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PhD thesis

On Analytic Torsion in Families of Arithmetic Manifolds

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Til Mette og Viggo

Abstract

This thesis concerns the asymptotics of analytic torsion when varying over families of arithmetic manifolds, and applies this to study the growth of torsion in the cohomology of the manifolds. The thesis is split into three chapters, the first being a preliminary section, the second an article on this topic for the group $G = \mathrm{SL}(n)$, and the third presenting ongoing work on generalizing the results of the article to a larger family of reductive groups.

The main results of Chapter 2 is the following: Let $\Gamma(N)$ be the principal congruence subgroup in $\mathrm{SL}(n, \mathbb{Z})$ of level N, and let $X(N)$ be the associated locally symmetric space. Let τ be a finite-dimensional irreducible representation of $\mathrm{SL}(n, \mathbb{R})$. Assume that τ is λ-strongly acyclic for a certain $\lambda > 0$. Then, as N goes to infinity, the analytic torsion $T_{X(N)}(\tau)$ of $X(N)$ is equal to the L^2-torsion $T^{(2)}_{X(N)}(\tau)$ of $X(N)$ up to an error term of the size $O(\mathrm{vol}(X(N))N^{-(n-1)}(\log N)^a)$ for some $a > 0$. We furthermore prove the existence of infinitely many λ-strongly acyclic representations of any semisimple Lie group of deficiency greater than or equal to 1.

In chapter 3, this theorem is appropriately generalized to principal congruence subgroups in reductive groups. This result is then applied to estimate the growth of torsion in the cohomology of the Borel-Serre compactification of congruence quotients of manifolds of the form $(\mathbb{H}^2)^b \times (\mathbb{H}^3)^c$.

Resumé

Denne afhandling omhandler vækst af analytisk torsion, når man varierer over en familie af aritmetiske mangfoldigheder, og den anvender dette til at studere væksten af torsion i mangfoldighedernes kohomologi. Afhandlingen er opdelt i tre kapitler, hvor det første er et indledende afsnit, det andet en artikel om dette emne for gruppen $G = \mathrm{SL}(n)$, og det tredje præsenterer igangværende arbejde med at generalisere artiklens resultater til en større familie af reduktive grupper.

Hovedresultaterne fra kapitel 2 er følgende: Lad $\Gamma(N)$ være den principale kongruensundergruppe i $\mathrm{SL}(n,\mathbb{Z})$ af niveau N, og lad $X(N)$ være det tilhørende lokalsymmetriske rum. Lad τ være en endeligdimensionel irreducibel repræsentation af $\mathrm{SL}(n,\mathbb{R})$. Antag, at τ er λ-stærkt acyklisk for et vist $\lambda > 0$. Da gælder, at når N går mod uendelig, er den analytiske torsion $T_{X(N)}(\tau)$ af $X(N)$ lig med L^2-torsionen $T^{(2)}_{X(N)}(\tau)$ af $X(N)$ op til et fejlled af størrelsen $O(\mathrm{vol}(X(N)N^{-(n-1)}(\log N)^a)$ for en vis konstant $a > 0$. Vi beviser yderligere eksistensen af uendeligt mange λ-stærkt acykliske repræsentationer af enhver semisimpel Lie gruppe med defekt større end eller lig 1.

I kapitel 3 generaliseres denne sætning på passende vis til hovedkongruensundergrupper i reduktive grupper. Dette resultat anvendes derefter til at estimere væksten af torsion i kohomologien af Borel-Serre-kompaktificeringen af kongruenskvotienter af mangfoldigheder af formen $(\mathbb{H}^2)^b \times (\mathbb{H}^3)^c$.

Thesis Statement

I am the sole author of the following work. A draft of Chapter 2 has been presented on the arXiv [Ber25].

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Contents

Introduction

Music is the arithmetic of sounds
as optics is the geometry of light.

— *Achille Claude Debussy*

Of significant importance in modern mathematics are results tying together different subjects of study, metaphorically the 'building of bridges'. Such bridges are a major theme of this thesis, in particular bridges between the worlds of arithmetic and geometry in their broadest interpretations. Below we relay four central bridges that permeate and motivate the thesis. This introduction is meant to be understandable for non-experts and hence sacrifices precision in favour of a larger heuristic picture.

Symmetric spaces

This thesis concerns itself with the topic of analytic torsion and its applications in number theory. Analytic torsion is an invariant of compact Riemannian manifolds defined by Ray and Singer [RS71], and is thus a geometric object. It is constructed from the heat operator, an operator of functions on the manifold, defined as the fundamental solution to the heat equation,

$$\begin{aligned} \Delta P_t &= -\frac{\partial}{\partial t} P_t, \quad t > 0, \\ P_0 &= \delta. \end{aligned}$$

Here t is the time variable, Δ is the Laplace operator and δ is the Dirac delta function. In this case, the heat operator is often written $e^{-t\Delta}$.

For a general compact Riemannian manifold, this is a story of geometry and analysis. Luckily, for the manifolds we are concerned with in this thesis, we have much more structure, allowing us to also leverage tools from other areas.

Symmetric spaces are manifolds constructed as quotients of Lie groups with compact subgroups. The representation theory of Lie groups and their algebras is rich and well understood in many regards, and it turns out that one may utilize this in describing differential operators on symmetric spaces. As our principal example, the Laplace operator can be described through the action of the Casimir element, an element of the universal enveloping algebra of the Lie algebra. This is the first bridge. At once, we have the tools and results of the representation theory of Lie algebras at our disposal. This is used to great effect in the thesis, primarily in estimating the heat operator to provide bounds on analytic torsion.

Cheeger-Müller formulas

The definition of analytic torsion was given to mirror another invariant called *Reidemeister torsion*, built from the cohomology of the manifold. Ray and Singer conjectured that the two invariants were equal, and the conjecture was proven independently by Jeff Cheeger [Che79] and Werner Müller [Mül78] for trivial coefficients, and later generalized in [Mül93] and [BZ94]. It is sometimes known as the *Cheeger-Müller formula*. This results in the extraordinary fact that one may study torsion cohomology by analytic means, our second bridge.

In 2013, Nicolas Bergeron and Akshay Venkatesh [BV13] applied this equality to study the growth of torsion in the cohomology of arithmetic groups. Let G be a semisimple Lie group with maximal compact subgroup K such that $\tilde{X} = G/K$ is a symmetric space, and let $d = \dim \tilde{X}$. We denote by $\delta(G) = \mathrm{rk}_{\mathbb{C}} G - \mathrm{rk}_{\mathbb{C}} K$ the *deficiency* of G, also sometimes called the *fundamental rank* of G. Consider a family of cocompact arithmetic lattices Γ_i, $i \in \mathbb{N}$, in G with $\bigcap_i \Gamma_i = \{0\}$ with associated locally symmetric spaces $X_i = \Gamma_i \backslash \tilde{X}$. Inspired by work in geometric group theory, they prove that one can approximate L^2-torsion of X_i using its analytic torsion. They further show that for $\delta(G) = 1$, the L^2-torsion is non-zero. As a consequence of this, they show exponential growth of analytic torsion in terms of the volume, and using the Cheeger-Müller formula, they derive as a consequence the result

$$\sum_{j=0}^{d} (-1)^{j+\frac{d-1}{2}} \frac{\log |H_j(\Gamma_i, M)_{\mathrm{tors}}|}{[\Gamma_1 : \Gamma_i]} > 0.$$

They require a technical assumption that the Γ_1-module M is *strongly acyclic*. This will be an important theme throughout the thesis.

The noncompact setting and trace formulas

In number theory, one is often interested in arithmetic lattices that are not cocompact, and so it is desirable to extend the result of Bergeron–Venkatesh in this direction. This was accomplished in part in a series of papers by Jasmin Matz and Werner Müller ([MM17],[MM20],[MM23]). The main difficulty in the non-compact setting is that the heat operator now may have continuous spectrum, while on a compact manifold it must be discrete. The previous construction of analytic torsion is no longer well defined, and so it is the primary focus of [MM17] to define and show convergence of a regularized version of analytic torsion. Their two later papers then show that one may again approximate L^2-torsion using analytic torsion, analogous to the result in [BV13].

