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Introduction.

It is the purpose of statistical mechanics and kinetic theory to give a description of the properties of continuous media, in particular their thermal behaviour, based on the atomic hypothesis, i.e. on the assumption that they consist of an enormous number of particles moving according to the laws of mechanics. Nobody has until now been able to derive the statistical mechanical laws without statistical arguments with no foundation in mechanics. Several different approaches have been suggested. The most famous of these uses the ergodic theorem and has received the most rigorous mathematical treatment. Khinchine gave a very clear exposition of the theory in his book [1949].

The main defects of the ergodic theory are the following: 1) the method is based on averages over infinite time which gives only a description of equilibrium states, no dynamical theory, and therefore no explanation of irreversible processes, 2) the ergodic theorem provides a so-called statistical statement (convergence almost everywhere) which can only be applied on statistical assumptions, 3) it is not used directly that the systems contain many particles.

Until recently a main argument against the theory was the lack of a proof of the ergodic hypothesis for mechanical systems. This objection is no longer valid after the contributions of the Russian school, see Sinai [1966].

We shall in this paper review the fundamental physical problem and suggest a method which leads directly from particle mechanics to statistical mechanics. The method uses approximation considerations of ordinary topological type and no statistical arguments, and it provides a dynamical as well as a steady state theory. The mechanical systems considered here are too simple to be of great physical interest, but the method can be generalized to more complex systems.

The results were announced in a paper read to the European Meeting of Statisticians in London 1966.

I Mechanical systems of independent particles.

1. The motion of a single particle.

The state of a single particle is described by a point in its phase space Γ. If the particle has no interior degrees of freedom Γ is a locally compact subset of R^{2n}, where R is the real line and n = 1, 2, or 3. The first half of the coordinates represent the position of the particle in R^n and the other half the corresponding velocities (not moments).

If the particle moves in a conservative field with an intensity proportional to its mass, the trajectory of the particle is independent of its mass and Liouville's theorem will hold. This is equivalent to considering a Borel measurable semigroup $(T_t)_{t \in R_+}$ of transformations of Γ which leave Lebesgue measure λ invariant ($R_+ = [0, +\infty[$). We shall furthermore assume that there is a subset Γ_1 of Γ on which all T_t are bijective and such that $\lambda\Gamma_1 = \lambda\Gamma$. Finally each transformation T_t is supposed to be continuous a.e. with respect to λ.

2. Systems of independent particles.

We consider now a mechanical system of N identical particles moving completely independent of each other, and each particle in the way described in 1. The phase space is Γ^N and the transformation from time zero to time t of the whole system is determined by

$$(\gamma_1, \ldots, \gamma_N) \mapsto (T_t\gamma_1, \ldots, T_t\gamma_N) \ \forall \ t \geq 0$$

where T_t is transformation for one particle.

If we consider these transformations for all N we have a complete description of the behaviour of any finite set of particles down to the last detail for every single particle. In statistical mechanics one is concerned with the macroscopic aspect of the system, i.e. the time development of mechanical properties to which all particles contribute in the same way. Mathematically we shall define a macroscopic property as a function depending only on the distribution of mass in Γ. To determine the evolution of the macroscopic properties it is sufficient to know the mass distribution for all t.

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If a system consists of N particles with mass $m_1, \dots, m_N$ and initial values $\gamma_1, \dots, \gamma_N$ the mass distribution M at time zero is determined by

$$MA = \sum_{i=1}^{N} m_i I_{\gamma_i} A \qquad \forall\ A \subset \Gamma$$

where

$$I_\gamma A = \begin{cases} 0 & \forall \gamma \in A \\ 1 & \forall \gamma \in \Gamma \setminus A. \end{cases}$$

At time t the particles are in $T_t\gamma_1, \dots, T_t\gamma_N$ respectively and the mass distribution M_t is given by

$$M_t A = \sum_{i=1}^{N} m_i I_{T_t\gamma_i} A = \sum_{i=1}^{N} m_i I_{\gamma_i} T_t^{-1} A$$

or

(1) $$M_t A = MT_t^{-1} A \qquad \forall\ A \subset \Gamma$$

which shows that M is transformed into M_t by T_t. (1) determines uniquely the time development of the mass distribution for any finite set of particles.

