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Sufficiency and Time Series Analysis

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1. Introduction

A standard problem of time series analysis is from observations of a part of the series, say $x_1, \ldots, x_n$, to infer about the structure and parameters of the probability mechanism generating the series and at the same time to predict future outcomes of the series.

The classical approache to time series analysis is to treat these problems of inference separately, i.e. the "structural" inference is performed using ordinary statistical methods, and the prediction is done under the assumption of complete knowledge of the probability structure of the series. This seems to be unsatisfactory and suggests a development of a theoretical framework allowing a simultaneous treatment of different aspects of inference in time series analysis.

Sufficiency is one of the basic concepts in classical statistical theory. This concept is clearly made up for the purpose of structural, i.e. parametric inference only as an answer to the question: how much can we reduce our data and still keep all available information about the parameters of our model ? In time series analysis, the corresponding question is: how much can we reduce the data and still keep all available information about parameters <u>and</u> about unobserved values of the series ?

It is the purpose of the present paper to define a concept of sufficiency corresponding to the last question and investigate some of its basic properties.

But first we shall consider an example, which will play a fundamental role throughout the paper showing some of the difficulties and results.

Example 1.1: Let $(X_t, t = 0, \pm 1, \pm 2, \ldots)$ be a normal stationary autoregressive process of order 1 and mean value 0, i.e. $X_t - \beta X_{t-1} = \xi_t$, where $(\xi_t, t = 0, \pm 1, \pm 2, \ldots)$ are independent and normally distributed with mean zero and variance σ^2, $|\beta| < 1$, β and σ^2 are unknown parameters.

Suppose that $x = (x_1, \ldots, x_n)$ are observed and we want to estimate β and σ^2 and predict $X_{n+1}, \ldots, X_{n+k}$. The likelihood function

$$(1) \quad L(\beta, \sigma^2, x) = c(\beta, \sigma^2) \cdot e^{-\frac{1}{2\sigma^2}\left(x_1^2 + x_n^2 + (1+\beta^2)\sum_{i=2}^{n-1} x_i^2 - 2\beta \sum_{i=2}^{n} x_i x_{i-1}\right)}$$

shows that the minimal sufficient statistic is

$$(2) \quad (t_1(x), t_2(x), t_3(x)) = \left(x_1^2 + x_n^2, \sum_{i=2}^{n-1} x_i^2, \sum_{i=2}^{n} x_i x_{i-1}\right)$$

The procedure would now be to find estimates $(\hat{\beta}, \hat{\sigma}^2)$ of (β, σ^2) based on the sufficient statistics and then use $(x_n, \hat{\beta})$ to predict X_{n+i} as

$$(3) \quad \hat{X}_{n+i} = \hat{\beta}^i \cdot x_n \qquad i = 1, \ldots, k.$$

So obviously, if you throw away x_n and keep only the sufficient statistic (t_1, t_2, t_3) you are not as well of as you would like to be. Similarly, if it was interesting to say something about the past, you would like to keep x_1.

Suppose also, that you have two statisticians, one of them observing $(x_1,\dots,x_k)$ and the other $(x_{k+1},\dots,x_n)$. If both statisticians reduce their data by sufficiency to respectively

$$(x_1^2+x_k^2, \sum_{i=2}^{k-1} x_i^2, \sum_{i=2}^{k} x_i x_{i-1})$$

and

$$(x_{k+1}^2+x_n^2, \sum_{i=k+2}^{n-1} x_1^2, \sum_{i=k+2}^{n} x_i x_{i-1})$$

you cannot from their reduced data get the sufficient reduction of $(x_1,\dots,x_n)$; this, however, is possible if both statisticians also keep the first and last observation from their data.

We shall later see, that the statistic $(x_1, \sum_{i=2}^{n-1} x_i^2, \sum_{i=2}^{n} x_i x_{i-1}, x_n)$ is what we call "totally sufficient".