The definition critically uses the geometric side of the *Arthur-Selberg trace formula*. This trace formula is a strong and flexible tool of modern number theory, introduced by James Arthur in a series of papers in the 80's, see [Art05] for a newer introduction. Given a test function f, the content of the Arthur-Selberg trace formula is another bridge, namely the equality of two different expressions in f known as the geometric side and the spectral side of the trace formula. This is the main tool used in the work of Matz–Müller as well as in this thesis.

The standard Cheeger-Müller formula no longer holds when our manifolds are non-compact, and it is an active area of study to find replacements for such manifolds. This has been accomplished in low rank cases, first for hyperbolic manifolds (see [Pfa14], [MR20]), i.e. the $\mathbb{R}$-rank 1 case, and later for $\mathbb{Q}$-rank 1 cases in [MR24]. In the following, formulas relating analytic torsion and Reidemeister torsion will also be known as Cheeger-Müller formulas. Such a formula is used in the final section of the thesis to provide results on the growth of torsion in the cohomology of certain locally symmetric spaces.

A torsion Langlands correspondence

We discuss some of the more heuristic motivation for this thesis. One of the most important problems in modern day mathematics is the Langlands program. It is a deep and far-reaching web of conjectures originally proposed by Robert Langlands in the late 1960's in his letter to André Weil ([Lan]). A central feature of the Langlands program is the Langlands correspondence, a bridge between the worlds of Galois theory and the theory of automorphic forms. In short, it predicts a one-to-one correspondence between two families of respectively automorphic representations and Galois representations, preserving many of the essential properties of these representations.

More recently, a conjecture by Avner Ash ([Ash92]) asserts a similar correspondence between systems of Hecke eigenvalues in the $\mathbb{F}_p$-cohomology of arithmetic groups and certain Galois representations. The conjecture was partially proven by Peter Scholze in [Sch15]. This implies that if one proves the existence of an abundance of torsion in the integral cohomology of arithmetic groups, one conjecturally shows the existence of many Galois representations.

Organization and results

The thesis consists of three chapters. The first gives a gentle introduction to a selection of the tools used in the thesis, namely how to express the Laplacian through representation theory, and the geometric side of the Arthur trace formula. The content should be well known to experts.

The second chapter is my article on analytic torsion when the group is $G = \mathrm{SL}(n, \mathbb{R})$, and the arithmetic subgroups are the principal congruence subgroups $\Gamma(N)$ of $\mathrm{SL}(n, \mathbb{Z})$. Let $X(N) = \Gamma(N)\backslash \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$, and let τ be a finite-dimensional irreducible representation of $\mathrm{SL}(n, \mathbb{R})$. We denote by $T_{X(N)}(\tau)$ and $T^{(2)}_{X(N)}(\tau)$ the analytic torsion, respectively the L^2-torsion, associated to $X(N)$ and τ. The main theorem of the paper is the following:

Theorem A. *Assume that τ is λ-strongly acyclic, for a certain λ depending only on n. Then there exists some $a > 0$ such that*

$$\log T_{X(N)}(\tau) = \log T^{(2)}_{X(N)}(\tau) + O(\mathrm{vol}(X(N))N^{-(n-1)} \log(N)^a)$$

as N tends to infinity.

This is Theorem 2.1.1 in the thesis. We furthermore show the existence of infinitely many λ-strongly acyclic representations for any $\lambda > 0$ in Proposition 2.3.2. The paper is self-contained and has been included with only only minor revisions from its preprint version available on arXiv [Ber25]. For this reason, there will be a cursory overlap between Chapters 1 and 2.

In the final chapter, I present ongoing work on generalizing the results from the paper to a much larger class of reductive groups, and give applications to torsion cohomology. To be precise, the above theorem is generalized for G a reductive group and $X(N)$, $N \geq 3$ a family of associated congruence manifolds.

Theorem B. *Let τ be an irreducible and λ-strongly acyclic representation of $G(\mathbb{R})^1$, for some $\lambda > 0$ only depending on G. Assume that G satisfies the properties (TWN) and (BD) (see [FLM15]). Assume either Conjecture 3.2.1 or that N varies over a prime-fixed set. Then there exists some $a > 0$ such that*

$$\log T_{X(N)}(\tau) = \log T^{(2)}_{X(N)}(\tau) + O(\mathrm{vol}(X(N))N^{-k(G)}(\log N)^a)$$

as N tends to infinity.

This is Theorem 3.2.2 in the thesis. We remark that $k(G)$ is defined in (3.1.10). This theorem is used to give new asymptotics on torsion in the cohomology of locally symmetric spaces associated to $\mathrm{SL}(2, F)$ for F a number field. Let r_1 and r_2 be the number of real, respectively pairs of complex embeddings of F. Let $G = \mathrm{res}_{F/\mathbb{Q}}(\mathrm{SL}(2)/F)$, and pick an embedding ρ of G into $\mathrm{GL}(n)$. Let ρ_∞ be the induced representation of $G(\mathbb{R})$. Let $X(N)$, $N \in \mathbb{N}$, $N \geq 3$ be a certain family of congruence manifolds, and $\overline{X(N)}$ their Borel-Serre compactifications. We associate a local system L_ρ of free $\mathbb{Z}$-modules over $X(N)$. Then the below result is a consequence of Theorem B.

Theorem C. *Assume that r_2 is odd, $r_1 > 0$ and $r_1 + r_2 > 2$. Assume further that ρ_∞ decomposes into a sum of λ-strongly acyclic representations, with $\lambda > 0$ chosen as in Theorem B. Then there exists a contant $a > 0$ such that*

$$\sum_{q=0}^{d}(-1)^{q+1}\log|H^q(\overline{X(N)}, L_\rho)| = 2 \cdot T^{(2)}_{X(N)}(\rho_\infty) + O(\mathrm{vol}(X(N))N^{-k(G)}\log(N)^a)$$

as N tends to infinity.

This is Theorem 3.3.2 in the thesis. The chapter concludes with Corollary 3.3.3, which is the specialization of the theorem above to deficiency strictly greater than 1, inferring new vanishing results.

Corollary D. *Assume further that $r_2 > 1$, such that $\delta(G) > 1$. Then $T^{(2)}_{X(N)}(\rho_\infty) = 0$, and hence,*

$$\sum_{q=0}^{d}(-1)^{q+1}\frac{\log|H^q(\overline{X(N)}, L_\rho)|}{[\Gamma(1):\Gamma(N)]} = O(N^{-k(G)}(\log N)^a)$$

as N tends to infinity.

The complete bibliography for the entire thesis is presented after Chapter 3.

1 Preliminaries

This chapter is meant as an introduction to two essential methods of the thesis.

The first section is an account of the interweaving of geometry and representation theory in the study of symmetric spaces, with a focus on the p-form Laplacian. It overlaps to some extent with Section 2.2, but is both much more introductory and more detailed. The primary references are [Kna96], [Hel62], [Mia80], but we also draw inspiration from [Olb02], Chapter 2 of [Hel84], [Lee12], [Woi13], and [Par18].

The second section focuses on the geometric side of the Arthur-Selberg trace formula. The presentation here is inspired by that of [Art05], but is more streamlined towards the coarse and fine geometric expansions, as well as including a few newer results. Again we must suffer a slight overlap with Chapter 2, here Section 2.5.

We will assume basic familiarity with manifolds, as well as Lie groups and Lie algebras, and later their representation theory. Proofs are included only in the case that they are both brief and insightful. I claim no originality over any of the results presented here.

1.1 Geometry and Representation Theory

Lie groups and Lie algebras occupy a central position in the intersection between geometry and algebra. On one hand, the algebraic structure allows the use of group theory, linear algebra, and representation theory. On the other hand, the manifold structure naturally lends itself to tools from measure theory, analysis, and topology. In the modern theory, these tools are woven together elegantly, often complimenting each other and allowing for new insights. Very often, we will see a problem on one side that, once translated to the other side, has a beautiful solution that can then be translated back. Crucially, this happens in both directions, meaning we are allowed new algebraic results by the use of geometric methods, as well as solving geometric problems using algebra.

The interlacing of these two different fields is the central topic of this section. The goal is to show how to interpret spaces of differential forms on symmetric spaces as L^2-spaces on the Lie group, and utilize this to describe form Laplacians. We will usually only include proofs that we deem sufficiently brief and insightful.