If all particles have the same mass m the number distribution is

$$\frac{1}{m} MA \qquad \forall\ A \subset \Gamma.$$

3. Arbitrary mass distributions.

Equation (1) defines a transformation of the set $\acute{M}$ of mass distributions with finite support. A straightforward generalization of (1) to any bounded measure μ on Γ is

$$\mu_t A = \mu T_t^{-1} A \qquad \forall \text{ Borel sets } A \subset \Gamma.$$

This definition can be considered as an extension by continuity. Let μ be a bounded measure on Γ, absolutely continuous with respect to λ, and let $M \in \acute{M}$ converge vaguely towards μ. Then $M\Gamma$ will tend to $\mu\Gamma$ and since T_t is continuous, a.e.[λ] MT_t^{-1} will tend vaguely towards μT_t^{-1} (Bourbaki [1965], p. 71 and 201).

In other words, if we choose to approximate our initial mass distribution by an absolutely continuous distribution we can use the transformation by T_t of this measure as an approximation to the mass distribution at time t.

We shall consider a few general measure theoretic results in this connection.

II Stable measures.

Let μ be a bounded measure on Γ and let $\acute{C}_1$ be the σ-field of T-invariant sets in Γ_1. $\mu_{\acute{C}_1}$ is the restriction of μ to $\acute{C}_1$. According to a theorem by Blum and Hanson [1960] there is at most one invariant measure ν such that $\mu_{\acute{C}_1} = \nu_{\acute{C}_1}$. If we assume that μ is absolutely continuous with respect to λ it follows that there is at most one invariant ν such that $\mu_{\acute{C}} = \nu_{\acute{C}}$.

We say that ν is the invariant measure determined by μ.

Theorem 1. Assume that λ is σ-finite on $\acute{C}$. Every bounded measure μ which is absolutely continuous with respect to λ determines an invariant measure given by

$$\nu A = \int \lambda_\gamma^{\acute{C}}(A)\mu(d\gamma)$$

for all Borel sets A. Here $\lambda_\gamma^{\acute{C}}$ is the value in γ of a version of the conditional probability of A for given $\acute{C}$ relative to λ.

ν is absolutely continuous with respect to λ and has as density the conditional expectation of ν's density for given $\acute{C}$ relative to λ.

Proof. For the first statement it is sufficient to show that the measure ν is invariant and that $\nu_{\acute{C}} = \mu_{\acute{C}}$.

For all $C \in \acute{C}$ is

$$\int_C \lambda_\gamma^{\acute{C}} T^{-1} A \, \lambda(d\gamma) = \lambda(T^{-1}A \cap C) = \lambda T^{-1}(A \cap C)$$

$$= \lambda(A \cap C) = \int_C \lambda_\gamma^{\acute{C}} A \, \lambda(d\gamma)$$

and therefore

$$\lambda_\gamma^{\acute{C}} T^{-1} A = \lambda_\gamma^{\acute{C}} A \quad [\lambda].$$

Let f be μ's density with respect to λ. Then

$$\nu T^{-1} A = \int \lambda_\gamma^{\acute{C}} T^{-1} A f(\gamma) \lambda(d\gamma) = \int \lambda_\gamma^{\acute{C}} A f(\gamma) \lambda(d\gamma) = \nu A.$$

For $C \in \acute{C}$ is IC measurable with respect to $\acute{C}$ and

$$\int_{C_1} I_\gamma C \, \lambda(d\gamma) = \lambda(C_1 \cap C) \quad \forall \, C_1 \in \acute{C}.$$

Therefore IC is a conditional probability of C for given $\acute{C}$ relative to λ, and so $\nu C = \int I_\gamma C f(\gamma) \lambda(d\gamma) = \mu C$.

The second statement follows from the fact that the density of ν with respect to μ is $\acute{C}$-measurable and the identities

$$\nu C = \mu C = \int_C f(\gamma) \, \lambda(d\gamma) = \int_C E_\gamma^{\acute{C},\lambda}(f) \, \lambda(d\gamma)$$

for $C \in \acute{C}$.

A measure μ has the measure ν as <u>equilibrium measure</u> if μT_t^{-1} tends vaguely to ν for $t \to \infty$.

A continuous function between locally compact spaces is said to be <u>proper</u> if the inverse images of compact sets are compact.

Theorem 2. Assume that $\acute{C}$ is generated by a proper continuous real function H on Γ, and let μ be a bounded measure absolutely continuous with respect to λ. Then λ is σ-finite on $\acute{C}$, and if μ has an equilibrium measure it is the invariant measure determined by μ.

Proof. The first statement is a trivial consequence of the σ-compactness of R. Let ν be the invariant measure determined by μ, and let ν' be the equilibrium measure. If h is a real continuous function on R with compact support then $h \circ H = h \circ H \circ T_t$ is continuous with compact support and so

$$\nu'(h \circ H) = \lim_{t \to \infty} T_t\mu(h \circ H) = \mu(h \circ H) = \nu(h \circ H).$$

It follows that $\nu'(h' \circ H) = \nu(h' \circ H)$ for all bounded Borel-measurable real functions h' on R and therefore $\nu'_C = \nu_C$.

An invariant measure ν is stable if every bounded measure absolutely continuous with respect to ν has an equilibrium measure.

Theorem 3. Let $\acute{D}$ be a vague convergence determining class of Borel subsets of Γ. A necessary and sufficient condition for an invariant measure λ to be stable is that

$$\lambda((T_t^{-1} D) \cap B)$$

has a limit for $t \to \infty$ for all $D \in \acute{D}$ and all B in a class of sets for which the indicators form a fundamental subset of the Lebesgue space $L_1(\lambda)$.

To prove the sufficiency let L_t be defined by

$$L_t(g) = \int I_\gamma(T_t^{-1} D)g(\gamma)\lambda(d\gamma)$$

for $g \in L_1(\lambda)$. Then L_t is a linear functional on $L_1(\lambda)$ and $|L_t(g)| \leq ||g||$. By assumption $L_t(g)$ has a limit as $t \to \infty$ for all g in a fundamental set and so by the principle of uniform boundedness it has a limit for all $g \in L_1(\lambda)$ (Dunford and Schwarts [1958], p. 55).

Let now μ be a measure on Γ which is bounded and absolutely continuous with respect to λ, and let f be its density. Then $f \in L_1(\lambda)$ and since

$$T_t\mu(D) = \int I_\gamma(T_t^{-1}D)f(\gamma)\lambda(d\gamma)$$

has a limit for $t \to \infty$ it follows that $T_t\mu(h)$ has a limit $\nu(h)$ for all $h \in K(\Gamma)$. ν is a positive linear functional on $K(\Gamma)$ and so a measure on Γ.

The necessity of the condition in the theorem is now trivial and so the proof is finished.

It is important to notice that the theorems can be proved without the ergodic theorem.

Our concept of a stable measure stands in close connection with the stable transformations considered by Maitra [1965], compare Renyi [1963].

III 1-dimensional gas.

1. Stability of Lebesgue measure.

Consider a particle moving back and forth in the interval $[0, V]$ with elastic reflexion at 0 and V ($V > 0$). This mechanical system satisfies the assumptions made in I.1, and the system consisting of a number of particles moving independently in this way we shall call a 1-dimensional gas.

Let us specify the system mathematically. The phase space of one particle is $\Gamma = [0, V] \times R$ which is a closed subset of R^2. We shall denote Lebesgue measure on Γ by λ. It is the product of arclength (Lebesgue measure) on $[0, V]$ and R.

Let φ be the real function on R which is even and periodic with period 2V, and which on $]0, V]$ equals the identity. φ is continuous and left differentiable. The derivative ψ is the real function on R which is uneven and periodic with period 2V, and which on $]0, V]$ is constant equal to 1.

For every $t \in R$ let T_t denote the transformation which takes (x, c) into $(\varphi(x+ct), c\psi(x+ct))$. The family $(T_t)_{t \in R_+}$ is a transformation semi-group on Γ describing the behaviour of one particle. The transformations are continuous $[\lambda]$.

All the transformations leave the set

$$\Gamma_1 = (]0, V] \times [0, +\infty[) \cup ([0, V[\times]-\infty, 0])$$

invariant and are bijective on Γ_1. Therefore $(T_t)_{t\in R}$ is a transformation group on Γ_1, but for mathematical convenience we shall consider the semi-group on Γ.

The kinetic energy (per unit mass) is the proper continuous function $H : \Gamma \to R_+ \; (x, c) \mapsto c^2)$. The surface of constant energy e is the set

$$H^{-1}(e) = [0, V] \times (\{\sqrt{e}\} \cup \{-\sqrt{e}\}).$$

The orbit of $(x, c) \in \Gamma$ is $H^{-1}H(x, c)$ and the σ-field $\acute{C}$ of invariant sets is the σ-field induced by H.