2. Totally Sufficient Subfields

Let $(\Omega,\mathfrak{A})$ be a measurable space, π a family of probability measures on $(\Omega,\mathfrak{A})$, T an arbitrary set and $(\xi_t, t\in T)$ a real valued stochastic process on $(\Omega,\mathfrak{A})$. Let $\mathfrak{A}_{T_0}$ denote the σ-field generated by $(\xi_t, t\in T_0)$ for $T_0 \subseteq T$ and suppose $\mathfrak{A}_T = \mathfrak{A}$. For $B \subseteq \Omega$, 1_B is indicator of B, i.e.

(1)
$$1_B(\omega) = \begin{cases} 1 & \text{for } \omega\in B \\ 0 & \text{for } \omega\notin B \end{cases}$$

For $\mathcal{S}$, $\mathcal{B}$ subfields of $\mathcal{A}$ with $\mathcal{S} \subseteq \mathcal{B}$ we write $\mathcal{S}$ suf $(\pi,\mathcal{B})$ if $\mathcal{S}$ is sufficient for π on $\mathcal{B}$, i.e. if $\forall B \in \mathcal{B}\ \exists \phi_B\ \mathcal{S}$-measurable $\forall p \in \pi$:

(2) $$\phi_B = E_p^{\mathcal{S}} 1_B \text{ a.s.}(p)$$

For $\mathcal{S}$, $\mathcal{B}$, $\mathcal{C}$ subfields of $\mathcal{A}$ we write $\mathcal{B} \perp_p^{\mathcal{S}} \mathcal{C}$ if $\mathcal{B}$ and $\mathcal{C}$ are conditionally independent given $\mathcal{S}$ according to p, i.e. if $\forall B \in \mathcal{B}\ \forall C \in \mathcal{C}$:

(3) $$E_p^{\mathcal{S}} 1_{B \cap C} = E_p^{\mathcal{S}} 1_B E_p^{\mathcal{S}} 1_C \text{ a.s.}(p)$$

In the following, we shall not always explicitety write "a.s. (p)", hoping that the reader will not be confused.

<u>Definition 2.1</u>: A subfield $\mathcal{S}_0 \subseteq \mathcal{A}_{T_0}$ is said to be <u>totally sufficient</u> for π relative to T_0, and we write $\mathcal{S}_0 \text{tsuf}(\pi, T_0)$, if

(a) $\mathcal{S}_0$ suf $(\pi, \mathcal{A}_{T_0})$

(b) $\forall p \in \pi$: $\mathcal{A}_{T_0} \perp_p^{\mathcal{S}_0} \mathcal{A}_{T \setminus T_0}$

Condition (a) and (b) ensures, that all information concerning π and $\mathcal{A}_{T \setminus T_0}$ available from $\mathcal{A}_{T_0}$ is summarized in $\mathcal{S}_0$. You might call (a) "parameter sufficiency" and (b) "prediction sufficiency".

<u>Proposition 2.1</u>: $\mathcal{A}_{T_0}$ tsuf (π, T_0)

<u>Proof</u>: (a) is true as we can use 1_A as ϕ_A in the definition of sufficiency; (b) is true as for $A \in \mathcal{A}_{T_0}$, $A^* \in \mathcal{A}_{T \setminus T_0}$:

$$E_p^{\mathcal{A}_{T_0}} 1_{A \cap A^*} = 1_A E_p^{\mathcal{A}_{T_0}} 1_{A^*}.$$

Before we proceed to more interesting propositions, we shall prove the following lemmas:

Lemma 2.1: Let $(\Omega,\mathcal{A},p)$ be a probability space, $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{S}_0$, $\mathcal{S}_1$ subfields of $\mathcal{A}$ so that $\mathcal{S}_0 \subseteq \mathcal{A}_0$, $\mathcal{S}_1 \subseteq \mathcal{A}_1$. Then

$$\mathcal{A}_0 \perp_p^{\mathcal{S}_0} \mathcal{A}_1 \wedge \mathcal{A}_0 \perp_p^{\mathcal{S}_1} \mathcal{A}_1 \Leftrightarrow \forall A_0 \in \mathcal{A}_0 \forall A_1 \in \mathcal{A}_1:$$

(3)
$$E_p^{\mathcal{S}_0 \vee \mathcal{S}_1} 1_{A_0 \cap A_1} = E_p^{\mathcal{S}_0} 1_{A_0} E_p^{\mathcal{S}_1} 1_{A_1}$$