Lie groups and Lie algebras

We begin with some basic theory on Lie groups and Lie algebras. Recall that a Lie algebra $\mathfrak{g}$ acts on itself by $\mathrm{ad}_X(Y) = [X, Y]$.

Definition 1.1.1. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field F, and let $X, Y \in \mathfrak{g}$. We define the *Killing form* of $\mathfrak{g}$ as

$$B(X, Y) \coloneqq \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y).$$

Taking this trace makes sense as ad_X and ad_Y are linear endomorphisms of the finite-dimensional vector space $\mathfrak{g}$. Then B is a symmetric bilinear form on $\mathfrak{g}$. It furthermore satisfies the property $B([X, Y], Z) = B(X, [Y, Z])$.

As a first application of the Killing form, we get a criteria for semisimplicity and solvability.

Proposition 1.1.2 (Cartan's Criteria)**.** *The following two critera holds:*

1. *The Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form B is nondegenerate.*

2. *The Lie algebra $\mathfrak{g}$ is solvable if and only if its Killing form B satisfies*

$$B(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0.$$

See ([Kna96], Theorem 1.45 and Proposition 1.46) for a proof.

For matrix Lie algebras, one can make the Killing form more explicit. As an example, let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, with $n \geq 2$. Then

$$B(X, Y) = 2n\,\mathrm{tr}(XY).$$

There is an elegant proof using a bit of theory. We argue the result for $\mathfrak{sl}(n, \mathbb{C})$, which then implies the result for $\mathfrak{sl}(n, \mathbb{R})$ by restriction. As $\mathfrak{sl}(n, \mathbb{C})$ is simple, it is a consequence of Schur's lemma that any two non-degenerate symmetric bilinear forms must be scalar multiples of each other. Since these properties hold for both the Killing form and the *trace form* $(X, Y) \mapsto \mathrm{tr}(XY)$, we know that $B(X, Y) = c\ \mathrm{tr}(XY)$. One then determines $c = 2n$ by computing any nonzero example.

Definition 1.1.3. A *Cartan involution* of $\mathfrak{g}$ is an idempotent Lie algebra automorphism θ of $\mathfrak{g}$, i.e. satisfying $\theta \circ \theta = \mathrm{id}_{\mathfrak{g}}$, such that the bilinear form

$$B_\theta(X, Y) \coloneqq -B(X, \theta Y)$$

is positive definite.

It is a standard result that any real semisimple Lie algebra has a Cartan involution and that it is unique up to inner automorphism ([Kna96], Corollaries 6.18 and 6.19). Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan involution θ. Then θ, as an involution on the vector space $\mathfrak{g}$, has an eigenspace decomposition with eigenvalues ± 1. We write

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{1.1.1}$$

with $\mathfrak{k}$ the 1-eigenspace and $\mathfrak{p}$ the -1-eigenspace. This is called the *Cartan decomposition* of $\mathfrak{g}$ with respect to θ. As θ is a Lie algebra endomorphism, it respects the Lie bracket, which implies that

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}. \tag{1.1.2}$$

As a consequence, for $X \in \mathfrak{k}$ and $Y \in \mathfrak{p}$ we see that $\mathrm{ad}_X \circ \mathrm{ad}_Y$ maps $\mathfrak{k}$ into $\mathfrak{p}$ and $\mathfrak{p}$ into $\mathfrak{k}$. In particular, it must have trace 0. Then by definition, we see that the Cartan decomposition is orthogonal with respect to the Killing form.

As $B_\theta = -B(X, \theta Y)$ is positive definite by definition of θ, we see that the Killing form B is positive definite on $\mathfrak{p}$ and negative definite on $\mathfrak{k}$.

Conversely, any decomposition (1.1.1) satisfying (1.1.2) and having the restriction of the Killing form be positive definite on $\mathfrak{p}$, respectively negative definite on $\mathfrak{k}$, induces a Cartan involution defined to be 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$.

There is an analogy of the Cartan involution on the level of Lie groups.

Proposition 1.1.4. *Let G be a non-compact semisimple Lie group, and let θ be a Cartan involution of its Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let K be the Lie subgroup with Lie algebra $\mathfrak{k}$. Then there exists an involution Θ of G, called the* global Cartan involution*, such that K is the set of fixpoints of Θ. Furthermore, the mapping $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k \cdot \exp(X)$ is a diffeomorphism. If we denote $P = \exp(\mathfrak{p})$, this gives a global Cartan decomposition $G = KP$.*

See ([Kna96], Theorem 6.31). We return to our example. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. This has a Cartan involution given by $\theta(X) = -X^T$. Indeed, it is clear that is is an involution of the vector space $\mathfrak{g}$, and it is an easy check to see that it preserves the Lie bracket. Furthermore, it follows directly that B_θ is positive definite given the explicit description of the Killing form above, using that XX^T is a positive semidefinite matrix for any real matrix X.

This Cartan involution obviously has as its 1-eigenspace the skew-symmetric matrices $\mathfrak{k} = \mathfrak{so}(n)$ and as -1-eigenspace the symmetric matrices $\mathfrak{p}$, giving us the decomposition

$$\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{p}.$$

On the level of Lie groups, we get the global Cartan involution $\Theta(g) = (g^{-1})^T$, and the global Cartan decomposition is the familiar polar decomposition $\mathrm{SL}(n, \mathbb{R}) = \mathrm{SO}(n) PSD(n)$, where $PSD(n)$ is the set of real positive definite $n \times n$ matrices.

Symmetric spaces

Let us define our central topics of this section.

Definition 1.1.5. Let M be a smooth Riemannian manifold.

1. We say that M is *Riemannian globally symmetric space* if every point $p \in M$ is an isolated fixed point of an involutive isometry s_p of M.

2. We say that M is *Riemannian locally symmetric space* if each point $p \in M$ has a normal neighborhood U_p on which the geodesic symmetry with respect to p is an isometry.

We will quickly disperse of the use of "Riemannian" and simply write globally or locally symmetric space. There is a more general notion of symmetric spaces that are not manifolds, but rather *orbifolds*. Although important, we will not need this generalization in the thesis, and will stick to the convention above.

The naming conventions of these properties is justified thusly: If M is a globally symmetric space, for every point $p \in M$ there exists a normal neighborhood U_p of p on which s_p is the geodesic symmetry (see Helgason, Lemma IV.3.1). Said in another way, the local isometries extend to global isometries. Thus, globally symmetric implies locally symmetric.

For M a Riemannian manifold, let $I(M)$ be the set of isometries of M. This is a group under composition, and furthermore a locally compact Hausdorff topological group under the compact open topology. For any point p, the stabilizer $\mathrm{Stab}(p) \subseteq I(M)$ is compact. We denote by $I_0(M)$ the subgroup with the underlying set being the connected component of the identity in $I(M)$.

We will need the following lemma.

Lemma 1.1.6. *Let G be a Lie group with closed subgroup $H \subseteq G$. Then G/H, the space of left cosets, equipped with the natural topology, has a unique smooth structure such that G acts smoothly trough left multiplication.*

See ([Hel62], Theorem II.4.2).

The following central results ties together the study of symmetric spaces and of Lie groups, combining the algebraic and geometric theories.

Proposition 1.1.7. *Let M be a globally symmetric space, and let $p \in M$.*

1. *$I(M)$ has an smooth structure compatible with the compact open topology, turning it into a Lie group.*

2. *Set $G = I_0(M)$ and $K = Stab(p)$. Then K is a compact subgroup of the Lie group G, and the set of cosets G/K with the induced smooth structure as in Lemma 1.1.6 is isomorphic to M under the mapping $gK \mapsto g.p$, $g \in G$.*

3. *In the same setting,* $\Theta : g \mapsto s_p g s_p$ *is an involution of* G *with* $K_\Theta \subseteq G$ *the subgroup of fixpoints of* Θ *such that*

$$(K_\Theta)_0 \subseteq K \subseteq K_\Theta.$$

4. *Denote by* $\mathfrak{g}$ *and* $\mathfrak{k}$ *the Lie algebras of* G *and* K*, respectively. Then* $\mathrm{Lie}(\Theta)$ *is a Cartan involution of* $\mathfrak{g}$*, and* $\mathfrak{k}$ *is its* 1*-eigenspace. With Cartan decomposition* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$*, for* $\pi : G \to M$ *given by* $g \mapsto g.p$*, its differential* $d\pi_p : \mathfrak{g} \to T_pM$ *maps* $\mathfrak{k}$ *to* $\{0\}$ *and* $\mathfrak{p}$ *isomorphically onto* T_pM*.*

See ([Hel62], Lemma IV.3.2, Theorem IV.3.2, and section IV.4). The following is a converse result, see ([Hel62], Theorem V.4.1).