Let ρ_e denote arclength on $H^{-1}(e)$ considered as a measure on Γ. Since λ is a product measure on Γ which is invariant under the transformation $(x, c) \mapsto (x, -c)$ it follows that the conditional distribution of λ for given H is $\frac{1}{2V}\rho_H$.

Theorem 4. In the 1-dimensional gasmodel Lebesgue measure is invariant and stable.

Proof. λ is invariant under T_t because T_t is piece-wise linear with determinant

$$\begin{vmatrix} \psi(x+ct) & 0 \\ t\,\psi(x+ct) & \psi(x+ct) \end{vmatrix} = 1.$$

To prove stability consider the class $\acute{D}$ of intervals of the form $[0, a] \times [b_1, b_2]$ where $0 < a \leq V$ and $b_1 < 0 < b_2$. $\acute{D}$ is clearly vague convergence determining, and also the indicators for $D \in \acute{D}$ form a set which is total in the Lebesgue space $L_1(\lambda)$.

Now let $D = [0, a] \times [b_1, b_2]$ and $D' = [0, a'] \times [b'_1, b'_2]$ both be in $\acute{D}$. Then for $t > 0$

$$\lambda(T_t^{-1} D \cap D')$$

$$= \int_0^V \int_{-\infty}^{\infty} I_{\varphi(x+ct)}[0, a] \; I_{c\psi(x+ct)}[b_1, b_2] \; I_x[0, a'] \; I_c[b'_1, b'_2] dc)dx$$

which by substituting $u = x+ct$ becomes

$$\int_0^{a'} I_t(x)dx$$

where

$$I_t(x) = \frac{1}{t}\int_{-\infty}^{\infty} I_{\varphi(u)}[0,a]\, I_{\frac{u-x}{t}\psi(u)}[b_1,b_2]\, I_{\frac{u-x}{t}}[b'_1,b'_2]du.$$

Let $A = \{u \in R \mid \psi(u) = 1\}$. Then $A^c = \{u \in R \mid \psi(u) = -1\}$ and so

$$I_t(x) = \frac{1}{t}\int_{-\infty}^{\infty} I_u A\; I_{\varphi(u)}[0,a]\; I_{\frac{u-x}{t}}[b_1,b_2]\; I_{\frac{u-x}{t}}[b'_1,b'_2]du$$

$$+\frac{1}{t}\int_{-\infty}^{\infty} I_u\; A^c\; I_{\varphi(u)}[0,a]\, I_{-\frac{u-x}{t}}[b_1,b_2]\, I_{\frac{u-x}{t}}[b'_1,b'_2]du$$

$$= \frac{1}{t}\int_{x+t\max(b_1,b'_1)}^{x+t\min(b_2,b'_2)} I_u A\; I_{\varphi(u)}[0,a_1]du$$

$$+\frac{1}{t}\int_{x+t\max(-b_2,b'_1)}^{x+t\min(-b_1,b'_2)} I_u A^c I_{\varphi(u)}[0,a_1]du.$$

In this expression both integrands are bounded periodic functions with period 2V, and the length of the integration interval is proportional to t. Therefore both addends tend to a limit for $t \to \infty$, and so we have proved that $I_t(x)$ has a limit for $t \to \infty$ for all x. Since $I_t(x)$ is uniformly bounded it follows that $\lambda(T_t^{-1} D \cap D')$ has a limit for $t \to \infty$, and the theorem now follows from theorem 3.

It follows from theorem 1, 2, and 4 that an initial distribution with density f will tend towards an equilibrium measure with density

$$f_\infty(x,c) = \int f(\gamma)\,\frac{1}{2V}\,\rho_{H(x,c)}(d\gamma) = \frac{1}{V}\cdot\frac{1}{2}(g(c)+g(-c))$$

where

$$g(c) = \int_0^V f(x,c)dx$$

is the initial marginal distribution of velocity. The equilibrium distribution is the product of the marginal distributions of position and velocity. The position distribution is uniform on $[0,V]$ and the velocity distribution is the symmetrized distribution corresponding to the initial velocity distribution.

The density in $(x,c) \in \Gamma$ at time t is $f \circ T_t^{-1}(x,c) = f(\varphi(x-ct),\ c\varphi(x-ct))$. For $c \neq 0$ this is a periodic function of t with period $\frac{2V}{|c|}$, and so the density does not converge pointwise to a limit for $t \to \infty$. This shows the importance of introducing the concept of vague convergence.

2. The Pressure.

Let x be an interior point in $[0,V]$. The <u>mean velocity</u> in x at time t is the conditional mean value of c given position equal to x relative to μT_t^{-1}. If μ has density f with respect to λ then μT_t^{-1} will have density $f_t = f \circ T_t^{-1}$, and the mean velocity in x is

$$\bar{c}_t(x) = \frac{1}{h_t(x)} \int c\ f_t(x,c)dc$$

where

$$h_t(x) = \int f_t(x,c)dc$$

is the marginal distribution of position. $\bar{c}_t(x)$ is well-defined when c is integrabel with respect to μT_t^{-1}. Then mean velocity is zero if f corresponds to an equilibrium distribution.

To define the pressure in x consider the <u>total flow of momentum</u> in x during the time interval $[0,t]\ (t > 0)$. This quantity is defined as

$$\Delta(t) = \int_\Gamma \Delta_{(x',c)}(t)\ \mu(dx'\, dc)$$

where the integrand is the finite sum

$$\Delta_{(x',c)} = \sum_{0 \leq s \leq t} (c\psi(x'+cs) - \bar{c}_t(x))I_{\varphi(x'+cs)}\{x\}.$$

Theorem 5. If μ has finite energy, $\Delta(t)$ exists and is finite for all $t \in R_+$, and Δ is of bounded variation.

Proof. The energy of μ is $\int H(\gamma)\mu(d\gamma) = \int c^2\mu(dxdc)$ and so $\bar{c}_t(x)$ exists.

Using the periodic character of φ it is seen that

$$|\Delta_{(x',c)}(t)| \leq (|c|+|\bar{c}_t(x)|)\sum_s I_{\varphi(x'+ct)}\{x\} \leq (|c|+|\bar{c}_t(x)|)2\left(\frac{t}{\frac{2V}{|c|}}+1\right)$$

or

$$|\Delta_{(x',c)}(t)| \leq (|c|+|\bar{c}_t(x)|)(t|c|+1)$$

which shows that Δ exists and is finite. The second statement follows by splitting the sum $\Delta_{(x',c)}(t)$ in its positive and negative parts which both are monotone in t.

Let μ be a measure with finite total energy. If Δ is absolutely continuous with derivative $t \mapsto p_t(x)$ we shall say that μ has <u>pressure</u> $p_t(x)$ in x at time t.

Theorem 6. Let μ be a measure with finite total energy. Assume that μ is absolutely continuous with respect to λ with density f, and put $f_t = f \circ T_t^{-1}$. Then μ has pressure

$$p_t(x) = \int (c-\bar{c}_t(x))^2 f_t(x,c)dc$$

in x at all time points t.

Proof. Splitting into half-periods one finds that

$$\Delta_{(x',c)}(t) = \sum_{k=-\infty}^{\infty}\sum_s (c-\bar{c}_t(x))(I_{x'+cs}]2kV,(2k+1)V]I_{x'+cs-2kV}\{x\}$$

$$-(c+\bar{c}_t(x)I_{x'+cs}](2k+1)V,2(k+1)V]I_{-x'-cs+2(k+1)V}\{x\})$$

$$=\sum_k\sum_s((c-\bar{c}_t(x))I_{x+cs-2kV}\{x\} - (c+\bar{c}_t(x))I_{-x'-cs+2(k+1)V}\{x\})$$

$$=\sum_k((c-\bar{c}_t(x))I_{x'-2kV}B(x,c)-(c+\bar{c}_t(x))I_{-x'+2(k+1)V}B(x,-c)),$$

where B(x, c) is the interval with endpoints x-ct and x. Therefore

$$\triangle(t) = \int \sum_k ((c-\bar{c}_t(x) \int_0^V I_{x',-2kV}\, B(x,c) f(x',c)dx'$$

$$- (c+\bar{c}_t(x) \int_0^V I_{-x',+2(k+1)V}\, B(x,-c) f(x',c)dx')dc)$$

$$= \int \sum_k ((c-\bar{c}_t(x)) \int_{2kV}^{(2k+1)V} I_y B(x,c) f(\varphi(y),c)dy$$