Proof: To prove "$\Rightarrow$" it is enough to show, that for

$$S_0 \in \mathcal{S}_0,\ S_1 \in \mathcal{S}_1,\ A_0 \in \mathcal{A}_0,\ A_1 \in \mathcal{A}_1:$$

(4)
$$E_p(1_{S_0 \cap S_1 \cap A_0 \cap A_1}) = E_p(1_{S_0 \cap S_1} \cdot E_p^{\mathcal{S}_0} 1_{A_0} \cdot E_p^{\mathcal{S}_1} 1_{A_1})$$

as the sets $(S_0 \cap S_1 | S_0 \in \mathcal{S}_0,\ S_1 \in \mathcal{S}_1)$ forms a semialgebra generating $\mathcal{S}_0 \vee \mathcal{S}_1$.

But

$$E_p(1_{S_0 \cap S_1 \cap A_0 \cap A_1}) = E_p(1_{A_0 \cap S_0} \cdot 1_{A_1 \cap S_1}) =$$

$$E_p(E_p^{\mathcal{S}_0}(1_{A_0 \cap S_0} \cdot 1_{A_1 \cap S_1})) \overset{*}{=} E_p(E_p^{\mathcal{S}_0} 1_{A_0 \cap S_0} \cdot E_p^{\mathcal{S}_0} 1_{A_1 \cap S_1}) =$$

$$E_p(E_p^{\mathcal{S}_0}(1_{A_1 \cap S_1} \cdot E_p^{\mathcal{S}_0} 1_{A_0 \cap S_0})) = E_p(1_{A_1 \cap S_1} \cdot E_p^{\mathcal{S}_0} 1_{A_0 \cap S_0}) \overset{*}{=}$$

$$E_p(E_p^{\mathcal{S}_1} 1_{A_1 \cap S_1} \cdot E_p^{\mathcal{S}_1}(E_p^{\mathcal{S}_0} 1_{A_0 \cap S_0})) =$$

$$E_p(E_p^{\mathcal{S}_1}(E_p^{\mathcal{S}_1} 1_{A_1 \cap S_1} E_p^{\mathcal{S}_0} 1_{A_0 \cap S_0})) =$$

$$E_p(1_{S_1} \cdot 1_{S_0} \cdot E_p^{\mathcal{S}_0} 1_{A_0} E_p^{\mathcal{S}_1} 1_{A_1})$$

and (4) is proved. The conditional independence is used only to establish the equalities marked $\overset{*}{=}$, otherwise only standard properties of conditional expectation are used.

If, on the other side, we have

$$(5) \qquad E_p^{\mathcal{S}_0 \vee \mathcal{S}_1} 1_{A_0 \cap A_1} = E_p^{\mathcal{S}_0} 1_{A_0} E_p^{\mathcal{S}_1} 1_{A_1}$$

we can put $A_0 = \Omega$ and get

$$(6) \qquad E_p^{\mathcal{S}_0} 1_{A_1} = E_p^{\mathcal{S}_0}(E_p^{\mathcal{S}_0 \vee \mathcal{S}_1} 1_{A_1}) = E_p^{\mathcal{S}_0} E_p^{\mathcal{S}_1} 1_{A_1}$$

and therefore, using $E_p^{\mathcal{S}_0}$ on both sides of (5)

$$(7) \qquad E_p^{\mathcal{S}_0} 1_{A_0 \cap A_1} = E_p^{\mathcal{S}_0} 1_{A_0} E_p^{\mathcal{S}_0} E_p^{\mathcal{S}_1} 1_{A_1} = E_p^{\mathcal{S}_0} 1_{A_0} E_p^{\mathcal{S}_0} 1_{A_1}$$

The conditional independence is established and the proof is complete.