Proposition 1.1.8. *Assume* G *is a connected Lie group with a closed subgroup* $K \subseteq G$ *and an involution* Θ *of* G *such that* $(K_\Theta)_0 \subseteq K \subseteq K_\Theta$*. Assume further that* $\mathrm{Ad}_G(K)$ *is compact (this is in particular satisfied if* K *is compact). Then* G/K *has a* G*-invariant Riemannian structure under which it is a globally symmetric space. If* G *is also assumed semisimple and acting faithfully on* G/K*, then* $G = I_0(G/K)$*.*

Given a connected semisimple Lie group G and its Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the Lie subgroup $K \subseteq G$ with Lie algebra $\mathfrak{k}$ is a maximal compact subgroup of G, and it is unique with this property up to conjugacy (because the Cartan involution is).

Proposition 1.1.9. *If* G *is a connected semisimple* noncompact *Lie group, the composition of the exponential map and quotient map* $\mathfrak{g} \to G \to G/K$ *restricts to a diffeomorphism of* $\mathfrak{p}$ *and* G/K*.*

See ([Hel62], Theorem VI.1.1).

The next result shows how to construct locally symmetric spaces, and give a reason why they are useful in number theory.

Proposition 1.1.10. *Let* G *be a connected semisimple Lie group with maximal compact subgroup* K*, such that* $M = G/K$ *is a globally symmetric space. Let* Γ *be a discrete torsion-free subgroup of* G*. Then* $\Gamma\backslash M$ *has a natural structure as a locally symmetric space.*

We sketch the proof here. First, we prove that the action of γ on G/K by left multiplication is free. Let $\gamma \in \Gamma$ and $gK \in G/K$. Then

$$\gamma g K = gK \quad \Leftrightarrow \quad \gamma \in gKg^{-1}.$$

Since K is compact, so is gKg^{-1}, and as Γ is discrete, the isotropy subgroup $\Gamma \cap gKg^{-1}$ is finite. But Γ is torsion-free, so any finite subgroup must be trivial, i.e. only the identity fixes gK, thus the action is free. In a similar vein, one can prove that if $p, q \in M$, there exists respective neighborhoods U, V such that for any $\gamma \in \Gamma$, if $\gamma(U)$ intersects V then $\gamma(p) = q$.

This then shows that $\Gamma\backslash M$, with the quotient topology, is Hausdorff, and the quotient map $M \to \Gamma\backslash M$ is a local homeomorphism. This induces a natural smooth structure on $\Gamma\backslash M$, and as Γ acts by isometries, also a Riemannian structure, such that the quotient map is a local diffeomorphism, and local diffeomorphisms preserves the property of being locally symmetric. □

For us, this means that $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is a globally symmetric space, and that the quotient with any discrete torsion-free subgroup is locally symmetric. Importantly, the class of such subgroups include the principal congruence subgroup $\Gamma(N)$ of $\mathrm{SL}(n,\mathbb{Z})$ for $N \geq 3$. For $n = 2$, we have a familiar model of the globally symmetric space given by the hyperbolic upper half plane $\mathbb{H}^2$. Indeed, one can check that $I_0(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$ acting by fractional linear transformations, and that the stabilizer of the point i is $\mathrm{Stab}(i) = \mathrm{PSO}(2)$, which means that

$$\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) = \mathrm{PSL}(2,\mathbb{R})/\mathrm{PSO}(2) \cong \mathbb{H}^2.$$

For principal congruence subgroups $\Gamma(N)$, the locally symmetric space

$$Y(N) = \Gamma(N)\backslash \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(n)$$

is then the usual modular curve.

We define a *classifying space* BG of a topological group G to be the quotient of a contractible space EG by a proper free action of G. Note that there is a more geometric equivalent definition, and that some replace contractible by *weakly contractible*.

By our proof sketch for Proposition 1.1.10 above, Γ acts properly and freely on $M = G/K$, and by Proposition 1.1.9, M is contractible (as it is homeomorphic to $\mathbb{R}^n$). Thus, $\Gamma\backslash M$ is a classifying space of Γ. Here is why that is interesting.

Proposition 1.1.11. *For a topological group G with classifying space BG, we have that $\pi_1(BG) = G$, and that the group homology of G is isomorphic to the singular homology of BG, i.e.*

$$H_*(G,\mathbb{Z}) = H_*(BG,\mathbb{Z})$$

A similar equality holds when taking coefficients in other G-modules as well.

Vector bundles and representations

For M a connected manifold with $\Gamma = \pi_1(M)$, there is a standard construction taking as input a (real or complex) finite dimensional representation (η, V_η)

and producing a flat vector bundle over M, as follows: Let $\tilde{M}$ be the universal covering of M. Consider the trivial bundle $\tilde{M} \times V_\eta \to \tilde{M}$. This has an obvious flat structure, and an action of Γ preserving the flat structure, given by

$$\gamma.(\tilde{m}, v) := (\gamma.\tilde{m}, \eta(\gamma)v).$$

Taking the quotient by this action yields a flat vector bundle, denoted F_η, over M.

We provide a similar construction to the above for Lie subgroups. Let G be a Lie group with $H \subseteq G$ a closed Lie subgroup. Given a finite-dimensional (real or complex) representation (ρ, V_ρ) of H, one may construct a G-homogeneous vector bundle $G \times_H V_\rho$ over G/H in the analogous way: Let H act on $G \times V_\rho$ by

$$(g, v) \xrightarrow{h} (gh, \rho(h^{-1})v).$$

We define $G \times_H V_\rho$ as the quotient by that action, and let G act by left multiplication on the first component. This clearly has a projection to G/H in the first component, and the fiber at any point is isomorphic to V_ρ. Homogeneous in this setting means that the projection $G \times_H V_\rho \to G/H$ intertwines the G-action on $G \times_H V_\rho$ and G/H.

Also attached to ρ we define a space of vector-valued functions

$$C^\infty(G, \rho) := \{f : G \to V_\rho \mid f \text{ smooth},\ f(gh) = v(h^{-1})f(g)\ \forall g \in G, h \in H\}.$$

The following is the crux of the section, where the interplay of representation theory and differential geometry shines. Let G be a non-compact connected semisimple Lie group with $K \subseteq G$ a maximal compact subgroup, and let $\Gamma \subseteq G$ be a discrete torsion-free subgroup.To align with later notation, we set $\tilde{X} = G/K$ and $X = \Gamma\backslash\tilde{X}$, which are then respectively a globally and and a locally symmetric space, the former the universal covering of the latter. Let (τ, V) be an irreducible finite-dimensional real or complex representation of G. Restricting to a representation of K, which we by abuse of notation also write as τ, we induce a homogeneous vector bundle $\tilde{E}_\tau$ on $\tilde{X}$.

Example 1.1.12. As we saw earlier, given our Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ associated to the choice of K we have a natural isomorphism of the tangent space of G/K at the identity eK and $\mathfrak{p}$. In fact, considering the adjoint representation of G on $\mathfrak{g}$ and restricting it to K, one sees that $\mathfrak{p}$ is an invariant subspace due to the bracket relations (1.1.2). Let $\pi : G \to G/K$ denote the quotient map. The subrepresentation $(\mathrm{Ad}|_K, \mathfrak{p})$ then corresponds to the tangent bundle using the construction above, with the explicit isomorphism

$$\begin{aligned} G \times_K \mathfrak{p} &\xrightarrow{\sim} T\tilde{X} \\ [g, P] &\mapsto \frac{d}{dt}\pi(g\exp(tP))|_{t=0}. \end{aligned}$$

Define now $E_\rho := \Gamma\backslash\tilde{E}_\rho$ as the locally homogeneous vector bundle over X given by the quotient of $\tilde{E}_\rho$ by left multiplication from Γ on the first coordinate. Furthermore, restricting τ to Γ, as $\Gamma = \pi_1(X)$, it induces a flat vector bundle F_τ on X. Perhaps unsurprisingly, this vector bundle is isomorphic to E_τ, i.e.

$$E_\tau \cong F_\tau, \tag{1.1.3}$$

and this is used in some of the following propositions.

We consider the Γ-invariant subspace of the space of smooth functions above:

$$C^\infty(\Gamma\backslash G, \tau) := \{f \in C^\infty(G,\tau) \mid f(\gamma g) = f(g) \;\; \forall g \in G, \gamma \in \Gamma\}.$$

Proposition 1.1.13. *Let $C^\infty(\tilde{X}, \tilde{E}_\tau)$ denote the smooth sections of the vector bundle $\tilde{E}_\tau$. We then have an isomorphism*

$$\tilde{A} : C^\infty(G,\tau) \xrightarrow{\sim} C^\infty(\tilde{X}, \tilde{E}_\tau), \tag{1.1.4}$$

$$(\tilde{A}f)(gK) := [g, f(g)]. \tag{1.1.5}$$

The K-equivariance of $f \in C^\infty(G,\tau)$ ensures exactly that the map $\tilde{A}f$ is well defined. The fact that it is a section is obvious, and smoothness follows directly from the definition. See ([Mia80], p. 4) for more details. Once a proper inner product on sections of $\tilde{E}_\tau$ is defined, this isomorphism extends to an isometry of corresponding L^2-spaces, i.e. replacing the assumption of smoothness with one of L^2-integrability. If we similarly set $C^\infty(X, E_\tau)$ the smooth sections of E_τ, then the isomorphism above furthermore descends to the Γ-invariant subspace, giving an isomorphism

$$C^\infty(\Gamma\backslash G, \tau) \xrightarrow{\sim} C^\infty(X, E_\tau), \tag{1.1.6}$$

which extends to an isometry of L^2-spaces as well.

There is a further isomorphism for the spaces in Proposition 1.1.13, giving another representation-theoretic characterization of the space of smooth sections. Consider the vector space $C^\infty(G) \otimes V_\tau$, and let K act by right translation on $C^\infty(G)$ and τ on V_τ. Let $[C^\infty(G) \otimes V_\tau]^K$ denote the K-fixpoints. Then

$$\tilde{B} : C^\infty(G,\tau) \xrightarrow{\sim} [C^\infty(G) \otimes V_\tau]^K. \tag{1.1.7}$$

We describe the map $\tilde{B}$. Take a basis $(v_1, \ldots, v_k)$ of V_τ, and for $f \in C^\infty(G,\tau)$ and $i \leq n$ let $f_i : G \to \mathbb{F}$ ($\mathbb{F}$ being either $\mathbb{R}$ or $\mathbb{C}$) be the smooth functions $f_i(g) := \langle f(g), v_i \rangle$. Then $\tilde{B}$ is the map

$$\tilde{B}(f) \mapsto \sum_i f_i \otimes v_i.$$

One can check that $\tilde{B}$ is independent of the choice of basis. The K-equivariance of f implies the K-invariance of $\tilde{B}(f)$, which is easily verified by direct computation.

We wish to use these isomorphisms to describe form Laplacians. We recall the definition of differential forms with values in a vector bundle.

Definition 1.1.14. Let M be a smooth manifold. For $E \to M$ a smooth vector bundle over M, the space of *E-valued differential p-forms* is the space $\Lambda^p(M, E)$ of smooth sections of the tensor product bundle $\Lambda^p(E) := E \otimes \Lambda^p T^*M$ of E and the p'th exterior power of the cotangent bundle $\Lambda^p T^*M$.

Using the de Rham differential d and its adjoint d^* in the setting of the definition, we define the *form Laplacian* as $dd^* + d^*d$. For our locally symmetric space X, we consider the space $\Lambda^p(X, F_\tau)$ of F_τ-valued p-forms, and denote by $\Delta_p(\tau)$ the associated form Laplacian (from now on *the Laplace operator* or simply the *Laplacian*). It is this operator we would like to understand, but for technical reasons, we will work with its lift to the universal covering.

Consider now the K-representation given by the tensor product of τ and the p'th exterior power of the coadjoint representation of K on $\mathfrak{p}^*$:

$$\Lambda^p \operatorname{Ad}^* \otimes \tau : K \to \operatorname{GL}(\Lambda^p \mathfrak{p}^* \otimes V_\tau).$$

We denote this representation by $v_{p,\tau}$. It is a direct check, essentially a consequence of Example 1.1.12, that this K-representation exactly corresponds to the vector bundle $\Lambda^p(\tilde{E}_\tau)$, which means that (1.1.3) and (1.1.6) gives us the isomorphisms

$$C^\infty(\Gamma \backslash G, \nu_{p,\tau}) \xrightarrow{\sim} \Lambda^p(X, F_\tau).$$

Let $\tilde{F}_\tau$ denote the pullback of F_τ to $\tilde{X}$. Then $\Delta_p(\tau)$ lifts to an operator $\tilde{\Delta}_p(\tau)$ on the space of p-forms $\Lambda^p(\tilde{X}, \tilde{F}_\tau)$ of $\tilde{X}$. We get another isomorphism

$$C^\infty(G, \nu_{p,\tau}) \xrightarrow{\sim} \Lambda^p(\tilde{X}, \tilde{F}_\tau).$$

This allows us to describe $\tilde{\Delta}_p(\tau)$ in terms of the Lie algebra. Indeed, let $\Omega \in \mathcal{Z}(\mathfrak{g})$ be the Casimir element of G, living in the center of the universal enveloping algebra of $\mathfrak{g}$, and let R be the regular representation of G on $C^\infty(G, \nu_{p,\tau})$. Then it is essentially Kuga's lemma (see e.g [MM63]) that in terms of the isomorphism above, we have

$$\tilde{\Delta}_p(\tau) = -R(\Omega) + \tau(\Omega). \tag{1.1.8}$$

The main strength of this identity is that we understand the behaviour of the Casimir element very well. In particular, as the Casimir element lives in the center of the universal enveloping algebra, Schur's lemma tells us that Ω acts by a scalar on any irreducible complex representation. We utilize this now. By the isomorphism (1.1.7), we get

$$\Lambda^p(\tilde{X}, \tilde{F}_\tau) \cong [C^\infty(G) \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau]^K.$$

In the same way as before, this extends to an isometry of L^2-spaces. The space on the right-hand side is then $[L^2(G) \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau]^K$. G acts on this space both on the left and on the right, by the left, respectively right regular representation on the first coordinate. Understanding the regular representation on $L^2(G)$ is a deep and important theory, and we will not touch on it here. We only remark that given an irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ occurring in the decomposition of $L^2(G)$ under the *right* regular representation, we can consider the G-representation

$$[\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau]^K.$$

Since $(\pi, \mathcal{H}_\pi)$ is irreducible, the Casimir element acts by a scalar on this space, and thus also on the K-fixed space above. The identity (1.1.8), once the Laplace operators have been properly extended to L^2-spaces (discussed in Chapter 2), then tells us that the Laplacian acts by an explicit scalar on this space. It is a major theme of Chapter 2 that by choosing τ correctly, one can in fact ensure that this scalar is sufficiently large, used for both convergence and vanishing results. This topic is continued in Section 2.2.

1.2 The Arthur-Selberg Trace Formula

The Arthur-Selberg trace formula is an important tool of modern number theory, ubiquitous in and surrounding the theory of automorphic forms. It was constructed as a generalization of the Selberg trace formula by James Arthur. It is also the primary tool applied in Chapters 2 and 3. In this section we give a short introduction to the Selberg trace formula, introduce the necessary theory of adelic groups, and finally present the Arthur-Selberg trace formula with a focus on the geometric side and its expansions.

A motivating example

Let $f : \mathbb{R} \to \mathbb{C}$ be a *Schwartz function*, i.e. smooth and with rapid decay at infinity for all its derivatives. We define its fourier transform as

$$\hat{f}(x) := \int_{\mathbb{R}} f(y) e^{2i\pi xy} dy.$$

Then we have the classical *Poisson summation formula*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

We will present a functional analysis-flavoured proof which showcases the formula as an example of the Selberg trace formula.

Define the operator

$$R(f) : L^2(\mathbb{R}/\mathbb{Z}) \to L^2(\mathbb{R}/\mathbb{Z})$$
$$(R(f)\varphi)(x) := \int_{\mathbb{R}} f(y)\varphi(x-y)dy.$$

We wish to compute the trace of this operator. We compute the trace in two ways, and the equality of the two resulting expressions will be the Poisson summation formula.

Firstly, the *geometric* side. Using the $\mathbb{Z}$-invariance of φ, we can write

$$\int_{\mathbb{R}} f(y)\varphi(x-y)dy = \int_{\mathbb{R}} f(x-y)\varphi(y)dy \tag{1.2.1}$$
$$= \int_{\mathbb{R}/\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} f(x-y+n)\varphi(y+n) \right) dy \tag{1.2.2}$$
$$= \int_{\mathbb{R}/\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} f(x-y+n) \right) \varphi(y)dy. \tag{1.2.3}$$

This shows that $R(f)$ is an integral operator with kernel $K(x,y) = \sum_{n\in\mathbb{Z}} f(x-y+n)$. One may compute the trace of an integral operator by integrating the kernel over the diagonal, i.e.

$$\operatorname{tr} R(f) = \int_{\mathbb{R}/\mathbb{Z}} K(y,y)dy = \int_{\mathbb{R}/\mathbb{Z}} \sum_{n\in\mathbb{Z}} f(n)dy = \sum_{n\in\mathbb{Z}} f(n). \tag{1.2.4}$$

Now we consider the *spectral* side. Recall that the functions $e_n(x) := e^{2\pi inx}$ for $n \in \mathbb{Z}$ constitute a orthonormal basis of $L^2(\mathbb{R}/\mathbb{Z})$. Furthermore, a substitution as above yields that

$$R(f)(e_n(x)) = \hat{f}(n)e_n(x).$$

In particular, we have the full set of eigenvalues of $R(f)$ given as $(\hat{f}(n))_{n\in\mathbb{Z}}$. Thus, we may compute the trace of $R(f)$ as the sum of its eigenvalues,

$$\operatorname{tr} R(f) = \sum_{n\in\mathbb{Z}} \hat{f}(n). \tag{1.2.5}$$

The equality of (1.2.4) and (1.2.5) is exactly the Poisson summation formula.

The Selberg trace formula

We continue in a more general, but totally analogous, setting of the previous section. Many details that was ignored will now be highlighted, and we will see how the story above fails in the general setup.

Let H be a unimodular topological group with $\Gamma \subseteq H$ a discrete subgroup. Then the space $\Gamma\backslash H$ has a right H-invariant Borel measure. We denote by R the unitary representation of H given by right translation on the Hilbert space $L^2(\Gamma\backslash H)$, i.e. for $x, y \in H$ and $\varphi \in L^2(\Gamma\backslash H)$,

$$(R(y)\varphi)(x) := \varphi(xy)$$

We will call this the *right regular representation* of H. Studying this representation is a central topic in representation theory and number theory. The Arthur-Selberg trace formula can be seen as a tool for this exact purpose. Following along these lines, the main idea is to study R through integrating it against certain test functions. Denote by $C_c(H)$ the continuous compactly supported functions from H to $\mathbb{C}$. Let $f \in C_c(H)$, and define the operator on $L^2(\Gamma\backslash H)$ given as

$$R(f) := \int_H f(y)R(y)dy.$$

This should be understood in the following way:

$$(R(f)\varphi)(x) = \int_H f(y)(R(y)\varphi)(x)dy.$$

By the exact same manipulations as in (1.2.1), we see that $R(f)$ is an integral operator with kernel

$$K(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).$$

The sum is convergent as it is in practice finite: It may be taken over the finite set $\Gamma \cap x\,\mathrm{supp}(f)y^{-1}$, where finiteness follows from intersecting a discrete set with a compact set. Now, assume $\Gamma\backslash H$ is compact, i.e. Γ is cocompact in H. This makes the analysis of $R(f)$ much simpler. In particular, the representation R decomposes discretely into irreducible representations of finite multiplicities, and so we may consider $R(f)$ as an operator on invariant subspaces. Furthermore, with reasonable further assumptions, such as H being a Lie group and f being smooth, we have that $R(f)$ is a trace class, and the trace can be computed as the integral over the kernel as in (1.2.4).

Given such reasonable assumptions, let $\{\Gamma\}$ be a set of representatives of conjugacy classes in Γ. For S any subset of H we denote by S_γ the centralizer of γ in S. Then we may write

$$\begin{aligned}
\operatorname{tr} R(f) &= \int_{\Gamma\backslash H} \sum_{\gamma\in\Gamma} f(y^{-1}\gamma y)dy \\
&= \int_{\Gamma\backslash H} \sum_{\gamma\in\{\Gamma\}} \sum_{\sigma\in\Gamma_\gamma\backslash\Gamma} f(y^{-1}\sigma^{-1}\gamma\sigma y)dy \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{\Gamma_\gamma\backslash H} f(y^{-1}\gamma y)dy \\
&= \sum_{\gamma\in\{\Gamma\}} \int_{H_\gamma\backslash H} \int_{\Gamma_\gamma\backslash H_\gamma} f(y^{-1}x^{-1}\gamma xy)dxdy \\
&= \sum_{\gamma\in\{\Gamma\}} \mathrm{vol}(\Gamma_\gamma\backslash H_\gamma) \int_{H_\gamma\backslash H} f(y^{-1}\gamma y)dy.
\end{aligned}$$

On the spectral side, instead of eigenvalues we have irreducible subrepresentations π. Denote the set of irreducible subrepresentations of R by $\hat{H}$. One may compute the trace as the sum of traces over invariant subspaces V_π with multiplicities $m(\pi)$, arriving at another decomposition of the trace.

The equality of these two methods is the *Selberg trace formula*,

$$\sum_{\gamma\in\{\Gamma\}} \mathrm{vol}(\Gamma_\gamma\backslash H_\gamma) \int_{H_\gamma\backslash H} f(y^{-1}\gamma y)dy = \sum_{\pi\in\hat{H}} m(\pi)\operatorname{tr}(\pi(f)). \tag{1.2.6}$$

Number theorists would really like to remove the assumption that Γ is cocompact. For example, the quotient $\Gamma(N)\backslash \mathrm{SL}(n,\mathbb{R})$ is not compact for any $N \geq 1$ and $n \geq 2$, where $\Gamma(N)$ is the principal congruence subgroup, and neither is the associated locally symmetric space $\Gamma(N)\backslash \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ [1]. However, as we hinted to in the sketch above, this formula breaks down without the compactness assumption. In particular, on the spectral side we are not guaranteed a discrete decomposition into irreducible subrepresentations, and on the geometric side both the volume term and the integral may be infinite.

It is the seminal work of Arthur through a long series of papers[2] that there exists an extension of this formula to the non-cocompact setting. Not only did he reinterpret both the spectral side and the geometric side to ensure convergence and prove the equality of the two sides, he also worked on expanding both sides to allow for explicit computation. This has cemented it as a strong and versatile modern tool in number theory. We restrict ourselves to mentioning two important applications. The Arthur-Selberg trace formula was a central tool in Kottwitz' proof of Weil's conjecture on Tamagawa numbers [Kot88], and on Lafforgue's proof of the Langlands correspondence for $\mathrm{GL}(n)$ over function fields [Laf02].

We will not attempt to give a detailed introduction to all aspects of the Arthur-Selberg trace formula (see rather [Art05]). Instead, we opt to present a sketch of the construction of the geometric side of the trace formula and some of its refinements, as this will be the most relevant part in this thesis.

One of the strengths of the Arthur-Selberg trace formula is its generality. Not only does it allow for the discrete subgroup to be non-cocompact, but it is furthermore defined for a general reductive algebraic group defined over a global field. This setting requires a bit of preparation which we do in the following section.

Linear algebraic groups

We recall some notions from algebraic geometry. Let k be a ring. An *affine group scheme* G over k is a group object in the category of affine k-schemes. Being a group object implies that any R-points $G(R)$ (for R a k-algebra) has a group structure, and in fact this is an equivalent characterization. Indeed, there is a one-to-one correspondence between affine group schemes and representable functors

$$k-\mathbf{algebra} \longrightarrow \mathbf{Group}.$$

This correspondence is given by considering the hom-functor of the coordinate ring of the scheme.

For example, one may think of $\mathrm{GL}(n)$ as such a functor, mapping any k-algebra R to the group of invertible $n \times n$ R-matrices. This is representable by the k-algebra

$$k[\{x_{ij}\}_{1\leq i,j\leq n}][y]/(\det((x_{ij}))\cdot y-1).$$

[1]Selberg himself did extend his formula to the spaces $\Gamma(N)\backslash \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, $N \geq 1$, and certain other rank 1 quotients.

[2]See the references in [Art05] for a comprehensive overview of the papers involved.

A morphism of affine group schemes (sometimes called a homomorphism) is then a morphism of schemes compatible with the group object structures. Equivalently, it is a natural transformation of the associated functors.

Definition 1.2.1. A morphism $H \to G$ of affine group schemes is called an *embedding* if the associated map on coordinate rings $\mathcal{O}(G) \to \mathcal{O}(H)$ is surjective.

Definition 1.2.2. An affine group scheme G is called *linear* if it admits an embedding $G \to \mathrm{GL}(n)$ for some n.

Thus, we have a notion of *linear algebraic groups* over k: It is an affine algebraic scheme (i.e. an affine scheme of finite type over k) with a group object structure admitting an embedding into $\mathrm{GL}(n)$. If we assume that k is a field, the definition simplifies: Indeed, one can show that an affine algebraic group scheme over a field k is automatically linear (see e.g. [Mil12], Theorem VIII.9.1). In the following, we assume that k is a perfect field.

Definition 1.2.3. An algebraic group G over k is called *smooth* if it is geometrically reduced, i.e. $\mathcal{O}(G) \otimes \overline{k}$ has no nilpotents (here $\overline{k}$ denotes the algebraic closure of k).

A more general definition of smoothness is available (see e.g. [Mil12], Section VI.8), from which one may view the above as a theorem.

We say that an algebraic group is connected if the underlying scheme is connected. Furthermore, a subgroup of an algebraic group is an affine subscheme with a group object structure such that the immersion is a morphism of group schemes.

We import a few concepts from linear algebra. We recall the definitions of semisimple, nilpotent and unipotent matrices: $x \in M_{n\times n}(\overline{k})$ is semisimple if it is $\mathrm{GL}(n,\overline{k})$-conjugate to a diagonal matrix, it is nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$, and it is unipotent if $x - I$ is nilpotent, for I the identity matrix. Let G be a linear algebraic group over k. We say that $g \in G(\overline{k})$ is respectively semisimple, nilpotent or unipotent if its image $\varphi(g)$ is so, for $\varphi : G(\overline{k}) \to \mathrm{GL}(n,\overline{k})$ a homomorphism induced by some (equivalently, any) embedding $G \to \mathrm{GL}(n)$.

Any matrix $x \in \mathrm{GL}(n,k)$ may be factored as $x = x_s \cdot x_u = x_u x_s$, with x_s semisimple and x_u unipotent, and moreover, this decomposition is unique - this is the Jordan decomposition. It is a theorem (see [Mil12], Theorem X.2.8) that any linear algebraic group over k also has Jordan decompositions, i.e. any $g \in G(k)$ has a unique factorization $g = g_s g_u = g_u g_s$ with g_s semisimple and g_u unipotent.

Definition 1.2.4. Let G be a connected smooth algebraic group over k.

1. The *radical* $R(G)$ of G is the maximal connected normal subgroup of G such that $R(G)(\overline{k})$ is solvable. We say that G is *semisimple* if $R(G)$ is trivial.

2. The *unipotent radical* $R_u(G)$ of G is the maximal connected normal subgroup of G such that $R_u(G)(\overline{k})$ consists of unipotent elements. We say that G is *reductive* if $R_u(G)$ is trivial.

As any unipotent subgroup is solvable, we have $R_u(G) \subseteq R(G)$, and thus semisimple implies reductive. Note that not everyone requires a reductive group to be connected.

In particular, $\mathrm{SL}(n)$, as a linear algebraic group, is semisimple, while $\mathrm{GL}(n)$ is reductive but not semisimple.

We will sometimes, when the context is clear, simply write "reductive group" for reductive linear algebraic groups, and similarly for semisimple groups.

As we saw in the latter sections of Chapter 1.1, we have many nice theorems on semisimple groups, and they do not necessarily extend to reductive groups. Thus it is reasonable to ask why we would want to make this generalization. The answer is threefold: Firstly, one can often get around working with a reductive group by working with a semisimple subgroup instead - we see an example of this in the next section. Secondly, as seen above, the class of reductive algebraic groups contains $\mathrm{GL}(n)$, the most important example of a linear algebraic group, as well as many other natural and important groups. Finally, the assumption of being reductive gives enough structure for a rich representation theory, which we will use in a moment.

Adelic symmetric spaces

Take F a number field and G a reductive group over F. For the analogy with the Selberg trace formula, we would like to replace Γ with $G(F)$. However, this does not embed discretely in $G(\mathbb{R})$ or $G(\mathbb{C})$, hence we need a larger ring. For this, we take the *ring of adeles* $\mathbb{A}_F$ of F. For simplicity, most of this section is presented for $F = \mathbb{Q}$, but the theory carries over completely with suitable modifications.

Let $|\cdot|_p$ be the p-adic absolute value, and $|\cdot|_\infty$ denote the usual archimedean absolute value. For $v \in \{p, \infty\}$ we denote by $\mathbb{Q}_v$ the completion of $\mathbb{Q}$ with respect to the associated absolute value. Hence $\mathbb{Q}_p$ is the p-adic numbers, and $\mathbb{Q}_\infty = \mathbb{R}$. We denote by $\mathbb{Z}_p$ the closed unit ball in $\mathbb{Q}_p$, which is also a subring and is called the p-adic integers. We call the collection of primes together with ∞ the *places* of $\mathbb{Q}$.

Definition 1.2.5. The *ring of adeles* of $\mathbb{Q}$ is the restricted product

$$\mathbb{A}_{\mathbb{Q}} = \prod_v{}' \mathbb{Q}_v$$

for v running over all the primes and ∞, where restricted means that for any $x = (x_v) \in \mathbb{A}_{\mathbb{Q}}$, all but finitely many x_p lies in $\mathbb{Z}_p$. It is equipped with componentwise addition and multiplication.

The reason to take the restricted product instead of the usual direct product is to ensure local compactness when equipped with the direct limit topology, in this case called the restricted product topology.

The field $\mathbb{Q}$ maps injectively into every completion $\mathbb{Q}_\nu$, and embeds diagonally into $\mathbb{A}_{\mathbb{Q}}$,

$$\begin{aligned} \mathbb{Q} &\longrightarrow \mathbb{A}_{\mathbb{Q}} \\ q &\mapsto (q, q, q, \dots), \end{aligned}$$

and this turns $\mathbb{A}_{\mathbb{Q}}$ into a $\mathbb{Q}$-algebra. This story can be done for a general number field F. It is an easy check to see that this diagonal embedding is discrete, and this property translates to points of G, i.e.

$$G(F) \hookrightarrow G(\mathbb{A}_F)$$

with induced topologies is a discrete embedding.

Note that we incur no loss of generality in working with $\mathbb{Q}$, since for any reductive group G over a number field F, we may take $\tilde{G}$ as the reductive group obtained by restriction of scalars from F to $\mathbb{Q}$, and in this case, $G(F) = \tilde{G}(\mathbb{Q})$ and $G(\mathbb{A}_F) = \tilde{G}(\mathbb{A}_{\mathbb{Q}})$. Hence, from now on we will write $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$.

