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Abstract

A Remark on the Supercritical Branching Process with Random Environments.

For the supercritical branching process with random environments, the rate of growth of the generation size Z_n is studied in the marginal distribution. It is shown that unless the environmental process yields a constant conditional expectation $E(Z_1|\zeta)$, the asymptotic distribution of

$$(Z_n \exp(-nE_\zeta(\log E(Z_1|\zeta))))^{1/\sqrt{n}}$$

is that of Ue^V where U and V are independent, $P(U=0) = 1-P(U=1) = P(Z_n \to 0)$ and V is normal $(0, V_\zeta(\log E(Z_1|\zeta))$.

1. Introduction.

In a supercritical Galton-Watson branching process $Z_0, Z_1, \ldots$ an important result states that if $E(Z_1 \log Z_1) < \infty$, then $Z_n/EZ_n \to W$ a.s. where $P\{W = 0\} = q = P\{Z_n \to 0\}$ and where otherwise the distribution of W is absolutely continuous (Kesten and Stigum 1966). In their recent treatment of the branching process with random environments, Athreya and Karlin (1971b, p. 1845) for the supercritical process "evaluate the rate of growth of Z_n on the set of nonextinction". They propose to study the limiting distribution of the random variable $W_n = Z_n/E(Z_n|\zeta_0, \ldots, \zeta_{n-1})$, the denominator denoting the conditional expectation of Z_n given the environmental process up to time n-1. Athreya and Karlin obtain results similar to those by Kesten and Stigum.

It seems, however, in some situations more appropriate to consider a "marginal" norming of Z_n which does not depend on the realized environmental process and it is therefore proposed in this note to study the asymptotics of the variables $X_n = Z_n e^{-n\mu}$, $\mu = E_\zeta(\log E(Z_1|\zeta))$ for the case of independent environmental variables with $\sigma^2 = V_\zeta(\log E(Z_1|\zeta)) < \infty$. If $\sigma^2 = 0$, $E(Z_1|\zeta)$ is a.s. constant and the results of the deterministic environment will essentially apply. If $\sigma^2 > 0$, however, there appear rather different results which can be informally stated as follows.

Apart from the event of extinction the fluctuations in X_n are determined by the fluctuations in $E(Z_n|\zeta_0, \ldots, \zeta_{n-1})$ which in turn can be described by the central limit theorem. Furthermore, the event of extinction is asymptotically independent of the latter fluctuations.

An application of the BPRE that has received some attention is demographic forecasting (Pollard 1968, Sykes 1969, Schweder and Hoem 1972). It appears that the variation due to the random environment is much greater than the intrinsic random variation given the environment. The result in this note illustrates these empirical findings.

2. Results.

Let $Z_0, Z_1, Z_2, \ldots$ be a branching process with random environments. The environmental process is denoted by $\overline{\zeta} = (\zeta_0, \zeta_1, \ldots)$ where it is assumed that $\zeta_0, \zeta_1, \ldots$ are independent and identically distributed. Assume $Z_0 = 1$ and define $m(\zeta) \equiv m_1(\zeta) = E(Z_1 | \zeta)$ and

$$m_n(\zeta_0, \zeta_1, \ldots, \zeta_{n-1}) = E(Z_n | \overline{\zeta}) = \prod_{i=0}^{n-1} m(\zeta_i).$$

For details, see Athreya and Karlin (1971a).

<u>Theorem</u>. Let $Z_n, n = 0,1,2,\ldots$ be a <u>supercritical</u> branching process with random environments, <u>i.e.</u> $E_\zeta(-\log P\{Z_1 > 0 | \zeta\}) < \infty$ and $E_\zeta(\log m(\zeta))^- < E_\zeta(\log m(\zeta))^+ \leqq \infty$ and assume that $E_\zeta(E(Z_1 \log Z_1 | \zeta)/m(\zeta)) < \infty$.

Let $\mu = E_\zeta(\log m(\zeta))$ and $\sigma^2 = V_\zeta(\log m(\zeta)) < \infty$.

(a) If $\sigma^2 = 0$, $Z_n e^{-n\mu} \to W$ a.s. as $n \to \infty$ where $P\{W = 0\} = q = P\{Z_n \to 0\}$.

(b) If $\sigma^2 > 0$, the asymptotic distribution of $(Z_n e^{-n\mu})^{\frac{1}{\sqrt{n}}}$ as $n \to \infty$ is that of Ue^V where U and V are independent, $P(U = 0) = 1 - P(U=1) = q$ and V is normal $(0, \sigma^2)$.

