the statistics for inference purpose ?

What comes out if we try to formulate statistical models as stochastic processes, where a given set of data observed correspond to a part of the indexset ? The concept of a statistical population seems to have a precise formulation in this framework. Problems related to nuisance-structural parameters and conditional inference also seem to be suited for a formulation like this.
We hope to work out answers for some of these questions in the future.

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References:

