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Mathematics of Topological Order

Superselection Sector Theory of Levin-Wen Models

PhD thesis

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Abstract

In this thesis, we study mathematical aspects of topological order in two-dimensional quantum spin systems. This is a phenomenon in models of quantum matter at very low temperatures with extraordinary properties. Theoretically, we aim to establish topological order as a well-defined feature of phases of matter, and to classify phases according to different kinds of topological order. Central mathematical challenges towards this aim are to give a definition of phases, to define topological order as an invariant of a phase, and to compute the invariant. Recent years have seen much progress on the former two challenges. In particular, superselection sector theory adapted to spin systems in the thermodynamic limit has been shown to define an invariant of topologically ordered phases with respect to quasi-local automorphism, corresponding to local perturbations of the dynamics. In this thesis, we compute this invariant for the class of Levin-Wen models which is widely regarded to be an exhaustive set of models of gapped phases admitting a gapped boundary. The invariant is a unitary modular tensor category, also known as a theory of anyons.

As a first case, we study the double semion state, the first example of a group-based quantum double model with a cohomological twist studied using superselection sector theory. This theory is shown to yield the representation category of the twisted Drinfeld double algebra $\mathcal{D}^\phi(\mathbb{Z}_2)$. The string operators of this model can be defined by elementary constructions on the lattice and give a representation of the anyon theory on the observable algebra.

This is no longer the case, when we proceed to study the Levin-Wen model based on arbitrary unitary fusion categories $\mathcal{C}$, with obstruction coming from K-theory. Here, the analysis of the ground state and its excitation as well as the construction of string operators requires a detailed understanding of the local Hilbert spaces of the Levin-Wen model in terms of skein theory. This relies on the connection between the Levin-Wen model and the Turaev-Viro topological quantum field theory, and enables the construction of string operators producing anyon representations. The main result is the unitary braided monoidal equivalence,

$$\mathsf{SSS}_f \simeq Z(\mathcal{C}),$$

between the full subcategory of superselection sectors of the Levin-Wen model with finite-dimensional endomorphism spaces, and the Drinfeld center of the category $\mathcal{C}$.

Sammenfatning

Denne afhandling omhandler matematiske aspekter af topologisk orden i kvante-spin-systemer i to dimensioner. Dette er et fænomen i modeller for kvantematerie ved meget lave temperaturer med ekstraordinære egenskaber. Som teoretisk mål søger man at etablere topologisk orden som en veldefineret egenskab for faser af materie og at klassificere faser alt efter forskellige typer af topologisk orden. Centrale matematiske udfordringer for denne ambition omfatter definitionen af faser, samt definitionen af topologisk orden som en invariant for faser, og ikke mindst beregningen af denne invariant. I de senere år har der været betydelig fremgang i forhold til at løse førstnævnte udfordringer. Navnlig er superselektionsteori, adapteret til spin systemer i den termodynamiske grænse, blevet etableret som en invariant for topologisk ordnede faser med hensyn til kvasilokale automorfier, svarende til lokale perturbationer af dynamikken. I denne afhandling beregnes denne invariant for klassen af Levin-Wen modeller, som er betragtet som værende repræsentativ for faser med et gab, og som tillader rand med gab. Invarianten er en unitær modulær tensorkategori, også kendt som en teori om anyoner.

Som et første tilfælde, studerer vi dobbel-semion modellen, det første eksempel på en kvante-dobbel-model baseret på en gruppe med et ikke-trivielt cohomologisk tvist studeret ved brug a superselektions sektor teori. Denne teori svarer til repræsentationskategorien for kvante-dobbelalgebraen $\mathcal{D}^{\phi}(\mathbb{Z}_2)$. Strenge-operatorerne for denne model kan defineres ved elementære betragtninger af lattice-modellen, og giver en repræsentation af anyon teorien på algebraen af observable.

Det samme gør sig ikke længere gældende, når vi går videre til at studere Levin-Wen modellen baseret på en arbitrær unitær fusionskategori $\mathcal{C}$, hvor en obstruktion er givet ved K-teori. Her afhænger analysen af en detaljeret forståelse af de lokale frihedsgrader for modellen i form af fed-teori (skein theory) og muliggør konstruktionen af strengeoperatorer, der producerer anyon repræsentationer. Hovedresultatet er den unitære flettede monoidale ækvivalens,

$$\mathsf{SSS}_f \simeq Z(\mathcal{C}),$$

mellem den fulde underkategori af superselektionssektorer for Levin-Wen modellen med endelig-dimensionelle endomorfirum, og Drinfeld centeret af kategorien $\mathcal{C}$.

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This thesis is based on the following manuscripts to which all named authors contributed equally.

Chapter 2 consists of

[BKM24] Alex Bols, Boris Kjær, and Alvin Moon. The double semion state in infinite volume. *Annales Henri Poincaré*, (3):1009–1053. `doi:10.1007/s00023-024-01445-y`.

A draft of Chapter 3 has been divided and presented on the arXiv in

[BK25] Alex Bols and Boris Kjær. Sector Theory of Levin-Wen Models I : Classification of Anyon Sectors, 2025. `arXiv:2511.21521`.

[BK26] Alex Bols and Boris Kjær. Sector Theory of Levin-Wen Models II : Fusion and Braiding, 2026. `arXiv:2603.01936`.

Concretely, Section 1 of Chapter 3 is based on the Introduction of [BK26], and includes all of [BK25] except for Section 1. Section 2.2 of [BK26] appears as Appendix C, and everything from Section 5 and onwards of [BK26] is included in full. Sections 3 and 4 of [BK26] are redundant and are omitted from Chapter 3.

During the course of the PhD programme this author has also contributed as a co-author of

[HKK25] Ulrik Thinggaard Hansen, Boris Kjær, and Frederik Ravn Klausen. The uniform even subgraph and its connection to phase transitions of graphical representations of the Ising model. *Communications in Mathematical Physics*, (6):124. `doi:10.1007/s00220-025-05297-3`.

Introduction

This thesis is a development of mathematical theory that is aimed at describing the physical world. As the main text is about mathematics, we give a brief account of how the theory is founded in physics, followed by an outline of the thesis and an overview of its main results.

Statistical Physics

Statistical physics is a broad term for physical theories that use probabilistic methods to describe emergent *macroscopic* phenomena in systems composed of many constituent bodies or components. Examples of such phenomena include temperature, pressure, magnetisation, (Bose-Einstein) condensation, (super-) conductance, and many more. Under more or less simplistic assumptions of the nature of constituent parts of a system, a statistical mechanical model is set up to describe the *microscopic* kinematics and interactions among these parts, with the goal of deriving quantities that describe the observable macroscopic behaviour.

Eager with excitement at first encounter with the subject, one could hope that our models of nature are rich with detail down to the subatomic scale and give valid predictions about the world as we see it. In practice, things are the other way round, with the most simplistic models being those best studied. The wisdom lies in the idea of *universality.* In a loose sense, the idea is that different microscopic models may give the same macroscopic predictions, and therefore the study reduces to the study of universality classes of models. With this perspective, it is advantageous to study the simplest possible model within a given universality class in order to access physical predictions and understanding.

Spin systems

Spin systems are an important part of (quantum) statistical mechanics. In the most direct physical interpretation, a spin system may be regarded as a mathematical model for a configuration of particles, say molecules, with fixed positions in space, each with some degree of freedom, generically referred to as spin. In classical mechanics the spin degree of freedom is modelled by a topological space - in many examples just a finite set. In quantum mechanics the spin degree of freedom is a Hilbert space, often taken to be finite-dimensional. The notion of 'spin' is historical, and refers to the relation between angular momentum and magnetic dipole moment of a charged body. Usually, we think of the spins as representing the magnetic moment of the particle at a particular point in space. In the famous Ising model, spins take values in the set with two points sometimes referred to as $\uparrow, \downarrow$. As argued by Lenz, who originally introduced the model, the structure of atoms in a crystal would have limited number of preferred orientations or spins and energetically prefer alignment among neighbouring atoms [BR97].

Spin configurations are the possible states of the spin system, which in the case of the Ising model, assigns the spin $\uparrow$ or $\downarrow$ to each point particle of the system. This is seen as a vanilla model for a collection of mini-magnets whose collective alignment (or absence thereof) may (or may not) produce a *macroscopic* magnetic field.

The Ising model describes the energy cost of misalignment of nearby spins. The statistical mechanical approach is to assign for every fixed value of the total energy, a probability distribution on the space of configurations that *in expectation* agrees with a fixed total amount of energy. The total energy is proportional to the temperature of the system, a deeply rooted concept in thermodynamics. The probability distribution assigned is the grand canonical ensemble, also named as the Gibbs state, with the defining property that it maximizes the entropy of the distribution. This is thought of as choosing, among the possible distributions that correspond with the given energy level, the one that assumes the least about the configurations, or in other words, pertains the highest degree of uncertainty about the concrete configuration in which we encounter the system. This philosophy aligns with the idea of not relying on the microscopic details when trying to understand the macroscopic behaviour.

Famously, the Ising model exhibits a phase transition in dimensions greater than one, which is quantified in terms of the emergence of order - as the material cools down, there is not enough energy to keep the spins pointing in all directions. They start to align.

This emergence of order is associated with a corresponding asymmetry. At the outset, there is no preference between $\uparrow$ or $\downarrow$, but the sudden alignment implies that there is a clear majority of spins voting for either $\uparrow$ or $\downarrow$. The symmetry is said to be broken in the cool phase. The correspondence between phases of matter and symmetries has vast implications and is referred to as the Landau paradigm.

Topological order

The microscopic nature of the world is better described by quantum mechanics at low energies, so ever since the advent of quantum mechanics, it has been clear that we ought to study *quantum* statistical mechanics. Nevertheless, classical statistical mechanics has ample application even in quantum field theory, so there is no argument for its dismissal.[1]

Naturally, the quantum mechanical theory generalises the ideas that were sketched before. But the discovery of topological phases of matter in quantum spin systems made it apparent that the Landau paradigm is insufficient. There are ordered phases that are not characterised by the breaking of symmetry in the sense of the Ising model - a group acting on every spin degree of freedom acting simultaneously. After a number of years in the grave, the Landau paradigm was only recently brought back to life with the only possible remedy - extending the notion of symmetry. 1-form symmetries are represented by operations that only alter configurations of the system along loop rather than everywhere at once. For this symmetry to detect non-trivial topological order, the space in which the system is extended should be *topologically non-trivial* meaning that it should have holes. Mathematically, this is easily described as a space that is not simply connected. Owing their name to the breaking of this kind of 1-form symmetry, topologically ordered states of quantum spin systems in two spatial dimensions have been paradigmatic examples for the development of the understanding of a generalised Landau paradigm.

In the simplest examples, the generalised symmetry object can be understood as a group related to the representations of the fundamental group of the physical space of the model. But the symmetry object can be more general, allowing in particular for non-invertible symmetries. A modern slogan is that generalised symmetries can be described by a topological quantum field theory in one dimension greater than the spatial dimension of the symmetric system.

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[1] Quantum and statistical mechanics were more or less contemporary so the label ‘classical’ should not be taken as in ‘the earlier theory’ but rather as the theory which disregards quantum mechanics for the sake of simplicity.

This thesis is part of the research programme of understanding and classifying phases of matter. Specifically, we study topologically ordered spin systems in two dimensions by extracting the generalised symmetry object which characterises the topological order. This is described by the anyon content of the model, thought of as particle-like excitations of the zero-energy state which satisfy fusion and braiding relations. These relations may be regarded as the transformations under the action of the generalised symmetry.

Outline and main results

This thesis comprises three papers and a preliminary chapter, with a total of three individually self-contained chapters. The papers, on which the thesis is based, were each prepared with equal contribution from the respective authors.

Preliminaries The first chapter gives a brief review of notions in mathematical physics that motivate and define phases of gapped quantum spin systems, paraphrasing results from the state of the art. As one can expect, the review is somewhat superficial and fashioned to set the scene for the forthcoming chapters.

The notion of topological order is introduced through the explicit example of lattice gauge theory based on a finite group G, also known as (Kitaev) quantum double models. We give an original proof of the key proposition that describes anyon sectors in this model as seen in finite volume. The purpose is twofold. Stating the precise result gives a way of discussing sectors and motivate their definition in thermodynamic limit. Second, the proof is elementary, in that it does not appeal to constructions from topological quantum field theory or the language of unitary fusion categories, which is heavily used in Chapter 3, where a more general proposition is proven. As such, it serves as a gateway for the uninitiated reader to appreciate the topics being presented in the thesis.

Finally, the section on superselection sector theory summarises the derivation of a braided C*-tensor category from a gapped ground state taken as the definition of an anyon theory. The rôle of Haag duality is emphasised, and weaker notions are discussed. We observe a sufficient criterion for defining the fusion of anyon sectors which is weaker than those found in the literature. While this observation may be original in its formulation, the techniques and results from which it derives all follow the referenced literature.

Double semion The second chapter consists of the published paper [BKM24] in its entirety. This work was the result of our early studies of the superselection sector approach to anyon theory of topological order. As such, the main results are subset of those presented in later chapters. Nonetheless, it was the first treatment of the superselection sector theory of a model beyond the quantum double models based on finite groups as reviewed in Chapter 1. The departure from the group case is, however, not very far, since the model is based on the group $\mathbb{Z}_2$ but twisted by a non-trival co-cycle $\phi \in H^3(\mathbb{Z}^2, U(1))$. This makes it a good case study, in that there are explicit constructions of string operators forming a group of automorphisms of the observable algebra. The analysis of the infinite volume ground state as well as the computation of fusion and braiding of string operators can be understood by direct calculations on the lattice.

The paper also constructs in detail a subcategory of the full category of localized and transportable endomorphisms, which is shown to be a unitary braided fusion category. This does not rely on the assumption of Haag duality or any weakening thereof, since the necessary property of localized intertwiners can be proven directly.

The main result of the paper, is the braided equivalence of this subcategory of superselection sectors with the category of finite-dimensional representations of the twisted Drinfeld double algebra $\mathcal{D}^{\phi}(G)$, which is established by the computation of F- and R-symbols of irreducible objects.

Levin-Wen The final chapter is based on a draft which was presented in two parts on the arXiv [BK25; BK26]. The content is largely identical to these manuscripts, except for a revised introduction and [BK26, Sections 2 and 3] that are redundant, except for Section 2.2 which has been turned into Appendix C of Chapter 3.

The Levin-Wen models [LW05] are believed to exhaust the class of gapped ground state phases that can be realised with a commuting projector Hamiltonian on a two-dimensional quantum spin system. In Chapter 3, we analyse the superselection sectors of Levin-Wen model of an arbitrary unitary fusion category $\mathcal{C}$, deriving, as expected, a unitary braided equivalence with the Drinfeld center of $\mathcal{C}$,

$$\mathsf{SSS}_f \simeq Z(\mathcal{C}).$$

This comprises the largest class of phases for which the superselection sector theory has been computed. Notably, it provides the first complete characterisation of the category of superselection sectors for a class of two-dimensional lattice models supporting anyons with non-integer quantum dimensions.

In order to achieve this, we first classify isomorphism classes of irreducible anyon sectors, showing that they are in one-to-one correspondence with equivalence classes of simple objects of the Drinfeld center $Z(\mathcal{C})$. This is obtained by making explicit how the Levin-Wen Hamiltonian stabilizes subspaces isomorphic to state spaces of the corresponding Turaev-Viro topological quantum field theory, and developing a detailed understanding of these state spaces on punctured disks. In particular, we construct Drinfeld insertion operators on such spaces that can move anyons between the punctures and can change their fusion channels. Using these Drinfeld insertions, we construct explicit string operators that excite anyons above the ground state. By constructing explicit isomorphisms between the fusion spaces of SSS and those of the Drinfeld center $Z(\mathcal{C})$, we show that these two categories have isomorphic F- and R-symbols and conclude the equivalence of unitary braided fusion categories.

Chapter 1

Preliminaries

Contents

This introduction describes the classification problem of gapped ground state phases of spin systems in some detail. We emphasise two key bounds of the theory: The adiabatic theorem and the Lieb-Robinson bound. By clever combination of these ideas Hastings and other authors formed the notion of quasi-local automorphisms, which transform between equivalent gapped ground states of spin systems. We review the notion of topological order and its relation to ground state phases, and end this introduction with a review of the braided C*-tensor category of superselection sectors as an invariant of gapped ground states up to quasi-local automorphism.

1 Quantum many-body systems and gapped phases

1.1 Quantum Dynamics

The instantaneous state space of a quantum mechanical system is described as (the set of rays in) a complex Hilbert space $\mathcal{H}$. The energy of the system is described by a densely defined self-adjoint operator $H : \mathcal{D} \to \mathcal{H}$, the Hamiltonian. This operator governs the dynamics via the Schrödinger equation,

$$i\frac{d}{dt}U(t) = HU(t),$$

with initial condition $U(0) = 1_{\mathcal{H}}$. The unique solution is given by $U(t) = e^{-itH}$, and by Stone's theorem this is a strongly continuous unitary group, meaning that the evolution of any state $\psi \in \mathcal{H}$,

$$t \mapsto U(t)\psi,$$

is continuous with respect to the Hilbert space norm, and $U(t)U(s) = U(t+s)$.

For many purposes, it is reasonable to assume the Hamiltonian to vary over time, if for example H depends on an external potential or represents an effective theory. We thus assume that $t \mapsto H(t)$ is strongly (i.e. point-wise) continuous on some interval I where each $H(t)$ is defined on a time-independent dense domain $\mathcal{D}$. Moreover, we assume that the variation to bounded, that is, $||H(t) - H(s)|| < \infty$ for all $t, s \in I$.

In this setting, we can only guarantee a weak solution to the corresponding time-dependent differential equation [NSY19, Proposition 2.4], namely

$$i\frac{d}{dt}\langle \phi, U(t,s)\psi\rangle = \langle H(t)\phi, U(t,s)\psi\rangle$$

for all $\phi \in \mathcal{D}$ and $\psi \in \mathcal{H}$, with initial condition $U(s,s) = 1_{\mathcal{H}}$. The solution is a unique strongly continuous unitary cocycle $\{U(t,s) \in B(\mathcal{H}) | \, t, s \in I\}$, i.e.

$$U(r,s)U(s,t) = U(r,t) \quad \text{for all } r \leq s \leq t \in I.$$

A notable difference between the dynamics of a Hamiltonian that depends on time and one that does not, is that the evolution of a time-independent Hamiltonian preserves eigenspaces of the Hamiltonian. This is no longer true in the time-dependent case. We can state this more generally in terms of spectral projections of $H(t)$ using the corresponding Heisenberg evolution. This is the transformation of observables $A \in B(\mathcal{H})$, defined for $t, s \in I$, by

$$\tau_{t,s}(A) = \mathrm{Ad}[U(t,s)^*](A) = U(t,s)^* A U(t,s).$$

Spectral projections of $H(t)$ are defined for isolated compact parts of the spectrum of $H(t)$, or equivalently, for any bounded $S \subset \sigma(H(t))$ such that $\gamma_S(t) := \mathrm{dist}(S, \sigma(H(t)) \setminus S) > 0$. One may consider situations where S varies continuously with t, but this will not be necessary here. Instead, we assume that the spectral projection for S is defined for all $H(t)$. Moreover, assume the uniform gap condition: That $\gamma_S(t) \geq \gamma$ for some $\gamma > 0$ independent of t. This defines a strongly continuous family of projections $t \mapsto P(t) \in B(\mathcal{H})$. Generically, assuming that the interaction does vary with time,

$$\tau_{t,s}(P_s) \neq P_t,$$

implying, for instance, that a gapped ground state at time s doesn't evolve into a gapped ground state at time t. But the evolved state could be close to a gapped ground state, and this issue is addressed by the adiabatic theorem.

Under suitable regularity conditions on H inherited by P, we can define the adiabatic Hamiltonian

$$H_a(t) = H(t) + i[P'(t), P(t)]. \tag{1}$$

The Heisenberg evolution as defined for this Hamiltonian $\tau^a_{t,s}$ satisfies

$$\tau^a_{t,s}(P(s)) = P(t), \tag{2}$$

and is called the adiabatic evolution or the spectral flow. The adiabatic theorem [Teu03, Theorem 1.2] gives a bound

$$||\tau_{t,s}(A) - \tau^a_{t,s}(A)|| \leq C(1 + |t-s|)||A||$$

for a constant C which depends on γ. The crucial feature of the theorem is that by slowing down the rate of change of H, considering instead the dynamics of $t \mapsto H(\epsilon t)$, the constant C is found to satisfy $C = O(\epsilon)$. Applied to $A = P(s)$, the adiabatic theorem and Eq. (2) show that the spectral projections are approximately mapped to one another as long as the interaction varies slow enough.

For this application, the adiabatic evolution is a tool to establish a bound the for the original dynamics. Later, in the study of gapped phases, we use the adiabatic theorem in the other direction to approximate a spectral flow.

1.2 Quantum many-body systems

The kinematics of two quantum mechanical systems combined into one whole is described in terms of $\mathcal{H}_1 \otimes \mathcal{H}_2$, where the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ describe the two systems individually. The (time-independet) energy of each individual system may be given by a Hamiltonian H_1, resp. H_2, while the Hamiltonian H of the combined system may include an interaction energy H_{12} between the two parts,

$$H = H_1 + H_2 + H_{12}.$$

Here we use the notation $H_1 = H_1 \otimes 1_{\mathcal{H}_2}$ and $H_2 = 1_{\mathcal{H}_1} \otimes H_2$. By extension, we define an interaction on a collection of 'bodies' indexed by Γ, i.e. Hilbert spaces $\{\mathcal{H}_v\}_{v\in\Gamma}$, to be a map that assigns to each finite subset $\Lambda \subset \Gamma$ a self-adjoint operator,

$$\Phi(\Lambda) \in B(\mathcal{H}_\Lambda)_{s.a.}, \quad \text{where} \quad \mathcal{H}_\Lambda = \bigotimes_{v\in\Lambda} \mathcal{H}_v.$$

Here we make the assumption that each interaction term is bounded, which in principle isn't necessary, but will be the case for our applications. The Hamiltonian associated to Λ is

$$H^\Phi_\Lambda = \sum_{X\subset\Lambda} \Phi(X),$$

where again we interpret this sum using the inclusions $B(\mathcal{H}_X) \subset B(\mathcal{H}_\Lambda)$, for any $X \subset \Lambda$, given by the tensor decomposition $\mathcal{H}_\Lambda = \mathcal{H}_X \otimes \mathcal{H}_{\Lambda\setminus X}$ and the identification of $A \in B(\mathcal{H}_X)$ with $A \otimes 1_{\mathcal{H}_{X\setminus\Lambda}} \in B(\mathcal{H}_\Lambda)$.

In principle, it would suffice to consider interactions on a finite number of bodies Γ. Indeed, statistical mechanics aims to make statements about systems of a large but finite number of bodies. Such statements are made with an explicit dependence on the number of bodies n or in the thermodynamic limit as $n \to \infty$. A standard example is to consider $\Gamma_n = [-n, n]^d \subset \mathbb{Z}^d$ with an interaction Φ_n defining a many-body system for each n. Then a statistical mechanical model associates an interaction energy $\Phi_n(\Lambda)$ on $\Lambda \subset \Gamma_n$, that is assumed to be independent of n[1]. This implies that the interactions Φ_n on Γ_n define an interaction Φ on $\Gamma = \mathbb{Z}^d$ and vice versa. When the interaction terms and corresponding local Hamiltonians depend on time $t \in I$, we write $\Phi(\Lambda, t)$ and $H^\Phi_\Lambda(t)$.

---

[1] As long as n is large enough. In many examples, different choices of boundary conditions may be considered in finite volumes. But the assumption is that the bulk interaction is independent of the boundary condition.

There is no direct generalisation of composite systems to infinitely many bodies. Instead, we consider the convergence of the evolution $\tau_{t,s}^{\Lambda,\Phi} : B(\mathcal{H}_\Lambda) \to B(\mathcal{H}_\Lambda)$ defined by $H_\Lambda^\Phi(\cdot)$ as Λ increases. Note that for fixed $\Lambda_0 \subset \Lambda$, the evolution of $\mathcal{H}_{\Lambda_0}$ under $\tau_{t,s}^{\Lambda,\Phi}$, considering $B(\mathcal{H}_{\Lambda_0}) \subset B(\mathcal{H}_\Lambda)$, may be regarded as the evolution of an open system, since information is allowed to dissipate from Λ_0 into Λ. This illustrates the need for defining the *quasi-local* algebra for studying the convergence of $\{\tau_{t,s}^{\Lambda,\Phi}\}_\Lambda$. For this we introduce the notation

$$\mathcal{A}_\Lambda = B(\mathcal{H}_\Lambda), \qquad \Lambda \subset \Gamma \text{ finite.}$$

As already discussed, there are inclusions $\mathcal{A}_\Lambda \subset \mathcal{A}_{\Lambda'}$ whenever $\Lambda \subset \Lambda'$. Thus, for any (infinite) $\Delta \subset \Gamma$, there is an algebra

$$\mathcal{A}_\Delta^{\text{loc}} = \bigcup_{\Lambda \subset \Delta} \mathcal{A}_\Delta,$$

since the set of finite subsets is directed with respect to the inclusion order, and all the inclusions of algebras are compatible. The operator norm(s) induces a norm on $\mathcal{A}_\Delta^{\text{loc}}$ and the completion in this norm is the quasi-local algebra $\mathcal{A}_\Delta = \overline{\mathcal{A}_\Delta^{\text{loc}}}$. The quasi-local algebra is a C^*-algebra. The natural inclusions moreover induce inclusions of quasi-local algebras $\mathcal{A}_\Delta \subset \mathcal{A}_{\Delta'}$ whenever $\Delta \subset \Delta'$.

One can show that for any Δ, there is a unique tensor product of C^*-algebras $\mathcal{A}_\Delta \otimes \mathcal{A}_{\Delta^c}$ which is isomorphic to $\mathcal{A} = \mathcal{A}_\Gamma$. This implies that the *local* evolutions $\tau_t^{\Lambda,\Phi}$ may be interpreted as automorphisms of $\mathcal{A}$ via $\tau_t^{\Lambda,\Phi}(A \otimes x) = \tau_t^{\Lambda,\Phi}(A) \otimes x$. We say that an interaction Φ defines an evolution in the thermodynamic limit if there is a strongly continuous family of automorphisms $\tau_t^\Phi \in \text{Aut}(\mathcal{A})$ such that

$$\tau_t^\Phi = \lim_{\Lambda \subset \Gamma} \tau_t^{\Phi,\Lambda},$$

where the limit is taken pointwise in norm over the net of finite subsets ordered by inclusion, and the convergence is uniform for t in compact intervals. A necessary condition is that there is a derivation δ_t^Φ defined on $\mathcal{A}^{\text{loc}}$ given by (see [BR97, Proposition 6.2.3] and following remark)

$$\delta_t^\Phi(A) = i \sum_X [\Phi(X,t), A] \qquad \text{for all } A \in \mathcal{A}^{\text{loc}}.$$

This condition is satisfied on the condition of finite energy,

$$\sum_{x \in \Lambda} ||\Phi(\Lambda,t)|| < \infty, \qquad \text{for all } x \in \Gamma, t \in I.$$

Note that summands of $\delta_t^\Phi(A)$ vanish if $A \in \mathcal{A}_\Lambda$ and $X \cap \Lambda = \emptyset$. There are several further general conditions on time-independent interactions Φ that are sufficient such that δ^Φ generates a strongly continuous family of automorphisms [BR97, Theorems 6.2.4, 6.2.6]. Here we shall focus on certain *geometric* locality conditions that imply Lieb-Robinson bounds, sufficient for the convergence of the thermodynamic evolution.

For that matter, assume that (Γ, d) is a countable metric space. Together with a quasi-local algebra as described above in terms of on-site Hilbert spaces, this defines the kinematics of what we refer to as a *(quantum) spin system*. We shall henceforth assume that the Hilbert spaces $\mathcal{H}_v$ are finite-dimensional for all $v \in \Gamma$. We also use the metric notation for the distance between subsets: $d(X,Y) = \inf\{d(x,y) | \, x \in X, y \in Y\}$, and write the diameter as $\text{diam}(X) = \sup\{d(x,y) | \, x,y \in X\}$.

An interaction Φ is said to satisfy a Lieb-Robinson bound if there are constants $v_\Phi, C > 0$ such that for all finite $\Lambda \subset \Gamma$, and all disjoint $X, Y \subset \Lambda$,

$$||[\tau_{t,s}^\Lambda(A), B]|| \leq C ||A|| \, ||B|| \, \min(|X|,|Y|) e^{v_\Phi |t-s| - d(X,Y)} \qquad \text{for all } A \in \mathcal{A}_X, B \in \mathcal{A}_Y, t,s \in I.$$

Lieb-Robinson bounds are satisfied by *suitably local* interactions. These are interactions whose terms decay with the size of X. For example Φ_t satisfies exponential decay if there are constants $a, C > 0$ such that $||\Phi_t(X)|| \leq Ce^{-a \cdot \text{diam}(X)}$ for all finite $X \subset \Gamma$. As a special case, many models are given by uniformly bounded and finite range interactions, meaning that $\Phi(X) = 0$ if $\text{diam}(X) > r$ for some fixed $r > 0$. A more sophisticated and general criterion on the decay of an interaction is given in terms of F-functions, that are also used to formulate stronger forms of Lieb-Robinson bounds [NSY19, Theorem 3.1]. We shall simply refer to interactions satisfying this criterion as suitably local, and note that it is sufficient for defining an evolution in the thermodynamic limit [NSY19, Theorem 3.5].

A Lieb-Robinson bound is often referred to as defining a light-cone in which the propagation of information is effectively contained. Another point of view is that for fixed t, s, we get estimates on the localisation of $\tau^{\Lambda}_{t,s}(A)$. A general map $T : \mathcal{A} \to \mathcal{A}$ is said to be *quasi-local* [NSY19, Section 5.1] if there is a constant $C > 0$ and a non-increasing function $G : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{x\to\infty} G(x) = 0$, such that for all finite $X, Y \subset \Gamma$,

$$||[T(A), B]|| \leq C|X|||A||\,||B||G(d(X,Y)) \qquad \text{for all } A \in \mathcal{A}_X, B \in \mathcal{A}_Y.$$

1.3 Gapped Phases

Having introduced quantum spin systems and their interactions defining time evolution in the thermodynamic limit, we revisit the adiabatic theorem in this context. The discussion of the adiabatic theorem in Section 1.1 was concerning a spectral flow between spectral projections of a time-dependent Hamiltonian on a Hilbert space. For spin systems on an infinite lattice, there are no spectral projections defined. However, we may consider spectral flows on finite volumes and study their convergence to an evolution in the thermodynamic limit.

Recall that $H^{\Phi}_{\Lambda}(t) \in \mathcal{A}_{\Lambda}$ is a self-adjoint bounded local Hamiltonian. Denote by

$$E^{\Phi}_{\Lambda}(t) = \min \sigma(H^{\Phi}_{\Lambda}(t)) \in \mathbb{R},$$

the local ground state energy.

Definition 1.1 ([NSY19, Definition 7.1]). *An interaction $\Phi(t)$ on a quantum spin system $(\Gamma, d, \mathcal{A})$ is uniformly gapped with gap $\gamma > 0$ if there is a non-increasing map that assigns $\epsilon(\Lambda) \geq 0$ to finite subsets $\Lambda \subset \Gamma$ such that $\lim_{\Lambda\subset\Gamma} \epsilon(\Lambda) = 0$, and the interval*

$$(E^{\Phi}_{\Lambda}(t) + \epsilon(\Lambda), E^{\Phi}_{\Lambda}(t) + \gamma - \epsilon(\Lambda))$$

is disjoint from $\sigma(H^{\Phi}_{\Lambda}(t))$ for all $t \in I$ and $\Lambda \subset \Gamma$.

The adiabatic theorem motivates the definition of gapped phases, which we can now state in the context of spin systems. The appropriate notion of spectral flow in this context is discussed subsequently.

Definition 1.2 ([NSY19, Definition 7.2]). *Gapped phases of a quantum spin system are the equivalence classes of uniformly gapped suitably local time-independent interactions under the equivalence relation that $\Phi_0 \simeq \Phi_1$ if there is a uniformly gapped suitably local time-dependent interaction defined on $[0,1]$ such that $\Phi_0 = \Phi(0)$ and $\Phi_1 = \Phi(1)$.*

That the relation of Definition 1.2 is transitive follows from considering suitably smooth nonlinear reparametrisations and concatenation. The remaining conditions for defining an equivalence relation are satisfied rather obviously.

We refer to the spectral projections of an interaction $\Phi(t)$ with a uniform gap corresponding to the intervals $[E^{\Phi}_{\Lambda}(t), E^{\Phi}_{\Lambda}(t)+\epsilon(\Lambda)]$ as the families of local ground state projections, denoted $P^{\Phi}_{\Lambda}(t)$. These projections are defined for all large enough Λ by the gap condition.

Let us for simplicity consider time in the interval $I=[0,1]$. Recall that an evolution α_t is a spectral flow for a generic strongly continuous family of spectral projections $P(t)$, if

$$\alpha_t(P(0)) = P(t) \quad \text{for all } t\in I. \tag{3}$$

This is satisfied by the evolution of the adiabatic Hamiltion Eq. (1). Suppose the spectral flow is generated by a derivation via the equation of motion

$$\frac{d}{dt}\alpha_t(A) = \alpha_t(\delta_t(A)) = \alpha_t(i[D(t),A]) = i[D(t),\alpha_t(A)],$$

then the spectral flow condition (3) implies

$$i[D(t),\alpha_t(P(0))] = i[D(t),P(t)] = \frac{d}{dt}P(t). \tag{4}$$

Kato's solution to this equation given by $D^K(t)=[P'(t),P(t)]$, may be considered a generic solution since it only depends on the family of projections. The solution $H(s)+D(s)$ provides a new solution for any given solution $D(s)$, and the adiabatic Hamiltonian is constructed thus.

In [HW05] Hastings and Wen considered an interaction with a uniform gapped ground state and the spectral projections $P^{\Phi}_{\Lambda}(t)$. They produced a solution $D^{\Phi}_{\Lambda}(t)$ for each $\Lambda\subset\Gamma$ (assuming $\epsilon(\Lambda)$ small enough), whose evolution in the thermodynamic limit would be quasi-local. We will not give the explicit formula, but refer to [NSY19] and adopt their terminology, referring to the solution $D^{\Phi}_{\Lambda}(t)$ as the Hastings generator[2].

Theorem 1.3. *[NSY19, Theorem 6.4] Given a suitably local interaction $\Phi(t)$ on a quantum spin system, the evolution of each of the Hastings generators $D^{\Phi}_{\Lambda}(t)$ for finte $\Lambda\subset\Gamma$ implement a spectral flow for the family of local ground state projections $P^{\Phi}_{\Lambda}(t)$. The Hastings generators are themselves the local Hamiltonians of a suitably local interaction on the same quantum spin system.*

The implication of Theorem 1.3 is that the local evolutions of the Hastings generators converge to a quasi-local evolution in the thermodynamic limit, that is, a family of quasi-local automorphisms α_t.

The following definition is also adapted from [NSY19, Section 7.1].

Definition 1.4. *For a uniformly bounded interaction $\Phi(t)$, define the set of asymptotic ground states at time $t\in I$,*

$$\mathcal{S}^{\Phi}(t) = \{\omega\in\mathcal{S}(\mathcal{A}) \,|\, \exists\Lambda_0 : \omega(P^{\Phi}_{\Lambda}(t)) = 1 \ \textit{for all}\ \Lambda\supset\Lambda_0\}.$$

where $\mathcal{S}(\mathcal{A})$ denotes the set of states on $\mathcal{A}$.

It is an easy consequence of the spectral flow properties and the convergence of the local evolutions to α_t, that the map $\omega(0)\mapsto\omega(t)=\omega_0\circ\alpha_t$ is an isomorphism $\mathcal{S}(0)\to\mathcal{S}(t)$ for every $t\in I$.

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[2] By considering the inverse flow α_t^{-1} generated by $-\delta_t(A) = -[D(t),A]$, one finds that the condition of Eq. (4) is in fact sufficient to define a spectral flow.

Corollary 1.5. *If $\Phi(t)$ is a uniformly gapped suitably local interaction on a quantum spin system, and there is a unique asymptotic ground state ω_0 for $\Phi(t)$ at $t = 0$, then there is a unique asymptotic ground state ω_1 at time $t = 1$ and a quasi-local automorphism α such that*

$$\omega_0 \circ \alpha = \omega_1.$$

Thus, if two interactions are equivalent $\Phi_0 \simeq \Phi_1$ and have unique gapped ground states then these states are related by a quasi-local automorphism of $\mathcal{A}$. Invariants of unique gapped ground states under quasi-local automorphism then provide (partial) invariants of gapped phases. In the Section 3, we introduce such an invariant.

Before the close of this section, we touch on the problem of whether the space of uniformly gapped and suitably local interactions is locally convex, or in physics terms, whether the gap of an interaction can be closed by an arbitrarily small perturbation. From the perspective of physics, gap stability is a very desirable property of a theory of gapped phases, and as such presents a problem to the stated definition. Several positive results are known describing classes of interactions for which the gap is stable [BHM10; NSY20]. The guiding example for these results, and of gapped phases in the first place, is given by models of topological order.

2 Topological order

The notion of topological order was coined by Wen already in [Wen90] where he argued, at a physics level of rigour, that certain topological quantum field theories (TQFTs) could be realised as describing the low-energy- or ground states of a Hamiltonian on a lattice. The degeneracy of the ground state when putting the model on a torus is a hallmark of topological order. Wen realised that the topological character of the TQFT implies a remarkable stability of the gap above the ground state with respect to perturbation.

In this section we describe the spin models known as (Kitaev) quantum double models based on a finite group G [Kit03], in order to be able to concretely discuss and motivate features of topological order. We take a bottom-up approach, starting with the development of skein modules based on discrete gauge theory on planar graphs. The quantum double model is then introduced as a spin model whose ground state space is described by such skein modules.

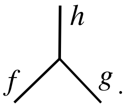
The main feature of this description of the ground state space is Proposition 2.3, which is an analogue of the key Proposition 4.6 of Chapter 3. We give detailed proof, staying within the realm of graph, group, and representation theory. In particular, we avoid the notions of fusion categories and topological quantum field theory that are used in Chapter 3. Yet, we attempt at giving the unfamiliar reader a first impression of diagrammatic calculus and skein modules, omitting the discussion of string operators. This makes our treatment differ from the standard, notably [Kit03], but aligns with the more general approach taken in Chapter 3, where string operators do not feature in their conventional form.

With an understanding of skein modules, the last part of this section gives a definition of anyon sectors as certain finite dimensional representations. Discussing properties of these representations, we introduce features of anyon sectors and conclude with remarks on ground state degeneracy and gap stability.

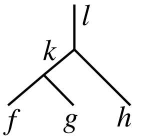
All definitions and results in this section should be regarded as well-known, in particular, as subset of results of [Kit03] and Chapter 3.

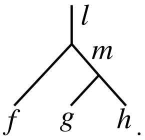
2.1 Discrete gauge theory

Let G be a group. Let us represent the relation $fg = h$ between group elements $f, g, h \in G$, as

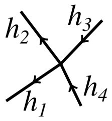


The combination of relations $fg = k$ and $kh = l$ to give the relation $(fg)h = l$ can be represented by stacking diagrams. The law of associativity is reflected by the uniqueness of the top label l in the two trees





By decorating lines with orientations we can express relations more flexibly. Here,



(5)

represents the relation $1 = h_1 h_2 h_3^{-1} h_4^{-1}$. When labels and orientations match, these diagrams may again be combined and used to derive new relations. This is the basic idea of a graphical calculus, which is heavily used in the setting of fusion categories in Chapter 3. In this Chapter, we stick to the group G and develop a form of lattice gauge theory setting up for the quantum double model of [Kit03].

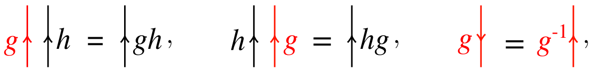
The relation implied on the product of edge labels (h_i) in Eq. (5) is referred to as the Gauss law. Note that the orientation of the plane defines the clockwise cyclic order of the product around the vertex, and the Gauss law is preserved under cyclic permutation of the product. By analogy, we may think of the product as a measure of the holonomy of a G-connection along a closed path in the dual graph encircling the vertex in the middle. The Gauss law is then instating flatness of the connection. Our object of interest will be (locally) flat G-connections, referred to as *string-nets* or *string diagrams*, on an oriented finite connected embedded planar graph $\Gamma = (V, E)$. The set of leaves $L \subset V$ are those vertices of degree *at most* one. As in the diagrams above, the Gauss law is not enforced at leaves, so define

$$\Sigma = \{\sigma : E \to G \,|\, \sigma \text{ satisfies the Gauss law at every } v \in V_0\},$$

where the vertices are partitioned into $V = V_0 \sqcup L$. Every leaf defines a unique edge that is incident to it, called the leaf edge. Since the value of a string-net is unconstrained at leaves, we refer to $b : L \to G$ as a boundary condition.[3] Note that the orientation of the edges, as well as the embedding into the plane, are used to define the Gauss law.

We refer to $\pi_0(\mathbb{R}^2 \backslash \Gamma)$, the set of connected components of Γ^c, as the *holes* of Γ (see Figure 1.1). We say a hole is populated if there is at least one leaf contained it its boundary, partitioning $\pi_0(\Gamma^c) = O \sqcup F$, where the set of populated holes is denoted by O, and the set of faces by F.

There is a group action on Σ which we refer to as the gauge action. We represent multiplication by elements $g \in G$ at an edge by



(6)

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[3] For this reason, leaves correspond to sources as referred to in other parts of the literature [HKK25].

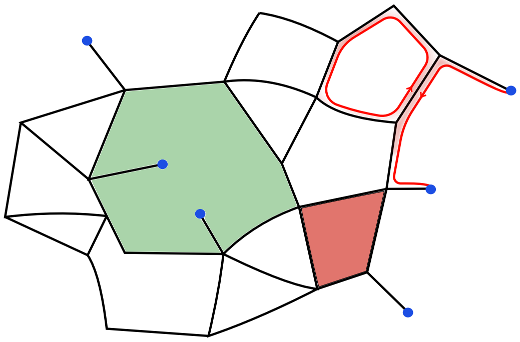
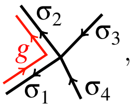


Figure 1.1: A finite connected embedded planar graph. One populated hole is shaded green and one face is shaded red. Leaves are marked with blue dots. An 'open' and a 'closed' ribbon are indicated with red lines and shade.

where the last relation is used to interpret the action when edges are oppositely aligned. It is easy to verify that by acting on some edge, the Gauss law is restored by acting on the appropriate neighbouring edges, as illustrated by



$$1 = \sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1} \iff 1 = \sigma_1 g^{-1} g\sigma_2\sigma_3^{-1}\sigma_4^{-1}.$$

Starting with the action of g at a given edge, this determines a sequence of edges and g-actions on those edges. The sequence of edges determines a path and an obvious notion of *side* of each of the edges along the path. This determines a *ribbon* (see Figure 1.1), and fixing the counter-clockwise orientation of the path, the g-action on each edge along the path agrees with the definitions in Eq. (6). This determines an action $\sigma \mapsto g_r\sigma$ defining the *gauge action* of $g \in G$ at the ribbon $r \in R$, denoting the set of ribbons.

Note that the path of a ribbon follows the boundary of a hole, and either forms a closed path around the edges of a face, or start and end at a leaf. Therefore, the set of ribbons R is in bijection with $L \sqcup F$, where a leaf is assigned to the unique ribbon starting at this leaf. The total gauge group is then $G^R = G^L \times G^F$, whose action preserves $\Sigma \subset G^E$.

For connected subgraphs $\tilde{\Gamma}, \tilde{\Gamma}_u \subset \Gamma$, we use the notation $\tilde{E}, \tilde{L}, \tilde{R}, \tilde{\Sigma}$ for the obvious definitions relative to the graph $\tilde{\Gamma}$, and $\tilde{E}_u, \tilde{L}_u, \tilde{R}_u, \tilde{\Sigma}_u$ ditto $\tilde{\Gamma}_u$. A string-net $\tilde{\sigma} \in \tilde{\Sigma}$ extends canonically to a string diagram on Γ, taking the value 1 on $E \setminus \tilde{E}$, as long as $\tilde{L} \subset L$. With this notion we consider the set of string-nets $\tilde{\Sigma} \subset \Sigma$. Note that if $\tilde{E}$ is empty, there is a single empty string-net in $\tilde{\Sigma}$ which extends to the constant string-net on non-empty E.

Lemma 2.1. *Any oriented finite connected embedded planar graph has connected subgraphs $\tilde{\Gamma} \subset \tilde{\Gamma}_u \subset \Gamma$ with inclusions of string-diagrams $\tilde{\Sigma} \subset \tilde{\Sigma}_u$ and $\tilde{\Sigma}_u \subset \Sigma$ that are equivariant with respect to gauge-actions and induce bijections of orbits. The subgraph $\tilde{\Gamma}_u$ has the same faces as Γ, but a unique leaf in every populated hole, while $\tilde{\Gamma}$ is either empty or has no faces. The equivariance of the inclusion is given by a homomorphism $f : G^{\tilde{R}} \to G^R$ such that*

$$\iota(\tilde{g}\tilde{\sigma}) = f(\tilde{g})\iota(\tilde{\sigma}) \in \Sigma \quad \textit{for all } g \in G^{\tilde{R}}, \tilde{\sigma} \in \tilde{\Sigma}. \tag{7}$$

Moreover, $\tilde{\Gamma}$ is minimal with the stated property (up to vertices of degree two).

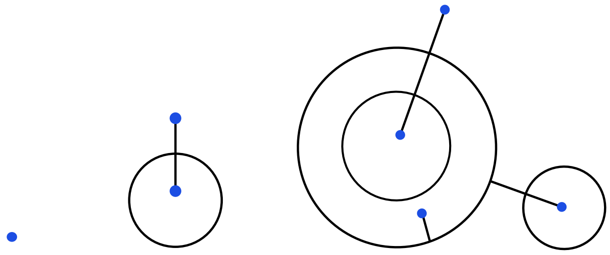


Figure 1.2: Three examples of connected graphs $\tilde{\Gamma}$ with no faces and a unique leaf in each hole. The total numbers of holes are $1, 2,$ and 4, respectively.

Proof : The equivariance of the inclusion can be understood as follows. A ribbon r follows a path of edges, and we abuse notation and refer to r as the set of edges in the trace of this path. There is a ribbon associated with either side of an edge. Given a ribbon r and an edge $e \in r$, we refer to r^e as the ribbon on the other side of e.

Let $e \in E$ and $r \in R$ with $e \in r$. Consider the graph $\tilde{\Gamma} = (V, E \setminus s(e))$, where $s(e)$ is the 'stretch' of e, the set of edges along a path of maximal length containing e and vertices of degree two along the interior[4]. Any ribbon $r' \in R$ with $r' \neq r, r^e$, is identified with a ribbon in $\tilde{R}$ since no edges have been excised from r'. There is a unique $\tilde{r} \in \tilde{R}$, such that $\tilde{r} \cup \{e\} = r \cup r^e$. This defines a map $R \to \tilde{R}$ sending $r, r^e \mapsto \tilde{r}$, which induces the homomorphism $f : G^{\tilde{R}} \to G^R$. This clearly satisfies Eq. (7).

The induced map on orbits is injective since, if $\tilde{\sigma}, \tilde{\sigma}' \in \tilde{\Sigma}$ and $g \in G^R$ satisfy $\iota(\tilde{\sigma}') = g\iota(\tilde{\sigma})$, then $g_r = g_{r^e}$ since $\iota(\tilde{\sigma})_e = \iota(\tilde{\sigma}')_e = 1$. Thus, g is in the range of f, so by Eq. (7), $\tilde{\sigma}'$ and σ belong to the same orbit.

So far, we have not assumed that $r^e \neq r$, which is indeed a possibility. That $r^e \neq r$ is necessary and sufficient for the inclusion map on orbits to be surjective. To see this, note that if $r^e = r$, the gauge action of the ribbon r is given by conjugation at the edge e. If $r^e \neq r$, it is given by multiplication from either the left or the right. So if $\sigma \in \Sigma$ and $\sigma_e \neq 1$, then σ_e can be gauged away precisely if $r^e \neq r$. Indeed, there is a gauge transformation g acting with multiplication by $g_r = \sigma_e^{-1}$ from the left or the right at e. Then $\tilde{\sigma} = g\sigma$ satisfies $\tilde{\sigma}_e = 1$, and by the Gauss law, $\tilde{\sigma}_{e'} = 1$ for all $e' \in s(e)$. So indeed, $\tilde{\sigma}$ is obtained as the inclusion of a string-net on $\tilde{\Gamma}$.

We complete the proof by induction using the preceding construction, and showing that if Γ has faces, or multiple leaves in the same hole, they can be removed, one at a time, by removing (stretches of) edges e with $r^e \neq r$.

If $e \in r$ is a 'leaf edge', i.e. connecting to a leaf $l \in L$, then $r^e = r$, if and only if, l is the unique leaf in the hole to which it belongs. To see this, it suffices to observe that r must start and terminate at a leaf belonging to the same hole, and it can visit no more than two leaf edges. One should care to remove isolated vertices after pruning the stretch of the leaf edge to obtain a connected graph $\tilde{\Gamma}_u$. In doing so, we do not remove any $l' \in L$ unless $l' = l$. In particular, the line graph is reduced to a single vertex.

We can reduce the number of faces of Γ by removing an edge of a face as long as it is not a bridge. Indeed, if e is a bridge, then there is only one connected component of Γ^c which contains e in its boundary, and so removing e does not change the number of connected components of

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[4] It is tempting to assume that Γ has no vertices of degree 2, but this is not a good alternative since removing edges reduces the degree of vertices.

Γ^c. For the same reason it is clear that $r^e \neq r$ if e is not a bridge.

Note that the removal of $s(e)$ may result in a new leaf. The stretch of the corresponding leaf edge e_l must also be removed along with any isolated vertices, except for vertices that were leaves in Γ. This does not affect the surjectivity of the map on orbits, because the Guass law implies that $\tilde{\sigma}_{e_l} = 1$ if, as above, $\tilde{\sigma}$ is the string-net obtained from gauging away the edge e. In the resulting graph, the number of populated holes as well as the set of leaves are preserved, while the number of faces is decreased by one.

Supposing that all edges of a face are bridges is absurd, since this would imply that the boundary were a tree, in particular containing leaves. By definition, there are no leaves in the boundary of a face.

Finally, as we have seen, it is not possible to get a surjective map on orbits from the subgraph $\tilde{\Gamma}$ where we remove an edge e for which $r^e = r$. This shows minimality of $\tilde{\Gamma}$, if $\tilde{\Gamma}$ has no faces and no multiple leaves (as long as all leaf edges have stretch of length one). ☐

Examples of minimal graphs $\tilde{\Gamma}$ of Lemma 2.1 are given in Figure 1.2. The set of orbits of the gauge action on Γ may be computed by considering the space $\tilde{\Sigma}$ on such graphs. By fixing the gauge, every string-net in $\tilde{\Sigma}$ is equivalent to a string-net supported on a subtree of $\tilde{\Gamma}$ with the same set of leaves. Such configurations correspond precisely to the kind of diagrams considered at the beginning of this section. Reading off the values on leaf edges in such a configuration defines an element in G^O. This defines a map on the set of orbits of Σ up to conjugation by the diagonal in G^O. Indeed, the remaining gauge freedom corresponds to the simultaneous gauge transformation along all ribbons on the tree.

Details of this argument are omitted to avoid moving too far astray, but we note one consequence that will be significant to us. Namely, that the stabiliser of the gauge action on a string-net is realised as a subgroup of the diagonal in G^O conjugated by the gauge fixing just described. In particular, the stabiliser involves gauge transformation on leaf edges *unless* there are no leaves. In the case where Γ has no leaves, $\tilde{\Gamma}$ is in fact empty. Defining $F_0 = F$, whenever Γ has leaves, and F_0 to be the set of bounded faces (excluding the single unbounded face) otherwise, the action of $G^{F_0} \subset G^R$ on Σ is seen to be free.

Remark 2.2. *For later use, observe that the gauge transformation $g \in G^R$ that relates $\sigma \in \Sigma$ to $g\sigma = \iota(\tilde{\sigma})$ may be taken to belong to G^L when $\tilde{\sigma} \in \tilde{\Sigma}_u$ of Lemma 2.1. This is a straightforward consequence of the procedure of gauging away σ_{e_l} for a leaf edge e_l described in the proof. Similarly, way may find $g \in G^{F_0}$ that relates $\sigma \in \tilde{\Sigma}_u$ to $g\sigma = \iota(\tilde{\sigma})$ for $\tilde{\sigma} \in \tilde{\Sigma}$.*

2.2 Skein modules

For the remainder of this section, assume that G is finite. Consider an oriented finite connected embedded planar graph Γ with subgraphs $\tilde{\Gamma} \subset \tilde{\Gamma}_u \subset \Gamma$ as given in Lemma 2.1. Consider the space of (continuous) $\mathbb{C}$-valued functions on Σ,

$$C(\Sigma) = \operatorname{span}(\{|\sigma\rangle \mid \sigma \in \Sigma\}),$$

regarded as a Hilbert space. There are natural inclusions $C(\tilde{\Sigma}) \subset C(\tilde{\Sigma}_u) \subset C(\Sigma)$ defining the projections $\tilde{Q}$, $\tilde{Q}_u$ onto $C(\tilde{\Sigma})$, resp. $C(\tilde{\Sigma}_u)$. We consider the actions of the gauge groups G^R, resp. $G^{\tilde{R}}$, defined by linear extension from their action on the canonical basis, $g\,|\sigma\rangle = |g\sigma\rangle$. By linear extension, this defines representations of the respective group algebras, $\mathbb{C}[G^R]$ and $\mathbb{C}[G^{\tilde{R}}]$. We also consider projections onto boundary conditions $b : L \to G$,

$$Q_b\,|\sigma\rangle = \left(\prod_{l \in L} \delta_{b_{e_l}, \sigma_{b_{e_l}}}\right) |\sigma\rangle\,.$$

These projections generate a representation of the algebra of functions $C(G^L)$ under pointwise multiplication, identifying leaves with leaf edges. Define the algebra generated by the gauge-actions and projections onto boundary conditions

$$T = \mathbb{C}[G^R] \vee C(G^L) \subset B(C(\Sigma)).$$

The algebra $\tilde{T} \subset B(C(\tilde{\Sigma}))$ is the corresponding algebra, defined on the graph $\tilde{\Sigma}$.

Let P be the projection onto the invariant subspace of the action of $G^{F_0} \subset G^R$, denoted $H = PC(\Sigma)$. P is given by

$$P = \frac{1}{|G^{F_0}|} \sum_{g \in G^{F_0}} g. \tag{8}$$

Clearly $gP = P = Pg$ for any $g \in G^{F_0}$, and $gP = Pg$ for all $g \in G^R$, so $P \in Z(T)$. It follows that T acts on H, generating the corner

$$S = PTP = TP = PT \subset B(C(\Sigma)). \tag{9}$$

The subspace H is referred to as a *skein module* for the algebra S. The following result may appear abstract, but it encodes the so-called topological properties of the spin model we describe below. We elaborate on the notion of Morita equivalence after proving the result.

Proposition 2.3. *The algebras S and $\tilde{T}$ are Morita equivalent, and their representations on H, resp. $C(\tilde{\Sigma})$, are equivalent.*

Proof : Observe that for any string-net $\sigma \in C(\Sigma)$, we have $\langle \sigma | P\sigma \rangle = |G^{F_0}|^{-1}$ since the action of G^{F_0} is free. This shows that $\tilde{Q}P\tilde{Q} = c\tilde{Q}$, where $c = |G^{F_0}|^{-1}$. It follows that $\tilde{Q}H = C(\tilde{\Sigma})$. We also observe that $SC(\tilde{\Sigma}) = PTC(\tilde{\Sigma}) = H$, since by Lemma 2.1, there is $g \in G^R$ and $\tilde{\sigma} \in \tilde{\Sigma}$, such that $g\tilde{\sigma} = \sigma$. These observations show that

$$(\tilde{Q}TP)H = C(\tilde{\Sigma}),$$
$$(PT\tilde{Q})C(\tilde{\Sigma}) = H.$$

As $PT\tilde{Q}$ is an S-$\tilde{T}$ bimodule and $\tilde{Q}TP$ a $\tilde{T}$-S bimodule, it remains to show that these bimodules are mutually inverse. To see that

$$(\tilde{Q}TP)(PT\tilde{Q}) = \tilde{T}, \tag{10}$$

we show that $\tilde{T} \subset \tilde{Q}PT\tilde{Q} = \tilde{Q}S\tilde{Q}$, since the opposite inclusion is clear. By equivariance Eq. (7), the homomorphism $f : G^{\tilde{R}} \to G^R$, given by Lemma 2.1, extends the action of a gauge transformation $\tilde{g} \in G^{\tilde{R}}$ to act in $C(\Sigma)$. If we identify $\tilde{g}$ with its representation in $\tilde{T}$, $f(\tilde{g})$ with its representation in T, and $\tilde{T}$ with the corner $\tilde{Q}B(C(\Sigma))\tilde{Q}$, we get $\tilde{g} = \tilde{Q}\tilde{g}\tilde{Q} = f(\tilde{g})\tilde{Q}$. Then we see that $\tilde{g} \in \tilde{Q}S\tilde{Q}$, by computing

$$\tilde{Q}Pf(\tilde{g})\tilde{Q} = \tilde{Q}P\tilde{Q}f(\tilde{g})\tilde{Q} = c\tilde{Q}f(\tilde{g})\tilde{Q} = \tilde{g}.$$

Since any projection on boundary conditions commutes with $\tilde{Q}$, we have that the generators of $\tilde{T}$ belong to $\tilde{Q}S\tilde{Q}$, showing the desired inclusion.

Next, we turn to

$$(PT\tilde{Q})(\tilde{Q}TP) = S, \tag{11}$$

which follows from showing $S \subset PT\tilde{Q}TP = S\tilde{Q}S$, since the opposite inclusion is clear. It then suffices to show $P \in S\tilde{Q}S$. First we show that $P\tilde{Q}P = cP\tilde{Q}_u$. By construction, $\tilde{\Gamma} \subset \tilde{\Gamma}_u$ so $\tilde{Q} \leq \tilde{Q}_u$. It follows from $P\tilde{Q}_u = \tilde{Q}_u P$ that $c^{-1}P\tilde{Q}P \leq P\tilde{Q}_u$. As $\tilde{Q}_u$ is diagonal in the string-net basis, we find the opposite relation by verifying that $cP\,|\sigma\rangle = P\tilde{Q}P\,|\sigma\rangle$, for arbitrary $\sigma \in \Sigma$ that satisfies $\tilde{Q}_u\,|\sigma\rangle = |\sigma\rangle$, i.e. $\sigma \in \tilde{\Sigma}_u$. By Lemma 2.1, there is a gauge transformation such that $g\sigma \in \tilde{\Sigma}$, and we may take $g \in G^{F_0}$, cf. Remark 2.2. Thus,

$$P\tilde{Q}P\,|\sigma\rangle = P\tilde{Q}P\,|g\sigma\rangle = P\tilde{Q}P\tilde{Q}\,|g\sigma\rangle = cP\tilde{Q}\,|g\sigma\rangle = cP\,|g\sigma\rangle = cP\,|\sigma\rangle\,.$$

Using Lemma 2.1 for $\tilde{\Gamma}_u \subset \Gamma$, we see that for any $\sigma \in \Sigma$ with $b = \sigma|_L : L \to G$, there is a gauge transformation on the leaves such that $k_b\sigma \in \tilde{\Sigma}_u$, where $k_b \in G^L$ by Remark 2.2. Writing $Q_{k_b b} = k_b Q_b k_b^{-1}$, we have $Q_{k_b b} \leq \tilde{Q}_u$. Assume k_b to be defined for all boundary conditions b. By resolving the identity with projections on boundary conditions, we find

$$P = P\sum_{b\in G^L} Q_b = \sum_{b\in G^L} Pk_b^{-1}Q_{k_b b}k_b = \sum_{b\in G^L} k_b^{-1}P\tilde{Q}_u Q_{k_b b}k_b = c^{-1}\sum_{b\in G^L} k_b^{-1}P\tilde{Q}PQ_{k_b b}k_b \in S\tilde{Q}S.$$

We have now established that the bimodules $PT\tilde{Q}$ and $\tilde{Q}TP$ are inverse to one another, completing the proof of Morita equivalence. ☐

Recall that Morita equivalence is an equivalence of categories of representations. It was announced that this Chapter would be light on category theory, so we state the concrete effects of the invertible bimodule $PT\tilde{Q}$ constructed in the recent proof.

Just as $H = (PT\tilde{Q})C(\tilde{\Sigma})$, we get an S-invariant subspace of $(PT\tilde{Q})V \subset H$ for any $\tilde{T}$-invariant subspace $V \subset C(\tilde{\Sigma})$. Since $PT\tilde{Q}$ is invertible in the sense of Eq. (10) and (11), this in fact gives a bijection between invariant subspaces of H and those of $C(\tilde{\Sigma})$.

Moreover, this bijection respects isomorphism types of subrepresentations. Recall that minimal central projections of $Z(S)$, resp. $Z(\tilde{T})$, label the irreducible representations up to isomorphism. We get an isomorphism $Z(S) \simeq Z(\tilde{T})$ from identifying a minimal projection $p \in Z(\tilde{T})$ with the projection $q \in Z(S)$ satisfying $q(PT\tilde{Q})V = (PT\tilde{Q})V$ for an irreducible invariant subspace $V = pV$.

Recall the bijection $R \simeq L \sqcup F$, and consider the partition $L = \bigsqcup_{o\in O} L_o$ where L_o is the set of leaves belonging to the boundary of $o \in O$. As such we identify $\mathbb{C}[G^{L_o}] \subset \mathbb{C}[G^R]$, and consider projections onto boundary conditions specified on L_o, generating

$$\mathrm{Tube}_{\Gamma_o} = \mathbb{C}[G^{L_o}] \vee C(G^{L_o}) \subset B(\mathcal{H}). \tag{12}$$

Note that Tube_{Γ_o} is supported in a subalgebra corresponding to a certain collar region around each hole. Define

$$\mathrm{Tube}_\Gamma = \bigotimes_{o\in O} \mathrm{Tube}_{\Gamma_o}, \tag{13}$$

each of whose factors commute as subalgebras of $B(\mathcal{H})$. Therefore, they generate a representation of Tube_Γ on $\mathcal{H}$ which preserves H and generates $S \subset B(H)$.

Corollary 2.4. *H decomposes into irreducible invariant subspaces,*

$$H \simeq \bigoplus_{V\in \mathrm{Irr}\, S} \mathbb{C}^{m_V} \otimes V,$$

where each $V \in \mathrm{Irr}\, S$ is given as $V \simeq \bigotimes_{o\in O} V_o$ for irreducible representations $V_o \in \mathrm{Irr}\,\mathrm{Tube}_{\Gamma_o}$.

Moreover, if Γ' is a finite connected embedded planar graph with populated holes O' and S'-skein module H', an identification of $O' \simeq O$ gives a Morita equivalence between S' and S, identifying H' and H, which in particular determines the multiplicities m_V.

Proof : The invariant subspaces of Tube_Γ agree with the invariant subspaces of S. These are semisimple algebras by the Artin-Wedderburn theorem, so H decomposes as a direct sum of irreducible invariant subspaces. An irreducible representation V of Tube_Γ is of the form $\bigotimes_{o\in O} V_o$ for irreducible representations V_o of Tube_{Γ_o}.

Consider graphs $\tilde{\Gamma}$ and $\tilde{\Gamma}'$ as given by Lemma 2.1. An identification $O \simeq O'$ gives an identification of ribbons $\tilde{R}' \simeq \tilde{R}$ and an isomorphism of ribbons string-net spaces $\tilde{\Sigma}' \simeq \tilde{\Sigma}$ (respecting the gauge action). Hence, $C(\tilde{\Sigma}')$ and $C(\tilde{\Sigma})$ are isomorphic skein modules, so the claim follows by Proposition 2.3. □

Corollary 2.4 is comparable to Proposition 3.5 of Chapter 3, treating a string-net model based on a unitary fusion category $\mathcal{C}$. In Chapter 3, there is no mention of Morita equivalence, but the multiplicity space is given explicitly as (with an isomorphism depending on data similar to a labelling of O)

$$Z(\mathcal{C})(\mathbb{1} \to V_1 \otimes \cdots \otimes V_{|O|}), \tag{14}$$

where V_i labels an irreducible representation of Tube_{Γ_1} for each $o_i \in O$, and a fixed graph Γ_1. This space is defined as a morphism space in the Drinfeld double category $Z(\mathcal{C})$. There is a well-known correspondence between $Z(\mathcal{C})$ and the category of finite-dimensional representations of $\mathcal{D}(G)$, the *Drinfeld double* or quantum double algebra of G, when $\mathcal{C} = \mathsf{Vec}_f G$ is the category of finite-dimensional G-graded vector spaces. As for the tensor product in $\mathsf{Rep}_f\, \mathcal{D}(G)$, one may in fact take the point of view, that the multiplicity space of the corresponding irreducible subspace of $C(\tilde{\Sigma})$ *defines* the fusion space of the tensor product in Eq. (14) (see [Kaw+24]). We expand on the relation between Tube_{Γ_1} and $\mathcal{D}(G)$ below.

2.3 Spin model

Now we describe a spin model whose ground state space is H [Kit03]. Let Γ and G be as above, and define a spin system from the set of *unconstrained* configurations

$$\mathcal{H} = \bigotimes_{e\in E} \mathcal{H}_e \simeq \mathrm{span}(\{|\sigma\rangle \mid \sigma : E \to G\}),$$

where have defined $\mathcal{H}_e = \mathbb{C}^G$ for all $e \in E$. The Gauss law defines a condition for each vertex $v \in V_0$ to be checked on incident edges $\{e \in E | \, v \in e\}$. This defines a local projection $A_v \in B(\bigotimes_{v\in e} \mathcal{H}_e)$ onto the span of configurations satisfying the Gauss law at v. Clearly $A_v A_w = A_w A_v$, so if Q is the projection onto $C(\Sigma) \subset \mathcal{H}$, then

$$Q = \prod_{v\in V} A_v.$$

The gauge action of $g \in G^R$ is also defined on $\mathcal{H}$, and it preserves the Gauss law just as well the failure of it, so $gA_v = A_v g$. We may also consider the projection P onto the subspace $H \subset C(\Sigma)$ to be defined on all of $\mathcal{H}$. The projection onto the symmetric subspace of $\mathcal{H}$, denoted $P^{\mathcal{H}}$, satisfies the relation Eq. (8), which can be rewritten as

$$P^{\mathcal{H}} = \frac{1}{|G|^{F_0}} \sum_{\vec{g}\in G^{F_0}} \vec{g} = \frac{1}{|G|^{F_0}} \prod_{f\in F_0} \sum_{g\in G} g_f = \prod_{f\in F_0} B_f,$$

where we use the notation g_f to represent the gauge action of g at the ribbon in the face f, and define $B_f = |G|^{-1} \sum_{g\in G} g_f$. Then we define the commuting projector Hamiltonian,

$$H_\Gamma^\Phi = -\sum_{v\in V} A_v - \sum_{f\in F_0} B_f,$$

and observe that H is the ground state subspace, projected onto by $P^{\mathcal{H}}Q = P$.

A quantum spin-system is defined on the edges of an infinite graph in the sense of Section 1.1 by setting $\mathcal{A}_e = B(\mathbb{C}^G)$ with the corresponding quasi-local algebra $\mathcal{A}$. For concreteness, let us consider the standard square lattice on vertices $\mathbb{Z}^2 \subset \mathbb{R}^2$, fixing an arbitrary orientation of the edges. Then the projections A_v, B_f define the terms of an interaction on this quantum spin system. The notions developed so far can be applied to finite connected graphs Γ, whose edges are considered as a subset of the edges of the lattice.

Given such a graph Γ, recall the definition of the algebra Tube_Γ in Eqs. (12)(13). This extends immediately to define a representation of Tube_Γ on $\mathcal{H}$. Under this representation, define

$$\mathcal{B}_{\mathrm{bulk}} = \mathrm{Tube}_\Gamma' \cap B(\mathcal{H}) = \{x \in B(\mathcal{H}) \,|\, xt = tx \text{ for all } t \in \mathrm{Tube}_\Gamma\}.$$

Proposition 2.5. *Consider the space* $\mathrm{End}_{\mathrm{Tube}_\Gamma}(H)$ *of intertwiners of the subrepresentation of* Tube_Γ *on* H*. Then*

$$P\mathcal{B}_{bulk}P = \mathrm{End}_{\mathrm{Tube}_\Gamma}(H)P.$$

In particular, if Γ *has no more than one populated hole, then*

$$P\mathcal{B}_{bulk}P = \mathbb{C}P. \tag{15}$$

Proof : The first part of the statement is clear.

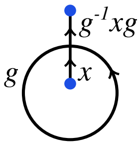
If Γ has no more than one populated hole, it follows from Lemma 2.1 that the set of gauge orbits of Σ is equivalent to the orbits of string-nets $\tilde{\Sigma}$ on a graph with no faces and at most one leaf. Such a graph is either empty, or has at most one vertex and no edges. In particular, it has a single orbit of the gauge action on $\tilde{\Sigma}$. It follows that $C(\tilde{\Sigma})$ is an irreducible representation of $\tilde{T}$, so by Proposition 2.3, the representation of S on H is irreducible. Recall that the representation of Tube_Γ on H generates the algebra S, so H is also irreducible as a representation of Tube_Γ. The result now follows by Schur's lemma. □

Note that if e is not contained in the exterior boundary of Γ, then $B(\mathcal{H}_e) \subset \mathcal{B}_{\mathrm{bulk}}$. Equation (15) is saying that all states given by density matrices supported on H, i.e. ground states, will agree in the *bulk*. This is known as the local topological order condition (number 1) [BHM10].

Remark 2.6. *By the Knill-Laflamme theorem, the local topological order condition Eq.* (15) *is equivalent to saying that* H *is an error correcting code for syndromes in* $\mathcal{B}_{bulk}$*. This relation plays is a key part in the overlap between theory of quantum computation and condensed matter.*

We move on to analyse the case where Γ has two populated holes which will be the basis for the discussion of anyon sectors.

Let Γ_1 an oriented connected embedded planar graph with three vertices, three edges, and two populated holes, and let Σ_1 denote the set of string-nets. There is only one Gauss law constraint, so any string net is of the form



,

where $g, x \in G$. That is, there is a bijection $\Sigma_1 \simeq G \times G$, taking $\sigma \mapsto (g, x)$. By composing with $(x, y) \mapsto x \otimes \delta_y \in \mathbb{C}[G] \otimes C(G)$ and extending linearly, we get a linear isomorphism of $C(\Sigma_1)$.

Now consider the algebra $\mathrm{Tube}_1 := \mathrm{Tube}_{(\Gamma_1)_o}$ corresponding to the inner hole o with a single leaf edge labelled x above. As vector spaces $\mathrm{Tube}_1 \simeq \mathbb{C}[G] \otimes C(G)$, and under this isomorphism the product in Tube_1 can be seen to satisfy

$$g\delta_x = \delta_{gxg^{-1}}g.$$

This relation is defining for the Drinfeld double algebra, establishing isomorphism $\mathcal{D}(G) \simeq \mathrm{Tube}_1$.

A similar relation is satisfied for the algebra $\mathrm{Tube}_{(\Gamma_1)_{o'}}$ acting on the outer hole, and one verifies that the isomorphism $\mathrm{Tube}_1 \simeq C(\Sigma_1)$ intertwines the regular representation of Tube_1 with the representation of Tube_{Γ_1}. Therefore,

$$C(\Sigma_1) \simeq \mathrm{Tube}_{\Gamma_1} \simeq \bigoplus_{V \in \mathrm{Irr}\,\mathrm{Tube}_1} B(V) \simeq \bigoplus_{V \in \mathrm{Irr}\,\mathrm{Tube}_1} V \otimes V^*, \tag{16}$$

where $\mathrm{Tube}_1 = \mathrm{Tube}_{(\Gamma_1)_o}$ and $\mathrm{Tube}_{(\Gamma_1)_{o'}}$ act on either factor in the decomposition on the right. This result underlies the definition of sectors.

2.4 Anyon sectors

Sectors The discussion will be somewhat less formal for the remainder of this section, as we illustrate the concepts characteristic of topological order based on the results obtained up until this point.

Consider a fixed (large) ambient graph Λ. Taking the subgraph Λ_n of the square lattice with vertices $[-n,n]^2 \cap \mathbb{Z}^2$ is a typical example. For a subset $E \subset E(\Lambda)$ of the edges of Λ, we consider a graph $\Gamma = (V, E)$ in which an edge $e \in E$ is a leaf edge unless the neighbourhoods of either vertex incident to e (as defined in Λ) are contained in E. We think of Γ as defined by the intersection with a nice subset $X \subset \mathbb{R}^2$ (see Figure 1.3)[5]. Most importantly, the definition of Γ is such that the reduced density matrix of a state in $H \subset \mathcal{H}_\Lambda$ is supported on $H_\Gamma \subset C(\Sigma_\Gamma) \subset \mathcal{H}_E$. Here, the subscript Γ implies that the definitions given above apply to Γ.

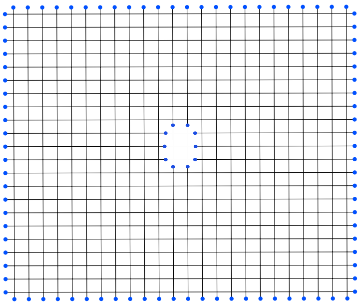


Figure 1.3: An example of a graph Γ defined by a subset $E \subset E(\Lambda)$. We consider Γ to be obtained by the excision of a contractible subset $D \subset \mathbb{R}^2$ in the middle of Λ.

Low-lying excitations are eigenstates of the Hamiltonian with small eigenvalues above the ground state energy. Therefore, they violate few of the local projection terms A_v, B_f, since these are simultaneously diagonalized. From another point of view, disturbances of the vacuum should

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[5] As such Γ is not a subgraph of Λ. This technicality shows up because we want to apply the formalism of string-nets on graphs. We will not address the issue of defining Γ any further, as it is obvious in examples of interest.

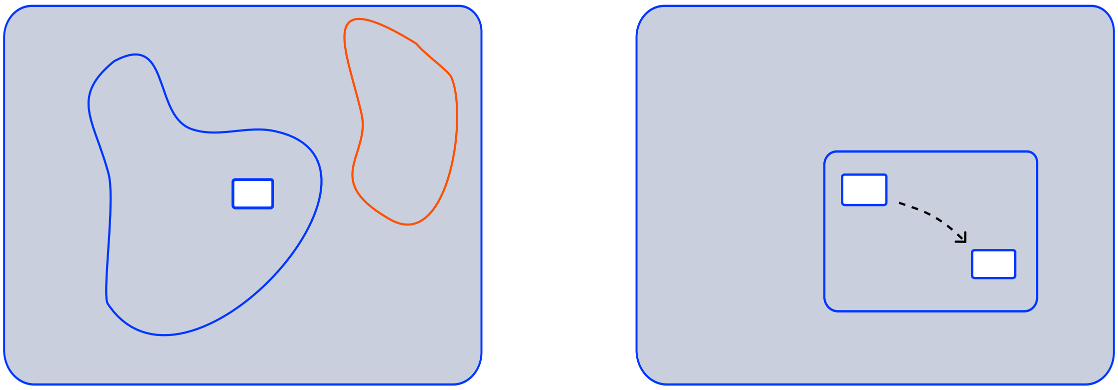


Figure 1.4: Considering anyon states on regions of the lattice corresponding to Λ with a contractible subset D removed from the middle. Left: The same sector is detected along all three blue loops. The vacuum sector is detected along the orange loop. Right: The localization of an anyon state in the region D can be transported to become localized in D' by an operator supported a larger region containing D and D'.

be considered as local with respect to the scale of the system. Either way, we are led to consider local excitations that violate the ground state projection terms only within a contractible region $D \subset \mathbb{R}^2$.

Let Γ be given by the excision of D from Λ, and consider the excitations that satisfy the ground state projection terms in Γ. Such states define reduced density matrices supported on H_Γ. The graph Γ has two populated holes and we refer to their boundaries as inner and outer, respectively. By Proposition 2.3 and Eq. (16), we have

$$H_\Gamma \simeq \bigoplus_{V \in \mathrm{Irr}\,\mathrm{Tube}_1} V_{\mathrm{inn}} \otimes V_{\mathrm{out}}, \tag{17}$$

for irreducible representations $V_{\mathrm{inn}} \otimes V_{\mathrm{out}}$ determined for each $V \in \mathrm{Irr}\,\mathrm{Tube}_{\Gamma_1}$ by the Morita equivalence of Proposition 2.3. Here, V_{inn} is an irreducible representation of $\mathrm{Tube}_{\Gamma_{\mathrm{inn}}}$, and there is a corresponding minimal central projection P^V_{inn} onto this representation. Similarly, there is a projection $P^V_{\mathrm{out}} \in \mathrm{Tube}_{\mathrm{out}}$, and the decomposition in Eq. (17) shows that these projections *agree* when acting on H_Γ. We refer to the summands of the decomposition in Eq. (17) as *sectors*, and conclude that the sector may be detected by observables supported along either boundary component. This can be seen as a manifestation of long-range entanglement. Indeed, the *measurement* of the sector on the inner boundary determines the outcome of the measurement of the sector on the outer boundary and vice versa. This is a characteristic phenomenon of topological order.

Non-triviality Note that by Proposition 2.5, H_Λ is an irreducible representation. The corresponding projection onto this representation is labelled $\mathbb{1}$, specifically $P^{\mathbb{1}}_{\mathrm{out}} \in \mathrm{Tube}_{\mathrm{out}}$. This is referred to as the vacuum sector.

Consider a local excitation of a ground state on Λ. That is, a state on H_Λ acted upon by an operator supported in a contractible region D, or more generally, by a channel with Krauss operators supported in D, producing a state ρ. The reduced density matrix ρ_Γ is supported on H_Γ since the operators supported in D commute with all ground state projection terms defined on Γ. Moreover, operators supported in D commute with $P^{\mathbb{1}}_{\mathrm{out}}$, showing that ρ_Γ belongs to the vacuum sector. If we refer to states supported in sectors distinct from the vacuum as *anyon states*, this shows that anyon states cannot be produced from the vacuum by local operations. This is the property which in the thermodynamic limit leads to disjoint unitary equivalence classes of anyon sectors referred to as non-trivial sector theory.

At the same time, a state ρ which restricts to an anyon state ρ_Γ is said to be localised in D. Indeed, it must violate some terms of the ground state projection, otherwise it would belong to the vacuum sector, but these violations are limited to D.

Indistinguishability We learn from Eq. (15) of Proposition 2.5 that if an anyon state is localised in the region D, and A is a region which is contractible *in the complement* of D, then the state cannot be distinguished from the states of the vacuum sector on any observable supported in A (see Figure 1.4). Indeed, if we associate a graph to the region $\Gamma(A)$ with a single populated hole corresponding to the complement of A, then any state in H_Γ restricts to a density matrix supported in $H_{\Gamma(A)}$.

However, if A is corresponds to a deformation retract of $X = B_\Lambda \setminus D$, where $B_\Lambda = [-n, n]^2 \subset \mathbb{R}^2$, then the restriction $\rho_{\Gamma(A)}$ of an anyon state ρ_Γ belongs to the same sector in $H_{\Gamma(A)}$ as the state on H_Γ. One can see this by considering retracts that preserve one of the two boundaries, say the inner boundary, observing that the projections P^V_{inn} identify sectors on the two spaces. If $P^V_{\mathrm{out}'}$ denotes the projection supported on the boundary of A, then this shows that the sector may be detected by operators supported in non-trivial topological regions in the bulk of Γ, see Figure 1.4.

The superselection criterion, stated precisely in the next section, is formulated with the preceding two observations in mind.

Transportability Consider a pure anyon state ρ defined on Λ and localised in D. Since ρ is pure and belongs to a definite sector, there is a minimal projection of $p \in \mathrm{Tube}_{\Gamma_{\mathrm{out}}}$ satisfied by ρ, i.e. $p\rho = \rho = \rho p$. Let Γ' be defined by the excision of another contractible region D'. Then there is a region A obtained by the excision of a contractible region $C \supset D \cup D'$, and corresponding graph $\Gamma(A)$ (see Figure 1.4). Since the action of $\mathrm{Tube}_{\Gamma(A)}$ is irreducible in the given sector of $H_{\Gamma(A)}$, it follows that $\mathrm{Tube}_{\Gamma(A)_{\mathrm{inn}}}$, together with the algebra of operators supported strictly on C, act transitively on the subspace of $\mathcal{H}_\Lambda$ projected onto by the ground state projection onto H_C as well as the projection p on the outer boundary. Hence, there is an operator supported in C that maps ρ to an anyon state localized in D', belonging to the same sector.

Fusion and braiding Using the of the same configuration $D, D' \subset C$, supposing D are D' are suitably separated, it is conceivable that there exists a state in which a sector V is detected on the boundary of D, and a sector V' on the boundary of D'. If the state satisfies ground state conditions everywhere else, we are led to consider the excision of both disks from Λ, producing a graph with three populated holes. Referring again to Corollary 2.4, we may interpret the invariant subspaces of H on this graph, as encoding the *fusion* of anyons (see also discussion after Corollary 2.4).

We refrain from describing braiding of anyons in this formalism, postponing the treatment for subsequent Chapters, but remark that it is implemented by transformations on the fusion space described above, implemented by the transportation of anyons (see also [Shi20]).

String operators Lastly, we remark upon the absence of string operators from the present discussion. Closed string operators are usually defined in a way that they generate the center

$Z(\mathrm{Tube}_{\Gamma_o})$ for some Γ with hole o. They have concrete descriptions, and aid the understanding anyon of anyon states. But this presentation has been given without introducing them, in line with the approach taken in Chapter 3. In Chapter 3, the analogue of Proposition 2.3 is used to *define* a version of string operators suitable for the general treatment of anyon sectors given therein.

2.5 Ground state degeneracy and gap stability

The fact that anyon sectors cannot be distinguished in contractible regions is often conceptualised as degeneracy of the ground state. As a realization of this, consider the model defined on a graph Γ_T embedded on a torus. By cutting the torus along a non-contractible loop in the dual graph, we recover a planar graph Γ with two populated holes. There is a basis of H_{Γ_T} given by the unique gauge-invariant state in each sector of H_Γ with respect to the action of the gauge group on the torus. This action is non-local when realised on H_Γ, but it gives a convenient way to eradicate the degrees of freedom associated with boundary conditions. Another way of doing so is to simply trace away the degrees of freedom along the boundary. This defines the so-called information convex set as studied in the entanglement bootstrap programme [SKK20]. Either way, the degeneracy of the ground state refers to the existence of non-trivial anyon sectors.

In addition to ground state degeneracy and local indistinguishability, there is a second condition of topological order identified in [BHM10]. It can be interpreted as a converse to previous observation, that the restriction of a ground state on Λ is a ground state on Γ. For this discussion, we need to assume that Γ and Λ have the same topology, i.e. Λ also has two populated holes. The condition is that such restriction must have full support on the corresponding sector in H_Γ. While this is not saying that any given state in H_Γ is the restriction of a state in H_Λ, it does imply that there are local operations that can transform a state that restricts to a density matrix supported on H_Γ to a state in H_Λ. In this sense, it is a requirement of extendability of local ground states. We omit the verification of this condition in the present model but note that the property is verified for the more general models in Lemma 4.9 of Chapter 3, which is used to show the existence (and uniqueness) of the frustration free ground state in the thermodynamic limit. Also note that in [QW20], the second condition of local topological order has been directly linked with the properties of Indistinguishability discussed in Section 2.4. These properties were formalised as the disk axiom and the annulus axiom in [QW20].

The model considered here is trivially gapped in the sense of Theorem 1.1 with $\gamma \geq 1$ since the interaction consists of commuting projections. Stability of the gap was proven in [BHM10] based on the second condition of topological order described before.

3 The category of superselection sectors

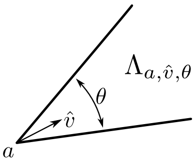
In the last section of this first chapter, we define an invariant of gapped phases in the thermodynamic limit which captures the notion of anyon sectors as objects of a braided C*-tensor category. The construction builds on the approach of Doplicher, Haag, and Roberts to charges and fields in the setting of algebraic quantum field theory [DHR69]. The adaptation to quantum spin systems was introduced in [Naa11], and the invariant as defined in here, in [Oga22]. All definitions and results in this section are found in these references, with the exception of Proposition 3.7 which is a generalisation of statements found in these references but whose proof follows the same arguments. Chapter 3 contains a more detailed exposition of the category of superselection sectors, treating in particular the assumption of bounded spread Haag duality.

Consider a quasi-local algebra $\mathcal{A}$. Recall that a state ω on $\mathcal{A}$ engenders a canonical GNS-representation on a Hilbert space $\mathcal{H}$ with a cyclic vector encoding the state. Throughout this section we consider the GNS-representation $\pi^{\mathbb{1}} : \mathcal{A} \to B(\mathcal{H})$, of a fixed gapped ground state referred to as the vacuum.

A physical interpretation from the perspective of quantum spin systems, is that $\mathcal{H}$ is generated from states that are produced by local operations from the vacuum. We can now formalise the idea from Section 2.4 conceived in finite volume, that anyons states cannot be produced from the vacuum by local operations, as follows. On an infinite spin system we say that the GNS representation of an anyon state is unitarily *in*equivalent to the vacuum representation. The characteristic features of localization, indistinguishability and transportability of anyon states are encoded at the level of representations in the Definition 3.1 of the superselection criterion.

First, define a *cone* $\Lambda \subset \mathbb{R}^2$ to be a subset of the plane of the form

$$\Lambda = \Lambda_{a,\hat{v},\theta} := \{x \in \mathbb{R}^2 \, : \, (x-a)\cdot \hat{v} > \|x-a\| \cos(\theta/2)\}$$



for some *apex* $a \in \mathbb{R}^2$, a unit vector $\hat{v} \in \mathbb{R}^2$ called the *axis*, and an *opening angle* $\theta \in (0, 2\pi)$. The closure of a cone is also called a cone with the same apex, axis, and opening angle.

Definition 3.1. *A representation $\pi : \mathcal{A} \to B(\mathcal{H})$ is said to be localized in a cone Λ (with respect to $\pi^{\mathbb{1}}$) if*

$$\pi(x) = \pi^{\mathbb{1}}(x) \qquad \text{for all } x \in \mathcal{A}_{\Lambda^c}.$$

The representation π satisfies the superselection criterion (with respect to $\pi^{\mathbb{1}}$) if for any cone Λ there is a unitarily equivalent representation localized in Λ. Representations satisfying the superselection criterion are called anyon representations, and their unitary equivalence classes, anyon sectors.

Anyon representations and intertwiners naturally form a unitary category. The solution to defining fusion of anyons as a tensor product is to find a subcategory of the following form.

Definition 3.2. *For a unital *-algebra $\mathcal{B}$, the set* $\mathrm{End}(\mathcal{B})$ *of unital *-endomorphisms of $\mathcal{B}$ are the objects of a monoidal dagger category with morphisms and tensor product*

$$\begin{aligned}
\mathrm{Hom}(\rho \to \sigma) &= \{T \in \mathcal{B} \, : \, T\rho(x) = \sigma(x)T \ \text{ for all } \ x \in \mathcal{B}\},\\
\rho \times \sigma &= \rho \circ \sigma, \quad \rho, \sigma \in \mathrm{End}(\mathcal{B})\\
S \times T &= S\rho(T) = \sigma(S)T, \quad S \in \mathrm{Hom}(\rho \to \sigma),\, T \in \mathrm{Hom}(\rho' \to \sigma').
\end{aligned}$$

*The unitary structure is given by the *-operation on $\mathcal{B}$.*

This suggests finding $\pi^{\mathbb{1}}(\mathcal{A}) \subset \mathcal{B} \subset B(\mathcal{H})$ with the property that any anyon representation π may be implemented by a (unique) endomorphism $\rho \in \mathrm{End}(\mathcal{B})$, in the sense that $\pi = \rho \circ \pi^{\mathbb{1}}$. Intertwiners must also belong to $\mathcal{B}$, so we make the following observation.

Given an anyon representation π and cones Λ_1, Λ_2, there is a representation π_i localized in Λ_i and an intertwiner $V_i : \pi \to \pi_i$ for $i = 1, 2$. Then we have an intertwiner $V = V_2 V_1^* : \pi_1 \to \pi_2$, and if $x \in \mathcal{A}_{\Lambda_1^c} \cap \mathcal{A}_{\Lambda_2^c}$, then

$$V\pi^{\mathbb{1}}(x) = V\pi_1(x) = \pi_2(x)V = \pi^{\mathbb{1}}(x)V. \tag{18}$$

In particular, $V \in \pi^{\mathbb{1}}(\mathcal{A}_{\widetilde{\Lambda}^c})'$ for any cone $\widetilde{\Lambda} \supset \Lambda_1 \cup \Lambda_2$. In fact, Eq. (18) shows that any intertwiner $V : \pi_1 \to \pi_2$ belongs to $\pi^{\mathbb{1}}(\mathcal{A}_{\widetilde{\Lambda}^c})'$.

Definition 3.3. *For a unit vector* $\hat{f} \in \mathbb{R}^2$ *and a cone* $\Lambda = \Lambda_{a,\hat{v},\theta}$*, we write* $\Lambda \perp \hat{f}$ *if* $\hat{v} \cdot \hat{f} \geq \cos(\theta/2)$*. With an arbitrary fixed choice of forbidden direction, say* $\hat{f} = (0,1)$*, we say a cone is* allowed *if* $\Lambda \perp \hat{f}$*. Define*

$$\widetilde{\mathcal{B}} = \overline{\bigcup_{\Lambda \perp \hat{f}} \pi^{\mathbb{1}}(\mathcal{A}_{\Lambda^c})'}^{||\cdot||}, \qquad \mathcal{B} = \overline{\bigcup_{\Lambda \perp \hat{f}} \pi^{\mathbb{1}}(\mathcal{A}_{\Lambda})''}^{||\cdot||}.$$

Since the set of allowed cones is directed under inclusion, both $\widetilde{\mathcal{B}}$ and $\mathcal{B}$ are C^*-algebras. The latter is referred to as the *allowed algebra.* Since $\pi^{\mathbb{1}}(\mathcal{A}_{\Lambda^c}) \subset \pi^{\mathbb{1}}(\mathcal{A})'$ for all Λ, we have

$$\pi^{\mathbb{1}}(\mathcal{A}) \subset \mathcal{B} \subset \widetilde{\mathcal{B}}.$$

Proposition 3.4. *For every anyon representation* π *there is a unique representation* $\mathcal{B} \to B(\mathcal{H})$ *such that* $\pi = \rho \circ \pi^{\mathbb{1}}$*. If* $\rho' : \mathcal{B} \to B(\mathcal{H})$ *is another representation such that* $\pi' = \rho' \circ \pi^{\mathbb{1}}$ *is an anyon representation, then* V *is an intertwiner* $\pi \to \pi'$*, if and only if,* V *is an intertwiner* $\rho \to \rho'$*. If moreover,* π *and* π' *are both localized in allowed cones, then* $V \in \widetilde{\mathcal{B}}$ *if* $V : \pi \to \pi'$*, and it holds that* $\rho(\mathcal{B}) \subset \widetilde{\mathcal{B}}$*.*

Proof : Let π be an anyon representation. For a given cone Λ, we construct an extension

$$\rho_\Lambda : \pi^{\mathbb{1}}(\mathcal{A}_\Lambda)'' \to B(\mathcal{H}),$$

satisfying $\rho_\Lambda \circ \pi^{\mathbb{1}}|_{\mathcal{A}_\Lambda} = \pi|_{\mathcal{A}_\Lambda}$, by localizing π in Λ^c. By hypothesis indeed, there is a unitary V_{Λ^c} which localizes π in Λ^c, that is,

$$V_{\Lambda^c}\pi(x) = \pi^{1}(x)V_{\Lambda^c} \quad \text{for all } x \in \mathcal{A}_\Lambda, \tag{19}$$

which is equivalent to $\pi|_{\mathcal{A}_\Lambda} = \mathrm{Ad}[V^*_{\Lambda^c}] \circ \pi^{1}|_{\mathcal{A}_\Lambda}$. This determines $\rho_\Lambda := \mathrm{Ad}[V^*_{\Lambda^c}]|_{\pi^{\mathbb{1}}(\mathcal{A}_\Lambda)''}$, independent on the choice of V_{Λ^c} satisfying (19) since $\mathrm{Ad}[V^*_{\Lambda^c}]$ is WOT-continuous and $\pi^{\mathbb{1}}(\mathcal{A}_\Lambda)''$ is the WOT-closure of $\pi^{\mathbb{1}}(\mathcal{A}_\Lambda)$ by von Neumanns bi-commutant theorem. So for two cones $\Lambda \subset \Lambda'$, the representation $\rho_{\Lambda'}$ extends the representation ρ_Λ. Since the set of allowed cones is directed, it follows that there is a unique extension $\rho : \mathcal{B} \to B(\mathcal{H})$ of the representations ρ_Λ to $\mathcal{B}$. Since $\pi^{1}(\mathcal{A}) \subset \mathcal{B}$, we also have

$$\pi = \rho \circ \pi^{1}.$$

For any $V \in B(\mathcal{H})$, the map $x \mapsto VxV^*$ is WOT-continuous. Since $\pi^{\mathbb{1}}(\mathcal{A}) \subset \mathcal{B}$ is WOT-dense, it is clear that V is an intertwiner of anyon sectors if and only if it is an intertwiner of the extension to $\mathcal{B}$.

Assume now that π is localized in an allowed cone Λ' and ρ is the unique extension to $\mathcal{B}$. Consider any allowed cone Λ. Since ρ_Λ is determined, independent of the choice of unitary satisfying (19), we fix an allowed cone $\Lambda^a \subset \Lambda^c$, and consider a unitary intertwiner V_{Λ^a} such that $\mathrm{Ad}[V_{\Lambda^a}] \circ \pi$ is localized in Λ^a. For an allowed cone $\widetilde{\Lambda} \supset \Lambda^a \cup \Lambda'$, it was shown in (18) that

$$V_{\Lambda^a} \in \pi^{\mathbb{1}}(\mathcal{A}_{\widetilde{\Lambda}^c})' \subset \widetilde{\mathcal{B}}.$$

This shows that $\rho_\Lambda\big(\pi^{\mathbb{1}}(\mathcal{A}_\Lambda)''\big) = V^*_{\Lambda^a}\big(\pi^{\mathbb{1}}(\mathcal{A}_\Lambda)''\big)V_{\Lambda^a} \subset \widetilde{\mathcal{B}}$, and the last part of the statement follows. □

Definition 3.5. *The representation* $\pi^{\mathbb{1}} : \mathcal{A} \to B(\mathcal{H})$ *satisfies Haag duality for cones, if for all cones* Λ*,*

$$\pi^{\mathbb{1}}(\mathcal{A}_\Lambda)'' = \pi^{\mathbb{1}}(\mathcal{A}_{\Lambda^c})'.$$

It is clear from the definitions that Haag duality implies $\mathcal{B} = \widetilde{\mathcal{B}}$. This also follows from the weaker notion of approximate Haag duality [Oga22, Lemma 2.4].

Definition 3.6. *Define* SSS *to be the full subcategory of* $\operatorname{End}(\mathcal{B})$ *whose objects* ρ *have the property that* $\rho \circ \pi^{\mathbb{1}}$ *is an anyon representation localized in an allowed cone. Define* SSS_f *to be the full subcategory of* SSS *whose objects have finite-dimensional endomorphism algebras.*

We refer to the objects of SSS as localized and transportable endomorphisms. It is easily verified that the product of localized endomorphisms is again localized, so it follows from Proposition 3.4, that:

Proposition 3.7. *Under the assumption that* $\mathcal{B} = \widetilde{\mathcal{B}}$*, the category* SSS *is a monoidal category whose underlying category is equivalent to the full subcategory of anyon representations localized in allowed cones.*

We remark that the assumption $\mathcal{B} = \widetilde{\mathcal{B}}$ may be regarded as a further weakening of Haag duality, sufficient for defining the fusion of anyon sectors. It does not, however, appear sufficient for the definition of the braiding.

The construction of the braiding of anyon sectors is due to [GF93]. The axis $\hat{v}$ of an allowed cone $\Lambda_{a,\hat{v},\theta}$ makes an angle $\phi(\hat{v}) \in (0, 2\pi)$ clockwise with the forbidden direction. We say a cone $\Lambda_{a',\hat{v}',\theta'}$ lies to the right of $\Lambda_{a,\hat{v},\theta}$, if $0 < \phi(\hat{v}) \leq \phi(\hat{v}') < 2\pi$, and we refer to the clockwise order of allowed cones.

Lemma 3.8. *Assume Haag duality for cones. If* $\rho, \rho' \in \operatorname{Ob}\mathsf{SSS}$ *are localized in disjoint cones, then* $\rho \times \rho' = \rho' \times \rho$*. If* $\rho, \sigma \in \operatorname{Ob}\mathsf{SSS}$ *and* ρ *is localized in* Λ*, and* Λ' *is an allowed cone disjoint from* Λ*, then there is* $\sigma' \in \operatorname{Ob}\mathsf{SSS}$ *localized in* Λ' *and an intertwiner* $V : \sigma \to \sigma'$*. The composition,*

$$\rho \times \sigma \xrightarrow{\mathrm{id}\times V} \rho \times \sigma' = \sigma' \times \rho \xrightarrow{V^* \times \mathrm{id}} \sigma \times \rho,$$

is given by the intertwiner $V^*\rho(V)$*. If* V *is unitary, then* $\epsilon^{\pm}(\rho, \sigma) := V^*\rho(V)$ *only depends on the clockwise order of* Λ *and* Λ'*. We let* ϵ^+ *denote the case when* Λ' *lies to the right of* Λ*.*

There is a braiding on SSS defined by

$$\beta^{\mathsf{SSS}}_{\rho,\sigma} := \epsilon^+(\rho, \sigma). \tag{20}$$

We refer to the proofs of this statement and Lemma 3.8 in [GF93; Oga22; Naa11] as well as Chapter 3, remarking only on the rôle of Haag duality. We say an operator $V \in B(\mathcal{H})$ is localized, if there is a region Γ such that $V \in \pi^{\mathbb{1}}(\mathcal{A}_\Gamma)''$. By Eq. (18), Haag duality implies that intertwiners are localized. This is used to show that $\rho \times \sigma' = \sigma' \times \rho$ in Lemma 3.8. Approximate Haag duality demands approximation of intertwiners by localized operators (see also [Bha+25] for the case of bounded spread Haag duality), and it turns out to be sufficient for defining the braiding. Approximate Haag duality is also used in the construction of subobjects.

Theorem 3.9 ([Oga22])**.** *Let* $\pi^{\mathbb{1}}$ *be the GNS representation of a pure gapped ground state* ω_π *of a uniformly bounded finite range interaction on* $\mathcal{A}$*, and assume that* $\pi^{\mathbb{1}}$ *satisfies approximate Haag duality for cones and has non-trivial sector theory. Then the category* SSS *is a braided* C^**-tensor category.*

Moreover, if ω_σ *is a pure gapped ground state with the same conditions as above, but for a different interaction, and there is a quasi-local automorphism* $\alpha \in \operatorname{Aut}(\mathcal{A})$ *satisfying* $\omega_\sigma = \omega_\pi \circ \alpha$*, then the* SSS*-category of the GNS representation of* ω_σ *is equivalent to that of* $\pi^{\mathbb{1}}$*, as braided* C^**-tensor categories.*

The braiding constructed in [Oga22] specialises to the braiding defined by Lemma 3.8 under the stricter assumption of Haag duality. Although we refrain from repeating their definitions here, this is obvious on inspection. In Chapter 3, we give a detailed account of the construction of the category SSS based on the intermediate notion of bounded spread Haag duality.

In light of Corollary 1.5, Theorem 3.9 shows that SSS is an invariant of those phases that are represented by a uniformly bounded finite range interaction with a unique asymptotic ground state satisfying the conditions of Theorem 3.9. It is quite possible that several of these conditions can be lifted or weakened extending the range of the invariant. The rest of this thesis is devoted to computing the category SSS_f up to braided monoidal equivalence.

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Chapter 2

The Double Semion State in Infinite Volume

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The Double Semion State in Infinite Volume

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Abstract. We describe in a simple setting how to extract a braided tensor category from a collection of superselection sectors of a two-dimensional quantum spin system, corresponding to abelian anyons. We extract from this category its fusion ring as well as its F and R-symbols. We then construct the double semion state in infinite volume and extract the braided tensor category describing its semion, anti-semion, and bound state excitations. We verify that this category is equivalent to the representation category of the twisted quantum double $\mathcal{D}^{\phi}(\mathbb{Z}_2)$.

1. Introduction

Gapped ground states of two-dimensional quantum lattice systems may support anyonic excitations. Paradigmatic examples include Kitaev's quantum double models [15] and, more generally, the string-net models of Levin and Wen [16]. It is widely believed that the types of anyons supported by a given ground state are organised in a unitary braided fusion category, whose simple objects are the anyon types and whose structure is described by F and R-matrices.

Recent years have seen a lot of progress towards justifying this belief, by adapting the DHR analysis of superselection sectors in algebraic quantum field theory [5,6,9–11] to the setting of microscopic lattice systems [3,19,21]. This body of work manages to associate with a large class of gapped ground states a strict braided C^*-tensor category whose objects are localised and transportable endomorphisms of the observable algebra, and shows that this category is a robust invariant for the gapped phase [1] to which the ground state belongs.

One can extract from any braided tensor category its fusion ring and F-symbols, which encode the tensor structure of the category, as well as R-symbols, which encode the braided structure of the category. This yields in particular a microscopic definition of fusion rules, F-symbols, and R-symbols,

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which are commonly employed in the physics literature to describe anyon theories. We note that a microscopic definition of F and R-symbols has been given before in [14]. Extracting the F and R-symbols from the DHR-type analysis has the advantage that it is clear that these data yield a robust invariant of gapped phases [3,21].

In Sect. 2 of the paper, we review the construction of a braided tensor category from a given ground state under some simplifying assumptions. We assume in particular that the anyons we consider are abelian. By, moreover, assuming that the set of anyon types is finite, and that each anyon has an anti-particle, we show that corresponding fusion ring is group-like and abelian and is equipped with F and R-symbols that satisfy the pentagon and hexagon equations.

Section 3 is devoted to the construction and analysis of the double semion state [16] in infinite volume. This is the simplest state that supports abelian anyons whose braided tensor category has non-trivial F-symbols. We show that the double semion ground state satisfies the assumptions of Sect. 2 with anyon content consisting of the vacuum, the semion, the anti-semion, and a semion–anti-semion bound state. We extract the fusion ring, F-symbols, and R-symbols from the corresponding braided tensor category. Finally, we use a result from [12] to conclude that the braided tensor category describing these anyons is equivalent to the category $\mathrm{Rep}_f(\mathcal{D}^{\phi}(\mathbb{Z}_2))$ of finite- dimensional representations of the twisted quantum double algebra of $\mathbb{Z}_2$. Since the double semion model is the string-net model defined by input fusion category $\mathcal{F} = \mathrm{Vec}_{\mathbb{Z}_2}^{\phi}$ with ϕ a non-trivial 3-cocycle of $\mathbb{Z}_2$, this partially verifies the conjecture that the resulting anyon theory is described by the Drinfeld centre $\mathcal{Z}(\mathrm{Vec}_{\mathbb{Z}_2}^{\phi}) = \mathrm{Rep}_f(\mathcal{D}^{\phi}(\mathbb{Z}_2))$, see [17].

In appendix, we show various properties of the double semion state that are used in Sect. 3. In particular, we introduce a new technique based on results from [13] to show that the double semion state we construct is pure. This technique is applicable to the construction and proof of purity of ground states and excited states of a large class of lattice spin models including all Levin–Wen string-net models.

2. Braided Tensor Category for Abelian Anyons

2.1. *Setup.*

2.1.1. Algebra of Observables and State. Consider a countable set $\Gamma \subset \mathbb{R}^2$ of sites in the plane. To each site $x \in \Gamma$, we associate an algebra $\mathcal{A}_x \simeq \mathrm{End}(\mathbb{C}^d)$ for some fixed $d \geq 2$. For any finite $X \subset \Gamma$, we set $\mathcal{A}_X = \bigotimes_{x \in X} \mathcal{A}_x$. If $X \subset Y$ are finite subsets of Γ, then there is a natural norm-preserving inclusion $\mathcal{A}_X \hookrightarrow \mathcal{A}_Y$ by tensoring with the identity.

For any infinite subset $Y \subset \Gamma$, we then have a local net of algebras $\mathcal{A}_X$ for finite $X \subset Y$, whose direct limit is $\mathcal{A}_{Y,\mathrm{loc}}$, the algebra of local observables supported in Y. Its norm completion is $\mathcal{A}_Y := \overline{\mathcal{A}_{Y,\mathrm{loc}}}^{\|\cdot\|}$, the algebra of quasi-local observables supported in Y, and we get inclusions $\mathcal{A}_X \hookrightarrow \mathcal{A}_Y$ also for infinite $X \subset Y$. We write $\mathcal{A}_{\mathrm{loc}} = \mathcal{A}_{\Gamma,\mathrm{loc}}$ and $\mathcal{A} = \mathcal{A}_\Gamma$ for the algebra of all local and all quasi-local observables, respectively. The *support* of an observable $O \in \mathcal{A}$ is the smallest set $X \subset \Gamma$ such that $O \in \mathcal{A}_X$.

Similarly, the support of an automorphism w of $\mathcal{A}$ is the smallest set Y such that $w|_{\mathcal{A}_{Y^c}} = \mathrm{id}_{\mathcal{A}_{Y^c}}$. Any subset $Z \subset \mathbb{R}^2$ of the plane determines a subset $\overline{Z} = Z \cap \Gamma$ of Γ, and we denote $\mathcal{A}_Z := \mathcal{A}_{\overline{Z}}$.

A major role is played by *cones*. The cone with apex at $a \in \mathbb{R}^2$, axis $\hat{v} \in \mathbb{R}^2$ of unit length, and opening angle $\theta \in (0, 2\pi)$ is

$$\Lambda_{a,\hat{v},\theta} := \{x \in \mathbb{R}^2 \,|\, (x-a)\cdot\hat{v} > \|x-a\| \cos(\theta/2)\}. \tag{1}$$

We will assume that $\overline{\Lambda}$ is infinite for any cone Λ. In particular, this holds if Γ is a lattice.

We will consider a pure state ω on $\mathcal{A}$ with GNS representation $(\pi_1, \mathcal{H}, \Omega)$. For any $X \subset \Gamma$, we put $\mathcal{R}(X) := (\pi_1(\mathcal{A}_X))''$, the von Neumann algebra associated with X. For $Z \subset \mathbb{R}^2$, we also write $\mathcal{R}(Z) = \mathcal{R}(\overline{Z})$. We note that if Λ is a cone, then $\mathcal{R}(\Lambda)$ is an infinite factor (Theorem 5.2 of [19]).

2.1.2. Superselection Sectors.

Definition 2.1. An irreducible representation $(\pi, \mathcal{H})$ of $\mathcal{A}$ is said to satisfy the superselection criterion with respect to π_1 if for any cone Λ there is a unitary $U \in \mathcal{B}(\mathcal{H})$ such that

$$U\pi(O)U^* = \pi_1(O) \quad \text{for all } O \in \mathcal{A}_{\Lambda^c} \tag{2}$$

If two representations π, π' are unitarily equivalent, then we write $\pi \simeq \pi'$.

We assume that we have a finite set of irreducible representations $\mathcal{O} = \{\pi_a \,|\, a \in I\}$ of $\mathcal{A}$ indexed by a labelling set I. We assume moreover that $\pi_a \simeq \pi_b$ if and only if $a = b$, so all sectors in $\mathcal{O}$ are truly distinct. Moreover, $1 \in I$ so that $\pi_1 \in \mathcal{O}$. We call π_1 the *vacuum sector*.

We will now make some additional Assumptions (1–4) on these sectors which will in particular imply that they satisfy the superselection criterion with respect to π_1. These assumptions are not generically satisfied by gapped ground states, but Assumptions (1–3) and the first part of Assumption 4 are verified for the Toric code model in [19] and for all abelian quantum double models in [8]. The second part of Assumption 4 can be shown for these models using similar methods. Below, we will verify these assumptions for the double semion model. More generally, the authors expect that these assumptions hold for all abelian string-net models [16].

Assumption 1. For any cone Λ and any $a \in I$, there is an automorphism $w_{a,\Lambda}$ supported on Λ such that

$$\pi_a \simeq \pi_1 \circ w_{a,\Lambda}. \tag{3}$$

In particular, we take $w_{1,\Lambda} = \mathrm{id}$ for all cones Λ.

The following assumption says that the anyons we study are abelian.

Assumption 2. For any $a, b \in I$ there is a unique $c \in I$ such that for any two cones Λ_1, Λ_2 we have $\pi_c \simeq \pi_1 \circ w_{a,\Lambda_1} \circ w_{b,\Lambda_2}$. We write $c = a \times b$.

The following assumption says that each anyon has an antiparticle.

Assumption 3. For each $a \in I$, there is an $a^* \in I$ such that $a \times a^* = 1$.

The final assumption is of a technical nature it plays an important role in constructing a tensor category in Sect. 2.2.

Assumption 4. Assumption 1 implies that if $\Lambda_1, \Lambda_2 \subset \Lambda$ are cones, then $\pi_1 \circ w_{a,\Lambda_1} \simeq \pi_1 \circ w_{a,\Lambda_2}$. We assume that any unitary $V \in \mathcal{B}(\mathcal{H})$ implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.

Similarly, it follows from Assumption 2 that $\pi_1 \circ w_{a,\Lambda} \circ w_{b,\Lambda} \simeq \pi_1 \circ w_{a \times b,\Lambda}$. We assume that any unitary $V \in \mathcal{B}(\mathcal{H})$ implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.

This assumption is implied by Haag duality (cf. the discussion at the end of Sect. 2.2.4) and can also often be proven directly if the automorphisms $w_{a,\Lambda}$ are known explicitly, for example, for the abelian quantum double models [8, 19] and for the double semion model treated below.

Assumptions 1–3 have a few elementary but important consequences.

Lemma 2.2. *The representations π_a, $a \in I$ satisfy the superselection criterion w.r.t. π_1.*

Proof. Fix a cone Λ. By Assumption 1, there is an automorphism $w_{a,\Lambda}$ supported in Λ such that $\pi_1 \circ w_{a,\Lambda} \simeq \pi_a$. i.e. there is a unitary $U \in \mathcal{B}(\mathcal{H})$ such that

$$U \pi_a(O) U^* = \pi_1(w_{a,\Lambda}(O)) \tag{4}$$

for all $O \in \mathcal{A}$. Since $w_{a,\Lambda}$ is supported in Λ, we have $w(O) = O$ for $O \in \mathcal{A}_{\Lambda^c}$ and therefore

$$U \pi_a(O) U^* = \pi_1(O) \quad \text{for all } O \in \mathcal{A}_{\Lambda^c}. \tag{5}$$

□

Lemma 2.3. *The binary operation $\times : I \times I \to I$ makes I into an abelian group with unit 1 and inverse $a^{-1} = a^*$.*

Proof. We first show that $\times$ is abelian. Take $a, b \in I$. Assumption 2 says that for any two cones Λ_1, Λ_2 there are automorphisms w_{a,Λ_1}, w_{b,Λ_2} such that $\pi_{a \times b} \simeq \pi_1 \circ w_{a,\Lambda_1} \circ w_{b,\Lambda_2}$. Exchanging the roles of a and b and of Λ_1 and Λ_2, we have $\pi_{b \times a} \simeq \pi_1 \circ w_{b,\Lambda_2} \circ w_{a,\Lambda_1}$. If we now take Λ_1 and Λ_2 to be disjoint, then certainly $w_{a,\Lambda_1} \circ w_{b,\Lambda_2} = w_{b,\Lambda_2} \circ w_{a,\Lambda_1}$ and therefore $\pi_{a \times b} \simeq \pi_{b \times a}$. But we assumed that two representations in $\mathcal{O}$ are unitarily equivalent only if they are the same, so $a \times b = b \times a$.

We now show that 1 is the identity for the product $\times$. Fix cones Λ_1 and Λ_2. By Assumptions 1 and 2, there are automorphisms $w_{1,\Lambda_1} = \mathrm{id}$ and w_{a,Λ_2} such that $\pi_{1 \times a} \simeq \pi_1 \circ \mathrm{id} \circ w_{a,\Lambda_2} = \pi_1 \circ w_{a,\Lambda_2} \simeq \pi_a$, hence $1 \times a = a$. We already know that $\times$ is abelian, so also $a \times 1 = a$.

Finally, Assumption 3 states that a^* is the inverse of a. □

We will often write $ab = a \times b$ for the product of elements $a, b \in I$.

2.2. *Braided Tensor Category.* It is well understood how to associate a braided tensor category with a pure state on a quantum spin system [3,19,21]. In this section, we recap this construction and use Assumptions 1–4 to identify a braided fusion subcategory with abelian fusion rules.

2.2.1. Tensor Category of Localised and Transportable Endomorphisms. Fix a unit vector $\hat{f} \in \mathbb{R}^2$, representing a 'forbidden direction'. We say a cone $\Lambda_{a,\hat{v},\theta}$ with axis $\hat{v}$ and opening angle θ is *forbidden* if it contains the forbidden direction $\hat{f}$, i.e. if $\hat{v} \cdot \hat{f} > \cos(\theta/2)$. A cone that is not forbidden is said to be *allowed*. We define an *allowed algebra* by

$$\mathcal{B} := \overline{\bigcup_{\text{allowed }\Lambda} \mathcal{R}(\Lambda)}^{\|\cdot\|} \subset \mathcal{B}(\mathcal{H}). \tag{6}$$

This is a C^*-algebra that contains $\pi_1(\mathcal{A})$ since any local observable is supported in some allowed cone.

Definition 2.4. We say an endomorphism $\bar{\rho}$ of $\mathcal{B}$ is localised on Λ if $\bar{\rho}(\pi_1(O)) = \pi_1(O)$ for all $O \in \mathcal{A}_{\Lambda^c}$, and that $\bar{\rho}$ is localised if it is localised on some allowed cone Λ. We say that an endomorphism $\bar{\rho}$ of $\mathcal{B}$ that is localised on an allowed cone Λ is transportable if for any allowed cone Λ' there is an endomorphism $\bar{\rho}'$ of $\mathcal{B}$, localised on Λ', and a unitary $U \in \mathcal{B}$ such that $U\,\bar{\rho}(\pi_1(O)) = \bar{\rho}'(\pi_1(O))\,U$ for all $O \in \mathcal{A}$ and such that $U \in \mathcal{R}(\widetilde{\Lambda})$ for any allowed cone $\widetilde{\Lambda}$ that contains the cones Λ and Λ'. We denote by Δ the set of all localised and transportable endomorphisms.

The localised and transportable endomorphisms of $\mathcal{B}$ are the objects of a $\mathbb{C}$-linear category with morphisms

$$(\bar{\rho}, \bar{\sigma}) := \{R \in \mathcal{B} \,:\, R\bar{\rho}(\pi_1(O)) = \bar{\sigma}(\pi_1(O))R \text{ for all } \ O \in \mathcal{A}\} \tag{7}$$

for any $\bar{\rho}, \bar{\sigma} \in \Delta$. Morphisms are referred to as intertwiners.

Direct sums of objects can be constructed as in Lemma 6.1 of [19]. Indeed, for any cone Λ, Corollary 5.3 of [19] shows that there are isometries $V_1, V_2 \in \mathcal{R}(\Lambda)$ such that $V_i^* V_j = \delta_{i,j}\, \mathbb{1}$ and $V_1 V_1^* + V_2 V_2^* = \mathbb{1}$. If $\overline{\rho}, \overline{\sigma}$ are localised on Λ, then $\mathrm{Ad}[V_1] \circ \overline{\rho} + \mathrm{Ad}[V_2] \circ \overline{\sigma}$ is a direct sum of $\overline{\rho}$ and $\overline{\sigma}$ which is still localised on Λ. Moreover, if $\overline{\rho}$ and $\overline{\sigma}$ are transportable, then for any allowed cone Λ' there are endomorphisms $\overline{\rho}', \overline{\sigma}'$ localised on Λ' and unitary morphisms $U_1 \in (\overline{\rho}, \overline{\rho}')$ and $U_2 \in (\overline{\sigma}, \overline{\sigma}')$ such that $U_1, U_2 \in \mathcal{R}(\widetilde{\Lambda})$ for any allowed cone $\widetilde{\Lambda} \supseteq \Lambda \cup \Lambda'$. Then take isometries $V_1', V_2' \in \mathcal{R}(\Lambda')$ as above and consider the direct sum $\mathrm{Ad}[V_1'] \circ \overline{\rho}' + \mathrm{Ad}[V_2'] \circ \overline{\sigma}'$ of $\overline{\rho}'$ and $\overline{\sigma}'$, which is localised on Λ'. The unitary $W = V_1' U_1 V_1^* + V_2' U_2 V_2^* \in \mathcal{R}(\widetilde{\Lambda})$ then intertwines the two direct sums, showing that direct sums of localised and transportable endomorphisms are again localised and transportable.

The category of localised and transportable endomorphisms of $\mathcal{B}$ can be equipped with a monoidal structure. For any $\bar{\rho}, \bar{\sigma} \in \Delta$, we define their tensor product by

$$\bar{\rho} \otimes \bar{\sigma} := \bar{\rho} \circ \bar{\sigma}, \tag{8}$$

and for any intertwiners $R \in (\bar{\rho}, \bar{\rho}')$ and $S \in (\bar{\sigma}, \bar{\sigma}')$ we define the tensor product by

$$R \otimes S := R\bar{\rho}(S) \in (\bar{\rho} \otimes \bar{\sigma}, \bar{\rho}' \otimes \bar{\sigma}'). \tag{9}$$

2.2.2. Subcategory of Abelian Anyons. We use Assumptions 1–4 to obtain a subcategory of Δ whose simple objects correspond to the anyon types $a \in I$.

The following lemma shows that the automorphisms $w_{a,\Lambda}$ of Assumption 1 can be extended to localised and transportable endomorphisms of the allowed algebra $\mathcal{B}$, i.e. they yield objects in the category Δ.

Lemma 2.5 (Proposition 4.6 [19]). *For each allowed cone Λ and each $a \in I$, the automorphism $w_{a,\Lambda}$ of Assumption 1 has a unique extension to an endomorphism $\overline{w}_{a,\Lambda}$ of $\mathcal{B}$ that is weakly continuous on $\mathcal{R}(\Lambda')$ for any allowed cone Λ'. If $\Lambda' \supset \Lambda$, then $\overline{w}_{a,\Lambda}(\mathcal{R}(\Lambda')) = \mathcal{R}(\Lambda')$. Moreover, $\overline{w}_{a,\Lambda}$ is localised on Λ, and is transportable. In particular, $\overline{w}_{a,\Lambda} \in \Delta$.*

Proof. Let $\Lambda' \supset \Lambda$ be any allowed cone that contains Λ. From Assumption 1, we have for any forbidden cone Λ'' with $\Lambda' \cap \Lambda'' = \emptyset$ that $\pi_1 \circ w_{a,\Lambda} \simeq \pi_1 \circ w_{a,\Lambda''}$. Let $V \in \mathcal{B}(\mathcal{H})$ be a unitary implementing this equivalence. We have for any $O \in \mathcal{A}_{\Lambda'}$ that

$$\pi_1\big(w_{a,\Lambda}(O)\big) = V\pi_1\big(w_{a,\Lambda''}(O)\big)V^* = V\pi_1(O)V^* \tag{10}$$

where we used that $w_{a,\Lambda''}$ is supported on Λ'', which is disjoint form Λ'. We define the action of $\overline{w}_{a,\Lambda}$ on $\mathcal{R}(\Lambda')$ by $\mathrm{Ad}(V)$, which is weakly continuous, and is uniquely determined by the action of $w_{a,\Lambda}$ on $\mathcal{A}_{\Lambda'}$. Clearly this action on $\mathcal{R}(\Lambda')$ does not depend on the choice of Λ''. It follows that the extensions to $\mathcal{R}(\Lambda')$ for different Λ' are consistent with each other. Together with the weak continuity, this shows that the extension $\overline{w}_{a,\Lambda}$ is well defined on all of $\mathcal{B}$. Moreover, we have $w_{a,\Lambda}(\mathcal{A}_{\Lambda'}) = \mathcal{A}_{\Lambda'}$ and weak continuity then implies $\overline{w}_{a,\Lambda}(\mathcal{R}(\Lambda')) = (w_{a,\Lambda}(\mathcal{A}_{\Lambda'}))'' = \mathcal{R}(\Lambda')$ as required. This implies the inclusion $\overline{w}_{a,\Lambda}(\mathcal{B}) \subseteq \mathcal{B}$, so $\overline{w}_{a,\Lambda}$ is indeed an endomorphism of $\mathcal{B}$.

The endomorphism $\overline{w}_{a,\Lambda}$ is localised on Λ by construction. To see that it is transportable, let Λ' be any allowed cone and let $\overline{w}_{a,\Lambda'}$ be the extension of the automorphism $w_{a,\Lambda'}$ to the allowed algebra $\mathcal{B}$. By Assumption 4, there is a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $U \in \mathcal{R}(\Lambda'')$ for any allowed cone $\Lambda'' \supset \Lambda \cup \Lambda'$ and

$$U\,\pi_1\big(w_{a,\Lambda}(O)\big) = \pi_1(w_{a,\Lambda'}(O))\,U \tag{11}$$

for all $O \in \mathcal{A}$. It follows from this and the construction of $\overline{w}_{a,\Lambda}$ and $\overline{w}_{a,\Lambda'}$ that $U \in (\overline{w}_{a,\Lambda}, \overline{w}_{a,\Lambda'})$, showing that $\overline{w}_{a,\Lambda}$ is transportable. □

Lemma 2.6. *The endomorphisms $\overline{w}_{a,\Lambda}$ are simple objects of Δ, and two such objects $\overline{w}_{a,\Lambda}$ and $\overline{w}_{b,\Lambda'}$ are isomorphic if and only if $a = b$. That is,*

$$(\overline{w}_{a,\Lambda}, \overline{w}_{b,\Lambda'}) \simeq \begin{cases} \mathbb{C}\,\mathbb{1} & \text{if } a = b \\ \{0\} & \text{otherwise.} \end{cases} \tag{12}$$

Proof. Suppose $R \in (\overline{w}_{a,\Lambda}, \overline{w}_{b,\Lambda'})$, i.e. the operator $R \in \mathcal{B}$ satisfies

$$R\,\overline{w}_{a,\Lambda}(\pi_1(O)) = \overline{w}_{b,\Lambda'}(\pi_1(O))R \tag{13}$$

for all $O \in \mathcal{A}$. By construction of $\overline{w}_{a,\Lambda}$, this implies

$$R\,(\pi_1 \circ w_{a,\Lambda})(O) = (\pi_1 \circ w_{b,\Lambda'})(O)\,R \tag{14}$$

for all $O \in \mathcal{A}$. By Assumption 1, we have $\pi_1 \circ w_{a,\Lambda} \simeq \pi_a$ and $\pi_1 \circ w_{b,\Lambda} \simeq \pi_b$, so there are unitaries $U_a, U_b \in \mathcal{B}(\mathcal{H})$ such that

$$RU_a\,\pi_a(O)\,U_a^* = U_b\,\pi_b(O)\,U_b^*R \tag{15}$$

for all $O \in \mathcal{A}$. We see that $U_b^*RU_a$ intertwines the irreducible representations π_a and π_b. By assumption, we have $\pi_a \simeq \pi_b$ if and only if $a = b$. So if $a = b$ then we must have $U_b^*RU_a \in \mathbb{C}\,\mathbb{1}$ which holds if and only if $R \in \mathbb{C}I$, and if $a \neq b$ then we must have $U_b^*RU_a = 0$ hence $R = 0$. ☐

Recall from Assumption 2 and Lemma 2.3 that the set of anyon types I is equipped with an abelian product $\times$.

Lemma 2.7. *If Λ and Λ' are allowed cones and $a, b \in I$, then $\overline{w}_{a,\Lambda} \otimes \overline{w}_{b,\Lambda'} \simeq \overline{w}_{a\times b,\Lambda''}$ for any allowed cone Λ''. If $\widetilde{\Lambda} \supseteq \Lambda \cup \Lambda' \cup \Lambda''$, then the intertwiners that realise this isomorphism are elements of $\mathcal{R}(\widetilde{\Lambda})$.*

Proof. By Assumption 4, we have $\pi_1 \circ w_{a,\Lambda} \simeq \pi_1 \circ w_{a,\Lambda''}$ implemented by unitaries $V \in \mathcal{R}(\widetilde{\Lambda})$. It follows from this and Lemma 2.6 that $(\overline{w}_{a,\Lambda}, \overline{w}_{a,\Lambda''})$ is spanned by a unitary $U_a \in \mathcal{R}(\widetilde{\Lambda})$. Similarly, $(\overline{w}_{b,\Lambda'}, \overline{w}_{b,\Lambda''})$ is spanned by a unitary $U_b \in \mathcal{R}(\widetilde{\Lambda})$.

Again by Assumption 4, we have $\pi_1 \circ w_{a,\Lambda''} \circ w_{b,\Lambda''} \simeq \pi_1 \circ w_{a\times b,\Lambda''}$ with unitary intertwiners all belonging to $\mathcal{R}(\Lambda'')$. It follows that $\overline{w}_{a,\Lambda''} \otimes \overline{w}_{b,\Lambda''}$ is isomorphic to $\overline{w}_{a\times b,\Lambda''}$ and is therefore simple. Moreover, $(\overline{w}_{a,\Lambda''} \otimes \overline{w}_{b,\Lambda''}, \overline{w}_{a\times b,\Lambda''})$ is spanned by a unitary $V_{a\times b} \in \mathcal{R}(\Lambda'') \subset \mathcal{R}(\widetilde{\Lambda})$.

We now find that $V_{a\times b}(U_a \otimes U_b) \in \mathcal{R}(\widetilde{\Lambda})$ is a unitary intertwiner in $(\overline{w}_{a,\Lambda} \otimes \overline{w}_{b,\Lambda'}, \overline{w}_{a\times b,\Lambda''})$. Since $\overline{w}_{a\times b,\Lambda''}$ is simple, so is $\overline{w}_{a,\Lambda} \otimes \overline{w}_{b,\Lambda'}$, and it follows that $(\overline{w}_{a,\Lambda} \otimes \overline{w}_{b,\Lambda'}, \overline{w}_{a\times b,\Lambda''})$ is actually spanned by $V_{a\times b}(U_a \otimes U_b)$. This proves the claim. ☐

We now identify a full subcategory of Δ whose isomorphism classes correspond to sums of anyons types $a \in I$.

For any allowed cone Λ, we fix isometries $V_1, V_2 \in \mathcal{R}(\Lambda)$ such that $V_i^*V_j = \delta_{i,j}\,\mathbb{1}$ and $V_1V_1^* + V_2V_2^* = \mathbb{1}$ (existence follows from Corollary 5.3 of [19]) and define for any $\bar{\rho}, \bar{\sigma} \in \Delta$ the concrete direct sum $\bar{\rho} \oplus_\Lambda \bar{\sigma} := \mathrm{Ad}[V_1]\circ\bar{\rho} + \mathrm{Ad}[V_2]\circ\bar{\sigma}$.

Definition 2.8. We let Δ_Λ^I be the full tensor subcategory of Δ generated by the simple objects $\{\overline{w}_{a,\Lambda}\}_{a\in I}$ using the tensor product $\otimes$ and the direct sum $\oplus_\Lambda$.

By construction, all objects of Δ_Λ^I are localised on Λ. Moreover, the category Δ_Λ^I is semisimple as show in the following lemma.

Lemma 2.9. *Every object of Δ_Λ^I is isomorphic to an object of the form*

$$\overline{\rho}_1 \oplus_\Lambda \overline{\rho}_2 \oplus_\Lambda \cdots \oplus_\Lambda \overline{\rho}_n \tag{16}$$

where the $\overline{\rho}_i$ are tensor products of the simple objects $\overline{w}_{a,\Lambda}$ and the isomorphism is given by a unitary in $\mathcal{R}(\Lambda)$.

(Note that the direct sum $\oplus_\Lambda$ is not associative, so the expression (16) is to be interpreted as being defined by some choice of bracketing. Different bracketings are isomorphic through a unitary in $\mathcal{R}(\Lambda)$).

Proof. For the duration of this proof, we write $\oplus = \oplus_\Lambda$. Denote by $V_1, V_2 \in \mathcal{R}(\Lambda)$ the isometries used to construct the direct sum $\oplus_\Lambda$, and for any $\overline{\rho} \in \Delta$ we write $\oplus_{\overline{\rho}}$ for the direct sum constructed using the isometries $\overline{\rho}(V_1), \overline{\rho}(V_2)$.

Suppose the claim is true for two objects $\overline{\rho}$ and $\overline{\sigma}$ of Δ_Λ^I, i.e. $\overline{\rho} \simeq \overline{\rho}_1 \oplus \cdots \oplus \overline{\rho}_n$ and $\overline{\sigma} \simeq \overline{\sigma}_1 \oplus \cdots \oplus \overline{\sigma}_m$ with isomorphisms implemented by unitaries in $\mathcal{R}(\Lambda)$, and where the $\overline{\rho}_i$ and $\overline{\sigma}_j$ are finite tensor products of the simple objects $\overline{w}_{a,\Lambda}$. The claim then holds trivially for $\overline{\rho} \oplus \overline{\sigma}$. Let us now show that the claim holds for the tensor product $\overline{\rho} \otimes \overline{\sigma}$. We have

$$\begin{aligned} \overline{\rho} \otimes \overline{\sigma} &\simeq (\overline{\rho}_1 \oplus \cdots \oplus \overline{\rho}_n) \otimes (\overline{\sigma}_1 \oplus \cdots \oplus \overline{\sigma}_m) \\ &= \bigoplus_{\kappa=1}^{n} (\overline{\rho}_\kappa \otimes \overline{\sigma}_1) \oplus_{\overline{\rho}_\kappa} \cdots \oplus_{\overline{\rho}_\kappa} (\overline{\rho}_\kappa \otimes \overline{\sigma}_m) \end{aligned}$$

where the isomorphism is implemented by a unitary in $\mathcal{R}(\Lambda)$. Noting that the direct sums $\oplus_{\overline{\rho}_\kappa}$ are isomorphic to $\oplus$ through a unitary in $\mathcal{R}(\Lambda)$ we obtain the required equivalence of $\overline{\rho} \otimes \overline{\sigma}$ to an object of the form (16). We have shown that if $\overline{\rho}$ and $\overline{\sigma}$ both satisfy the claim, then so do $\overline{\rho} \oplus \overline{\sigma}$ and $\overline{\rho} \otimes \overline{\sigma}$. Since the claim holds trivially for the simple objects $\{\overline{w}_{a,\Lambda}\}_{a \in I}$ and the category Δ_Λ^I is by definition generated by these simple objects using $\otimes$ and $\oplus$, we conclude that every object of Δ_Λ^I is isomorphic to an object of the form (16) through a unitary in $\mathcal{R}(\Lambda)$, as we wanted to show. □

Lemma 2.10. *Let Λ be an allowed cone and $\Lambda_1, \Lambda_2 \subset \Lambda$ two allowed subcones. For any two objects $\overline{\rho} \in \Delta_{\Lambda_1}^I$ and $\overline{\sigma} \in \Delta_{\Lambda_2}^I$, we have $(\overline{\rho}, \overline{\sigma}) \subset \mathcal{R}(\Lambda)$.*

Proof. Let $\overline{\rho}_1, \overline{\rho}_2 \in \Delta_{\Lambda_1}^I$ and $\overline{\sigma}_1, \overline{\sigma}_2 \in \Delta_{\Lambda_2}^I$ and suppose $(\overline{\rho}_k, \overline{\sigma}_l) \subset \mathcal{R}(\Lambda)$ for $k, l \in \{1, 2\}$. We will show that $(\overline{\rho}_1 \oplus_{\Lambda_1} \overline{\rho}_2, \overline{\sigma}_1 \oplus_{\Lambda_2} \overline{\sigma}_2) \subset \mathcal{R}(\Lambda)$. According to Lemma 2.9, the result then follows by induction on the number of summands in (16), since it holds for the simple objects $\{\overline{w}_{a,\Lambda_1}\}_{a \in I}$ and $\{\overline{w}_{a,\Lambda_2}\}_{a \in I}$ by Lemma 2.6 and for finite tensor products of these by repeated application of Lemma 2.7.

For $i \in \{1, 2\}$, let $V_1^{(i)}, V_2^{(i)} \in \mathcal{R}(\Lambda_i)$ be the isometries used to define $\oplus_{\Lambda_i}$, and let $p_k^{(i)} = V_k^{(i)} (V_k^{(i)})^*$ for $k = 1, 2$. Then $(V_k^{(i)})^* p_l^{(i)} = \delta_{k,l} (V_k^{(i)})^*$ and $p_k^{(i)} V_l^{(i)} = \delta_{k,l} V_k^{(i)}$. Suppose $R \in (\overline{\rho}_1 \oplus_{\Lambda_1} \overline{\rho}_2, \overline{\sigma}_1 \oplus_{\Lambda_2} \overline{\sigma}_2)$, then

$$\begin{aligned} &R\big(\mathrm{Ad}[V_1^{(1)}] \circ \overline{\rho}_1(O) + \mathrm{Ad}[V_2^{(1)}] \circ \overline{\rho}_2(O)\big) \\ &\quad = \big(\mathrm{Ad}[V_1^{(2)}] \circ \overline{\sigma}_1(O) + \mathrm{Ad}[V_2^{(2)}] \circ \overline{\sigma}_2(O)\big) R \end{aligned} \tag{17}$$

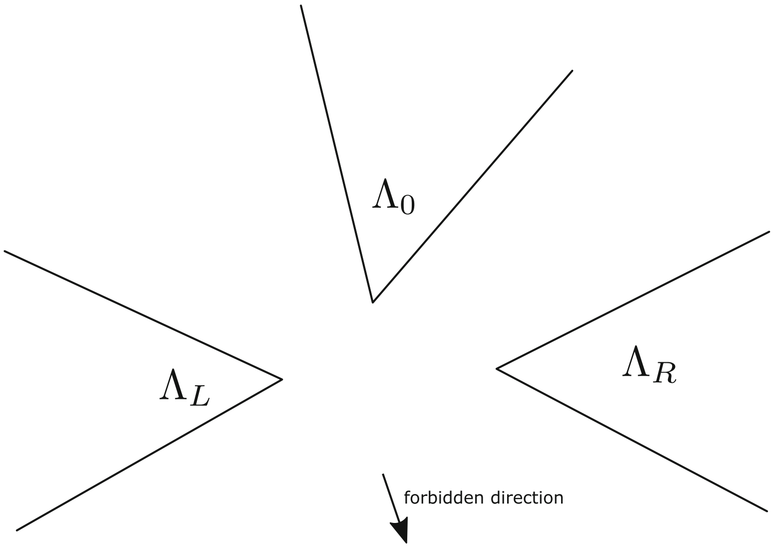

Figure 1. Cones Λ_0, Λ_L and Λ_R used in the definition of the braiding intertwiners $\epsilon(\rho,\sigma)$

for all $O \in \mathcal{A}$. Multiplying from the left with $(V_k^{(2)})^*$ and from the right with $V_l^{(1)}$ yields

$$(V_k^{(2)})^* R\, V_l^{(1)}\, \overline{\rho}_l(O) = \overline{\sigma}_k(O)\, (V_k^{(2)})^* R\, V_l^{(1)} \tag{18}$$

for all $O \in \mathcal{A}$, so $(V_k^{(2)})^* R\, V_l^{(1)} \in (\overline{\rho}_l, \overline{\sigma}_k)$. So then by hypothesis, $(V_k^{(2)})^* R\, V_l^{(1)} \in \mathcal{R}(\Lambda)$ for all $k, l = 1, 2$, and also $p_k^{(2)} R\, p_l^{(1)} \in \mathcal{R}(\Lambda)$. Therefore, $R = \sum_{k,l} p_k^{(2)} R\, p_l^{(1)} \in \mathcal{R}(\Lambda)$ as required. □

2.2.3. Braided Structure. Fix an allowed cone Λ_0 and consider two endomorphisms $\bar{\rho}, \bar{\sigma} \in \Delta^I_{\Lambda_0}$. Pick allowed cones Λ_L and Λ_R as in Fig. 1. i.e. the disjoint allowed cones Λ_R, Λ_0 and Λ_L are arranged in a counterclockwise order from the forbidden direction, and there are allowed cones $\widetilde{\Lambda}_L \supset \Lambda_L \cup \Lambda_0$ and $\widetilde{\Lambda}_R \supset \Lambda_R \cup \Lambda_0$ such that $\widetilde{\Lambda}_L \cap \Lambda_R = \widetilde{\Lambda}_R \cap \Lambda_L = \emptyset$. We say Λ_L is to the left of Λ_0, and Λ_R is to the right of Λ_0. By transportability, there are endomorphisms $\bar{\rho}_L \in \Delta^I_{\Lambda_L}$ and $\bar{\sigma}_R \in \Delta^I_{\Lambda_R}$, and unitary intertwiners $U \in (\bar{\rho}, \bar{\rho}_L)$ and $V \in (\bar{\sigma}, \bar{\sigma}_R)$ such that $U \in \mathcal{R}(\widetilde{\Lambda}_L)$ and $V \in \mathcal{R}(\widetilde{\Lambda}_R)$.

Definition 2.11. The braiding intertwiner $\epsilon(\bar{\rho}, \bar{\sigma}) \in (\bar{\rho} \otimes \bar{\sigma}, \bar{\sigma} \otimes \bar{\rho})$ is given by

$$\epsilon(\bar{\rho}, \bar{\sigma}) := (V^* \otimes U^*)(U \otimes V) = V^* \bar{\rho}(V). \tag{19}$$

To get the last equality, we use $\bar{\sigma}_R(U) = U$ which holds because $\bar{\sigma}_R$ is localised on Λ_R while $U \in \mathcal{R}(\widetilde{\Lambda}_L)$. Using $\bar{\rho}_L \otimes \bar{\sigma}_R = \bar{\sigma}_R \otimes \bar{\rho}_L$, one easily verifies that $\epsilon(\bar{\rho}, \bar{\sigma})$ is indeed an intertwiner from $\bar{\rho} \otimes \bar{\sigma}$ to $\bar{\sigma} \otimes \bar{\rho}$.

Lemma 2.12. *The braiding $\epsilon(\bar{\rho}, \bar{\sigma})$ is independent of the choice of cones Λ_L, Λ_R, the choice of objects $\bar{\rho}_L \in \Delta^I_{\Lambda_L}$ and $\bar{\sigma}_R \in \Delta^I_{\Lambda_R}$ and the choice of intertwiners $U \in (\bar{\rho}, \bar{\rho}_L)$ and $V \in (\bar{\sigma}, \bar{\sigma}_R)$.*

Proof. Choose different cones Λ_L' and Λ_R' to the left and to the right of Λ_0. Then there are allowed cones $\Lambda_L'' \supseteq \Lambda_L \cup \Lambda_L'$ and $\Lambda_R'' \supseteq \Lambda_R \cup \Lambda_R'$ that are also to the left and to the right of Λ_0, respectively. Choose objects $\bar{\rho}_L' \in \Delta^I_{\Lambda_L'}$ and $\bar{\sigma}_R' \in \Delta^I_{\Lambda_R'}$, as well as morphisms $U' \in (\bar{\rho}, \bar{\rho}_L')$ and $V' \in (\bar{\sigma}, \bar{\sigma}_R')$. Then $V'V^* \in (\bar{\sigma}_R, \bar{\sigma}_R') \subset \mathcal{R}(\Lambda_R'')$ by Lemma 2.10. The new choice $\bar{\rho}_L', \bar{\sigma}_R'$ leads to a braiding intertwiner

$$\begin{aligned}\epsilon'(\bar{\rho}, \bar{\sigma}) &= (V'^* \otimes U'^*)(U' \otimes V') = V'^* \bar{\rho}(V') \\ &= V^*(V'V^*)^* \bar{\rho}\big((V'V^*)V\big) = V^* \bar{\rho}(V) \\ &= \epsilon(\bar{\rho}, \bar{\sigma})\end{aligned}$$

where we used $\bar{\rho}(V'V^*) = V'V^*$ since $V'V^* \in \mathcal{R}(\Lambda_R'')$ and $\bar{\rho}$ is supported in Λ_0, which is disjoint from Λ_R''. □

Lemma 2.13. *The braiding intertwiners satisfy the braid equations*

$$\begin{aligned}\epsilon(\rho \otimes \sigma, \tau) &= \big(\epsilon(\rho, \tau) \otimes 1_\sigma\big)\big(1_\rho \otimes \epsilon(\sigma, \tau)\big) \\ \epsilon(\rho, \sigma \otimes \tau) &= \big(1_\sigma \otimes \epsilon(\rho, \tau)\big)\big(\epsilon(\rho, \sigma) \otimes 1_\tau\big)\end{aligned}$$

where $1_\rho = \mathbb{1} \in (\rho, \rho)$.

Proof. Let us prove the first equation, the second is shown in the same way. Choose ρ_L, σ_L supported in Λ_L and morphisms $U_\rho \in (\rho, \rho_L)$, $U_\sigma \in (\sigma, \sigma_L)$. Choose τ_R supported in Λ_R and a morphism $V_\tau \in (\tau, \tau_R)$. Then

$$\begin{aligned}\epsilon(\rho \otimes \sigma, \tau) &= \big(V_\tau^* \otimes (U_\rho \otimes U_\sigma)^*\big)\big((U_\rho \otimes U_\sigma) \otimes V_\tau\big) \\ &= V_\tau^* \overline{\rho \otimes \sigma}(V_\tau) = V_\tau^* \bar{\rho}\big(\bar{\sigma}(V_\tau)\big) \\ &= V_\tau^* \bar{\rho}(V_\tau) \bar{\rho}(V_\tau^*) \bar{\rho}\big(\bar{\sigma}(V_\tau)\big) \\ &= \big(\epsilon(\rho, \tau) \otimes 1_\sigma\big)\big(\bar{\rho}(V_\tau^* \bar{\sigma}(V_\tau))\big) \\ &= \big(\epsilon(\rho, \tau) \otimes 1_\sigma\big)\big(1_\rho \otimes \epsilon(\sigma, \tau)\big).\end{aligned}$$

□

2.2.4. Relation to Previous Work.

The state ω is said to satisfy Haag duality for cones if $\mathcal{R}(\Lambda) = \mathcal{R}(\Lambda^c)'$ for all cones Λ. Haag duality for cones has been verified for abelian quantum double models in [8,20]. We believe that the double semion state introduced below can also be shown to satisfy Haag duality for cones using similar methods.

Under the assumption that the pure state ω satisfies (approximate) Haag duality for cones, it is shown in [21] that the category of localised and transportable endomorphisms Δ is a braided C*-tensor category with isomorphism classes of simple objects in one-to-one correspondence with (equivalence classes of) irreducible representations of the observable algebra that satisfy the superselection criterion (Definition 2.1).

The braided tensor categories Δ^I_Λ constructed above are full subcategories of Δ. If ω satisfies Haag duality for cones, then Δ is equipped with a braiding given by Definition 4.11 of [21]. The restriction of this braiding to Δ^I_Λ agrees with the braiding defined in 2.11. It follows that the category Δ^I_Λ

completely describes the closed system of abelian anyons corresponding to the irreducible representations $\{\pi_a\}_{a\in I}$ in a way that is consistent with the theory of [21]. These abelian anyons form a subset of all anyon types present in the model (anyon types correspond to irreducible representations of the observables algebra that satisfy the superselection criterion). In contrast, the theory of [21] captures all anyon types, in particular also all non-abelian anyons.

The reason that (approximate) Haag duality allows a description of all anyon types is twofold. Most importantly, Haag duality allows one to construct for any representation π that satisfies the superselection criterion with respect to π_1, and for any cone Λ, a transportable endomorphism $\overline{\rho}_{\pi,\Lambda}$ localised on Λ such that $\pi \simeq \overline{\rho}_{\pi,\Lambda} \circ \pi_1$ (Definition 2.13 and Lemma 2.14 of [21]). This ensures that Δ contains objects corresponding to any superselection sector. In our setting, we obtain localised and transportable endomorphisms for the superselection sectors $\{\pi_a\}_{a\in I}$ using Assumptions 1 and 4, see Lemma 2.5.

Haag duality also allows a braiding to be defined for the entire tensor category Δ. Note that for the braiding of definition 2.11 to be well defined, we must show that it is independent of the choice of intertwiners U and V used in the definition. This is done in Lemma 2.12 using the fact that if $\bar{\rho} \in \Delta^I_{\Lambda_1}$ and $\bar{\sigma} \in \Delta^I_{\Lambda_2}$, then all morphisms in $(\bar{\rho}, \bar{\sigma})$ are elements of $\mathcal{R}(\Lambda)$ for any allowed cone $\Lambda \supseteq \Lambda_1 \cup \Lambda_2$. In our setting, this follows from Assumption 4 (Lemma 2.10). With Haag duality for cones, this locality property of morphisms follows immediately. Indeed, suppose $\overline{\rho}$ and $\overline{\sigma}$ are both localised on an allowed cone Λ, and suppose $R \in (\overline{\rho}, \overline{\sigma})$. Then for any $O \in \mathcal{A}_{\Lambda^c}$ we have

$$\overline{\rho}(\pi_1(O)) = \overline{\sigma}(\pi_1(O)) = \pi_1(O), \tag{20}$$

and hence

$$R\pi_1(O) = R\overline{\rho}(\pi_1(O)) = \overline{\sigma}(\pi_1(O))R = \pi_1(O)R. \tag{21}$$

We see that $R \in \pi_1(\mathcal{A}_{\Lambda^c})' = \mathcal{R}(\Lambda^c)' = \mathcal{R}(\Lambda)$ by Haag duality.

Using the same argument, one can use Haag duality for cones to prove Assumption 4, as mentioned above.

2.3. *Fusion Ring, F-Symbols, and R-Symbols.*

2.3.1. Fusion Ring of Δ^I_Λ.

To any semisimple tensor category, one may associate its fusion ring, see Section 4.5 of [7] for details. For the category Δ^I_Λ, the construction goes as follows. First, note that the isomorphism classes of simple objects of Δ^I_Λ are labelled by the elements of I, they are precisely the classes $[\overline{w}_{a,\Lambda}]$ for $a \in I$. Since Δ^I_Λ is semisimple, any object $\bar{\rho}$ of Δ^I_Λ is isomorphic to a direct sum of simple objects $\overline{w}_{a,\Lambda}$. The number of times that $\overline{w}_{a,\Lambda}$ appears in such a direct sum decomposition is independent on the particular choice of direct sum decomposition and is called the multiplicity of a in $\bar{\rho}$, and denoted by $[\bar{\rho} : a]$. The fusion ring of Δ^I_Λ is the free abelian group generated by the isomorphism classes of simple objects $\{[\overline{w}_{a,\Lambda}]\}_{a\in I}$, which we can identify with the elements of I. Addition in this group corresponds to the direct sum in the category. The isomorphism class of a general object $\bar{\rho} \in \Delta_{\Lambda}^{I}$ corresponds to an element of fusion ring given by

$$[\bar{\rho}] := \sum_{a \in I} [\bar{\rho} : a] \, a. \tag{22}$$

The multiplication of the fusion ring is given by the tensor product of the category. It is sufficient to define the multiplication on the generators by

$$a \times b := [\overline{w}_{a,\Lambda} \otimes \overline{w}_{b,\Lambda}] = [\overline{w}_{a \times b,\Lambda}] = ab, \tag{23}$$

which corresponds precisely to the multiplication on I introduced in Assumption 2 and Lemma 2.3. We see that the fusion ring of Δ_{Λ}^{I} is given by $\mathbb{Z}[I]$, the ring of polynomials over the abelian group I with integer coefficients.

Two tensor categories that are monoidally equivalent have isomorphic fusion rings. The converse certainly does not hold; there are inequivalent tensor categories that have isomorphic fusion rings (for example, all categories Vec_G^{ω} for 3-cocycle $\omega : G^{\times 3} \to \mathbb{C}^{\times}$ have fusion ring $\mathbb{Z}[G]$). It turns out that the fusion ring together with certain cohomological data is sufficient information to characterise a tensor category up to monoidal equivalence, see, for example, Proposition 1.1 of [23]. Below, we will describe these cohomological data for the category Δ_{Λ}^{I} in terms of ‘F-symbols’, a nomenclature common in the physics literature. Since we are interested in *braided* tensor categories, the question arises to what extent the fusion ring together with the F-symbols characterises a braided tensor category. The braiding induces more structure on the fusion ring in the form of ‘R-symbols’, which will be described for our Δ_{Λ}^{I} below. To the best of the authors’ knowledge, it is not known if a fusion ring together with F and R-symbols is enough information to characterise a braided tensor category completely. In the special case where the fusion ring takes the form $\mathbb{Z}[G]$ for a finite abelian group G however, this does turn out to be the case, see Proposition 7.5.2 of [12].

2.3.2. Fusion and F-Symbols. Fix an allowed cone Λ_0 and write $\overline{w}_a := \overline{w}_{a,\Lambda_0}$. Pick unitary intertwiners $\Omega(a,b) \in (\overline{w}_a \otimes \overline{w}_b, \overline{w}_{a \times b}) \subset \mathcal{R}(\Lambda_0)$ called fusion operators. Note that the $\Omega(a,b)$ are unique up to phase. The unitaries

$$\Omega(ab,c)(\Omega(a,b) \otimes 1_c) = \Omega(ab,c)\Omega(a,b)$$
$$\Omega(a,bc)(1_a \otimes \Omega(b,c)) = \Omega(a,bc)\overline{w}_a(\Omega(b,c))$$

are both intertwiners from $\overline{w}_a \otimes \overline{w}_b \otimes \overline{w}_c$ to $\overline{w}_{abc}$. Since $(\overline{w}_a \otimes \overline{w}_b \otimes \overline{w}_c, \overline{w}_{abc})$ is one-dimensional, there are phases $F(a,b,c) \in U(1)$ such that

$$\Omega(ab,c)\Omega(a,b) = F(a,b,c) \times \Omega(a,bc)\overline{w}_a(\Omega(b,c)). \tag{24}$$

These $F(a,b,c)$ are the F-symbols. Figure 2 gives a graphical representation of Eq. (24).

The F-symbols satisfy a pentagon equation, which in our setting of abelian anyons takes the form of a cocycle relation.

Proposition 2.14. *The F-symbols satisfy*

$$(\mathrm{d}F)(a,b,c,d) := \frac{F(a,b,c)F(a,bc,d)F(b,c,d)}{F(ab,c,d)F(a,b,cd)} = 1. \tag{25}$$

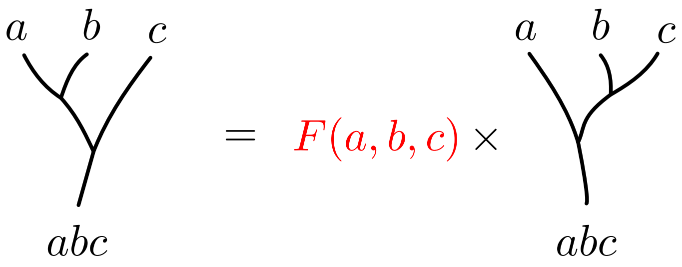

FIGURE 2. Graphical representation of Eq. (24), defining the F-symbols $F(a,b,c)$. Each node represents a fusion operator. The diagrams represent two different compositions of fusion operators both yielding intertwiners from $w_a \otimes w_b \otimes w_c$ to w_{abc}

Proof. A graphical proof is shown in Fig. 3. In equations, we have

$$\begin{aligned}
&\Omega(abc,d)\Omega(ab,c)\Omega(a,b)\\
&\quad = F(ab,c,d) \times \Omega(ab,cd)\overline{w}_{ab}\big(\Omega(c,d)\big)\Omega(a,b)\\
&\quad = F(ab,c,d) \times \Omega(ab,cd)\Omega(a,b)\overline{w}_a\big(\overline{w}_b\big(\Omega(c,d)\big)\big)\\
&\quad = F(ab,c,d)F(a,b,cd) \times \Omega(a,bcd)\overline{w}_a(\Omega(b,cd))\overline{w}_a(\overline{w}_b(\Omega(c,d)))\\
&\quad = F(ab,c,d)F(a,b,cd) \times \Omega(a,bcd)\overline{w}_a\big(\Omega(b,cd)\overline{w}_b(\Omega(c,d))\big)
\end{aligned}$$

but also

$$\begin{aligned}
&\Omega(abc,d)\Omega(ab,c)\Omega(a,b)\\
&\quad = F(a,b,c) \times \Omega(abc,d)\Omega(a,bc)\overline{w}_a(\Omega(b,c))\\
&\quad = F(a,b,c)F(a,bc,d) \times \Omega(a,bcd)\overline{w}_a(\Omega(bc,d))\overline{w}_a(\Omega(b,c))\\
&\quad = F(a,b,c)F(a,bc,d) \times \Omega(a,bcd)\overline{w}_a\big(\Omega(bc,d)\Omega(b,c)\big)\\
&\quad = F(a,b,c)F(a,bc,d)F(b,c,d) \times \Omega(a,bcd)\overline{w}_a\big(\Omega(b,cd)\overline{w}_b(\Omega(c,d))\big).
\end{aligned}$$

And the desired equality follows. □

2.3.3. Braiding and R-Symbols. We simply set $\epsilon(a,b) := \epsilon(\overline{w}_a, \overline{w}_b)$ for any $a, b \in I$. The unitaries $\Omega(b,a)\epsilon(a,b)$ and $\Omega(a,b)$ are both intertwiners from $\overline{w}_a \otimes \overline{w}_b$ to $\overline{w}_{ab}$. Since $(\overline{w}_a \otimes \overline{w}_b, \overline{w}_{ab})$ is one-dimensional, there exist phases $R(a,b) \in U(1)$ such that

$$\Omega(b,a)\epsilon(a,b) = R(a,b) \times \Omega(a,b). \tag{26}$$

The phases $R(a,b)$ are the R-symbols. Figure 4 gives a graphical representation of Eq. (26).

2.3.4. Yang–Baxter Equation. The braidings $\epsilon(a,b)$ and fusions $\Omega(a,b)$ satisfy the Yang–Baxter equations, see Fig. 5.

Proposition 2.15. *We have*

$$\overline{w}_c\big(\Omega(a,b)\big)\epsilon(a,c)\overline{w}_a\big(\epsilon(b,c)\big) = \epsilon(ab,c)\Omega(a,b) \tag{27}$$

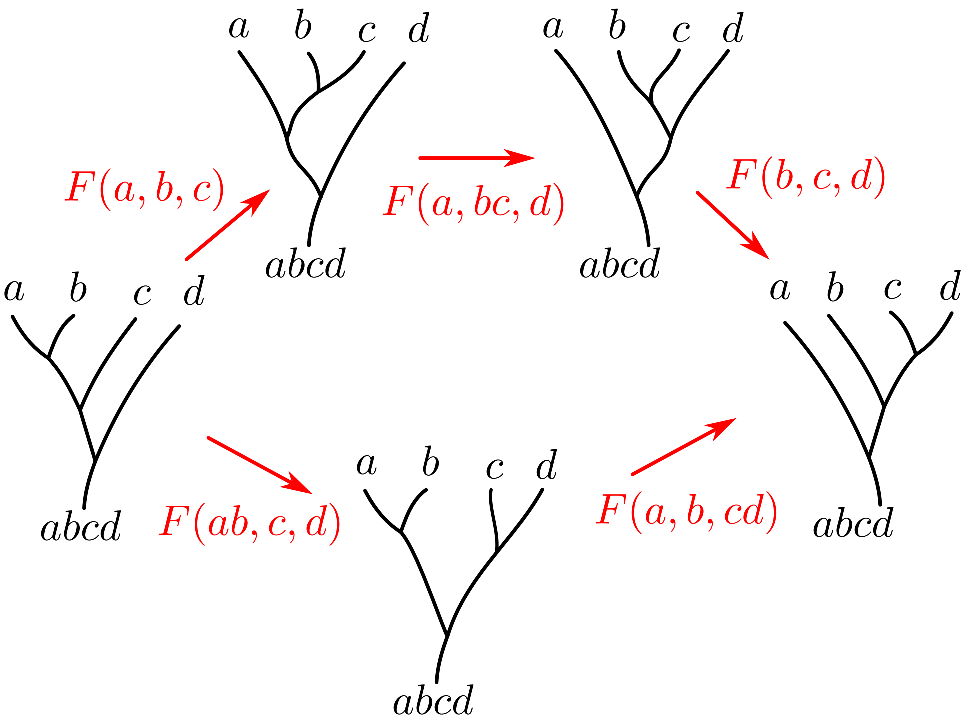

FIGURE 3. A graphical proof of the Pentagon equation

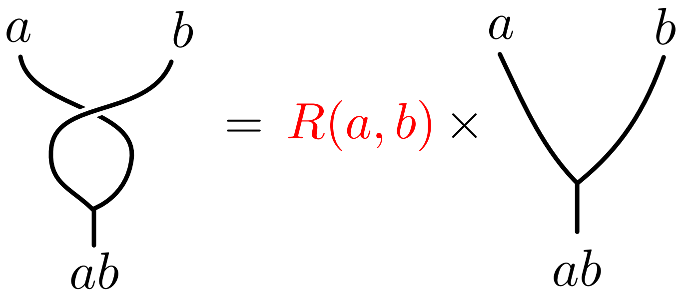

FIGURE 4. Graphical representation of Eq. (26), defining the R-symbols $R(a,b)$. The point where the a-line passes under the b-line represents the braiding intertwiner $\epsilon(a,b)$

and

$$\Omega(b,c)\overline{w}_b\big(\epsilon(a,c)\big)\epsilon(a,b) = \epsilon(a,bc)\overline{w}_a\big(\Omega(b,c)\big). \tag{28}$$

Proof. We prove Eq. (27); the proof of Eq. (28) is similar. Choose a cone Λ_R, an endomorphism $\overline{w}_c^R \in \Delta^I_{\Lambda_R}$, and a unitary $V \in (\overline{w}_c, \overline{w}_c^R)$ as in Definition 2.11. Then

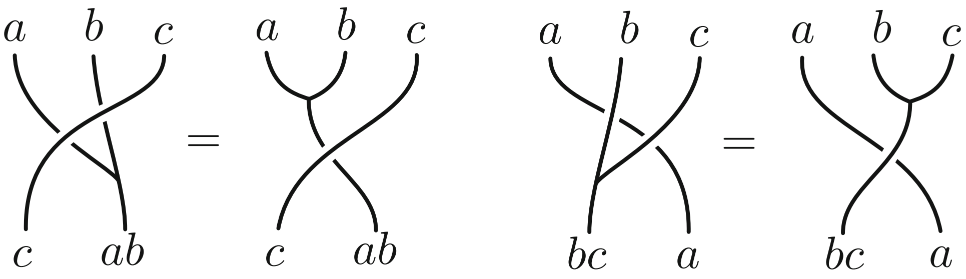

FIGURE 5. Graphical representation of the Yang–Baxter equations

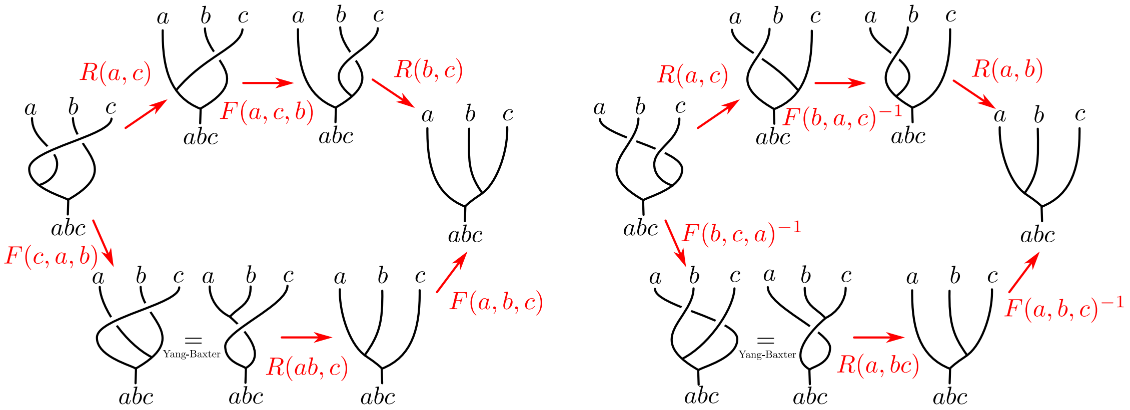

FIGURE 6. Graphical representations of the first and second hexagon equations

$$\begin{aligned}
\epsilon(ab,c)\Omega(a,b) &= \epsilon(\overline{w}_{ab},\overline{w}_c)\Omega(a,b)\\
&= V^*\,\overline{w}_{ab}\big(V\big)\Omega(a,b)\\
&= V^*\Omega(a,b)\overline{w}_a\big(\overline{w}_b(V)\big)\\
&= V^*\overline{w}_c^R\big(\Omega(a,b)\big)\overline{w}_a(V)\,\overline{w}_a\big(V^*\overline{w}_b(V)\big)\\
&= \overline{w}_c\big(\Omega(a,b)\big)\,V^*\overline{w}_a(V)\,\overline{w}_a\big(V^*\overline{w}_b(V)\big)\\
&= \overline{w}_c\big(\Omega(a,b)\big)\,\epsilon(a,c)\,\overline{w}_a\big(\epsilon(b,c)\big)
\end{aligned}$$

where we used $\Omega(a,b) \in (\overline{w}_a \otimes \overline{w}_b, \overline{w}_{ab})$ to obtain the third line. We used the fact that $\Omega(a,b) \in \mathcal{R}(\Lambda_0)$ so $\overline{w}_c^R(\Omega(a,b)) = \Omega(a,b)$ to obtain the fourth line. The fifth line follows from $V \in (\overline{w}_c, \overline{w}_c^R)$ and the fact that V is unitary, and the final line follows by the definition of the braidings $\epsilon(a,c)$ and $\epsilon(b,c)$. ☐

2.3.5. Hexagon Equation. Using the Yang–Baxter equation, we obtain the Hexagon equation, see Fig. 6.

Proposition 2.16. *The F and R-symbols satisfy the hexagon equations*

$$\frac{F(a,b,c)F(c,a,b)}{F(a,c,b)} = \frac{R(a,c)R(b,c)}{R(ab,c)} \tag{29}$$

and

$$\frac{F(a,b,c)F(b,c,a)}{F(b,a,c)} = \frac{R(a,bc)}{R(a,b)R(a,c)}. \tag{30}$$

Proof. The left diagram in Fig. 6 suggests the following two equalities of morphisms:

$$\begin{aligned}
&\Omega(ca,b)\Omega(c,a)\epsilon(a,c)\overline{w}_a(\epsilon(b,c))\\
&\quad = R(a,c) \times \Omega(ca,b)\Omega(a,c)\overline{w}_a(\epsilon(b,c))\\
&\quad = R(a,c)F(a,c,b) \times \Omega(a,cb)\overline{w}_a(\Omega(c,b))\overline{w}_a(\epsilon(b,c))\\
&\quad = R(a,c)F(a,c,b) \times \Omega(a,cb)\overline{w}_a\big(\Omega(c,b)\epsilon(b,c)\big)\\
&\quad = R(a,c)F(a,c,b)R(b,c) \times \Omega(a,cb)\overline{w}_a(\Omega(b,c))
\end{aligned}$$

and

$$\begin{aligned}
&\Omega(ca,b)\Omega(c,a)\epsilon(a,c)\overline{w}_a(\epsilon(b,c))\\
&\quad = F(c,a,b) \times \Omega(c,ab)\overline{w}_c(\Omega(a,b))\epsilon(a,c)\overline{w}_a(\epsilon(b,c))\\
&\quad = F(c,a,b) \times \Omega(c,ab)\epsilon(ab,c)\Omega(a,b)\\
&\quad = F(c,a,b)R(ab,c) \times \Omega(ab,c)\Omega(a,b)\\
&\quad = F(c,a,b)R(ab,c)F(a,b,c) \times \Omega(a,bc)\overline{w}_a(\Omega(b,c))
\end{aligned}$$

where we used the Yang–Baxter equation to obtain the second line. The coefficients of the right-hand sides must be equal, yielding the first hexagon equation.

The second hexagon equation is obtained in exactly the same way, following the right diagram in Fig. 6. □

2.3.6. Dependence of F and R-Symbols on the Choice of $\boldsymbol{\Lambda_0}$ and the Phases of $\boldsymbol{\Omega(a,b)}$. Suppose we chose different phases for the intertwiners $\Omega(a,b)$, i.e. we consider

$$\Omega'(a,b) = \chi(a,b)\Omega(a,b) \tag{31}$$

for phases $\chi(a,b)$. This yields new F-symbols by

$$\Omega'(ab,c)\Omega'(a,b) = F'(a,b,c) \times \Omega'(a,bc)\overline{w}_a(\Omega'(b,c)) \tag{32}$$

which are related to the original F-symbols by

$$F'(a,b,c) = (\mathrm{d}\chi)(a,b,c)F(a,b,c) = \frac{\chi(b,c)\chi(a,bc)}{\chi(ab,c)\chi(a,b)}F(a,b,c). \tag{33}$$

i.e. F' is related to F by the coboundary $\mathrm{d}\chi$. It follows that only the cohomology class $[F] \in H^3(I,U(1))$ is well defined.

The R-symbols are also affected by the different choice of phases. Indeed, the new R-symbols defined by

$$\Omega'(b,a)\epsilon(a,b) = R'(a,b) \times \Omega'(a,b) \tag{34}$$

are related to the old by

$$R'(a,b) = \frac{\chi(b,a)}{\chi(a,b)}R(a,b). \tag{35}$$

It follows that the self-statistics $R(a,a)$ and the double braidings $R(a,b)R(b,a)$ are invariants.

Next, we investigate the dependence of the F and R-symbols on the choice of allowed cone Λ_0. We will find no additional ambiguity beyond the one just discussed.

Let Λ_0' be another allowed cone. Then there is an allowed cone $\widetilde{\Lambda}_0$ containing $\Lambda_0 \cup \Lambda_0'$. Denote $\overline{w}_a' = \overline{w}_{a,\Lambda_0'}$. Then there are unitaries $W_a \in \mathcal{R}(\widetilde{\Lambda}_0)$ such that

$$W_a \in (\overline{w}_a, \overline{w}_a'). \tag{36}$$

These unitaries are unique up to phase.

This leads to new fusion intertwiners $\Omega'(a,b) \in (\overline{w}_a' \otimes \overline{w}_b', \overline{w}_{ab}')$ given by

$$\Omega'(a,b) = W_{ab}\Omega(a,b)(W_a \otimes W_b)^*. \tag{37}$$

The new F-symbols are determined by

$$\Omega'(ab,c)\Omega'(a,b) = F'(a,b,c) \times \Omega'(a,bc)\bar{w}_a'(\Omega'(b,c)). \tag{38}$$

Using Eq. 37, we compute

$$\Omega'(ab,c)\Omega'(a,b) = W_{abc}\Omega(ab,c)\Omega(a,b)\bar{w}_a\big(\bar{w}_b(W_c^*)W_b^*\big)W_a^* \tag{39}$$

and

$$\Omega'(a,bc)\bar{w}_a'(\Omega'(b,c)) = W_{abc}\Omega(a,bc)\bar{w}_a(\Omega(b,c))\bar{w}_a\big(\bar{w}_b(W_c^*)W_b^*\big)W_a^*. \tag{40}$$

It follows that $F'(a,b,c) = F(a,b,c)$ for all $a,b,c \in I$.

Recall that the braiding intertwiners $\epsilon(a,b)$ are defined in terms of endomorphisms $\overline{w}_a$ localised on Λ_0, $\overline{w}_a^L$ localised on Λ_L and $\overline{w}_a^R$ localised on Λ_R as follows. Pick intertwiners (unique up to phase) $U_a \in (\overline{w}_a, \overline{w}_a^L)$ and $V_a \in (\overline{w}_a, \overline{w}_a^R)$, then

$$\epsilon(a,b) = (V_b^* \otimes U_a^*)(U_a \otimes V_b) = V_b^*\overline{w}_a(V_b). \tag{41}$$

It is shown in Lemma 2.12 that this braiding intertwiner is independent of the choice of cones Λ_L, Λ_R. Moreover, $\epsilon(a,b)$ is independent of the choice of phase for the intertwiners U_a and V_a.

In order to make the comparison with the braiding on $\Delta^I_{\widetilde{\Lambda}_0}$, let us choose the left and right cones Λ_L and Λ_R such that they are to the left and right of both Λ_0 and $\widetilde{\Lambda}_0$.

With the new endomorphisms $\overline{w}_a'$ related to the old $\overline{w}_a$ by Eq. (36), we get new intertwiners $U_a' = U_a W_a^* \in (\overline{w}_a', \overline{w}_a^L)$ and $V_a' = V_a W_a^* \in (\overline{w}_a', \overline{w}_a^R)$ and therefore new braiding intertwiners

$$\epsilon'(a,b) = ((V_b')^* \otimes (U_a')^*)(U_a' \otimes V_b') = (V_b')^*\bar{w}_a'\big(V_b'\big). \tag{42}$$

A short computation relates this to the braiding $\epsilon(a,b)$ as

$$\epsilon'(a,b) = W_b V_b^* W_a V_b \epsilon(a,b)\overline{w}_a(W_b^*)W_a^*. \tag{43}$$

The new R-symbol is determined by

$$\Omega'(b,a)\epsilon'(a,b) = R'(a,b) \times \Omega'(a,b). \tag{44}$$

Using Eqs. (37) and (43), the left-hand side becomes

$$\Omega'(b,a)\epsilon'(a,b) = W_{ba}\Omega(b,a)\epsilon(a,b)\overline{w}_a(W_b^*)W_a^* \tag{45}$$

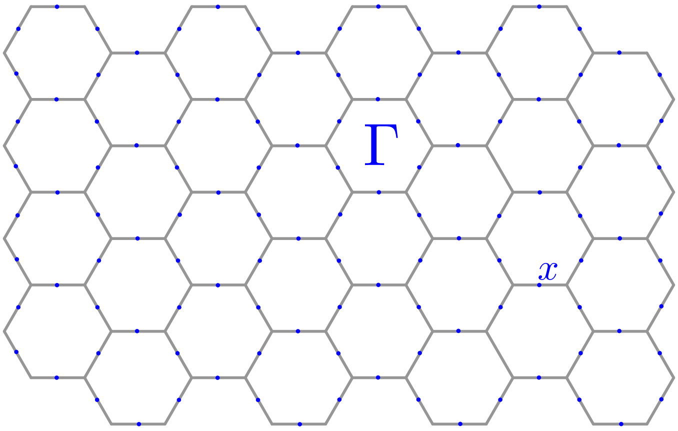

Figure 7. Degrees of freedom of the double semion state live on the edges of a hexagonal lattice

and

$$\Omega'(a,b) = W_{ab}\Omega(a,b)\overline{w}_a(W_b^*)W_a^* \tag{46}$$

so, noting that $ab = ba$, we find $R'(a,b) = R(a,b)$.

We conclude that Eqs. (33) and (35) are the only ambiguities in the F and R-symbols.

We call two sets of F and R-symbols (F,R) and (F',R') on the fusion ring $\mathbb{Z}(I)$ gauge equivalent if they are related by (33) and (35) for some phases $\chi(a,b)$.

3. The Double Semion State

We construct an infinite volume version of the ground state of the double semion model, first introduced in [16]. We identify superselection sectors corresponding to semion, anti-semion, and bound state anyons and find that the braided fusion category describing these anyons corresponds to the representation category of a twisted quantum double algebra $\mathcal{D}^{\phi}(\mathbb{Z}_2)$.

3.1. *Construction of the Double Semion State.* Let $\Gamma^V \subset \mathbb{R}^2$ be the vertices of the hexagonal lattice. We take $\Gamma = \Gamma^E$ to be the (midpoints of the) edges of the hexagonal lattice (Fig. 7) and to each edge $e \in \Gamma$ we associate a degree of freedom $\mathcal{A}_e \simeq \mathrm{End}(\mathbb{C}^2)$. We fix Pauli matrices $\sigma_e^X, \sigma_e^Y, \sigma_e^Z$ in each $\mathcal{A}_e$. We denote by $\Gamma^F = (\Gamma^V)^*$ be the set of faces of the hexagonal lattice.

We say an edge $e \in \Gamma$ belongs to a hexagon $p \in \Gamma^F$ and write $e \in p$ if e is one of the six boundary edges of p. We write ∂p for the set of six edges that belong to p. For any subset $\Pi \subset \Gamma^F$, we write $\Pi^E = \cup_{p\in\Pi}\partial p$ for all edges that belong to some hexagon in Π and by $\partial\Pi = \Pi^E \cap (\Pi^c)^E$ the collection of edges that belong to exactly one hexagon in Π. That is, $\partial\Pi$ is the boundary of Π.

We interpret $\sigma_e^Z = -1$ as the edge e being occupied by a string, while $\sigma_e^Z = 1$ means that the edge is unoccupied.

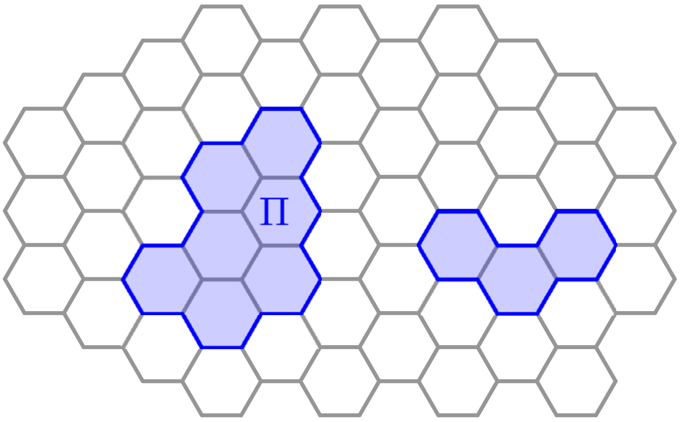

FIGURE 8. Π is the set of edges of the hexagons shaded blue. Acting with A_Π on the state ω_0 yields a string configuration with two connected components (color figure online)

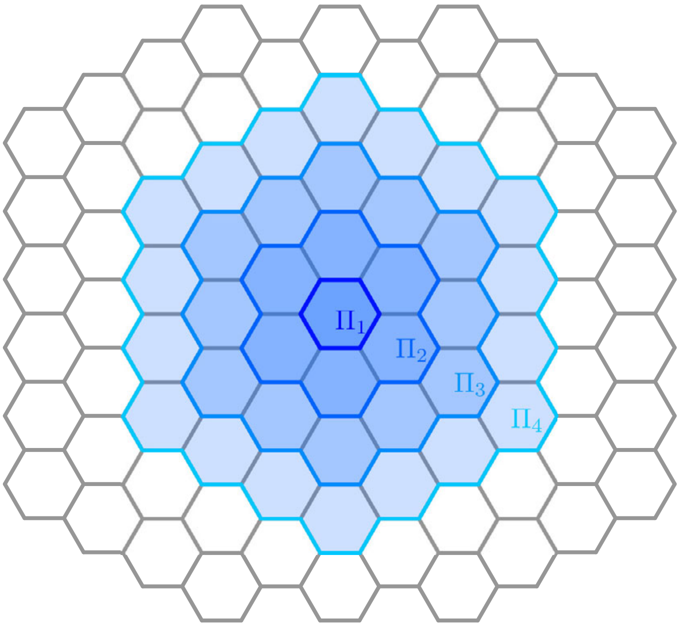

FIGURE 9. Increasing sequence of balls Π_n depicted in the primal hexagonal lattice

For any hexagon $p \in \Gamma^F$, let

$$A_p = \prod_{e \in \partial p} \sigma_e^X \tag{47}$$

and for any finite set Π of hexagons, let

$$A_\Pi = \prod_{p \in \Pi} A_p = \prod_{e \in \partial \Pi} \sigma_e^X . \tag{48}$$

Note that A_Π produces a string around the region Π when it acts on ω_0, see Fig. 8.

Let us fix a hexagon $p_0 \in \Pi^F$ as an origin and define $\Pi_n = \{p \in \Pi^F : \mathrm{dist}(p, p_0) \leq n\}$, where $\mathrm{dist}(\cdot, \cdot)$ is the graph distance for Π^F, see Fig. 9.

Let ω_0 be the pure product state without any strings, i.e. $\omega_0(\sigma_e^Z) = 1$ for all $e \in \Gamma$. Let $(\pi_0, \mathcal{H}_0, \Omega_0)$ be the GNS triple for ω_0, and let

$$\Omega_n := \sqrt{\frac{1}{2^{|\Pi_n|}}} \sum_{\Pi \subset \Pi_n} (-1)^{\sharp \Pi} A_\Pi \Omega_0 \tag{49}$$

where $\sharp\Pi$ is the number of connected components of Π. i.e. Ω_n is a superposition of closed string configurations supported in Π_n^E, with phases determined by the parity of the number of components of the string configuration.

The vectors Ω_n determine a sequence of pure states ω_n on $\mathcal{A}$. The following theorem is proved in Appendix A.

Theorem 3.1. *The sequence ω_n converges in the weak-* topology to a pure state ω.*

We call this pure state ω the *double semion state* and denote its GNS triple by $(\pi_1, \mathcal{H}, \Omega)$.

3.2. *String Operators.* An oriented edge is a pair (v_0, v_1) of neighbouring vertices of Γ^V. We say v_0 (v_1) is the initial (final) vertex of (v_0, v_1). A *path* P is a collection of oriented edges such that there is a sequence of vertices $(\cdots, v_{i-1}, v_i, v_{i+1} \cdots)$ such that each oriented edge in P is of the form $e = (v_i, v_{i+1})$ for some i. Such a sequence is called a vertex sequence for P. The set of vertices appearing in any vertex sequence for P is uniquely determined by P and denoted by P^V. We call P^V the vertex set of P; it is the set of vertices that are the initial or final vertex of some edge in P. We, moreover, require paths to be self-avoiding in the sense that any vertex sequence for P consists of distinct vertices, except for possibly the initial vertex, which may be equal to the final vertex of the sequence, if these exist. In the latter case, we say the path P is closed. Similarly, if a vertex sequence for P is bi-infinite, we also say P is closed. If P has finite vertex sequence with all its vertices being distinct, then the vertex sequence is uniquely determined by P and we denote by $\partial_0 P$ the first vertex of the vertex sequence and by $\partial_1 P$ the final vertex of the vertex sequence. We say an edge e belongs to P if P contains an oriented edge corresponding to e. With slight abuse of notation, we write $e \in P$ if e belongs to P.

Let P be a path. An edge e is said to be a *leg* of a path P at $v \in P^V$ if e is the unique edge with endpoint v such that e does not belong to P. If e is a leg of P, then e either lies to the left or to the right of P w.r.t. the orientation of P and the standard orientation of the plane. A leg that lies to the left is called an L-leg of P, and a leg of P that lies to the right is called an R-leg of P. Let P^V be the vertex set of P. If $v \in P^V$ is the endpoint of an L$-$leg of P, then we say v is an L$-$vertex of P. Similarly, if $v \in P^V$ is the endpoint of an R-leg of P, then we say v is an R-vertex of P, see Fig. 10.

Following [16], we define three types of non-trivial string operators.

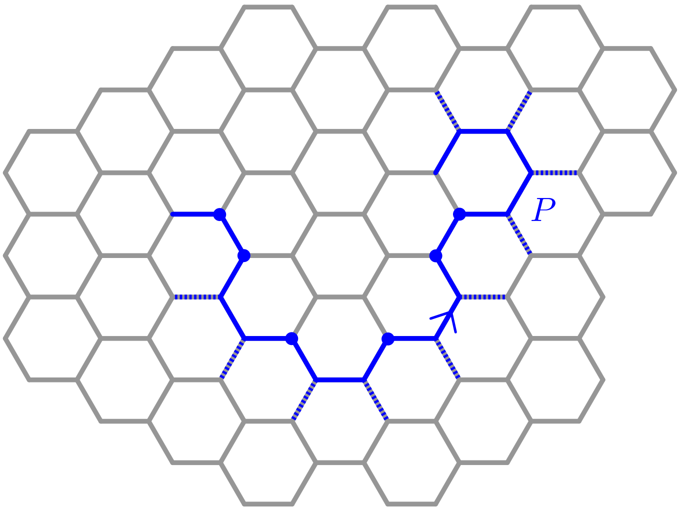

FIGURE 10. An oriented path P in solid blue with its L-vertices fattened and its R-legs marked with dotted lines (color figure online)

The *semion string* is given by

$$W_S[P] := \left(\prod_{e \in P} \sigma_e^X\right)\left(\prod_{\text{R-legs}\, e} \mathrm{i}^{\frac{1-\sigma_e^Z}{2}}\right)\left(\prod_{\text{L-vertices}\, v} (-1)^{s_v}\right) \tag{50}$$

where $s_v = \frac{1}{4}(1-\sigma_e^Z)(1+\sigma_{e'}^Z)$ and e, e' are the edges of P that go in and out of the vertex v, respectively.

The *anti-semion* string is given by

$$W_{\bar{S}}[P] := \left(\prod_{e \in P} \sigma_e^X\right)\left(\prod_{\text{R-legs}\, e} (-\mathrm{i})^{\frac{1-\sigma_e^Z}{2}}\right)\left(\prod_{\text{L-vertices}\, v} (-1)^{s_v}\right) \tag{51}$$

and the *bound-state string* is given by

$$W_B[P] := \left(\prod_{\text{R-legs}\, e} \sigma_e^Z\right). \tag{52}$$

We further define string operators for the vacuum sector $W_1[P] = \mathbb{1}$. We set $I = \{1, S, \bar{S}, B\}$, and we denote by $w_a[P]$ the automorphism defined by conjugation with the (possibly formal) unitary $W_a[P]$. We say $W_a[P]$ or $w_a[P]$ is a closed string operator whenever P is a closed path.

One easily checks that $W_{\bar{S}}[P] = W_S[P]W_B[P] = W_B[P]W_S[P]$, and $W_B[P]^2 = \mathbb{1}$. We will see later that this implies that the anyons $1, S, \bar{S}$ and B satisfy fusion rules $S \times B = \bar{S}$, $\bar{S} \times B = S$ and $B \times B = 1$. The fusion rules $S \times S = 1$ and $\bar{S} \times \bar{S} = 1$ do not follow so simply. For example, $W_S[P]^2 \neq \mathbb{1}$, see Eq. (54). This failure of the string operators to form a strict representation of the fusion rules is the origin of the non-trivial F-symbols of the double semion model, see Sect. 3.3.

These string operators have the following important property:

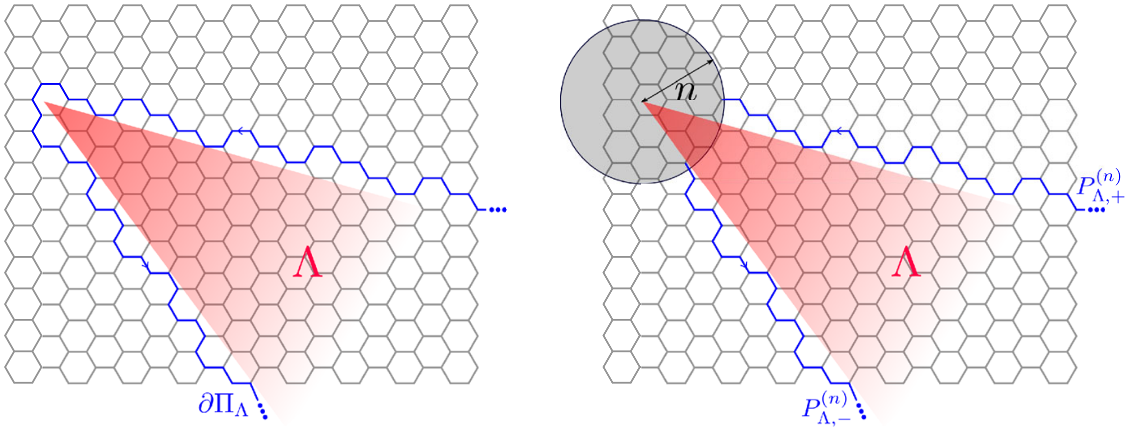

FIGURE 11. A cone Λ with the oriented path $\partial\Pi_\Lambda$ going around it in a counterclockwise direction. The paths $P^{(n)}_{\Lambda,+}$, $P^{(n)}_{\Lambda,-}$ straddle the left and right side of Λ, respectively

Proposition 3.2. *Closed string operators leave the ground state invariant. i.e. if P is a closed string, then*

$$\omega \circ w_a[P] = \omega \tag{53}$$

for all $a \in I$.

The proof is in Appendix B.

3.2.1. Definition of $w_{a,\Lambda}$. For any set $Z \subset \mathbb{R}^2$, let Π_Z be the set of hexagons (regarded as open subsets of $\mathbb{R}^2$) that have some overlap with the set Z. For a cone Λ, we interpret the boundary $\partial\Pi_\Lambda$ as an infinite closed path oriented counterclockwise around Λ. Assuming the opening angle of Λ is less than π, the edges of $\partial\Pi_\Lambda$ whose centre lies a distance further than $n > 2$ from the apex of Λ form two half-infinite oriented paths $P^{(n)}_{\Lambda,+}$ and $P^{(n)}_{\Lambda,-}$, as shown in Fig. 11.

Let Λ be a cone, and let $\Lambda^{(L)}$ and $\Lambda^{(R)}$ be its left- and right half-cones, see Fig. 12. Take $n > 2$ sufficiently large such that $w_{a,\Lambda} := w_a[P^{(n)}_{\Lambda^{(R)},+}]$ and $v_{a,\Lambda} := w_a[P^{(n)}_{\Lambda^{(L)},-}]$ are supported in Λ for all $a \in I$. Denote $P_\Lambda := P^{(n)}_{\Lambda^{(R)},+}$ and $\overline{P}_\Lambda := P^{(n)}_{\Lambda^{(L)},-}$.

3.2.2. Fusion Rules. In this section, we show the fusion rules for the string operators $w_{a,\Lambda}$. In particular, we will show that $w_{a,\Lambda} \circ w_{b,\Lambda}$ is unitarily equivalent to $w_{ab,\Lambda}$.

Let us begin with semion-semion fusion. For any oriented path P, the automorphism $w_S[P] \circ w_S[P]$ is given by conjugation with the formal unitary

$$W_S[P]^2 = \Omega_{S,S}[P] := \left(\prod_{\text{R-legs } e} \sigma^Z_e \right) \left(\prod_{\substack{\text{L-vertices} \\ v=(e,e')}} \sigma^Z_e \sigma^Z_{e'} \right). \tag{54}$$

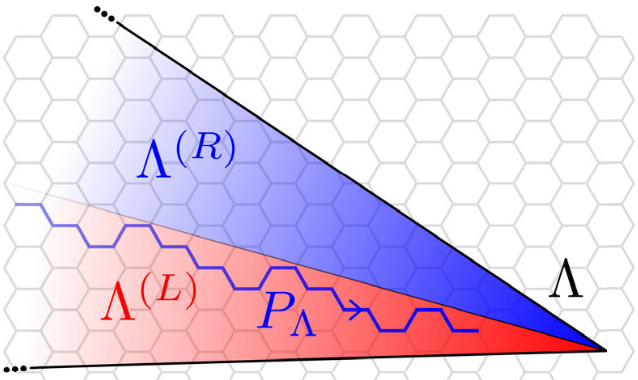

FIGURE 12. A cone Λ divided into its left and right cones $\Lambda^{(L)}$ and $\Lambda^{(R)}$. The path P_Λ is the largest part of $\partial\Pi_{\Lambda^{(R)}}$ such that $w_a[P_\Lambda]$ is supported in Λ

Let e_i and e_f be the initial and final edges of the path P (if they exist) and let $u[P]$ be the automorphism given by conjugation with

$$U[P] := \Omega_{S,S}[P] \times \sigma^Z_{e_i}\sigma^Z_{e_f}. \tag{55}$$

Lemma 3.3. *We have $\omega \circ u[P] = \omega$ for any path P.*

Proof. We first take P finite and show that $u[P]$ leaves any ω_n invariant.

Recall that ω_n is a vector state in the GNS representation of ω_0, given by

$$\Omega_n = \sqrt{\frac{1}{2^{|\Pi_n|}}} \sum_{\Pi\subset\Pi_n} (-1)^{\sharp\Pi} A_\Pi\Omega_0. \tag{56}$$

We have

$$U[P]A_\Pi\Omega_0 = A_\Pi\Omega_0. \tag{57}$$

Indeed, A_Π is a product of σ^X_e for all $e \in \partial\Pi$, a finite closed path. Now, any such closed path supports an even number of factors σ^Z_e of the unitary $U[P]$. Indeed, if $\partial\Pi$ travels along P, then the two edges of P along an R-leg carry no σ^Z, while the two edges along an L-vertex both have a σ^Z. The closed path $\partial\Pi$ must enter/leave the path P an even number of times. If it enters through an R-leg, it picks up a σ^Z from the R-leg. If it enters through an L-vertex, then it picks up exactly one of the σ^Z's of the two edges of P next to the L-vertex. Finally, if $\partial\Pi$ enters P through an endpoint of P, then the factors σ^Z_i, σ^Z_f at the initial/final edges ensure that a factor σ^Z is picked up. In all, we see that $U[P]A_\Pi = A_\Pi U[P]$, because the computation involves an even number of commutations of a σ^X with a σ^Z. Obviously $U[P]\Omega_0 = \Omega_0$ so $U[P]A_\Pi\Omega_0 = A_\Pi\Omega_0$ and $U[P]\Omega_n = \Omega_n$. It follows that $\omega_n \circ u[P] = \omega_n$ for any n and hence $\omega \circ u[P] = \omega$ for any finite P.

If P is infinite, then for any strictly local observable O we can find a finite P' such that $u[P](O) = u[P'](O)$ so $(\omega \circ u[P])(O) = (\omega \circ u[P'])(O) = \omega(O)$. Since the strictly local observables are dense in $\mathcal{A}$, this proves the claim. □

Lemma 3.4. *If $u[P]$ is supported in a cone Λ, then $\pi_1 \circ u[P] \simeq \pi_1$, and the unitary implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.*

Proof. The unitary equivalence $\pi_1 \circ u[P] \simeq \pi_1$ follows immediately from Lemma 3.3. Let U be the unitary implementing this equivalence, i.e.

$$UOU^* = u[P](O) \tag{58}$$

for all $O \in \mathcal{A}$, and $U\Omega = \Omega$. (We identify $\mathcal{A}$ with its image under the faithful representation π_1.)

If P is finite, then actually $U \in \mathcal{A}_\Lambda \subset \mathcal{R}(\Lambda)$. If P is infinite, let P_n be the path consisting of edges of P whose midpoints lie in Π_n. Then $U[P_n] \in \mathcal{A}_\Lambda$ has $U[P_n]\Omega = \Omega$ for all n, and for any strictly local observables O, O' we have

$$\begin{aligned}\langle O\Omega, UO'\Omega\rangle &= \langle O\Omega, u[P](O')U\Omega\rangle = \lim_{n\uparrow\infty} \langle O\Omega, u[P_n](O')\Omega\rangle \\ &= \lim_{n\uparrow\infty} \langle O\Omega, U[P_n]O'\Omega\rangle.\end{aligned} \tag{59}$$

Since the vectors $O\Omega$, $O'\Omega$ for O, O' strictly local observables are dense in $\mathcal{H}$, this shows that the sequence $U[P_n]$ converges weakly to U. Since $U[P_n] \in \mathcal{A}_\Lambda$ for all n, it follows that $U \in \mathcal{R}(\Lambda)$. □

Lemma 3.5. *For any cone Λ, we have that $\pi_1 \circ w_{S,\Lambda} \circ w_{S,\Lambda} \simeq \pi_1$, and the unitary U_Λ implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.*

Proof. By definition, $w_{S,\Lambda} = w_S[P_\Lambda]$ so $w_{S,\Lambda} \circ w_{S,\Lambda} = \mathrm{Ad}(\sigma^Z_f) \circ u[P_\Lambda]$ where e_f is the final edge of the half-infinite path P_Λ (cf. Eqs. (54) and (55)). From Lemma 3.4, we find that there exists a unitary $U_\Lambda \in \mathcal{R}(\Lambda)$ such that $u[P_\Lambda] = \mathrm{Ad}(U_\Lambda)$, hence

$$\pi_1 \circ w_{S,\Lambda} \circ w_{S,\Lambda} = \pi_1 \circ \mathrm{Ad}(\sigma^Z_{e_f} U_\Lambda), \tag{60}$$

proving the claim. □

We can now easily show

Proposition 3.6. *For each cone Λ, there are unitaries $\Omega(a,b) \in \mathcal{R}(\Lambda)$ such that*

$$\mathrm{Ad}(\Omega(a,b)) \circ w_{a,\Lambda} \circ w_{b,\Lambda} = w_{a\times b,\Lambda} \tag{61}$$

for all $a, b \in I = \{1, S, \bar{S}, B\}$ and where $\times$ is an abelian product on I given by

$\times$	1	S	$\bar{S}$	B
1	1	S	$\bar{S}$	B
S	S	1	B	$\bar{S}$
$\bar{S}$	$\bar{S}$	B	1	S
B	B	$\bar{S}$	S	1

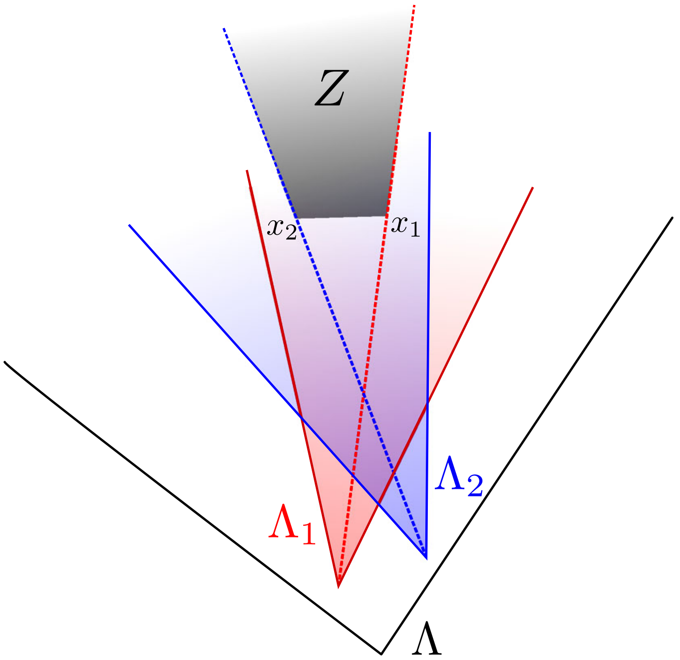

FIGURE 13. Axis of Λ_1 points to the right of the axis of Λ_2 relative to the cone Λ. The region Z has $w_a[\partial\Pi_Z]$ supported in Λ. Moreover, the path $\partial\Pi_Z$ differs from the union of the paths P_{Λ_1}, $\overline{P}_{\Lambda_2}$ by a finite number of edges

Proof. Lemma 3.5 shows that the claim holds for $S \times S = 1$. The rest of the claim follows from this case and

$$w_{\bar{S},\Lambda} = w_{S,\Lambda} \circ w_{B,\Lambda} = w_{B,\Lambda} \circ w_{S,\Lambda}, \quad w_{B,\Lambda} \circ w_{B,\Lambda} = \mathrm{id}. \tag{62}$$

□

3.2.3. Transportability.

Lemma 3.7. *If Λ_1 and Λ_2 are cones with axes $\hat{w}_1$ and $\hat{w}_2$, both contained in a cone Λ and such that $\hat{w}_1$ points to the right of $\hat{w}_2$ relative to Λ (see Fig. 13). Then $\pi_1 \circ w_{a,\Lambda_1} \simeq \pi_1 \circ v^{-1}_{a,\Lambda_2}$ and the unitary implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.*

Proof. Fix points x_1, x_2 on the central axes of the cones Λ_1, Λ_2 such that the region Z bounded by the half-infinite parts of these axes starting at x_1, x_2, and the line between x_1 and x_2 is convex and has $w_{a,\partial\Pi_Z}$ supported in Λ, see Fig. 13.

By construction, $w_{a,\partial\Pi_Z}$ differs from $w_{a,\Lambda_1} \circ v_{a,\Lambda_2}$ by the action of a local unitary W supported on Λ. Since $\partial\Pi_Z$ is a closed path, it follows from Proposition 3.2 that there exists a unitary $V \in \mathcal{B}(\mathcal{H})$ such that $\pi_1 \circ w_{a,\partial\Pi_Z} = \mathrm{Ad}(V) \circ \pi_1$ and $V\Omega = \Omega$; hence,

$$\pi_1 \circ w_{a,\Lambda_1} = \mathrm{Ad}(VW) \circ \pi_1 \circ v^{-1}_{a,\Lambda_2}. \tag{63}$$

This shows the required unitary equivalence. It remains to show that $V \in \mathcal{R}(\Lambda)$.

Let $Z_n = Z \cap B_n$ where B_n is the disk of radius n centred at the origin of $\mathbb{R}^2$. Then $\partial\Pi_{Z_n}$ are closed paths and the automorphisms $w_{\partial\Pi_{Z_n}} = \mathrm{Ad}(W_a[\partial\Pi_{Z_n}])$ leave the ground state invariant, and are supported in Λ. In particular, there exist phases ϕ_n such that $V_n := \phi_n W_a[\partial\Pi_{Z_n}]$ satisfies $V_n\Omega = \Omega$. For any strictly local observables $O, O' \in \pi_1(\mathcal{A})$ we have

$$\begin{aligned}\langle O\Omega, VO'\Omega\rangle &= \langle O\Omega, w_{a,\partial\Pi_Z}(O')V\Omega\rangle = \lim_{n\uparrow\infty}\langle O\Omega, w_{a,\partial\Pi_{Z_n}}(O')V_n\Omega\rangle \\ &= \lim_{n\uparrow\infty}\langle O\Omega, V_nO'\Omega\rangle. \end{aligned} \tag{64}$$

Since the vectors $O\Omega, O'\Omega$ are dense in $\mathcal{H}$, this shows that V_n converges weakly to V. Since each V_n is in $\mathcal{A}_\Lambda$, we conclude that $V \in \mathcal{R}(\Lambda)$. □

Proposition 3.8. *If Λ_1 and Λ_2 are cones both contained in a cone Λ, then $\pi_1 \circ w_{a,\Lambda_1} \simeq \pi_1 \circ w_{a,\Lambda_2}$ and the unitary implementing this equivalence belongs to the von Neumann algebra $\mathcal{R}(\Lambda)$.*

Proof. Let $\hat{w}_1, \hat{w}_2$ be the axes of the cones Λ_1, Λ_2 and take a cone $\Lambda_3 \subset \Lambda$ such that its axis $\hat{w}_3$ points to the right of both $\hat{w}_1$ and $\hat{w}_2$ relative to Λ. Then, Lemma 3.7 implies that there are unitaries $V_1, V_2 \in \mathcal{R}(\Lambda)$ such that

$$\pi_1 \circ w_{a,\Lambda_1} = \mathrm{Ad}(V_1) \circ \pi_1 \circ v_{a,\Lambda_3}^{-1}, \quad \pi_1 \circ w_{a,\Lambda_2} = \mathrm{Ad}(V_2) \circ \pi_1 \circ v_{a,\Lambda_3}^{-1}, \tag{65}$$

hence

$$\pi_1 \circ w_{a,\Lambda_1} = \mathrm{Ad}(V_2^*V_1) \circ \pi_1 \circ w_{a,\Lambda_2}. \tag{66}$$

Since $V_2^*V_1 \in \mathcal{R}(\Lambda)$, this proves the claim. □

3.2.4. Distinct Sectors.

Fix a cone Λ_0 with axis $(0, 1)$ and let $\pi_a := \pi_1 \circ w_{a,\Lambda_0}$ for $a \in I$.

Proposition 3.9. *For all $a, b \in I$, we have $\pi_a \simeq \pi_b$ if and only if $a = b$.*

Proof. For any n large enough such that the endpoint of P_{Λ_0} is contained in Π_{n-2}, consider the S-matrix

$$S_{ab} := \frac{1}{2}(\omega \circ w_{a,\Lambda_0})(W_b[\partial\Pi_n]). \tag{67}$$

An easy calculation shows that these quantities are independent of n and given by

$$S = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}. \tag{68}$$

It follows that for any $a \neq b$ there is a c such that $(\omega \circ w_{a,\Lambda_0})(W_c[\partial\Pi_n]) = -(\omega \circ w_{b,\Lambda_0})(W_c[\partial\Pi_n])$ for all n sufficiently large. Corollary 2.6.11 of [2] then implies that π_a and π_b are disjoint. □

Table 1. Fusion intertwiners $\Omega(a,b)$ for the double semion state

$\Omega(a,b)$	1	S	$\bar{S}$	B
1	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$
S	$\mathbb{1}$	$U\sigma^Z_{e_f}$	$U\sigma^Z_{e_f}$	$\mathbb{1}$
$\bar{S}$	$\mathbb{1}$	$U\sigma^Z_{e_f}$	$U\sigma^Z_{e_f}$	$\mathbb{1}$
B	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$

3.2.5. Verification of Assumptions. The four faithful irreducible representations $\pi_1, \pi_S, \pi_{\bar{S}}, \pi_B$ defined by $\pi_a = \pi_1 \circ w_{a,\Lambda_0}$ for $a \in \{1, S, \bar{S}, B\} = I$ are pairwise disjoint by Proposition 3.9.

For any cone Λ and any $a \in I$, we defined an automorphism $w_{a,\Lambda}$ supported in Λ. This collection of automorphisms satisfies Assumption 1 by Proposition 3.8. Assumptions 2 and 3 are verified by Proposition 3.6. Finally, Assumption 4 holds by Propositions 3.6 and 3.8.

3.3. *Computation of F-Symbols.* Having fixed the cone Λ_0 with axis $(0,1)$, we use for all $a \in I$ the shorthand notations $w_a := w_{a,\Lambda_0}$ and $\overline{w}_a := \overline{w}_{a,\Lambda_0}$, where the latter are the extensions of w_a to the allowed algebra $\mathcal{B}$ constructed in Lemma 2.5.

Let e_f be the final edge of the path P_{Λ_0} and let $U \in \mathcal{R}(\Lambda_0)$ be the unitary such that $\pi_1 \circ u[P_{\Lambda_0}] = \mathrm{Ad}(U) \circ \pi_1$ and $U\Omega = \Omega$ provided by Lemma 3.4. The proof of Lemma 3.5 shows that

$$\mathrm{Ad}(\Omega(S,S)) \circ (\overline{w}_S \circ \overline{w}_S) = \overline{w}_1 \tag{69}$$

with $\Omega(S,S) = \sigma^Z_{e_f} U$.

Using

$$w_{\bar{S}} = w_B \circ w_S = w_S \circ w_B, \quad w_B \circ w_B = \mathrm{id}, \tag{70}$$

we find that

$$\mathrm{Ad}(\Omega(a,b)) \circ (w_a \circ w_b) = w_{a\times b} \tag{71}$$

for all $a, b \in I$ with fusion intertwiners $\Omega(a,b)$ given in Table 1. It follows that $\Omega(a,b) \in (\overline{w}_a \otimes \overline{w}_b, \overline{w}_{a\times b})$ for all $a, b \in I$.

In order to compute the F-symbols, we first show

Lemma 3.10.

$$\overline{w}_S(U\sigma^Z_{e_f}) = -U\sigma^Z_{e_f}, \quad \overline{w}_B(U\sigma^Z_{e_f}) = U\sigma^Z_{e_f}, \quad \overline{w}_{\bar{S}}(U\sigma^Z_{e_f}) = -U\sigma^Z_{e_f}. \tag{72}$$

Proof. Since e_f is the final edge of the path P_{Λ_0}, we have $w_S(\sigma^Z_{e_f}) = w_{\bar{S}}(\sigma^Z_{e_f}) = -\sigma^Z_{e_f}$ and $w_B(\sigma^Z_{e_f}) = \sigma^Z_{e_f}$. It remains to show that $\overline{w}_S(U) = \overline{w}_{\bar{S}}(U) = \overline{w}_B(U) = U$.

Since U is the weak limit of the sequence $U_n = U[P_n]$ where P_n is the path consisting of edges of P_{Λ_0} whose midpoints lie in Π_n (cf. the proof of Lemma 3.4), it is sufficient to show $w_S(U_n) = w_{\bar{S}}(U_n) = w_B(U_n) = U_n$. This follows similar to the argument in the proof of Lemma 3.4. Since U_n is a product of σ^Z's, we have that $w_B(U_n) = U_n$, and

$$w_S(U_n) = w_{\bar{S}}(U_n) = \left(\prod_{e \in P_n} \sigma_e^X \right) U_n \left(\prod_{e \in P_n} \sigma_e^X \right). \tag{73}$$

By design, the unitary U_n has an even number of σ^Z's on the path P_n. Indeed, there are two factors of σ^Z for every L-vertex, zero for every R-leg, and another two for the endpoints. We conclude that $w_S(U_n) = W_{\bar{S}}(U_n) = U_n$ for all n.

□

We can now start computing the F-symbols. If in Eq. (24) we take $a = 1$, then

$$\Omega(b, c)\Omega(1, b) = F(1, b, c)\Omega(1, bc)\Omega(B, C). \tag{74}$$

Since $\Omega(1, b) = \Omega(1, bc) = \mathbb{1}$, we find that $F(1, b, c) = 1$ for all b, c.

Similarly we find $F(a, 1, c) = F(a, b, 1) = 1$ for all a, b, c.

Let us now consider F-symbols that involve the bound state B, for example,

$$\Omega(Bb, c)\Omega(B, b) = F(B, b, c)\Omega(B, bc)\overline{w}_B(\Omega(b, c)). \tag{75}$$

Since $\Omega(B, b) = \Omega(B, bc) = \mathbb{1}$, this reduces to

$$\Omega(Bb, c) = F(B, b, c)\overline{w}_B(\Omega(b, c)). \tag{76}$$

If $b = B$ or $c = B$, then then $\Omega(Bb, b) = \Omega(b, c) = \mathbb{1}$ so $F(B, b, c) = 1$. If $b, c \in \{S, \bar{S}\}$, then $\Omega(Bb, c) = \Omega(b, c) = U\sigma^Z_{e_f}$, so using Lemma 3.10 we find again $F(B, b, c) = 1$ for all b, c.

Similar considerations show that $F(B, b, c) = F(a, B, c) = F(a, b, B) = 1$ for all a, b, c.

Finally, we consider the case where $a, b, c \in \{S, \bar{S}\}$. Then since $ab, bc \in \{1, B\}$ we have $\Omega(ab, c) = \Omega(a, bc) = \mathbb{1}$ so

$$U\sigma^Z_{e_f} = F(a, b, c)\overline{w}_a(U\sigma^Z_{e_f}). \tag{77}$$

Using Lemma 3.10, we conclude that $F(a, b, c) = -1$ for $a, b, c \in \{S, \bar{S}\}$.

3.4. *Computation of R-Symbols.*

Choose cones Λ_L with axis $(-1, 0)$ and Λ_R with axis $(1, 0)$, both disjoint from Λ_0 as in Fig. 14. Let $\widetilde{\Lambda}_L \supseteq \Lambda_0 \cup \Lambda_L$ and $\widetilde{\Lambda}_R \supseteq \Lambda_0 \cup \Lambda_R$ be allowed cones such that $\widetilde{\Lambda}_L \cap \Lambda_R = \widetilde{\Lambda}_R \cap \Lambda_L = \emptyset$.

To compute the braiding intertwiners $\epsilon(a, b) = \epsilon(\overline{w}_a, \overline{w}_b)$, set $\overline{v}^L_a = \overline{v}_a [\Lambda_L]^{-1}$, $\overline{v}^R_a = \overline{v}_a[\Lambda_R]^{-1}$ as well as $\overline{w}^L_a = \overline{w}_{a,\Lambda_L}$ and $\overline{w}^R_a = \overline{w}_{a,\Lambda_R}$ for all $a = 1, S, \bar{S}, B$. (Recall that the automorphisms $v_{a,\Lambda}$ are defined in Sect. 3.2.1.)

It follows from Lemma 3.7 that there are unitaries $U_a \in (\overline{w}_a, \overline{v}^L_a)$ and $V_b \in (\overline{w}_b, \overline{v}^R_b)$ such that $U_a \in \mathcal{R}(\widetilde{\Lambda}_L)$ and $V_b \in \mathcal{R}(\widetilde{\Lambda}_R)$. By the same lemma, there are unitaries $U'_a \in (\overline{v}^L_a, \overline{w}^L_a)$ and $V'_b \in (\overline{v}^R_b, \overline{w}^R_b)$ such that $U'_a \in \mathcal{R}(\Lambda_L)$ and $V'_b \in \mathcal{R}(\Lambda_R)$. We therefore have unitary morphisms $U'_a U_a \in (\overline{w}_a, \overline{w}^L_a)$ and

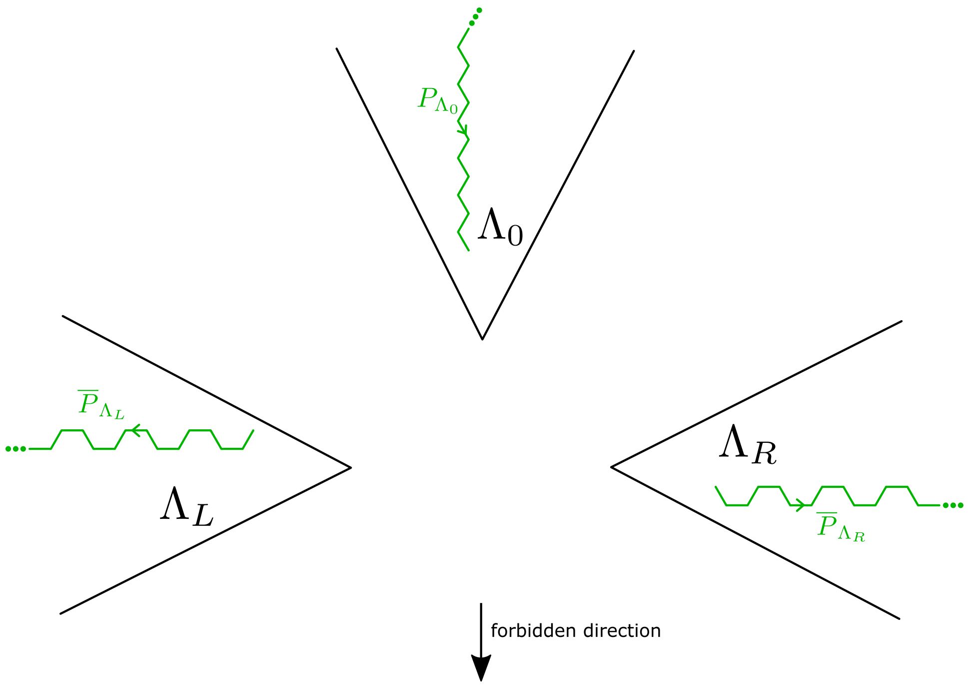

FIGURE 14. Cones Λ_0, Λ_L and Λ_R used to define the braiding intertwiners

$V_b' V_b \in (\overline{w}_b, \overline{w}_b^R)$ with $U_a' U_a \in \mathcal{R}(\widetilde{\Lambda}_L)$ and $V_b' V_b \in \mathcal{R}(\widetilde{\Lambda}_R)$. By definition 2.11 and the fact that $\overline{w}_a(V_b') = V_b'$, we have

$$\epsilon(a, b) = V_b^* \, \overline{w}_a\big(V_b\big). \tag{78}$$

In order to compute $\overline{w}_a(V_b)$, let us realise V_b as the weak limit of a sequence of strictly local unitaries.

Let K be the cone whose legs coincide with the central axes of Λ_0 and Λ_R, see Fig. 15. Then the path $\partial\Pi_K$ contains P_{Λ_0} and $\overline{P}_{\Lambda_R}$ and the path $Q = \partial\Pi_K \backslash (P_{\Lambda_0} \cup P_{\Lambda_R})$ is finite. For each n, let $K_n = K \cap B_n$ where $B_n \subset \mathbb{R}^2$ is the disk of radius n centred at the origin of $\mathbb{R}^2$. Consider the sequence of paths $P_n = \partial\Pi_{K_n} \setminus Q$ and set $V_b^{(n)} := W_b[P_n]$.

Lemma 3.11. *There are phases ϕ_n such that the sequence $\phi_n V_b^{(n)}$ converges weakly to V_b.*

Proof. Consider first the sequence of finite closed paths $\partial\Pi_{K_n} = P_n \cup Q$ and corresponding string operators $W_b[\partial\Pi_{K_n}]$. By Proposition 3.2, the unitaries $W_b[\partial\Pi_{K_n}]$ leave the ground state invariant up to a phase, so there are phases ϕ_n such that $\widetilde{W}^{(n)} := \phi_n W_b[\partial\Pi_{K_n}]$ satisfy $\widetilde{W}^{(n)}\Omega = \Omega$.

Since $\partial\Pi_K$ is a closed path, the automorphism $w_b[\partial\Pi_K]$ leaves the grounds state invariant by Proposition 3.2. It follows that there is a unique unitary $\widetilde{W} \in \mathcal{B}(\mathcal{H})$ such that $\widetilde{W}\Omega = \Omega$ and $w_b[\partial\Pi_K] = \mathrm{Ad}(\widetilde{W})$ (as automorphisms on $\pi_1(\mathcal{A})$).

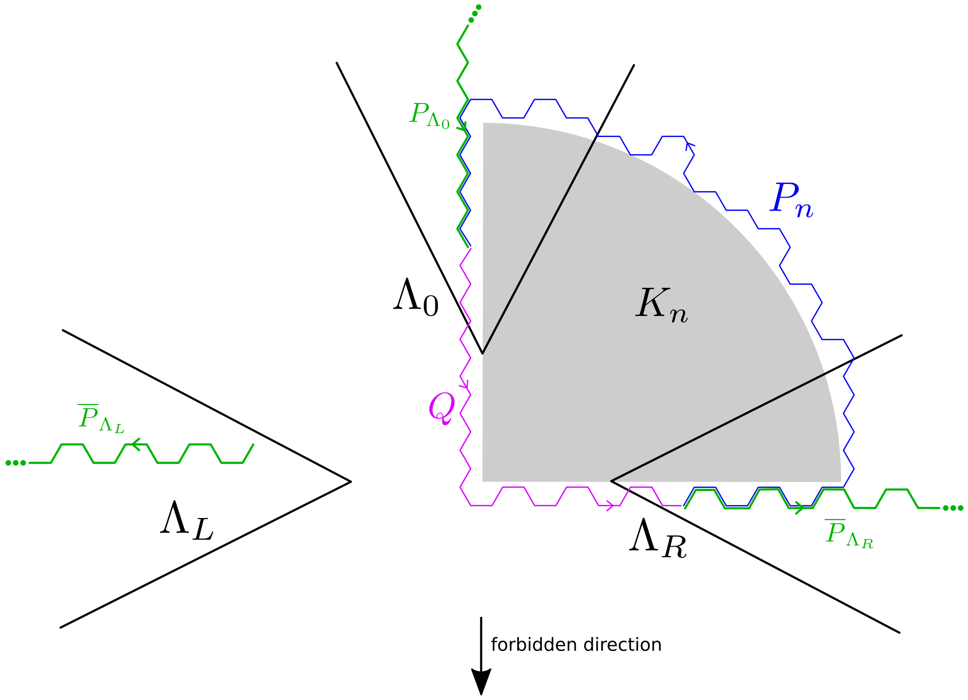

FIGURE 15. Sets K_n and the paths P_n and Q used in the construction of the sequence of unitaries $V_b^{(n)}$ that converge weakly to the intertwiner V_b

Now, for any strictly local observable O we have $w_b[\partial\Pi_K](O) = w_b[\partial\Pi_{K_n}](O)$ for all n large enough. Thus, for any strictly local operators O, O' we have

$$\begin{aligned}\langle O\Omega, \widetilde{W}\, O'\Omega\rangle &= \langle O\Omega, w_b[\partial\Pi_K](O')\, \widetilde{W}\Omega\rangle\\ &= \lim_{n\uparrow\infty} \langle O\Omega, w_b[\partial\Pi_{K_n}](O')\,\Omega\rangle\\ &= \lim_{n\uparrow\infty} \langle O\Omega, \widetilde{W}^{(n)} O'\Omega\rangle,\end{aligned}$$

showing that the sequence $\widetilde{W}^{(n)}$ converges weakly to $\widetilde{W}$.

Now note that the paths $\partial\Pi_{K_n}$ and $P_n = \partial\Pi_{K_n}\backslash Q$ differ by the same path $Q = \partial\Pi_{K_n}\backslash P_n$ for all n. It follows that the corresponding string operators $W_b[\partial\Pi_{K_n}]$ and $V_b^{(n)} = W_b[P_n]$ satisfy $V_b^{(n)}(W_b[\widetilde{P}_n])^* = W$ for a unitary W that is independent of n and is supported on the path Q and edges adjacent to Q. (In fact, W is equal to $W_b[Q]^*$ up local operators supported near the endpoints of the path Q.) Therefore, $V_b^{(n)} = W W_b[\partial\Pi_{K_n}]$ and $\phi_n V_b^{(n)} = W\widetilde{W}^{(n)}$.

Since $\widetilde{W}^{(n)}$ converges weakly to $\widetilde{W}$, it follows that the sequence $\phi_n V_b^{(n)} = W\widetilde{W}^{(n)}$ converges weakly to $W\widetilde{W}$. By construction, $\mathrm{Ad}(W\widetilde{W}) = w_b \circ (w_b^R)^{-1} = \mathrm{Ad}(V_b)$ so $W\widetilde{W} = \mu V_b$ for some phase μ. We then find that the sequence $\mu^*\phi_n V_n^{(n)}$ converges weakly to V_b. This proves Lemma. □

We can now compute the braiding intertwiners.

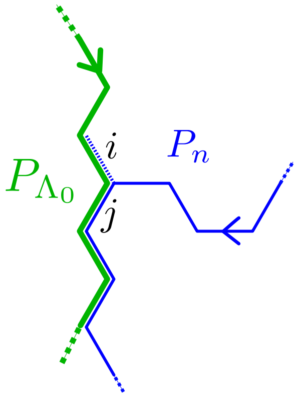

Figure 16. Edges e and e' playing a role in the computation of $\epsilon(S,S)$

Obviously $\overline{w}_1 = \mathrm{id}$ and $V_1 = I$ so $\epsilon(1,a) = \epsilon(a,1) = \mathbb{1}$ for all $a \in I$. It is also easy to see that

$$w_B(V_a^{(n)}) = V_a^{(n)} \tag{79}$$

for any $a \in \{1, S, \bar{S}, B\}$, so $\epsilon(B,a) = \mathbb{1}$ for all a, while

$$w_S(V_B^{(n)}) = w_{\bar{S}}(V_B^{(n)}) = -V_B^{(n)}, \tag{80}$$

because the path P_{Λ_0} contains a single R-leg of the path P_n. So $\epsilon(S,B) = \epsilon(\bar{S},B) = -\mathbb{1}$.

Let us now compute $\epsilon(S,S)$. Note that the path P_n enters the path P_{Λ_0} at an L-vertex of P_{Λ_0}. Let (e,e') be the edges of P_{Λ_0} before and after this L-vertex, see Fig. 16. We find

$$\begin{aligned}&(V_S^{(n)})^* \, w_S(V_S^{(n)}) \\ &\quad = \left(\sigma^X_{e'}\left(\mathrm{i}^{\frac{1-\sigma^Z_e}{2}}\right)\right)\left(\sigma^X_e\sigma^X_{e'}(-1)^{s_I}\right)\left(\left(\mathrm{i}^{\frac{1-\sigma^Z_e}{2}}\right)^*\sigma^X_{e'}\right)\left((-1)^{s_I}\sigma^X_{e'}\sigma^X_e\right) = \mathrm{i}\,\mathbb{1},\end{aligned}$$

which implies $\epsilon(S,S) = \mathrm{i}\,\mathbb{1}$.

We now use the braid equations (Lemma 2.13)

$$\begin{aligned}\epsilon(\rho, \sigma\otimes\tau) &= (\mathbb{1}_\sigma \otimes \epsilon(\rho,\tau))(\epsilon(\rho,\sigma)\otimes\mathbb{1}_\tau)\\ \epsilon(\rho\otimes\sigma, \tau) &= (\epsilon(\rho,\tau)\otimes\mathbb{1}_\sigma)(\mathbb{1}_\rho)\otimes\epsilon(\sigma,\tau))\end{aligned}$$

(where I_ρ denotes the identity intertwiner from ρ to itself) to compute

$$\begin{aligned}\epsilon(\bar{S},S) &= \epsilon(S\times B, S) = \epsilon(S,S)\overline{w}_S(\epsilon(B,S)) = \mathrm{i}\,\mathbb{1}\\ \epsilon(S,\bar{S}) &= \epsilon(S, S\times B) = \overline{w}_S(\epsilon(S,B))\epsilon(S,S) = -\mathrm{i}\,\mathbb{1}\\ \epsilon(\bar{S},\bar{S}) &= \epsilon(S\times B,\bar{S}) = \epsilon(S,\bar{S})\overline{w}_S(\epsilon(B,S)) = -\mathrm{i}\,\mathbb{1}\,.\end{aligned}$$

Thus, we have computed all braiding intertwiners,see Table 2 for a summary.

The R-symbols are defined (26) by

$$\Omega(b,a)\epsilon(a,b) = R(a,b)\times\Omega(a,b). \tag{81}$$

Since $\Omega(a,b) = \Omega(b,a)$ for all a,b, we find that the R-symbols are shown in Table 3.

Table 2. Braiding intertwiners $\epsilon(a,b)$ for the double semion state

$\epsilon(a,b)$	1	S	$\bar{S}$	B
1	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$
S	$\mathbb{1}$	$\mathrm{i}\,\mathbb{1}$	$-\mathrm{i}\,\mathbb{1}$	$-\mathbb{1}$
$\bar{S}$	$\mathbb{1}$	$\mathrm{i}\,\mathbb{1}$	$-\mathrm{i}\,\mathbb{1}$	$-\mathbb{1}$
B	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$

Table 3. R-symbols $R(a,b)$ for the double semion state

$R(a,b)$	1	S	$\bar{S}$	B
1	1	1	1	1
S	1	i	$-\mathrm{i}$	-1
$\bar{S}$	1	i	$-\mathrm{i}$	-1
B	1	1	1	1

One can verify that the F and R-symbols indeed satisfy the pentagon and hexagon equations.

3.5. *Anyons are Described by* $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$. In Sect. 3.2.5, we verified Assumptions 1–4 for the superselection sectors π_a with $a \in I = \{1, S, \bar{S}, B\}$ of the double semion state. Let us denote by $\Delta^I_{\Lambda_0}$ the corresponding braided tensor category constructed in Sect. 2.2.

In this section, we show that $\Delta^I_{\Lambda_0}$ is braided monoidal equivalent to the representation category $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$, where ϕ is a non-trivial 3-cocycle on $\mathbb{Z}_2$. We will do this by showing that both categories have equivalent fusion rings, F-symbols, and R-symbols, and appealing to Proposition 7.5.2 of [12].

This identification is relevant, because it partially verifies a conjecture about string-net models [16]. A given string-net model is defined by an input fusion category $\mathcal{F}$, and the topological order of the model is conjectured [17] to correspond to the Drinfeld centre $\mathcal{Z}(\mathcal{F})$ of $\mathcal{F}$. In the case where $\mathcal{F} = \mathrm{Vec}^\phi_G$ for a finite group G and a 3-cocycle ϕ of G, we have $\mathcal{Z}(\mathcal{F}) = \mathrm{Rep}_f\mathcal{D}^\phi(G)$ (cf. [18]), where $\mathcal{D}^\phi(G)$ is the twisted quantum double algebra first described in [4].

3.5.1. The Braided Fusion Ring of $\Delta^I_{\Lambda_0}$. We have extracted from the braided tensor category Δ^I_Λ its fusion ring, generated by the elements of $I = \{1, S, \bar{S}, B\}$ with abelian fusion rules given by the group structure on I described in Proposition 3.6. In other words, the fusion ring of $\Delta^I_{\Lambda_0}$ is isomorphic to $\mathbb{Z}(I)$. In Sect. 3.3, we obtained the F-symbols, and in Sect. 3.4 we obtained the R-symbols, Cf. Table 3 derived from the braided tensor category $\Delta^I_{\Lambda_0}$. It follows from Proposition 7.5.2 of [12] that these data completely determine the category $\Delta^I_{\Lambda_0}$ up to braided monoidal equivalence.

3.5.2. Description of $\mathbf{Rep}_f\,\mathcal{D}^\phi(\mathbb{Z}_2)$. We describe the quasi Hopf algebra $\mathcal{D}^\phi(\mathbb{Z}_2)$ first introduced in [4]. We follow the presentation in [22].

Let $\phi : (\mathbb{Z}_2)^3 \to U(1)$ be the normalised representative of the non-trivial class in $H^3(\mathbb{Z}_2, U(1))$:

$$\phi(-,-,-) = -1, \quad \text{all other components equal to 1.} \tag{82}$$

Let

$$c_x(f,g) := (\iota_x \phi)(f,g) = \frac{\phi(x,f,g)\phi(f,g,x)}{\phi(f,x,g)}. \tag{83}$$

for all $x, f, g \in \mathbb{Z}_2$. For each $x \in \mathbb{Z}_2$ the map $c_x : (\mathbb{Z}_2)^2 \to U(1)$ is a 2-cocycle, it satisfies

$$c_x(f,g)c_x(fg,h) = c_x(f,gh)c_x(g,h). \tag{84}$$

The quasi-quantum double $\mathcal{D}^\phi(\mathbb{Z}_2)$ is an algebra spanned by $\{P_x f\}_{x,f\in\mathbb{Z}_2}$ with multiplication

$$(P_x f)(P_y g) = \delta_{x,y}(P_x fg)\, c_x(f,g). \tag{85}$$

The unit for this multiplication is $\sum_{x\in\mathbb{Z}_2}(P_x 1)$.

The quasi-quantum double is, moreover, equipped with a coproduct $\Delta : \mathcal{D}^\phi(\mathbb{Z}_2) \to \mathcal{D}^\phi(\mathbb{Z}_2) \otimes \mathcal{D}^\phi(\mathbb{Z}_2)$ given by

$$\Delta(P_x f) = \sum_{yz=x} c_f(y,z)(P_y f) \otimes (P_z f). \tag{86}$$

Associativity and quasicoassociativity follow readily from Eq. (84), in particular

$$(\mathrm{id} \otimes \Delta)\Delta(P_x f) = \varphi \cdot (\Delta \otimes \mathrm{id})\Delta(P_x f) \cdot \varphi^{-1} \tag{87}$$

with $\varphi = \sum_{f,g,h\in\mathbb{Z}_2} \phi^{-1}(f,g,h)\,(P_f 1) \otimes (P_g 1) \otimes (P_h 1)$. That Δ is an algebra morphism follows from the identity

$$\frac{c_x(f,g)c_y(f,g)}{c_{xy}(f,g)} \times \frac{c_f(x,y)c_g(x,y)}{c_{fg}(x,y)} = 1. \tag{88}$$

There is a counit $\epsilon : \mathcal{D}^\phi(\mathbb{Z}_2) \to \mathbb{C}$ and an antipode $S : \mathcal{D}^\phi(\mathbb{Z}_2) \to \mathcal{D}^\phi(\mathbb{Z}_2)$ given by

$$\epsilon(P_x f) = \delta_{x,1}, \quad S(P_x f) = (P_{x^{-1}} f^{-1})\, c_{x^{-1}}(f, f^{-1})^{-1} c_f(x, x^{-1})^{-1}. \tag{89}$$

These give $\mathcal{D}^\phi(\mathbb{Z}_2)$ the structure of a quasi Hopf algebra. This quasi-Hopf algebra is, moreover, quasitriangular with universal R-matrix

$$R = \sum_{x,y} (P_x 1) \otimes (P_y x). \tag{90}$$

3.5.3. Category of Representations and Its Fusion Ring. Since $\mathcal{D}^\phi(\mathbb{Z}_2)$ is a quasitriangular Hopf algebra, its category of finite-dimensional representations $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$ is a braided tensor category. We extract the fusion ring, F-symbols, and R-symbols of this braided tensor category. See [18] for a more in depth analysis of this category of representations.

There are four irreducible representations of $\mathcal{D}^\phi(\mathbb{Z}_2)$, labelled by pairs $(x, \chi) \in \mathbb{Z}_2 \times \mathbb{Z}_2^*$. ($\mathbb{Z}_2^*$ consists of the characters of $\mathbb{Z}_2$, namely 1 and sgn.) They are given by

$$\Pi_{(x,\chi)}(P_y f) = \delta_{x,y}\, \varepsilon_x(f)\chi(f) \tag{91}$$

with

$$\varepsilon_x(f) := \exp\left(\frac{\pi \mathrm{i}}{2}[x].[f]\right) \tag{92}$$

where $[x], [f]$ are the additive representation of x and f. i.e. $\varepsilon_-(-) = \mathrm{i}$ and all other components are equal to one. ε_x is a cocycle and

$$c_x(f,g) = (d\varepsilon_x)(f,g) = \frac{\varepsilon_x(fg)}{\varepsilon_x(f)\varepsilon_x(g)}. \tag{93}$$

Since we have a coproduct, we have the following tensor product of representations:

$$(\Pi_1 \otimes \Pi_2)(P_x f) := ((\Pi_1 \otimes \Pi_2) \circ \Delta)(P_x f) = \sum_{yz=x} c_f(y,z)\, \Pi_1(P_y f) \otimes \Pi_2(P_z f). \tag{94}$$

One easily verifies that

$$\Pi_{(x,\chi)} \otimes \Pi_{(y,\sigma)} = \Pi_{(xy,\chi\sigma)}. \tag{95}$$

The representation $\Pi_{(1,1)}$ is an identity for this tensor product (with trivial left and right unitors). In particular, the fusion ring of the representation category is $\mathbb{Z}(G)$ with G the abelian group with elements $\{(x,\chi)\}_{(x,\chi)\in\mathbb{Z}_2\times\mathbb{Z}_2^*}$ and group multiplication given by $(x,\chi)\cdot(y,\sigma) = (xy, \chi\sigma)$.

With this tensor product, the representations of $\mathcal{D}^\phi(\mathbb{Z}_2)$ form a tensor category with simple objects $\Pi_{(x,\chi)}$ and associators between simple objects

$$\alpha_{(x,\chi),(y,\sigma),(z,\tau)} : (\Pi_{(x,\chi)} \otimes \Pi_{(y,\sigma)}) \otimes \Pi_{(z,\tau)} \to \Pi_{(x,\chi)} \otimes (\Pi_{(y,\sigma)} \otimes \Pi_{(z,\tau)}) \tag{96}$$

given by multiplication with $\phi(x,y,z)$. This shows that the F-symbols of the representation category $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$ are given by the 3-cocycle α on G given by $\alpha((x,\chi),(y,\sigma),(z,\tau)) = \phi(x,y,z)$.

The braiding $\epsilon_{(x,\chi),(y,\sigma)} : \Pi_{(x,\chi)} \otimes \Pi_{(y,\sigma)} \to \Pi_{(y,\sigma)} \otimes \Pi_{(x,\chi)}$ of simple objects of $\mathcal{D}^\phi(\mathbb{Z}_2)$ is given by multiplication with

$$(\Pi_{(x,\chi)} \otimes \Pi_{(y,\sigma)})(R) = \varepsilon_y(x)\sigma(x), \tag{97}$$

where R is the universal R-matrix given in Eq. (90). These braidings are summarised in Table 4, and it follows from (95) that the R-symbols of $\mathcal{D}^\phi(\mathbb{Z}_2)$ are given by the same table.

Table 4. Braiding isomorphisms of $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$ for simple objects

$\epsilon_{(x,\chi),(y,\sigma)}$	$(1,1)$	$(-1,1)$	$(-1,\mathrm{sgn})$	$(1,\mathrm{sgn})$
$(1,1)$	1	1	1	1
$(-1,1)$	1	i	$-$i	-1
$(-1,\mathrm{sgn})$	1	i	$-$i	-1
$(1,\mathrm{sgn})$	1	1	1	1

3.5.4. Braided Monoidal Equivalence of $\Delta^I_{\Lambda_0}$ and $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$. By the identification

$$(1,1)\leftrightarrow 1,\quad (-1,1)\leftrightarrow S,\quad (-1,\mathrm{sgn})\leftrightarrow \bar{S},\quad (1,\mathrm{sgn})\leftrightarrow B. \tag{98}$$

we see that the groups G and I and therefore the fusion rings $\mathbb{Z}(G)$ and $\mathbb{Z}(I)$ of $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$ and $\Delta^I_{\Lambda_0}$ are isomorphic.

Under this identification, the F-symbols of $\Delta^I_{\Lambda_0}$ computed in Sect. 3.3 match precisely with the F-symbols α of the representation category. Furthermore, comparing Tables 3 and 4 we see that also the R-symbols match precisely. Thus, the braided tensor category $\Delta^I_{\Lambda_0}$ and the representation category $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$ have the same fusion rings $\mathbb{Z}(G)\simeq\mathbb{Z}(I)$, the same F-symbols, and the same R-symbols. It follows from Proposition 7.5.2 of [12] that $\Delta^I_{\Lambda_0}$ and $\mathrm{Rep}_f\mathcal{D}^\phi(\mathbb{Z}_2)$ are isomorphic as braided monoidal categories.

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Appendix A. Purity of the Double Semion State

In appendix, we prove Theorem 3.1, stating that the double semion state ω constructed in Sect. 3.1 is pure.

A.1. *Restrictions of* ω *to finite regions.* Denote by Π_n^E the set of edges belonging to some hexagon of Π_n and set $\mathcal{M}_n = \mathcal{A}_{\Pi_n^E}$. We investigate the restrictions $\omega|_n := \omega|_{\mathcal{M}_n}$.

Lemma A.1. *For any* $m > n$*, we have that* $\omega|_n = \omega_m|_n$*.*

Proof. It is sufficient to note that ω_m has the same expectation value for any operator in $\mathcal{M}_n$ as ω does. Indeed, for any $A \in \mathcal{M}_n$ we have $\omega(A) = \lim \omega_m(A)$, and the latter sequence becomes constant as soon as Π_m contains all hexagons containing edges in the support of A. i.e. $\omega_m(A) = \omega(A)$ for all $m \geq n+1$. □

Thus, we can restrict our attention to states $\omega_m|_n$. Recall that ω_m is given by the expectation value in the vector state

$$\Omega_m = \sqrt{\frac{1}{2^{|\Pi_m|}}} \sum_{\Pi \subset \Pi_m} (-1)^{\sharp \Pi} A_\Pi \Omega_0 = \sqrt{\frac{1}{2^{|\Pi_m|}}} \sum_{\Pi \subset \Pi_m} (-)^{\sharp \partial \Pi} |\partial \Pi\rangle \tag{99}$$

where $\sharp\partial\Pi$ is the number of closed loops in the loop soup $\partial\Pi$, and we chose to write A_Π instead of $\pi_0(A_\Pi)$ because the representation π_0 is faithful, and $|\partial\Pi\rangle$ is the product state with all degrees of freedom spin up, except those on the edges along the path $\partial\Pi$, which are spin down.

Note that every closed path α supported on Π_m^E is of the form $\partial\Pi$ for a unique $\Pi \subset \Pi_m$, so we have written Ω_m as a uniform superposition over all closed-loop soups supported on Π_m^E. Moreover, for closed paths α and β we have $|\alpha\rangle = |\beta\rangle$ if and only if $\alpha = \beta$, and these states are orthogonal otherwise.

We will show that $\omega_m|_n$ is a mixed state which is an equal-weight convex combination of pure states $\eta_n(b)$ where b is a *boundary condition*, namely an assignment of up-or down to each out edge of the region Π_n such that an even number of edges are up, see Fig. 17

The state $\eta_n(b)$ is then given by a uniform superposition of all loop soups that satisfy the boundary condition b, weighed by ± 1 depending on whether a fixed 'closure' of the boundary condition has an even or an odd number of closed loops.

Lemma A.2. *There are* $2^{|\Pi_n|}$ *such loop soups for each boundary condition* b*.*

Proof. For the boundary condition with all spins up, this is obvious, because then the loop soups are precisely closed-loop soups in Π_n^E.

To obtain loop soups for an arbitrary boundary condition b, act on any closed-loop soup with A_p on the hexagons between pairs of boundary edges where b forces a loop to end (choose one of two possible pairings). This yields a loop soup that satisfies the boundary condition, and two different closed-loop soups give two different loop soups satisfying the boundary condition. Conversely, every loop soup satisfying the boundary condition arises in this

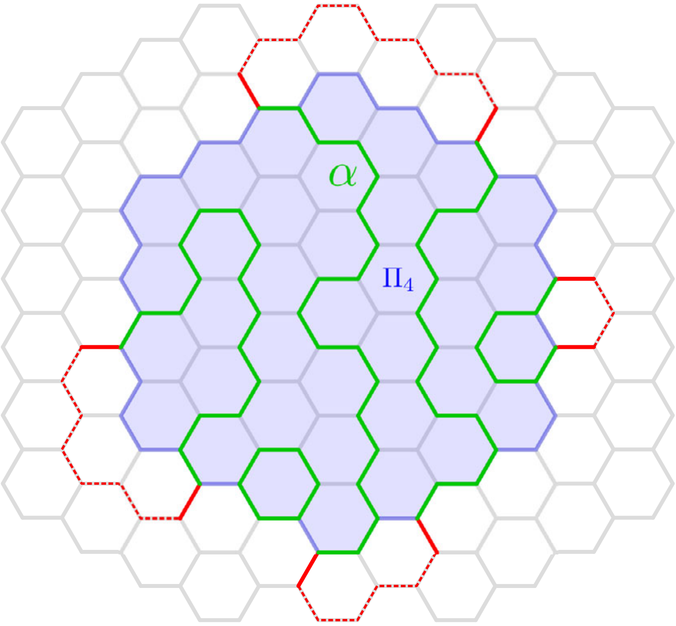

Figure 17. A loop soup $\alpha \in \mathcal{P}_4^{(b)}$ with boundary condition b corresponding to the red edges. The dotted red paths indicate one of two ways of pairing neighbouring red edges, resulting in a closed-loop soup (color figure online)

way, because acting on loop soups satisfying b with A_v's on the vertices between pairs of boundary edges where b forces a loop to end yields a closed-loop soup. □

Write $\mathcal{P}_n^{(b)}$ for the loop soups in Π_n^E that satisfy the boundary condition b. For a given boundary condition b, any $\alpha \in \mathcal{P}_n^{(b)}$ can be 'closed up' in precisely two ways by connecting neighbouring marked edges using edges in $\Pi_{n+1}^E \backslash \Pi_n^E$, see Fig. 17. Pick one such 'pairing' of marked boundary edges, and let $\sharp\alpha$ be the number of loops of α closed up with the chosen pairing. Then we have normalised vectors

$$|\eta_n^{(b)}\rangle = \sqrt{\frac{1}{2^{|\Pi_n|}}} \sum_{\alpha \in \mathcal{P}_n^{(b)}} (-1)^{\sharp\alpha} |\alpha\rangle. \tag{100}$$

We have $\langle \eta_n^{(b)}, \eta_n^{(b')} \rangle = \delta_{b,b'}$, i.e. these vectors form an orthonormal set. Denote by $\eta_n^{(b)}$ the pure state on $\mathcal{M}_n$ corresponding to the vector $|\eta_n^{(b)}\rangle$.

Proposition A.3. *For $m > n \geq 1$,*

$$\omega_m|_n = \frac{1}{2^{6n-1}} \sum_b \eta_n^{(b)}. \tag{101}$$

Since the $|\eta_n^{(b)}\rangle$ form an orthonormal set, this is a Schmidt decomposition of $\omega_m|_n$.

Here, 2^{6n-1} is the number of boundary conditions b. Indeed, there are $6n$ outer edges where the boundary condition either forces or does not force a string to pass, and the number of edges where a string is forced to end must be even. There are as many even boundary conditions as there are odd boundary conditions. Indeed, flipping a fixed edge gives a bijection.

Proof. By Lemma A.1, it is sufficient to consider $m = n+1$. The state ω_{n+1} on Π^E_{n+1} is a uniform superposition of closed-loop soups in Π^E_{n+1}. Any such loop soup α defines a boundary condition $b(\alpha)$ by the outer edges of Π_n that are occupied by strings of α. We can therefore organise the α according to which boundary condition they induce:

$$|\Omega_{n+1}\rangle = \sqrt{\frac{1}{2^{|\Pi_{n+1}|}}} \sum_b \sum_{\alpha : b(\alpha)=b} (-1)^{\sharp\alpha} |\alpha\rangle. \tag{102}$$

The states $|\alpha\rangle$ are orthonormal product states. If O is supported on Π^E_n, then the matrix elements $\langle\beta, O\alpha\rangle$ only depend on the configuration of α and β on Π^E_n. This information still allows us to deduce the boundary conditions $b(\alpha)$ and $b(\beta)$. Moreover, the matrix element vanishes if $b(\alpha) \neq b(\beta)$, hence

$$\begin{aligned}
\omega_{n+1}(O) = \langle\Omega_{n+1}, O\Omega_{n+1}\rangle &= \frac{1}{2^{|\Pi_{n+1}|}} \sum_{\alpha,\beta} (-1)^{\sharp\alpha+\sharp\beta} \langle\beta, O\alpha\rangle \\
&= \frac{1}{2^{|\Pi_{n+1}|}} \sum_b \sum_{\substack{\alpha : b(\alpha)=b \\ \beta : b(\beta)=b}} (-1)^{\sharp\alpha+\sharp\beta} \langle\beta, O\alpha\rangle \\
&= \frac{2}{2^{|\Pi_{n+1}|}} \sum_b \sum_{\alpha',\beta' \in \mathcal{P}^{(b)}_n} (-1)^{\sharp\alpha'+\sharp\beta'} \langle\beta', O\alpha'\rangle \\
&= \frac{1}{2^{6n-1}} \sum_b \frac{1}{2^{|\Pi_n|}} \sum_{\alpha',\beta' \in \mathcal{P}^{(b)}_n} (-1)^{\sharp\alpha'+\sharp\beta'} \langle\beta', O\alpha'\rangle \\
&= \frac{1}{2^{6n-1}} \sum_b \eta^{(b)}_n(O).
\end{aligned}$$

The factor of 2 appearing in the third line is the number of choices of completing a loop soup α' in Π^E_n with boundary condition b to a closed-loop soup α in Π^E_{n+1}. The phase $(-)^{\sharp\alpha'+\sharp\beta'}$ does not depend on which (common) completion is chosen. Indeed, changing the completion changes both $\sharp\alpha'$ and $\sharp\beta'$ by an odd amount if the number of marked edges is a multiple of 4, and both by an even amount otherwise (Lemma A.4). To get the fourth line, we used that $|\Pi_{n+1}| - |\Pi_n| = 6n - 1$. □

Lemma A.4. *Given $\alpha \in \mathcal{P}^{(b)}_n$, denote by $\sharp_1\alpha$ and $\sharp_2\alpha$ the number of loops in the two possible completions. Then $\sharp_1\alpha - \sharp_2\alpha$ is odd if the number of marked points for b is a multiple of 4, and even otherwise.*

Proof. Assume first that α has no closed loops. Let the number of marked points be $2n$. The following construction is illustrated in Fig. 18. Abstract the region Π_n to a disk with the marked points sitting on the boundary. Then the two completions correspond to two sets of n intervals that 'interlace' along the

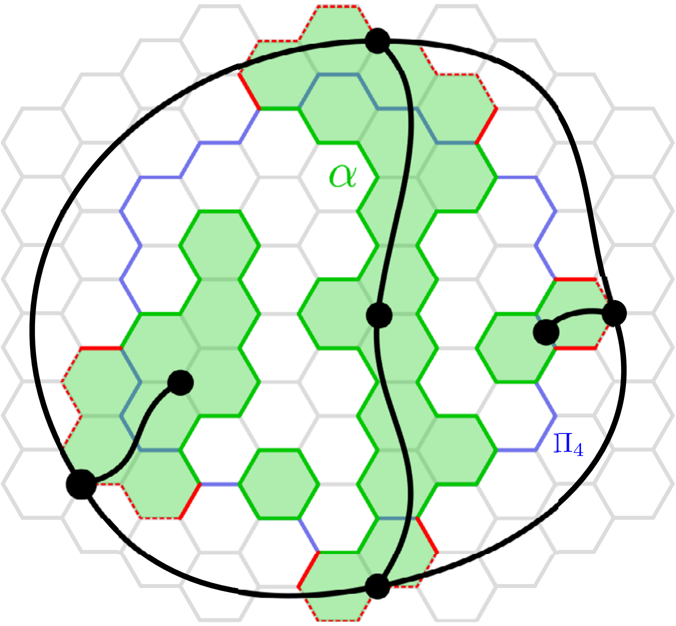

FIGURE 18. Red dotted paths completing α to a closed-loop soup are marked with vertices (black), and so are the closed regions (green) resulting from this completion. The white regions correspond one-to-one to faces of the black graph. Each such white region corresponds to a loop of the alternative completion of α to a closed-loop soup (color figure online)

boundary of the disk. Choose one of them. The loop soup α connects these n intervals into groups. The number of groups g is the number of closed loops in this completion, say $\sharp_1\alpha = g$. Put a vertex on each interval for this completion, and add a vertex in each group. Connect this vertex by edges to the vertices of the intervals in the group. Finally, connect the vertices on the intervals by edges along the boundary of the disk. This gives a connected graph with $V = n + g$ vertices and $E = 2n$ edges. By the Euler formula, this graph has $F = 1 - V + E = 1 + n - g$ internal faces. The number of internal faces corresponds precisely to $\sharp_2\alpha$, and we find

$$\sharp_1\alpha - \sharp_2\alpha = g - (1 + n - g) = 2g - n - 1, \tag{103}$$

which is odd if n is even and vice versa.

Any closed loops of α remain connected components of both completions, so internal loops do not contribute to $\sharp_1\alpha = \sharp_2\alpha$.

□

We further show

Lemma A.5. *For any boundary condition b and any O supported on Π^E_{n-1}, we have $\eta^{(b)}_n(O) = \omega(O)$.*

Proof. From Lemma A.1, it is sufficient to show that $\eta^{(b)}_n(O) = \omega_n(O)$. Note that $\omega_n = \eta^{\emptyset}_n$, where $\emptyset$ stand for the trivial boundary condition.

For any other boundary condition b, let A_b be the product of A_p operators over hexagons between pairs of marked edges of b. Clearly, A_b is supported outside Π^E_{n-1}, so $A_b^* O A_b = O$, and since A_b bijectively maps loop soups satisfying b to closed-loop soups, we find

$$\eta_n^{(b)}(O) = \eta_n^{(b)}(A_b^* O A_b) = \eta_n^{\emptyset}(O) = \omega_n(O) = \omega(O). \tag{104}$$

□

A.2. *Purity of the Limit State.* We will now show that ω is a pure state by making use of the following lemma, which is a special case of Lemma 2.1. of [13].

Lemma A.6 (Lemma 2.1 of [13]). *A state ω on a UHF algebra realised as the inductive limit of a sequence of finite matrix algebras $\{\mathcal{M}_m\}$ is pure if the following holds:*

For each n, there exists $m > n$ such that if ρ is a linear functional on $\mathcal{M}_m$ that satisfies

$$\omega|_{\mathcal{M}_m} \geq \rho \geq 0, \tag{105}$$

then

$$\rho|_{\mathcal{M}_n} = \lambda \omega|_{\mathcal{M}_n} \tag{106}$$

for some $\lambda \in \mathbb{R}$.

In applying this theorem to our setting, we take $\mathcal{M}_n$ to be the algebra supported on Π^E_n.

Fix n and take $m \geq n+1$. Let ρ be a linear functional on $\mathcal{M}_m$ such that Eq. (105) is satisfied. From Proposition A.3 and Lemma A.1, we have

$$\omega|_m = \omega_{m+1}|_m = \frac{1}{2^{6m-1}} \sum_b \eta_m^{(b)}, \tag{107}$$

which is a Schmidt decomposition for $\omega|_m$. The assumption $\omega|_m \geq \rho \geq 0$ implies that ρ is a mixture of pure states in the span of the $\eta_m^{(b)}$'s (Lemma A.7). It then follows from Lemma A.5 and $m > n$ that

$$\rho|_n = \lambda \omega|_n \tag{108}$$

for some $0 \leq \lambda \leq 1$.

We conclude by Lemma A.6 that ω is pure.

We have used the following lemma:

Lemma A.7. *Let ω and ρ be linear functionals on a finite matrix algebra such that $\omega \geq \rho \geq 0$. Suppose*

$$\omega = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha| \tag{109}$$

is a Schmidt decomposition of ω. Then any Schmidt decomposition

$$\rho = \sum_\beta q_\beta |\phi_\beta\rangle\langle\phi_\beta| \tag{110}$$

satisfies

$$\operatorname{span}\{|\phi_\beta\rangle\}_\beta \subset \operatorname{span}\{|\psi_\alpha\rangle\}_\alpha. \tag{111}$$

i.e. the Schmidt states of ρ *span a subspace of the space spanned by the Schmidt states of* ω*.*

Proof. Suppose the conclusion is false, so one of the ϕ_β, say ϕ_1, lies outside of $\mathcal{V} = \operatorname{span}\{\psi_\alpha\}$. Then there is a vector χ orthogonal to $\mathcal{V}$ and such that $c = \langle \phi_1, \chi\rangle \neq 0$.

Consider now the positive operator $P = |\chi\rangle\langle\chi|$. We have $\omega(P) = 0$ and

$$\rho(P) = \sum_\beta q_\beta |\langle \phi_\beta, \chi\rangle|^2 > 0 \tag{112}$$

where all terms are non-negative and at least the term $\beta = 1$ is strictly positive.

It follows that $(\omega - \rho)(P) < 0$, violating the assumption. □

Appendix B. Properties of String Operators

For each vertex v, let

$$A_v := \frac{1}{2}\left(1 + \prod_{e\sim v} \sigma_e^Z\right) \tag{113}$$

where the product runs over the three edges connected to v. For each hexagon p, regarded as a closed loop with counterclockwise orientation, let

$$B_p := \frac{1}{2}\left(1 + W_S[\partial p]\right)\left(\prod_{v\in p} A_v\right). \tag{114}$$

The operators B_p and A_v are orthogonal projections, and they all commute with each other.

Let Π_n be a finite set of hexagons as in Fig. 9. Let Π_n^V be the set of vertices belonging to some hexagon in Π_n. We set

$$H_{\Pi_n} := \sum_{v\in\Pi_n^V} (1 - A_v) + \sum_{p\in\Pi_n} B_p. \tag{115}$$

Let us also introduce terms imposing boundary conditions:

$$H_{\partial\Pi_n} = \sum_{e\in\partial\Pi_n} \frac{1}{2}\left(1 - \sigma_e^Z\right). \tag{116}$$

This is also a sum of orthogonal projections, and they all commute with each other and with the B_p and A_v appearing in H_{Π_n}.

We now consider the commuting projection Hamiltonians

$$H_n := H_{\Pi_n} + H_{\partial\Pi_n}. \tag{117}$$

Let $\widetilde{\Pi}_n$ be the collection of edges that have an endpoint in Π_n^V. Then $\mathcal{H}_n \in \mathcal{A}_{\widetilde{\Pi}_n}$. Moreover, the state ω_n restricts to $\mathcal{A}_{\widetilde{\Pi}_n}$ as a pure state. Let us continue to denote this restriction by ω_n. We have

Lemma B.1. *The state* ω_n *on* $\mathcal{A}_{\widetilde{\Pi}_n}$ *is the unique ground state of* H_n*.*

Proof. The state ω_n is defined by the expectation in the vector state

$$\Omega_n = \sqrt{\frac{1}{2^{|\Pi_n|}}} \sum_{\Pi \subset \Pi_n} (-1)^{\sharp(\Pi)} A_\Pi \Omega_0 \tag{118}$$

where Ω_0 has all $\sigma_e^Z = 1$.

The state Ω_n is a superposition of closed string configurations in Π_n. Each such closed string configuration satisfies

$$(1 - A_v) A_\Pi \Omega_0 = 0, \quad \frac{1}{2}(1 - \sigma_e^Z) A_\Pi \Omega_0 = 0 \tag{119}$$

for all $v \in \Pi_n^V$, all $e \in \partial \Pi_n$ and all $\Pi \subset \Pi_n$.

To see that Ω_n is a ground state of H_n it remains to show that it is in the kernel of all B_p for $p \in \Pi_n$. One can check that

$$W_S[\partial p] A_\Pi \Omega_0 = \phi(p, \Pi) A_{p \triangle \Pi} \Omega_0 \tag{120}$$

where $\phi(p, \Pi) = -1$ if $p \triangle \Pi$ has the same parity of connected components as Π, and $\phi(p, \Pi) = 1$ otherwise. i.e.

$$\phi(p, \Pi) = (-1)^{\sharp \Pi + \sharp (p \triangle \Pi) + 1} \tag{121}$$

It follows that $W_S[\partial p] \Omega_n = -\Omega_n$ for any $p \in \Pi_n$, hence $B_p \Omega_n = 0$.

To see that Ω_n is the unique ground state, observe that any ground state must be in the kernel of all the $1 - A_v$ for $v \in \Pi_n^V$ and all the $\frac{1}{2}(1 - \sigma_e^Z)$ for $e \in \partial \Pi_n$. The space of states that are simultaneously in the kernels of all these projections is spanned by the closed string states

$$A_\Pi \Omega_0, \quad \Pi \subset \Pi_n. \tag{122}$$

We must find in this space a state that is in the kernel of all the B_p, equivalently a -1 eigenstate of all the $W_S[\partial p]$ for $p \in \Pi_n$. Consider a general state

$$\Psi = \sum_{\Pi \subset \Pi_n} \psi(\Pi) A_\Pi \Omega_0 \tag{123}$$

where $\psi(\Pi) \in \mathbb{C}$ are arbitrary. Then

$$W_S[\partial p] \Psi = \sum_{\Pi \subset \Pi_n} \psi(\Pi) \phi(p, \Pi) A_{p \triangle \Pi} \Omega_0, \tag{124}$$

so $W_S[\partial p] \Psi = -\Psi$ only if

$$\psi(\Pi)(-1)^{\sharp \Pi} - \psi(p \triangle \Pi)(-1)^{\sharp (p \triangle \Pi)}. \tag{125}$$

If any of the $\psi(\Pi)$ is non-zero (which must be the case, otherwise $\Psi = 0$), then this enforces

$$\psi(\Pi') = (-1)^{\sharp \Pi + \sharp \Pi'} \psi(\Pi') \tag{126}$$

for all $\Pi' \subset \Pi_n$. Indeed, and Π can be related to any Π' by a sequence of symmetric differences with elementary hexagons p. This shows that $\Psi \simeq \Omega_n$, so Ω_n is indeed the unique ground state of H_n on $\mathcal{A}_{\widetilde{\Pi}_n}$. □

Lemma B.2. *If P is a closed path entirely contained in Π_n, then $W_a[P]$ commutes with H_n.*

Proof. This is shown for the string operators of any Levin–Wen model using a graphical representation of the string operators in [16]. In our case of the double semion model, we can also show it by brute force. That $W_S[P]$ commutes with the star operators A_v and with the boundary terms in $H_{\partial\Pi_n}$ is obvious. Let us show that $W_S[P]$ commutes with B_p for $p \in \Pi_n$.

To this end, note simply that if Q is the path, possibly consisting of multiple components, made up of edges of P that are also edges or R-legs of p, oriented with the same orientation as P, then

$$W_S[P] W_S[\partial p] W_S[P]^* = W_S[\partial p] U[Q] \tag{127}$$

where the string operators $U[Q]$ are defined in Eq. (55). Since $U[Q](\prod_{v\in p} A_v) = \prod_{v\in p} A_v$, and all A_v's commute with $W_S[P]$ we find

$$\begin{aligned} W_S[P]\left(W_S[\partial p]\Big(\prod_{v\in p} A_v\Big)\right) W_S[P]^* &= W_S[\partial p] U[Q] \left(\prod_{v\in p} A_v\right) \\ &= W_S[\partial p]\left(\prod_{v\in p} A_v\right). \end{aligned} \tag{128}$$

The claim for semion string operators $W_S[P]$ follows.

The required result is easy to verify for bound state strings $W_B[P]$, and since $W_{\bar{S}}[P] = W_S[P] W_B[P]$, the claim also holds for the anti-semion string operators. □

Lemma B.3. *If P is a finite closed string, then*

$$\omega \circ w_a[P] = \omega \tag{129}$$

for any $a \in \{1, S, \bar{S}, B\}$.

Proof. By Lemmas B.1 and B.2, we have

$$W_a[P]\Omega_n \sim \Omega_n \tag{130}$$

for n sufficiently large. Hence, $\omega_n \circ w_a[P] = \omega_n$ for n sufficiently large. The double semion state ω is by definition the weak-* limit of the sequence ω_n so

$$\omega \circ w_a[P] = \lim_{n\uparrow\infty} \omega_n \circ w_a[P] = \lim_{n\uparrow\infty} \omega_n = \omega. \tag{131}$$

□

We are now ready to give the

Proposition 3.2. Let O be a strictly local observable. Then we can find a finite closed loop P' such that $w_a[P](O) = w_a[P'](O)$. From Lemma B.3, we then find $(\omega \circ w_a[P])(O) = (\omega \circ w_a[P'])(O) = \omega(O)$. Since the strictly local operators are dense in $\mathcal{A}$, it follows that $\omega \circ w_a[P] = \omega$.

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Chapter 3

Sector Theory of Levin-Wen Models

1 Introduction

The ambition to understand gapped phases of quantum spin systems has generated a vast literature. Within mathematical physics, this program is cast as a classification problem; we want to establish a complete set of invariants of gapped phases so that if two gapped ground states have the same invariant, then they belong to the same gapped phase. An important invariant for this classification problem in two dimensions is the *anyon content* of a gapped ground state, or more generally, the topological defects admitted by the ground state. It has become clear over the past decade that these topological defects, together with their fusion and braiding properties, can be captured rigorously using *sector theory* [Naa11; Oga22; Bha+25; Jon24; KVW24; RO24].

In this paper we study the sector theory of Levin-Wen models on the infinite square lattice based on a unitary fusion category (UFC) $\mathcal{C}$. These models are believed to represent all gapped phases in two spatial dimensions that admit a gapped boundary.

The first main result is the classification of the irreducible anyon sectors of Levin-Wen models, namely the unitary equivalence classes of irreducible representations of the observable algebra $\mathcal{A}$ that satisfy the superselection criterion with respect to the GNS-representation $\pi^{\mathbb{1}}$ of the unique Levin-Wen ground state $\omega^{\mathbb{1}}$. Concretely, for each simple object $X \in \operatorname{Irr} Z(\mathcal{C})$ of the *Drinfeld center* of $\mathcal{C}$ we construct an endomorphism $\rho^X \in \operatorname{End}(\mathcal{A})$ so that $\{\pi^X := \pi^1 \circ \rho^X\}_{X \in \operatorname{Irr} Z(\mathcal{C})}$ is a complete set of disjoint irreducible anyon representations.

These anyon representations are the objects of the category SSS of *superselection sectors* associated to $\omega^{\mathbb{1}}$. Under the assumption of (approximate) Haag duality, SSS is a braided C*-tensor category [Naa11; FN15; Oga22]. This construction is an adaptation of the sector theory of algebraic quantum field theory [DHR69; DHR71; BF82; FGM90] to the setting of lattice spin systems. The category SSS rigorously captures the anyon content of the theory, describing the fusion and braiding of anyons, and establishes these data as an invariant of gapped phases [NSY19; Oga22].

The classification of irreducible anyon sectors imply that SSS_f, the full semisimple subcategory of SSS generated by its simple objects, is linearly equivalent to $Z(\mathcal{C})$. The fusion and braiding properties are captured by constructing isomorphisms

$$\Phi^Z_{XY} : Z(\mathcal{C})(X \otimes Y \to Z) \to \mathsf{SSS}_f(\rho^X \otimes \rho^Y \to \rho^Z),$$

and showing that these isomorphisms preserve F- and R-symbols. In Appendix D, we review the details of the familiar statement that the F- and R-symbols determine semisimple braided monoidal categories up to equivalence.

Under the assumption of bounded spread Haag duality for the ground state of the Levin-Wen model (Assumption 2.3 below), we obtain:

Theorem 1.1. *There is a unitary braided monoidal equivalence*

$$Z(\mathcal{C}) \simeq \mathsf{SSS}_f \,.$$

The proof appears in Section 11. Note that this equivalence pushes the unitary modular tensor category (UMTC) structure of $Z(\mathcal{C})$ forward to SSS_f, showing in particular that anyons of the Levin-Wen model have conjugates (antiparticles).

A similar result has recently been obtained [BV25; Bol+26] for Kitaev's quantum double models [Kit03] on the plane. For these models one can explicitly construct localized and transportable *amplimorphisms*, originally introduced in [Naa15], representing all equivalence classes of objects in SSS_f. The arguments in [Bol+26] rely crucially on the fact that these amplimorphisms moreover provide an action of the model's anyon theory, namely the representation category of the quantum double $\mathcal{D}(G)$ of the gauge group, on the observable algebra. It follows from the discussion in [Che+22, Section 5.1] that the analogous property *cannot* hold for string operators of the Levin-Wen models covered here (indeed, $Z(\mathcal{C})$ may have non-integer quantum dimensions). Our methods for computing the braided monoidal structure of SSS_f for Levin-Wen models therefore necessarily differ significantly from the strategy pursued in [Bol+26]. We believe that the strategy presented here can be used to compute the sector theory of representative ground states of *all* gapped phases of two dimensional spin systems [Sop23].

The chapter is organised as follows. In section 2, we describe the Levin-Wen model based on a unitary fusion category $\mathcal{C}$ in infinite volume. In Section 3, we introduce and analyse *skein modules* and various algebras acting on them, in particular the Tube-algebras. These skein modules are the state spaces of a Turaev-Viro TQFT. Section 4 makes explicit how the Levin-Wen Hamiltonian stabilizes *skein subspaces* isomorphic to the skein modules of Section 3. In Section 5, with a good understanding of skein subspaces in hand, we construct, for every simple object X of the Drinfeld center $Z(\mathcal{C})$ and any edge e of the lattice, a pure state ω_e^X interpreted as an anyon excitation of type X sitting near e. We construct string operators in Section 6, and show in Section 7 that these string operators yield the GNS representations π_e^X of the anyon states ω_e^X, when composed with the vacuum representation of the model. We use these string operators to show that the π_e^X satisfy the superselection criterion. In Section 8, we show that any irreducible anyon representation is equivalent to one of the π_e^X, concluding that the irreducible anyon sectors of the model are in one-to-one correspondence with equivalence classes of simple objects of $Z(\mathcal{C})$. In Section 9, we construct the isomorphisms Φ_{XY}^Z of fusion spaces and show that they preserve F-symbols. In Section 10, we establish that they also preserve R-symbols. Finally, we summarise the results in Section 11, assembling the proof of the main Theorem 1.1. Here we appeal to the result of Appendix D, which reviews how the maps Φ_{XY}^Z can be used to construct a unitary braided monoidal equivalence between $Z(\mathcal{C})$ and SSS_f. Appendices A and B contain proofs of certain basic properties of skein modules and skein subspaces, and Appendix C gives a general exposition of the category SSS.

Contents

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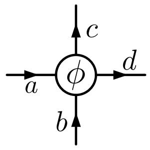
2 The Levin-Wen model and its anyon sectors

2.1 Local degrees of freedom

Fix a unitary fusion category (UFC) $\mathcal{C}$ with a representative set of simple objects $\operatorname{Irr}\mathcal{C}$. (See Section 3.1 below for details.) To each site $v \in \mathbb{Z}^2 \subset \mathbb{R}^2$ of the square lattice we associate a local degree of freedom

$$\mathcal{H}_v = \bigoplus_{a,b,c,d \in \operatorname{Irr}\mathcal{C}} \mathcal{C}(a \otimes b \to c \otimes b).$$

An element $\phi \in \mathcal{H}_v$ in the subspace $\mathcal{C}(a \otimes b \to c \otimes d)$ will be represented graphically by

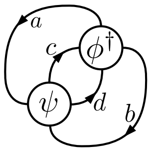


, (1)

where the morphism ϕ is associated to the site $v \in \mathbb{Z}^2$, and the diagram is read from bottom left to top right.

Following [Kon14; Chr+23; Gre+24], the degrees of freedom $\mathcal{H}_v$ are equipped with the *skein inner product* given for $\phi, \psi \in \mathcal{C}(a \otimes b \to c \otimes d)$ by

$$\langle \phi, \psi \rangle := \frac{\operatorname{tr}\{\phi^\dagger \circ \psi\}}{\sqrt{d_a d_b d_c d_d}} = \frac{1}{\sqrt{d_a d_b d_c d_d}}$$



. (2)

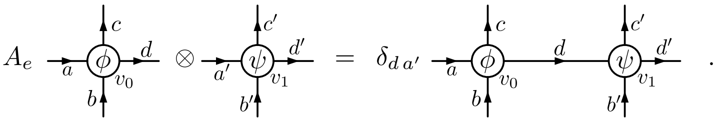
For any $v \in \mathbb{Z}^2$ we put $\mathcal{A}_v = \operatorname{End}(\mathcal{H}_v)$, and for finite $V \subset\subset \mathbb{Z}^2$ we put $\mathcal{H}_V = \bigotimes_{v \in V} \mathcal{H}_v$ and $\mathcal{A}_V = \bigotimes_{v \in V} \mathcal{A}_v \simeq \operatorname{End}(\mathcal{H}_V)$. If $V \subset W$ is an inclusion of finite subsets of $\mathbb{Z}^2$ then there is a natural embedding $\mathcal{A}_V \hookrightarrow \mathcal{A}_W$ by tensoring with the identity of $\mathcal{A}_{W \setminus V}$. For any, possibly infinite, $X \subset \mathbb{Z}^2$ these inclusions make $\{\mathcal{A}_V\}_{V \subset\subset X}$ into a directed system of matrix algebras. We denote its direct limit by $\mathcal{A}^{\text{loc}}_X$ and put $\mathcal{A}_X = \overline{\mathcal{A}^{\text{loc}}_X}^{\|\cdot\|}$. The *-algebra $\mathcal{A}^{\text{loc}}_X$ is called the algebra of local observables on X, and the C*-algebra $\mathcal{A}_X$ is called the quasi-local algebra on X. We also write $\mathcal{A} = \mathcal{A}_{\mathbb{Z}^2}$ and $\mathcal{A}^{\text{loc}} = \mathcal{A}^{\text{loc}}_{\mathbb{Z}^2}$, called the quasi-local algebra and the local algebra respectively.

For any $S \subset \mathbb{R}^2$ let $\overline{S} = S \cap \mathbb{Z}^2$ and write $\mathcal{A}_S := \mathcal{A}_{\overline{S}}$. If $\overline{S}$ is finite, we also write $\mathcal{H}_S := \mathcal{H}_{\overline{S}}$.

2.2 The Levin-Wen Hamiltonian

Let $\boldsymbol{e}_1 = (1,0)$ and $\boldsymbol{e}_2 = (0,1)$ be unit vectors in $\mathbb{Z}^2 \subset \mathbb{R}^2$. We denote by $\vec{\mathcal{E}} = \{(v_0, v_1) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : \operatorname{dist}(v_0, v_1) = 1\}$ the set of oriented edges of $\mathbb{Z}^2$ and by $\mathcal{E} = \{(v_0, v_1) \in \mathbb{Z}^2 \times \mathbb{Z}^2 : v_1 - v_0 \in \{\boldsymbol{e}_1, \boldsymbol{e}_2\}\}$ the set of edges of $\mathbb{Z}^2$ that are oriented to the top right. We identify $\mathcal{E}$ with the set of unoriented edges of $\mathbb{Z}^2$. For $e = (v_0, v_1) \in \vec{\mathcal{E}}$ we write $\bar{e} = (v_1, v_0)$ for its orientation reversal. We also write $\partial_{\mathfrak{i}} e = v_0$ and $\partial_{\mathfrak{f}} e = v_1$.

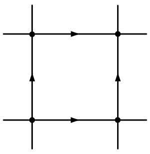
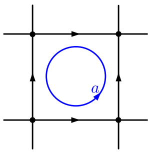
For each $e = (v_0, v_1) \in \vec{\mathcal{E}}$ we define a projector A_e acting on $\mathcal{H}_{v_0} \otimes \mathcal{H}_{v_1}$ by



(3)

Clearly $A_e = A_{\bar{e}}$. We say A_e enforces the *string-net constraint* at edge e.

Denote by $\mathcal{F}$ the set of faces of $\mathbb{Z}^2$. For any face $f \in \mathcal{F}$ we let $\mathcal{H}_f = \bigotimes_{v\in f} \mathcal{H}_v$ and define an orthogonal projector B_f on $\mathcal{H}_f$ which annihilates the orthogonal complement of $\prod_{e\in f} A_e \mathcal{H}_f$ and acts on $\prod_{e\in f} A_e\mathcal{H}_f$ by inserting the regular element of $\mathcal{C}$ and using local relations to bring the diagram back into product form:

B_f  $= \dfrac{1}{\mathcal{D}^2} \sum_{a\in \mathrm{Irr}\,\mathcal{C}} d_a$ 

. (4)

Here d_a is the quantum dimension of the object a, and $\mathcal{D}^2 = \sum_a d_a^2$. A precise definition will be given in Section 4.5 below.

It was shown that B_f is an orthogonal projector in [Gre+24, page 10]. Moreover, all $\{B_f\}_{f\in\mathcal{F}}$ commute with each other, as has been shown in [Zha17, Proposition 5.14]. This fact will follow from Lemma 4.5 below. The Levin-Wen Hamiltonian is the formal commuting projector Hamiltonian

$$H_{LW} = -\sum_{f\in\mathcal{F}} B_f.$$

A state $\omega : \mathcal{A} \to \mathbb{C}$ is a frustration free ground state of H_{LW} if $\omega(B_f) = 1$ for all $f \in \mathcal{F}$. The following Proposition has been proved in [Jon+23] (see section 2.3 and Theorem 4.8 of that paper):

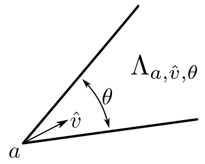
Proposition 2.1. *The Levin-Wen Hamiltonian has a unique frustration free ground state which we denote by $\omega^{\mathbb{1}}$. This frustration free ground state is pure.*

We present an independent proof at the end of section 5.2 below. We let $(\pi^{\mathbb{1}}, \mathcal{H}, \Omega)$ be the GNS triple of $\omega^{\mathbb{1}}$. Since $\omega^{\mathbb{1}}$ is pure, the *vacuum representation* $\pi^{\mathbb{1}}$ is irreducible.

2.3 The category of anyon sectors

The cone with apex at $a \in \mathbb{R}^2$, axis $\hat{v} \in \mathbb{R}^2$ of unit length, and opening angle $\theta \in (0, 2\pi)$ is the open subset of $\mathbb{R}^2$ given by

$$\Lambda_{a,\hat{v},\theta} := \{x \in \mathbb{R}^2 \,:\, (x-a)\cdot\hat{v} > \|x-a\| \cos(\theta/2)\}.$$



Any subset $\Lambda \subset \mathbb{R}^2$ of this form will be called a cone. The closure of a cone is also called a cone with the same apex, axis, and opening angle.

Definition 2.2. *A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies the superselection criterion w.r.t. the vacuum representation $\pi^{\mathbb{1}}$ if for any cone Λ, there is a unitary equivalence*

$$\pi|_\Lambda \simeq_{u.e.} \pi^{\mathbb{1}}|_\Lambda$$

of representations of $\mathcal{A}_\Lambda$. We will call such a representation an anyon representation. *A unitary equivalence class of anyon representations is called an* anyon sector. *An anyon sector is called irreducible if any and therefore all of its representative representations is irreducible.*

For any cone Λ we put $\mathcal{R}_\Lambda := \pi^{\mathbb{1}}(\mathcal{A}_\Lambda)''$ which we call the *cone algebra* of Λ. Define $\Lambda^{+s} := \Lambda_{b,\hat{v},\theta}$ with $b = a - s\sin(\theta/2)^{-1} \cdot \hat{v}$ for the cone obtained by moving the bounding rays of $\Lambda = \Lambda_{a,\hat{v},\theta}$ perpendicularly outward by a distance s. The following notion is adapted from [Jon24].

Assumption 2.3 (Bounded spread Haag duality)**.** *There is $s \geq 0$ so that*

$$\mathcal{R}'_{\Lambda^c} \subseteq \mathcal{R}_{\Lambda^{+s}}$$

holds for all cones Λ.

Remark 2.4. *(Bounded spread) Haag duality was proven for a wide class of commuting projector models based on C*-weak Hopf algebras in [OPR25]. This class includes models that are believed to be in the same phase as the Levin-Wen model considered here. That is, the Levin-Wen ground state can be transformed into the ground state of one of the models of [OPR25] by finite depth quantum circuits and adding/removing decoupled ancillas. Since bounded spread Haag duality is stable with respect to such transformations, we expect that the Levin-Wen ground state considered here does satisfy bounded spread Haag duality. Note, moreover, that the results presented in this paper suffice to derive a meaningful result on the anyon theory even without making the assumption of bounded spread Haag duality; see Remark 2.7 below.*

Fix a unit vector $\hat{f} \in \mathbb{R}^2$ that we call the *forbidden direction*. We say that a cone $\Lambda = \Lambda_{a,\hat{v},\theta}$ is *allowed* if $\hat{v} \cdot \hat{f} < \cos(\theta/2)$, in which case we write $\Lambda \perp \hat{f}$. Define the *allowed algebra*

$$\mathcal{B} := \overline{\bigcup_{\Lambda \perp \hat{f}} \mathcal{R}_\Lambda}^{\|\cdot\|}. \tag{5}$$

Note that $\pi^{\mathbb{1}}(\mathcal{A}) \subset \mathcal{B}$ is WOT-dense.

Definition 2.5. *An endomorphism $\rho \in \mathrm{End}(\mathcal{B})$ is localized in a cone Λ if $\rho \circ \pi^{\mathbb{1}}|_{\mathcal{A}_{\Lambda^c}} = \pi^{\mathbb{1}}|_{\mathcal{A}_{\Lambda^c}}$. The endomorphism ρ is transportable if for any cone Δ there is an endomorphism $\rho' : \mathcal{B} \to \mathcal{B}$ localized in Δ that is unitarily equivalent to ρ.*

The category of superselection sectors SSS is the C-category whose objects are endomorphisms of $\mathcal{B}$ which are localized in an allowed cone and are transportable. The morphisms of SSS are intertwiners. For any $\rho, \rho' \in \mathrm{Ob}\,\mathsf{SSS}$ we denote the space of intertwiners from ρ to ρ' by*

$$\mathsf{SSS}(\rho \to \rho') = \{V \in B(\mathcal{H}) \,|\, V\rho(x) = \rho'(x)V \text{ for all } x \in \mathcal{B}\}.$$

Define SSS_f as the full subcategory of SSS whose objects ρ have a finite-dimensional endomorphism space $\mathsf{SSS}(\rho \to \rho)$.

In Appendix C, we review the fact that localized anyon representations extend to unique localized and transportable endomorphisms in SSS, and that SSS is a braided C*-tensor category under the assumption of bounded spread Haag duality.

Remark 2.6. *The reason for introducing SSS_f is that we want to prove an equivalence with the Drinfeld center $Z(\mathcal{C})$, all of whose objects have finite dimensional endomorphism spaces. Since SSS admits infinite direct sums, restricting to SSS_f is necessary for the equivalence to hold. We believe that all infinite objects of SSS are infinite direct sums or integrals of objects of SSS_f.*

Remark 2.7. *In Section 6 we give explicit constructions of localized and transportable endomorphisms of $\mathcal{B}$ representing each isomorphism class of simple objects in SSS. It can be shown directly that the construction indeed gives endomorphisms of $\mathcal{B}$ without using Haag duality (see e.g.*

[Naa11, Proposition 4.6]). Transportability of these endomoprhisms also holds independently of Haag duality. Indeed, we construct explicit transporters in Section 10.1 that are localized in the sense of Lemma C.5.

The subcategory $\mathcal{D} \subset$ SSS generated by all finite products and direct sums of those explicitly constructed endomorphisms naturally has the structure of a braided C^*-tensor category. Indeed, Eq. (89) can be adapted to define the braiding, direct sums exist by virtue of the cone algebras being properly infinite (cf. Section C.3), while Lemma 9.9, proven below, can be applied to construct subobjects. See [BKM24] for a similar approach.

Having equipped $\mathcal{D}$ with the structure of a braided C^*-tensor category in this way, the arguments in this paper show that $\mathcal{D}$ is unitary braided monoidally equivalent to $Z(\mathcal{C})$, without any (weakened) assumption of Haag duality.

3 Tube algebras and skein modules

3.1 Unitary fusion category and Drinfeld center

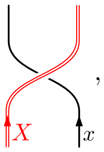
Throughout this paper we consider a fixed unitary fusion category (UFC) $\mathcal{C}$ with its canonical spherical structure. See [Eti+15; TV17; Pen21; Pen20] for general introductions. We write $\mathcal{C}(x \to y)$ for the vector space of morphisms from $x \in \operatorname{Ob}\mathcal{C}$ to $y \in \operatorname{Ob}\mathcal{C}$. Each $\mathcal{C}(x \to x)$ is equipped with a spherical trace tr and the quantum dimension of a non-zero object x is $d_x := \operatorname{tr}\operatorname{id}_x > 0$. We choose a representative set of simple objects $\operatorname{Irr}\mathcal{C}$ which contains the tensor unit $\mathbb{1}$. The total quantum dimension of $\mathcal{C}$ is $\mathcal{D} := \sqrt{\sum_{a\in\operatorname{Irr}\mathcal{C}} d_a^2}$.

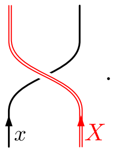
The dual of an object $x \in \operatorname{Ob}\mathcal{C}$ is denoted x^* and its evaluation and coevaluation morphisms are $\operatorname{ev}_x : x^* \otimes x \to \mathbb{1}$ and $\operatorname{coev}_x : \mathbb{1} \to x \otimes x^*$. For each $a \in \operatorname{Irr}\mathcal{C}$ there is a unique $\bar{a} \in \operatorname{Irr}\mathcal{C}$ such that $\bar{a}$ is isomorphic to a^*.

The unitary structure consists of dagger maps $\dagger : \mathcal{C}(x \to y) \to \mathcal{C}(y \to x)$ written as $f \mapsto f^\dagger$. Each morphism space $\mathcal{C}(x \to y)$ is a Hilbert space with the trace inner product $(f, g)_{\mathrm{tr}} := \operatorname{tr}\{f^\dagger \circ g\}$.

We freely use the graphical calculus to represent and manipulate morphisms of $\mathcal{C}$. See [TV17, Chapter I.2] for a good introduction. As usual, we suppress associators, unitors, and pivotal isomorphisms.

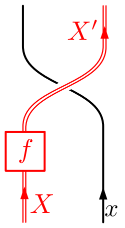
The *Drinfeld center* of $\mathcal{C}$ is the category $Z(\mathcal{C})$ whose objects are pairs (X, σ) of $X \in \operatorname{Ob}\mathcal{C}$ and a *half-braiding* $\sigma : X \otimes - \implies - \otimes X$ whose component morphisms and their inverses we represent by

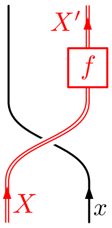
$\sigma_x =$ 

, $\sigma_x^{-1} =$ 

. (6)

The morphisms $Z(\mathcal{C})((X, \sigma) \to (X', \sigma'))$ are precisely those morphisms $f \in \mathcal{C}(X \to X')$ for which



$=$ 

(7)

for all $x \in \operatorname{Ob}\mathcal{C}$.

The Drinfeld center of $\mathcal{C}$ is a unitary modular tensor category whose pivotal and dagger structures coincide with that of $\mathcal{C}$ [Müg03]. (See [HPT16a, Proposition 2.3] for the statement about the pivotal structure in the case where $\mathcal{C}$ is not strict pivotal.) In particular, we can choose a finite set of representative simple objects $\operatorname{Irr} Z(\mathcal{C})$ containing the tensor unit $\mathbb{1}$, and for each $X \in \operatorname{Irr} Z(\mathcal{C})$ there is a unique $\bar{X} \in \operatorname{Irr} Z(\mathcal{C})$ which is isomorphic to X^*.

3.2 String diagrams and skein modules

We introduce skein modules on decorated surfaces. See [KKR10, Appendix A], [Kir11, Section 2], [Wal21], [Wal06] for similar setups. We work in the category of piecewise linear manifolds throughout. A *decorated 1-manifold* is a compact oriented 1-manifold (possibly with boundary) $\mathcal{N}$ together with a *decoration*, which consists of a finite collection of *signed marked points* $m_{\mathcal{N}} \subset \mathcal{N}$. A *decorated surface* is an oriented surface Σ together with a decoration of its boundary $\partial\Sigma$. We will often make reference to the topology of the underlying surface of Σ and for example say that Σ is homeomorphic to a sphere with m holes removed.

A *string diagram* on Σ is an embedded graph Γ in Σ whose edges are labelled by objects of $\mathcal{C}$ and whose internal vertices (i.e. not the marked boundary points) are labelled by morphisms of $\mathcal{C}$ in the Hom space determined by the labels of the attaching edges by the usual rules of the graphical calculus. Using the forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$, string diagrams may also be labelled by objects and morphisms of $Z(\mathcal{C})$. The graph meets the boundary of Σ transversally at the marked boundary points, with edges oriented into the boundary where they meet positive boundary points, and oriented away from the boundary where they meet negative boundary points. The labels of the edges attaching to $\partial\Sigma$ constitute the *boundary condition* $b : m_{\partial\Sigma} \to \operatorname{Ob}\mathcal{C}$ of the string diagram. We say the boundary condition is *simple* if all these labels belong to $\operatorname{Irr}\mathcal{C}$. We write $\operatorname{Bd}(\Sigma)$ for the set of connected boundary components of Σ, seen as decorated 1-manifolds. Given a boundary condition b and a decorated submanifold $\mathcal{N} \subset \partial\Sigma$, we write $b_{\mathcal{N}}$ for the restriction of b to the marked points in $\mathcal{N}$. Given a string diagram x on Σ, we also write $x_{\mathcal{N}}$ for the boundary condition on $\mathcal{N}$ induced by x.

Given a decorated 1-manifold $\mathcal{N}$ we let $\hat{\mathcal{N}}$ be the decorated 1-manifold obtained from $\mathcal{N}$ by reversing orientation and flipping the signs of all marked points. Given a decorated surface Σ we let $\hat{\Sigma}$ be the decorated surface obtained from Σ by reversing the orientation and flipping the signs of all marked boundary points. Then $\partial\hat{\Sigma} = \widehat{\partial\Sigma}$ as decorated 1-manifolds. Given a string diagram x on Σ we obtain a string diagram $\hat{x}$ on $\hat{\Sigma}$ by reversing the orientations of all edges and replacing the label of each internal vertex by its dagger.

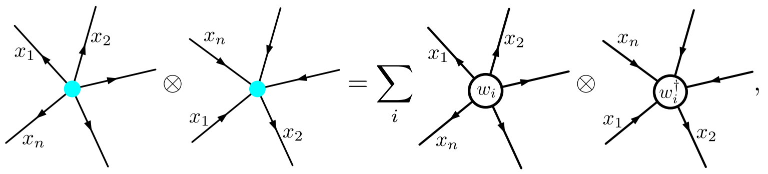
Let $\mathcal{S}(\Sigma; b)$ denote the set of all string diagrams on Σ with boundary condition b. We define $A(\Sigma; b) := \mathbb{C}[\mathcal{S}(\Sigma; b)]/\sim$, the space of all finite formal $\mathbb{C}$-linear combinations of string diagrams on Σ with boundary condition b, modded out by the equivalence relation $\sim$ generated by

- isotopy of string diagrams in Σ keeping $\partial\Sigma$ fixed.
- local relations of the graphical calculus applied inside contractible disks in Σ, to string diagrams whose edges cross the boundary of this disk transversally.

We call $A(\Sigma) := \bigoplus_{b\in\mathcal{B}(\Sigma)} A(\Sigma; b)$ the *skein module* on Σ. Here, $\mathcal{B}(\Sigma)$ is the finite set of all simple boundary conditions for string diagrams on Σ.

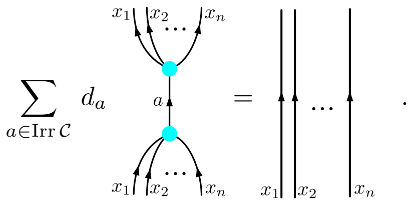
Given a string diagram x on a decorated surface Σ, we write $[x]_\Sigma \in A(\Sigma)$ for its equivalence class in the skein module. When the decorated surface is clear from context, we often simply write $[x]$.

Convention 3.1. *For any $x_1, \cdots, x_n \in \operatorname{Ob}\mathcal{C}$, a pair of coloured vertices,*



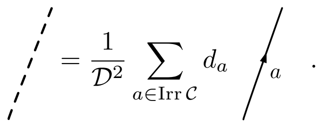
will stand for a sum over an orthonormal basis $\{w_i\}$ of $\mathcal{C}(\mathbb{1} \to x_1 \otimes \cdots \otimes x_n)$ and its dagger. This object does not depend on the choice of basis $\{w_i\}$.

With this convention, we have the following well known local relation of the graphical calculus ([KB10, Lemma 1.1]):



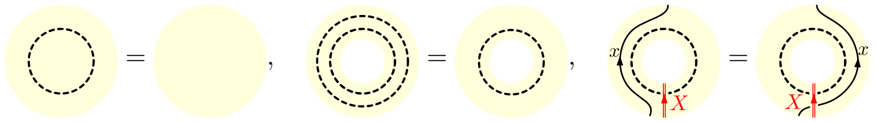
(8)

Convention 3.2. *We allow string diagrams to have edges coloured by a dotted line, defined by*



(9)

This dotted line satisfies the following local relations for any $x \in \operatorname{Ob}\mathcal{C}$ and any $X \in \operatorname{Ob} Z(\mathcal{C})$ ([Kir11, Corollary 3.5], [KB10, Lemma 2.2]):



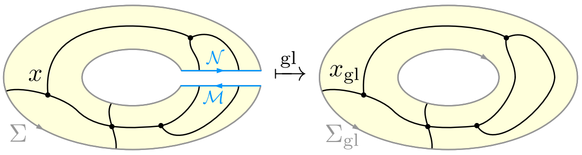
(10)

on embedded disks and annuli in any decorated surface. These are called the normalization, projector, and cloaking properties of the dotted line.

3.3 Gluing

Let Σ be a decorated surface and let $\mathcal{N}, \mathcal{M} \subset \partial\Sigma$ be two disjoint decorated submanifolds of the boundary of Σ. We say $\mathcal{N}$ can be glued to $\mathcal{M}$ if $\mathcal{N} \simeq \hat{\mathcal{M}}$ as decorated 1-manifolds. If this is the case then there is an orientation reversing homeomorphism $\psi : \mathcal{N} \to \mathcal{M}$ which maps marked boundary points to marked boundary points. We let Σ_ψ be the extended surface obtained from Σ by identifying $\mathcal{N}$ and $\mathcal{M}$ under the map ψ. The resulting decorated surface Σ_ψ depends only on the isotopy class of ψ ([BK01, Lemma 4.1.1]).

Let the identification ψ be fixed and write Σ_{gl} for the glued surface. Given a string diagram x on Σ we say the boundary conditions $x_{\mathcal{M}}$ and $x_{\mathcal{N}}$ match if all corresponding pairs of marked boundary points $(m, \psi(m)) \in m_{\mathcal{N}} \times m_{\mathcal{M}}$ are labelled by the same object. Suppose this is the case for x. Let us write x_{gl} for the string diagram obtained by interpreting x as a string diagram on Σ_{gl}:



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Setting $[x_{\mathrm{gl}}] = 0$ if the boundary conditions do not match, the gluing of string diagrams descends to a well-defined gluing map $\mathrm{gl} : A(\Sigma) \to A(\Sigma_{\mathrm{gl}})$ by setting $\mathrm{gl}([x]) := [x_{\mathrm{gl}}]$ and extending to $A(\Sigma)$ by linearity.

3.4 Cylinder algebras and their actions on skein modules

Let $\mathcal{N}$ be a decorated 1-manifold, then $\mathcal{N}$ is a disjoint union of decorated circles and decorated intervals. By taking the product with the closed unit interval I we get a decorated surface $\mathcal{N} \times I$ which is a union of decorated cylinders and disks. We let $\partial_b(\mathcal{N} \times I) = (\hat{\mathcal{N}} \times \{0\})$ be the *bottom* and $\partial_t(\mathcal{N} \times I) = (\mathcal{N} \times \{1\})$ the *top* of the decorated surface $\mathcal{N} \times I$.

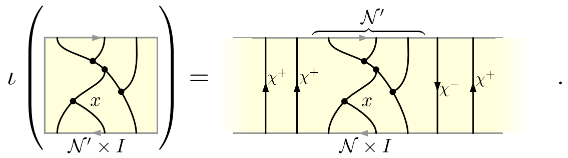
The skein module $A(\mathcal{N}) := A(\mathcal{N} \times I)$ has the structure of a C*-algebra with multiplication of two elements $[x]$ and $[y]$ defined by gluing the bottom of $[x]$ to the top of $[y]$, and with *-operation given by $[x]^* = [f(\hat{x})]$ where $f : \mathcal{N} \times I \to \mathcal{N} \times I : (\theta, r) \mapsto (\theta, 1-r)$. That is, the string diagram is flipped upside down, orientations of all strands are reversed, and all morphisms are replaced by their daggers. We call $A(\mathcal{N})$ the *cylinder algebra* on $\mathcal{N}$. If $\mathcal{S}$ is a decorated circle then $\mathrm{Tube}_{\mathcal{S}} = A(\mathcal{S})$ is called the *Tube algebra* on $\mathcal{S}$ [Izu00; Izu01; Müg03].

If $\mathcal{I}$ is a decorated *interval* then its marked points $m_{\mathcal{I}} = (m_1, \cdots, m_n)$ have a linear order following the opposite orientation of $\mathcal{I}$. For any labelling $\underline{a} : m_{\mathcal{I}} \to \mathrm{Irr}\,\mathcal{C}$ we write $\otimes\underline{a} = a(m_1)^{\sigma_1} \otimes \cdots \otimes a(m_n)^{\sigma_n}$ where $\sigma_i \in \{+, -\}$ is the sign of the marked point m_i, and $a^+ = a$ while $a^- = a^*$ for any $a \in \mathrm{Ob}\,\mathcal{C}$. Let $\chi = \bigoplus_{a \in \mathrm{Irr}\,\mathcal{C}} a$. We further define the object

$$\chi^{\otimes\mathcal{I}} := \bigotimes_{i=1}^{n} \chi^{\sigma_i} \simeq \bigoplus_{\underline{a}: m_{\mathcal{I}} \to \mathrm{Irr}\,\mathcal{C}} \otimes\underline{a}. \tag{11}$$

There is an obvious identification $A(\mathcal{I}) \simeq \mathcal{C}(\chi^{\otimes\mathcal{I}} \to \chi^{\otimes\mathcal{I}})$.

If $\mathcal{N}' \subset \mathcal{N}$, then $\mathcal{N}' \times I \subset \mathcal{N} \times I$, and we define an embedding $\iota : A(\mathcal{N}') \hookrightarrow A(\mathcal{N})$ by



(12)

Let Σ be a decorated surface with compact boundary and let $\mathcal{N} \subset \partial\Sigma$ be a decorated submanifold of the boundary of Σ. Then there is a left action $\triangleright_{\mathcal{N}}$ of $A(\mathcal{N})$ on $A(\Sigma)$ given by gluing the bottom of $[x] \in A(\mathcal{N})$ onto $\mathcal{N} \subset \partial\Sigma$. Similarly, there is a left action $\triangleright^{\mathrm{op}}_{\mathcal{N}}$ of $A(\hat{\mathcal{N}})^{\mathrm{op}}$ on $A(\Sigma)$ given by gluing the top of $[x] \in A(\hat{\mathcal{N}})$ onto $\mathcal{N} \subset \partial\Sigma$. There is an isomorphism of C*-algebras $A(\hat{\mathcal{N}}) \simeq A(\mathcal{N})^{\mathrm{op}}$ induced by flipping string diagrams upside down. We denote the image of $a \in A(\mathcal{N})$ under this isomorphism by a^{op}. Then we have $a \triangleright^{\mathrm{op}}_{\mathcal{N}} = a^{\mathrm{op}} \triangleright_{\hat{\mathcal{N}}}$.

If Σ has compact boundary, $\partial\Sigma$ is the disjoint union of connected boundary components $\mathcal{S} \in \mathrm{Bd}(\Sigma)$, each of which is a decorated circle. We have

$$A(\partial\Sigma) = \bigotimes_{\mathcal{S} \in \mathrm{Bd}(\Sigma)} \mathrm{Tube}_{\mathcal{S}}\,.$$

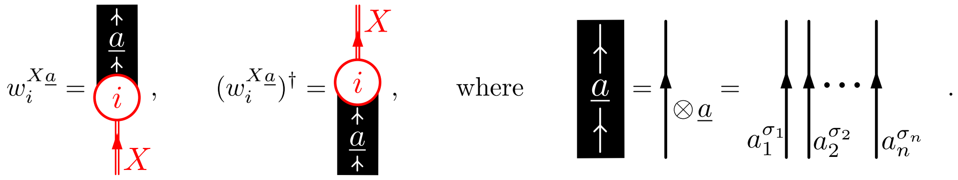
3.5 Matrix units for $\mathrm{Tube}_{\mathcal{S}}$

Let $\mathcal{N}$ be a decorated 1-manifold and pick a *fiducial point* on $\mathcal{N}$ which is distinct from all the marked points. A decorated 1-manifold equipped with a fiducial point is called an *extended 1-manifold*. A decorated surface of which each connected boundary component is equipped with a fiducial point is call an *extended surface*. Fiducial points of extended 1-manifolds will always be depicted by an X-mark in figures.

Let $\mathcal{S}$ be an extended circle and assume that $\mathcal{S}$ has n marked points $m_{\mathcal{S}} \subset \mathcal{S}$. The fiducial point induces an enumeration of the marked points $m_{\mathcal{S}} = (m_1, \cdots, m_n)$ starting at the fiducial point and following the *opposite* orientation of $\mathcal{S}$. Let $\sigma_i \in \{+, -\}$ denote the sign of the marked point m_i for $i = 1, \cdots, n$.

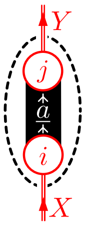
For a boundary condition $\underline{a} : m_{\mathcal{S}} \to \operatorname{Irr} \mathcal{C}$ we write $\otimes \underline{a} = a(m_1)^{\sigma_1} \otimes \cdots \otimes a(m_n)^{\sigma_n}$. We also write $d_{\underline{a}} := d_{\otimes \underline{a}}$ for the quantum dimension of the boundary condition $\underline{a}$, and set $\chi^{\otimes \mathcal{S}} = \bigoplus_{\underline{a}: m_{\mathcal{S}} \to \operatorname{Irr}\mathcal{C}} \otimes \underline{a}$.

For every $X \in \operatorname{Irr} Z(\mathcal{C})$ and every boundary condition $\underline{a}$ we fix an orthonormal basis $\{w_i^{X\,\underline{a}}\}_i$ of $\mathcal{C}(X \to \otimes \underline{a})$ with respect to the trace inner product. We will represent these morphisms graphically as follows:



These morphisms satisfy the following useful identity:

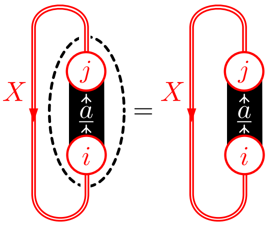
Lemma 3.3. *We have*



$$= \delta_{X,Y}\, \delta_{i,j}\, \frac{\mathrm{id}_X}{d_X}. \tag{13}$$

Proof : The cloaking property of the dotted line makes the left hand side of the claimed equality an element of $Z(\mathcal{C})(X \to Y)$. Since X and Y are irreducible it follows that the morphism represented by the left hand side equals $\delta_{X,Y}\, \lambda\, \mathrm{id}_X$ for some $\lambda \in \mathbb{C}$. Assuming $X = Y$ and taking the trace yields

$$\lambda d_X =$$

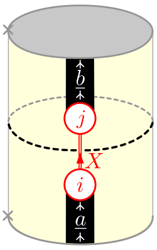


$$= \delta_{i,j}.$$

Here we used sphericity of $\mathcal{C}$ in the second step to unwrap the dotted line. ☐

Proposition 3.4. *The* $\mathrm{Tube}_\mathcal{S}$*-elements*

$$E^X_{\underline{b},j;\underline{a},i} := d_X$$



form a complete set of matrix units for $\mathrm{Tube}_\mathcal{S}$*. We have*

$$E^Y_{\underline{c},k;\underline{b}',j'}E^X_{\underline{b},j;\underline{a},i} = \delta_{XY}\delta_{\underline{b}\,\underline{b}'}\delta_{jj'}\,E^X_{\underline{c},k;\underline{a},i}, \quad \text{and} \quad \sum_{X,\underline{a},i} E^X_{\underline{a},i;\underline{a},i} = \mathrm{id}_{\mathrm{Tube}_\mathcal{S}}. \tag{14}$$

In particular, the minimal central projections of $\mathrm{Tube}_\mathcal{S}$ *are given by*

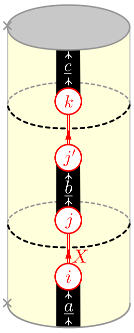
$$P^X = \sum_{\underline{a},i} E^X_{\underline{a},i;\underline{a},i}$$

for each $X \in \mathrm{Irr}\, Z(\mathcal{C})$.

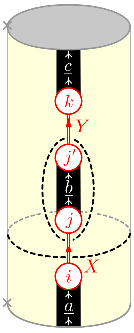
We will say that a projector $p \in \mathrm{Tube}_\mathcal{S}$ is of *type* $X \in \mathrm{Irr}\, Z(\mathcal{C})$ if it is dominated by the central projector P^X.

Proof of Proposition 3.4 : Using the cloaking property of one dotted loop to pull the other dotted loop to the front of the cylinder and using the relation Eq. (13) we find

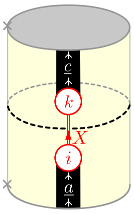
$$E^Y_{\underline{c},k;\underline{b}',j'}E^X_{\underline{b},j;\underline{a},i} = \delta_{\underline{b}\,\underline{b}'}\, d_X d_Y$$



$$= \delta_{\underline{b}\,\underline{b}'}\, d_X d_Y$$



$$= \delta_{X,Y}\delta_{\underline{b}\,\underline{b}'}\delta_{j\,j'}\, d_X$$



$$= \delta_{XY}\delta_{\underline{b}\,\underline{b}'}\delta_{j\,j'}\, E^X_{\underline{c},k;\underline{a},i}\ ,$$

so the $E^X_{\underline{b},j;\underline{a},i}$ do indeed form a system of matrix units. Linear independence of the $E^X_{\underline{b},j;\underline{a},i}$ also follows from this.

That this set of matrix units is complete follows from the identification of $\mathrm{Tube}_\mathcal{S}$ with

$$\bigoplus_{a \in \mathrm{Irr}\,\mathcal{C}} \mathcal{C}(a \otimes \chi^{\otimes \mathcal{S}} \to \chi^{\otimes \mathcal{S}} \otimes a) \simeq \bigoplus_{X \in \mathrm{Irr}\, Z(\mathcal{C})} \mathcal{C}(\chi^{\otimes \mathcal{S}} \to X^*) \otimes \mathcal{C}(X \to \chi^{\otimes \mathcal{S}}),$$

where this isomorphism is provided in Lemma 7.4 of [KB10]. □

3.6 The TQFT inner product

The skein modules $A(\Sigma)$ are the state spaces of the Turaev-Viro-Barrett-Westbury TQFT [TV92; BW96]. Since $\mathcal{C}$ is unitary one can use the TQFT partition function Z to equip these skein modules with an inner product as follows (see [Wal21; Wal06]).

Recall that the partition function Z assigns a complex number $Z(M; x)$ to any 3-manifold M with a string diagram $x \in \mathcal{S}(\partial M)$ on its boundary. In fact, $Z(M; x)$ depends only on the equivalence class of x in $A(\partial M)$ so we have a linear map $Z_M : A(\partial M) \to \mathbb{C}$ for each 3-manifold M. Since any 3-manifold admits a handle decomposition, the partition function is completely determined by the following normalisation, gluing, and product properties:

- For the 3-ball B and a string diagram x on the 2-sphere ∂B we have

  $$Z_B(x) = \mathrm{ev}(x)\,,$$

  where $\mathrm{ev}(x)$ interprets x as a diagram in the plane and uses the graphical calculus of $\mathcal{C}$ to evaluate that diagram to a number ([KB10, Theorem 2.4], [BW96]).

- If M_{gl} is obtained from a 3-manifold M with boundary $\partial M = N \cup \overline{N} \cup L$ by gluing N to $\overline{N}$ then [Wal21, Section 4]

  $$Z_{M_{\mathrm{gl}}}(x_{\mathrm{gl}}) = \sum_e \frac{Z_M(x \cup e \cup \hat{e})}{(e, e)}\,, \tag{15}$$

  where the sum ranges over an orthogonal basis of $A(N; b)$, with b the boundary condition induced by x.

- If $M = M_1 \sqcup M_2$ is a disjoint union and $x = x_1 \sqcup x_2$ is a string diagram on ∂M with x_1 on ∂M_1 and x_2 on ∂M_2 then

  $$Z_M(x) = Z_{M_1}(x_1) Z_{M_2}(x_2).$$

We can use the partition function to define an inner product on the skein module $A(\Sigma)$ as follows. Whenever Σ is a surface, $\Sigma \times I$ is defined to be the pinched product of Σ and $I = [0, 1]$ (i.e. the Cartesian product is pinched by retracting $\partial\Sigma \times I \simeq \partial\Sigma$ so that $\partial(\Sigma \times I) = \Sigma \cup \hat{\Sigma}$). Given $[x] \in A(\Sigma; b)$ and $[y] \in A(\Sigma; b')$ we put

$$([x], [y])_{A(\Sigma)} := \delta_{b\,b'}\, Z_{\Sigma \times I}(\hat{x} \cup y),$$

where $\hat{x} \cup y$ is the string diagram on $\partial(\Sigma \times I)$ consisting of string diagram $\hat{x}$ on $\hat{\Sigma}$ and string diagram y on Σ. This yields a well-defined inner product on $A(\Sigma)$ which we will call the TQFT inner product.

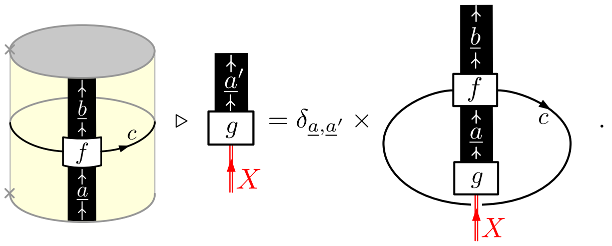
It is straightforward to verify that the Tube actions on the various boundary components of Σ are *-actions w.r.t. the TQFT inner product. This gives $A(\Sigma)$ the structure of a unitary $A(\partial\Sigma)$-module.

3.7 Characterisation of skein modules on punctured spheres

We characterise the skein module $A(\Sigma)$ where Σ is an extended surface homeomorphic to the sphere with m holes cut out. The characterization will depend on a choice of *anchor* for the extended surface Σ. (see [HPT16b] for a similar notion). An anchor ⚓ for Σ is an embedded graph in Σ consisting of one vertex called the anchor point, and for each boundary component

$\mathcal{S} \in \mathrm{Bd}(\Sigma)$ an edge running from the anchor point to that boundary component, attaching transversally at an attachment point on $\mathcal{S}$ which is distinct from the fiducial point on that boundary component. The edges of the graph are moreover linearly ordered going clockwise around the anchor point. This induces an enumeration $\{\mathcal{S}^{⚓}_{\kappa}\}_{\kappa=1}^{m}$ of $\mathrm{Bd}(\Sigma)$ which we call the enumeration induced by ⚓. Two anchors are equivalent if one can be deformed into the other by isotopy in Σ which keeps the fiducial points fixed on all boundary components.

The characterization of $A(\Sigma)$ is given in terms of morphism spaces of $\mathcal{C}$ and $Z(\mathcal{C})$, all of which we regard as Hilbert spaces equipped with the trace inner product. For any extended circle $\mathcal{S}$ and for any $X \in \mathrm{Ob}\, Z(\mathcal{C})$ the space $\mathcal{C}(X \to \chi^{\otimes \mathcal{S}})$ is a unitary left $\mathrm{Tube}_{\mathcal{S}}$-module with action given by



(16)

If $X \in \mathrm{Irr}\, Z(\mathcal{C})$, then the module $\mathcal{C}(X \to \chi^{\otimes \mathcal{S}})$ is irreducible.

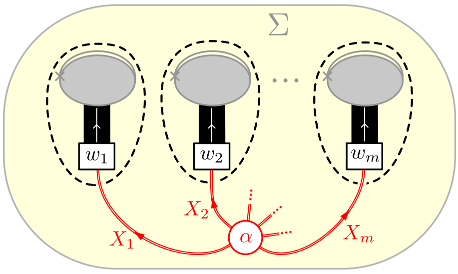
We will show that $A(\Sigma)$ is isomorphic as a unitary $A(\partial\Sigma)$-module to

$$\begin{aligned}\mathcal{C}_{⚓}(\Sigma) := \bigoplus_{X_1,\dots,X_m \in \mathrm{Irr}(Z(\mathcal{C}))} & Z(\mathcal{C})(\mathbb{1} \to X_1 \otimes \cdots \otimes X_m) \\ & \otimes \mathcal{C}(X_1 \to \chi^{\otimes \mathcal{S}^{⚓}_1}) \otimes \cdots \otimes \mathcal{C}(X_m \to \chi^{\otimes \mathcal{S}^{⚓}_m}).\end{aligned} \tag{17}$$

For each $\mathcal{S}^{⚓}_{\kappa}$ there is a unitary $\mathrm{Tube}_{\mathcal{S}^{⚓}_{\kappa}}$ action on $\mathcal{C}_{⚓}(\Sigma)$ given by Eq. (16) which we denote by $\triangleright_{\kappa}$. Recalling that $A(\partial\Sigma) = \bigotimes_{\mathcal{S}\in\mathrm{Bd}(\Sigma)} \mathrm{Tube}_{\mathcal{S}}$, we see that these actions give $\mathcal{C}_{⚓}(\Sigma)$ the structure of a unitary $A(\partial\Sigma)$-module.

Given an anchor ⚓ for Σ we define a linear map $\Phi^{⚓}_{\Sigma} : \mathcal{C}_{⚓}(\Sigma) \to A(\Sigma)$ by

$$\Phi^{⚓}_{\Sigma} : \alpha \otimes w_1 \otimes \cdots \otimes w_m \mapsto \left(\prod_{\kappa=1}^{m} d_{X_\kappa}\right)^{1/2} \mathcal{D}^{m-1}$$



, (18)

where in the string diagram on the right hand side, the morphism α sits at the anchor point of ⚓, and the strand labelled X_κ lies along the κ^{th} edge of ⚓ until it resolves into the morphisms w_κ near the boundary component $\mathcal{S}^{⚓}_{\kappa}$. If two anchors $⚓ \sim ⚓'$ are equivalent, then $\Phi^{⚓}_{\Sigma} = \Phi^{⚓'}_{\Sigma}$.

Proposition 3.5. *Let Σ be an extended surface homeomorphic to the sphere with m holes cut out, and let ⚓ be an anchor for Σ. The map*

$$\Phi^{⚓}_{\Sigma} : \mathcal{C}_{⚓}(\Sigma) \to A(\Sigma)$$

is a unitary isomorphism of $A(\partial\Sigma)$-modules.

The proof appears in Appendix A.

Remark 3.6. *There is a well-known description of Tube algebras in terms of χ and the adjoint of the forgetful functor $F : Z(\mathcal{C}) \to \mathcal{C}$ (see [Müg03], and [Kaw+24; KL20] for a nice overview in the context of Levin-Wen models). Following [HPT16a], we denote the adjunction $F \dashv \mathrm{Tr}$. Then*

$$\mathrm{Tube}_{\mathcal{S}} \simeq \mathrm{End}_{Z(\mathcal{C})}(\mathrm{Tr}\,\chi^{\otimes \mathcal{S}}),$$

with the isomorphism provided by the adjunction once the vector space of the Tube algebra is realised as $\mathrm{Tube}_{\mathcal{S}} \simeq \bigoplus_{a \in \mathrm{Irr}\,\mathcal{C}} \mathcal{C}(a \otimes \chi^{\otimes \mathcal{S}} \otimes a^* \to \chi^{\otimes \mathcal{S}})$. On $\mathrm{Tube}_{\mathcal{S}}$-representations the adjunction gives the isomorphisms

$$\mathcal{C}(X \to \chi^{\otimes \mathcal{S}}) \simeq Z(\mathcal{C})(X \to \mathrm{Tr}\,\chi^{\otimes \mathcal{S}}),$$

where the right hand side is a natural representation of $\mathrm{End}_{Z(\mathcal{C})}(\mathrm{Tr}\,\chi^{\otimes \mathcal{S}})$ given by composition of morphisms. In this description, Proposition 3.5 can be recast as

$$A(\Sigma) \simeq Z(\mathcal{C})(1 \to \mathrm{Tr}\,\chi^{\otimes \mathcal{S}_1} \otimes \cdots \otimes \mathrm{Tr}\,\chi^{\otimes \mathcal{S}_m}). \tag{19}$$

For later use, we investigate how the map $\Phi_{\Sigma}^{\text{⚓}}$ changes when we modify the anchor using Dehn twists around the holes of Σ.

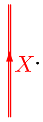
Lemma 3.7. *Let Σ be a decorated surface homeomorphic to the sphere with m holes cut out. Let ⚓ and ⚓$'$ be anchors for Σ so that ⚓$'$ is obtained from ⚓ by a Dehn twist around boundary component $\mathcal{S}_{\kappa}^{\text{⚓}}$. Then*

$$\Phi_{\Sigma}^{\text{⚓}'}\big(\alpha \otimes w_1 \otimes \cdots \otimes w_m\big) = \theta_{X_\kappa} \times \Phi_{\Sigma}^{\text{⚓}}\big(\alpha \otimes w_1 \otimes \cdots \otimes w_m\big) \tag{20}$$

for any $\alpha \in Z(\mathcal{C})(\mathbb{1} \to X_1 \otimes \cdots \otimes X_m)$ and any $w_\kappa \in \mathcal{C}(X_\kappa \to \chi^{\otimes \mathcal{S}_\kappa^{\text{⚓}}})$ for $\kappa = 1, \cdots, m$.

Here $\theta_{X_\kappa} \in U(1)$ is the twist of $X_\kappa \in \mathrm{Irr}\,Z(\mathcal{C})$, defined for any $X \in \mathrm{Ob}\,Z(\mathcal{C})$ by

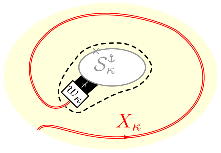
$$= \theta_X \times$$



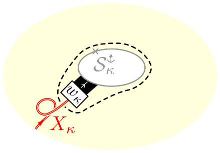
. (21)

Proof : Writing $K = \left(\prod_{\kappa=1}^{m} d_{X_\kappa}\right)^{1/2} \mathcal{D}^{m-1}$ we find

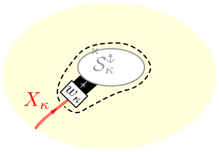
$$\Phi_{\Sigma}^{\text{⚓}'}\big(\alpha \otimes w_1 \otimes \cdots \otimes w_m\big) = K \times$$



$$= K \times$$



$$= K \times \theta_{X_\kappa} \times$$



$$= \theta_{X_\kappa} \times \Phi_{\Sigma}^{\text{⚓}}\big(\alpha \otimes w_1 \otimes \cdots \otimes w_m\big),$$

where we used the cloaking property of the dotted line in the second step. □

4 Skein subspaces

4.1 Regions and their string-net subspaces

Let $C^{\mathbb{Z}^2}$ be the cell complex with vertices $C_0^{\mathbb{Z}^2} = \mathbb{Z}^2$, all edges $\mathcal{E}$ of $\mathbb{Z}^2$ thought of as open intervals in $\mathbb{R}^2$ connecting neighbouring vertices, and all faces $\mathcal{F}$ of $\mathbb{Z}^2$ thought of as open unit squares in $\mathbb{R}^2$.

A *region* is a subcomplex C of $C^{\mathbb{Z}^2}$. In particular, if C contains a face f then it also contains all edges on the boundary of that face, and if C contains an edge e then it also contains the vertices at the endpoints of that edge. We write C_0, C_1 for the 0-skeleton and 1-skeleton of C respectively, considered as subcomplexes of C. We also write V_C, E_C, and F_C for the set of vertices, edges, and faces of C respectively. Also, $\vec{E}_C = \{e \in \vec{\mathcal{E}} \,:\, e \in E_C \,\text{or}\, \bar{e} \in E_C\}$ is the set of oriented edges of C, and $\mathcal{A}_C = \mathcal{A}_{C_0}$ is the algebra of observables supported on C. The *oriented boundary* of C is

$$\vec{\partial} C := \{e \in \vec{\mathcal{E}} \,:\, \partial_{\mathsf{f}} e \in V_C \,\text{and}\, \partial_{\mathsf{i}} e \notin V_C\},$$

and we write

$$\vec{\mathcal{E}}_C := \vec{\partial} C \cup \vec{E}_C = \{e \in \vec{\mathcal{E}} \,:\, \partial_{\mathsf{f}} e \in V_C\}.$$

A *labelling* of C is an assignment $l : \vec{\mathcal{E}}_C \to \operatorname{Irr} \mathcal{C}$ of simple objects from $\operatorname{Irr} \mathcal{C}$ to oriented edges of C. An *internal labelling* of C is an assignment $l : \vec{E}_C \to \operatorname{Irr} \mathcal{C}$ and a *boundary labelling* of C is an assignment $l : \vec{\partial} C \to \operatorname{Irr} \mathcal{C}$. We denote by $\mathcal{L}(C), \mathcal{L}^{\text{in}}(C)$ and $\mathcal{L}^{\partial}(C)$ respectively the sets of labellings, internal labellings, and boundary labellings of C.

An (internal) labelling l of C is called a (internal) *string-net labelling* if $l(e) = l(\bar{e})$ for all $e \in \vec{E}_C$. We denote by $\mathcal{L}_{SN}(G)$ the set of all string-net labellings of C and by $\mathcal{L}_{SN}^{\text{in}}(C)$ the set of all internal string-net labellings of C.

We can identify any vertex v with the subcomplex $C^v = C_0^v = \{v\}$. Then $\vec{\partial} v = \{e_1, e_2, e_3, e_4\}$ where e_1, e_2, e_3, e_4 are the four oriented edges pointing into v. Given a boundary labelling $l : \vec{\partial} v \to \operatorname{Irr} \mathcal{C}$ we let $\mathcal{H}_v(l) \subset \mathcal{H}_v$ be the subspace of $\mathcal{H}_v$ corresponding to the direct summand $\mathcal{C}(l(e_1) \otimes l(e_2) \to l(e_3) \otimes l(e_4))$. Then $\mathcal{H}_v = \bigoplus_{l \in \mathcal{L}^{\partial}(v)} \mathcal{H}_v(l)$.

Given a finite region C we write

$$\mathcal{H}_C := \mathcal{H}_{V_C} = \bigoplus_{l \in \mathcal{L}(C)} \mathcal{H}_C(l), \quad \text{where} \quad \mathcal{H}_C(l) := \bigotimes_{v \in V_C} \mathcal{H}_v(l|_{\vec{\partial} v}).$$

The *string-net subspace* associated to C is

$$H_{C_1} := \bigoplus_{l \in \mathcal{L}_{SN}(C)} \mathcal{H}_C(l) = \left(\prod_{e \in E_C} A_e \right) \mathcal{H}_C,$$

i.e. the string-net constraints are imposed along all edges of C. As the notation reflects, the string-net subspace of C depends only on the 1-skeleton C_1.

We further decompose the string-net subspace according to boundary labellings. For $b \in \mathcal{L}^{\partial}(C)$ we denote by $\mathcal{L}^b(C)$ the set of labellings l of C such that $l|_{\vec{\partial} C} = b$. Similarly, $\mathcal{L}_{SN}^b(C)$ is the set of string-net labellings l of C such that $l|_{\vec{\partial} C} = b$. Then we have

$$H_{C_1} = \bigoplus_{b \in \mathcal{L}^{\partial}(C)} H_{C_1}^b \quad \text{with} \quad H_{C_1}^b := \bigoplus_{l \in \mathcal{L}_{SN}^b(C)} \mathcal{H}_C(l).$$

4.2 Extended surfaces assigned to regions

To any region C we assign an extended surface Σ_C as follows. The surface Σ_C is the $1/3$-fattening of $C \subset \mathbb{R}^2$. That is, $\Sigma_C := \{v \in \mathbb{R}^2 \, : \, \mathrm{dist}(v, C) \leq 1/3\}$ where $\mathrm{dist}(v, C) = \inf_{w \in C} \|w - v\|_\infty$ and we regard C as a closed subset of the plane. The marked boundary points are the points of intersection of $\partial \Sigma_C$ with edges of $\mathbb{Z}^2$. Such a marked boundary point has *positive* sign if the corresponding edge $e \in \mathcal{E}$ (which is oriented towards the top left) points out of the surface Σ_C, and it has *negative* sign if the corresponding edge $e \in \mathcal{E}$ points into the surface. Finally, each connected boundary component $\mathcal{N} \in \mathrm{Bd}(\Sigma_C)$ is the union of some horizontal and vertical line segments. If $\mathcal{N}$ is compact then we place the fiducial point at the upper extremity of top leftmost vertical segment. If $\mathcal{N}$ is not compact we pick an arbitrary fiducial point. When C is finite and connected there is a distinct *outer* boundary component of Σ_C. All other boundary components are referred to as *inner*.

Note that Σ_C and Σ_{C_1} have the same marked boundary points. Indeed, the surface Σ_{C_1} has all the same boundary components as Σ_C, and in addition a boundary component for each face of C, but there are no marked boundary points on the boundary components of Σ_{C_1} corresponding to these faces.

Boundary conditions for string diagrams on Σ_C or Σ_{C_1} are equivalent to boundary labellings of the region C itself, and we will identify these concepts.

4.3 Isomorphism of string-net subspaces with skein modules

Let C be a finite region. There is a natural way to regard product vectors in the string-net subspace H_{C_1} as string diagrams on Σ_{C_1}. For a given string-net labelling $l \in \mathcal{L}_{SN}(C)$, consider unit vectors $f_v \in \mathcal{H}_v(l)$ for all $v \in C_0$. Recall that f_v is a morphism in $\mathcal{C}$. Denoting $f = (f_v)_{v \in C_0}$, we define $\phi_f = \bigotimes_{v \in C_0} f_v \in \mathcal{H}_C(l)$. By construction, ϕ_f belongs to the string-net subspace H_{C_1}, and H_{C_1} is spanned by such product vectors. We define the corresponding string diagram x_f on Σ_{C_1} whose graph is the intersection of the graph of $\mathbb{Z}^2$ with the surface Σ_{C_1} (with edges directed towards the top right), which has edges labelled according to l, and vertices $v \in C_0$ labelled by the morphisms f_v. We define $\pi_{C_1} : H_{C_1} \to A(\Sigma_{C_1})$ by

$$\pi_{C_1}(\phi_f) = [x_f]_{\Sigma_{C_1}}, \tag{22}$$

where we interpret the string diagram x_f as a string diagram on Σ_{C_1}.

The following lemma extends Lemma 5.3 of [Kir11] to include surfaces with boundary. Their proof generalizes mutatis mutandis.

Lemma 4.1 ([Kir11])**.** *For any finite region C, the map $\pi_{C_1} : H_{C_1} \to A(\Sigma_{C_1})$ is an isomorphism of vector spaces.*

Convention 4.2. *We will freely use the isomorphism π_{C_1} to represent states in H_{C_1} by string diagrams on Σ_{C_1}, where we can use the graphical calculus. Whenever a string diagram on a surface Σ_{C_1} is interpreted as an element of H_{C_1}, this will implicitly be done using π_{C_1}.*

If x is a string diagram on Σ_{C_1}, then we write $\phi_x := \pi_{C_1}^{-1}([x])$ for the corresponding vector in the string-net subspace H_{C_1}.

Let us also introduce the map $\sigma_{C_1} : H_{C_1} \to A(\Sigma_{C_1})$ defined on product vectors by

$$\sigma_{C_1}(\phi_f) := d_{\partial l}^{-1/4} \, [x_f]_{\Sigma_{C_1}}, \tag{23}$$

and extended linearly to all of H_{C_1}. Here, $d_{\partial l} = \prod_{e \in \vec{\partial} C} d_{l(e)}$ is the product of the quantum dimensions of the objects assigned by l to the boundary of C. Note that by construction we have $\sigma_C(H_{C_1}^b) \subset A(\Sigma_{C_1}; b)$ for any boundary condition b.

The reason for modifying the map π_{C_1} with weights as in Eq. (23) is to obtain an isomorphism of Hilbert spaces, see Lemma B.1.

4.4 Actions of Tube algebras

Let C be a region such that Σ_C has compact boundary. To each connected boundary component $\mathcal{S} \in \mathrm{Bd}(\Sigma_C)$ we associate a *collar region* $C^{\mathcal{S}}$, defined to be the smallest subregion of C such that $\mathcal{S} \in \mathrm{Bd}(\Sigma_{C^{\mathcal{S}}})$. Note that if C contains a face f and C' is obtained from C by removing f, then $\Sigma_{C'}$ has a connected boundary component at f which we denote by $\mathcal{S}_f$. We denote the corresponding collar region by $C^f := C^{\mathcal{S}_f}$.

More generally, if $\mathcal{N} \subset \mathcal{S} \in \mathrm{Bd}(\Sigma_C)$ then we let $C^{\mathcal{N}}$ be the smallest subregion of C so that $\mathcal{N} \subset \partial\Sigma_{C^{\mathcal{N}}}$. The region $C^{\mathcal{N}}$ does not depend on $\mathcal{S}$ in the sense that if $\mathcal{N} \subset \mathcal{S}' \mathrm{in} \mathrm{Bd}(\Sigma_{C'})$ for some other region C', then $C^{\mathcal{N}}$ is also the smallest subregion of C' such that $\mathcal{N} \subset \partial\Sigma_{C^{\mathcal{N}}}$.

For each $\mathcal{S} \in \mathrm{Bd}(\Sigma_C)$ we define a representation $\mathfrak{t}_{\mathcal{S}}$ of $\mathrm{Tube}_{\mathcal{S}}$ on $\mathcal{H}_{C^{\mathcal{S}}}$ as follows. First, for any $a \in \mathrm{Tube}_{\mathcal{S}}$, the operator $\mathfrak{t}_{\mathcal{S}}(a)$ acts as zero on the orthogonal complement of the string-net subspace $H_{C^{\mathcal{S}}}$. That is, any $\mathfrak{t}_{\mathcal{S}}(a)$ enforces string-net constraints on the collar region $C^{\mathcal{S}}$.

On the string-net subspace, the action is given by

$$\mathfrak{t}_{\mathcal{S}}(a)\,\phi := \sigma_{C^{\mathcal{S}}}^{-1}\big(a \triangleright_{\mathcal{S}} \sigma_{C^{\mathcal{S}}}(\phi)\big) \tag{24}$$

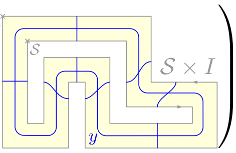
for any $\phi \in H_{C^{\mathcal{S}}}$. It follows from Lemma B.1 that each $\mathfrak{t}_{\mathcal{S}}$ is a *-representation of $\mathrm{Tube}_{\mathcal{S}}$. Let us also define a *-representation of $\mathrm{Tube}_{\hat{\mathcal{S}}}^{\mathrm{op}}$ on $\mathcal{H}_{C^{\mathcal{S}}}$ by

$$\mathfrak{t}_{\mathcal{S}}^{\mathrm{op}}(a) := \mathfrak{t}_{\mathcal{S}}(a^{\mathrm{op}}) \tag{25}$$

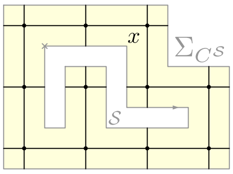
for any $a \in \mathrm{Tube}_{\hat{\mathcal{S}}}$.

Unpacking definitions and making use of Convention 4.2, the action of $\mathfrak{t}_{\mathcal{S}}([y])$ on $H_{C^{\mathcal{S}}}$ is represented graphically by

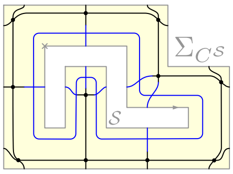
$$\mathfrak{t}_{\mathcal{S}}\Bigg($$



$$\Bigg)$$



$$= \delta_{x_{\mathcal{S}}, y_{\hat{\mathcal{S}}}} \left(\frac{d_{y_{\mathcal{S}}}}{d_{x_{\mathcal{S}}}}\right)^{1/4}$$



$$. \tag{26}$$

Here we represented $\mathcal{S} \times I$ as a subset of the plane with the bottom boundary component identified with the outer boundary in the figure.

More generally, if $\mathcal{N} \subset \mathcal{S} \in \mathrm{Bd}(\Sigma_C)$ then we define a *-representation $\mathfrak{t}_{\mathcal{N}}$ of $A(\mathcal{N})$ on $\mathcal{H}_{C^{\mathcal{N}}}$ by letting $\mathfrak{t}_{\mathcal{N}}(a)$ annihilate the orthogonal complement of the string-net subspace $H_{C^{\mathcal{N}}}$, and letting $\mathfrak{t}_{\mathcal{N}}(a)$ act on this subspace by

$$\mathfrak{t}_{\mathcal{N}}(a)\phi := \sigma_{C^{\mathcal{N}}}^{-1}\big(a \triangleright_{\mathcal{N}} \sigma_{C^{\mathcal{N}}}(\phi)\big) \tag{27}$$

for any $\phi \in H_{C^{\mathcal{N}}}$. We also define the $A(\hat{\mathcal{N}})^{\mathrm{op}}$-action $\mathfrak{t}_{\mathcal{N}}^{\mathrm{op}}(a) := \mathfrak{t}_{\mathcal{N}}(a^{\mathrm{op}})$. It follows immediately from the definitions that

Lemma 4.3. *If $\mathcal{N} \subset \mathcal{S} \in \mathrm{Bd}(\Sigma_C)$ for some finite region C then*

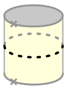
$$\mathfrak{t}_{\mathcal{N}}(a)\mathfrak{t}_{\mathcal{S}}(\mathrm{id}) = \mathfrak{t}_{\mathcal{S}}(\mathrm{id})\mathfrak{t}_{\mathcal{N}}(a) = \mathfrak{t}_{\mathcal{S}}(\iota(a)),$$

where $\iota : A(\mathcal{N}) \to A(\mathcal{S})$ is the inclusion (12).

4.5 Definition of B_f projectors

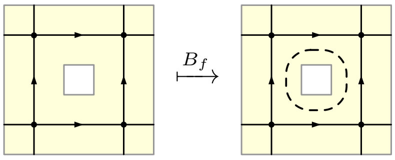
For a boundary component $\mathcal{S}_f$ that corresponds to a face $f \in \mathcal{F}$ we write $\mathfrak{t}_f := \mathfrak{t}_{\mathcal{S}_f}$ for the corresponding $\mathrm{Tube}_{\mathcal{S}_f}$-representation on $\mathcal{H}_{C^f}$. We define the orthogonal projector

$$B_f = \mathfrak{t}_f(P^{\mathbb{1}}), \quad \text{where} \quad P^{\mathbb{1}} =$$



$$\in \mathrm{Tube}_{\mathcal{S}_f} . \tag{28}$$

That is, B_f enforces string-net constraints and inserts a dotted loop around the face f. Using Convention 4.2, we can also describe the action of B_f on H_{C^f} graphically as



. (29)

4.6 Skein subspaces

Let C be a finite region. We define its associated *skein subspace* by

$$H_C := B_C H_{C_1} = P_C \mathcal{H}_C, \tag{30}$$

where $B_C := \prod_{f \in F_C} B_f$ and $P_C := \prod_{f \in F_C} B_f \prod_{e \in E_C} A_e$. This subspace inherits the skein inner product from $\mathcal{H}_C$.

The inclusion $\Sigma_{C_1} \subset \Sigma_C$ induces a map

$$\iota_C : A(\Sigma_{C_1}) \to A(\Sigma_C) : [y]_{\Sigma_{C_1}} \mapsto [y]_{\Sigma_C}$$

which is well-defined since local relations in Σ_{C_1} are also local relations in Σ_C, and isotopy fixing the boundary of Σ_{C_1} extends to isotopy fixing the boundary of Σ_C. The map ι_C is surjective because any string diagram on Σ_C can be deformed by isotopy to be contained in $\Sigma_{C_1} \subset \Sigma_C$.

Recalling definitions (22)(23), define

$$\begin{aligned} \pi_C &= \iota_C \circ \pi_{C_1} &&: H_{C_1} \to A(\Sigma_C), \\ \sigma_C &= \mathcal{D}^{-|F_C|} \times \iota_C \circ \sigma_{C_1} &&: H_{C_1} \to A(\Sigma_C). \end{aligned}$$

The normalization of σ_C is so that its restriction to H_C yields an isomorphism of Hilbert spaces.

Proposition 4.4. *The map $\sigma_C|_{H_C} : H_C \to A(\Sigma_C)$ is a unitary isomorphism. Moreover, for any connected boundary component $\mathcal{S} \in \mathrm{Bd}(\Sigma_C)$ of Σ_C, we have*

$$a \triangleright_{\mathcal{S}} \sigma_C(\psi) = \sigma_C\big(\mathfrak{t}_{\mathcal{S}}(a)\psi\big)$$

for all $\psi \in H_C$ and all $a \in \mathrm{Tube}_{\mathcal{S}}$. In particular, the skein subspace H_C equipped with the Tube*-actions given by the representations $\{\mathfrak{t}_{\mathcal{S}}\}$ is a unitary $A(\partial\Sigma_C)$-module, and $\sigma_C|_{H_C}$ is an intertwiner of unitary $A(\partial\Sigma_C)$-modules.*

The proof appears in Appendix B. As a corollary, we obtain the *commutativity lemma*:

Lemma 4.5. *Let C be a finite region and let $\mathcal{S}, \mathcal{S}' \in \mathrm{Bd}(\Sigma_C)$ be distinct boundary components. Then we have*

$$\mathfrak{t}_{\mathcal{S}}(a)\mathfrak{t}_{\mathcal{S}'}(b) = \mathfrak{t}_{\mathcal{S}'}(b)\mathfrak{t}_{\mathcal{S}}(a),$$

for all $a \in \mathrm{Tube}_{\mathcal{S}}$ and all $b \in \mathrm{Tube}_{\mathcal{S}'}$. In particular, all B_f projectors commute with one another.

Proof : Let $C^{\mathcal{S}}$ and $C^{\mathcal{S}'}$ be the collar regions of the boundary components $\mathcal{S}$ and $\mathcal{S}'$ and put $\widetilde{C} = C^{\mathcal{S}} \cup C^{\mathcal{S}'}$. Then $\mathcal{S}$ and $\mathcal{S}'$ are also boundary components of $\Sigma_{\widetilde{C}}$ with the same corresponding Tube-representations $\mathfrak{t}_{\mathcal{S}}$ and $\mathfrak{t}_{\mathcal{S}'}$. We regard $\mathfrak{t}_{\mathcal{S}}(a)$ and $\mathfrak{t}_{\mathcal{S}'}(b)$ as operators on $\mathcal{H}_{\widetilde{C}}$. On the subspace $H_{\widetilde{C}} \subset \mathcal{H}_{\widetilde{C}}$, commutativity of $\mathfrak{t}_{\mathcal{S}}(a)$ and $\mathfrak{t}_{\mathcal{S}'}(b)$ now follows from Proposition 4.4 and the fact that Tube-actions corresponding to different boundary components on the skein module $A(\Sigma_{\widetilde{C}})$ all commute with each other. On the orthogonal complement of $H_{\widetilde{C}}$ both $\mathfrak{t}_{\mathcal{S}}(a)\mathfrak{t}_{\mathcal{S}'}(b)$ and $\mathfrak{t}_{\mathcal{S}'}(b)\mathfrak{t}_{\mathcal{S}}(a)$ vanish because at least one string-net constraint on one of the collar regions $C^{\mathcal{S}}$ or $C^{\mathcal{S}'}$ is violated.

The claim about the B_f's now follows by recalling that $B_f = \mathfrak{t}_f(P^{\mathbb{1}})$ and considering, for any two faces f_1 and f_2, the region $\widetilde{C} = C^{f_1} \cup C^{f_2}$. □

A boundary condition for H_C is an assignment $\underline{p} = \{p_{\mathcal{S}}\}_{\mathcal{S} \in \mathrm{Bd}(\sigma_C)}$ of a projector $p_{\mathcal{S}} \in \mathrm{Tube}_{\mathcal{S}}$ to each connected boundary component $\mathcal{S} \in \mathrm{Bd}(\Sigma_C)$. We define

$$H_C(\underline{p}) := \left(\prod_{\mathcal{S} \in \mathrm{Bd}(\Sigma_C)}^{m} \mathfrak{t}_{\mathcal{S}}(p_{\mathcal{S}}) \right) H_C. \tag{31}$$

Given an enumeration $\{\mathcal{S}_\kappa\}_{\kappa=1}^m$ of $\mathrm{Bd}(\Sigma_C)$ we will also write this as

$$H_C(p_{\mathcal{S}_1}, \cdots, p_{\mathcal{S}_m}) = H_C(\underline{p}).$$

We further define the following collections of density matrices:

$$\begin{aligned}
\mathcal{D}_C &:= \{\text{density matrices } \rho \in \mathcal{B}(\mathcal{H}_C) \text{ supported on } H_C\}, \\
\mathcal{D}_C(\underline{p}) &:= \{\text{density matrices } \rho \in \mathcal{B}(\mathcal{H}_C) \text{ supported on } H_C(\underline{p})\}, \\
&= \{\rho \in \mathcal{D}_C \, : \, \mathrm{Tr}\{\rho \, \mathfrak{t}_{\mathcal{S}}(p_{\mathcal{S}})\} = 1 \text{ for all } \mathcal{S} \in \mathrm{Bd}(\Sigma_C)\}.
\end{aligned}$$

Given an enumeration $\{\mathcal{S}_\kappa\}_{\kappa=1}^m$ of $\mathrm{Bd}(\Sigma_C)$ we will also write this as

$$\mathcal{D}_C(p_{\mathcal{S}_1}, \cdots, p_{\mathcal{S}_m}) = \mathcal{D}_C(\underline{p}).$$

A density matrix $\rho \in \mathcal{D}_C$ is said to satisfy *maximally mixed boundary conditions* at boundary component $\mathcal{S} \in \mathrm{Bd}(\Sigma_C)$ if

$$\mathfrak{t}_{\mathcal{S}}(u) \rho \mathfrak{t}_{\mathcal{S}}(u^*) = \rho$$

for all unitaries $u \in \mathrm{Tube}_{\mathcal{S}}$. This condition is denoted by $\star$. If in addition

$$\mathrm{Tr}\{\rho \mathfrak{t}_{\mathcal{S}}(P^X)\} = 1,$$

then we say ρ satisfies the maximally mixed boundary condition $\star^X$ of type X at boundary component $\mathcal{S}$.

We allow a boundary condition $\underline{p} = \{p_{\mathcal{S}}\}_{\mathcal{S}}$ to have components $p_{\mathcal{S}} = \star$ or $p_{\mathcal{S}} = \star^X$ for some $X \in \mathrm{Irr}\, Z(\mathcal{C})$, and define $\mathcal{D}_C(\underline{p})$ to consist of those density matrices $\rho \in \mathcal{D}_C$ such that for each $\mathcal{S} \in \mathrm{Bd}(\Sigma_C)$ we have $\mathrm{Tr}\{\rho p_{\mathcal{S}}\} = 1$ in case $p_{\mathcal{S}}$ is a projector, and such that ρ satisfies the maximally mixed boundary conditions at $\mathcal{S}$ in case $p_{\mathcal{S}} = \star$, or maximally mixed boundary conditions of type X at $\mathcal{S}$ if $p_{\mathcal{S}} = \star^X$.

4.7 Characterization of skein subspaces

Let C be a finite connected region. Then there is $m \in \mathbb{N}_0$ so that Σ_C is homeomorphic to the sphere with $m+1$ holes cut out. An anchor ⚓ for Σ_C is said to be an anchor for C if the induced enumeration $\{\mathcal{S}_\kappa^{⚓}\}_{\kappa=0}^m$ is such that $\mathcal{S}_0^{⚓}$ is the outer boundary of σ_C, the outer boundaries of the surfaces Σ_C play a distinguished role and we find it convenient to replace $\mathcal{C}_{⚓}(\Sigma_C)$ by

$$\mathcal{C}^*_{⚓}(\Sigma_C) := \bigoplus_{X_0,\cdots,X_m \in \mathrm{Irr}(Z(\mathcal{C}))} Z(\mathcal{C})(X_0^* \to X_1 \otimes \cdots \otimes X_m) \otimes \mathcal{C}(X_0 \to \chi^{\otimes \mathcal{S}_0^{⚓}}) \otimes \cdots \otimes \mathcal{C}(X_m \to \chi^{\otimes \mathcal{S}_m^{⚓}}), \tag{32}$$

which is naturally isomorphic to $\mathcal{C}_{⚓}(\Sigma_C)$ by composing elements of the factor $Z(\mathcal{C})(X_0^* \to X_1 \otimes \cdots \otimes X_m)$ with a duality morphism. Composing with the isomorphism of Proposition 3.5, this defines a map $\Phi_{\Sigma_C}^{⚓,*} : \mathcal{C}^*_{⚓}(\Sigma_C) \to A(\Sigma_C)$, with

$$\Phi_{\Sigma_C}^{⚓,*}\big(\alpha \otimes w_0 \otimes \cdots \otimes w_m\big) := \Phi_{\Sigma_C}^{⚓}\big(((\mathrm{id}_{X_0} \otimes \alpha) \circ \mathrm{coev}_{X_0}) \otimes w_0 \otimes \cdots \otimes w_m\big) \tag{33}$$

for any $\alpha \in Z(\mathcal{C})(X_0^* \to X_1 \otimes \cdots \otimes X_m)$, and any boundary conditions $w_\kappa \in \mathcal{C}(X_\kappa \to \chi^{\otimes \mathcal{S}_\kappa^{⚓}})$ for $\kappa = 0, \cdots, m$. Combined with the isomorphism of Proposition 4.4, $H_C \simeq A(\Sigma_C)$, this gives:

Proposition 4.6. *Let C be a finite connected region and let ⚓ be an anchor for C. Then the map*

$$\Psi_C^{⚓} := \sigma_C^{-1} \circ \Phi_C^{⚓,*} : \mathcal{C}^*_{⚓}(\Sigma_C) \to H_C \tag{34}$$

is a unitary isomorphism of $A(\partial\Sigma_C)$-modules.

We can now characterise the spaces $\mathcal{D}_C(\underline{p})$ as follows:

Lemma 4.7. *Let C be a finite region and let $\underline{p} = \{p_{\mathcal{S}}\}_{\mathcal{S} \in \mathrm{Bd}(\Sigma_C)}$ with each $p_{\mathcal{S}}$ either a minimal projector of type $X_{\mathcal{S}} \in \mathrm{Irr}\, Z(\mathcal{C})$, or a maximally mixed boundary condition $\star^{X_{\mathcal{S}}}$ of type $X_{\mathcal{S}}$. Let ⚓ be an anchor for C. Then $\mathcal{D}_C(\underline{p})$ is isomorphic to the space of density matrices on $Z(\mathcal{C})(X^*_{\mathcal{S}_0^{⚓}} \to \bigotimes_{\kappa=1}^m X_{\mathcal{S}_\kappa^{⚓}})$.*

Proof : If each $p_{\mathcal{S}}$ is a minimal projector of type $X_{\mathcal{S}}$, then Proposition 4.6 yields an isomorphism,

$$H_C(\underline{p}) \overset{\Psi_C^{⚓}}{\simeq} Z(\mathcal{C})(X_{\mathcal{S}_0^{⚓}} \to \bigotimes_{\kappa=1}^m X_{\mathcal{S}_\kappa^{⚓}}),$$

and the claim follows.

Let us now consider the case with one maximally mixed boundary condition $p_{\mathcal{S}_*} = \star^{X_{\mathcal{S}_*}}$ and all other $p_{\mathcal{S}}$ given by minimal projections of types $X_{\mathcal{S}}$. Proposition 4.6 yields an isomorphism of Tube-modules

$$H_C(\underline{p}) \overset{\Psi_C^{⚓}}{\simeq} Z(\mathcal{C})(X^*_{\mathcal{S}_0^{⚓}} \to \bigotimes_{\kappa=1}^m X_{\mathcal{S}_\kappa^{⚓}}) \otimes \mathcal{C}(X_{\mathcal{S}_*} \to \chi^{\otimes \mathcal{S}_*}).$$

Density matrices $\rho \in \mathcal{D}_C(\underline{p})$ are supported on $H_C(\underline{p})$ and therefore in one-to-one correspondence with density matrices $\rho' = \big(\Psi_C^{⚓}|_{H_C(\underline{p})}\big)^{-1} \rho\, \Psi_C^{⚓}|_{H_C(\underline{p})}$ acting on $Z(\mathcal{C})(X^*_{\mathcal{S}_0^{⚓}} \to \bigotimes_{\kappa=1}^m X_{\mathcal{S}_\kappa^{⚓}}) \otimes \mathcal{C}(X_{\mathcal{S}_*} \to \chi^{\otimes \mathcal{S}_*})$. The algebra $\mathrm{Tube}_{\mathcal{S}_*}$ acts on the latter space by arbitrary unitaries of the form $\mathbb{1} \otimes U$ since the representation on $\mathcal{C}(X_{\mathcal{S}_*} \to \chi^{\otimes \mathcal{S}_*})$ is irreducible. The maximally mixed boundary condition $\star^{X_{\mathcal{S}_*}}$ asserts that ρ' commutes with all unitaries of this form. This implies that $\rho' = \sigma \otimes \mathbb{1}$ for a density matrix σ supported on the space $Z(\mathcal{C})(X^*_{\mathcal{S}_0^{⚓}} \to \bigotimes_{\kappa=1}^m X_{\mathcal{S}_\kappa^{⚓}})$. This shows that the space of density matrices $\mathcal{D}_C(\underline{p})$ is isomorphic to the space of density matrices on $Z(\mathcal{C})(X^*_{\mathcal{S}_0^{⚓}} \to \bigotimes_{\kappa=1}^m X_{\mathcal{S}_\kappa^{⚓}})$, as required.

The case with multiple maximally mixed boundary conditions is obtained similarly. □

Remark 4.8. *The spaces $\mathcal{D}_C(\underline{p})$, with each $p_{\mathcal{S}} = \star$ given by the maximally mixed boundary condition, are precisely the information convex sets of the entanglement bootstrap program [SKK20]. Indeed, the information convex sets of [SKK20] consist of density matrices supported on some region C, obtained by tracing out the boundary degrees of freedom of density matrices in $\mathcal{D}_{\widetilde{C}}$, where $\widetilde{C}$ is a slight 'fattening' of C. It follows from Lemma 4.9 below that such density matrices satisfy maximally mixed boundary conditions on all boundary components.*

By applying Lemma 4.7 to annuli and twice punctured disks, we therefore obtain the simple anyon types and the fusion rules of the Levin-Wen model in the framework of the entanglement bootstrap program.

4.8 Restrictions yield maximally mixed boundary conditions

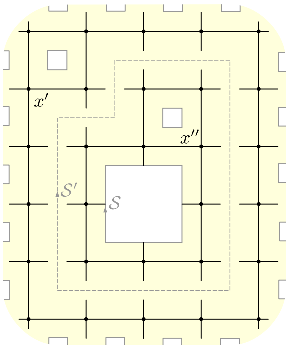
Maximally mixed boundary conditions arise when density matrices supported on skein subspaces of some region are restricted to certain smaller regions.

Lemma 4.9. *Let $C' \subset C$ be finite regions such that $\Sigma_C \setminus \Sigma_{C'}$ is an annulus. Then there are unique boundary components $\mathcal{S} \in \mathrm{Bd}(\Sigma_C) \setminus \mathrm{Bd}(\Sigma_{C'})$, and $\mathcal{S}' \in \mathrm{Bd}(\Sigma_{C'}) \setminus \mathrm{Bd}(\Sigma_C)$. Suppose $\rho \in \mathcal{D}_C$ satisfies boundary conditions of type $X \in \mathrm{Irr}\, Z(\mathcal{C})$ at boundary component $\mathcal{S}$. Let $\rho' = \mathrm{Tr}_{C_0 \setminus C_0'}\{\rho\}$ be the restriction of ρ to the region C'. Then $\rho' \in \mathcal{D}_{C'}$ satisfies maximally mixed boundary conditions of type X at boundary component $\mathcal{S}'$.*

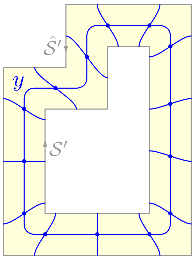
Proof : By hypothesis $\Sigma_C \setminus \Sigma_{C'}$ has two connected boundary components $\mathcal{S}$ and $\hat{\mathcal{S}}'$ corresponding to those of Σ_C and $\Sigma_{C'}$ as stated. Let C'' be the region consisting of all faces, edges, and vertices contained in the annulus $\Sigma_C \setminus \Sigma_{C'}$.

Consider a product state $\phi = \phi_{x'} \otimes \phi_{x''} \in H_{C_1'} \otimes H_{C_1''}$ corresponding to string diagrams x' on $\Sigma_{C_1'}$ and x'' on $\Sigma_{C_1''}$, and $a \in \mathrm{Tube}_{\mathcal{S}'}$ corresponding to a string diagram y on $\mathcal{S}' \times I$. In the case of an inner boundary, we may illustrate the situation as follows (recall that we use Convention 4.2 to interpret string diagrams as vectors in H_{C_1}):

$$\phi :=$$



$$, \quad a := \left(\frac{d_{y_{\mathcal{S}'}}}{d_{y_{\hat{\mathcal{S}}'}}}\right)^{1/4} \times$$

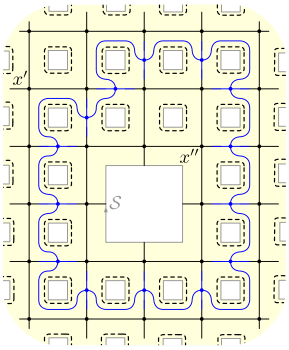


.

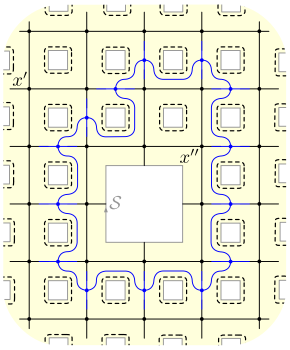
The normalization factor of a is chosen to cancel the factor depending on boundary conditions

(Eq. (26)) coming from the actions $\mathfrak{t}_{\mathcal{S}'}$ and $\mathfrak{t}^{\mathrm{op}}_{\hat{\mathcal{S}}'}$, so that we can represent,

$$B_C \mathfrak{t}_{\mathcal{S}'}(a)\phi =$$



$$, \quad B_C \mathfrak{t}^{\mathrm{op}}_{\hat{\mathcal{S}}'}(a)\phi =$$



.

Note that both these vectors are zero if any of the boundary conditions of the string diagrams x', x'' and y do not match where they are glued together. By using the cloaking property of the dotted loops within each face that sits on $\mathcal{S}'$, we find that $B_C \mathfrak{t}_{\mathcal{S}'}(a)\phi = B_C \mathfrak{t}^{\mathrm{op}}_{\hat{\mathcal{S}}'}(a)\phi$. By linearity, this equality extends to any $a \in \mathrm{Tube}_{\mathcal{S}'}$ and any $\phi \in H_{C_1'} \otimes H_{C_1''}$. In fact, this equality holds for any $\phi \in \mathcal{H}_C$ because if any string-net constraint in C' or in C'' is violated, then ϕ is annihilated either by a Tube-action or by B_C. We conclude that

$$B_C \mathfrak{t}_{\mathcal{S}'}(a) = B_C \mathfrak{t}^{\mathrm{op}}_{\hat{\mathcal{S}}'}(a). \tag{35}$$

By taking adjoints, we find that also $\mathfrak{t}_{\mathcal{S}'}(a) B_C = \mathfrak{t}^{\mathrm{op}}_{\hat{\mathcal{S}}'}(a) B_C$. Using these relations, the fact that $\rho = \rho B_C = B_C \rho$, and the fact that $\mathfrak{t}_{\mathcal{S}'}(a)$ is supported on C' while $\mathfrak{t}^{\mathrm{op}}_{\hat{\mathcal{S}}'}(a)$ is supported on C'', we find that

$$\begin{aligned} \rho' \, \mathfrak{t}_{\mathcal{S}'}(a) &= \mathrm{Tr}_{\mathcal{H}_{C''}}\{\rho \mathfrak{t}_{\mathcal{S}'}(a)\} = \mathrm{Tr}_{\mathcal{H}_{C''}}\{\rho B_C \mathfrak{t}_{\mathcal{S}'}(a)\} = \mathrm{Tr}_{\mathcal{H}_{C''}}\{\rho B_C \mathfrak{t}^{\mathrm{op}}_{\hat{\mathcal{S}}'}(a)\} \\ &= \mathrm{Tr}_{\mathcal{H}_{C''}}\{\mathfrak{t}^{\mathrm{op}}_{\hat{\mathcal{S}}'}(a) B_C \rho\} = \mathrm{Tr}_{\mathcal{H}_{C''}}\{\mathfrak{t}_{\mathcal{S}'}(a)\rho\} = \mathfrak{t}_{\mathcal{S}'}(a)\rho' \end{aligned}$$

for any $a \in \mathrm{Tube}_{\mathcal{S}'}$. Thus ρ' belongs to the commutant of $\mathfrak{t}_{\mathcal{S}'}(\mathrm{Tube}_{\mathcal{S}'})$ in $\mathcal{B}(\mathcal{H}_{C'})$ which immediately implies $\rho' = \mathfrak{t}_{\mathcal{S}'}(u)\rho' \, \mathfrak{t}_{\mathcal{S}'}(u^*)$ for any unitary $u \in \mathrm{Tube}_{\mathcal{S}'}$.

Let us finally show that ρ' satisfies boundary conditions of type X at the boundary component $\mathcal{S}'$. By assumption, we have $\mathrm{Tr}_{\mathcal{H}_C}\{\rho \mathfrak{t}_{\mathcal{S}}(P^X)\} = 1$. Writing $\rho'' = \mathrm{Tr}_{\mathcal{H}_{C'}}\{\rho\}$ for the restriction of ρ to the annular region C'', we immediately obtain that $\mathrm{Tr}_{\mathcal{H}_{C''}}\{\rho'' \, \mathfrak{t}_{\mathcal{S}}(P^X)\} = 1$. It follows that ρ'' belongs to $\mathcal{D}_{C''}(P^X, \mathrm{id})$, where the first slot corresponds to the boundary component $\mathcal{S}$ and the second slot corresponds to the boundary component $\hat{\mathcal{S}}'$. By Proposition 4.6, in fact $\rho'' \in \mathcal{D}_{C''}(P^X, P^{\bar{X}})$, meaning that $\mathrm{Tr}_{\mathcal{H}_{C''}}\{\rho'' \, \mathfrak{t}_{\hat{\mathcal{S}}'}(P^{\bar{X}})\} = 1$.

Using (35), and the fact that $(P^X)^{\mathrm{op}} = P^{\bar{X}}$ for any minimal central projector P^X of a Tube algebra, we get

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}_{C'}}\{\rho' \, \mathfrak{t}_{\mathcal{S}'}(P^X)\} &= \mathrm{Tr}_{\mathcal{H}_C}\{\rho \, B_C \, \mathfrak{t}_{\mathcal{S}'}(P^X)\} = \mathrm{Tr}_{\mathcal{H}_C}\{\rho \, B_C \, \mathfrak{t}^{\mathrm{op}}_{\hat{\mathcal{S}}'}(P^X)\} \\ &= \mathrm{Tr}_{\mathcal{H}_C}\{\rho \, B_C \, \mathfrak{t}_{\hat{\mathcal{S}}'}(P^{\bar{X}})\} = \mathrm{Tr}_{\mathcal{H}_{C''}}\{\rho'' \, \mathfrak{t}_{\hat{\mathcal{S}}'}(P^{\bar{X}})\} = 1 \end{aligned}$$

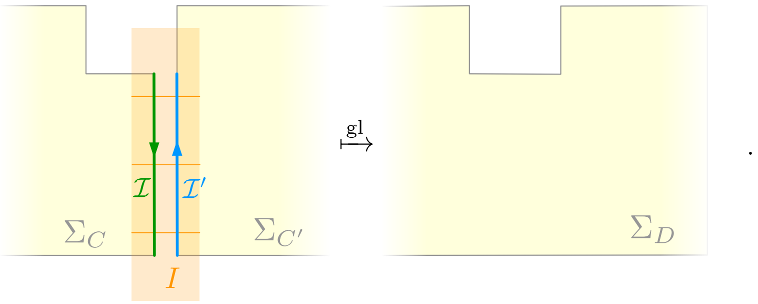
as desired. □

4.9 Gluing along intervals

A dual path $I = \{f_i\}_{i=1}^{l}$ is a sequence of faces such that f_i neighbours f_{i+1} for all $i = 1, \cdots, l-1$. This dual path is self-avoiding if all the f_i are distinct. We write $\partial_{\mathfrak{i}} I = f_1$ and $\partial_{\mathfrak{f}} I = f_l$. We also write $I_{\mathrm{in}} = \{f_i\}_{i=2}^{l-1}$ for the dual path obtained from I by removing the first and last face. (I_{in} can be empty). We write $\mathcal{E}_I \subset \mathcal{E}$ for the set of edges which lie between successive faces of I.

Let D be a finite connected region. We say D can be cut along a self-avoiding dual path I if all faces of I_{in} and all edges in $\mathcal{E}_I$ belong to D, and the initial and final faces $\partial_{\mathfrak{i}} I$ and $\partial_{\mathfrak{f}} I$ of I are distinct and sit on the outer boundary of D. Write $D \setminus I$ for the region obtained from D by removing the faces in I_{in} as well as all the edges between successive faces of I. Then $D \setminus I$ is the union of two uniquely determined disjoint finite connected regions C and C'. Moreover, each inner boundary component of Σ_D is either an inner boundary component of C or an inner boundary component of C'. We write $D = C \sqcup_I C'$ and say D is obtained by gluing C and C' along I.

The subregions C and C' have associated surfaces Σ_C and $\Sigma_{C'}$ whose outer boundary components both contain a boundary interval (see Section 3.3) $\mathcal{I}$ and $\mathcal{I}'$ respectively, determined by I as follows. Let $m_{C,I}$ be the set of marked boundary points of Σ_C which lie on one of the edges of $\mathcal{E}_I$. Define $m_{C',I}$ similarly. Then we may define $\mathcal{I}$ as the smallest closed subinterval of $\partial\Sigma_C$ that contains $B_{1/3}(m)$ for each $m \in m_{C,I}$, where $B_r(x) \subset \mathbb{R}^2$ is the closed ball or radius r centred on $x \in \mathbb{R}^2$. The boundary interval $\mathcal{I}'$ is defined similarly. Moreover, $\mathcal{I}$ can be glued to $\mathcal{I}'$ to obtain a new surface homeomorphic to Σ_D as extended surfaces. We will identify the glued surface with Σ_D. Denote by $\mathrm{gl} : A(\Sigma_C \sqcup \Sigma_{C'}) \simeq A(\Sigma_C) \otimes A(\Sigma_{C'}) \to A(\Sigma_D)$ the associated gluing map as described in Section 3.3:



Let $B_I = \prod_{f \in I_{\mathrm{in}}} B_f$.

Lemma 4.10. *Let $D = C \sqcup_I C'$ be as above. For any $\phi \in H^b_{C_1}$ and $\phi' \in H^{b'}_{C'_1}$ we have*

$$\big(\mathrm{gl} \circ (\sigma_C \otimes \sigma_{C'})\big)(\phi \otimes \phi') = \mathcal{D}^{|I_{\mathrm{in}}|}\, d_{b_{\mathcal{I}}}^{-1/2}\, \sigma_D\big(B_I\, \phi \otimes \phi'\big). \tag{36}$$

Proof : By construction of the gluing map gl and Lemma B.4 we have

$$\mathrm{gl} \circ (\pi_C \otimes \pi_{C'}) = \pi_D = \pi_D \circ B_D.$$

To finish the proof it therefore suffices to verify that normalization factors relating the σ's to the π's depending on boundary conditions agree. For $\phi \in H^b_{C_1}$ and $\phi' \in H^{b'}_{C'_1}$ we find

$$\sigma_C(\phi) = \mathcal{D}^{-|F_C|}\, d_b^{1/4}\, \pi_C(\phi), \quad \sigma_{C'}(\phi') = \mathcal{D}^{-|F_{C'}|}\, d_{b'}^{1/4}\, \pi_{C'}(\phi),$$

and

$$(\sigma_D \circ B_D)(\phi \otimes \phi') = \delta_{b_{\mathcal{I}}, b'_{\mathcal{I}'}}\, \mathcal{D}^{-|F_D|} \left(\frac{d_b d_{b'}}{d_{b_{\mathcal{I}}}^2} \right)^{1/4} (\pi_D \circ B_D)(\phi \otimes \phi').$$

The claim follows by noting that $|F_D| - |F_C| - |F_{C'}| = |I_{\mathrm{in}}|$ and that gl also imposes compatible boundary conditions $\delta_{b_{\mathcal{I}}, b'_{\mathcal{I}'}}$. □

5 Anyon States

5.1 Unique anyon states from local constraints

Let C be an infinite region such that Σ_C is homeomorphic to $\mathbb{R}^2$ with an open disk removed. The associated surface Σ_C has a single connected boundary component $\mathcal{S}$. We denote the corresponding $\mathrm{Tube}_{\mathcal{S}}$ action by $\mathfrak{t} = \mathfrak{t}_{\mathcal{S}}$ for the remainder of this section.

Note that Lemma 4.5 implies that $\mathfrak{t}(a)$ commutes with B_f for all $a \in \mathrm{Tube}_{\mathcal{S}}$ and all $f \in F_C$.

Definition 5.1. *Let $\mathcal{S}_C$ be the set of states ω on $\mathcal{A}_C$ for which $\omega(B_f) = 1$ for all $f \in F_C$.*

For any projector $p \in \mathrm{Tube}_{\mathcal{S}}$ we let $\mathcal{S}_C^p$ be the set of states ω on $\mathcal{A}_C$ such that $\omega \in \mathcal{S}_C$ and $\omega(\mathfrak{t}(p)) = 1$.

We further define $\mathcal{S}_C^{\star}$ to be the set of states $\omega \in \mathcal{S}_C$ for which $\omega = \omega \circ \mathrm{Ad}[\mathfrak{t}(u)]$ for all unitaries $u \in \mathrm{Tube}_{\mathcal{S}}$.

For any $X \in \mathrm{Irr}\, Z(\mathcal{C})$ we let $\mathcal{S}_C^{\star^X} = \mathcal{S}_C^{\star} \cap \mathcal{S}_C^{P^X}$. We say that $\omega \in \mathcal{S}_C^{\star^X}$ satisfies maximally mixed boundary conditions of type X.

Proposition 5.2. *If $p \in \mathrm{Tube}_{\mathcal{S}}$ is a minimal projector, then $\mathcal{S}_C^p$ consists of a single pure state. Similarly, the state space $\mathcal{S}_C^{\star^X}$ consists of a single state (which however need not be pure).*

Proof : We first prove the claim about $\mathcal{S}_C^p$. For any $R > 0$, denote by $C(R)$ the subregion of C consisting of all vertices, edges and faces of C that are contained in $B_R = \{v \in \mathbb{R}^2 \, : \, \|v\|_\infty \leq R\}$.

There is an R_0 such that for all $R \geq R_0$ large enough, the associated surface $\Sigma_{C(R)}$ is homeomorphic to an annulus. The inner boundary component of $\Sigma_{C(R)}$ is the same as the unique boundary component of Σ_C, and so comes with the same $\mathrm{Tube}_{\mathcal{S}}$ representation $\mathfrak{t}$. Let $\mathcal{S}_R$ be the outer boundary component of $\Sigma_{C(R)}$ and denote by $\mathfrak{t}_R = \mathfrak{t}_{\mathcal{S}_R}$ the $\mathrm{Tube}_{\mathcal{S}_R}$ action associated to that boundary component.

Since p is minimal, it has definite type X for some $X \in \mathrm{Irr}\, Z(\mathcal{C})$. By Lemma 4.7 there is a unique density matrix in $\mathcal{D}_{C(R)}(p, \star^{\bar{X}})$ defining a state ψ_R on $\mathcal{A}_{C(R)}$ for all $R \geq R_0$. By Lemma 4.9, $\psi_{R'}|_{C(R)} = \psi_R$ whenever $R' > R$, so it is clear that $\psi(O) = \lim_{R \uparrow \infty} \psi_R(O)$ defines a state on $\mathcal{A}_C^{\mathrm{loc}}$ which extends uniquely to a state $\psi : \mathcal{A}_C \to \mathbb{C}$. By construction $\psi_R(\mathfrak{t}(p)) = 1$ for all $R > R_0$, and $\psi_R(B_f) = 1$ for $f \in F_C$ whenever R is large enough, so $\psi \in \mathcal{S}_C^p$.

To see that ψ is the only state in $\mathcal{S}_C^p$ it suffices to note that the restriction of any state in $\mathcal{S}_C^p$ to the region $C(R)$ for $R \geq R_0$ corresponds to the unique density matrix in $\mathcal{D}_{C(R)}(p, \star^X)$.

To see that ψ is pure, suppose $\psi = \lambda \psi_1 + (1-\lambda)\psi_2$ for some $\lambda \in (0, 1)$. For any projector q, if $\psi(q) = 1$ then $\lambda \psi_1(q) + (1-\lambda)\psi_2(q) = 1$, which can only hold if $\psi_1(q) = \psi_2(q) = 1$. Applying this to $q = \mathfrak{t}(p)$ and $q = B_f$ it follows that ψ_1 and ψ_2 both belong to $\mathcal{S}_C^p$ as well. By the uniqueness we conclude that $\psi_1 = \psi_2 = \psi$, so ψ is pure.

The claim about $\mathcal{S}_C^{\star^X}$ is shown in exactly the same way, using that the spaces $\mathcal{D}_{C(R)}(\star^X, \star^{\bar{X}})$ all contain a unique density matrix by Lemma 4.7. (Purity of the unique state in $\mathcal{S}_C^{\star^X}$ does not follow in this case because, unlike $\mathcal{S}_C^p$, the space $\mathcal{S}_C^{\star^X}$ is not defined by commuting projector constraints). □

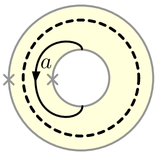
5.2 Pure anyons at punctures

Consider the infinite region $C^{(e)}$ obtained from $C^{\mathbb{Z}^2}$ by removing the edge e and the two neighbouring faces. This region satisfies the requirements of Proposition 5.2. In particular, the associated surface $\Sigma_{C^{(e)}}$ has a single connected boundary component $\mathcal{S}_e$, referred to as the puncture at e, with an associated collar region $C^{\mathcal{S}_e}$ and $\mathrm{Tube}_{\mathcal{S}_e}$-representation denoted by $\mathfrak{t}_e = \mathfrak{t}_{\mathcal{S}_e}$.

It follows from Proposition 5.2 that for any minimal projector $p \in \mathrm{Tube}_{\mathcal{S}_e}$ we have a unique pure state $\omega_e^p \in \mathcal{S}^p_{C(e)}$. Note that $\mathcal{A}_{C(e)} = \mathcal{A}$, so ω_e^p is a state on the full quasi-local algebra.

Let us finally argue how we get the existence, uniqueness, and purity of the frustration free ground state $\omega^{\mathbb{1}}$ (Proposition 2.1) as a corollary of Proposition 5.2. Let

$$p^{\mathbb{1}} := vv^*, \quad \text{where} \quad v := \frac{1}{\mathcal{D}} \sum_{a \in \operatorname{Irr} \mathcal{C}} d_a^{1/2}$$



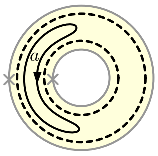
$$\in \mathrm{Tube}_{\mathcal{S}_e}, \tag{37}$$

where we present $\mathcal{S}_e \times I$ as an annulus whose outer boundary is identified with the bottom of $\mathcal{S}_e \times I$.

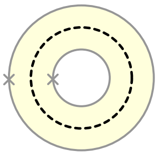
Lemma 5.3. *The element $p^{\mathbb{1}} \in \mathrm{Tube}_{\mathcal{S}_e}$ is a minimal projector of type $\mathbb{1}$.*

Proof : We compute

$$v^*v = \frac{1}{\mathcal{D}^2} \sum_a d_a$$



$$=$$

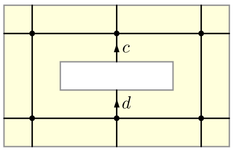


$$, \tag{38}$$

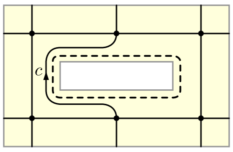
which is a minimal projector of type $\mathbb{1}$ by Proposition 3.4. This implies that v is a partial isometry, and that $p^{\mathbb{1}} = vv^*$ is unitarily equivalent to v^*v, and is therefore also a minimal projector of type $\mathbb{1}$. □

Lemma 5.4. *Let f_1 and f_2 be the faces neighbouring a given edge e. Then $\mathfrak{t}_e(p^{\mathbb{1}}) = B_{f_1} B_{f_2}$.*

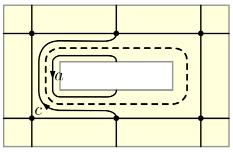
Proof : Write $V_e = \mathfrak{t}_e(v)$, then (using Convention 4.2)



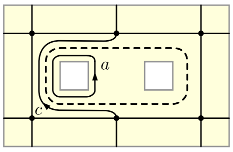
$$\xmapsto{V_e^*} \frac{\delta_{c,d}}{\mathcal{D}}$$



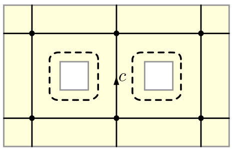
$$\xmapsto{V_e} \frac{\delta_{c,d}}{\mathcal{D}^2} \sum_a d_a$$



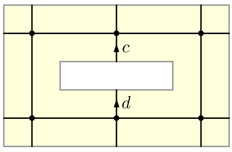
$$= \frac{\delta_{c,d}}{\mathcal{D}^2} \sum_a d_a$$



$$= \delta_{c,d}$$



$$= B_{f_1} B_{f_2}$$



$$,$$

where in applying V_e^* we noted that the factor $d_c^{1/2}$ coming from the definition of v cancels the factor $d_c^{-1/2}$ due to the dependence of $\mathfrak{t}_e$ on boundary conditions (see Eq. (26)), and in applying V_e we get one factor of $d_a^{1/2}$ from the definition of v and other factor of $d_a^{1/2}$ from the boundary conditions. This shows that $\mathfrak{t}_e(p^{\mathbb{1}}) = V_e V_e^* = B_{f_1} B_{f_2}$ on the string-net subspace. If any of the string net constraints is violated, left and right hand sides of the claimed equality both evaluate to zero. This proves the Lemma. ☐

Proof of Proposition 2.1 : It follows from Lemma 5.4 that the space $\mathcal{S}^{p^{\mathbb{1}}}_{C^{(e)}}$ consists of those states ω on $\mathcal{A}$ for which

$$1 = \omega(B_f) = \omega(B_{f_1} B_{f_2})$$

for all faces f belonging to $C^{(e)}$, and where f_1 and f_2 are the two faces neighbouring the edge e. Using the Cauchy-Schwarz inequality we find

$$|\omega(B_{f_1} - B_{f_1}B_{f_2})|^2 = |\omega(B_{f_1}(\mathbb{1} - B_{f_1}B_{f_2}))|^2 \leq \omega(B_{f_1})\omega(\mathbb{1} - B_{f_1}B_{f_2}) = 0,$$

where we used that B_{f_1} and B_{f_2} are commuting projectors (Lemma 4.5). It follows that $\omega(B_{f_1}) = \omega(B_{f_1}B_{f_2}) = 1$ and therefore also $\omega(B_{f_2}) = 1$. We conclude that $\mathcal{S}^{p^{\mathbb{1}}}_{C^{(e)}}$ consists precisely of the frustration free ground states of the Levin-Wen Hamiltonian. By Proposition 5.2 and Lemma 5.3 we find that $\mathcal{S}^{p^{\mathbb{1}}}_{C^{(e)}}$ contains a single pure state $\omega^{p^{\mathbb{1}}}_e$, which is therefore the unique frustration free ground state of the Levin-Wen Hamiltonian. ☐

5.3 Restrictions of infinite volume states

Restrictions lead to maximally mixed boundary conditions also for infinite volume states. We formalise this in the present setting.

Lemma 5.5. *Let C be an infinite region so that Σ_C is homeomorphic to $\mathbb{R}^2$ with an open disk removed. Let $C' \subset C$ be such that $\Sigma_C \setminus \Sigma_{C'}$ is homeomorphic to an annulus. If $p \in \mathrm{Tube}_{\partial C}$ is a projection of type $X \in \operatorname{Irr} Z(\mathcal{C})$ and $\omega \in \mathcal{S}^p_C$, then $\omega|_{\mathcal{A}_{C'}} \in \mathcal{S}^{\star^X}_{C'}$.*

Proof : It suffices to show that $\omega|_{\mathcal{A}_{C'}}$ satisfies maximally mixed boundary conditions at the unique $\mathcal{S}' \in \mathrm{Bd}(\Sigma_{C'})$. This follows by applying Lemma 4.9 to the density matrix corresponding to the restriction of ω to a finite region $\widetilde{C} \subset C$, chosen such that $\mathcal{S} \in \mathrm{Bd}(\Sigma_{\widetilde{C}})$ and such that if $\widetilde{C}' = \widetilde{C} \cap C'$ then $\Sigma_{\widetilde{C}} \setminus \Sigma_{\widetilde{C}'}$ is an annulus and $\mathcal{S}' \in \mathrm{Bd}(\Sigma_{\widetilde{C}'})$. ☐

6 String Operators

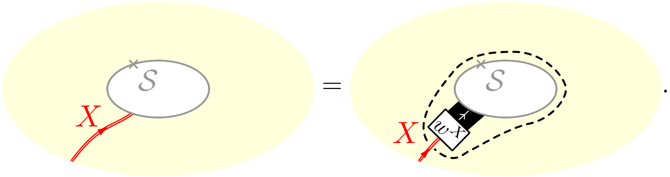
6.1 Drinfeld insertions

For each $X \in \operatorname{Irr} Z(\mathcal{C})$ and each extended circle $\mathcal{S}$ we fix a unit vector $w^X_{\mathcal{S}} \in \mathcal{C}(X \to \chi^{\otimes \mathcal{S}})$ with respect to the trace inner product. We write $p^X_{\mathcal{S}} \in \mathrm{Tube}_{\mathcal{S}}$ for the corresponding minimal projector (see Proposition 3.4). In the case where $X = \mathbb{1}$ and $\mathcal{S} = \mathcal{S}_e$, we specify

$$w^{\mathbb{1}}_{\mathcal{S}_e} = \frac{1}{\mathcal{D}} \sum_{a \in \operatorname{Irr} \mathcal{C}} d_a^{1/2} \qquad (39)$$

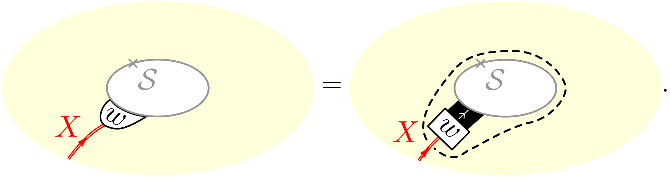
so that the corresponding minimal projector $p^{\mathbb{1}}_{\mathcal{S}_e} = p^{\mathbb{1}}$ is the projector introduced in Eq. (37). (Note that all boundary components $\mathcal{S}_e$ for $e \in \mathcal{E}$ are isomorphic as extended circles). Since the extended circle $\mathcal{S}$ will always be clear from context, it will always be dropped from the notation, so that w^X stands for $w^X_{\mathcal{S}}$ and p^X stands for $p^X_{\mathcal{S}}$.

Convention 6.1. *Having fixed the boundary conditions* w^X *for* $X \in \operatorname{Irr} Z(\mathcal{C})$ *we introduce the following graphical convention for string diagrams:*



(40)

Given an arbitrary boundary condition $w \in \mathcal{C}(X \to \chi^{\otimes \mathcal{S}})$ *we also depict*

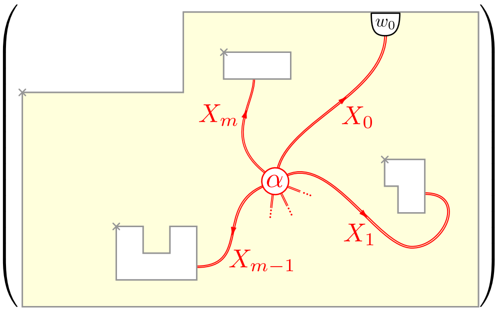


(41)

Let C be a finite connected region and let ⚓ be an anchor for C. Proposition 4.6 implies that vectors of the the form

$$\Psi_C^{⚓}(\alpha \otimes w_0 \otimes w^{X_1} \otimes \cdots \otimes w^{X_m})$$

$$= \left(\prod_{\kappa=0}^{m} d_{X_\kappa} \right)^{1/2} \mathcal{D}^m \times \sigma_C^{-1} \left(\vphantom{\prod} \right),$$



for $X_0 \in \operatorname{Irr} Z(\mathcal{C})$, $w_0 \in \mathcal{C}(X_0 \to \chi^{\otimes \mathcal{S}_0^{⚓}})$, and $\alpha \in Z(\mathcal{C})(X_0^* \to X_1 \otimes \cdots \otimes X_m)$ span the subspace

$$H_C^{(X_1,\cdots,X_m)} := H_C(\mathrm{id}, p^{X_1}, \cdots, p^{X_m}) \tag{42}$$

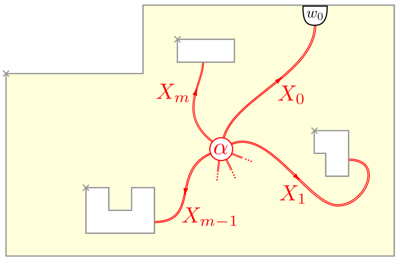
defined in Eq. (31). Here we use the enumeration of connected boundary components induced by ⚓.

For any $\beta \in Z(\mathcal{C})(X_1 \otimes \cdots \otimes X_m \to Y_1 \otimes \cdots \otimes Y_m)$ we define the Drinfeld insertion $\mathrm{Dr}_C^{⚓}[\beta] \in \mathcal{A}_C$ which acts on $H_C^{(X_1,\cdots,X_m)}$ as

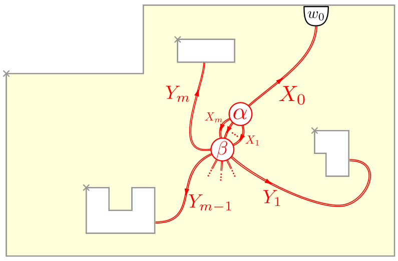
$$\begin{aligned} \mathrm{Dr}_C^{⚓}[\beta] : \Psi_C^{⚓}(\alpha \otimes w_0 \otimes w^{X_1} \otimes \cdots \otimes w^{X_m}) \\ \mapsto \Psi_C^{⚓}\big((\beta \circ \alpha) \otimes w_0 \otimes w^{Y_1} \otimes \cdots \otimes w^{Y_m}\big), \end{aligned} \tag{43}$$

and annihilates the orthogonal complement of $H_C^{(X_1,\cdots,X_m)} \subset \mathcal{H}_C$. Graphically, the action of

$\sigma_C \circ \mathrm{Dr}_C^{⚓}[\beta] \circ \sigma_C^{-1} : A(\Sigma_C) \to A(\Sigma_C)$ is given by



$$\mapsto \left(\prod_{\kappa=1}^{m} \frac{d_{Y_\kappa}}{d_{X_\kappa}} \right)^{1/2}$$



. (44)

Lemma 6.2 (Multiplicativity). *Let C be a finite connected region with $|\mathrm{Bd}(\Sigma_C)| = m+1$ and ⚓ an anchor for C. If $\Theta = \bigoplus_{X\in\mathrm{Irr}(Z(\mathcal{C}))} X$, then $\mathrm{Dr}_C^{⚓}$ defines a *-representation of $\mathrm{End}_{Z(\mathcal{C})}(\Theta^{\otimes m})$ on*

$$\bigoplus_{X_1,\ldots,X_m\in\mathrm{Irr}(Z(\mathcal{C}))} H_C^{(X_1,\cdots,X_m)}. \tag{45}$$

Proof : We have that

$$\mathrm{End}_{Z(\mathcal{C})}(\Theta^m) \simeq \bigoplus_{\substack{X_1,\ldots,X_m\in\mathrm{Irr}(Z(\mathcal{C}))\\ Y_1,\ldots,Y_m\in\mathrm{Irr}(Z(\mathcal{C}))}} Z(\mathcal{C})\left(X_1\otimes\cdots\otimes X_m \to Y_1\otimes\cdots\otimes Y_m\right).$$

Let β be a morphism belonging to one of these direct summands. It is clear that $\mathrm{Ran}\,\mathrm{Dr}_C^{⚓}[\beta] \subset H_C^{(Y_1,\cdots,Y_m)}$, and using that $\Psi_C^{⚓}$ is an isomorphism of Hilbert spaces we find $(\mathrm{Dr}_C^{⚓}[\beta])^* = \mathrm{Dr}_C^{⚓}[\beta^\dagger]$. Multiplicativity is immediate from the definition (43), and $\mathrm{Dr}_C^{⚓}[\mathrm{id}_{\Theta^{\otimes m}}]$ is the projection onto the subspace in (45). ☐

We say two anchors are equivalent up to $\mathcal{S} \in \mathrm{Bd}(\Sigma)$, if they induce the same enumeration on $\mathrm{Bd}(\Sigma)$ and if, after removing the edge connected to $\mathcal{S}$ from either anchor, the remaining subgraphs are equivalent in the sense of equivalence of anchors. That is, one *subanchor* may be transformed into the other by isotopy of Σ which fixes all fiducial points.

Lemma 6.3. *If C is a finite connected region and ⚓, ⚓′ are anchors on Σ_C, equivalent up to the outer boundary $\mathcal{S}_0 \in \mathrm{Bd}(\Sigma_C)$, then $\mathrm{Dr}_C^{⚓} = \mathrm{Dr}_C^{⚓'}$.*

Proof : Let $\epsilon_{⚓} \subset ⚓, \epsilon_{⚓'} \subset ⚓'$ denote the edges connecting the anchor point to $\mathcal{S}_0$ in either anchor. Since $\Psi_C^{⚓}$ depends only on the equivalence class of the anchor, the same is true for the representation $\mathrm{Dr}_C^{⚓}$. Therefore, without loss of generality, we may assume that $⚓ \backslash \epsilon_{⚓} = ⚓' \backslash \epsilon_{⚓'}$. Moreover, we may assume that the edges $\epsilon_{⚓}, \epsilon_{⚓'}$ have the same attaching point on $\mathcal{S}_0$. Cutting Σ_C along the common subanchor will connect all the interior boundary components, producing a surface homeomorphic to an annulus. Since ⚓, ⚓′ induce the same enumeration of $\mathrm{Bd}(\Sigma_C)$, the edges $\epsilon_{⚓}, \epsilon_{⚓'}$ have the same attaching point on the unique interior boundary of the cut surface. Since the circle S^1 is a deformation retract of the annulus, the homotopy-classes of paths between two fixed points on either boundary component of the annulus is a $\pi_1(S^1)$-torsor. The generator is (the isotopy class of) a Dehn twist around $\mathcal{S}_0$. It follows by Lemma 3.7, that there is a $z \in \mathbb{Z}$ such that

$$\Psi_C^{⚓'}\left(\alpha\otimes w_0\otimes w^{X_1}\otimes\cdots\otimes w^{X_m}\right) = \theta_{X_0}^{z} \times \Psi_C^{⚓}\left(\alpha\otimes w_0\otimes w^{X_1}\otimes\cdots\otimes w^{X_m}\right)$$

for all $\alpha \in Z(\mathcal{C})(X_0^* \to X_1 \otimes \cdots \otimes X_m)$ and all $w_0 \in \mathcal{C}(X_0 \to \chi^{\otimes \mathcal{S}_0^{⚓}})$.

That $\mathrm{Dr}_C^{⚓}[\beta] = \mathrm{Dr}_C^{⚓\prime}[\beta]$ for any $\beta \in Z(\mathcal{C})(X_1 \otimes \cdots \otimes X_m \to Y_1 \otimes \cdots \otimes Y_m)$ now follows from the fact that these operators do not change the object X_0 assigned to the outer boundary component. ☐

Lemma 6.4. *The actions* $\mathrm{Dr}_C^{⚓}$ *and* $\mathfrak{t}_{\mathcal{S}_0^{⚓}}$ *commute for any finite connected region* C *with anchor* ⚓. *That is,* $[\mathrm{Dr}_C^{⚓}[\beta], \mathfrak{t}_{\mathcal{S}_0^{⚓}}(a)] = 0$ *for all* $a \in \mathrm{Tube}_{\mathcal{S}_0^{⚓}}$ *and* $\beta \in \mathrm{End}_{Z(\mathcal{C})}(\Theta^{\otimes m})$*, where* $|\mathrm{Bd}(\Sigma_C)| = m+1$*. The same holds if* $\mathfrak{t}_{\mathcal{S}_0^{⚓}}$ *is replaced by* $\mathfrak{t}_{\mathcal{N}}$ *for a decorated submanifold* $\mathcal{N} \subset \mathcal{S}_0^{⚓}$*.*

Proof : Let us write $\mathcal{S} = \mathcal{S}_0^{⚓}$. We first show $\left[\mathrm{Dr}_C^{⚓}[\beta], \mathfrak{t}_{\mathcal{S}}(a)\right] = 0$ for any $a \in \mathrm{Tube}_{\mathcal{S}}$. On the orthogonal complement of the skein subspace H_C the commutator vanishes because $\mathrm{Dr}_C^{⚓}[\beta]$ annihilates $H_C^{\perp}$, and $\mathfrak{t}_{\mathcal{S}}(a) H_C^{\perp} \subset H_C^{\perp}$. It therefore remains to verify that the commutator vanishes on H_C. Since $\Psi_C^{⚓}$ is an isomorphism of unitary Tube-modules it is sufficient to show that

$$\left(\Psi_C^{⚓}\right)^{-1} \mathrm{Dr}_C^{⚓}[\beta]\, \Psi_C^{⚓} : \mathcal{C}_{⚓}^*(\Sigma_C) \to \mathcal{C}_{⚓}^*(\Sigma_C)$$

commutes with the Tube action $\triangleright_0$ on $\mathcal{C}_{⚓}^*$ corresponding under $\Psi_C^{⚓}$ to $\mathfrak{t}_{\mathcal{S}}$. But $\triangleright_0$ acts only on the tensor factor $\mathcal{C}(X_0 \to \chi^{\otimes \mathcal{S}_0^{⚓}})$ of $\mathcal{C}_{⚓}^*$, while $\left(\Psi_C^{⚓}\right)^{-1} \mathrm{Dr}_C^{⚓}[\beta]\, \Psi_C^{⚓}$ acts as identity on that factor by definition. We conclude that $\left[\mathrm{Dr}_C^{⚓}[\beta], \mathfrak{t}_{\mathcal{S}}(a)\right] = 0$.

Finally, let $\mathcal{N} \subset \mathcal{S}_0^{⚓}$ and $a \in A(\mathcal{N})$. As before, $[\mathrm{Dr}_C^{⚓}[\beta], \mathfrak{t}_{\mathcal{N}}(a)]$ vanishes on $H_C^{\perp}$, and $\mathfrak{t}_{\mathcal{N}}(a) = \mathfrak{t}_{\mathcal{S}_0^{⚓}}(\iota(a))$ when acting on H_C (see Lemma 4.3) so the result for $\mathfrak{t}_{\mathcal{N}}$ follows from that of $\mathfrak{t}_{\mathcal{S}_0^{⚓}}$. ☐

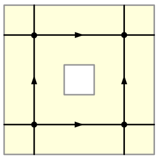
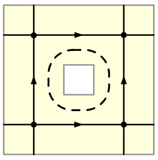
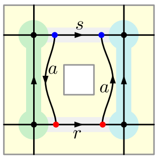
Lemma 6.5. *Let* C *be a finite connected region with anchor* ⚓ *and let* f *be a face such that one of the following holds:*

- *none of the vertices of* f *belong to* C*,*
- f *belongs to* C*,*
- f *sits on the outer boundary of* C*.*

Then B_f *commutes with the action* $\mathrm{Dr}_C^{⚓}$*, i.e.* $[\mathrm{Dr}_C^{⚓}[\beta], B_f] = 0$*.*

Proof : The first two cases are obvious.

Write $\mathcal{S}_0 = \mathcal{S}_0^{⚓}$ for the outer boundary. If f sits on the outer boundary then the region C^f decomposes into two non-empty regions C^{in} and C^{out}, where C^{in} consists of the vertices and edges of C^f that are entirely contained in Σ_C, and C^{out} consists of the vertices and edges that are entirely contained in Σ_C^c. Let $\mathcal{N}_{\mathrm{in}} := \partial\Sigma_{C^{\mathrm{in}}} \cap \partial\Sigma_C$ be the decorated submanifold shared by $\partial\Sigma_{C^{\mathrm{in}}}$ and $\partial\Sigma_C$. Let us specialise to the case where C_0^{in} consists of the left two vertices of the face f and the edge between them. By writing B_f as

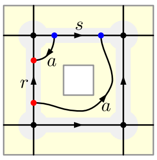
B_f  $=$  $= \dfrac{1}{\mathcal{D}^2} \displaystyle\sum_{a \in \mathrm{Irr}\,\mathcal{C}} d_a \sum_{r,s \in \mathrm{Irr}\,\mathcal{C}} d_r d_s$  , (46)

where we indicated C^{in} in green and C^{out} in blue, we see that B_f belongs to the algebra generated by $\mathfrak{t}_{\mathcal{N}_{\mathrm{in}}}(A(\mathcal{N}_{\mathrm{in}}))$ and $\mathfrak{t}_{\partial\Sigma_{C^{\mathrm{out}}}}(A(\partial\Sigma_{C^{\mathrm{out}}}))$. Since $\mathrm{Dr}_C^{⚓}[\beta]$ commutes with $\mathfrak{t}_{\mathcal{N}_{\mathrm{in}}}(A(\mathcal{N}_{\mathrm{in}}))$ by

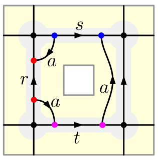
Lemma 6.4, and $\mathrm{Dr}_C^{\text{⚓}}[\beta]$ commutes with $\mathfrak{t}_{\partial\Sigma_{C\mathrm{out}}}(\mathrm{Tube}_{\partial\Sigma_{C\mathrm{out}}})$ by disjoint supports, we find that $[\mathrm{Dr}_C^{\text{⚓}}[\beta], B_f] = 0$.

All other cases can be treated in the same way, by decomposing the insertion of an a-loop as above, or in one of the following ways:

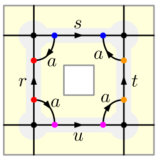
$\sum_{r,s\in\mathrm{Irr}\,\mathcal{C}} d_r d_s$



$= \sum_{r,s,t\in\mathrm{Irr}\,\mathcal{C}} d_r d_s d_t$



$= \sum_{r,s,t,u\in\mathrm{Irr}\,\mathcal{C}} d_r d_s d_t d_u$



,

or in a rotated version of one of these. □

Remark 6.6. *Lemma 6.5 also follows from Lemma 6.4, Lemma 4.10, and the statement that* gl *descends to an isomorphism* $A(\Sigma_C) \otimes_{A(\mathcal{I})} A(\Sigma_{C'}) \simeq A(\Sigma_C \sqcup_{\mathcal{I}} \Sigma_{C'})$*, where* $\mathcal{I}$ *is a common gluing boundary (see [Wal06, Theorem 5.2.10]).*

6.2 Inclusion Lemma

Let $D = C \sqcup_I C'$ be a finite connected region obtained by gluing C and C' along a self-avoiding dual path I as described in Section 4.9. Then all inner boundary components of Σ_C are also inner boundary components of Σ_D. Let ⚓_C be an anchor of Σ_C whose attachment point to the outer boundary component $\mathcal{S}_0^{\text{⚓}_C}$ of Σ_C also lies on the outer boundary component of Σ_D. Let ⚓_D be an anchor for Σ_D such that $\mathcal{S}_0^{\text{⚓}_D}$ is the outer boundary of Σ_D and such that $\mathcal{S}_\kappa^{\text{⚓}_C} = \mathcal{S}_{\kappa+\lambda}^{\text{⚓}_D}$ for $\kappa = 1, \cdots, m$ and some fixed *offset* $\lambda \in \{0, 1, \cdots, n-m\}$. If the graph of ⚓_D moreover contains the graph of ⚓_C as a subgraph, then we say ⚓_D *extends* ⚓_C with offset λ. More generally, any anchor $\text{⚓}'_D \sim \text{⚓}_D$ is also said to extend ⚓_C with offset λ.

Write $P_D^{(X_1,\cdots,X_n)}$ for the orthogonal projector onto the skein subspace $H_D^{(X_1,\cdots,X_n)} \subset \mathcal{H}_D$ defined in Eq. (42).

Lemma 6.7 (Inclusion)**.** *Let D be obtained by gluing C and C' along a self-avoiding dual path I, and let ⚓_D extend ⚓_C with offset λ. For any $X_1, \cdots, X_n, Y_1, \cdots, Y_n \in \mathrm{Irr}\, Z(\mathcal{C})$ and any $\beta \in Z(\mathcal{C})\big(X_{\lambda+1} \otimes \cdots \otimes X_{\lambda+m} \to Y_1 \otimes \cdots \otimes Y_m\big)$ we have*

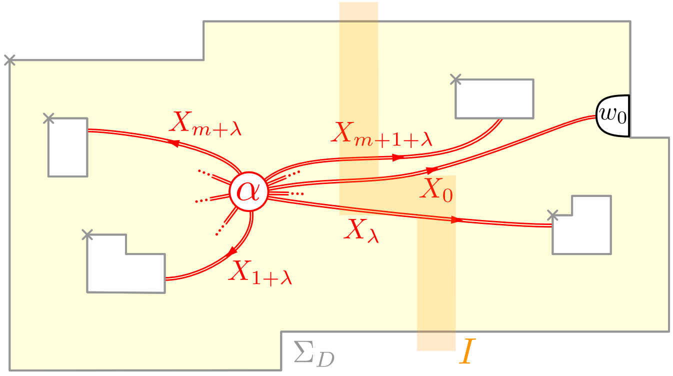
$$\mathrm{Dr}_C^{\text{⚓}_C}[\beta] P_D^{(X_1,\cdots,X_n)} = \mathrm{Dr}_D^{\text{⚓}_D}[\mathrm{id}_{X_1\otimes\cdots\otimes X_\lambda} \otimes \beta \otimes \mathrm{id}_{X_{\lambda+m+1}\otimes\cdots\otimes X_n}].$$

Proof : Let

$$\psi = \Psi_D^{\text{⚓}_D}\big(\alpha \otimes w_0 \otimes w^{X_1} \otimes \cdots \otimes w^{X_n}\big)$$

for arbitrary $X_0 \in \mathrm{Irr}\, Z(\mathcal{C})$, $\alpha \in Z(\mathcal{C})(X_0^* \to X_1 \otimes \cdots \otimes X_m)$, and $w_0 \in \mathcal{C}(X_0 \to \chi^{\otimes \mathcal{S}_0})$. By Proposition 4.6, the range of $P_D^{(X_1,\cdots,X_n)}$ is spanned by vectors of this form. In order to compute the action of $\mathrm{Dr}_C^{\text{⚓}_C}[\beta]$ on ψ we will write ψ as a linear combination of pure tensors according to the decomposition $\mathcal{H}_D = \mathcal{H}_C \otimes \mathcal{H}_{C'}$. By replacing ⚓_D with an equivalent anchor, we may without loss of generality assume that the anchor point of ⚓_D lies in Σ_C and that its attachment point to the exterior boundary component also belongs to the exterior boundary of $\Sigma_{C'}$. Schematically,

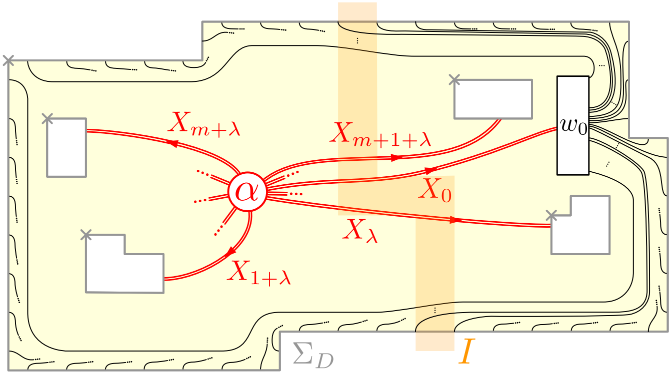
$\sigma_D(\psi) =$



,

where the Drinfeld center strands lie on the anchor $⚓_D$. By unpacking Convention 6.1 for the exterior boundary condition w_0 and noting that the corresponding dotted line can be contracted to a point, we obtain

$\sigma_D(\psi) =$



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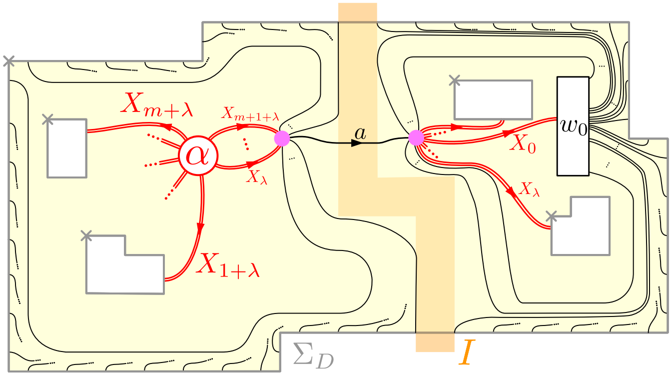
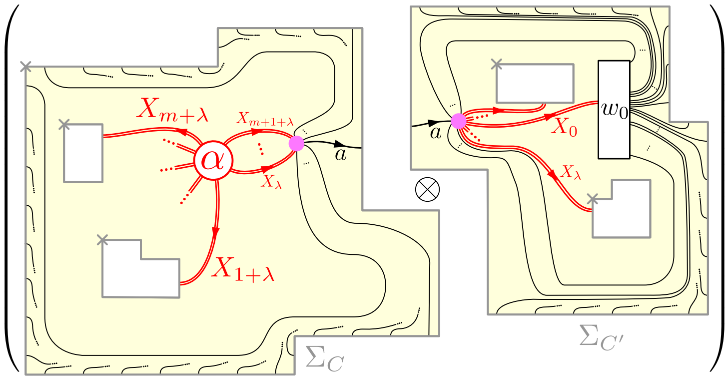
This illustrates that by isotopy we may assume that anchor lines connected to inner boundary components of C' as well as all the edges attaching to $\mathcal{S}_0^{⚓_C}$ intersect the cut I exactly once, and no other edges of the string diagram intersect I. Since I is contractible, we may apply a decomposition (8) to get a sum over string diagrams which have a single strand, labelled by a simple object $a \in \operatorname{Irr} \mathcal{C}$, crossing I along some edge in $\mathcal{E}_I$. (Note that the assumptions on I made in Section 4.9 guarantee that $\mathcal{E}_I$ is not empty.) With this arrangement of the string diagrams we find that each summand factorises under gluing. That is, the restrictions to $\Sigma_C, \Sigma_{C'}$ are valid string diagrams. Schematically,

$$\sigma_D(\psi) = \sum_{a\in\operatorname{Irr}\mathcal{C}} d_a$$



$$= \sum_{a\in\operatorname{Irr}\mathcal{C}} d_a \,\mathrm{gl}$$



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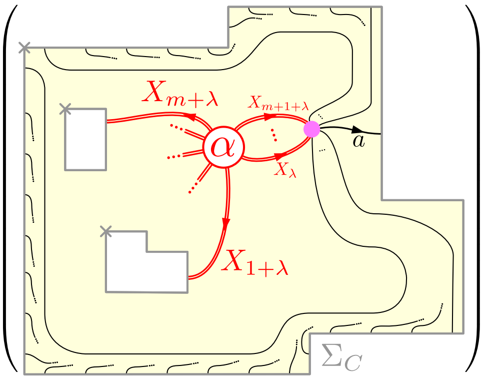
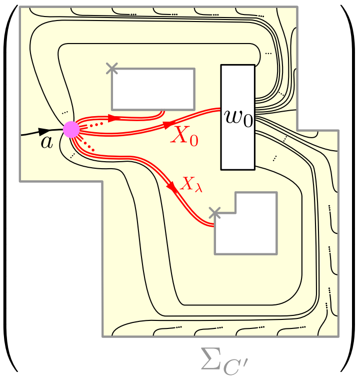
where $\mathrm{gl} : A(\Sigma_C) \otimes A(\Sigma_{C'}) \to A(\Sigma_D)$ is the gluing map of Section 3.3. Using Lemma 4.10 we express ψ as the image under B_I of a sum of product vectors in $\mathcal{H}_C \otimes \mathcal{H}_{C'}$:

$$\psi = \mathcal{D}^{|I_{\mathrm{in}}|}\, B_I \times \sum_a d_a^{1/2}$$

$$\sigma_C^{-1}$$



$$\otimes\, \sigma_{C'}^{-1}$$



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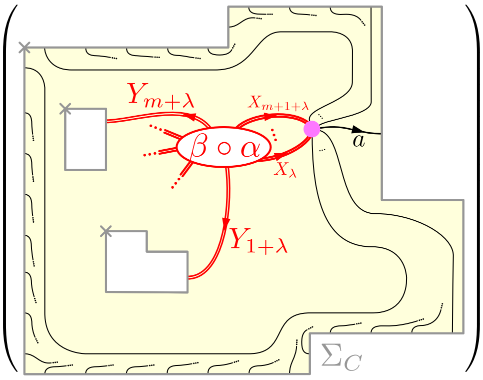
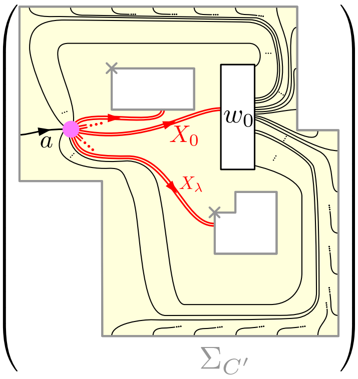
Lemma 6.5 implies that $\mathrm{Dr}_C^{\text{⚓}C}[\beta]$ commutes with B_I, so we can evaluate $\mathrm{Dr}_C^{\text{⚓}C}[\beta]$ directly on the first factors of each summand. Here we can reinstate a dotted line and push it towards the outer boundary so, possibly after decomposing the identity on $X_{m+1+\lambda} \otimes \cdots \otimes X_\lambda$ in $Z(\mathcal{C})$, these string diagrams are of the form (44). That is, the action is given by (43), and we obtain

$$\mathrm{Dr}_C^{\text{⚓}_C}[\beta]\psi = \mathcal{D}^{|I_{\mathrm{in}}|}\, B_I \times \left(\frac{\prod_{\kappa=1}^{m} d_{Y_\kappa}}{\prod_{\kappa=1}^{m} d_{X_{\kappa+\lambda}}} \right)^{1/2} \sum_a d_a^{1/2}$$

$$\times\ \sigma_C^{-1}$$



$$\otimes\ \sigma_{C'}^{-1}$$



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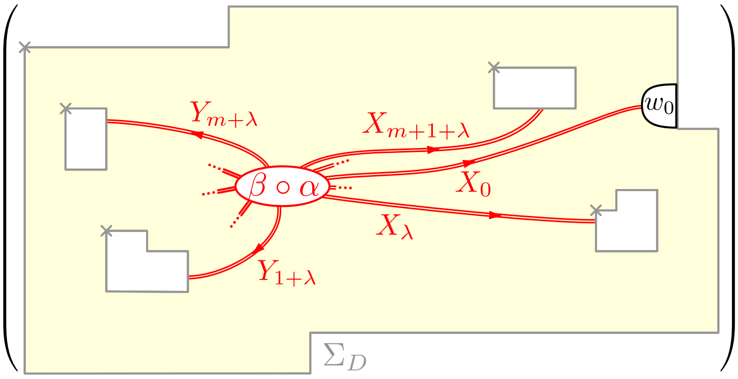
The action of $\mathrm{Dr}_C^{\text{⚓}_C}[\beta]$ results in a string diagram for every summand with the morphism $\beta \circ \alpha$ at the anchor point, while the boundary condition at $S_0^{\text{⚓}_C}$ is left unchanged. The tensor product decomposition can now be undone, resulting in a string diagram on Σ_D which is again of the form (44), it is the image under $\Psi_D^{\text{⚓}_D}$ of a product vector in $\mathcal{C}^*_{\text{⚓}}(\Sigma_D)$. We find $\mathrm{Dr}_C^{\text{⚓}_C}[\beta]\psi = \mathrm{Dr}_D^{\text{⚓}_D}[\mathrm{id}_{X_1 \otimes \cdots \otimes X_\lambda} \otimes \beta \otimes \mathrm{id}_{X_{\lambda+m+1} \otimes \cdots \otimes X_n}]\psi$ by comparing boundary conditions and the morphism at the anchor point. That is,

$$\mathrm{Dr}_C^{\text{⚓}_C}[\beta]\psi = \left(\frac{\prod_{\kappa=1}^{m} d_{Y_\kappa}}{\prod_{\kappa=1}^{m} d_{X_{\kappa+\lambda}}} \right)^{1/2} \sigma_D^{-1}$$



$$= \mathrm{Dr}_D^{\text{⚓}_D}[\mathrm{id}_{X_1 \otimes \cdots \otimes X_\lambda} \otimes \beta \otimes \mathrm{id}_{X_{\lambda+m+1} \otimes \cdots \otimes X_n}]\psi.$$

Since the operator $\mathrm{Dr}_D^{\text{⚓}_D}[\mathrm{id}_{X_1 \otimes \cdots \otimes X_\lambda} \otimes \beta \otimes \mathrm{id}_{X_{\lambda+m+1} \otimes \cdots \otimes X_n}]$ annihilates the orthogonal complement of $P_D^{(X_1, \cdots, X_n)}$, this proves the lemma. ☐

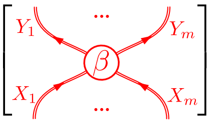
Remark 6.8. *With care, the Inclusion Lemma can be extended with more flexible ways of defining the extension of anchors. We mention one trivial but useful relation that is not covered by the Inclusion Lemma as stated.*

Let C be a finite connected region with anchor ⚓ and assume that $\mathcal{S}_{k+1}^{\text{⚓}} = \mathcal{S}_e$ for some edge e and number k. Let $C' \supset C$ be the finite subregion obtained from C by filling in the puncture at e, and let ⚓′ be the unique subanchor of ⚓ which is an anchor on C'. It follows immediately from the definitions and the choice of vacuum boundary condition $p^{\mathbb{1}}$ in Eq. (37) that

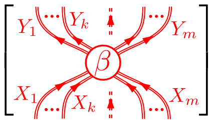
$$P_{C'}^{(X_1, \cdots, X_m)} = P_C^{(X_1, \cdots, X_k, \mathbb{1}, X_{k+1}, \cdots, X_m)},$$

where $m+1 = |\mathrm{Bd}(\Sigma_{C'})|$*. Moreover, given a morphism* $\beta : X_1 \otimes \cdots \otimes X_m \to Y_1 \otimes \cdots \otimes Y_m$ *we have*

$$\mathrm{Dr}^{⚓'}_{C'}$$



$$P_{C'}^{(X_1,\cdots,X_m)} = \mathrm{Dr}^{⚓}_{C}$$



$$P_C^{(X_1,\cdots,X_k,\mathbb{1},X_{k+1},\cdots,X_m)}.$$

6.3 Links

A *link* is a finite dual path $L = (f_1, \cdots, f_l)$ consisting of at least five faces, such that

- f_i and f_j share an edge if and only if $j = i \pm 1$ (or $i = j$). In particular, L is self-avoiding.
- the first four faces of L lie on a straight line, as do the last four faces of L.

Let $e_{\mathsf{i}} =: \partial_{\mathsf{i}} L$ denote the edge between faces f_1 and f_2 and $e_{\mathsf{f}} =: \partial_{\mathsf{f}} L$ the edge between the last two faces f_{l-1} and f_l. The puncture at e_{i}, resp. e_{f}, is called the *initial*, resp. *final*, puncture of L. To a link L we associate a region C^L consisting of all vertices and edges belonging to the faces constituting the link except for e_{i} and e_{f}, and all the faces $f_3, \cdots, f_{l-2}$, which we call the *bulk faces* of the link. The associated surface $\Sigma_L := \Sigma_{C^L}$ is a twice punctured disk, with punctures at the edges e_{i} and e_{f}. The assumptions on L guarantee that C^L can be cut along some path in between the two punctures, for example cutting transversally across the third face.

Let $C^{L,r} \subset C^L$ be the subregion consisting of all vertices and edges on the right side of C^L with respect to the direction of L. We call the surface $\Sigma_{L,r} := \Sigma_{C^{L,r}}$ the *right strip* of L.

We fix an anchor $⚓_L$ for C^L whose equivalence class up to the outer boundary component $\mathcal{S}_0^{⚓_L}$ is uniquely determined by the condition that the underlying graph of $⚓_L$ lies in the right strip $\Sigma_{L,r}$. In particular, $\mathcal{S}_1^{⚓_L} = \mathcal{S}_{e_{\mathsf{i}}}$ is the initial puncture, and $\mathcal{S}_2^{⚓_L} = \mathcal{S}_{e_{\mathsf{f}}}$ is the final puncture of L. To see that this condition indeed determines the equivalence class up to $\mathcal{S}_0^{⚓_L}$ uniquely, note first that $\Sigma_{L,r}$ is a disk, so any two anchors with the same attachment points in $\Sigma_{L,r}$ are equivalent by isotopy in $\Sigma_{L,r}$ keeping $\partial\Sigma_{L,r}$ fixed. The boundary components $\mathcal{S}_1^{⚓_L}$ and $\mathcal{S}_2^{⚓_L}$ are rectangles with fiducial points sitting on a corner, and $\partial\Sigma_{L,r}$ contains exactly one straight line of each of these boundary components. It follows that different choices of attachment point to these boundary components are also related to each other by isotopy of Σ_L keeping the fiducial points fixed. Different choices of attachment point to the outer boundary are related by isotopy of Σ_L, where we need not keep the fiducial point on $\mathcal{S}_0^{⚓_L}$ fixed.

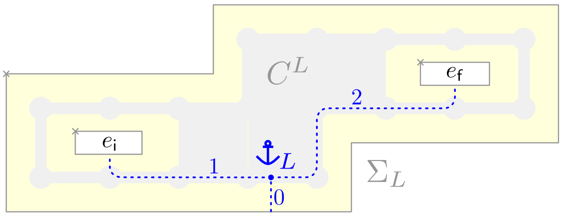


Figure 3.1: The anatomy of a link L. The region C^L is indicated in light grey, the surface Σ_L in light yellow, and the anchor $⚓_L$ in blue.

We write $\mathrm{supp}\,(L) = V_{C^L}$ for the *support* of the link L. We will often write L instead of C^L in notations depending on the region C^L. For example, $\mathcal{A}_L := \mathcal{A}_{C^L}, \mathcal{H}_L := \mathcal{H}_{C^L}$, $H_L := H_{C^L}$, $\sigma_L := \sigma_{C^L}$, $\Psi_L^{⚓} := \Psi_{C^L}^{⚓}$, etc. Consistent with this notation, we have $F_L = F_{C^L} = \{f_3, \cdots, f_{l-2}\}$. We also suppress the standardised choice of anchor $⚓_L$, writing $\mathrm{Dr}_L[\beta] = \mathrm{Dr}_{C^L}^{⚓_L}[\beta]$. The boundary components of Σ_L will always have the linear ordering induced by ⚓_L, so that $H_L(p_0, p_1, p_2) = H_L(\underline{p})$ with $p_\kappa = p_{\mathcal{S}_\kappa^{\text{⚓}_L}}$ for $\kappa = 0, 1, 2$. We also write $\mathfrak{t}_\kappa^L := \mathfrak{t}_{\mathcal{S}_\kappa^{\text{⚓}_L}}$ for the Tube-actions on these three boundary components.

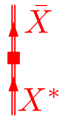
Two links $L_1 = (f_1, \cdots, f_k)$ and $L_2 = (f'_1 \cdots, f'_l)$ are *composable* if $\partial_{\mathfrak{i}}(L_1) = \partial_{\mathfrak{f}}(L_2)$ and the composite $L_2 \wedge L_1 := (f'_1, \cdots, f'_{l-1}, f'_l, f_2, \cdots, f_k)$ is again a link. Being composable is *not* a symmetric relation. Note that the right strip of $L_2 \wedge L_1$ is the union of the right strips of L_1 and L_2.

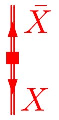
6.4 Unitary gates

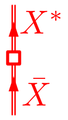
We construct unitaries which produce and annihilate anyon pairs on the punctures of a given link, or move an anyon from one puncture to another. The construction extends the hopping operators of [Chr+23; Gre+24].

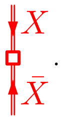
Partial isometries for pair creation and hopping

For each $X \in \operatorname{Irr} Z(\mathcal{C})$ we fix a (non-canonical) unitary isomorphism $\zeta_X : X^* \to \bar{X}$. We represent ζ_X in the graphical calculus by a solid box, and $\zeta_X^\dagger$ by an empty box:

$$\zeta_X =$$


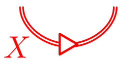
$$=$$


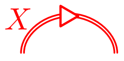
$$, \qquad \zeta_X^{-1} = \zeta_X^\dagger =$$


$$=$$


$$. \qquad (47)$$

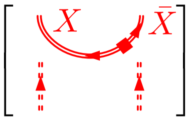
We will moreover make use of the morphisms $\mathrm{ev}_X^\dagger$ and $\mathrm{coev}_X^\dagger$, for which we introduce the following graphical representations:

$$\mathrm{ev}_X^\dagger =$$


$$, \quad \mathrm{coev}_X^\dagger =$$


$$. \qquad (48)$$

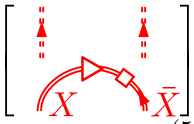
Let L be a link. We define an operator which creates an $X\bar{X}$ anyon pair at the punctures of L by

$$\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1} \to X\bar{X})} := d_X^{-1/2}\, \mathrm{Dr}_L[(\mathrm{id}_X \otimes \zeta_X) \circ \mathrm{coev}_X] = d_X^{-1/2}\, \mathrm{Dr}_L$$


$$, \qquad (49)$$

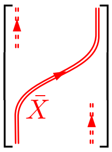
where we indicate explicitly the incoming and outgoing identity strands using dotted double lines in the diagrams, so that source and target objects of the morphisms are clear. These identity strands are attached to the rest of the diagram using unitors in an arbitrary way, all ways representing the same morphism by Mac Lane's coherence theorem.

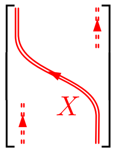
The adjoint of this operator annihilates an $X\bar{X}$ pair:

$$\mathrm{Dr}_L^{(X\bar{X} \to \mathbb{1}\,\mathbb{1})} := (\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1} \to X\bar{X})})^* = d_X^{-1/2}\, \mathrm{Dr}_L[\mathrm{coev}_X^\dagger \circ (\mathrm{id}_X \otimes \zeta_X^\dagger)] = d_X^{-1/2}\, \mathrm{Dr}_L$$


$$. \qquad (50)$$

Similarly, we define operators which move an X anyon between the initial and final punctures of L:

$$\mathrm{Dr}_L^{(X\,\mathbb{1} \to \mathbb{1}\,X)} := \mathrm{Dr}_L$$


$$, \quad \mathrm{Dr}_L^{(\mathbb{1}\,X \to X\,\mathbb{1})} := (\mathrm{Dr}_L^{(\mathbb{1}\,X \to X\,\mathbb{1})})^* = \mathrm{Dr}_L$$


$$. \qquad (51)$$

Write $B_L = B_{C_L} = \prod_{f \in F_L} B_f$.

Lemma 6.9. *The pair creation operator* $\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})} \in \mathcal{A}_L$ *is a partial isometry between* $H_L(P^{\mathbb{1}}, p^{\mathbb{1}}, p^{\mathbb{1}})$ *and* $H_L(P^{\mathbb{1}}, p^X, p^{\bar{X}})$*, with*

$$P_L^{\mathbb{1}\,\mathbb{1}} := \mathfrak{t}_1^L(p^{\mathbb{1}})\mathfrak{t}_2^L(p^{\mathbb{1}})B_L = (\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})})^*\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})}, \tag{52}$$

$$P_L^{X\bar{X}} := \mathfrak{t}_0^L(P^{\mathbb{1}})\mathfrak{t}_1^L(p^X)\mathfrak{t}_2^L(p^{\bar{X}})B_L = \mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})}(\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})})^*. \tag{53}$$

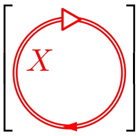
Similarly, the hopping operator $\mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)} \in \mathcal{A}_L$ *is a partial isometry between* $H_L(P^{\bar{X}}, p^X, p^{\mathbb{1}})$ *and* $H_L(P^{\bar{X}}, p^{\mathbb{1}}, p^X)$*, with*

$$P_L^{X\,\mathbb{1}} := \mathfrak{t}_1^L(p^X)\mathfrak{t}_2^L(p^{\mathbb{1}})B_L = (\mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)})^*\mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)}, \tag{54}$$

$$P_L^{\mathbb{1}\,X} := \mathfrak{t}_1^L(p^{\mathbb{1}})\mathfrak{t}_2^L(p^X)B_L = \mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)}(\mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)})^*. \tag{55}$$

Proof : Note that $\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})}$ annihilates the orthogonal complement of $H_L(\mathbb{1}, p^{\mathbb{1}}, p^{\mathbb{1}}) = H_L(P^{\mathbb{1}}, p^{\mathbb{1}}, p^{\mathbb{1}})$ by definition. The operator

$$(\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})})^*\,\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})} = d_X^{-1}\,\mathrm{Dr}_L$$

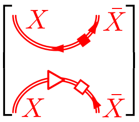


acts as identity on this subspace. Indeed, the morphism in the argument of the right hand side is precisely the trace of id_X (see for example [Pen20, Remark 3.14]). This shows the first claim.

Similarly, $(\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})})^*$ annihilates the orthogonal complement of $H_L(\mathrm{id}, p^X, p^{\bar{X}})$ by definition, and it also annihilates $H_L(P^Y, p^X, p^{\bar{X}})$ for $Y \neq \mathbb{1}$. Indeed, this subspace is mapped into $H_L(P^Y, p^{\mathbb{1}}, p^{\mathbb{1}}) = \{0\}$ which is trivial by Proposition 4.6 and the fact that $Z(\mathcal{C})(Y^* \to \mathbb{1}) = \{0\}$.

By noting that the operator

$$\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})}(\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})})^* = d_X^{-1}\mathrm{Dr}_L$$



acts as identity on the remaining subspace $H_L(P^{\mathbb{1}}, p^X, p^{\bar{X}})$ (again see [Pen20, Remark 3.14]), we obtain the second claim.

The hopping operator $\mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)}$ annihilates the orthogonal complement of $H_L(\mathrm{id}, p^X, p^{\mathbb{1}}) = H_L(P^{\bar{X}}, p^X, p^{\mathbb{1}})$ by definition, and the operator

$$(\mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)})^*\mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)} = \mathrm{Dr}_L\,[\mathrm{id}_{X\otimes\mathbb{1}}]$$

acts on this subspace as the identity. This shows the third equality. The proof of the last equality is identical to that of the third. □

Unitary gates and concatenation

For any link L we have the projectors $P_L^{\mathbb{1}\,\mathbb{1}}, P_L^{\mathbb{1}\,X}, P_L^{X\,\mathbb{1}}, P_L^{X\bar{X}} \in \mathcal{B}(\mathcal{H}_L)$ defined in Lemma 6.9. Note that all these projectors are all orthogonal to each other. We write $P_L^{\mathbb{1}?} := P_L^{\mathbb{1}\,\mathbb{1}} + P_L^{\mathbb{1}\,X}$,

$P_L^{X?} := P_L^{X\,\mathbb{1}} + P_L^{X\bar{X}}$, and $P_L^{??} = P_L^{X?} + P_L^{\mathbb{1}?} = P^{\mathbb{1}\,\mathbb{1}} + P_L^{X\,\mathbb{1}} + P_L^{\mathbb{1}\,X} + P_L^{X\bar{X}}$. The definition of $P_L^{\mathbb{1}?}$ and $P_L^{??}$ depends on the object X, which will always be clear from context.

For any $X \in \operatorname{Irr} Z(\mathcal{C})$ and any link L we define a self-adjoint unitary u_L^X by

$$u_L^X := \mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})} + \mathrm{Dr}_L^{(X\bar{X}\to\mathbb{1}\,\mathbb{1})} + \mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)} + \mathrm{Dr}_L^{(\mathbb{1}\,X\to X\,\mathbb{1})} + (\mathbb{1} - P_L^{??}). \tag{56}$$

Note that $u_L^{\mathbb{1}} = \mathbb{1}$. It is immediate from Lemma 6.9 that the unitary u_L^X satisfies the following intertwining properties:

$$u_L^X P_L^{\mathbb{1}\,\mathbb{1}} = P_L^{X\bar{X}} u_L^X, \qquad u_L^X P_L^{\mathbb{1}\,X} = P_L^{X\,\mathbb{1}} u_L^X, \tag{57}$$

which combine into

$$u_L^X P_L^{\mathbb{1}?} = P_L^{X?} u_L^X. \tag{58}$$

We also see that if L_1 and L_2 are composable links, then

$$u_{L_1}^X P_{L_2}^{\mathbb{1}\,\mathbb{1}} = P_{L_2}^{\mathbb{1}?} u_{L_1}^X P_{L_2}^{\mathbb{1}\,\mathbb{1}}. \tag{59}$$

Combined, these equations show that

$$u_{L_2}^X u_{L_1}^X P_{L_2}^{\mathbb{1}\,\mathbb{1}} = P_{L_2}^{X?} u_{L_2}^X u_{L_1}^X P_{L_2}^{\mathbb{1}\,\mathbb{1}},$$

which illustrates a key feature of the unitary gates: when acting sequentially along a *chain* of composable links on a state that satisfies ground state constraints on all but finitely many of those links, they will eventually start moving an X anyon to infinity.

We also have the following concatenation property.

Lemma 6.10 (Concatenation). *Let $X \in \operatorname{Irr} Z(\mathcal{C})$ and let L_1 and L_2 be composable links with $L = L_2 \wedge L_1$. Then*

$$u_{L_2}^X u_{L_1}^X P_L^{\mathbb{1}?} = u_L^X P_L^{\mathbb{1}?}.$$

Proof : Using Lemmas 6.5 and 6.9 we find

$$u_{L_2}^X u_{L_1}^X P_L^{\mathbb{1}?} = \mathrm{Dr}_{L_2}^{(\mathbb{1}\,X\to X\,\mathbb{1})} \mathrm{Dr}_{L_1}^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})} P_L^{\mathbb{1}\,\mathbb{1}} \quad + \quad \mathrm{Dr}_{L_2}^{(\mathbb{1}\,X\to X\,\mathbb{1})} \mathrm{Dr}_{L_1}^{(\mathbb{1}\,X\to X\,\mathbb{1})} P_L^{\mathbb{1}\,X},$$

and

$$u_L^X P_L^{\mathbb{1}?} = \mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})} P_L^{\mathbb{1}\,\mathbb{1}} \quad + \quad \mathrm{Dr}_L^{(\mathbb{1}\,X\to X\,\mathbb{1})} P_L^{\mathbb{1}\,X}.$$

It is therefore sufficient to show that

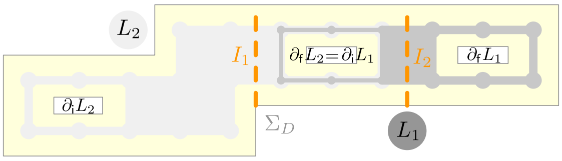
$$\mathrm{Dr}_{L_2}^{(\mathbb{1}\,X\to X\,\mathbb{1})} \mathrm{Dr}_{L_1}^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})} P_L^{\mathbb{1}\,\mathbb{1}} = \mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})} P_L^{\mathbb{1}\,\mathbb{1}}, \tag{60}$$

and

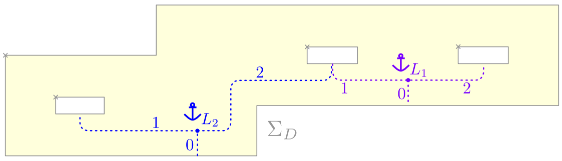
$$\mathrm{Dr}_{L_2}^{(\mathbb{1}\,X\to X\,\mathbb{1})} \mathrm{Dr}_{L_1}^{(\mathbb{1}\,X\to X\,\mathbb{1})} P_L^{\mathbb{1}\,X} = \mathrm{Dr}_L^{(\mathbb{1}\,X\to X\,\mathbb{1})} P_L^{\mathbb{1}\,X}. \tag{61}$$

Let's first derive Eq. (60). Consider the region $D = C^{L_1} \cup C^{L_2}$. We may describe D as the region obtained from C^L by removing the faces $L_1 \cap L_2$ as well as the edge shared between these faces. Since $L = L_2 \wedge L_1$ is a link, the region D has an associated surface Σ_D which is homeomorphic to a disk with three holes cut out, corresponding to the punctures of L_1 and L_2 where the initial puncture of L_1 is the same as the final puncture of L_2.

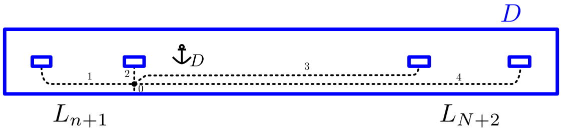
For invoking the inclusion lemma, we regard D as being obtained by gluing C^{L_1} and a uniquely determined region $C^{L_1,c}$ along the dual path I_1 cutting across the third face of L_2:



Similarly we may regard D as being obtained by gluing C^{L_2} and a uniquely determined region $C^{L_2,c}$ along the dual path I_2 cutting across the third to last face of L_1. The anchors $⚓_{L_1}$ and $⚓_{L_2}$ lie on Σ_D as follows:



We pick an anchor $⚓_D$ for D in the equivalence class up to the outer boundary, uniquely determined by the property that $⚓_D$ extends the anchor $⚓_{L_2}$ with offset 0, and is supported on the right strip of L. Then the anchor $⚓_D$ for D also extends the anchor $⚓_{L_1}$ with offset 1:



As we get L from filling the middle puncture of D,

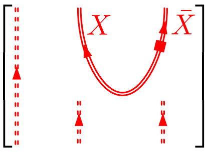
$$B_L = B_D \mathfrak{t}_{\mathcal{S}_2^{⚓_D}}(p^1) = \mathfrak{t}_{\mathcal{S}_2^{⚓_D}}(p^1) B_D.$$

Since P_L^{11} and $P_L^{\bar{X}1}$ are both dominated by B_L, we can invoke the inclusion lemma 6.7 twice to get

$$\mathrm{Dr}_{L_2}^{(\mathbb{1}\,X\to X\,\mathbb{1})}\mathrm{Dr}_{L_1}^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})}P_L^{\mathbb{1}\,\mathbb{1}} = \mathrm{Dr}_D^{(\mathbb{1}\,X\bar{X}\to X\,\mathbb{1}\,\bar{X})}\mathrm{Dr}_D^{(\mathbb{1}\,\mathbb{1}\,\mathbb{1}\to\mathbb{1}\,X\bar{X})}P_L^{\mathbb{1}\,\mathbb{1}}, \tag{62}$$

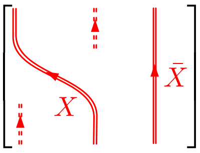
where

$$\mathrm{Dr}_D^{(\mathbb{1}\,\mathbb{1}\,\mathbb{1}\to\mathbb{1}\,X\bar{X})} = \mathrm{Dr}_D$$



,

$$\mathrm{Dr}_D^{(X\bar{X}\,\mathbb{1}\to X\,\mathbb{1}\,\bar{X})} = \mathrm{Dr}_D$$



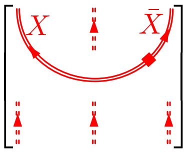
.

By the choice of boundary condition w^1, Eq. (39) (see Remark 6.8), we have

$$\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})}P_L^{\mathbb{1}\,\mathbb{1}} = \mathrm{Dr}_D^{(\mathbb{1}\,\mathbb{1}\,\mathbb{1}\to X\,\mathbb{1}\,\bar{X})}P_L^{\mathbb{1}\,\mathbb{1}}, \tag{63}$$

where we define

$$\mathrm{Dr}_D^{(\mathbb{1}\,\mathbb{1}\,\mathbb{1}\to X\,\mathbb{1}\,\bar{X})} = \mathrm{Dr}_D$$



.

Equation (60) now follows from Eqs. (62) and (63), and the fact that Dr_D is a representation (The Multiplicativity Lemma 6.2). With similar definitions, we derive (61) by

$$\begin{aligned}\mathrm{Dr}_{L_2}^{(\mathbb{1}\,X\to X\,\mathbb{1})}\mathrm{Dr}_{L_1}^{(\mathbb{1}\,X\to X\,\mathbb{1})}P_L^{\mathbb{1}\,X} &= \mathrm{Dr}_D^{(\mathbb{1}\,X\,\mathbb{1}\to X\,\mathbb{1}\,\mathbb{1})}\mathrm{Dr}_D^{(\mathbb{1}\,\mathbb{1}\,X\to\mathbb{1}\,X\,\mathbb{1})}P_L^{\mathbb{1}\,X}\\ &= \mathrm{Dr}_D^{(\mathbb{1}\,\mathbb{1}\,X\to X\,\mathbb{1}\,\mathbb{1})}P_L^{\mathbb{1}\,X}\\ &= \mathrm{Dr}_L^{(\mathbb{1}\,X\to X\,\mathbb{1})}P_L^{\mathbb{1}\,X}.\end{aligned}$$

□

6.5 String operators

A *chain*[1] $\mathscr{C} = (L_n)_{n\in\mathbb{N}}$ is a half-infinite sequence of links L_n such that $L^{\mathscr{C}}_{n\to m} := L_n \wedge L_{n-1} \wedge \cdots \wedge L_m$ is a well-defined link for all natural numbers $m \leq n$. We write $\mathscr{C}[n \to m] = (L_i)_{i=m}^{n}$ for the finite subchains of $\mathscr{C}$ as well as $\mathscr{C}_n := \mathscr{C}[n \to 1]$, and $\partial_{\mathfrak{i}}\mathscr{C} = \partial_{\mathfrak{f}} L_1$ for the initial puncture of the chain, which is the final puncture of the first link. We also write $\mathrm{supp}\,(\mathscr{C}) = \bigcup_{n\in\mathbb{N}} \mathrm{supp}\,(L_n)$ for the support of the chain $\mathscr{C}$. We say a face $f \in \mathcal{F}$ belongs to the chain $\mathscr{C}$ if it belongs to one of its links.

For any $X \in \mathrm{Irr}\, Z(\mathcal{C})$ and natural numbers $m \leq n$ we define unitaries

$$U^X_{\mathscr{C}[n\to m]} = u^X_{L_n} \times \cdots \times u^X_{L_m}$$

and automorphisms $\rho^X_{\mathscr{C}_n} := \mathrm{Ad}[(U^X_{\mathscr{C}[n\to 1]})^*]$.

Lemma 6.11. *Let $\mathscr{C}$ be a chain. For any $x \in \mathcal{A}^{\mathrm{loc}}$ there is n_0 large enough so that $\rho^X_{\mathscr{C}_n}(x) = \rho^X_{\mathscr{C}_{n_0}}(x)$ for all $n \geq n_0$. In particular, the limit*

$$\rho^X_{\mathscr{C}} := \lim_{n\uparrow\infty} \rho^X_{\mathscr{C}_n},$$

*exists and defines a unital *-endomorphism of $\mathcal{A}$. Moreover, $\rho^X_{\mathscr{C}}$ is supported on $\mathrm{supp}\,(\mathscr{C})$ in the sense that if $x \in \mathcal{A}_{\mathrm{supp}\,(\mathscr{C})^c}$ then $\rho^X_{\mathscr{C}}(x) = x$.*

Proof : Given $x \in \mathcal{A}^{\mathrm{loc}}$, there is a maximal n_0 such that $\mathrm{supp}\,(u^X_{L_{n_0}}) \cap \mathrm{supp}\,(x) \neq \emptyset$ (unless x is a multiple of the identity, in which case $n_0 = 1$). It is clear from the definition of $\rho_{\mathscr{C}_n}$ that n_0 has the desired property. The sequence $\rho^X_{\mathscr{C}_n}(x)$ is eventually constant with limit $\rho^X_{\mathscr{C}}(x) := \rho^X_{\mathscr{C}_{n_0}}(x)$. This yields a well-defined unital *-endomorphism of $\mathcal{A}^{\mathrm{loc}}$ which extends to the whole of $\mathcal{A}$ by continuity, and which is evidently supported on $\mathrm{supp}\,(\mathscr{C})$. □

7 Irreducible Anyon Representations

Recall that $(\pi^{\mathbb{1}}, \mathcal{H}, \Omega)$ is the GNS triple of the unique frustration free ground state $\omega^{\mathbb{1}}$ of the Levin-Wen Hamiltonian. We define a new representation

$$\pi^X_{\mathscr{C}} := \pi^{\mathbb{1}} \circ \rho^X_{\mathscr{C}},$$

where $\rho^X_{\mathscr{C}}$ is the endomorphism defined in Lemma 6.11. Our goal in this section is to show that $\pi^X_{\mathscr{C}}$ is an irreducible anyon representation, that $\pi^X_{\mathscr{C}} \simeq \pi^X_{\mathscr{C}'}$ for any two chains $\mathscr{C}, \mathscr{C}'$, and that $\pi^X_{\mathscr{C}}$ and $\pi^Y_{\mathscr{C}'}$ are disjoint if $X \neq Y$.

Fix a chain $\mathscr{C}$ supported in an allowed cone and a simple object $X \in \mathrm{Irr}\, Z(\mathcal{C})$. Throughout this section we employ the following notation.

---

[1] visualize a bicycle chain

Notation 7.1. *Let the fixed chain be given as* $\mathcal{C} = (L_n)$ *and denote* $e_n = \partial_{\mathsf{i}} L_n$*. We will write* $\rho^X_n = \rho^X_{\mathcal{C}_n}$*,* $\rho^X = \rho^X_{\mathcal{C}}$ *for the corresponding endomorphisms,* $u^X_n = u^X_{L_n}$*,* $u^X_{n\to m} = u^X_{L^{\mathcal{C}}_{n\to m}}$*,* $U^X_n = U^X_{\mathcal{C}_n}$*,* $U^X_{n\to m} = U^X_{\mathcal{C}[n\to m]}$ *for the unitary gates and circuits, and* $P^{\bullet\bullet}_n = P^{\bullet\bullet}_{L_n}$*, as well as* $P^{\bullet\bullet}_{n\to m} = P^{\bullet\bullet}_{L^{\mathcal{C}}_{n\to m}}$ *for natural numbers* $m \leq n$*.*

7.1 Pure anyon states belonging to $\pi^X_{\mathcal{C}}$

Recall from Section 5.2 that for any $e \in \mathcal{E}$ and any minimal projector $p \in \mathrm{Tube}_{\mathcal{S}_e}$ we have a pure state ω^p_e uniquely characterized by the constraints

$$\omega^p_e(\mathfrak{t}_e(p)) = \omega^p_e(B_f) = 1 \quad \forall f \in F_{C^{(e)}}. \tag{64}$$

In Section 6.1 we fixed for each $X \in \mathrm{Irr}\,\mathcal{C}$ a minimal projector $p^X \in \mathrm{Tube}_{\mathcal{S}_e}$ of type X. Let us write $\omega^X_e := \omega^{p^{\bar{X}}}_e$. We give this state the label X because it is in the range of $\mathfrak{t}_{\partial\Sigma_{\mathbb{D}}}(P^X)$ for any sufficiently large region $\mathbb{D}$ homeomorphic to a disk with a puncture at e. This is the correct notion of charge for the state ω^X_e from the point of view of sector theory.

Lemma 7.2. *We have*

$$\omega^X_e = \omega^{\mathbb{1}} \circ \rho^X.$$

Proof : It is sufficient to show that the state $\omega^1 \circ \rho^X$ satisfies the constraints of Eq. (64) with $p = p^{\bar{X}}$. Applying the Concatenation Lemma 6.10 inductively shows that for all $m < n \in \mathbb{N}$,

$$U^X_{n\to m} P^{11}_{n\to m} = u^X_n \cdots u^X_m P^{11}_{n\to m} = u^X_{n\to m} P^{11}_{n\to m}. \tag{65}$$

Since ω^1 satisfies all ground state constraints, we have that

$$\begin{aligned}\omega^1(\rho^X_n(x)) &= \omega^1(P^{11}_{n\to 1}\, \rho^X_n(x)\, P^{11}_{n\to 1}) \\ &= \omega^1((U^X_n P^{11}_{n\to 1})^* x U^X_n P^{11}_{n\to 1}) = \omega^1((u^X_{n\to 1})^*\, x\, u^X_{n\to 1})\end{aligned}$$

for all $x \in \mathcal{A}$. Using the intertwining property (57), we have

$$\mathfrak{t}_e(p^{\bar{X}}) u^X_{n\to 1} P^{11}_{n\to 1} = \mathfrak{t}_e(p^{\bar{X}}) P^{X\bar{X}}_{n\to 1} u^X_{n\to 1} = u^X_{n\to 1} P^{11}_{n\to 1}.$$

It follows that $\omega^{\mathbb{1}}\big(\rho^X_n(\mathfrak{t}_e(p^{\bar{X}}))\big) = 1$ for all n, so $\omega^{\mathbb{1}}\big(\rho^X(\mathfrak{t}_e(p^{\bar{X}}))\big) = 1$.

Let f be a face which is not adjacent to e. Then by Lemma 6.5, B_f commutes with $u^X_{n\to 1}$ unless f is adjacent to $\partial_{\mathsf{i}} L^{\mathcal{C}}_{n\to 1}$. So there is an N such that B_f commutes with $u^X_{n\to 1}$ for all $n \geq N$, and again we find that $\omega^{\mathbb{1}}\big(\rho^X(B_f)\big) = 1$. □

Let us denote by $(\pi^X_e, \mathcal{H}^X_e, \Omega^X_e)$ the GNS triple of ω^X_e. Since ω^X_e is a pure state (Proposition 5.2), the representation π^X_e is irreducible.

7.2 Unitary equivalence of $\pi^X_{\mathcal{C}}$ and π^X_e

Denote by $(\pi^X_e, \mathcal{H}^X_e, \Omega^X_e)$ the GNS triple of the pure state ω^X_e. Recall that $\partial_{\mathsf{i}}\mathcal{C} = e$. In this section we prove the following proposition.

Proposition 7.3. *The representation* $\pi^X_{\mathcal{C}} = \pi^{\mathbb{1}} \circ \rho^X_{\mathcal{C}}$ *is irreducible, and there is a unitary equivalence,*

$$\pi^X_{\mathcal{C}} \simeq \pi^X_e.$$

The strategy for showing that $\pi_{\mathcal{C}}^X$ is irreducible is to show that all of its vector states are pure. In fact, it is sufficient to show this for vector states corresponding to a dense subset of the unit ball of $\mathcal{H}$.

Let us consider a unit vector in $\mathcal{H}$ of the form $|\Psi\rangle = \pi^{\mathbb{1}}(O)|\Omega\rangle$ for some $O \in \mathcal{A}^{\mathrm{loc}}$. This defines a vector state ψ in $\pi_{\mathcal{C}}^X$ by

$$\psi(x) := \langle \Psi, \pi_{\mathcal{C}}^X(x)\Psi\rangle = \langle \Omega, \pi^{\mathbb{1}}\big(O^* \rho^X(x) O\big)\, \Omega\rangle$$

for all $x \in \mathcal{A}$. Our goal is to show that ψ is also a vector state of the GNS representation π_e^X.

First note that the state ψ can be obtained as the limit of a sequence of vector states of the vacuum representation $\pi^{\mathbb{1}}$ as follows. For $n \in \mathbb{N}$ we let

$$|\Psi_n\rangle := \pi^{\mathbb{1}}(U_n^X)|\Psi\rangle.$$

These are unit vectors in $\mathcal{H}$ which correspond under the vacuum representation to pure states ψ_n defined by

$$\psi_n(x) = \langle \Psi_n, \, \pi^{\mathbb{1}}(x)\, \Psi_n\rangle, \quad x \in \mathcal{A}.$$

Lemma 7.4. *In the w^*-topology, $\psi_n \to \psi$.*

Proof : Take $x \in \mathcal{A}^{\mathrm{loc}}$. By Lemma 6.11 there is n_0 large enough so that

$$\pi_{\mathcal{C}}^X(x) = \pi^{\mathbb{1}}(\rho^X(x)) = \pi^{\mathbb{1}}(\rho_n^X(x)) = \pi^{\mathbb{1}}\big((U_n^X)^*\, x\, U_n^X\big)$$

for all $n \geq n_0$. It follows that

$$\begin{aligned} \psi(x) &= \langle \Psi, \pi_{\mathcal{C}}^X(x)\, \Psi\rangle = \langle \Psi, \pi^{\mathbb{1}}\big((U_n^X)^* x U_n^X\big)\, \Psi\rangle \\ &= \langle \Psi_n, \pi^{\mathbb{1}}(x)\, \Psi_n\rangle = \psi_n(x) \end{aligned}$$

for all $n \geq n_0$. The claim now follows by density. □

In a similar way we can approximate the anyon state ω_e^X by a sequence of vector states of the vacuum representation as follows. For each $n \in \mathbb{N}$ we let

$$|\Omega_n\rangle = \pi^{\mathbb{1}}(U_n^X)|\Omega\rangle$$

and obtain pure states ω_n defined by

$$\omega_n(x) := \langle \Omega_n, \pi^{\mathbb{1}}(x)\, \Omega_n\rangle = \omega^{\mathbb{1}}((U_n^X)^* x U_n^X), \quad x \in \mathcal{A}.$$

Lemma 7.5. *In the w^*-topology, $\omega_n \to \omega_e^X$.*

Proof : Using Lemma 7.2, the proof is identical to that of Lemma 7.4. □

Before continuing towards the proof of Proposition 7.3, let us introduce some regions that will be used often below.

Recall that for any $R > 0$ we denote by $B_R = \{x \in \mathbb{R}^2 \, : \, \|x\|_\infty \leq R\}$ the closed box of side length $2R$ centered on the origin. We define the region $\mathbb{D}_R$ to consist of all faces, edges, and vertices of $C^{\mathbb{Z}^2}$ that are contained in B_R. The associated surface $\Sigma_{\mathbb{D}_R}$ is homeomorphic to a disk. For any $0 < R_1 < R_2 - 1$ we define the annular region $\mathbb{A}_{R_1,R_2}$ to consist of all faces, edges, and vertices contained in the closure of $B_{R_2} \setminus B_{R_1}$. For any edge $e \in \mathcal{E}$ we moreover define regions $\mathbb{D}_R^{(e)}$ obtained from $\mathbb{D}_R$ by removing e and its two neighbouring faces, and similarly $\mathbb{A}_{R_1,R_2}^{(e)}$ obtained from $\mathbb{A}_{R_1,R_2}$ by removing e and its two neighbouring faces. If e belongs to $\mathbb{D}_R$ at a suitable distance from the boundary then $\Sigma_{\mathbb{D}_R^{(e)}}$ is homeomorphic to an annulus and we simply say $\mathbb{D}_R^{(e)}$ is an annulus.

Now we argue that for n sufficiently large, the states ω_n and ψ_n agree on all observables supported outside of some box. To this end, let us introduce the following state spaces:

Definition 7.6. *For any $R > 0$, any $e \in \mathcal{E}$, and any $X \in \operatorname{Irr} Z(\mathcal{C})$ we let $\mathcal{S}_{>R}^{(e,X)}$ be the space of states $\psi : \mathcal{A} \to \mathbb{C}$ for which*

$$\psi(\mathfrak{t}_e(p^X)) = \psi(B_f) = 1$$

for all faces $f \subset B_R^c$ such that e is not adjacent to f.

That is, $\mathcal{S}_{>R}^{(e,X)}$ consists of states satisfying ground state constraints outside of the box of size R, except at the puncture at e where instead the projector $\mathfrak{t}_e(p^X)$ is satisfied.

Recall that $\mathcal{C} = \{L_n\}_{n\in\mathbb{N}}$ and put $e_n = \partial_{\mathfrak{i}} L_n = \partial_{\mathfrak{f}} L_{n+1}$ for all $n \in \mathbb{N}$.

Lemma 7.7. *There is $R > 0$ and $n_1 \in \mathbb{N}$ such that ω_n and ψ_n both belong to $\mathcal{S}_{>R}^{(e_n,X)}$ for all $n \geq n_1$.*

Proof : Recall that a link is a sequence of faces. We say a link is contained in $S \subset \mathbb{R}^2$ if all of its faces are subsets of S. Let $R_O > 0$ be such that $\operatorname{supp} O \subset B_{R_O}$. Let n_0 be maximal such that $L_{n_0} \cap B_{R_O} \neq \emptyset$. Then $\partial_{\mathfrak{i}} L_{n_0}$ necessarily lies outside B_{R_O}. We choose $R > R_O$ such that L_{n_0} is contained in B_R and set $n_1 = n_0 + 1$. The point is now that $|\Psi\rangle$ satisfies ground state constraints along the support of any subchain starting from n_1, i.e. $\pi^1(P^{\mathbb{1}\mathbb{1}}_{n\to n_1})|\Psi\rangle = |\Psi\rangle$. The support of the links L_k where $k < n_0 = n_1 - 1$ is disjoint from this subchain, so for any $n \geq n_1$ we have

$$\pi^{\mathbb{1}}(P^{\mathbb{1}\mathbb{1}}_{n\to n_1})|\Psi_{n_0-1}\rangle = \pi^{\mathbb{1}}(P^{\mathbb{1}\mathbb{1}}_{n\to n_1} U^X_{n_0-1})|\Psi\rangle = |\Psi_{n_0-1}\rangle.$$

The link L_{n_0} overlaps with L_{n_1} at the puncture at $\partial_{\mathfrak{f}} L_{n_1}$ but as expressed by (59), $u^X_{n_0}$ will either preserve the ground state constraint or produce an X anyon at the puncture when acting on $|\Psi_{n_0-1}\rangle$. That is,

$$|\Psi_{n_0}\rangle = \pi^{\mathbb{1}}(u^X_{n_0})|\Psi_{n_0-1}\rangle = \pi^{\mathbb{1}}(P^{\mathbb{1}?}_{n\to n_1})|\Psi_{n_0}\rangle.$$

Now it follows, again by inductively applying the Concatenation The Concatenation Lemma 6.10, that

$$|\Psi_n\rangle = \pi^{\mathbb{1}}\left(u^X_{n\to n_1} P^{\mathbb{1}?}_{n\to n_1}\right)|\Psi_{n_0}\rangle = \pi^{\mathbb{1}}(P^{X?}_{n\to n_1})|\Psi_n\rangle,$$

where the last equality uses the intertwining property (58). This implies that $\psi_n(\mathfrak{t}_{e_n}(p^X)) = 1$ and $\psi_n(B_f) = 1$ for all bulk faces f of $L^{\mathcal{C}}_{n\to n_1}$. For any other face $f \subset B_R^c$ which is not adjacent to e_n, the projector B_f commutes with O and with U_n^X (by Lemma 6.5) so $\psi_n(B_f) = 1$. (Note in particular that the faces adjacent to $e_{n_0} = \partial_{\mathfrak{i}} L_{n_0}$ belong to L_{n_0} which is contained in B_R.) Thus $\psi_n \in \mathcal{S}_{>R}^{(e_n,X)}$. Taking $O = 1$ shows $\omega_n \in \mathcal{S}_{>R}^{(e_n,X)}$ as well. □

Lemma 7.8. *There is $R \geq 0$ and a number $n_2 \in \mathbb{N}$ such that $\omega_n|_{B_R^c} = \psi_n|_{B_R^c}$ for all $n \geq n_2$.*

Proof : By Lemma 7.7 there is $R > 0$ and $n_1 \in \mathbb{N}$ such that ω_n and ψ_n both belong to $\mathcal{S}_{>R}^{(e_n,X)}$ for all $n \geq n_1$.

Let $\mathcal{S}$ be the outer boundary of an arbitrary finite connected region C. Then $\mathcal{S}$ is the unique connected boundary component of the disk-like region D obtained from C by filling in its inner holes, and the restriction of $\omega^{\mathbb{1}}$ to $\mathcal{A}_D$ corresponds to a density matrix in $\mathcal{D}_D = \mathcal{D}_D(P^{\mathbb{1}})$, where we used Lemma 4.7. This shows that $\omega^{\mathbb{1}}(\mathfrak{t}_{\mathcal{S}}(P^{\mathbb{1}})) = 1$ for any such $\mathcal{S}$.

For any $n \geq n_1$, take $R' > R$ large enough so that the support of $\mathcal{C}[1, n]$ is contained in $B_{R'}$ (recall that $\operatorname{supp} O \subset B_R$). Then the region $\mathbb{A}^{(e_n)}_{R,R'}$ has associated surface homeomorphic to a thrice punctured sphere. (two connected boundary components from the original annulus and one from the puncture at e_n.) Moreover, denoting by $\mathcal{S}_{R'}$ the outer boundary component of $\Sigma_{\mathbb{A}_{R,R'}}$, we have that $\omega_n(\mathfrak{t}_{\mathcal{S}_{R'}}(a)) = \psi_n(\mathfrak{t}_{\mathcal{S}_{R'}}(a)) = \omega^{\mathbb{1}}(\mathfrak{t}_{\mathcal{S}_{R'}}(a))$ for any $a \in \mathrm{Tube}_{\mathcal{S}_{R'}}$. Since $\omega^{\mathbb{1}}(\mathfrak{t}_{\mathcal{S}_{R'}}(P^{\mathbb{1}})) = 1$, it follows that the restrictions of ω_n and ψ_n to $\mathbb{A}^{(e_n)}_{R,R'}$ are given by density matrices in

$$\mathcal{D}_{\mathbb{A}^{(e_n)}_{R,R'}}(P^{\mathbb{1}}, \mathrm{id}, p^X) = \mathcal{D}_{\mathbb{A}^{(e_n)}_{R,R'}}(P^{\mathbb{1}}, P^{\bar{X}}, p^X),$$

where the equality follows from Lemma 4.7. Here, the first slot gives the boundary condition on the outer boundary of the annulus, the second slot gives the boundary condition on the inner boundary of the annulus, and the final slot gives the boundary condition at the puncture at e_n.

Choosing $n_2 \geq n_1$ such that whenever $n \geq n_2$ and $R' > R + 1$ is large enough, then $\mathbb{A}^{e_n}_{R+1,R'-1}$ is also a thrice punctured sphere, we find by Lemma 4.9 that the restrictions of ψ_n, ω_n to $\mathbb{A}^{e_n}_{R+1,R'-1}$ correspond to density matrices in $\mathcal{D}_{\mathbb{A}^{(e_n)}_{R,R'}}(\star^{\mathbb{1}}, \star^{\bar{X}}, p^X)$. But this is a singleton according to Lemma 4.7, so

$$\psi_n|_{B_{R'-1} \setminus B_{R+1}} = \omega_n|_{B_{R'-1} \setminus B_{R+1}}.$$

As R' can be chosen arbitrarily large this proves the claim by density. ☐

Lemma 7.9. *There is a unitary $V \in \mathcal{A}^{\mathrm{loc}}$ such that $\psi = \omega_e^X \circ \mathrm{Ad}[V]$.*

Proof : Let R and n_2 be as in the statement of Lemma 7.8. Pick $n_3 \geq n_2$ such that L_n is disjoint from B_R for all $n \geq n_3$. The vacuum Hilbert space has tensor product structure $\mathcal{H} = \mathcal{H}_{B_R} \otimes \mathcal{H}_{B_R^c}$ (because $\mathcal{H}_{B_R}$ is finite dimensional), and Lemma 7.8 implies that the unit vectors $|\Omega_{n_3}\rangle$ and $|\Psi_{n_3}\rangle$ have the same expectation values for any observable in $\mathcal{B}(\mathcal{H}_{B_R^c})$. That is, the vectors $|\Omega_{n_3}\rangle$ and $|\Psi_{n_3}\rangle$ are both purifications of the same state on $\mathcal{A}_{B_R^c}$, with finite dimensional purification space $\mathcal{H}_{B_R}$. It follows that there exists a unitary $W \in \mathcal{B}(\mathcal{H}_{B_R})$ such that $W|\Psi_{n_3}\rangle = |\Omega_{n_3}\rangle$.

Let $n > n_3$. Using that W has support disjoint from $U^X_{n \to n_3}$ we find

$$|\Psi_n\rangle = \pi^1(U^X_{n \to n_3+1}) |\Psi_{n_3}\rangle = \pi^1(U^X_{n \to n_3+1}) W |\Omega_{n_3}\rangle = W \pi^1(U^X_{n \to n_3+1}) |\Omega_{n_3}\rangle = W |\Omega_n\rangle .$$

Since $B_R \cap \mathbb{Z}^2$ is finite, there is a unitary $V \in \mathcal{A}_{B_R}$ such that $W = \pi^{\mathbb{1}}(V)$. We then have $\psi_n = \omega_n \circ \mathrm{Ad}[V]$ for all $n > n_3$. Together with Lemmas 7.4 and 7.5 this implies that $\psi = \omega_e^X \circ \mathrm{Ad}[V]$, as required. ☐

Proof of Proposition 7.3 : It follows from the previous Lemma that any state ψ, corresponding under the representation $\pi^X_{\mathcal{C}}$ to a unit vector in $\mathcal{H}$ of the form $|\Psi\rangle = \pi^{\mathbb{1}}(O)|\Omega\rangle$ for some $O \in \mathcal{A}^{\mathrm{loc}}$, is a vector state for the GNS representation π^X_e. Indeed, a representing vector is given by $\pi^X_e(V)\, |\Omega^X_e\rangle$ where $|\Omega^X_e\rangle$ denotes the cyclic vector of the GNS representation corresponding to the state ω^X_e, and $V \in \mathcal{A}$ is the unitary granted by Lemma 7.9. In particular, any such ψ is a pure state.

Since $\pi^1(\mathcal{A}^{\mathrm{loc}})\, |\Omega\rangle$ is dense in $\mathcal{H}$, it follows that $\pi^X_{\mathcal{C}}$ is irreducible. By Lemma 7.2 the state ω^X_e is a vector state for both irreducible representations $\pi^X_{\mathcal{C}}$ and π^X_e, so we conclude that $\pi^X_{\mathcal{C}} \simeq_{u.e.} \pi^X_e$.
☐

7.3 Disjointness

We no longer keep $\mathfrak{C}$ and X fixed. Recall that π_e^X is the GNS representation of the pure state ω_e^X.

Proposition 7.10. *Let $X, Y \in \operatorname{Irr} Z(\mathcal{C})$ and $e, e' \in \mathcal{E}$. Then $\pi_e^X \simeq_{u.e.} \pi_{e'}^Y$ if and only if $X = Y$.*

Proof : The statement follows from showing that $\pi_e^X \simeq_{u.e.} \pi_{e'}^X$, and that $X \neq Y$ implies that π_e^X and π_e^Y are disjoint. Assume $X \neq Y$. Let $R > 0$ be such that $\Sigma_{\mathbb{D}_R^{(e)}}$ is homeomorphic to an annulus. As noted in the proof of Proposition 5.2, the restriction $\omega_e^X|_{\mathcal{A}_{B_R}}$ corresponds to the unique density matrix in $\mathcal{D}_{\mathbb{D}_R^{(e)}}(p^{\bar{X}}, \star^X)$. In particular $\omega_e^X(\mathfrak{t}_R(P^X)) = 1$, writing $\mathfrak{t}_R$ for the Tube action on the outer boundary of $\mathbb{D}_R^{(e)}$. Similarly, $\omega_e^Y(\mathfrak{t}_R(P^Y)) = 1$. By orthogonality, $P^X P^Y = 0$ so

$$\left|\omega_e^X\left(\mathfrak{t}_R(P^X)\right) - \omega_e^Y\left(\mathfrak{t}_R(P^X)\right)\right| = 1.$$

Since R can be taken arbitrarily large, it follows from Corollary 2.6.11 of [BR12] that the GNS representations π_e^X and π_e^Y are disjoint.

Now we show that $\pi_e^X \simeq_{u.e.} \pi_{e'}^X$. Assume that there is a link L with $\partial_{\mathfrak{i}} L = e$ and $\partial L_{\mathfrak{f}} = e'$. Then $\omega_{e'}^X = \omega_e^X \circ \operatorname{Ad}[(u_L^X)^*]$, so the GNS representations of these pure states are unitarily equivalent. In case there is no such link we obtain the result by considering an intermediate equivalence with $\pi_{e''}^X$ for a suitable edge e'' such that there are links L_1 and L_2 with $e = \partial_{\mathfrak{i}} L_1, e'' = \partial_{\mathfrak{f}} L_1 = \partial_{\mathfrak{i}} L_2$ and $e' = \partial_{\mathfrak{f}} L_2$. Such an intermediate e'' can always be found. □

7.4 Superselection criterion

Proposition 7.11. *The representations π_e^X for all $e \in \mathcal{E}$ and $X \in \operatorname{Irr} Z(\mathcal{C})$ satisfy the superselection criterion.*

Proof : We must show that for any cone Λ there is a unitary equivalence

$$\pi_e^X|_\Lambda \simeq_{u.e} \pi^{\mathbb{1}}|_\Lambda.$$

Let $\mathfrak{C}$ be a chain supported in the complement of Λ, so $\rho_{\mathfrak{C}}^X(x) = x$ for all $x \in \mathcal{A}_\Lambda$. Then we have

$$\pi_{\mathfrak{C}}^X|_\Lambda = (\pi^{\mathbb{1}} \circ \rho_{\mathfrak{C}}^X)|_\Lambda = \pi^{\mathbb{1}}|_\Lambda.$$

Let $e' = \partial_{\mathfrak{i}} \mathfrak{C}$. The unitary equivalence $\pi_{\mathfrak{C}}^X \simeq_{u.e.} \pi_{e'}^X$ from Proposition 7.3 together with $\pi_e^X \simeq_{u.e.} \pi_{e'}^X$ from Proposition 7.10 gives the desired equivalence. □

8 Completeness

We show that every irreducible anyon representation π is isomorphic to one of the anyon representations π_e^X constructed in Section 7.

8.1 Anyon representations contain locally excited vector states

Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be an irreducible anyon representation.

For any $S \subset \mathbb{R}^2$, let $S^F \subset \mathcal{F}$ be the set of faces such that $B_f \in \mathcal{A}_S$. For bounded $S \subset \mathbb{R}^2$, define

$$P_S := \prod_{f \in S^F} B_f.$$

We now construct analogous projectors for infinite regions S in the von Neumann algebra $\pi(\mathcal{A})'' = \mathcal{B}(\mathcal{H})$.

We say a non-decreasing sequence $\{S_n\}$ of subsets of $\mathbb{R}^2$ converges to $S \subset \mathbb{R}^2$ if $\bigcup_n S_n = S$. Let $\{S_n\}$ be such a non-decreasing sequence of bounded subsets of $\mathbb{R}^2$ which converges to a possibly unbounded $S \subset \mathbb{R}^2$. Then the sequence of projectors $\{\pi(P_{S_n})\}$ is non-increasing and converges in the strong operator topology to the orthogonal projector p_S onto the intersection of the ranges of the $\pi(P_{S_n})$, see for example [Wei80, Thm 4.32(a)]. In particular, p_S is independent of the particular sequence $\{S_n\}$, and if S is bounded then $p_S = \pi(P_S)$.

The following proposition is an adaptation of [BV25, Proposition 5.2] to Levin-Wen models.

Proposition 8.1. *Any anyon representation $\pi : \mathcal{A} \to \mathcal{H}$ has a vector state ψ for which there is $R \geq 0$ such that*

$$\psi(B_f) = 1$$

for all $f \in \mathcal{F}$ such that $B_f \in \mathcal{A}_{B_R^c}$.

Proof : Let Λ_1, Λ_2 be two cones such that $\Lambda_1 \cup \Lambda_2 = \mathbb{R}^2$, and such that any B_f belongs to $\mathcal{A}_{\Lambda_1}$ or $\mathcal{A}_{\Lambda_2}$ (or both).

Since π satisfies the superselection criterion, there are unitaries $U_i : \mathcal{H} \to \mathcal{H}$ for $i = 1, 2$ such that

$$\pi|_{\Lambda_i} = (\mathrm{Ad}[U_i] \circ \pi^{\mathbb{1}})|_{\Lambda_i}.$$

It follows that

$$\omega^{\mathbb{1}}(x) = \langle \Omega^{\mathbb{1}}, \pi^{\mathbb{1}}(x)\, \Omega^{\mathbb{1}} \rangle = \langle \Omega^{\mathbb{1}}, U_i^* \pi(x) U_i\, \Omega^{\mathbb{1}} \rangle$$

for all $x \in \mathcal{A}_{\Lambda_i}$. Define states ω_i by $\omega_i(x) := \langle \Omega_i, \pi(x)\Omega_i \rangle$ for $x \in \mathcal{A}$, where $|\Omega_i\rangle = U_i |\Omega^{\mathbb{1}}\rangle \in \mathcal{H}$. The states ω_i are vector states of π and satisfy $\omega_i(x) = \omega^{\mathbb{1}}(x)$ for all $x \in \mathcal{A}_{\Lambda_i}$.

Let $\Lambda_i^{>n} := \Lambda_i \setminus B_n$ and $\Lambda_i^{n,n+m} := \Lambda_i^{>n} \setminus \Lambda_i^{>n+m}$ for all $m, n \in \mathbb{N}$. Then the sequence $m \mapsto \Lambda_i^{n,n+m}$ is a non-decreasing sequence of bounded sets converging to $\Lambda_i^{>n}$. We have

$$1 = \omega^{\mathbb{1}}\big(P_{\Lambda_i^{n,n+m}}\big) = \omega_i\big(P_{\Lambda_i^{n,n+m}}\big) = \langle \Omega_i, \pi\big(P_{\Lambda_i^{n,n+m}}\big)\Omega_i \rangle,$$

where we used that all these projectors are supported in Λ_i. It follows that

$$\langle \Omega_i, p_{\Lambda_i^{>n}}\, \Omega_i \rangle = 1.$$

From Corollary 2.6.11 of [BR12] we obtain an $N \in \mathbb{N}$ such that

$$|\omega_1(O) - \omega_2(O)| \leq \frac{1}{2}\|O\|$$

for all $O \in \mathcal{A}^{\mathrm{loc}} \cap \mathcal{A}_{B_N^c}$. This implies that $\omega_1(P_{\Lambda_2^{N,n}}) > 1/2$ for all $n > N$, so by continuity $\langle \Omega_1, p_{\Lambda_2^{>N}} \Omega_1 \rangle \geq 1/2$. Since $B_N^c \subset \Lambda_1^{>N} \cup \Lambda_2^{>N}$, we have $p_{B_N^c} \geq p_{\Lambda_2^{>N}} p_{\Lambda_1^{>N}}$ so

$$\langle \Omega_1, p_{B_N^c} \Omega_1 \rangle \geq \langle \Omega_1, p_{\Lambda_2^{>N}} p_{\Lambda_1^{>N}} \Omega_1 \rangle \geq \frac{1}{2}.$$

It follows that $p_{B_N^c} |\Omega_1\rangle \neq 0$, so we can define a normalized vector

$$|\Psi\rangle := \frac{p_{B_N^c} |\Omega_1\rangle}{|| p_{B_N^c} |\Omega_1\rangle ||} \in \mathcal{H}.$$

The corresponding vector state of π is defined by $\psi(x) := \langle \Psi, \pi(x)\, \Psi \rangle$ for all $x \in \mathcal{A}$. To finish the proof, we verify that $\psi(B_f) = 1$ whenever $B_f \in \mathcal{A}_{B_N^c}$. We have

$$\psi(B_f) = \frac{\langle \Omega_1, p_{B_N^c} \pi(B_f) p_{B_N^c} \Omega_1 \rangle}{\| p_{B_N^c} |\Omega_1\rangle \|^2} = \frac{\langle \Omega_1, p_{B_N^c}\, \Omega_1 \rangle}{\| p_{B_N^c} |\Omega_1\rangle \|^2} = 1.$$

This concludes the proof. □

8.2 Proof of completeness

Proposition 8.2. *Let* $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ *be an irreducible anyon representation. Then there is a unique* $X \in \operatorname{Irr} Z(\mathcal{C})$ *so that* $\pi \simeq_{u.e.} \pi_e^X$ *for any* $e \in \mathcal{E}$*.*

Proof : By Proposition 8.1, π has a vector state ψ for which there is $R \geq 1$ such that $\psi(B_f) = 1$ for all faces f whose vertices belong to B_R^c. Consider the annular regions $\mathbb{A}_{R,R'}$ for $R' \geq R+1$ and denote by $\mathfrak{t}_R$ the Tube actions on the inner boundary of annuli $\Sigma_{\mathbb{A}_{R,R'}}$. Note that $\mathfrak{t}_R(1) = \sum_X \mathfrak{t}_R(P^X)$ is the projector which enforces string-net constraints along the boundary of B_R^c. This constraint is also enforced by $\prod_{f \in I_R} B_f$ where $I_R \subset \mathcal{F}$ is the finite set of faces on the boundary of B_R^c, so

$$\sum_X \psi(\mathfrak{t}_R(P^X)) = \psi\left(\prod_{f \in I_R} B_f\right) = 1.$$

Since each of the terms of this sum is positive, there is some $X \in \operatorname{Irr}(Z(\mathcal{C}))$ such that

$$\psi_X(\bullet) = \psi(\mathfrak{t}_R(P^{\bar{X}}))^{-1}\psi(\mathfrak{t}_R(P^{\bar{X}}) \bullet \mathfrak{t}_R(P^{\bar{X}})),$$

is a well-defined state. It is clear that ψ_X is also a vector state of π. It satisfies

$$\psi_X(B_f) = \psi_X\big(\mathfrak{t}_R(P^{\bar{X}})\big) = 1$$

for all faces f whose vertices belong to B_R^c. It follows that the restriction of ψ_X to any annular region $\mathbb{A}_{R,R'}$ corresponds to a density matrix in $\mathcal{D}_{\mathbb{A}_{R,R'}}(P^{\bar{X}}, P^X)$. It then follows from Lemma 4.9 that the restriction of ψ_X to any annular region $\mathbb{A}_{R+1,R'}$ for $R' \geq R+2$ corresponds to a density matrix in $\mathcal{D}_{\mathbb{A}_{R+1,R'}}(\star^{\bar{X}}, \star^X)$. Since R' can be chosen arbitrarily large, it follows that $\psi_X|_{B_{R+1}^c}$ belongs to $\mathcal{S}_{C_{>R+1}}^{\star^{\bar{X}}}$ (recall Definition 5.1), where $C_{>R+1}$ is the infinite region consisting of all faces, edges, and vertices belonging to the closure of B_{R+1}^c.

Let $e \in \mathcal{E}$ be given. By choosing $R > 0$ above large enough we may without loss of generality assume that $\mathbb{D}_R^{(e)}$ is an annulus. Then the definition of $\omega_e^X = \omega_e^{p^{\bar{X}}}$ as the unique state in $\mathcal{S}_{C^{(e)}}^{p^{\bar{X}}}$ together with Lemma 5.5 implies that $\omega_e^X|_{\mathcal{A}_{B_{R+1}^c}} \in \mathcal{S}_{C_{>R+1}}^{\star^{\bar{X}}}$. By Proposition 5.2, the state space $\mathcal{S}_{C_{>R+1}}^{\star^{\bar{X}}}$ is a singleton, so ψ_X and ω_e^X have identical restrictions to B_{R+1}^c. Since π and π_e^X are irreducible it now follows from Corollary 2.6.11 of [BR12] that $\pi \simeq_{u.e.} \pi_e^X$. Uniqueness of X follows from disjointness, Proposition 7.10. ☐

8.3 The classification of anyon sectors

We are now ready to prove the first part of the main result.

Theorem 8.3. *The irreducible anyon sectors with respect to the vacuum representation of the Levin-Wen model over* $\mathcal{C}$ *are in one-to-one correspondence with equivalence classes of simple objects of the Drinfeld center* $Z(\mathcal{C})$*.*

Proof : Fix an edge $e \in \mathcal{E}$. By Proposition 7.11 the representations $\{\pi_e^X\}_{X \in \operatorname{Irr} Z(\mathcal{C})}$ satisfy the superselection criterion. Since these representations are the GNS representations of pure states ω_e^X, they are irreducible anyon representations. Moreover, by Proposition 7.10 all these representations are disjoint from each other. We therefore have for each $X \in \operatorname{Irr} Z(\mathcal{C})$ a distinct irreducible anyon sector $[\pi_e^X]$. Finally, by Proposition 8.2 any irreducible anyon representation belongs to one of these sectors. ☐

9 Fusion

In this section, we turn to the category SSS_f of localized and transportable endomorphisms of the allowed algebra $\mathcal{B}$. See Appendix C for a detailed exposition of SSS and the implications of the assumption of bounded spread Haag duality. In particular, we study the unique extensions of anyon representations $\pi^X_{\mathscr{C}}$ to representations $\bar\rho^X_{\mathscr{C}} : \mathcal{B} \to B(\mathcal{H})$, satisfying

$$\bar\rho^X_{\mathscr{C}} \circ \pi^{\mathbb{1}} = \pi^{\mathbb{1}} \circ \rho^X_{\mathscr{C}} ,$$

for any chain $\mathscr{C}$ supported in an allowed cone and $X \in \operatorname{Irr} Z(\mathcal{C})$ (Lemma C.1). Recall that $\pi^{\mathbb{1}}(\mathcal{A})$ is WOT-dense in $\mathcal{B}$, so intertwiners among the extensions $\{\bar\rho^X_{\mathscr{C}}\}_{X\in\operatorname{Irr} Z(\mathcal{C})}$ are exactly the same as the intertwiners among the anyon representations $\{\pi^X_{\mathscr{C}}\}_{X\in\operatorname{Irr} Z(\mathcal{C})}$. Therefore, we get an equivalent formulation of Theorem 8.3.

Corollary 9.1. *Let $\mathscr{C}$ be a chain supported in an allowed cone. Then the set of endomorphisms $\{\bar\rho^X_{\mathscr{C}}\}_{X\in\operatorname{Irr} Z(\mathcal{C})}$ is a complete set of representative simple objects for SSS_f. In particular, under the assumption of bounded spread Haag duality (Assumption 2.3), SSS_f is finite semisimple.*

This result implies that $Z(\mathcal{C})$ and SSS_f are isomorphic as semisimple unitary categories. In this section, we show that $Z(\mathcal{C})$ and SSS_f have *isomorphic F-symbols*. It is well known that this implies that $Z(\mathcal{C})$ and SSS_f are monoidally equivalent. We will actually be able to conclude that they are unitarily monoidally equivalent by Proposition D.1.

Throughout this section we fix an arbitrary chain $\mathscr{C}$ supported in an allowed cone and employ Notation 7.1, abbreviating also the notation for the extensions as $\bar\rho^X = \bar\rho^X_{\mathscr{C}}$.

9.1 Local states and anyon creation

We start with some basic general observations that will be useful in the rest of the manuscript.

Local states and eventually constant sequences of operators

Definition 9.2. *Let $\mathcal{K} \subset \mathcal{H}$ be a subspace of a Hilbert space $\mathcal{H}$. A sequence $(B_n) \subset B(\mathcal{H})$ is* eventually constant on $\mathcal{K}$ *if for every $|\Psi\rangle \in \mathcal{K}$ there is $N \in \mathbb{N}$, such that for all $n \geq N$,*

$$B_n |\Psi\rangle = B_N |\Psi\rangle . \tag{66}$$

Lemma 9.3. *If $(A_n) \subset B(\mathcal{K})$ and $(B_n) \subset B(\mathcal{H})$ are eventually constant sequences on $\mathcal{K} \subset \mathcal{H}$ then, for any $k \in \mathbb{Z}$, the sequence $(B_{n+k}A_n)$ is eventually constant on $\mathcal{K}$. If moreover, $\overline{\mathcal{K}} = \mathcal{H}$ and the sequences $(A_n), (B_n)$ are uniformly bounded, then the limits $A = \lim_n A_n$ and $B = \lim_n B_n$ exist in the strong operator topology, and*

$$BA = \lim_n B_{n+k}A_n. \tag{67}$$

Proof : We prove the claim for $k \geq 0$. The case $k < 0$ is treated analogously. Let $|\Psi\rangle \in \mathcal{K}$, and take N satisfying (66) for (A_n). Since $A_N |\Psi\rangle \in \mathcal{K}$, there is M such that $B_m A_N |\Psi\rangle = B_M A_N |\Psi\rangle$ for all $m \geq M$. Then for all $n \geq n' \geq N$ such that $n' + k \geq M$,

$$B_{n+k}A_n |\Psi\rangle = B_{n+k}A_N |\Psi\rangle = B_{n'+k}A_N |\Psi\rangle = B_{n'+k}A_{n'} |\Psi\rangle , \tag{68}$$

proving the first claim. The limits are clearly defined pointwise on $\mathcal{K}$. This defines a bounded operator on $\mathcal{K}$ assuming a uniform bound for each sequence which extends to a bounded operator on $\mathcal{H}$ if $\mathcal{K}$ is dense. Then Eq. (67) follows by evaluating BA on the dense subspace $\mathcal{K}$ as in Eq. (68). □

Definition 9.4. *A sequence* $(A_n) \subset \mathcal{A}^{\mathrm{loc}}$ *is* eventually constant on local states *if* $(\pi^{\mathbb{1}}(A_n))$ *is eventually constant on* $\pi^1(\mathcal{A}^{\mathrm{loc}})\,|\Omega\rangle$*.*

Recall that string operators are defined on $x \in \mathcal{A}^{\mathrm{loc}}$ by the eventually constant sequence

$$\rho^X_{\mathcal{C}_n}(x) = (U^X_{\mathcal{C}_n})^* x U^X_{\mathcal{C}_n}.$$

Trivially, this sequence is eventually constant on local states.

Long strings create anyons

Referring to the puncture at the initial edge e_{n-1} of a link L_{n-1}, the unitary gate u^X_{n-1} either does not do anything or creates an excitation of type X at e_{n-1} when acting on an arbitrary state that satisfies the vacuum conditions at e_{n-1}. This is by design so that

$$U^X_n P^{11}_n = P^{X?}_n U^X_n P^{11}_n = p^X_n U^X_n P^{11}_n. \tag{69}$$

This fact was used in the proof of Lemma 7.7 as part of showing that string operators produce irreducible anyon representations. The following lemma is a convenient packaging of the same fact applied to multiple string operators acting on local states.

Recall that we fixed a chain $\mathcal{C} = (L_n)$ and use the simplified Notation 7.1.

Lemma 9.5 (Excitation)**.** *Let* $|\Psi\rangle = \pi^{\mathbb{1}}(x)\,|\Omega\rangle$ *for some* $x \in \mathcal{A}^{\mathrm{loc}}$*. Then there is* N *such that for all* $k \geq 1$ *and* $n \geq N + k$*, and for all* $X_0, \ldots, X_k \in \operatorname{Irr} Z(\mathcal{C})$*,*

$$\pi^{\mathbb{1}}(U^{X_0}_n \cdots U^{X_k}_{n+k})\,|\Psi\rangle = \pi^{\mathbb{1}}(P^{(X_k,\ldots,X_0)}_D U^{X_0}_n \cdots U^{X_k}_{n+k})\,|\Psi\rangle\,, \tag{70}$$

where $D = \bigcup_{\kappa=1}^k C^{L_{n+\kappa}}$ *with the boundary components enumerated so that* $\mathcal{S}_\kappa$ *is the puncture at* $\partial_{\mathsf{i}} L_{n+k-\kappa+1}$ *for* $\kappa = 1, \ldots, k+1$*. Moreover,* (70) *holds if* D *is replaced by any finite region* D' *as long as* D' *has the same inner boundary components as* D *and is disjoint from the support of* $\mathcal{C}[n-1 \to 1]$ *and the support of* x*.*

Remark 9.6. *The enumeration of boundary components above is chosen to match the enumeration induced by the standard anchor on the link* L_{n+1} *in the case* $k = 1$ *where* $D = C^{L_{n+1}}$*. For example, if* $X_0 = \mathbb{1}$ *the lemma states that* $\pi^{\mathbb{1}}(U^{X_1}_{n+1})\,|\Psi\rangle = \pi^{\mathbb{1}}(P^{X_1\,\mathbb{1}}_{L_{n+1}} U^{X_1}_{n+1})\,|\Psi\rangle$*, noting that* $P^{X_1\,\mathbb{1}}_{L_{n+1}} = P^{(X_1,\mathbb{1})}_{L_{n+1}}$ *with the chosen enumeration.*

Proof : For any $X \in \operatorname{Irr} Z(\mathcal{C})$, let us write $p^X_n = \mathfrak{t}_{\mathcal{S}_{e_n}}(p^X)$ for all $n \in \mathbb{N}$, where $e_n = \partial_{\mathsf{i}} L_n$. Let n be arbitrary. Using Eq. (69) we find for any $\kappa < n$ that

$$U^X_{n-\kappa} P^{\mathbb{1}\,\mathbb{1}}_{n+\kappa \to n-\kappa} = P^{\mathbb{1}\,X}_{n+\kappa \to n-\kappa+1} U^X_{n-\kappa} P^{\mathbb{1}\,\mathbb{1}}_{n+\kappa \to n-\kappa}.$$

By iterations of the Concatenation Lemma 6.10 and the intertwining properties of unitary gates (57),

$$\begin{aligned}
U^X_{n+\kappa} P^{\mathbb{1}\,\mathbb{1}}_{n+\kappa \to n-\kappa} &= U^X_{n+\kappa \to n-\kappa+1} U^X_{n-\kappa} P^{\mathbb{1}\,\mathbb{1}}_{n+\kappa \to n-\kappa} \\
&= U^X_{n+\kappa \to n-\kappa+1} P^{\mathbb{1}\,X}_{n+\kappa \to n-\kappa+1} U^X_{n-\kappa} P^{\mathbb{1}\,\mathbb{1}}_{n+\kappa \to n-\kappa} \\
&= u^X_{L_{n+\kappa \to n-\kappa+1}} P^{\mathbb{1}\,X}_{n+\kappa \to n-\kappa+1} U^X_{n-\kappa} P^{\mathbb{1}\,\mathbb{1}}_{n+\kappa \to n-\kappa} \\
&= P^{X\,\mathbb{1}}_{n+\kappa \to n-\kappa+1} u^X_{L_{n+\kappa \to n-\kappa+1}} U^X_{n-\kappa} P^{\mathbb{1}\,\mathbb{1}}_{n+\kappa \to n-\kappa} \\
&= P^{X\,\mathbb{1}}_{n+\kappa \to n-\kappa+1} U^X_{n+\kappa} P^{\mathbb{1}\,\mathbb{1}}_{n+\kappa \to n-\kappa} \\
&= P^{\mathbb{1}\,\mathbb{1}}_{n+\kappa-1 \to n-\kappa+1} p^X_{n+\kappa} U^X_{n+\kappa} P^{\mathbb{1}\,\mathbb{1}}_{n+\kappa \to n-\kappa}.
\end{aligned}$$

It follows that for $n > k$,

$$U_n^{X_0} \cdots U_{n+k}^{X_k} P^{\mathbb{1}\mathbb{1}}_{n+k\to n-k} = p_n^{X_0} \cdots p_{n+k}^{X_k} U_n^{X_0} \cdots U_{n+k}^{X_k} P^{\mathbb{1}\mathbb{1}}_{n+k\to n-k}.$$

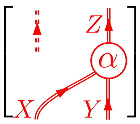
Since x has finite support, we may take N such that $L_n \cap \operatorname{supp}(x) = \emptyset$ for all $n \geq N$. Then $\pi^{\mathbb{1}}\big(P^{\mathbb{1}\mathbb{1}}_{n+k\to n-k}\big)\,|\Psi\rangle = |\Psi\rangle$ for all $n \geq N + k$. Now the statement follows if we take D' as under the hypothesis for given $k \geq 1$ and $n \geq N + k$, since $\pi^{\mathbb{1}}(P_{D'})\,|\Psi\rangle = |\Psi\rangle$ because D' has disjoint support from x, and since $P_{D'}$ commutes with $U^{X_\kappa}_{n+\kappa}$ for all $\kappa = 0, \ldots, k$. Indeed, this is true of B_f for every face in D' by Lemma 6.5, and it is easily extended to cover the case of edges in D' that do not belong to a face of D'. □

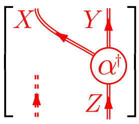
9.2 Isomorphisms of fusion spaces

We now define maps

$$\Phi^Z_{XY} : Z(\mathcal{C})(X \otimes Y \to Z) \to \mathsf{SSS}(\bar{\rho}^X \otimes \bar{\rho}^Y \to \bar{\rho}^Z)$$

for all $X, Y, Z \in \operatorname{Irr}(Z(\mathcal{C}))$ as follows. For $\alpha : X \otimes Y \to Z$, and every $n \geq m$, we define for any link (with its standard anchor)

$$\mathrm{Dr}_L[\alpha] = \mathrm{Dr}_L$$ 

, $$\mathrm{Dr}_L[\alpha^\dagger] = \mathrm{Dr}_L$$ 

and we use the following shorthand notation:

$$\mathrm{Dr}_n[\alpha] = \mathrm{Dr}_{L_n}[\alpha], \qquad \mathrm{Dr}_{n\to m}[\alpha] = \mathrm{Dr}_{L^{\mathfrak{e}}_{n\to m}}[\alpha].$$

Then define

$$\Phi_n[\alpha] := (U^Z_n)^* \mathrm{Dr}_{n+1}[\alpha] U^Y_n U^X_{n+1}. \tag{71}$$

Lemma 9.7. *Let $X, Y, Z \in \operatorname{Irr} Z(\mathcal{C})$ and $\alpha : X \otimes Y \to Z$. The sequence $(\Phi_n[\alpha])_{n\in\mathbb{N}}$ is eventually constant on local states. The limit,*

$$\Phi^Z_{XY}[\alpha] := \lim_n \pi^1(\Phi_n[\alpha]) \in \pi^1(\mathcal{A}_\Lambda)'',$$

is an intertwiner $\Phi^Z_{XY}[\alpha] : \bar{\rho}^X \times \bar{\rho}^Y \to \bar{\rho}^Z$.

Also the sequence $(\Phi_n[\alpha]^*)_{n\in\mathbb{N}}$ is eventually constant on local states, and

$$\Phi^{XY}_Z[\alpha^\dagger] := \big(\Phi^Z_{XY}[\alpha]\big)^* = \lim_n \pi^1(\Phi_n[\alpha]^*).$$

Proof : Let $|\Psi\rangle \in \pi^1(\mathcal{A}^{\mathrm{loc}})\,|\Omega\rangle$, and take N as per The Excitation Lemma 9.5, such that for all $n \geq N + 1$,

$$\pi^{\mathbb{1}}(U^Y_n U^X_{n+1})\,|\Psi\rangle = \pi^{\mathbb{1}}(P^{(X,Y)}_{L_{n+1}} U^Y_n U^X_{n+1})\,|\Psi\rangle\,. \tag{72}$$

Now let $n \geq N + 3$ and consider the region D obtained from $L_{n+1\to N+2}$ with additional punctures corresponding to punctures of L_{N+2} and L_{n+1}, and enumerate the boundary components according to the punctures $(e_{n+1}, e_n, e_{N+2}, e_{N+1})$. See Figure 3.2.

From the Excitation Lemma 9.5 we obtain

$$\pi^{\mathbb{1}}(U^Y_{N+1} U^X_{N+2})\,|\Psi\rangle = \pi^{\mathbb{1}}(P^{(\mathbb{1},\mathbb{1},X,Y)}_D U^Y_{N+1} U^X_{N+2})\,|\Psi\rangle\,.$$

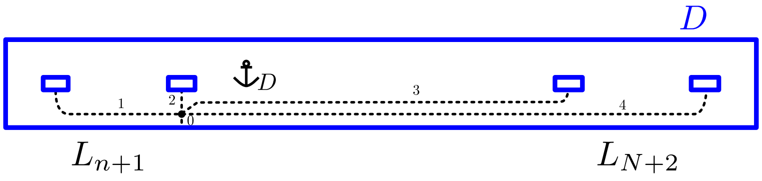
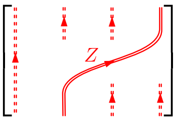


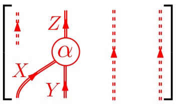
Figure 3.2: Schematic representation of the region D and its anchor ⚓_D. The enumeration of the punctures of D induced by ⚓_D corresponds to the ordering $(e_{n+1}, e_n, e_{N+2}, e_{N+1})$. The anchor ⚓_D extends the standard anchor $\text{⚓}_{L_{n+1}}$ and the standard anchor $\text{⚓}_{L_{N+2}}$ with offset 2.

Take an anchor ⚓_D for D as in Figure 3.2. By repeated use of the Concatenation Lemma 6.10 and the Inclusion Lemma 6.7, as well as Remark 6.8, we find

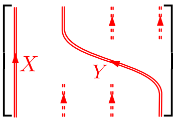
$$\begin{aligned}(U^Z_{n\to N+2})^*\mathrm{Dr}_{n+1}\big[\alpha\big]U^Y_{n\to N+2}U^X_{n+1\to N+3}P_D^{(\mathbb{1},\mathbb{1},X,Y)}\\ =&\mathrm{Dr}^{(Z1\to 1Z)}_{n\to N+2}\mathrm{Dr}_{n+1}\big[\alpha\big]\mathrm{Dr}^{(1Y\to Y1)}_{n\to N+2}\mathrm{Dr}^{(1X\to X1)}_{n+1\to N+3}P_D^{(\mathbb{1},\mathbb{1},X,Y)}\\ =&\mathrm{Dr}_D^{\text{⚓}_D}\end{aligned}$$



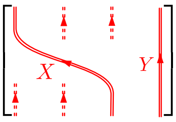
$$\times\mathrm{Dr}_D^{\text{⚓}_D}$$



$$\times\mathrm{Dr}_D^{\text{⚓}_D}$$



$$\times\mathrm{Dr}_D^{\text{⚓}_D}$$



$$P_D^{(\mathbb{1},\mathbb{1},X,Y)}$$

$$=\mathrm{Dr}_{N+2}\big[\alpha\big]P_D^{(\mathbb{1},\mathbb{1},X,Y)}.$$

In combination with

$$\begin{aligned}(U^Z_n)^* &= (U^Z_{N+1})^*(U^Z_{n\to N+2})^*,\\ U^Y_n U^X_{n+1} &= U^Y_{n\to N+2}U^X_{n+1\to N+3}U^Y_{N+1}U^X_{N+2},\end{aligned}$$

we obtain

$$\pi^{\mathbb{1}}(\Phi_n[\alpha])\,|\Psi\rangle = \pi^{\mathbb{1}}(\Phi_{N+1}[\alpha])\,|\Psi\rangle$$

for all $n \geq N+3$. Since $|\Psi\rangle$ was arbitrary, we conclude that $\Phi_n[\alpha]$ is eventually constant on local states.

We previously noted that $\rho^Y(x)$ is the limit of a sequence which is eventually constant on local states. Since $\rho^Y(x) \in \mathcal{A}^{\mathrm{loc}}$, also $\rho^X(\rho^Y(x))$ is the limit of a sequence eventually constant on local states. So we compute for $n \geq N$ such that $\mathrm{Dr}_n[\alpha]$ commutes with x,

$$\begin{aligned}\Phi_n[\alpha]\rho^X_{n+1}(\rho^Y_n(x)) &= (U^Z_n)^*\mathrm{Dr}_{n+1}\left[\alpha\right]U^Y_n U^X_{n+1}(U^X_{n+1})^*(U^Y_n)^* x U^Y_n U^X_{n+1} \\ &= (U^Z_n)^* x \mathrm{Dr}_{n+1}\left[\alpha\right]U^Y_n U^X_{n+1} \\ &= (U^Z_n)^* x (U^Z_n)^* U^Z_n \mathrm{Dr}_{n+1}\left[\alpha\right]U^Y_n U^X_{n+1} \\ &= \rho^Z_n(x)\Phi_n[\alpha],\end{aligned}$$

Using Lemma 9.3 we conclude

$$\Phi^Y_{XY}[\alpha]\bar{\rho}^X(\bar{\rho}^Y(x)) = \bar{\rho}^Z(x)\Phi^Y_{XY}[\alpha].$$

This extends to all of $\mathcal{A}$.

Finally, note that by the Multiplicativity Lemma 6.2, $\Phi_n\left[\alpha\right]^* = (U^X_{n+1})^*(U^Z_n)^*\mathrm{Dr}_{n+1}\left[\alpha^\dagger\right]U^Z_n$. The proof showing that $\Phi_n[\alpha]^*$ is eventually constant on local states runs completely analogously to the above, and is left to the reader. It follows that $\left(\Phi^Z_{XY}[\alpha]\right)^* = \Phi^{XY}_Z[\alpha^\dagger]$. □

Remark 9.8. *Observe that* $\pi^1(\Phi_n[\alpha]) \to \Phi^Z_{XY}[\alpha]$ *converges in the SOT-closed subalgebra* $\pi^1(\mathcal{A}_\Lambda)''$ *and that* $\bar{\rho}^W$ *is SOT-continuous on this subalgebra for any* $W \in \operatorname{Irr} Z(\mathcal{C})$*. Therefore,*

$$\bar{\rho}^W(\Phi^Z_{XY}[\alpha]) = \lim_n \bar{\rho}^W(\pi^1(F_n[\alpha])) = \lim_n \pi^1\left[(U^W_{n+1})^*\Phi_n[\alpha]U^W_{n+1}\right]. \tag{73}$$

Moreover, the sequence $\left((U^W_{n+1})^*\Phi_n[\alpha]U^W_{n+1}\right)_{n\in\mathbb{N}}$ *is eventually constant on local states. We omit the proof which is identical to the above with the additional presence of* W *excitations.*

The first observation about the maps Φ^Z_{XY} is that they provide an isomorphism of fusion rules (Proposition 9.10). This will follow from the following Lemma.

Lemma 9.9. *For each* $X, Y, Z \in \operatorname{Irr} Z(\mathcal{C})$ *and* $\alpha, \delta : X \otimes Y \to Z$ *we have*

$$\Phi^Z_{XY}\left[\alpha\right]\Phi^{XY}_Z\left[\delta^\dagger\right] = d_Z^{-1}\operatorname{tr}\left\{\alpha\circ\delta^\dagger\right\}\times 1_{\mathcal{H}},$$

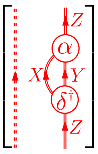
and

$$\Phi^{XY}_Z\left[\delta^\dagger\right]\Phi^Z_{YX}\left[\alpha\right] = \lim_n \pi^1\Big((U^X_{n+1})^*(U^Y_n)^*\mathrm{Dr}_{n+1}\left[\delta^\dagger\circ\alpha\right]U^Y_n U^X_{n+1}\Big).$$

Proof : Throughout this proof we identify $\mathcal{A}$ with its image $\pi^1(\mathcal{A})$ and drop π^1 from the notation. We verify the identity on the dense subspace of local states. Let $|\Psi\rangle \in \pi^1(\mathcal{A}^{\mathrm{loc}})\,|\Omega\rangle$. From the Excitation Lemma 9.5, Remark 9.6 we find $U^Z_n\,|\Psi\rangle = P^{\mathbb{1}\,Z}_{n+1}U^Z_n\,|\Psi\rangle$ for all n large enough. Using Lemma 9.3 and the Multiplicativity Lemma 6.2 we compute,

$$\Phi^Z_{XY}\left[\alpha\right]\Phi^{XY}_Z\left[\delta^\dagger\right]|\Psi\rangle = (U^Z_n)^*\mathrm{Dr}_{n+1}\left[\alpha\right]U^Y_n U^X_{n+1}(U^X_{n+1})^*(U^Y_n)^*\mathrm{Dr}_{n+1}\left[\delta^\dagger\right]U^Z_n\,|\Psi\rangle$$

$$= (U^Z_n)^*\mathrm{Dr}_{n+1}$$



$$U^Z_n\,|\Psi\rangle$$

$$\begin{aligned} &= d_Z^{-1}\operatorname{tr}\left\{\alpha\circ\delta^\dagger\right\}\times(U^Z_n)^*\mathrm{Dr}_{n+1}\left[\mathrm{id}_{\mathbb{1}}\otimes\mathrm{id}_Z\right]P^{\mathbb{1}\,Z}_{n+1}U^Z_n\,|\Psi\rangle \\ &= d_Z^{-1}\operatorname{tr}\left\{\alpha\circ\delta^\dagger\right\}\times|\Psi\rangle,\end{aligned}$$

where we use that $\alpha\circ\delta^\dagger = d_Z^{-1}\operatorname{tr}\left\{\alpha\circ\delta^\dagger\right\}\times\mathrm{id}_Z$ and the face that $\mathrm{Dr}_{n+1}[\mathrm{id}_{\mathbb{1}}\otimes\mathrm{id}_Z]P^{\mathbb{1}\,Z}_{n+1} = P^{\mathbb{1}\,Z}_{n+1}$.

Similarly, by The Excitation Lemma 9.5 we have $U_n^Y U_{n+1}^X |\Psi\rangle = P_{n+1}^{(X,Y)} U_n^Y U_{n+1}^X |\Psi\rangle$ for all n large enough. Arguing by Lemma 9.3 and The Multiplicativity Lemma 6.2 we find that

$$\begin{aligned}\Phi_Z^{XY}\left[\delta^\dagger\right]\Phi_{XY}^Z\left[\alpha\right]|\Psi\rangle &= (U_{n+1}^X)^*(U_n^Y)^*\mathrm{Dr}_{n+1}\left[\delta^\dagger\right]U_n^Z(U_n^Z)^*\mathrm{Dr}_{n+1}\left[\alpha\right]U_n^Y U_{n+1}^X|\Psi\rangle \\ &= (U_{n+1}^X)^*(U_n^Y)^*\mathrm{Dr}_{n+1}\left[\delta^\dagger\circ\alpha\right]U_n^Y U_{n+1}^X|\Psi\rangle\,,\end{aligned}$$

finishing the proof. ☐

Proposition 9.10. *Let* $X, Y \in \operatorname{Irr} Z(\mathcal{C})$ *and let*

$$\{\pi_{Z,\kappa} : X\otimes Y \to Z\}_{Z\in\operatorname{Irr} Z(\mathcal{C}),\kappa=1,\cdots,N_{XY}^Z}$$

provide an orthogonal direct sum decomposition of $X\otimes Y$*. Then*

$$\{\Phi_{XY}^Z\left[\pi_{Z,\kappa}\right] : \bar\rho^X\times\bar\rho^Y\to\bar\rho^Z\}_{Z\in\operatorname{Irr} Z(\mathcal{C}),\kappa=1,\cdots,N_{XY}^Z}$$

provides an orthogonal direct sum decomposition of $\bar\rho^X\times\bar\rho^Y$*. In particular,* SSS_f *is closed under the tensor product and* $\{\Phi_{XY}^Z\}$ *provides and isomorphism of fusion rules as defined in Section D.1.*

Proof : It is immediate from Lemma 9.9, that

$$\sum_{Z,\kappa}\Phi_{XY}^Z\left[\pi_{Z,\kappa}\right]^*\Phi_{XY}^Z\left[\pi_{Z,\kappa}\right] = \sum_{Z,\kappa}\Phi_Z^{XY}\left[\pi_{Z,\kappa}^\dagger\right]\Phi_{XY}^Z\left[\pi_{Z,\kappa}\right] = \mathrm{id}_{\bar\rho^X\times\bar\rho^Y}\,,$$

and

$$\Phi_Z^{XY}\left[\pi_{Z,\kappa}\right]\Phi_{Z'}^{XY}\left[\pi_{Z',\lambda}\right]^* = \Phi_{XY}^Z\left[\pi_{Z,\kappa}\right]\Phi_{Z'}^{XY}\left[\pi_{Z',\lambda}^\dagger\right] = \delta_{Z,Z'}\delta_{i,i'}\times\mathrm{id}_{\bar\rho^Z}$$

for all $Z, Z' \in \operatorname{Irr} Z(\mathcal{C})$. ☐

9.3 Isomorphism of F-symbols

F-symbols for $Z(\mathcal{C})$ are unitary maps defined by the commuting diagram

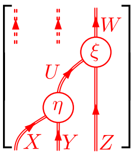
$$\begin{tikzcd}
Z(\mathcal{C})(X\otimes(Y\otimes Z)\to W) \arrow[r, "\simeq"] \arrow[d, "-\circ\alpha_{X,Y,X}"'] & \bigoplus\limits_{V\in\operatorname{Irr}(Z(\mathcal{C}))} Z(\mathcal{C})(X\otimes V\to W)\otimes Z(\mathcal{C})(Y\otimes Z\to V) \arrow[d, "F_{XYZ}^W"] \\
Z(\mathcal{C})((X\otimes Y)\otimes Z\to W) \arrow[r, "\simeq"] & \bigoplus\limits_{U\in\operatorname{Irr}(Z(\mathcal{C}))} Z(\mathcal{C})(U\otimes Z\to W)\otimes Z(\mathcal{C})(X\otimes Y\to U)
\end{tikzcd}$$

where the left vertical arrow is given by the precomposition with the associator α, and horizontal isomorphisms from right to left are given component-wise by composition: $\xi\otimes\eta\mapsto\xi\circ(\mathrm{id}_X\otimes\eta)$. F-symbols are defined similarly for SSS_f, where the associator is the identity.

The following Lemma shows that $\{\Phi_{XY}^Z\}$ intertwines the F-symbols of $Z(\mathcal{C})$ and SSS_f in the sense described in Appendix D.

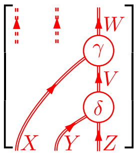
Lemma 9.11. *Let* $D_n = C^{L_n} \cup C^{L_{n+1}}$*, and* $U, V, W, X, Y, Z \in \operatorname{Irr} Z(\mathcal{C})$*. For all* $\xi : U \otimes Z \to W$ *and* $\eta : X \otimes Y \to U$*, and all* $\gamma : X \otimes V \to W$ *and* $\delta : Y \otimes Z \to V$*,*

$$\Phi^W_{UZ}[\xi] \circ (\Phi^U_{XY}[\eta] \times \mathrm{id}_{\bar\rho^Z}) = \lim_n \pi^1 \left((U_n^W)^* \mathrm{Dr}_{D_{n+1}} \right.$$



$$\left. U_n^Z U_{n+1}^Y U_{n+2}^X \right), \qquad (74)$$

$$\Phi^W_{XV}[\gamma] \circ (\mathrm{id}_{\bar\rho^X} \times \Phi^V_{YZ}[\delta]) = \lim_n \pi^1 \left((U_n^W)^* \mathrm{Dr}_{D_{n+1}} \right.$$



$$\left. U_n^Z U_{n+1}^Y U_{n+2}^X \right), \qquad (75)$$

where the limits are in the SOT-topology.

Proof : We verify both identities on the dense subspace of local states and omit $\pi^{\mathbb{1}}$ from notation throughout this proof. Let $|\Psi\rangle \in \pi^1(\mathcal{A}^{\mathrm{loc}}) |\Omega\rangle$, and take n large enough according to The Excitation Lemma 9.5 such that

$$U_n^Z U_{n+1}^Y U_{n+2}^X |\Psi\rangle = P_{D_{n+1}}^{(X,Y,Z)} U_n^Z U_{n+1}^Y U_{n+2}^X |\Psi\rangle .$$

Using this together with $\Phi^U_{XY}[\eta] \times \mathrm{id}_{\bar\rho^Z} = \Phi^U_{XY}[\eta]$ and Lemma 9.3 we find that the left hand side of (74) becomes

$$\begin{aligned}
\Phi^W_{UZ}[\xi]\Phi^U_{XY}[\eta] |\Psi\rangle &= (U_n^W)^* \mathrm{Dr}_{n+1}\big[\xi\big] U_n^Z U_{n+1}^U (U_{n+1}^U)^* \mathrm{Dr}_{n+2}\big[\eta\big] U_{n+1}^Y U_{n+2}^X |\Psi\rangle \\
&= (U_n^W)^* \mathrm{Dr}_{n+1}\big[\xi\big] \mathrm{Dr}_{n+2}\big[\eta\big] P_{D_{n+1}}^{(X,Y,Z)} U_n^Z U_{n+1}^Y U_{n+2}^X |\Psi\rangle \\
&= (U_n^W)^* \mathrm{Dr}_{D_{n+1}}\big[\xi \circ (\eta \otimes \mathrm{id}_Z)\big] U_n^Z U_{n+1}^Y U_{n+2}^X |\Psi\rangle ,
\end{aligned}$$

where we used The Multiplicativity Lemma 6.2 in combination with the Inclusion Lemma 6.7 applied to $L_n, L_{n+1} \subset D_{n+1}$ with an anchor $\text{⚓}_{D_{n+1}}$ extending ⚓_{L_n} and $\text{⚓}_{L_{n+1}}$. This shows the first identity.

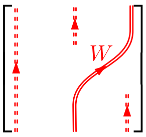
For the second identity we have $\mathrm{id}_{\bar\rho^X} \times \Phi^V_{YZ}[\delta] = \bar\rho^X(\Phi^V_{YZ}[\delta])$, so in this case we find

$$\begin{aligned}
&\Big(\Phi^W_{XV}[\gamma] \circ (\mathrm{id}_{\bar\rho^X} \times \Phi^V_{YZ}[\delta]) \Big) |\Psi\rangle \\
&= (U_{n+1}^W)^* \mathrm{Dr}_{n+2}\big[\gamma\big] U_{n+1}^V U_{n+2}^X (U_{n+2}^X)^* (U_n^V)^* \mathrm{Dr}_{n+1}\big[\delta\big] U_n^Z U_{n+1}^Y U_{n+2}^X |\Psi\rangle \\
&= (U_n^W)^* \big(u_{n+1}^W \mathrm{Dr}_{n+2}\big[\gamma\big] u_{n+1}^V \mathrm{Dr}_{n+1}\big[\delta\big] P_{D_{n+1}}^{(X,Y,Z)} \big) U_n^Z U_{n+1}^Y U_{n+2}^X |\Psi\rangle \\
&= (U_n^W)^* \mathrm{Dr}_{D_{n+1}}\big[\gamma \circ (\mathrm{id}_X \otimes \delta)\big] U_n^Z U_{n+1}^Y U_{n+2}^X |\Psi\rangle .
\end{aligned}$$

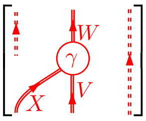
Here, the second equality is a simple algebraic reduction using definitions and disjointness of support. The last equality follows from a repeated application of the Inclusion Lemma 6.7 and the Multiplicativity Lemma 6.2, which shows:

$$u^W_{n+1}\mathrm{Dr}_{n+2}\big[\gamma\big]u^V_{n+1}\mathrm{Dr}_{n+1}\big[\delta\big]P^{(X,Y,Z)}_{D_{n+1}}$$

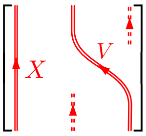
$$=\mathrm{Dr}^{\text{⚓}_{D_{n+1}}}_{D_{n+1}}$$



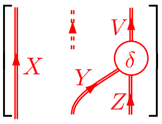
$$\times\mathrm{Dr}^{\text{⚓}_{D_{n+1}}}_{D_{n+1}}$$



$$\times\mathrm{Dr}^{\text{⚓}_{D_{n+1}}}_{D_{n+1}}$$



$$\times\mathrm{Dr}^{\text{⚓}_{D_{n+1}}}_{D_{n+1}}$$



$$P^{(X,Y,Z)}_{D_{n+1}}$$

$$=\mathrm{Dr}_{D_{n+1}}\big[\gamma\circ(\mathrm{id}_X\otimes\delta)\big]P^{(X,Y,Z)}_{D_{n+1}}.$$

□

10 Braiding

In this section, we show that the identification $\{\Phi^Z_{XY}\}$ of fusion spaces also intertwines R-symbols. This will imply that $Z(\mathcal{C})$ and SSS_f are equivalent as braided C*-tensor categories by Proposition D.1.

Note that the notation 7.1 is no longer in force because we will be dealing with more than one chain.

10.1 Transporters

In order to compute the braiding of SSS_f from Eq. (89), we construct explicit unitary transporters between $\bar\rho^X_{\mathscr{C}}$ and $\bar\rho^X_{\mathscr{C}'}$, for arbitrary $X\in\operatorname{Irr}Z(\mathcal{C})$ and sufficiently nice half-infinite chains $\mathscr{C}$ and $\mathscr{C}'$. Although it would suffice to do so for the fixed fiducial chain and a single auxilliary chain, we describe the general construction for completeness, cf. Remark 2.7.

Isotopy Lemma

The central ingredient for the transportation of string operators is a form of isotopy invariance enjoyed by the unitary gates of the string operators.

Definition 10.1. *There is a simple isotopy between links L and L' if $\partial_{\mathsf{i}}L=\partial_{\mathsf{i}}L'$ and $\partial_{\mathsf{f}}L=\partial_{\mathsf{f}}L'$, and there is a region E, such that*

1. $C^L, C^{L'}\subset E$,
2. *There is a region $C\subset E$ such that E is obtained by gluing L to C along a self-avoiding dual path (see Section 6.2),*

3. there is a region $C' \subset E$ such that E is obtained by gluing L' to C' along a self-avoiding dual path,

4. There is an anchor $⚓_E$ for E which extends both $⚓_L$ and $⚓_{L'}$ with offset zero.

The region E is said to support a simple isotopy between L and L'.

The definition is designed so that the Inclusion Lemma 6.7 can be applied. Note that it follows from the assumptions that Σ_E is a twice punctured disk whose punctures coincide with $\partial_{\mathsf{i}} L$ and $\partial_{\mathsf{f}} L$. The subregions C and C' glued to L and L' respectively to obtain E have associated regions Σ_C and $\Sigma_{C'}$ which are disks. See Figure 3.3.

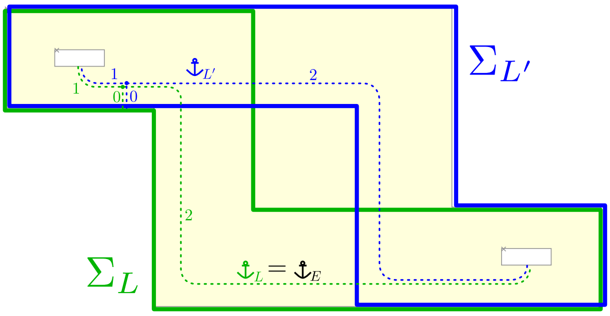


Figure 3.3: Isotopy of links L and L' supported by region E. The regions $\Sigma_L, \Sigma_{L'}$ and Σ_E are shown in green, blue, and light yellow respectively. A choice of anchors $⚓_E = ⚓_L$ and $⚓_{L'}$ is also shown, together with the enumeration of their strands.

Recall that for any region E, the projector $P_E \in \mathcal{A}_E$ is the orthogonal projector onto the skein subspace $H_E \subset \mathcal{H}_E$ associated to E. See Section 4.6.

Lemma 10.2 (Isotopy)**.** *If there is a simple isotopy between links L and L' supported by a region E, then*

$$u_L^X P_E = u_{L'}^X P_E$$

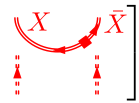
for any $X \in \operatorname{Irr} Z(\mathcal{C})$.

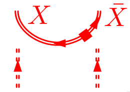
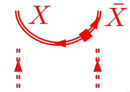
Proof : Recall the definition Eq. (56) of the unitary u_L^X. The Drinfeld insertions appearing in that definition are partial isometries with initial and final projections given by Eqs. (52)-(55). It is clear that $P_L^{\bullet\bullet} P_E = P_{L'}^{\bullet\bullet} P_E$ for all these projectors, so it is sufficient to show the following four equalities:

$$\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})} P_E = \mathrm{Dr}_{L'}^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})} P_E, \quad \mathrm{Dr}_L^{(X\bar{X}\to\mathbb{1}\,\mathbb{1})} P_E = \mathrm{Dr}_{L'}^{(X\bar{X}\to\mathbb{1}\,\mathbb{1})} P_E,$$
$$\mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)} P_E = \mathrm{Dr}_{L'}^{(X\,\mathbb{1}\to\mathbb{1}\,X)} P_E, \quad \mathrm{Dr}_L^{(\mathbb{1}\,X\to X\,\mathbb{1})} P_E = \mathrm{Dr}_{L'}^{(\mathbb{1}\,X\to X\,\mathbb{1})}.$$

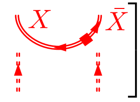
Let us show the first of these equalities. The proof of the others is completely analogous. We use the enumeration of boundary components induced by the anchors $⚓_L, ⚓_{L'}$ and $⚓_E$. Since E is obtained by gluing a disk C to the link L along a self-avoiding dual path, and $⚓_E$ extends $⚓_L$

with no offset, the inclusion Lemma 6.7 yields

$\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})}P_E = d_X^{-1/2}\,\mathrm{Dr}_L\Bigg[$  $\Bigg]P_E^{(\mathbb{1}\,\mathbb{1})}$

$= d_X^{-1/2}\,\mathrm{Dr}_E^{\text{⚓}_E}\Bigg[$  $\Bigg]P_E^{(\mathbb{1}\,\mathbb{1})} = d_X^{-1/2}\,\mathrm{Dr}_E^{\text{⚓}_E}\Bigg[$  $\Bigg]P_E$

where in the first and last equalities we noted that the Drinfeld insertions are supported on $P_L^{(\mathbb{1}\,\mathbb{1})}$ and $P_E^{(\mathbb{1}\,\mathbb{1})}$, and that $P_L^{\mathbb{1}\,\mathbb{1}}P_E = P_E^{(\mathbb{1}\,\mathbb{1})}$. For the same reason we have

$\mathrm{Dr}_L^{(\mathbb{1}\,\mathbb{1}\to X\bar{X})}P_E = d_X^{-1/2}\,\mathrm{Dr}_E^{\text{⚓}_E}\Bigg[$  $\Bigg]P_E.$

Combining these result yields the first equality above. □

In order to apply this isotopy below, we will also need the following variation of the concatenation lemma:

Lemma 10.3. *Let $X \in \operatorname{Irr} Z(\mathcal{C})$ and let L_1 and L_2 be composable links with $L = L_2 \wedge L_1$. Then*

$$u_{L_1}^X u_{L_2}^X P_L^{X\,\mathbb{1}} = u_L^X P_L^{X\,\mathbb{1}}.$$

Proof : As in the proof of the Concatenation Lemma 6.10, we have

$$u_{L_1}^X u_{L_2}^X P_L^{X\,\mathbb{1}} = \mathrm{Dr}_{L_1}^{(X\,\mathbb{1}\to\mathbb{1}\,X)}\mathrm{Dr}_{L_2}^{(X\,\mathbb{1}\to\mathbb{1}\,X)}P_L^{X\,\mathbb{1}}$$

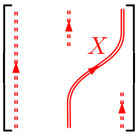
and

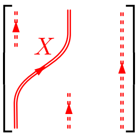
$$u_L^X P_L^{X\,\mathbb{1}} = \mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)}P_L^{X\,\mathbb{1}}.$$

Consider the region $D = C^{L_1} \cup C^{L_2}$, and let ⚓_D be as in the proof of the Concatenation Lemma 6.10, so that ⚓_D extends the anchor ⚓_{L_1} with offset 0, and ⚓_D extends the anchor ⚓_{L_1} with offset 1. As we get L from filling in the middle puncture of D, we have

$$B_L = B_D\mathfrak{t}_{\mathcal{S}_2^{\text{⚓}_D}}(p^1) = \mathfrak{t}_{\mathcal{S}_2^{\text{⚓}_D}}(p^1)B_D.$$

Making use of this observation as well as the Inclusion Lemma 6.7 and the Multiplicativity Lemma 6.2 we find

$\mathrm{Dr}_{L_1}^{(X\,\mathbb{1}\to\mathbb{1}\,X)}\mathrm{Dr}_{L_2}^{(X\,\mathbb{1}\to\mathbb{1}\,X)}P_L^{X\,\mathbb{1}} = \mathrm{Dr}_D\Bigg[$  $\Bigg]$

$\times\mathrm{Dr}_D\Bigg[$  $\Bigg]P_L^{X\,\mathbb{1}}$

$= \mathrm{Dr}_L^{(X\,\mathbb{1}\to\mathbb{1}\,X)}P_L^{X\,\mathbb{1}},$

where the last equality is evident as in Remark 6.8. □

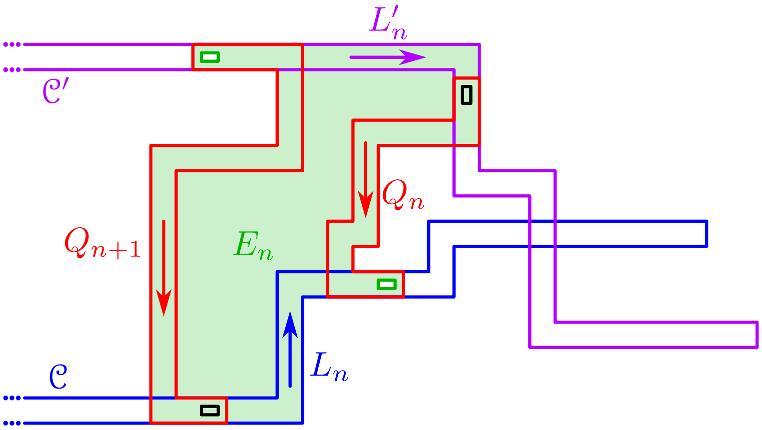


Figure 3.4: Part of a bridge $(\{Q_n\}, \{E_n\})$ between chains $\mathcal{C} = (L_n)$ and $\mathcal{C}' = (L'_n)$.

Construction of transporters

We now construct explicit transporters between string operators $\bar{\rho}^X_{\mathcal{C}}$ and $\bar{\rho}^X_{\mathcal{C}'}$ supported on chains $\mathcal{C}, \mathcal{C}'$ which admit a so-called *bridge*, see Figure 3.4

Definition 10.4. *Let $\mathcal{C} = \{L_n\}_{n\in\mathbb{N}}$ and $\mathcal{C}' = \{L'_n\}_{n\in\mathbb{N}}$ be chains. A bridge from $\mathcal{C}$ to $\mathcal{C}'$ is a pair $(\{Q_n\}_{n\geq N}, \{E_n\}_{n\geq N}\})$ for some $N \in \mathbb{N}$, where each Q_n is a link and each E_n is a region such that for all $n \geq N$,*

1. *L_n and Q_{n+1} are composable, i.e. $Q_{n+1} \wedge L_n$ is a link,*
2. *Q_n and L'_n are composable, i.e. $L'_n \wedge Q_n$ is a link,*
3. *The region E_n supports simple isotopy between $Q_{n+1} \wedge L_n$ and $L'_n \wedge Q_n$,*
4. *For any finite region C there is $n_0 \geq N$ so that $E_n \cap C = \emptyset$ if $n \geq n_0$.*
5. *The region E_n is disjoint from the supports of $\mathcal{C}[n-2 \to 1]$ and $\mathcal{C}'[n-2 \to 1]$.*

We say the bridge $(\{Q_n\}_{n\geq N}, \{E_n\}_{n\geq N})$ is supported in a cone Λ if the supports of all links $\{L_n, L'_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\geq N}$ are subsets of Λ.

Proposition 10.5. *Let $\mathcal{C}$ and $\mathcal{C}'$ be chains and suppose $(\{Q_n\}_{n\geq N}, \{E_n\}_{n\geq N})$ is a bridge from $\mathcal{C}$ to $\mathcal{C}'$. Let $X \in \operatorname{Irr} Z(\mathcal{C})$. For each $n \geq N$, define the unitary*

$$T_n = (U^X_{\mathcal{C}'_n})^* u^X_{Q_{n+1}} U^X_{\mathcal{C}_n}. \tag{76}$$

Then the sequence $\{T_n\}_{n\geq N}$is eventually constant on local states and the limit

$$T := \lim_{n\uparrow\infty} \pi^{\mathbb{1}}(T_n) \tag{77}$$

is a unitary intertwiner from $\bar{\rho}^X_{\mathcal{C}}$ to $\bar{\rho}^X_{\mathcal{C}'}$.

Moreover, if the bridge $(\{Q_n\}_{n\geq N}, \{E_n\}_{n\geq N})$ is supported in a cone Λ, then $T \in \mathcal{R}(\Lambda)$.

Proof : Throughout this proof we identify $\mathcal{A}$ with its image $\pi^{\mathbb{1}}(\mathcal{A})$ and drop $\pi^{\mathbb{1}}$ from the notation. Since the object $X \in \operatorname{Irr} Z(\mathcal{C})$ is fixed, we also drop it from the notation where no ambiguity arises.

Consider a local state $|\Psi\rangle := x\,|\Omega\rangle$ for some $x \in \mathcal{A}^{\mathrm{loc}}$.

Let $n \geq \max\{N+2, n_0\}$ where N is given by the Excitation Lemma 9.5 depending on x, and n_0 granted by item 4 of Definition 10.4 for the support of x. Taking into account item 5 of Definition 10.4, we see that the hypothesis of the Excitation Lemma 9.5 is satisfied for E_n (applying the lemma with values $n-1$ and $k=1$ with $X_0 = X$ and $X_1 = \mathbb{1}$, where the region $D = L_{(n-1)+1} \subset E_n$). This gives

$$U_{\mathcal{C}_{n-1}}\,|\Psi\rangle = P_{E_n}^{(\mathbb{1},X)} U_{\mathcal{C}_{n-1}}\,|\Psi\rangle\,. \tag{78}$$

Then we compute,

$$\begin{aligned}
T_n\,|\Psi\rangle &= U^*_{\mathcal{C}'_n}\,u_{Q_{n+1}}\,U_{\mathcal{C}_n}\,|\Psi\rangle = U^*_{\mathcal{C}'_n}\,u_{Q_{n+1}}\,u_{L_n}\,U_{\mathcal{C}_{n-1}}\,|\Psi\rangle \\
&\overset{(78)}{=} U^*_{\mathcal{C}'_n}\,u_{Q_{n+1}}\,u_{L_n}\,P_{E_n}^{(\mathbb{1},X)}\,U_{\mathcal{C}_{n-1}}\,|\Psi\rangle \overset{Lem.10.3}{=} U^*_{\mathcal{C}'_n}\,u_{Q_{n+1}\wedge L_n}\,P_{E_n}^{(\mathbb{1},X)}\,U_{\mathcal{C}_{n-1}}\,|\Psi\rangle \\
&\overset{Lem.10.2}{=} U^*_{\mathcal{C}'_n}\,u_{L'_n\wedge Q_n}\,P_{E_n}^{(\mathbb{1},X)}\,U_{\mathcal{C}_{n-1}}\,|\Psi\rangle \overset{Lem.10.3}{=} U^*_{\mathcal{C}'_n}\,u_{L'_n}\,u_{Q_n}\,P_{E_n}^{(\mathbb{1},X)}\,U_{\mathcal{C}_{n-1}}\,|\Psi\rangle \\
&= U^*_{\mathcal{C}'_{n-1}}\,u_{Q_n}\,U_{\mathcal{C}_{n-1}}\,|\Psi\rangle = T_{n-1}\,|\Psi\rangle\,.
\end{aligned}$$

Since this holds for all $n \geq \max\{N+2, n_0\}$, we conclude that $(T_n)_{n\geq N}$ is eventually constant on local states.

Let us now show that the limit $T = \lim_n T_n$ is an intertwiner from $\bar{\rho}^X_{\mathcal{C}}$ to $\bar{\rho}^X_{\mathcal{C}'}$. Fix $y \in \mathcal{A}^{\mathrm{loc}}$ and note that for all $n \in \mathbb{N}$ large enough so that $\operatorname{supp} Q_{n+1} \cap \operatorname{supp}(y) = \emptyset$ we have

$$T_n\left(U^*_{\mathcal{C}_n} y U_{\mathcal{C}_n}\right) = U_{\mathcal{C}'_n}\,u_{Q_{n+1}}\,y U_{\mathcal{C}_n} = U_{\mathcal{C}'_n}\,y\,u_{Q_{n+1}} U_{\mathcal{C}_n} = \left(U_{\mathcal{C}'_n} y U^*_{\mathcal{C}'_n}\right) T_n.$$

Since (T_n), $(U^*_{\mathcal{C}_n} y U_{\mathcal{C}_n})$, and $(U^*_{\mathcal{C}'_n} y U_{\mathcal{C}'_n})$ are sequences which are eventually constant on local states, converging to $T, \bar{\rho}^X_{\mathcal{C}}(y)$, and $\bar{\rho}^X_{\mathcal{C}'}(y)$ respectively, we conclude from Lemma 9.3 that

$$T\,\bar{\rho}^X_{\mathcal{C}}(y)\,|\Psi\rangle = \bar{\rho}^X_{\mathcal{C}'}(y)\,T\,|\Psi\rangle$$

for all $y \in \mathcal{A}^{\mathrm{loc}}$ and any local state $|\Psi\rangle \in \mathcal{A}^{\mathrm{loc}}\,|\Omega\rangle$. We then conclude that $T \in \mathsf{SSS}_f(\bar{\rho}^X_{\mathcal{C}} \to \bar{\rho}^X_{\mathcal{C}'})$ by density.

To see that T is unitary, we first note that by changing the orientation of the links $\{Q_n\}$ we obtain a bridge from $\mathcal{C}'$ to $\mathcal{C}$. Then it follows in the same way as above that the sequence $(T_n^*)_{n\geq N}$ is eventually constant on local states and therefore converges strongly to T^*. Using unitarity of T_n we find

$$T^*T\,|\Psi\rangle = \lim_n T_n^* T_n\,|\Psi\rangle = |\Psi\rangle\,,$$

and similarly $TT^*\,|\Psi\rangle = |\Psi\rangle$ for any local state $|\Psi\rangle \in \mathcal{A}^{\mathrm{loc}}\,|\Omega\rangle$ by Lemma 9.3. By density we conclude that T is unitary.

Finally, if $\Lambda \subset \mathbb{R}^2$ contains the supports of all links $\{L_n, L'_n\}_{n\in\mathbb{N}}$ and all links $\{Q_n\}_{n\geq N}$, then each T_n is supported in Λ and it follows that $T \in \mathcal{R}(\Lambda)$. □

Remark 10.6. *The endomorphisms $\bar{\rho}^Y_{\mathcal{C}}$ are SOT-continuous on $\pi^1(\mathcal{A}_\Lambda)$ for all chains $\mathcal{C}$, cones Λ, and $Y \in \operatorname{Irr} Z(\mathcal{C})$, so we observe that in the case of the above lemma where the SOT-convergence $T_n \to T$ takes place in $\pi^1(\mathcal{A}_\Lambda)''$ for an allowed cone Λ, then*

$$\bar{\rho}^Y_{\mathcal{C}}(T) = \lim_n \bar{\rho}^Y_{\mathcal{C}}(\pi^1(T_n)) = \lim_n \pi^1\Big[(U^Y_{\mathcal{C}_{n+1}})^* T_n U^Y_{\mathcal{C}_{n+1}}\Big]. \tag{79}$$

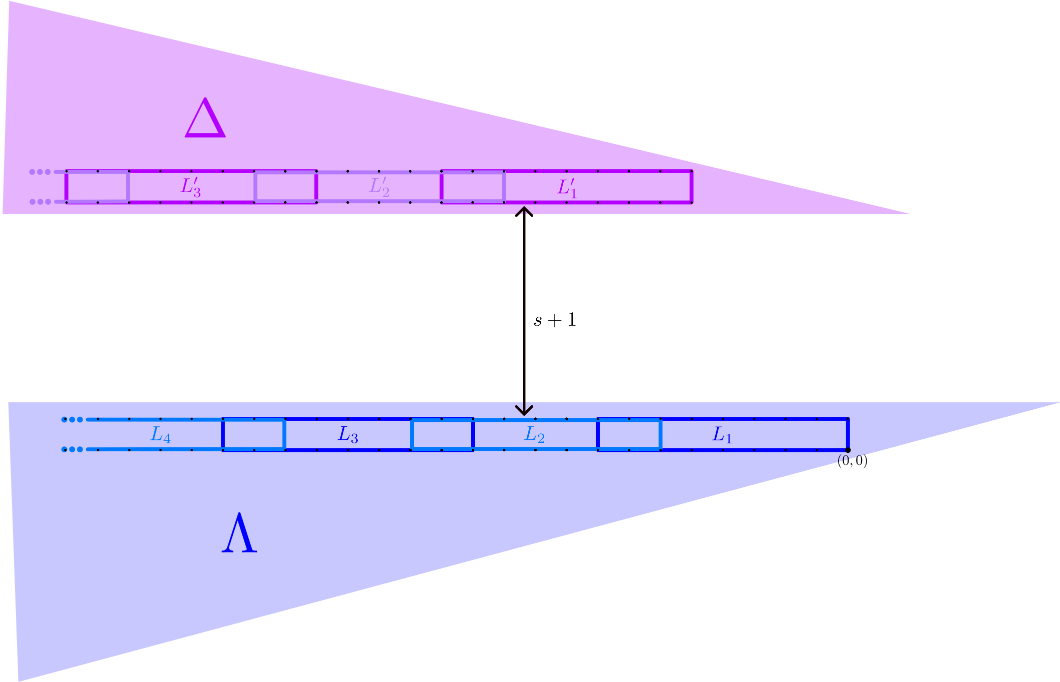


Figure 3.5: The fiducial chain $\mathscr{C}$ in blue, separated a distance $s+1$ from the right chain $\mathscr{C}'$ in purple above it. Also shown are cones Λ and Δ satisfying the assumptions of the braiding construction in Section C.2.

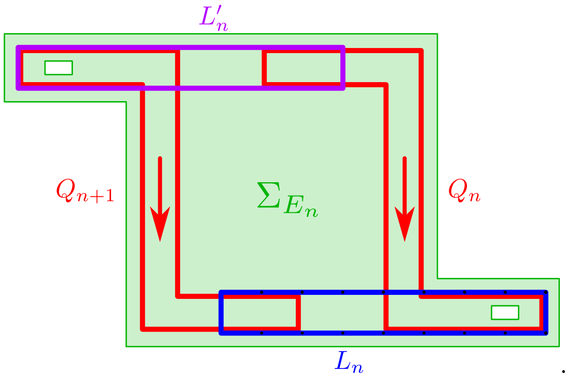
The sequence $\big((U^Y_{\mathscr{C}_{n+1}})^* T_n U^Y_{\mathscr{C}_{n+1}}\big)_{n \geq N}$ *is eventually constant on local states. A similar statement holds for* T^**. We omit the proof, with the remark that it is identical to the above with the additional presence of* Y *excitations.*

10.2 Isomorphism of R-symbols

For this subsection we fix a *fiducial chain* $\mathscr{C} = (L_n)$ as follows. Identify the set of faces $\mathcal{F}$ of $\mathbb{Z}^2$ with the dual lattice $(\mathbb{Z}^2)^*$ so that the face f whose bottom left vertex is $(x, y) \in \mathbb{Z}^2$ is given the coordinate $(x, y) \in (\mathbb{Z}^2)^*$. For each $n \in \mathbb{N}$ we take $L_n = ((-6n - 2 + i, 0))_{i=0}^{7}$, directed to the right. See figure 3.5. With the fiducial chain $\mathscr{C}$ fixed, we write $\bar{\rho}^X := \bar{\rho}^X_{\mathscr{C}}$ for any $X \in \operatorname{Irr} Z(\mathcal{C})$, and let $\Phi^Z_{XY} : Z(\mathcal{C})(X \otimes Y \to Z) \to \mathsf{SSS}_f(\bar{\rho}^X \times \bar{\rho}^Y \to \bar{\rho}^Z)$ be the maps constructed in Section 9.2 for the fiducial chain $\mathscr{C}$.

In order to compute R-symbols for SSS_f, we must compute the braidings $b_{Y,X} := b_{\bar{\rho}^Y, \bar{\rho}^X} : \bar{\rho}^Y \times \bar{\rho}^X \to \bar{\rho}^X \times \bar{\rho}^Y$ for any $X, Y \in \operatorname{Irr} Z(\mathcal{C})$. In order to do this, we fix a second chain $\mathscr{C}' = (L'_n)$ with links L'_n obtained from L_n by translating $s + 2$ lattice spacings up and five lattice spacings to the left. This guarantees that the endomorphisms $\bar{\rho}^X_{\mathscr{C}}$ and $\bar{\rho}^X_{\mathscr{C}'}$ are localized in allowed cones Λ and Δ respectively, satisfying the assumptions of the construction of the braiding in Section C.2, see Figure 3.5.

To compute the braiding $b_{Y,X}$ we require a unitary transporter $T^X : \bar{\rho}^X \to \bar{\rho}^X_{\mathscr{C}'}$. We will take T^X to be the transporter constructed in Proposition 10.5 using a bridge $(\{Q_n\}_{n \in \mathbb{N}}, \{E_n\}_{n \in N})$ between $\mathscr{C}$ and $\mathscr{C}'$ with the links Q_n and regions E_n specified for each $n \in \mathbb{N}$ as follows:



One easily checks that this indeed provides a bridge from $\mathcal{C}$ to $\mathcal{C}'$.

We then have

$$b_{Y,X} = (T^X)^* \,\bar{\rho}^Y(T^X). \tag{80}$$

Recall from appendix D that the R-symbols of SSS_f and $Z(\mathcal{C})$ are simply the actions of $- \circ b_{X,Y}$ and $- \circ \beta_{X,Y}$ on fusion spaces. The following lemma shows that the maps $\{\Phi^Z_{XY}\}$ intertwine the R-symbols in the sense described in Appendix D.

Lemma 10.7. *For any $X, Y, W \in \operatorname{Irr} Z(\mathcal{C})$, and $\alpha : X \otimes Y \to W$,*

$$\Phi^W_{XY}[\alpha] \circ b_{Y,X} = \Phi^W_{YX}\big[\alpha \circ \beta_{Y,X}\big].$$

Proof : From Lemma 9.7 we see that the claim is equivalent to

$$b^*_{Y,X} \circ \Phi^{XY}_W[\gamma] = \Phi^{YX}_W\big[\beta^\dagger_{Y,X} \circ \gamma\big],$$

where $\gamma = \alpha^\dagger : W \to X \otimes Y$. It suffices to verify this identity on the dense subspace $\pi^{\mathbb{1}}(\mathcal{A}^{\mathrm{loc}})\,|\Omega\rangle$ of local states. We shall henceforth omit $\pi^{\mathbb{1}}$ from notation and consider a local state $|\Psi\rangle = x\,|\Omega\rangle$ with $x \in \mathcal{A}^{\mathrm{loc}}$.

From The Excitation Lemma 9.5 and Definition 10.4 we can find an $N \in \mathbb{N}$ large enough so that the support of x is disjoint from E_n for all $n \geq N$ and

$$U^W_{\mathcal{C}_{n-1}}\,|\Psi\rangle = P^{(\mathbb{1},W)}_{E_n} U^W_{\mathcal{C}_{n-1}}\,|\Psi\rangle \tag{81}$$

for all $n \geq N$. We aim to show

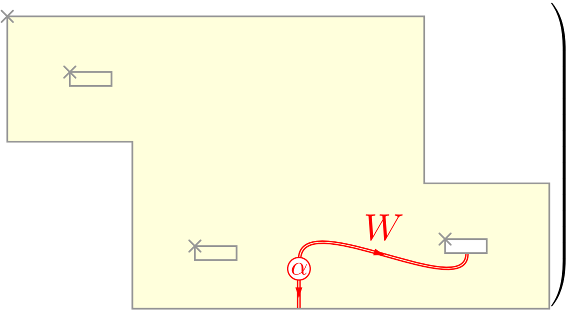
$$(u^X_{Q_n})^* (u^X_{L'_n})^* u^Y_{L_n} u^X_{Q_{n+1}} \mathrm{Dr}_{L_n}\big[\gamma\big] P^{(\mathbb{1},W)}_{E_n} = \mathrm{Dr}_{L_n}\big[\beta^\dagger_{Y,X} \circ \gamma\big] P^{(\mathbb{1},W)}_{E_n} \tag{82}$$

for all such n. Fix $n \geq N$, and let D_n be the region obtained from E_n by placing an additional puncture at $\partial_{\mathfrak{i}} L_n$, so Σ_{D_n} becomes a disk with three punctures. Write $K_n := Q_{n+1} \wedge L_n$ and $K'_n := L'_n \wedge Q_n$ so the region E_n supports a simple isotopy between K_n and K'_n. As in the discussion below Definition 10.1, we may take $\text{⚓}_{E_n} = \text{⚓}_{K_n}$. Note that we have $P^{(\mathbb{1},\mathbb{1},W)}_{D_n} = P^{(\mathbb{1},W)}_{E_n}$ where we use the ordering $(\partial_{\mathfrak{i}} L'_n, \partial_{\mathfrak{i}} L_n, \partial_{\mathfrak{f}} L_n)$ of the three punctures of D_n.

Recall from Section 6.1 that the range of $P_{D_n}^{(\mathbb{1},\mathbb{1},W)}$ is spanned by vectors of the form

$$\psi = \mathcal{D}^{-3} \times \Psi_{D_n}^{\text{⚓}_1}\left(\alpha \otimes w_0 \otimes w^{\mathbb{1}} \otimes w^{\mathbb{1}} \otimes w^{W}\right) \tag{83}$$

$$= \left(d_{X_0} d_W\right)^{1/2} \sigma_{D_n}^{-1} \Bigg($$

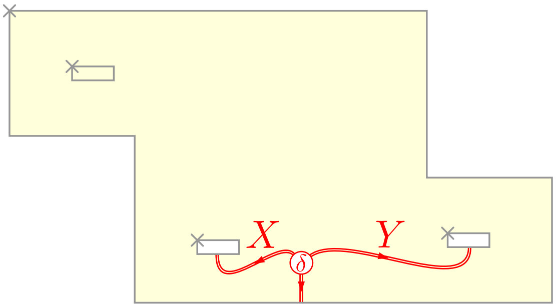


$$\Bigg) ,$$

for arbitrary boundary conditions $w_0 : \bar{W} \to \chi^{\otimes \mathcal{S}_0^{\text{⚓}_1}}$. Here, the anchor ⚓_1 is as shown in Figure 3.6 and we use the graphical representation introduced in Eq. (47) for the unitary $\zeta_W : W^* \to \bar{W}$. We adopt the convention that we do not indicate vacuum lines and that the vacuum boundary condition $w^{\mathbb{1}}$ is imposed on boundary components with no (= vacuum) attaching strands (cf. Convention 6.1). We also fill in the punctures at such boundary components to emphasize that they are cloaked and any strand of a string diagram can be pulled across them by isotopy without changing without changing the vector being represented. We also drop the specification of the outer boundary condition w_0 from the figures as it will be fixed throughout the upcoming computation.

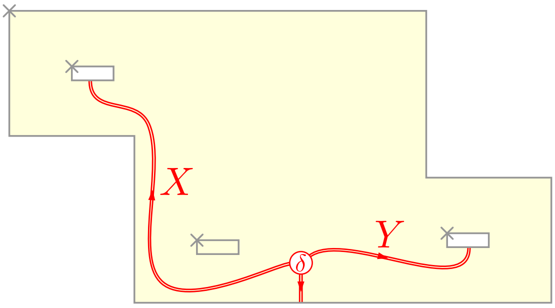
Further dropping the notation for the isomorphism σ_{D_n}, we compute the action of the operator $(u_{Q_n}^X)^* (u_{L_n'}^X)^* u_{L_n}^Y u_{Q_{n+1}}^X \mathrm{Dr}_{L_n}[\gamma]$ on a vector ψ of this form. First, using the Inclusion Lemma 6.7 for $L_n \subset D_n$ with anchors ⚓_{L_n} and ⚓_1 respectively, we get

$$\psi \xmapsto{\mathrm{Dr}_n[\gamma]} \left(d_{X_0} d_X d_Y\right)^{1/2}$$



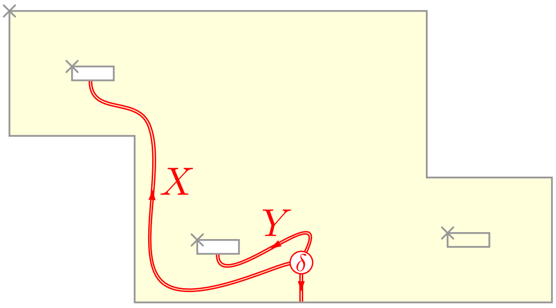
Applying the Inclusion Lemma 6.7 to $Q_{n+1} \subset D_n$ using anchors $\text{⚓}_{Q_{n+1}}$, and ⚓_1 respectively, now yields

$$\xmapsto{u_{Q_{n+1}}^X} \left(d_{X_0} d_X d_Y\right)^{1/2}$$



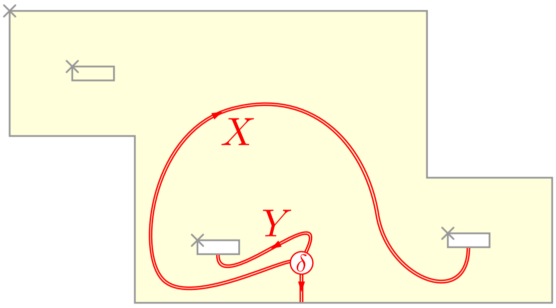
A similar application of the Inclusion Lemma 6.7 shows

$$\xmapsto{u^Y_{L_n}} \left(d_{X_0} d_X d_Y\right)^{1/2}$$



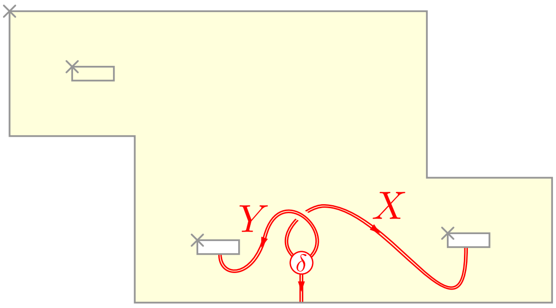
Note that the resulting vector is in the range of $P^{X\,\mathbb{1}}_{K'_n}$. We can therefore apply Lemma 10.3 and self-adjointness of $u^X_{L'_n}$ and u_{Q_n} to see that the action of $(u^X_{Q_n})^*(u^X_{L'_n})^*$ on this vector is the same as the action of $u^X_{K'_n}$. In order to apply the Inclusion Lemma 6.7 to understand the action of $u^X_{K'_n}$, we consider the anchor ⚓_2 shown in Figure 3.6 which extends $\text{⚓}_{K'_n}$. The corresponding $\mathrm{Dr}^{\text{⚓}_2}_{D_n}$-action is defined in terms of $\Psi^{\text{⚓}_2}_{D_n}$ and it is clear how the given state is parametrised under $\Psi^{\text{⚓}_2}_{D_n}$ (in the sense of Eq. (83)), so we obtain

$$\xmapsto{(u^X_{Q_n})^*(u^X_{L'_n})^*} \left(d_{X_0} d_X d_Y\right)^{1/2}$$



Finally, using isotopy and Eq. (10), and recalling Convention 6.1, we note that that the X-strand can be pulled through the puncture at $\partial_{\mathsf{i}} L_n$, ending up below the Y-strand as follows:

$$= \left(d_{X_0} d_X d_Y\right)^{1/2}$$



$$= \mathrm{Dr}_{L_n}\left[\beta^\dagger_{Y,X} \circ \gamma\right] \psi,$$

where we again used the Inclusion Lemma 6.7 in the last step. Since $\psi \in \mathrm{Ran} P^{(\mathbb{1},\mathbb{1},W)}_{D_n} = \mathrm{Ran} P^{(\mathbb{1},W)}_{E_n}$ was arbitrary, this proves (82).

Let us now use the results (81) and (82) to prove the claim of the lemma. From Remark 10.6 we have that the operator $\rho^Y((T^X)^*)$ is the limit of a sequence which is eventually constant on

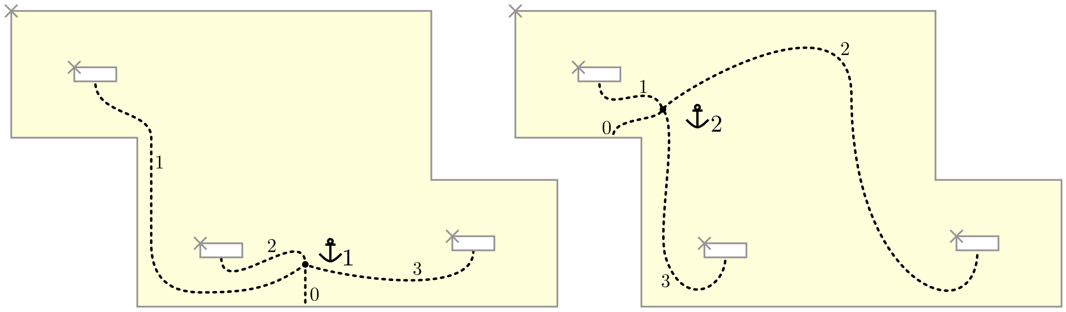


Figure 3.6: Anchors used in the proof of Lemma 10.7.

local states. So, using Lemma 9.3, we compute

$$\begin{aligned}
&\bar{\rho}^Y \left(T^X\right)^* T^X \, \Phi_W^{XY}[\gamma] \, |\Psi\rangle \\
&= (U^Y_{\mathcal{C}_n})^* (U^X_{\mathcal{C}_{n-1}})^* (u^X_{Q_n})^* U^X_{\mathcal{C}'_{n-1}} U^Y_{\mathcal{C}_n} \times (U^X_{\mathcal{C}'_n})^* u^X_{Q_{n+1}} U^X_{\mathcal{C}_n} \times (U^X_{\mathcal{C}_n})^* (U^Y_{\mathcal{C}_{n-1}})^* \mathrm{Dr}_{L_n}[\gamma] U^W_{\mathcal{C}_{n-1}} \, |\Psi\rangle \\
&= (U^Y_{\mathcal{C}_n})^* (U^X_{\mathcal{C}_{n-1}})^* (u^X_{Q_n})^* (u^X_{L'_n})^* u^Y_n \, u^X_{Q_{n+1}} \mathrm{Dr}_{L_n}[\gamma] U^W_{\mathcal{C}_{n-1}} \, |\Psi\rangle \\
&= (U^Y_{\mathcal{C}_n})^* (U^X_{\mathcal{C}_{n-1}})^* \mathrm{Dr}_{L_n}\left[\beta^\dagger_{Y,X} \circ \gamma\right] U^W_{\mathcal{C}_{n-1}} \, |\Psi\rangle \\
&= \Phi_W^{YX}\left[\beta^\dagger_{Y,X} \circ \gamma\right] |\Psi\rangle \, .
\end{aligned}$$

Since the local state $|\Psi\rangle$ was arbitrary, the claim now follows by density. ☐

11 Proof of Theorem 1.1

Proof : Fix the fiducial chain $\mathcal{C}$ as in Section 10.2 and write $\bar{\rho}^X = \bar{\rho}^X_{\mathcal{C}}$ for all $X \in \operatorname{Irr} Z(\mathcal{C})$.

Recall from Section C that under Assumption 2.3 the category SSS_f is a unitary braided tensor category. Corollary 9.1 moreover implies that SSS_f is *finite* semisimple, the set $\operatorname{Irr} \mathsf{SSS}_f := \{\bar{\rho}^X\}_{X \in \operatorname{Irr} Z(\mathcal{C})}$ being a representative set of simple objects for SSS_f. The map $X \mapsto \bar{\rho}^X$ provides a bijection from $\operatorname{Irr} Z(\mathcal{C})$ to $\operatorname{Irr} \mathsf{SSS}_f$. By Lemmas 9.9 and 9.11 we have that the maps $\{\Phi^Z_{XY}\}_{X,Y,Z \in \operatorname{Irr} Z(\mathcal{C})}$ are intertwiners of F and R-symbols of $Z(\mathcal{C})$ and SSS_f with respect to these sets of simples. Proposition 9.10 says that $\{\Phi^Z_{XY}\}$ preserve orthogonal direct sum decompositions. It follows from Proposition D.1 that $Z(\mathcal{C})$ and SSS_f are equivalent as unitary braided tensor categories. ☐

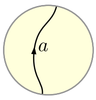
A Proof of Proposition 3.5

A.1 Orthonormal basis for skein modules on disks and annuli

In the course of the proof of Proposition 3.5, as well as in Appendix B, we will often make use of the gluing law Eq. (15). In preparation, let us here give some explicit bases of some skein modules on disks and annuli.

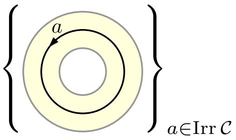
1. The disk D_1 with one marked boundary point : Suppose the boundary point is labelled by $a \in \mathrm{Irr}\,\mathcal{C}$. If $a \neq \mathbb{1}$ then any string diagram for this skein module evaluates to zero, i.e. $A(D_1; a) = \{0\}$ if $a \neq \mathbb{1}$. If $a = \mathbb{1}$ then any string diagram evaluates to a complex number and we have $A(D_1; \mathbb{1}) \simeq \mathbb{C}$. A unit vector is represented by the empty string diagram on the disk.
2. The disk D_2 with one incoming and one outgoing boundary point : Suppose the boundary points are labelled by $a, b \in \mathrm{Irr}\,\mathcal{C}$. If $a \neq b$ then any string diagram for this skein module evaluates to zero and we have $A(D_2, a \cup b) = \{0\}$. If $a = b$ then any string diagram can be interpreted as a morphism from a to a, so $A(D_2; a \cup a) \simeq \mathbb{C}$ and is spanned by the unit vector

$$\frac{1}{d_a^{1/2}}$$



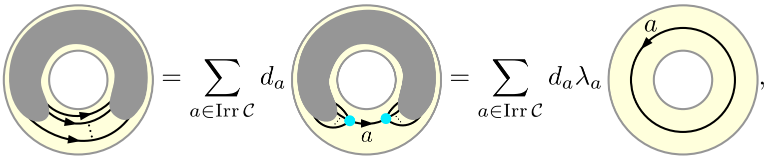
. (84)

3. The annulus $\mathbb{A}$ with no marked boundary points : An orthonormal basis is given by



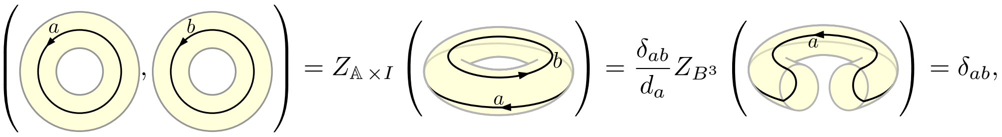
$$\left\{ \quad \right\}_{a \in \mathrm{Irr}\,\mathcal{C}} . \tag{85}$$

Indeed, any string diagram on $\mathbb{A}$ can by isotopy be put in the form of the left hand side of



$$= \sum_{a \in \mathrm{Irr}\,\mathcal{C}} d_a \quad = \sum_{a \in \mathrm{Irr}\,\mathcal{C}} d_a \lambda_a \quad ,$$

where the grey blob contains some arbitrary string diagram. The first equality is (8), and for the second equality we note that the grey blob together with the coloured vertices evaluates to some $\lambda_a \mathrm{id}_a \in \mathcal{C}(a \to a) \simeq \mathbb{C}$. We conclude that Eq. (85) is indeed a basis. To see that it is orthonormal, compute



$$\left(\quad , \quad \right) = Z_{\mathbb{A} \times I} \left(\quad \right) = \frac{\delta_{ab}}{d_a} Z_{B^3} \left(\quad \right) = \delta_{ab},$$

where we evaluate $Z_{\mathbb{A} \times I}$ using the gluing law (15) by cutting the solid torus transversally to the a and b strings. The summation of (15) runs over the unique basis vector (84).

A.2 Orthonormal basis of skein modules on punctured spheres

Let Σ be an extended surface homeomorphic to a sphere with m holes cut out, and let ⚓ be an anchor for Σ. Let $\mathbf{X} : \mathrm{Bd}(\Sigma) \to \mathrm{Irr}(Z(\mathcal{C}))$, writing $X_\kappa = \mathbf{X}(\mathcal{S}_\kappa^{\text{⚓}})$. Let $\underline{\mathbf{a}}$ be a boundary condition on $A(\Sigma)$ specified by a function on $\mathrm{Bd}(\Sigma)$ defined by

$$\underline{a}_\kappa = \underline{\mathbf{a}}(\mathcal{S}_\kappa^{\text{⚓}}) : m_{\mathcal{S}_\kappa^{\text{⚓}}} \to \mathrm{Irr}(\mathcal{C}).$$

For given $\mathbf{X}, \underline{\mathbf{a}}$, let $\mathbf{i} : \mathrm{Bd}(\Sigma) \to \mathbb{N}$ be such that $i_\kappa = \underline{\mathbf{i}}(\mathcal{S}_\kappa^{\text{⚓}})$ belongs to the index set of the chosen basis of $\mathcal{C}(X_\kappa \to \otimes \underline{a}_\kappa)$. The set of such functions $\mathbf{i}$ label the corresponding product basis of

$$\mathcal{C}(X_1 \to \otimes \underline{a}_1) \otimes \cdots \otimes \mathcal{C}(X_m \to \otimes \underline{a}_m),$$

and we refer to $\mathbf{i}$ as indexing the $(\mathbf{X}, \underline{\mathbf{a}})$-basis. For every m-tuple consisting of $X_1, \ldots, X_m \in \mathrm{Irr}\, Z(\mathcal{C})$, fix an orthonormal basis $\{\alpha_l\}$ of $Z(\mathcal{C})(\mathbb{1} \to X_1 \otimes \cdots \otimes X_m)$ with respect to the trace inner product.

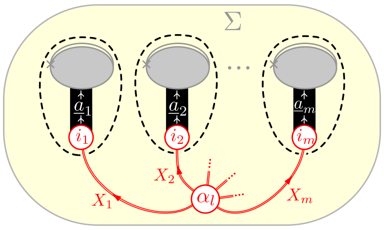
Lemma A.1. *The elements,*

$$e_{\mathbf{i},l}^{\text{⚓};\mathbf{X},\underline{\mathbf{a}}} := \Phi_\Sigma^{\text{⚓}}(\alpha_l \otimes w_{i_1}^{X_1\,\underline{a}_1} \otimes \cdots \otimes w_{i_m}^{X_m\,\underline{a}_m}) \tag{86}$$

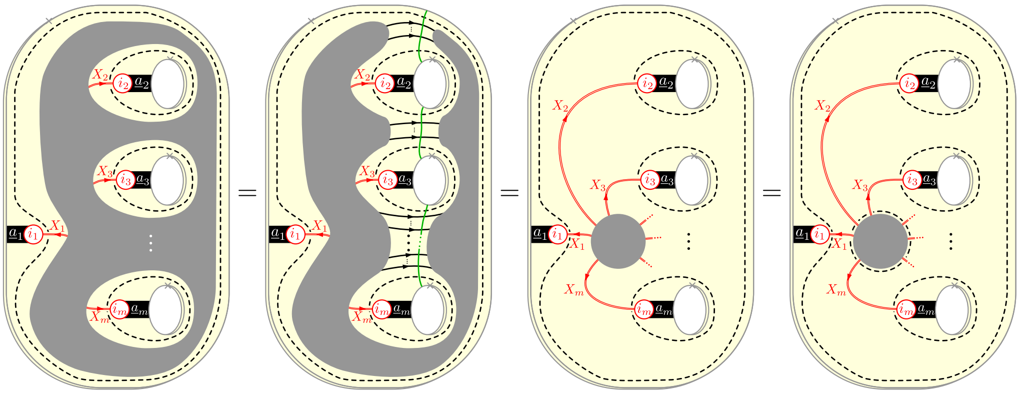
ranging over all $\mathbf{X} : \mathrm{Bd}(\Sigma) \to \mathrm{Irr}\, Z(\mathcal{C})$, *all boundary conditions* $\underline{\mathbf{a}}$ *on* Σ, *all* $\mathbf{i}$ *indexing the* $(\mathbf{X}, \underline{\mathbf{a}})$-*basis, and all* l *indexing the basis of* $Z(\mathcal{C})(\mathbb{1} \to X_1 \otimes \cdots \otimes X_m)$, *form an orthonormal basis of* $A(\Sigma)$ *w.r.t. the TQFT inner product. In particular, the map* $\Phi_\Sigma^{\text{⚓}}$ *of Proposition 3.5 is an isomorphism of Hilbert spaces.*

Proof : Throughout this proof, we drop the anchor from notation. For any $\mathbf{X}, \underline{\mathbf{a}}, \mathbf{i}, l$, let

$$\tilde{e}_{\mathbf{i},l}^{\mathbf{X},\underline{\mathbf{a}}} = \left(\prod_{\kappa=1}^{m} d_{X_\kappa}^{-1/2}\right) \mathcal{D}^{1-m} \times e_{\mathbf{i},l}^{\mathbf{X},\underline{\mathbf{a}}} = \quad \in A(\Sigma) \tag{87}$$



Using the resolution of identity from Eq. (14) we find that an arbitrary element of $A(\Sigma)$ can always be written as a linear combination of elements of the form on the left hand side of Eq. (88), where we present Σ as a disk with $m-1$ holes cut out of the interior, and the grey blob contains an arbitrary string diagram.

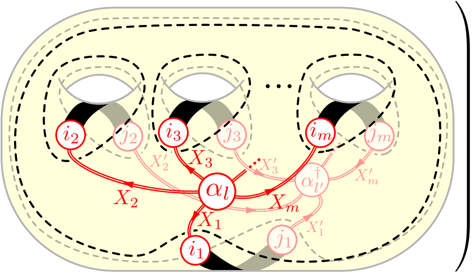


(88)

The string diagram contained in the grey blob may be moved past the punctures by isotopy and the cloaking property. In more detail, draw green lines connecting boundary components $\mathcal{S}_\kappa$ and $\mathcal{S}_{\kappa+1}$ for $\kappa = 1, \cdots, m-1$ such that these green lines do not intersect the anchor. This condition makes the green lines unique up to isotopy of Σ, keeping the anchor fixed. By isotopy invariance one can always arrange the string diagram such that the green lines do not contain vertices of the string diagram and such that edges of the string diagram intersect the green lines transversally, as shown in the second panel of Eq. (88). In the next step we use the cloaking property of the dotted lines and isotopy to deform the grey blob to a contractible region as in the third panel. Finally, we can insert a new small dotted loop and use the cloaking property of the pre-existing dotted lines to obtain a string diagram as in the final panel of Eq. (88). In this final diagram, the grey blob surrounded by the dotted loop represents a morphism in $Z(\mathcal{C})(\mathbb{1} \to X_1 \otimes \cdots \otimes X_m)$ which can be written as a linear combination of the basis elements $\{\alpha_l\}$. In all, we conclude that vectors $\tilde{e}_{\mathbf{i},l}^{\mathbf{X},\underline{\mathbf{a}}}$ span the skein module $A(\Sigma)$.

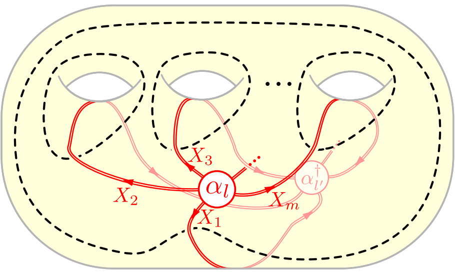
Let us now show that these vectors form an orthogonal family w.r.t. the TQFT inner product. This follows from the following computation, where we again present Σ as a disk with $m-1$ holes cut out:

$$\left(\tilde{e}_{\mathbf{j},l'}^{\mathbf{X}',\underline{\mathbf{a}}'}, \tilde{e}_{\mathbf{i},l}^{\mathbf{X},\underline{\mathbf{a}}}\right)_{A(\Sigma)} = \left(\prod_{\kappa=1}^{m} \delta_{\underline{a}_\kappa,\underline{a}'_\kappa}\right) Z_{\Sigma\times I} \left(\vphantom{\prod} \right) .$$



For each connected boundary component of Σ we now have a pair of dotted lines going around the corresponding hole of $\Sigma \times I$. Using the cloaking property of the dotted lines we pull one of each pair over the other to encircle the $w_{i_\kappa}^{X_\kappa \underline{a}_\kappa}$ and $(w_{j_\kappa}^{X'_\kappa \underline{a}'_\kappa})^\dagger$ morphisms, and use Eq. (13) to obtain

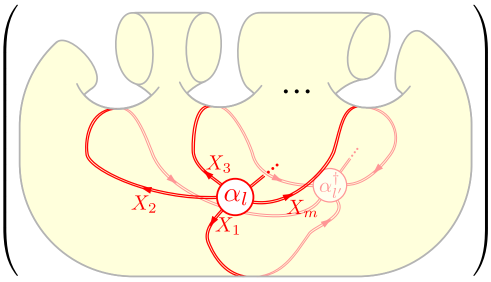
$$= \left(\prod_{\kappa=1}^{m} \frac{\delta_{\underline{a}_\kappa,\underline{a}'_\kappa}\, \delta_{X_\kappa,X'_\kappa}\, \delta_{i_\kappa\, j_\kappa}}{d_{X_\kappa}}\right) Z_{\Sigma\times I} \left(\vphantom{\prod} \right) .$$



The resulting diagram only contains dotted lines and lines labelled by objects of $Z(\mathcal{C})$. We may contract the outer dotted line to a point using the cloaking property and so get rid of it. Now evaluate $Z_{\Sigma\times I}$ using the gluing formula (15) by cutting $\Sigma \times I$ along $m-1$ disks transversal to the remaining dotted loops to obtain a 3-ball and summing over bases Eq. (84). We find

$$= \frac{1}{\mathcal{D}^{2(m-1)}} \left(\prod_{\kappa=1}^{m} \frac{\delta_{\underline{a}_\kappa,\underline{a}'_\kappa}\,\delta_{X_\kappa,X'_\kappa}\,\delta_{i_\kappa,j_\kappa}}{d_{X_\kappa}} \right)$$

$$\times Z_{B^3}$$



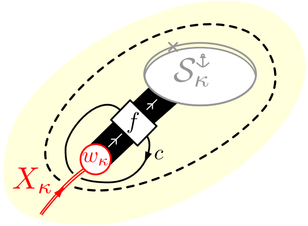
$$= \frac{\delta_{ll'}}{\mathcal{D}^{2(m-1)}} \left(\prod_{\kappa=1}^{m} \frac{\delta_{\underline{a}_\kappa,\underline{a}'_\kappa}\,\delta_{X_\kappa,X'_\kappa}\,\delta_{i_\kappa,j_\kappa}}{d_{X_\kappa}} \right).$$

Comparing to (87), we conclude that $e^{\mathbf{X},\underline{\mathbf{a}}}_{\mathbf{i},l}$ are the elements of an orthonormal basis of $A(\Sigma)$ with respect to the TQFT inner product. □

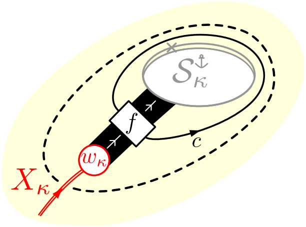
A.3 Intertwining Tube-actions

Proof of Proposition 3.5 : By Lemma A.1 we have that $\Phi_\Sigma^{⚓}$ is an isomorphism of Hilbert spaces. It remains to verify that $\Phi_\Sigma^{⚓}$ intertwines the Tube-actions on $A(\Sigma)$ described in Section 3.4 with those on the morphism spaces given by given by Eq. (16). This can be seen by using the cloaking property of the dotted line as follows. For $\mathcal{S}_\kappa^{⚓} \in \mathrm{Bd}(\Sigma)$, take $a \in \mathrm{Tube}_{\mathcal{S}_\kappa^{⚓}}$ corresponding to an $f \in \mathcal{C}(c \otimes \chi^{\otimes \mathcal{S}_\kappa^{⚓}} \to \chi^{\otimes \mathcal{S}_\kappa^{⚓}} \otimes c)$ as on the left-hand side of Eq. (16). Such a span the Tube algebra. Take further $\alpha \in Z(\mathcal{C})(\mathbb{1} \to X_1 \otimes \cdots \otimes X_m)$ and $w_\kappa \in \mathcal{C}(X_\kappa \to \chi^{\otimes \mathcal{S}_\kappa^{⚓}})$, and write $K := \left(\prod_{\kappa=1}^m d_{X_\kappa}^{1/2}\right) \mathcal{D}^{m-1}$. Then

$$\Phi_\Sigma^{⚓}\left(\alpha \otimes w_1 \otimes \cdots \otimes \left(a \triangleright w_\kappa\right) \otimes \cdots \otimes w_m\right) = K \times$$



$$= K \times$$



$$= a \triangleright_\kappa \Phi_\Sigma^{⚓}\left(\alpha \otimes w_1 \otimes \cdots \otimes w_\kappa \otimes \cdots \otimes w_m\right),$$

where we only depicted the string diagrams near the boundary component $\mathcal{S}_\kappa^{⚓}$. □

B Proof of Proposition 4.4

In order to prove the Proposition, we must compare the TQFT inner product on A_{Σ_C} with the skein inner product on H_C. To do this, we first compare the TQFT inner product on $A(\Sigma_{C_1}) \simeq H_{C_1}$ with the skein inner product on the string-net subspace H_{C_1}:

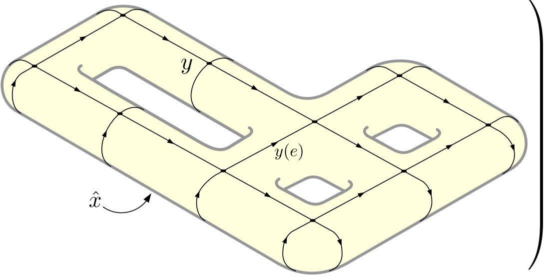
Lemma B.1. *Let C be a finite region and let $\phi, \psi \in H_{C_1}$. Then*

$$\big(\sigma_{C_1}(\phi), \sigma_{C_1}(\psi)\big)_{A(\Sigma_{C_1})} = \langle \phi, \psi \rangle.$$

i.e. $\sigma_{C_1} : H_{C_1} \to A(\Sigma_{C_1})$ is an isometry.

Proof : Let $\phi_x = \bigotimes_{v \in V_C} \phi_{x(v)}$ and $\phi_y = \bigotimes_{v \in V_C} \phi_{y(v)}$ be product states in $H^b_{C_1}$ corresponding under π_{C_1} to string diagrams x and y on Σ_{C_1}. We represent

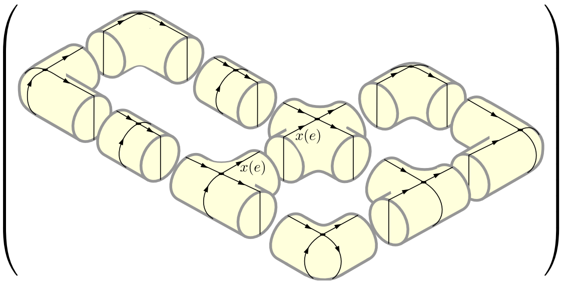
$$(x, y)_{A(\Sigma_{C_1})} = Z_{\Sigma_{C_1} \times I}\left(\vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right).$$

We cut $\Sigma_{C_1} \times I$ transversal to edges of C_1 and use the gluing formula (15), summing over bases Eq (84) to obtain

$$(x, y)_{A(\Sigma_{C_1})} = \left(\prod_{e \in E_C} \frac{\delta_{x(e), y(e)}}{d_{x(e)}} \right) Z_{\bigsqcup_{v \in V_C} B^3}\left(\vphantom{\int} \right.$$



$$\left. \vphantom{\int} \right).$$

Using the product property of the partition function we now find

$$\begin{aligned}
(x, y)_{A(\Sigma_{C_1})} &= \left(\prod_{e \in E_C} \frac{\delta_{x(e), y(e)}}{d_{x(e)}} \right) \prod_{v \in V_C} \left(\prod_{e \in \vec{\partial} v} d^{1/2}_{x(e)} \right) \langle \phi_{x(v)}, \phi_{y(v)} \rangle \\
&= \left(\prod_{e \in \vec{\partial} C} d^{1/2}_{x(e)} \right) \langle \phi_x, \phi_y \rangle = d^{1/2}_{\partial x} \langle \phi_x, \phi_y \rangle.
\end{aligned}$$

Recall that $\sigma_{C_1}(\phi_x) = d^{-1/4}_{\partial x} [x]_{\Sigma_{C_1}}$ and likewise for ϕ_y. Since $H^b_{C_1}$ is spanned by vectors of this form, and since both inner products vanish whenever boundary labellings of ϕ and ψ do not match, we conclude that σ_{C_1} is an isometry. ☐

We proceed to consider the TQFT inner product on $A(\Sigma_C)$.

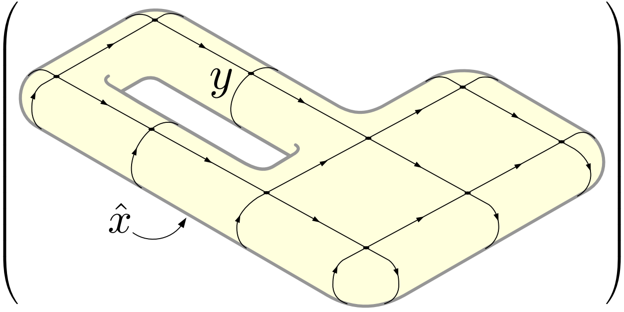
Lemma B.2. *Let C be a finite region and $\phi, \psi \in H_C$. Then*

$$\big(\sigma_C(\phi), \sigma_C(\psi)\big)_{A(\Sigma_C)} = \langle \phi, \psi \rangle,$$

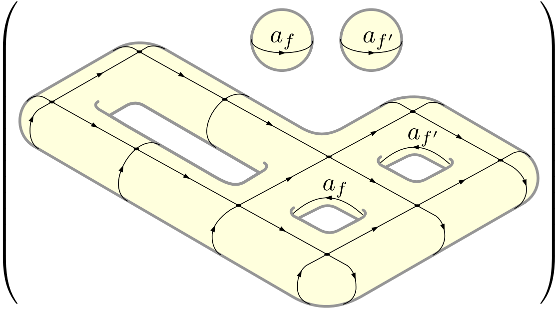
i.e. $\sigma_C|_{H_C} : H_C \to A(\Sigma_C)$ is an isometry.

Proof : Let $\phi_x = \bigotimes_{v\in V_C} \phi_{x(v)}$ and $\phi_y = \bigotimes_{v\in V_C} \phi_{y(v)}$ be product states in $H^b_{C_1}$ corresponding under π_{C_1} to string diagrams x and y on Σ_{C_1} with boundary condition b. We relate the TQFT inner product on $A(\Sigma_C)$ to the one on $A(\Sigma_{C_1})$ by using the gluing formula (15). Indeed $\Sigma_C \times I$ is obtained from $\Sigma_{C_1} \times I$ by patching the holes corresponding to the internal faces F_C. A hole is patched by gluing a 3-ball along an annulus around the hole, so the gluing formula gives rise to a summation over basis elements of the annulus with no marked points (85) for each hole. Using the product property of the partition function, the partition function on $\Sigma_C \times I$ is in this way written as a sum of products of a partition function on Σ_{C_1} with partition functions on the glued in 3-balls. The 3-balls have basis elements (85) placed on a belt on their surfaces, which evaluate to the quantum dimensions d_a. In this way we find

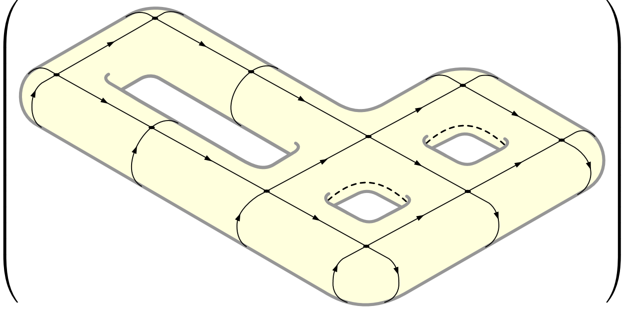
$$([x],[y])_{A(\Sigma_C)} = Z_{\Sigma_C \times I}$$



$$= \sum_{(a_f)\in(\mathrm{Irr}\,\mathcal{C})^{F_C}} Z_{(\Sigma_{C_1}\times I)\sqcup B^3\sqcup\cdots\sqcup B^3}$$



$$= \mathcal{D}^{2|F_C|}\, Z_{\Sigma_{C_1}\times I}$$



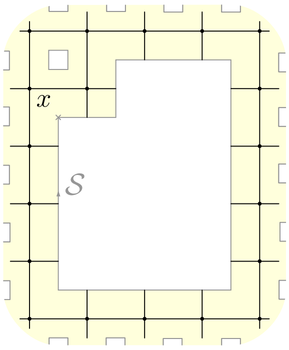
$$= \mathcal{D}^{2|F_C|} d_{\partial x}^{1/2}\, \langle \phi_x, B_C \phi_y \rangle\,,$$

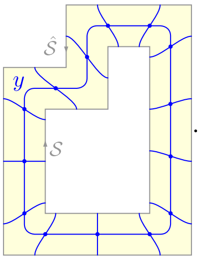
where we used Lemma B.1 in the last step to relate the TQFT inner product on $A(\Sigma_{C_1})$ to the skein inner product. Recalling the definition of σ_C, we obtain $(\sigma_C(\phi_x), \sigma_C(\phi_y))_{A(\Sigma_C)} = \langle B_C\phi_x, B_C\phi_y\rangle$. Since any $\phi, \psi \in H_C^b$ can be expressed as linear combinations of vectors of the form ϕ_x and ϕ_y, and both inner products vanish whenever the boundary conditions do not match, this concludes the proof. □

Lemma B.3. *Let C be a finite region. For all $\phi \in H_{C_1}$, and all $a \in \mathrm{Tube}_{\mathcal{S}}$ with $\mathcal{S} \in \mathrm{Bd}(\Sigma_C)$,*

$$\sigma_C\big(\mathfrak{t}_{\mathcal{S}}(a)\phi\big) = a \triangleright_{\mathcal{S}} \sigma_C(\phi).$$

Proof : Recall that σ_C is defined on the whole of H_{C_1}. Consider a product state $\phi_x \in H_{C_1}$ corresponding to a string diagram x on Σ_{C_1}. Let $a = [y] \in \mathrm{Tube}_{\mathcal{S}}$ corresponding to a string diagram y with boundary condition $y_{\hat{\mathcal{S}}}$ matching $x_{\mathcal{S}}$. Schematically,

$\phi_x =$ 

, $a =$ 

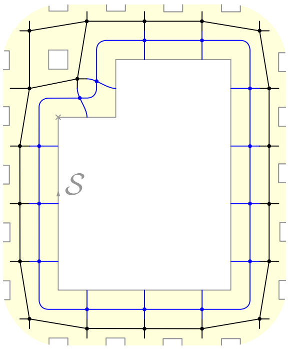
.

Since ϕ_x is a product state it factorises as $\phi_x = \phi_{\mathcal{S}} \otimes \phi_{\mathcal{S}^c}$, where $\phi_{\mathcal{S}} \in H_{C^{\mathcal{S}}}$, and $\phi_{\mathcal{S}^c} \in \mathcal{H}_{C_0 \setminus C_0^{\mathcal{S}}}$. The locality of local relations implies,

$$\mathfrak{t}_{\mathcal{S}}(a)\phi_x = \sigma^{-1}_{C_1^{\mathcal{S}}}\big(a \triangleright_{\mathcal{S}} \sigma_{C^{\mathcal{S}}}(\phi_{\mathcal{S}})\big) \otimes \phi_{\mathcal{S}^c} = \sigma^{-1}_{C_1}\big(a \triangleright_{\mathcal{S}} \sigma_{C_1}(\phi_x)\big).$$

In the graphical representation (Convention 4.2), this is illustrated by

$$\mathfrak{t}_{\mathcal{S}}(a)\phi_x = \left(\frac{d_{y_{\mathcal{S}}}}{d_{x_{\mathcal{S}}}}\right)^{1/4}$$



.

As a result,

$$\sigma_C\big(\mathfrak{t}_{\mathcal{S}}(a)\phi_x\big) = \mathcal{D}^{-|F_C|} d_{x_{\mathcal{S}}}^{-1/4}\big(a \triangleright_{\mathcal{S}} [x]_{\Sigma_C}\big) = a \triangleright_{\mathcal{S}} \sigma_C(\phi_x).$$

If the boundary conditions $x_{\mathcal{S}}$ and $y_{\hat{\mathcal{S}}}$ do not match, then this equality still holds because either side vanishes. The claim now follows because vectors of the form ϕ_x span H_{C_1}. □

Lemma B.4. *Let C be a finite region, then*

$$\sigma_C(\phi) = \sigma_C(B_C \phi)$$

for all $\phi \in H_{C_1}$. The same holds true with σ_C replaced by π_C.

Proof : Let us first prove the claim for π_C. It is sufficient to show that $\pi_C(\phi_x) = \pi_C(B_C \phi_x)$ for any string-net diagram x on Σ_{C_1}.

Recalling that B_f acts on the string-net subspace H_{Cf} as a Tube action, we find using Lemma B.3 that $B_C \phi_x = \pi_{C_1}^{-1}\big([\tilde{x}]_{\Sigma_{C_1}}\big)$ where $\tilde{x}$ is the string diagram on Σ_{C_1} obtained from x by inserting dotted loops around the holes corresponding to the faces of C. Since $\pi_C = \iota_C \circ \pi_{C_1}$,

$$\pi_C\big(B_C \phi_x\big) = \iota_C([\tilde{x}]_{\Sigma_{C_1}}) = [\tilde{x}]_{\Sigma_C} = [x]_{\Sigma_C} = \pi_C(\phi),$$

where in the penultimate step we noted that, in Σ_C, we can contract all the dotted loops to a point, i.e. remove them by local relations on Σ_C.

The claim for σ_C now follows by noting that B_C preserves boundary labellings, and σ_C differs from π_C only by factors depending on the boundary labelling. □

Proof of Proposition 4.4 : As a composition of surjective maps ι_C and σ_{C_1}, σ_C is also surjective. Surjectivity of the restriction $\sigma_C|_{H_C}$ follows from Lemma B.4. Lemma B.2 shows that $\sigma_C|_{H_C}$ is an isometry, and Lemma B.3 shows that σ_C is an intertwiner. □

C The category of superselection sectors

For completeness, we review the construction of the braided C*-tensor category of superselection sectors, in detail, under the assumption of bounded spread Haag duality. See also [Bha+25, Section 6.1] for a slightly different presentation.

Note that while this section deals with the SSS category associated with the ground state $\omega^{\mathbb{1}}$ of our Levin-Wen model, the discussions in this section are actually valid for any pure gapped ground state $\omega^{\mathbb{1}}$ on a two-dimensional spin system.

C.1 Extending anyon representations to endomorphisms

Anyon sectors capture the types of anyonic excitations that are supported by the ground state $\omega^{\mathbb{1}}$. In order to investigate the fusion of these anyons, we need a way to compose anyon representations. It turns out that under the assumption of bounded spread Haag duality (Assumption 2.3), any anyon representation can be extended from $\mathcal{A}$ to an *endomorphism* of the allowed algebra defined in Eq. (5) (in fact the weaker notion of approximate Haag duality is sufficient [Oga22]). The fusion of anyons then corresponds to the composition of such endomorphisms.

Lemma C.1. *Let $\pi : \mathcal{A} \to B(\mathcal{H})$ be an anyon representation. There is a unique WOT-continuous representation $\rho : \mathcal{B} \to B(\mathcal{H})$ such that $\rho \circ \pi^{\mathbb{1}} = \pi$. We call ρ the extension of π to $\mathcal{B}$.*

Proof : Since π is an anyon representation there is a unitary V_Λ for every cone Λ, such that $\mathrm{Ad}[V_\Lambda] \circ \pi^{\mathbb{1}}$ is localized in Λ^c. Equivalently, $\pi|_{\mathcal{A}_\Lambda} = \mathrm{Ad}[V_\Lambda] \circ \pi^{\mathbb{1}}|_{\mathcal{A}_\Lambda}$. Note that if $\Lambda \subset \Lambda'$ then

$$\mathrm{Ad}[V_{\Lambda'}] \circ \pi^{\mathbb{1}}|_{\mathcal{A}_\Lambda} = \pi|_{\mathcal{A}_\Lambda} = \mathrm{Ad}[V_\Lambda] \circ \pi^{\mathbb{1}}|_{\mathcal{A}_\Lambda}.$$

Since the adjoint action of a unitary is WOT-continuous, this implies that $\mathrm{Ad}[V_\Lambda]|_{\mathcal{R}_\Lambda} = \mathrm{Ad}[V_{\Lambda'}]|_{\mathcal{R}_\Lambda}$ whenever $\Lambda \subset \Lambda'$. Noting that the set of allowed cones, partially ordered by inclusion, is directed, this implies that we can consistently define ρ on $\bigcup_{\Lambda \perp \hat{f}} \mathcal{R}_\Lambda$ by setting $\rho(x) = \mathrm{Ad}[V_{\Lambda'}](x)$ whenever $x \in \mathcal{R}_{\Lambda'}$ for an allowed cone Λ'. Finally, ρ extends to $\mathcal{B}$ by norm-continuity.

Any $x \in \mathcal{A}^{\mathrm{loc}}$ belongs to some $\mathcal{A}_\Lambda$ for an allowed cone Λ, so we have $\rho(\pi^{\mathbb{1}}(x)) = (\mathrm{Ad}[V_\Lambda] \circ \pi^{\mathbb{1}})(x) = \pi(x)$. The equality $\rho \circ \pi^{\mathbb{1}} = \pi$ therefore holds on $\mathcal{A}^{\mathrm{loc}}$, and it extends to the whole of $\mathcal{A}$ by continuity. □

The Assumption 2.3 is sufficient to ensure that the extensions of anyon representations are endomorphisms of $\mathcal{B}$.

Lemma C.2. *Let π be an anyon representation localized in an allowed cone Λ, and let $\rho : \mathcal{B} \to B(\mathcal{H})$ be its extension to $\mathcal{B}$. Under Assumption 2.3 we have $\rho(\mathcal{R}_\Delta) \subset \mathcal{R}_{\Delta^{+s}}$ for any allowed cone Δ that contains Λ. In particular, ρ is an endomorphism of $\mathcal{B}$.*

Proof : By Lemma C.1 there is a unique WOT-continuous representation $\rho : \mathcal{B} \to B(\mathcal{H})$ such that $\rho \circ \pi^{\mathbb{1}} = \pi$. Suppose Δ is an allowed cone containing Λ, and take $x \in \mathcal{A}_\Delta$. For any $y \in \mathcal{A}_{\Delta^c}$, we have

$$[\rho(\pi^{\mathbb{1}}(x)), \pi^{\mathbb{1}}(y)] = [\pi(x), \pi(y)] = 0,$$

where we used that π is localized in Λ. This implies $\rho(\pi^{\mathbb{1}}(\mathcal{A}_\Delta)) \subseteq \pi^{\mathbb{1}}(\mathcal{A}_{\Delta^c})' \subseteq \mathcal{R}_{\Delta^{+s}}$, where the last inclusion is the assumption of bounded spread Haag duality. By continuity, we obtain $\rho(\mathcal{R}_\Delta) \subseteq \mathcal{R}_{\Delta^{+s}}$ for any allowed cone Δ containing Λ. Again, by continuity, this implies $\rho(\mathcal{B}) \subseteq \mathcal{B}$ as required. □

Extensions of anyon representations satisfy the following localization and transportability conditions:

Lemma C.3. *Let π be an anyon representation localized in an allowed cone Λ. Then the extension ρ of π to $\mathcal{B}$ is localized in Λ in the sense that $\rho \circ \pi^{\mathbb{1}}|_{\mathcal{A}_{\Lambda^c}} = \pi|_{\mathcal{A}_{\Lambda^c}} = \pi^{\mathbb{1}}|_{\mathcal{A}_{\Lambda^c}}$. Moreover, the extension ρ is transportable in the sense that for any allowed cone Δ there is a representation $\rho' : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ localized in Δ which is unitarily equivalent to ρ.*

Proof : The localization claim is immediate from $\rho \circ \pi^{\mathbb{1}} = \pi$. For transportability, since π is an anyon representation there is a unitary V_Δ so that $\pi' := \mathrm{Ad}[V_\Delta] \circ \pi$ is localized in Δ. The representation π' is an anyon representation, and by Lemma C.1 it has a unique WOT-continuous extension ρ' to $\mathcal{B}$ which is localized in Δ. The unitary V_Δ intertwines π and π', and by continuity this implies that V_Δ intertwines ρ and ρ' as well. That is, ρ and ρ' are unitarily equivalent. This concludes the proof. □

These preliminaries motivate the Definition 2.5 of the category SSS.

Remark C.4. *If $\rho \in \mathrm{Ob}\,\mathsf{SSS}$ is localized in an allowed cone Λ then $\rho \circ \pi^{\mathbb{1}}$ is an anyon representation localized in Λ, of which ρ is the unique extension to $\mathcal{B}$. It therefore follows from Lemma C.3 that, under the assumption of bounded spread Haag duality, the objects of SSS are precisely the extensions of anyon representations localized in allowed cones.*

C.2 Tensor product and braiding

We show how the category SSS of localized and transportable endomorphisms is naturally equipped with a tensor product and a braiding. Let us first note the following locality property of intertwiners.

Lemma C.5. *Let ρ and ρ' be endomorphisms of $\mathcal{B}$ both localized in an allowed cone Λ. If Assumption 2.3 holds, then $\mathsf{SSS}(\rho \to \rho') \subset \mathcal{R}_{\Lambda^{+s}} \subset \mathcal{B}$.*

Proof : Let $T \in \mathsf{SSS}(\rho \to \rho')$. Take $y \in \mathcal{A}_{\Lambda^c}$. Since ρ and ρ' are both localized in Λ we have

$$T\pi^{\mathbb{1}}(y) = T\rho(\pi^{\mathbb{1}}(y)) = \rho'(\pi^{\mathbb{1}}(y))T = \pi^{\mathbb{1}}(y)T,$$

which shows that $T \in \pi^{\mathbb{1}}(\mathcal{A}_{\Lambda^c})' = \mathcal{R}'_{\Lambda^c} \subseteq \mathcal{R}_{\Lambda^{+s}}$, where the last inclusion is the assumption of bounded spread Haag duality. $\square$

After realising that the composition of localized endomorphisms is again localized, it follows immediately from this lemma that the following strict tensor product on SSS is well-defined:

$$\begin{aligned} &\rho \times \sigma := \rho \circ \sigma && \rho, \sigma \in \operatorname{Ob}\mathsf{SSS} \\ &S \times T := S\rho(T) = \rho'(T)S, && S \in \mathsf{SSS}(\rho \to \rho'), T \in \mathsf{SSS}(\sigma \to \sigma'). \end{aligned}$$

Remark C.6. *It will follow from Proposition 9.10 below that SSS_f is closed under taking tensor products.*

The tensor product of endomorphisms that are localized in disjoint cones that are suitably far apart commutes.

Lemma C.7. *Let ρ and σ be endomorphisms of $\mathcal{B}$ localized in allowed cones Λ and Δ respectively, so that $\Lambda^{+s} \cap \Delta^{+s} = \emptyset$. Under Assumption 2.3 we have $\rho \times \sigma = \sigma \times \rho$.*

Proof : If $x \in \pi^{\mathbb{1}}(\mathcal{A}_{\Lambda^c \cap \Delta^c})$ then by localization we have $\rho(x) = x$ and $\sigma(x) = x$, so $\rho(\sigma(x)) = \rho(x) = x = \sigma(x) = \sigma(\rho(x))$ as required.

If $x \in \pi^{\mathbb{1}}(\mathcal{A}_\Delta)$ then by localization $\rho(x) = x$ and therefore $\sigma(\rho(x)) = \sigma(x)$. By Lemma C.2 we have $\sigma(x) \in \mathcal{R}_{\Delta^{+s}}$, and again by localization $\rho(\sigma(x)) = \sigma(x)$. This shows $\rho(\sigma(x)) = \sigma(\rho(x))$ in this case as well. We find $\rho(\sigma(x)) = \sigma(\rho(x))$ for $x \in \pi^{\mathbb{1}}(\mathcal{A}_\Lambda)$ in the same way.

Since $\rho \circ \sigma$ and $\sigma \circ \rho$ are endomorphisms and $\mathcal{A}$ is generated by $\mathcal{A}_\Lambda, \mathcal{A}_\Delta$, and $\mathcal{A}_{\Lambda^c \cap \Delta^c}$, it follows from the above that $\rho \circ \sigma = \sigma \circ \rho$ on $\pi^{\mathbb{1}}(\mathcal{A})$. Finally, the equality extends to the whole of $\mathcal{B}$ by continuity. $\square$

For the braiding, note first that for any allowed cone Λ there is an allowed cone Δ lying clockwise from Λ w.r.t. the forbidden direction and such that $\Lambda^{+s} \cap \Delta^{+s} = \emptyset$, see Figure 7. Given $\rho, \sigma \in \operatorname{Ob}\mathsf{SSS}$ localized in Λ and assuming bounded spread Haag duality (Assumption 2.3), Lemmas C.3 and C.2 provide an endomorphism $\sigma' \in \operatorname{Ob}\mathsf{SSS}$ localized in Δ and a unitary $V \in \mathsf{SSS}(\sigma \to \sigma')$. Then we may define

$$b_{\rho,\sigma} := V^*\rho(V) \in \mathsf{SSS}(\rho \times \sigma \to \sigma \times \rho). \tag{89}$$

To see that $b_{\rho,\sigma}$ intertwines $\rho \times \sigma$ and $\sigma \times \rho$, compute for any $x \in \mathcal{B}$

$$\begin{aligned} b_{\rho,\sigma}\,(\rho \times \sigma)(x) &= V^*\rho(V)\rho(\sigma(x)) = V^*\rho(V\sigma(x)) = V^*\rho(\sigma'(x))\rho(V) = V^*\sigma'(\rho(x))\rho(V) \\ &= \sigma(\rho(x))V^*\rho(V) = (\sigma \times \rho)(x)\, b_{\rho,\sigma} \end{aligned}$$

where we used Lemma C.7 in the fourth step.

Lemma C.8. *Under Assumption 2.3, the morphism defined in Eq. (89) does not depend on the choice of Δ, σ', or V.*

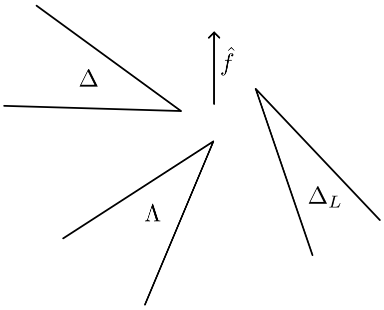


Figure 7: An allowed cone Λ together with allowed cones Δ and Δ_L, both disjoint from Λ^{+s} and lying respectively clockwise and counter clockwise from Λ w.r.t. the forbidden direction $\hat{f}$.

Proof : Suppose Δ', σ'', and W satisfy the same hypotheses as Δ, σ' and V above.

Assume there is an allowed cone $\Gamma \supset \Delta, \Delta'$ such that $\Gamma^{+s} \cap \Lambda^{+s} = \emptyset$ and lies clockwise from Λ w.r.t. the forbidden direction. Now, $VW^* \in \mathsf{SSS}(\sigma'' \to \sigma')$, and both σ' and σ'' are localized in Γ. It therefore follows from Lemma C.5 that $VW^* \in \mathcal{R}_{\Gamma^{+s}}$. Since ρ is localized in Λ, we have $(VW^*)^* \rho(VW^*) = 1$. Using this, we obtain

$$W^* \rho(W) = W^*(VW^*)^* \rho(VW^*W) = V^* \rho(V),$$

as required.

In general, one can always find a *zig-zag* from Δ to Δ' which is disjoint from Λ^{+s} [Bha+25]. That is, a sequence of allowed cones $\Delta_1 = \Delta, \Delta_2, \cdots, \Delta_{n-1}, \Delta_n = \Delta'$ and allowed cones $\Gamma_1, \cdots, \Gamma_{n-1}$ such that $\Gamma_i^{+s} \cap \Lambda^{+s} = \emptyset$ for each $i = 1, \cdots, n-1$, and such that each Γ_i lies clockwise from Λ w.r.t. the forbidden direction, and contains Δ_i and Δ_{i+1}. The result then follows from a repeated application of the above. □

Lemma C.9. *Under Assumption 2.3, the morphisms given by Eq. (89) define a unitary braiding on* SSS*. That is, for all* $\rho, \sigma \in \mathrm{Ob}\,\mathsf{SSS}$ *the morphism* $b_{\rho,\sigma}$ *is unitary, and we have*

$$b_{\rho',\sigma'}\, S \times T = T \times S\, b_{\rho,\sigma}$$

for any $\rho', \sigma' \in \mathrm{Ob}\,\mathsf{SSS}$*,* $S \in \mathsf{SSS}(\rho \to \rho')$*, and* $T \in \mathsf{SSS}(\sigma \to \sigma')$*, and*

$$\begin{aligned} b_{\rho\times\sigma,\tau} &= (b_{\rho,\tau} \times \mathbb{1}_\sigma)\,(\mathbb{1}_\rho \times b_{\sigma,\tau}) \\ b_{\rho,\sigma\times\tau} &= (\mathbb{1}_\sigma \times b_{\rho,\tau})\,(b_{\rho,\sigma} \times \mathbb{1}_\tau) \end{aligned}$$

for all $\rho, \sigma, \tau \in \mathrm{Ob}\,\mathsf{SSS}$.

Proof : Unitarity follows immediately from the definition. To show $b_{\rho',\sigma'} S \times T = T \times S b_{\rho,\sigma}$, we suppose that the endomorphisms ρ, ρ', σ and σ' are all localized in an allowed cone Λ, and fix an allowed cone Δ that lies clockwise from Λ so that $\Lambda^{+s} \cap \Delta^{+s} = \emptyset$, as well as endomorphisms σ_R, σ'_R localized in Δ and unitaries $V \in \mathsf{SSS}(\sigma \to \sigma_R)$ and $W \in \mathsf{SSS}(\sigma' \to \sigma'_R)$.

Let us first show the special case where $\sigma = \sigma'$ and $T = \mathbb{1}_\sigma$. then $S \times T = S$ and $T \times S = \sigma(S)$. We must therefore show $b_{\rho',\sigma} S = \sigma(S) b_{\rho,\sigma}$. This we do as follows:

$$b_{\rho',\sigma} S = V^* \rho'(V) S = V^* S \rho(V) = V^* \sigma_R(S) \rho(V) = \sigma(S) V^* \rho(V) = \sigma(S) b_{\rho,\sigma},$$

where we noted that Lemma C.5 and the localization of σ_R implies that $\sigma_R(S) = S$.

Let us now show the special case where $\rho = \rho'$ and $S = \mathbb{1}_\rho$. Then $S \times T = \rho(T)$ and $T \times S = T$, so we must show $b_{\rho,\sigma'}\rho(T) = Tb_{\rho,\sigma}$. To do this, note first that we can pick an allowed cone Δ_L which lies counter clockwise from Λ and such that $\Delta_L^{+s} \cap \Lambda^{+s} = \emptyset$ (see Figure 7), as well as an endomorphism ρ_L localized on Δ_L and a unitary $U \in \mathsf{SSS}(\rho \to \rho_L)$. The previous special case then yields

$$b_{\rho,\sigma} = \sigma(U^*)b_{\rho_L,\sigma}U \quad \text{and} \quad b_{\rho,\sigma'} = \sigma'(U^*)b_{\rho_L,\sigma'}U.$$

Using this, we compute

$$b_{\rho,\sigma'}\rho(T) = \sigma'(U^*)b_{\rho_L,\sigma'}U\rho(T) = \sigma'(U^*)W^*\rho_L(W)\rho_L(T)U = \sigma'(U^*)TS$$

where we used Lemma C.5 and the localization of ρ_L to conclude that $\rho_L(T) = T$ and $\rho_L(W) = W$. Similarly,

$$Tb_{\rho,\sigma} = T\sigma(U^*)b_{\rho_L,\sigma}U = \sigma'(U^*)TV^*\rho_L(V)U = \sigma'(U^*)TS,$$

so both special cases are now proven. The general case follows because

$$\begin{aligned} b_{\rho',\sigma'}\, S \times T &= b_{\rho',\sigma'}(S \times \mathbb{1}_{\sigma'})(\mathbb{1}_\rho \times T) = (\mathbb{1}_{\sigma'} \times S)b_{\rho,\sigma'}(\mathbb{1}_\rho \times T) \\ &= (\mathbb{1}_{\sigma'} \times S)(T \times \mathbb{1}_\rho)b_{\rho,\sigma} = T \times S\, b_{\rho,\sigma}. \end{aligned}$$

Fix ρ', σ', τ' localized in Δ with unitary transporters $V_\rho \in \mathsf{SSS}(\rho \to \rho')$, $V_\sigma \in \mathsf{SSS}(\sigma \to \sigma')$, and $V_\tau \in \mathsf{SSS}(\tau \to \tau')$. Then

$$(b_{\rho,\tau} \times \mathbb{1}_\sigma)(\mathbb{1}_\rho \times b_{\sigma,\tau}) = \beta_{\rho,\tau}\, \rho(\beta_{\sigma,\tau}) = V_\tau^*\rho(V_\tau)\, \rho(V_\tau^*\sigma(V_\tau)) = V_\tau^*(\rho \times \sigma)(V_\tau) = b_{\rho\times\sigma,\tau},$$

which shows the first hexagon identity. For the second, note that $V_{\sigma\times\tau} := V_\sigma \times V_\tau = V_\sigma\sigma(V_\tau) \in \mathsf{SSS}(\sigma \times \tau \to \sigma' \times \tau')$ and compute

$$\begin{aligned} (\mathbb{1}_\sigma \times b_{\rho,\tau})(b_{\rho,\sigma} \times \mathbb{1}_\tau) &= \sigma\big(V_\tau^*\rho(V_\tau)\big)\, V_\sigma^*\rho(V_\sigma) = \sigma(V_\tau)^*V_\sigma^*\, \sigma'\big(\rho(V_\tau)\big)\rho(V_\sigma) \\ &= \big(V_\sigma\sigma(V_\tau)\big)^*\, \rho\big(\sigma'(V_\tau)V_\sigma\big) = V_{\sigma\times\tau}^*\, \rho\big(V_{\sigma\times\tau}\big) = b_{\rho,\sigma\times\tau} \end{aligned}$$

where we used Lemma C.7 in the third step. □

C.3 Direct sums and subobjects

Since $\omega^{\mathbb{1}}$ is pure and is a gapped ground state of a local Hamiltonian, it follows from [Oga22, Lemma 5.3] that all cone algebras $\mathcal{R}_\Lambda$ are infinite factors.

Let $\rho_1, \cdots, \rho_n \in \operatorname{Ob}\mathsf{SSS}$. Then there is an allowed cone Λ so that each ρ_i is localized in Λ. Since the cone algebra $\mathcal{R}_\Lambda$ is properly infinite we can use the Halving Lemma [KR97, Lemma 6.3.3] to construct isometries $\{v_i\}_{i=1}^n$ generating an isomorphic copy of the Cuntz algebra $\mathcal{O}_n$. One then easily checks that

$$\rho := \sum_{i=1}^n \operatorname{Ad}[v_i] \circ \rho_i$$

is an endomorphism of $\mathcal{B}$ which is localized in Λ and transportable. This shows that SSS has direct sums. It is clear from the construction that SSS_f is closed under direct sums.

Now let $\rho \in \operatorname{Ob}\mathsf{SSS}$ be localized in an allowed cone Λ and let $p \in \mathsf{SSS}(\rho \to \rho)$ be a non-trivial orthogonal projection. Assuming bounded spread Haag duality (Assumption 2.3), Lemma C.5 yields $p \in \mathcal{R}_{\Lambda^{+s}}$, so $\tilde{\sigma}(x) := p\rho(x) = \rho(x)p$ defines a (non-unital) endomorphism of $\mathcal{B}$. In particular, $\tilde{\sigma}$ restricts to a non-zero representation of $\pi^{\mathbb{1}}(\mathcal{A})$ on $p\mathcal{H}$. Since $\mathcal{A}$ is simple, this representation is faithful. This implies that p must be infinite. Since $\mathcal{R}_{\Lambda^{+s}}$ is a properly infinite factor, it follows that $p \sim \mathbb{1}$, that is, there is a partial isometry $w \in \mathcal{R}_{\Lambda^{+s}}$ such that $p = ww^*$ and $\mathbb{1} = w^*w$. Consider now the endomorphism $\sigma = \mathrm{Ad}[w] \circ \rho$. One easily checks that σ is localized in Λ^{+s}, and that it is transportable. This shows that SSS admits subobjects. Note finally that if $\rho \in \mathrm{Ob}\,\mathsf{SSS}_f$, then also $\sigma \in \mathrm{Ob}\,\mathsf{SSS}_f$, so SSS_f is closed under taking subobjects.

In summary, we conclude that if Assumption 2.3 holds, then both SSS and SSS_f are braided C*-tensor categories, and SSS_f is moreover semisimple.

D F- and R-symbols determine braided tensor structure

We give a detailed account of this well-known statement.

D.1 Isomorphism of F- and R-symbols

Let $(\mathcal{C}, \otimes, \alpha, \mathbb{1}, l, r)$ be a semisimple monoidal category and let $\mathrm{Irr}\,\mathcal{C}$ be a set of representative simple objects containing $\mathbb{1}$. The F-symbols of $\mathcal{C}$ with respect to $\mathrm{Irr}\,\mathcal{C}$ are the invertible linear maps

$$F^l_{ijk} : \bigoplus_{n \in \mathrm{Irr}\,\mathcal{C}} \mathcal{C}(i \otimes n \to l) \otimes \mathcal{C}(i \otimes k \to n) \to \bigoplus_{m \in \mathrm{Irr}\,\mathcal{C}} \mathcal{C}(m \otimes k \to l) \otimes \mathcal{C}(i \otimes j \to m)$$

defined for all $i, j, k, l \in \mathrm{Irr}\,\mathcal{C}$ by the commuting diagram

$$\begin{tikzcd}
\mathcal{C}(i \otimes (j \otimes k) \to l) \arrow[r, "\simeq"] \arrow[d, "- \circ \alpha_{i,j,k}"'] & \bigoplus_{n \in \mathrm{Irr}\,\mathcal{C}} \mathcal{C}(i \otimes n \to l) \otimes \mathcal{C}(j \otimes k \to n) \arrow[d, "F^l_{ijk}"] \\
\mathcal{C}((i \otimes j) \otimes k \to l) \arrow[r, "\simeq"] & \bigoplus_{m \in \mathrm{Irr}\,\mathcal{C}} \mathcal{C}(m \otimes k \to l) \otimes \mathcal{C}(i \otimes j \to m).
\end{tikzcd}$$

If $\mathcal{C}$ is equipped with a braiding β, then the R-symbols of $\mathcal{C}$ with respect to $\mathrm{Irr}\,\mathcal{C}$ are the invertible morphisms

$$R^k_{ij} : \mathcal{C}(j \otimes i \to k) \xrightarrow{- \circ \beta_{i,j}} \mathcal{C}(i \otimes j \to k),$$

defined for all $i, j, k \in \mathrm{Irr}\,\mathcal{C}$.

Suppose now $(\mathcal{C}, \otimes, {}^{\mathcal{C}}\alpha, \mathbb{1}, {}^{\mathcal{C}}l, {}^{\mathcal{C}}r)$ and $(\mathcal{D}, \otimes, {}^{\mathcal{D}}\alpha, \mathbb{1}, {}^{\mathcal{D}}l, {}^{\mathcal{D}}r)$ are semisimple monoidal categories with representative sets of simples $\mathrm{Irr}\,\mathcal{C}$ and $\mathrm{Irr}\,\mathcal{D}$, respectively. In addition, suppose that there is a bijection $f : \mathrm{Irr}\,\mathcal{C} \to \mathrm{Irr}\,\mathcal{D}$ such that $f(\mathbb{1}) = \mathbb{1}$, which implies that $\mathcal{C}$ and $\mathcal{D}$ are equivalent as linear categories. We say $\mathcal{C}$ and $\mathcal{D}$ have the same fusion rules if

$$\mathcal{C}(i \otimes j \to k) \simeq \mathcal{D}(f(i) \otimes f(j) \to f(k))$$

for all $i, j, k \in \mathrm{Irr}\,\mathcal{C}$, in which case f provides an isomorphism of fusion rules.

Let ${}^{\mathcal{C}}F$ and ${}^{\mathcal{D}}F$ be the F-symbols of $\mathcal{C}$ and $\mathcal{D}$ with respect to the representative sets of simples $\mathrm{Irr}\,\mathcal{C}$ and $\mathrm{Irr}\,\mathcal{D}$. Writing $\underline{i} = f(i)$ to lighten notation, we say a collection of morphisms

$$\phi^k_{i,j} : \mathcal{C}(i \otimes j \to k) \to \mathcal{D}(\underline{i} \otimes \underline{j} \to \underline{k}),$$

intertwines the F-symbols if the following diagram commutes for all $i, j, k, l \in \operatorname{Irr}\mathcal{C}$:

$$\begin{tikzcd}
\bigoplus_n \mathcal{C}(i \otimes n \to l) \otimes \mathcal{C}(j \otimes k \to n) \arrow[r, "{}^{\mathcal{C}}F^{l}_{ijk}"] \arrow[d, "\bigoplus_n \phi^l_{i,n} \otimes \phi^n_{j,k}"] & \bigoplus_m \mathcal{C}(m \otimes k \to l) \otimes \mathcal{C}(i \otimes j \to m) \arrow[d, "\bigoplus_m \phi^l_{m,k} \otimes \phi^m_{i,j}"] \\
\bigoplus_n \mathcal{D}(\underline{i} \otimes \underline{n} \to \underline{l}) \otimes \mathcal{D}(\underline{j} \otimes \underline{k} \to \underline{n}) \arrow[r, "{}^{\mathcal{D}}F^{\underline{l}}_{\underline{ijk}}"] & \bigoplus_m \mathcal{D}(\underline{m} \otimes \underline{k} \to \underline{l}) \otimes \mathcal{D}(\underline{i} \otimes \underline{j} \to \underline{m})
\end{tikzcd} \tag{90}$$

If each $\phi^k_{i,j}$ is moreover invertible, then we say $\{\phi^k_{i,j}\}$ provides an isomorphism of F-symbols, which in particular implies that f provides an isomorphism of fusion rules.

If $\mathcal{C}$ and $\mathcal{D}$ are also braided and have R-symbols ${}^{\mathcal{C}}R$ and ${}^{\mathcal{D}}R$ with respect to $\operatorname{Irr}\mathcal{C}$ and $\operatorname{Irr}\mathcal{D}$ respectively, then the maps $\{\phi^k_{i,j}\}_{i,j,k \in \operatorname{Irr}\mathcal{C}}$ are said to intertwine the R-symbols if the following diagram commutes for all $i, j, k \in \operatorname{Irr}\mathcal{C}$:

$$\begin{tikzcd}
\mathcal{C}(j \otimes i \to k) \arrow[r, "{}^{\mathcal{C}}R^{k}_{i\,j}"] \arrow[d, "\phi^k_{j,i}"'] & \mathcal{C}(i \otimes j \to k) \arrow[d, "\phi^k_{i,j}"] \\
\mathcal{D}(\underline{j} \otimes \underline{i} \to \underline{k}) \arrow[r, "{}^{\mathcal{D}}R^{\underline{k}}_{\underline{i\,j}}"] & \mathcal{D}(\underline{i} \otimes \underline{j} \to \underline{k})
\end{tikzcd} \tag{91}$$

If each $\phi^k_{i,j}$ is moreover invertible, then we say $\{\phi^k_{i,j}\}$ provides an isomorphism of R-symbols.

We say $\mathcal{C}$ and $\mathcal{D}$ have isomorphic F-symbols if there is an isomorphism of F-symbols. We say $\mathcal{C}$ and $\mathcal{D}$ have isomorphic F- and R-symbols if there is an isomorphism of F-symbols that intertwines the R-symbols.

This appendix is devoted to proving the following result, whose proof appears in Section D.4 below.

Proposition D.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be semisimple monoidal categories. If $\mathcal{C}$ and $\mathcal{D}$ have isomorphic F-symbols, then they are monoidally equivalent. If $\mathcal{C}$ and $\mathcal{D}$ are moreover braided and have isomorphic F- and R-symbols, then they are braided monoidally equivalent.*

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are unitary. We say that $\{\phi^k_{i,j}\}_{i,j,k \in \operatorname{Irr}\mathcal{C}}$ preserve orthogonal direct sum decompositions if $(\phi^{k_\kappa}_{i,j}(\pi_\kappa) : \underline{i} \otimes \underline{j} \to \underline{k_\kappa})$ is an orthogonal direct sum decomposition of $\underline{i} \otimes \underline{j}$ for any orthogonal direct sum decomposition $(\pi_\kappa : i \otimes j \to k_\kappa)$ of $i \otimes j$ into simple objects for any $i, j \in \operatorname{Irr}\mathcal{C}$. If this is the case and $\{\phi^k_{i,j}\}_{i,j,k \in \operatorname{Irr}\mathcal{C}}$ is an isomorphism of F-symbols, then there is a unitary monoidal equivalence $\mathcal{C} \simeq \mathcal{D}$. If $\mathcal{C}$ and $\mathcal{D}$ are moreover unitarily braided and $\{\phi^k_{ij}\}$ also intertwines R-symbols, then there is a braided unitary monoidal equivalence $\mathcal{C} \simeq \mathcal{D}$.

Note that if $\mathcal{C}$ and $\mathcal{D}$ are unitary fusion categories equipped with their canonical spherical traces then their morphism spaces are Hilbert spaces equipped with the trace inner product. Unitarity of the equivalence can then be conditioned on the unitarity of $\phi^k_{i,j}$.

D.2 Functors of semisimple categories and their natural transformations

Recall that a semisimple category $\mathcal{C}$ has direct sum decompositions. That is, if $\operatorname{Irr}\mathcal{C}$ is a representative set of simples for $\mathcal{C}$, then each object $a \in \operatorname{Ob}\mathcal{C}$ admits a finite set of morphisms $\pi_\kappa : a \to i_\kappa$ and $\iota_\kappa : i_\kappa \to a$ for $\kappa = 1, \cdots, n$ to and from simples objects $i_\kappa \in \operatorname{Irr}\mathcal{C}$ such that

$$\pi_\lambda \circ \iota_\kappa = \delta_{\lambda,\kappa}\, \mathrm{id}_{i_\kappa} \qquad \text{and} \qquad \sum_\kappa \iota_\kappa \circ \pi_\kappa = \mathrm{id}_a.$$

If $\mathcal{C}$ is unitary, then the direct sum decomposition can be taken to be *orthogonal*, meaning that $\iota_\kappa = \pi_\kappa^\dagger$ for all κ. The collection of maps $(\pi_\kappa, \iota_\kappa)_\kappa$ is called a (orthogonal) direct sum decomposition of a. The fact that any object of a semisimple category can be decomposed into simples implies that functors of semisimple categories and their natural transformations are determined by their values on simple objects. This hinges on the fact that morphisms are determined by their compositions which is expressed by the Yoneda Lemma. This subsection is devoted to reviewing these facts.

Given a linear category $\mathcal{C}$ and any object $c \in \operatorname{Ob}\mathcal{C}$ we get a functor $\mathcal{C}(c \to -) : \mathcal{C} \to \mathsf{Vec}$ which sends an object a to the vector space $\mathcal{C}(c \to a)$ and sends a morphism $f : a \to b$ to the linear transformation $\mathcal{C}(c \to a) \to \mathcal{C}(c \to b)$ given by post-composition with f. Similarly we get a functor $\mathcal{C}(- \to c) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}$ which sends morphisms to their pre-composition.

These assignments yield the *Yoneda embeddings*

$$\text{よ}_{\mathcal{C}}^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathsf{Fun}(\mathcal{C} \to \mathsf{Vec}), \quad \text{よ}_{\mathcal{C}} : \mathcal{C} \to \mathsf{Fun}(\mathcal{C}^{\mathrm{op}} \to \mathsf{Vec})$$

which send morphisms in $\mathcal{C}$ to *natural transformations* given by pre- and post-composition respectively. These maps sending morphisms to natural transformations are invertible by the *Yoneda Lemma* (see for example [Pen23, Lemma 2.9.1]):

Lemma D.2 (Yoneda Lemma)**.** *Let $F : \mathcal{C} \to \mathsf{Vec}$ be a linear functor. For each $c \in \operatorname{Ob}\mathcal{C}$ the map*

$$\mathsf{Nat}(\mathcal{C}(c \to -) \implies F) \to F(c) : \eta \mapsto \eta_c(\mathrm{id}_c)$$

is an isomorphism. Similarly, if $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Vec}$ is a linear functor, then

$$\mathsf{Nat}(\mathcal{C}(- \to c) \implies F) \to F(c) : \eta \mapsto \eta_c(\mathrm{id}_c)$$

is an isomorphism. In both cases, the inverse maps $v \in F(c)$ to the natural transformation given by $\eta_a(f) = F(f)\big(v\big)$.

By applying the Yoneda Lemma to $F = \mathcal{C}(b \to -)$, resp. $F = \mathcal{C}(- \to b)$, it implies that the Yoneda embeddings are fully faithful (see for example [Pen23, Corollary 2.9.3]).

Lemma D.3. *Let $\mathcal{C}$ be semisimple with representative set of simples $\operatorname{Irr}\mathcal{C}$. Any morphism $f : a \to b$ is completely determined by its action by pre-composition on the spaces $\mathcal{C}(b \to i)$ for $i \in \operatorname{Irr}\mathcal{C}$.*

Proof : Since the Yoneda embedding is faithful, f is determined by its action on morphism spaces given by precomposition. Since $\mathcal{C}$ is semisimple, the action by precomposition with f on any $\mathcal{C}(b \to c)$ is completely determined by the actions on $\mathcal{C}(b \to i)$ for $i \in \operatorname{Irr}\mathcal{C}$. Indeed, let $(\iota_\kappa, \pi_\kappa)$ be a direct sum decomposition of c. Then we find for any $g \in \mathcal{C}(b \to c)$ that

$$g \circ f = \mathrm{id}_c \circ g \circ f = \sum_\kappa \pi_\kappa \circ (\iota_\kappa \circ g) \circ f.$$

□

Lemma D.4. *Let $F, G : \mathcal{C} \to \mathcal{D}$ be linear functors between semisimple categories $\mathcal{C}$ and $\mathcal{D}$. Let $\operatorname{Irr}\mathcal{C}$ be a representative set of simples for $\mathcal{C}$. Any collection of morphisms $\{\eta_i : F(i) \to G(i)\}_{i \in \operatorname{Irr}\mathcal{C}}$ uniquely determines a natural transformation $\eta : F \implies G$ by the formula*

$$\eta_a = \sum_\kappa G(\iota_\kappa) \circ \eta_{i_\kappa} \circ F(\pi_\kappa)$$

for any direct sum decomposition $(\pi_\kappa, \iota_\kappa)$ *of* a *into simples* $i_\kappa \in \operatorname{Irr}\mathcal{C}$*.*

If the $\{\eta_i\}_{i \in \operatorname{Irr}\mathcal{C}}$ *are invertible, then* η *is a natural isomorphism. If* F *and* G *are dagger functors and the* $\{\eta_i\}_{i \in \operatorname{Irr}\mathcal{C}}$ *are unitary, then* η *is a unitary natural isomorphism.*

Proof : Note first that the η_i satisfy naturality conditions amongst themselves. That is, for any $f : i \to j$ we have that

$$\begin{tikzcd} F(i) \arrow[r, "\eta_i"] \arrow[d, "F(f)"'] & G(i) \arrow[d, "G(f)"] \\ F(j) \arrow[r, "\eta_j"] & G(j) \end{tikzcd} \tag{92}$$

commutes. Indeed, if $i \neq j$ then $f = 0$ and the diagram commutes. If $i = j$ then $f = \lambda \mathrm{id}_i$ for some $\lambda \in \mathbb{C}$ and both compositions in the diagram yield $\lambda \eta_i$.

For every $a \in \operatorname{Ob}\mathcal{C}$, fix a direct sum decomposition $(\pi^a_\kappa, \iota^a_\kappa)_\kappa$ into simples $i^a_\kappa \in \operatorname{Irr}\mathcal{C}$. We define

$$\eta_a := \sum_\kappa G(\iota^a_\kappa) \circ \eta_{i^a_\kappa} \circ F(\pi^a_\kappa)$$

for all $a \in \operatorname{Ob}\mathcal{C}$. We now verify that this indeed defines a natural transformation. For $f : a \to b$ we have

$$\begin{aligned} G(f) \circ \eta_a &= \sum_\kappa G(f) \circ G(\iota^a_\kappa) \circ \eta_{i^a_\kappa} \circ F(\pi^a_\kappa) \\ &= \sum_{\kappa,\lambda} G(\iota^b_\lambda) \circ G(\pi^b_\lambda \circ f \circ \iota^a_\kappa) \circ \eta_{i^a_\kappa} \circ F(\pi^a_\kappa) \\ &= \sum_{\kappa,\lambda} G(\iota^b_\lambda) \circ \eta_{i^b_\lambda} \circ F(\pi^b_\lambda \circ f \circ \iota^a_\kappa) \circ F(\pi^a_\kappa) \\ &= \eta_b \circ F(f) \end{aligned}$$

where we used Eq. (92) in the third step.

It remains to verify that the η_a are independent of the choice of direct sum decomposition. This follows from a similar computation as the one used to verify naturality, and is left to the reader.

Finally, if the $\{\eta_i\}$ are invertible then $\sum_\kappa F(\iota^a_\kappa) \circ \eta^{-1}_{i^a_\kappa} \circ G(\pi^a_\kappa)$ is an inverse of η_a. Similarly, if F and G are dagger functors and the $\{\eta_i\}$ are unitary, then we can use an orthogonal direct sum decomposition $\iota^a_\kappa = (\pi^a_\kappa)^\dagger$ of a to find that $\eta_a^\dagger = \sum_\kappa F(\iota^a_\kappa) \circ \eta^\dagger_{i^a_\kappa} \circ G(\pi^a_\kappa)$ is the inverse of η_a. □

Lemma D.5. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be semisimple. Given an assignment* $f : \operatorname{Irr}\mathcal{C} \to \operatorname{Ob}\mathcal{D}$*, there is a linear functor* $F : \mathcal{C} \to \mathcal{D}$*, unique up to a canonical natural isomorphism, which extends* f *on objects. If* $\mathcal{C}$ *and* $\mathcal{D}$ *are moreover unitary categories, then* F *may be taken to be a dagger functor. In either case, if* $f : \operatorname{Irr}\mathcal{C} \to \operatorname{Irr}\mathcal{D}$ *is a bijection, then* F *is an equivalence.*

Proof : For each object $a \in \operatorname{Ob}\mathcal{C}$, fix a direct sum decomposition $(\pi^a_\kappa, \iota^a_\kappa)$ of a into simples $i^a_\kappa \in \operatorname{Irr}\mathcal{C}$ and define

$$F(a) = \bigoplus_\kappa f(i^a_\kappa),$$

where the right hand side is an arbitrary choice of direct sum of the objects $f(i^a_\kappa)$ in $\mathcal{D}$, given by a direct sum decomposition $(\tilde{\pi}^a_\kappa, \tilde{\iota}^a_\kappa)$. By functoriality, F is determined on morphisms $f : a \to b$

by

$$F(f) = \sum_{\kappa,\lambda} \tilde{\iota}^b_\kappa \circ F(\pi^b_\kappa \circ f \circ \iota^a_\lambda) \circ \tilde{\pi}^a_\lambda, \tag{93}$$

where we note that $\pi^b_\kappa \circ f \circ \iota^a_\lambda$ is either zero or a multiple of the identity. With this definition on morphisms, one easily checks that $F(\mathrm{id}_a) = \mathrm{id}_{F(a)}$ and that $F(g) \circ F(f) = F(g \circ f)$ whenever g and f are composable. Suppose G is another linear functor that extends f on objects. Then by Lemma D.4 the identity maps $\eta_i := \mathrm{id}_{F(i)} : F(i) \to G(i)$ for all $i \in \operatorname{Irr}\mathcal{C}$ determine a canonical natural isomorphism $\eta : F \implies G$.

For the unitary case, note that a unitary semisimple category admits all orthogonal direct sums. We can therefore assume that all direct sum decompositions in the construction above are orthogonal. That F is a dagger functor then follows by noting that $F(\pi^b_\kappa \circ f \circ \iota^a_\lambda)^\dagger = F(\pi^a_\lambda \circ f^\dagger \circ \iota^b_\kappa)$, since the dagger of this morphism is given by complex conjugation. It is a simple exercise to check that F is faithful and that if $f : \operatorname{Irr}\mathcal{C} \to \operatorname{Irr}\mathcal{D}$ is a bijection, then F is essentially surjective and full. □

D.3 Construction of a tensorator

Under the assumptions of Proposition D.1, Lemma D.5 gives linear equivalence $F : \mathcal{C} \to \mathcal{D}$ such that $F(i) = f(i)$ for each $i \in \operatorname{Irr}\mathcal{C}$, which we may take to be a unitary equivalence in the case where $\mathcal{C}$ and $\mathcal{D}$ are unitary.

By Lemma D.4, the given isomorphism of F-symbols $\{\phi^k_{ij}\}_{i,j,k \in \operatorname{Irr}\mathcal{C}}$ uniquely determines a natural transformation

$$\phi^-_{i,j} : \mathcal{C}(i \otimes j \to -) \implies \mathcal{D}(F(i) \otimes F(j) \to F(-))$$

for every pair $i, j \in \operatorname{Irr}\mathcal{C}$. The component morphisms are given by

$$\phi^c_{i,j}(\alpha) = \sum_\kappa F(\iota_\kappa) \circ \phi^{i_\kappa}_{i,j}(\pi_\kappa \circ \alpha) \tag{94}$$

for any direct sum decomposition $(\pi_\kappa, \iota_\kappa)_\kappa$ of c into simples $i_\kappa \in \operatorname{Irr}(\mathcal{C})$.

The Yoneda Lemma gives a canonical isomorphism

$$\begin{aligned} \mathsf{Nat}\big(\mathcal{C}(i \otimes j \to -) \implies \mathcal{D}(F(i) \otimes F(j) \to F(-))\big) &\xrightarrow{\simeq} \mathcal{D}(F(i) \otimes F(j) \to F(i \otimes j)) \\ \rho &\mapsto \rho_{i \otimes j}(\mathrm{id}_{i \otimes j}). \end{aligned}$$

Applying this to the natural transformation ϕ^-_{ij} yields the morphism

$$\tau_{i,j} := \phi^{i \otimes j}_{i,j}(\mathrm{id}_{i \otimes j}) : F(i) \otimes F(j) \to F(i \otimes j). \tag{95}$$

It follows from the Yoneda isomorphism (or the naturality of $\phi^-_{i,j}$) that

$$F(g) \circ \tau_{i,j} = \phi^k_{i,j}(g) \tag{96}$$

for all $g \in \mathcal{C}(i \otimes j \to k)$.

By Lemma D.4 the maps $\{\tau_{i,j}\}_{i,j \in \operatorname{Irr}\mathcal{C}}$ uniquely determine a natural transformation

$$\tau : F(-) \otimes F(-) \implies F(- \otimes -).$$

We will show that this τ can serve as a tensorator to equip the equivalence F with the structure of a monoidal equivalence.

Lemma D.6. *The natural transformation τ satisfies the following commutative diagram for all $a, b, c \in \operatorname{Ob}\mathcal{C}$:*

$$\begin{tikzcd}
(F(a)\otimes F(b))\otimes F(c) \arrow[r, "{}^{\mathcal{D}}\alpha_{F(a),F(b),F(c)}"] \arrow[d, "\tau_{a,b}\otimes \mathrm{id}_{F(c)}"'] & F(a)\otimes(F(b)\otimes F(c)) \arrow[d, "\mathrm{id}_{F(a)}\otimes\tau_{b,c}"] \\
F(a\otimes b)\otimes F(c) \arrow[d, "\tau_{a\otimes b,c}"'] & F(a)\otimes F(b\otimes c) \arrow[d, "\tau_{a,b\otimes c}"] \\
F((a\otimes b)\otimes c) \arrow[r, "F({}^{\mathcal{C}}\alpha_{a,b,c})"] & F(a\otimes(b\otimes c))
\end{tikzcd} \tag{97}$$

Proof : The two paths from top left to bottom right are components of natural transformations $(F(-)\otimes F(-))\otimes F(-) \implies F(-\otimes(-\otimes -))$ which by Lemma D.4 are completely determined by their simple components. It is therefore sufficient to show that (97) commutes for simple objects. This will follow from the assumption that the $\{\phi^k_{i,j}\}$ provide an isomorphism of F-symbols.

Let $i, j, k \in \operatorname{Irr}\mathcal{C}$, and note that by Lemma D.3, the diagram (97) with $a = i, b = j$, and $c = k$, commutes, if and only if, the top square of

$$\begin{tikzcd}
\mathcal{D}\big((F(i)\otimes F(j))\otimes F(k)\to F(l)\big) & \mathcal{D}\big(F(i)\otimes(F(j)\otimes F(k))\to F(l)\big) \arrow[l, "-\circ{}^{\mathcal{D}}\alpha_{\underline{i},\underline{j},\underline{k}}"'] \\
\mathcal{D}\big(F((i\otimes j)\otimes k)\to F(l)\big) \arrow[u, "-\circ\tau_{i\otimes j,k}\circ(\tau_{i,j}\otimes\mathrm{id}_{\underline{k}})"'] & \mathcal{D}\big(F(i\otimes(j\otimes k))\to F(l)\big) \arrow[l, "-\circ F({}^{\mathcal{C}}\alpha_{i,j,k})"'] \arrow[u, "-\circ\tau_{i,j\otimes k}\circ(\mathrm{id}_{\underline{i}}\otimes\tau_{j,k})"'] \\
\mathcal{C}\big((i\otimes j)\otimes k\to l\big) \arrow[u, "F"'] & \mathcal{C}\big(i\otimes(j\otimes k)\to l\big) \arrow[l, "-\circ{}^{\mathcal{C}}\alpha_{i,j,k}"'] \arrow[u, "F"']
\end{tikzcd} \tag{98}$$

commutes for all $l \in \operatorname{Irr}\mathcal{C}$. The bottom square commutes by functoriality of F. Let $l \in \operatorname{Irr}\mathcal{C}$ be given, and observe that for any $n, m \in \operatorname{Irr}\mathcal{C}$, any $\eta : i\otimes j \to m$ and $\xi : m\otimes k \to l$, we have

$$\begin{aligned}
F(\xi\circ(\eta\otimes\mathrm{id}_k))\circ\tau_{i\otimes j,k}\circ(\tau_{i,j}\otimes\mathrm{id}_k) &= F(\xi)\circ\tau_{m,k}\circ(F(\eta)\otimes\mathrm{id}_{\underline{k}})\circ(\tau_{i,j}\otimes\mathrm{id}_{\underline{k}}) \\
&= \phi^l_{m,k}(\xi)\circ\big(\phi^m_{i,j}(\eta)\otimes\mathrm{id}_{\underline{k}}\big),
\end{aligned}$$

where we used naturality of τ in the first step, and Eq. (96) in the last. This shows that the following diagram commutes:

$$\begin{tikzcd}
\begin{array}{c}\bigoplus_m \mathcal{D}\big(F(m)\otimes F(k)\to F(l)\big)\\ \otimes\mathcal{D}\big(F(i)\otimes F(j)\to F(m)\big)\end{array} \arrow[r, "\simeq"] & \mathcal{D}\big((F(i)\otimes F(j))\otimes F(k)\to F(l)\big) \\
 & \mathcal{D}\big(F((i\otimes j)\otimes k)\to F(l)\big) \arrow[u, "-\circ\tau_{i\otimes j,k}\circ(\tau_{i,j}\otimes\mathrm{id}_{\underline{k}})"'] \\
\bigoplus_m \mathcal{C}(m\otimes k\to l)\otimes\mathcal{C}(i\otimes j\to m) \arrow[uu, "\bigoplus_m \phi^l_{m,k}\otimes\phi^m_{i,j}"'] \arrow[r, "\simeq"] & \mathcal{C}\big((i\otimes j)\otimes k\to l\big). \arrow[u, "F"']
\end{tikzcd} \tag{99}$$

An analogous computation shows commutativity of

$$\begin{tikzcd}
\mathcal{D}\big(F(i)\otimes(F(j)\otimes F(k))\to F(l)\big) & \begin{array}{c}\bigoplus_n \mathcal{D}\big(F(i)\otimes F(n)\to F(l)\big)\\ \otimes\mathcal{D}\big(F(j)\otimes F(k)\to F(n)\big)\end{array} \arrow[l, "\simeq"] \\
\mathcal{D}\big(F(i\otimes(j\otimes k))\to F(l)\big) \arrow[u, "-\circ\tau_{i,j\otimes k}\circ(\mathrm{id}_{\underline{i}}\otimes\tau_{j,k})"'] & \\
\mathcal{C}\big(i\otimes(j\otimes k)\to l\big) \arrow[u, "F"'] & \bigoplus_n \mathcal{C}(i\otimes n\to l)\otimes\mathcal{C}(j\otimes k\to n). \arrow[l, "\simeq"] \arrow[uu, "\bigoplus_n \phi^l_{i,n}\otimes\phi^n_{j,k}"']
\end{tikzcd} \tag{100}$$

Placing diagrams (99) and (100) on the sides of diagram (98), we find that the latter commutes if and only if the outer square of the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}\big((F(i)\otimes F(j))\otimes F(k)\to F(l)\big) & \mathcal{D}\big(F(i)\otimes(F(j)\otimes F(k))\to F(l)\big) \arrow[l, "-\circ{}^{\mathcal{D}}\alpha_{\underline{i},\underline{j},\underline{k}}"'] \\
\begin{array}{c}\bigoplus_m \mathcal{D}\big(F(m)\otimes F(k)\to F(l)\big)\\ \otimes\mathcal{D}\big(F(i)\otimes F(j)\to F(m)\big)\end{array} \arrow[u, "\simeq"'] & \begin{array}{c}\bigoplus_n \mathcal{D}\big(F(i)\otimes F(n)\to F(l)\big)\\ \otimes\mathcal{D}\big(F(j)\otimes F(k)\to F(n)\big)\end{array} \arrow[u, "\simeq"'] \arrow[l, "{}^{\mathcal{D}}F^{\underline{l}}_{\underline{i}\,\underline{j}\,\underline{k}}"'] \\
\bigoplus_m \mathcal{C}(m\otimes k\to l)\otimes\mathcal{C}(i\otimes j\to m) \arrow[u, "\bigoplus_m \phi^l_{m,k}\otimes\phi^m_{i,j}"'] \arrow[d, "\simeq"'] & \bigoplus_n \mathcal{C}(i\otimes n\to l)\otimes\mathcal{C}(j\otimes k\to n) \arrow[l, "{}^{\mathcal{C}}F^{l}_{i\,j\,k}"'] \arrow[u, "\bigoplus_n \phi^l_{i,n}\otimes\phi^n_{j,k}"'] \arrow[d, "\simeq"'] \\
\mathcal{C}\big((i\otimes j)\otimes k\to l\big) & \mathcal{C}\big(i\otimes(j\otimes k)\to l\big) \arrow[l, "-\circ{}^{\mathcal{C}}\alpha_{i,j,k}"']
\end{tikzcd}$$

But here the top and bottom squares are the definition of the F-symbols, and commutativity of the middle square is precisely the assumption that $\{\phi^k_{i,j}\}$ intertwines F-symbols. ☐

Lemma D.7. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are equipped with braidings ${}^{\mathcal{C}}\beta, {}^{\mathcal{D}}\beta$ and the maps $\{\phi^k_{i,j}\}$ provide an isomorphism of the corresponding R-symbols. Then the natural transformation τ satisfies the following commutative diagram for all $a, b \in \operatorname{Ob}\mathcal{C}$:*

$$\begin{tikzcd}
F(a) \otimes F(b) \arrow[r, "{}^{\mathcal{D}}\beta_{F(a),F(b)}"] \arrow[d, "\tau_{a,b}"'] & F(b) \otimes F(a) \arrow[d, "\tau_{b,a}"] \\
F(a \otimes b) \arrow[r, "F({}^{\mathcal{C}}\beta_{a,b})"] & F(b \otimes a)
\end{tikzcd} \tag{101}$$

Proof : The two paths from top left to bottom right of the diagram (101) are components of natural transformations $F(-) \otimes F(\bullet) \implies F(\bullet \otimes -)$ which, by Lemma D.4 are completely determined by their simple components. It is therefore sufficient to show commutativity of (101) in the case where $a = i, b = j \in \operatorname{Irr}\mathcal{C}$ are simple. In this case, and using By Lemma D.3, commutativity of (101) is equivalent to commutativity of the top square of

$$\begin{tikzcd}
\mathcal{D}\big(F(i) \otimes F(j) \to F(k)\big) & \mathcal{D}\big(F(j) \otimes F(i) \to F(k)\big) \arrow[l, "- \circ {}^{\mathcal{D}}\beta_{F(i),F(j)}"'] \\
\mathcal{D}\big(F(i \otimes j) \to F(k)\big) \arrow[u, "- \circ \tau_{i,j}"] & \mathcal{D}\big(F(j \otimes i) \to F(k)\big) \arrow[l, "- \circ F\left({}^{\mathcal{C}}\beta_{i,j}\right)"'] \arrow[u, "- \circ \tau_{j,i}"'] \\
\mathcal{C}(i \otimes j \to k) \arrow[u, "F"] & \mathcal{C}(j \otimes i \to k) \arrow[l, "- \circ {}^{\mathcal{C}}\beta_{i,j}"'] \arrow[u, "F"']
\end{tikzcd} \tag{102}$$

for all $i, j, k \in \operatorname{Irr}\mathcal{C}$. The bottom square commutes by functoriality of F.

By Eq. (96) we see that the vertical compositions in this diagram are precisely the maps $\phi^k_{i,j}$ and $\phi^k_{j,i}$. Recalling further that the R-symbols are defined by precomposition with the braiding, commutativity the diagram (102) is precisely the statement that the $\{\phi^k_{i,j}\}$ intertwine the R-symbols. ☐

It remains to show that τ is a natural isomorphism, and in the unitary case, that τ is unitary.

Lemma D.8. *The natural transformation τ is a natural isomorphism.*

Proof : We must check that all components $\tau_{a,b}$ are invertible. By Lemma D.4 it is sufficient to verify this for the simple components $\{\tau_{i,j}\}_{i,j \in \operatorname{Irr}\mathcal{C}}$.

By assumption, the maps $\phi^k_{i,j}$ are invertible. The functor F is an equivalence, in particular it acts invertibly on Hom-spaces. Together with Eq. (96) we conclude that precomposition with $\tau_{i,j}$ acts invertibly on $\mathcal{D}(F(i \otimes j) \to F(k))$ for any $k \in \operatorname{Irr}\mathcal{C}$. By Lemma D.4 we find that pre-composition by $\tau_{i,j}$ is an *invertible natural* transformation. This natural transformation is precisely the Yoneda embedding $\text{よ}^{\mathrm{op}}_{\mathcal{D}}(\tau_{i,j})$. Since the Yoneda embedding is fully faithful it follows that $\tau_{i,j}$ is invertible. ☐

Lemma D.9. *If $\mathcal{C}$ and $\mathcal{D}$ are unitary monoidal categories and the isomorphisms $\{\phi^k_{i,j}\}_{i,j,k \in \operatorname{Irr}\mathcal{C}}$ preserve direct sum decompositions, then τ is a unitary natural isomorphism.*

Proof : We must check that all components $\tau_{a,b}$ are unitary. By Lemma D.4 it is sufficient to verify this for the simple components $\{\tau_{i,j}\}_{i,j \in \operatorname{Irr}\mathcal{C}}$. Writing $\tau_{i,j}$ in terms of Eq. (95) by specialising the formula (94) to an orthogonal direct sum decomposition (π_κ) of $i \otimes j$, we get

$$\tau_{i,j} \circ \tau^\dagger_{i,j} = \sum_{\kappa,\lambda} F(\pi^\dagger_\kappa) \circ \phi^{i_\kappa}_{i,j}(\pi_\kappa) \circ \phi^{i_\lambda}_{i,j}(\pi_\lambda)^\dagger \circ F(\pi_\lambda) = \sum_\kappa F(\pi^\dagger_\kappa) \circ F(\pi_\kappa) = \mathrm{id}_{i \otimes j},$$

and

$$\tau_{i,j}^{\dagger} \circ \tau_{i,j} = \sum_{\kappa,\lambda} \phi_{i,j}^{i_\lambda}(\pi_\lambda)^{\dagger} \circ F(\pi_\lambda) \circ F(\pi_\kappa^{\dagger}) \circ \phi_{i,j}^{i_\kappa}(\pi_\kappa) = \sum_{\kappa} \phi_{i,j}^{i_\kappa}(\pi_\kappa)^{\dagger} \circ \phi_{i,j}^{i_\kappa}(\pi_\kappa) = \mathrm{id}_{i \otimes j},$$

using that $(\phi_{i,j}^{k}(\pi_\kappa)$ is an orthogonal direct sum decomposition of $\underline{i} \otimes \underline{j}$. □

D.4 Proof of Proposition D.1

Proof : From Lemmas D.6, D.7, D.8, and D.9 it follows that τ is an appropriate tensorator for the functor F in all cases. A compatible unitor isomorphism $\upsilon : \mathbb{1} \to F(\mathbb{1})$ exists and is uniquely determined by [Eti+15, Proposition 2.4.3]. □

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