Proof. Recall from Athreya and Karlin (1971b, Theorem 1) the following properties under the stated assumptions: $W_n = Z_n/m_n(\zeta_0,\zeta_1,\ldots,\zeta_{n-1}) \to W$ a.s. as $n \to \infty$. $E(W|\bar{\zeta}) \neq 1$ and $P\{W = 0|\bar{\zeta}\} = q(\bar{\zeta}) = P\{Z_n \to 0|\bar{\zeta}\} < 1$ a.s. It follows that $\{W = 0\} = \{Z_n \to 0\}$ a.s. and that these events have probability $q = E_{\bar{\zeta}}q(\bar{\zeta})$.

Proof of (a). When $\sigma^2 = 0$, $m(\zeta) = m$ a.s., $\mu = \log m$ and $m_n(\zeta_0,\ldots,\zeta_{n-1}) = e^{n\mu}$ a.s. so that the result is a trivial corollary of the just stated facts.

Proof of (b). Write $Z_n e^{-n\mu} = W_n Y_n$ so that $Y_n = m_n(\zeta_0,\ldots,\zeta_{n-1})e^{-n\mu}$.

Let k_n be a sequence of integers such that $k_n \to \infty$ and $k_n n^{-\frac{1}{2}} \to 0$.

Define $U_n = I\{Z_{k_n} \neq 0\}$ and $X_n = m(\zeta_{k_n})\ldots m(\zeta_{n-1})e^{-n\mu}$. Since U_n depends on $(Z_0,\zeta_0,\ldots,Z_{k_n-1},\zeta_{k_n-1},Z_{k_n})$ only it is seen that U_n and X_n are independent. The proof will now be carried out by proving

(i) $U_n - W_n^{\frac{1}{\sqrt{n}}} \to 0$ a.s. and hence in probability,

(ii) $X_n^{\frac{1}{\sqrt{n}}} - Y_n^{\frac{1}{\sqrt{n}}} \xrightarrow{P} 0$,

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and

$$\text{(iii)} \qquad \frac{1}{\sqrt{n}} \log Y_n \text{ is asymptotically normal } (0,\sigma^2).$$

The proof will then be complete by the remark that by (i) and (ii), the limiting distribution of the pair $(W_n^{\frac{1}{\sqrt{n}}}, Y_n^{\frac{1}{\sqrt{n}}})$ is the same as that of the pair of independent variables $(U_n, X_n^{\frac{1}{\sqrt{n}}})$. But $U_n \to I\{Z_n \to \infty\}$ a.s. with $P\{Z_n \to \infty\} = 1 - P\{Z_n \to 0\} = 1 - q$ and the limiting distribution of $X_n^{\frac{1}{\sqrt{n}}}$ is, by (ii), the same as that of $Y_n^{\frac{1}{\sqrt{n}}}$ given by (iii).

Proof of (i). It follows from the results quoted above that $W_n \to W$ a.s. and $P\{W = 0\} = q$.

On the set $\{W = 0\}$, $W_n = 0$ from some n on a.s., and it follows that

$W_n^{\frac{1}{\sqrt{n}}} \to 0$ a.s. On the set $\{W > 0\}$, since $E(W|\zeta) = 1$, it is true a.s. that $0 < W < \infty$ and hence that $W_n^{\frac{1}{\sqrt{n}}} \to 1$. The proof is complete since

$\{W = 0\} = \{Z_n \to 0\}$ a.s.

Proof of (iii).

$$n^{-\frac{1}{2}} \log Y_n = \frac{\sum_{i=0}^{n-1} \log m(\zeta_i) - n\mu}{\sqrt{n}}$$

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and the result is immediate from the central limit theorem.

Proof of (ii).

$$Y_n^{\frac{1}{\sqrt{n}}} - X_n^{\frac{1}{\sqrt{n}}} = (m_n(\zeta_0,\zeta_1,\ldots,\zeta_{n-1})e^{-n\mu})^{\frac{1}{\sqrt{n}}} \; (1 - m_{k_n}(\zeta_0,\ldots,\zeta_{k_n-1})^{-\frac{1}{\sqrt{n}}}).$$

Since $$\frac{\log m_{k_n}(\zeta_0,\ldots,\zeta_{k_n-1}) - k_n\mu}{\sqrt{k_n}}$$

is asymptotically normal $(0,\sigma^2)$ by the central limit theorem cf. the proof above, and $k_n n^{-\frac{1}{2}} \to 0$, $m_{k_n}(\zeta_0,\ldots,\zeta_{k_n-1})^{-\frac{1}{\sqrt{n}}} \xrightarrow{P} 1$.

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