<u>Lemma 2.2</u>: Let $(\Omega, \mathcal{A}, p)$ be a probability space, $\mathcal{A}_0, \mathcal{A}_1, \mathcal{S}_0, \mathcal{B}_1$ subfields of $\mathcal{A}$ so that $\mathcal{S}_0 \subseteq \mathcal{A}_0$, $\mathcal{B}_1 \subseteq \mathcal{A}_1$ and $\mathcal{A}_0 \perp_p^{\mathcal{S}_0} \mathcal{A}_1$. Then if $A_0 \in \mathcal{A}_0$, $A_1 \in \mathcal{A}_1$

$$(8) \qquad E_p^{\mathcal{S}_0 \vee \mathcal{B}_1} 1_{A_0 \cap A_1} = E_p^{\mathcal{S}_0} 1_{A_0} E_p^{\mathcal{S}_0 \vee \mathcal{B}_1} 1_{A_1}$$

<u>Proof:</u> We have

$$(9) \qquad E_p^{\mathcal{S}_0 \vee B_1} 1_{A_0 \cap A_1} = E_p^{\mathcal{S}_0 \vee \mathcal{B}_1}(E_p^{\mathcal{S}_0 \vee A_1} 1_{A_0 \cap A_1})$$

Using lemma 2.1 with $\mathcal{S}_1 = \mathcal{A}_1$, (9) is equal to

(10) $$E_p^{\mathcal{S}_0 \vee \mathcal{B}_1}(E_p^{\mathcal{S}_0} 1_{A_0} E_p^{\mathcal{A}_1} 1_{A_1}) = E_p^{\mathcal{S}_0} 1_{A_0} E_p^{\mathcal{S}_0 \vee \mathcal{B}_1} 1_{A_1}$$

and (8) is proved.

Now we are able to show a proposition on combination of totally sufficient subfields.

<u>Proposition 2.2</u>: Let $T_1, T_2 \subseteq T$, $T_1 \cap T_2 = \emptyset$. If $\mathcal{S}_1$ tsuf(π, T_1) $\mathcal{S}_2$ tsuf(π, T_2) and $E_p^{\mathcal{S}_1} 1_{A_1} = \phi^1{}_{A_1}$, $E_p^{\mathcal{S}_2} = \phi^2{}_{A_2}$, where ϕ^1 and ϕ^2 can be chosen as regular conditional probabilities on $\mathcal{A}_1$ resp. $\mathcal{A}_2$, then $\mathcal{S}_1 \vee \mathcal{S}_2$ tsuf $(\pi, T_1 \cup T_2)$.

<u>Proof</u>: For $A_1 \in \mathcal{A}_{T_1}$, $A_2 \in \mathcal{A}_{T_2}$, lemma 2.1 gives:

(11) $$E_p^{\mathcal{S}_1 \vee \mathcal{S}_2} 1_{A_1 \cap A_2} = \phi^1_{A_1} \cdot \phi^2_{A_2}$$

The sets $(A_1 \cap A_2,\ A_1 \in \mathcal{A}_{T_1},\ A_2 \in \mathcal{A}_{T_2})$ form a semialgebra generating $\mathcal{A}_{T_1 \cap T_2}$. As $\phi^1_{\cdot}(\omega)$ and $\phi^2_{\cdot}(\omega)$ are probability measures for all ω, the function

(12) $$\phi^{12}_{A_1 \cap A_2} = \phi^1_{A_1}\ \phi^2_{A_2}$$

uniquely extends to $\mathcal{A}_{T_1 \cup T_2}$. It is trivial to verify, that for $S_{12} \in \mathcal{S}_{12}$

(13) $$\int_{S_{12}} \phi^{12}_{A_{12}}\, dp = p(A_{12} \cap S_{12}),$$

as the finite measure on $\mathcal{A}_{T_1 \cup T_2}$

(14) $$p^{S_{12}}(A_{12}) = p(A_{12} \cap S_{12})$$

is determined on sets of the form

$$(A_1 \cap A_2 \mid A_1 \in \hat{\mathcal{A}}_1,\ A_2 \in \hat{\mathcal{A}}_2).$$

We have now proved the "parameter sufficiency" of $\hat{S}_1 \vee \hat{S}_2$.

Let $A_1 \in \hat{\mathcal{A}}_{T_1}$, $A_2 \in \hat{\mathcal{A}}_{T_2}$, $A \in \hat{\mathcal{A}}_{T \setminus (T_1 \cup T_2)}$. Lemma 2.2 gives

$$(15) \qquad E_p^{\hat{S}_1 \vee \hat{S}_2} 1_{A_1 \cap A_2 \cap A^*} = E_p^{\hat{S}_1} 1_{A_1} E_p^{\hat{S}_1 \vee \hat{S}_2} 1_{A_2 \cap A^*}$$

but, again from lemma 2.2

$$(16) \qquad E_p^{\hat{S}_1 \vee \hat{S}_2} 1_{A_2 \cap A^*} = E_p^{\hat{S}_2} 1_{A_2} E_p^{\hat{S}_1 \vee \hat{S}_2} 1_{A^*}.$$

Combining (11), (15) and (16), we have

$$(15) \qquad E_p^{\hat{S}_1 \vee \hat{S}_2} 1_{A_1 \cap A_2 \cap A^*} = E_p^{\hat{S}_1 \vee \hat{S}_2} 1_{A_1 \cap A_2} \cdot E_p^{\hat{S}_1 \vee \hat{S}_2} 1_{A^*}$$

Again the function $p(\circ, A^*, S_{12})$, defined as

$$(16) \qquad p(A_{12}, A^*, S_{12}) = \int_{S_{12}} \phi^{12}_{A_{12}} E_p^{\hat{S}_1 \vee \hat{S}_2} 1_{A^*} dp$$

is a finite measure on $\hat{\mathcal{A}}_{T_1 \cup T_2}$ determined on sets of the form $(A_1 \cap A_2 \mid A_1 \in \hat{\mathcal{A}}_{T_1}, A_2 \in \hat{\mathcal{A}}_{T_2})$, so that

$$(17) \qquad E_p^{\hat{S}_1 \vee \hat{S}_2} 1_{A_{12} \cap A^*} = \phi^{12}_{A_{12}} E_p^{\hat{S}_1 \vee \hat{S}_2} 1_{A^*},$$

and the prediction sufficiency is proved.

The above proposition ensures the possibility of combining totally sufficient reductions of different observations of data concerning the same problem, even if these observations happen to be dependent. The next proposition shows, that the condition (b) in the definition of total sufficiency with mild restrictions on π is necessary to ensure a "calculus of sufficient reduction".

<u>Proposition 2.3</u>: Let $(\Omega,\mathcal{A})$ be a measurable space, π a family of probability measures on $(\Omega,\mathcal{A})$, $\mathcal{A}_1$, $\mathcal{A}_2$ subfields of $\mathcal{A}$, $\mathcal{S}_1 \subseteq \mathcal{A}_1$, $\mathcal{S}_2 \subseteq \mathcal{A}_2$ so that $\mathcal{S}_1$ suf $(\pi,\mathcal{A}_1)$, $\mathcal{S}_2$ suf $(\pi,\mathcal{A}_2)$ and $\mathcal{S}_1 \vee \mathcal{S}_2$ suf $(\pi,\mathcal{A}_1 \vee \mathcal{A}_2)$. Then

$$(18) \qquad [\exists p \in \pi: \mathcal{A}_1 \perp_p^{\mathcal{S}_1} \mathcal{A}_2 \wedge \mathcal{A}_1 \perp_p^{\mathcal{S}_2} \mathcal{A}_2] \Rightarrow$$

$$[\forall p \in \pi: \mathcal{A}_1 \perp_p^{\mathcal{S}_1} \mathcal{A}_2 \wedge \mathcal{A}_1 \perp_p^{\mathcal{S}_2} \mathcal{A}_2]$$

<u>Proof</u>: Suppose $\mathcal{A}_1 \perp_{p_0}^{\mathcal{S}_1} \mathcal{A}_2$ and $\mathcal{A}_1 \perp_{p_0}^{\mathcal{S}_2} \mathcal{A}_2$. Then, from lemma 2.1

$$(19) \qquad E_{p_0}^{\mathcal{S}_1 \vee \mathcal{S}_2} 1_{A_1 \cap A_2} = E_{p_0}^{\mathcal{S}_1} 1_{A_1} \, E_{p_0}^{\mathcal{S}_2} 1_{A_2}$$

for $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$;

The sufficiency of $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_1 \vee \mathcal{S}_2$ gives, that the right and left sides of (19) are independent of p_0, so that

$$(20) \qquad \forall p \in \pi: E_p^{\mathcal{S}_1 \vee \mathcal{S}_2} 1_{A_1 \cap A_2} = E_p^{\mathcal{S}_1} 1_{A_1} E_p^{\mathcal{S}_2} 1_{A_2}$$

Using lemma 2.1, we have

$$(21) \qquad \forall p \in \pi: \mathcal{A}_1 \perp_p^{\mathcal{S}_1} \mathcal{A}_2 \wedge \mathcal{A}_1 \perp_p^{\mathcal{S}_2} \mathcal{A}_2$$

and the proof is complete.

If, for example, $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent for just one $p \in \pi$, the theorem applies, i.e. you must have prediction sufficiency to be able to combine parameter sufficient subfields.

3. Relation between sufficiency and total sufficiency.

Spring 1972, Barndorff-Nielsen pointed out to me, that he together with Skibinsky worked out a concept of adequate subfields [1], which in a later paper by Skibinsky [3] was generalized to something seemingly equivalent to total sufficiency.

Apart from a slightly different motivation and framework, the two concepts can easily be proved to be identical, i.e.

(1) $\mathcal{S}_0$ tsuf $(\pi, T_0) \Leftrightarrow \mathcal{S}_0$ adq $(\mathcal{A}_{T_0}; \mathcal{A}_{T \setminus T_0}, \pi)$.

Skibinsky proves the equivalence between adequacy and sufficiency with respect to a certain family of conditional distributions (Theorems 1 and 2 of [2]). In this formulation, roughly speaking, $\mathcal{S}_0$ is totally sufficient for π relative to T_0 iff $\mathcal{S}_0$ is sufficient for the family of (regular) conditional probabilities $(p^{\mathcal{A}_{T \setminus T_0}}, p \in \pi)$ on $\mathcal{A}_{T_0}$. This equivalence gives rize to the "translation" of most standard theorems on sufficiency to adequacy. Results like proposition 2.2 and 2.3 are not available from this equivalence as the structure of the index set and the subfields of $\mathcal{A}$ are used.

We shall briefly mention some definitions analogous to the theory of sufficiency.

Definition 3.1: $\mathcal{S}_0 \subseteq \mathcal{A}_{T_0}$ is said to be minimal totally sufficient for π relative to T_0, and we write $\mathcal{S}_0$ min tsuf (π, T_0) if $\mathcal{S}_0$ tsuf (π, T_0) and

$$\mathcal{S}_0^* \text{ tsuf } (\pi, T_0) \Rightarrow \mathcal{S}_0 \subseteq \mathcal{S}_0^*[\mathcal{A}_{T_0}, \pi]$$

Definition 3.2: An $\mathcal{A}_{T_0}$-measurable statistic $t_0: (\Omega, \mathcal{A}) \to (\Omega^*; \mathcal{A}^*)$ where $(\Omega^*; \mathcal{A}^*)$ is a measurable space is said to be totally sufficient relative to T_0 if $t_0^{-1}(\mathcal{A}^*)$ tsuf (π, T_0).

A minimal totally sufficient statistic can be defined analogously to a minimal sufficient statistic, and in regular cases, where a minimal totally sufficient statistic t_{T_0} for any T_0 exists and generates a min tsuf subfield, proposition 2.2 gives for $T_0 \cap T_1 = \emptyset$.

$$t_{T_0 \cup T_1}(\xi_s, s \in T_0 \cup T_1) = \phi(t_{T_0}(\xi_s, s \in T_0), t_{T_1}(\xi_s, s \in T_1))$$

or in words, that the min tsuf statistic for the union of two disjoint subsets of T is a function of the min tsuf statistics for the subsets.

This property seems to make the theory applicable in classical statistical theory, where a lot of interesting models should be seen included in a "pattern of repetition" more complicated than independent repetitions of the same experiment.