$$- (c-\bar{c}_t(x)) \int_{(2k+1)V}^{2(k+1)V} I_y B(x,-c) f(\varphi(y),c)dy)dc$$

$$= \iint (c-\bar{c}_t(x)) I_y B(x,c) I_y A f(\varphi(y), c\psi(y))dydc$$

$$+ \iint (-c-\bar{c}_t(x)) I_y B(x,-c) I_{-y} A f(\varphi(y), -c\psi(y))dc$$

$$= \iint (c-\bar{c}_t(x)) I_y B(x,y) f(\varphi(y), c\psi(y))dydc$$

$$= \int (\int_0^t (c-\bar{c}_t(x)) f(\varphi(x-cs), c\psi(x-cs))ds)dc = \int_0^t (\int (c-\bar{c}_t(x))^2 f_s(x,c)dc)ds.$$

Corollary. An equilibrium distribution with finite energy and density f with respect to λ has the pressure

$$p = \frac{1}{V} \int H(\gamma) f(\gamma) d\gamma$$

in all interior points.

It is now natural to ask: Does the pressure in x at time t approach the equilibrium pressure as t tends to infinity? Since the pressure is defined as a Radon-Nikodym derivative we can not in general expect to prove more than some kind of vague convergence. This can actually be done by generalizing the methods of III 1, but we shall not go into details.

3. The Equation of State.

Let μ be a measure with finite energy and density f. The <u>temperature</u> in x at time t is defined as proportional to the conditional expectation of the kinetic energy of a particle given the position relative to μT_t^{-1}

$$kT(x) = m \frac{\int (c-\bar{c}_t(x))^2 f_t(x,c)dc}{h_t(x)}$$

(k is a constant). It follows that

$$p_t(x) = \frac{h_t(x)}{m} kT(x)$$

the local equation of state. If μ is an equilibrium measure, T(x) is independent of x and

$$pV = NkT$$

which is the equation of state for an ideal gas

4. A Theorem of Boltzmann's Type.

It is a consequence of Liouville's theorem that Boltzmann's H-function

$$\int f_t(\gamma) \log f_t(\gamma)\lambda(d\gamma)$$

is constant in t, but there are other similar functions which are strictly decreasing. Here is an example.

Theorem 7. Let μ have density $(x,c) \mapsto \frac{1}{V} g(c)$, and let g_t and g_∞ be the marginal velocity distribution at time t and equilibrium respectively. The function

$$B(t) = \int (-\log \frac{g_t(c)}{g_\infty(c)}) g_\infty(c)dc$$

is strictly decreasing in t except when g is symmetric and therefore $g = g_\infty$.

Proof. By defintion

$$g_t(c) = \frac{1}{V}\int_0^V g(c\psi(x-ct))dx$$

and so

$$g_t(c) = \frac{1}{V}\,(g(c)\lambda A(c) + g(-c)(V-\lambda A(c)))$$

where $A(c)=\{x \in [0,V] \mid \psi(x-ct) = 1\}$ and λ is Lebesgue measure on $[0,V]$. By Jensen's inequality it follows that

$$(1)\quad B(t) \leqq \int \frac{1}{V}(\lambda A(c)(-\log \frac{g(c)}{g_\infty(c)} + (V-\lambda A(c))(-\log \frac{g(-c)}{g_\infty(c)}))\; g_\infty(c)dc.$$

Since $x \in A(c)$ if and only if $V-x \in A(-c)$, $\lambda A(c) = \lambda A(-c)$, and so

$$B(t) \leqq \int \frac{1}{V}(\lambda A(c)(-\log \frac{g(c)}{g_\infty(c)})\; g_\infty(c) + (V-\lambda A(-c))(-\log \frac{g(-c)}{g_\infty(-c)})\; g_\infty(-c))dc$$

$$= \int(-\log \frac{g(c)}{g_\infty(c)})\; g_\infty(c)dc = H(0).$$

Equality holds in (1) exactly when $g(c) = g(-c)$ almost surely with respect to Lebesgue measure.

IV Concluding remarks.

The equation of state has been derived directly from the mechanical model without extra assumptions. The importance of the example lies in the demonstration that irreversible behaviour is compatible with the laws of mechanics and is peculiar to many-particle systems. Our definitions should be compared with the classical definitions in kinetic theory (see e.g. Chapman and Cowling [1952], chap. 2).

The computations in III can be carried through for systems of particles moving freely in containers of different shapes and reflected at the walls. By refining the approximation considerations in I.3 it is possible to treat interacting systems of particles.

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