4. Some examples.

Example 4.1: Put $T = \mathbb{Z}$ and let $(X_t, t \in \mathbb{Z})$ be a stationary autoregressive process of order 1 (see Ex. 1.1) with $|\beta| < 1$ and

$\sigma^2 > 0$ unknown.

For $T_0 =' \{1,\ldots,n\}$ a min tsuf statistic is easily seen to be

(1) $$t_{T_0}(X_1,\ldots,X_n) = (X_1, \sum_{i=1}^{n} X_i^2, \sum_{i=2}^{n} X_i X_{i-1}, X_n) = (t_{T_0}^{(1)}, t_{T_0}^{(2)}, t_{T_0}^{(3)}, t_{T_0}^{(4)})$$

Put $T_1 =' \{n+1,\ldots,m\}$ and notice, that

$$t_{T_0 \cup T_1}^{(1)}(X_1,\ldots,X_m) = t_{T_0}^{(1)}(X_1,\ldots,X_n)$$

$$t_{T_0 \cup T_1}^{(2)}(X_1,\ldots,X_m) = t_{T_0}^{(2)}(X_1,\ldots,X_n)+t_{T_1}^{(2)}(X_{n+1},\ldots,X_m);$$

and further with a short notation

$$t_{T_0 \cup T_1}^{(3)} = t_{T_0}^{(3)} + t_{T_0}^{(4)} \quad t_{T_1}^{(1)} + t_{T_1}^{(3)}$$

$$t_{T_0 \cup T_1}^{(4)} = t_{T_1}^{(4)},$$

which illustrates the effect of proposition 2.2 (compare with example 1.1). It is easy to see, that all autoregressive processes admit proper totally sufficient reductions for T_0 being an interval of $\mathbb{Z}$.

The following example shows, that the existence of a $p \in \pi$ giving conditional independence is not a superfluous condition in proposition 2.3.

<u>Example 4.2</u>: Let $(X_1,\ldots,X_n)$, $(Y_1,\ldots,Y_n)$ be normally distributed random variables with $EX_i = \xi, EY_i = \eta$, $Var(X_i)=Var(Y_i)=1$

$\mathrm{Cov}(X_i, Y_i) = \rho\delta_{ij}$ where $\xi, \eta \in \mathcal{R}$ are unknown, and ρ is a fixed, known correlation between X_i and Y_i. Let $\mathcal{S}_1 = \sigma(\sum_{i=1}^{n} X_i)$, $\mathcal{S}_2 = \sigma(\sum_{i=1}^{n} Y_i)$, $\mathcal{A}_1 = \sigma(X_1,\ldots,X_n)$, $\mathcal{A}_2 = \sigma(Y_1,\ldots,Y_n)$, π the family of probability measures on $\mathcal{R}^{2n}$ defined above. It is clear, that $\mathcal{S}_1$ suf $(\pi, \mathcal{A}_1)$, $\mathcal{S}_2$ suf $(\pi, \mathcal{A}_2)$ and $\mathcal{S}_1 \vee \mathcal{S}_2$ suf $(\pi, \mathcal{A}_1 \vee \mathcal{A}_2)$. If however, $\rho \neq 0$ it is not true, that $\mathcal{A}_1 \perp^{\mathcal{S}_1} \mathcal{A}_2$ for any value of ξ, η.

5. Questions to be discussed.

The present paper is to be seen as a preliminary report on some ideas about total sufficiency. A lot of open questions is still left to investigation, some of them being of a probabilistic and some of statistical nature.

For $T = \mathbb{Z}$ or $T = Z^n$, what kind of families π on $\mathcal{R}^T$ admit proper totally sufficient reductions of dimension p, whenever T_0 is something nice, say an interval ? Exactly when will total sufficiency and sufficiency coincide ?

How shall we use the statistics for inference purpose ?

What comes out if we try to formulate statistical models as stochastic processes, where a given set of data observed correspond to a part of the indexset ? The concept of a statistical population seems to have a precise formulation in this framework. Problems related to nuisance-structural parameters and conditional inference also seem to be suited for a formulation like this.

We hope to work out answers for some of these questions in the future.

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