One of the strongest tools in investigating quotients of $G(\mathbb{A})$ is *strong approximation.* For S a finite subset of places of $\mathbb{Q}$ containing ∞, we write

$$G(\mathbb{Q}_S) = \prod_{v \in S} G(\mathbb{Q}_v).$$

This is then a locally compact group. We set also $\mathbb{A}^S = \prod'_{v \notin S} \mathbb{Q}_v$, which we include as a subring in $\mathbb{A}$ in the obvious way. Take K^S an open compact subgroup of $G(\mathbb{A}^S)$. We say that G is simply connected if $G(\mathbb{C})$ is simply connected.

Theorem 1.2.6 (Strong approximation)**.** *Assume G is simply connected, and that for every simple factor G' of G over $\mathbb{Q}$, we have that $G'(\mathbb{Q}_S)$ is noncompact. Then*

$$G(\mathbb{A}) = G(\mathbb{Q}) \cdot G(\mathbb{Q}_S) K^S.$$

If we instead make the weaker assumption that $G'(\mathbb{Q}_S)$ is noncompact for every simple quotient G' of G over $\mathbb{Q}$, then

$$G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{Q}_S)K^S \tag{1.2.7}$$

is a finite set of double cosets.

Proof. See [Kne65] for a sketch. □

As $\mathrm{GL}(n, \mathbb{C})$ is not simply connected, the theorem does not apply ($\mathrm{GL}(n)$ instead satisfies *weak approximation*, see [PRR94], Section 7.1). However, it is satisfied for $\mathrm{SL}(n)$. We will use it to great effect in a moment.

Assuming G satisfies the latter half of the theorem, by (1.2.7) we may take a finite set of representatives $g_1, \ldots, g_k$ of the double coset, allowing us to write

$$G(\mathbb{A}) = \bigsqcup_{i=1}^{k} G(\mathbb{Q}) \cdot g_1 \cdot G(\mathbb{Q}_S)K^S.$$

It turns out that in this setting, the quotient $G(\mathbb{Q})\backslash G(\mathbb{A})$ is not always the easiest space to work with. If we furthermore allow taking a quotient by the compact subgroup K^S, the above identity yields a useful characterization,

$$\begin{aligned} G(\mathbb{Q})\backslash G(\mathbb{A})/K^S &= \bigsqcup_{i=1}^{k} G(\mathbb{Q})\backslash(G(\mathbb{Q}) \cdot g_1 \cdot G(\mathbb{Q}_S)K^S)/K^S \\ &= \bigsqcup_{i=1}^{k} \Gamma_S^i\backslash G(\mathbb{Q}_S), \end{aligned}$$

where $\Gamma_S^i \subseteq G(\mathbb{Q}_S)$ is the discrete subgroup

$$\Gamma_S^i = G(\mathbb{Q}_S) \cap (G(\mathbb{Q}) \cdot g_i \cdot K^S \cdot g_i^{-1}). \tag{1.2.8}$$

One particular choice of S will be very important to us, namely $S = \{\infty\}$. In this case, we have $\mathbb{Q}_S = \mathbb{R}$, and $\mathbb{A}^S = \mathbb{A}_{\text{fin}}$, the *finite adeles*. Set also $K^S = K_{\text{fin}}$. The above decomposition then becomes

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_{\text{fin}} = \bigsqcup_{i=1}^{k} \Gamma^i\backslash G(\mathbb{R}),$$

with each Γ^i a discrete subgroup of $G(\mathbb{R})$. If one further fixes a compact subgroup K_∞ of $G(\mathbb{R})$, we get the decomposition

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_{\text{fin}} = \bigsqcup_{i=1}^{k} \Gamma^i\backslash G(\mathbb{R})/K_\infty. \tag{1.2.9}$$

These two descriptions extend directly to $G(\mathbb{R})$-isomorphisms of Hilbert spaces once normalizations of measures are fixed,

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_{\text{fin}}) \cong \bigoplus_{i=1}^{n} L^2(\Gamma^i\backslash G(\mathbb{R})),$$
$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_{\text{fin}}) \cong \bigoplus_{i=1}^{n} L^2(\Gamma^i\backslash G(\mathbb{R})/K_\infty).$$

The spaces on the right-hand side of (1.2.9) bear a striking resemblance to the locally symmetric spaces we presented in Section 1.1, and this is no coincidence. Indeed, once proper assumptions are in order we will be able to import our results on locally symmetric spaces to analyze these quotients of adelic spaces, another important tool of Chapter 2.

We define K_{fin} to be *neat* if the eigenvalues of any element in Γ^i, $1 \leq i \leq n$, generate a torsion free subgroup of $\mathbb{C}^\times$. Let K_{fin} be neat. We assume G is semisimple, such that $G(\mathbb{R})$ is a semisimple Lie group, and require K_∞ to be a maximal compact subgroup. If this is all satisfied, then the right-hand side above is truly a disjoint union of locally symmetric spaces, and we can analyze each part separately.

Our primary example of this situation, which is also the setting of Chapter 2, is taking $G = \mathrm{SL}(n)$, $K_\infty = \mathrm{SO}(n)$, and $K_{\text{fin}} = K(N) := \prod_p K_p(p^{v_p(N)})$ for $N \in \mathbb{N}$. Here $K_p(p^e) \subseteq \mathrm{SL}(n, \mathbb{Q}_p)$ is the subgroup defined as

$$K_p(p^e) = \ker(\mathrm{SL}(n, \mathbb{Z}_p) \to \mathrm{SL}(n, \mathbb{Z}_p/p\mathbb{Z}_p)),$$

in some sense a p-adic analogue of the principal congruence subgroups $\Gamma(N)$ of $\mathrm{SL}(n, \mathbb{Z})$. Note that $K(N)$ is neat if $N \geq 3$. This turns out the be a very good analogy, since under these assumptions the decomposition (1.2.9) becomes

$$\mathrm{SL}(n, \mathbb{Q})\backslash \mathrm{SL}(n, \mathbb{A})/(\mathrm{SO}(n)K(N)) = \Gamma(N)\backslash \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n). \tag{1.2.10}$$

Here we only have one factor on the right hand side as $\mathrm{SL}(n)$ satisfies the first identity of Theorem 1.2.6. The fact that the associated discrete group is $\Gamma(N)$ is an easy computation using the definition (1.2.8).

We wish to do the same for $\mathrm{GL}(n)$, which as mentioned is not a priori possible as we do not have strong approximation. However, if we consider a certain semisimple subgroup, we can construct an associated space that will work for our purposes. First we return to a general reductive group G.

Let A_G denote the maximal central subgroup in G that is a $\mathbb{Q}$-split torus, i.e. A_G is isomorphic over $\mathbb{Q}$ to $\mathrm{GL}(1)^k$ for some $k \in \mathbb{N}_0$. We define

$$X(G)_\mathbb{Q} := \{\chi : G \to \mathrm{GL}(1) \mid \chi \text{ a } \mathbb{Q}\text{-homomorphism of algebraic groups}\}$$

We further define $\mathfrak{a}_G := \mathrm{Hom}_\mathbb{Z}(X(G)_\mathbb{Q}, \mathbb{R})$. As $X(G)_\mathbb{Q}$ is a free abelian group of rank k, this turns $\mathfrak{a}_G$ into a real vector space of dimension k. Write also

$\mathfrak{a}_G^* = X(G)_{\mathbb{Q}} \otimes_{\mathbb{Z}} \mathbb{R}$ for the dual vector space. Finally, for $g \in G(\mathbb{A})$ and $\chi \in X_{\mathbb{Q}}(G)$ we define

$$H_G : G(\mathbb{A}) \longrightarrow \mathfrak{a}_G, \tag{1.2.11}$$
$$H_G(g)(\chi) := \log |\chi(g)|. \tag{1.2.12}$$

Then we have a decomposition of $G(\mathbb{A})$ as the direct product of $A_G(\mathbb{R})^0$, the connected component of $A_G(\mathbb{R})$, with the normal subgroup $G(\mathbb{A})^1$, defined as

$$G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) \mid H_G(g) = 0\}.$$

We will also use the notation on the real points: set $G(\mathbb{R})^1 = G(\mathbb{R}) \cap G(\mathbb{A})^1$. The point is then that $G(\mathbb{R})^1$ is a semisimple Lie group.

For $\mathrm{GL}(n)$, we may pick A_G given by the embedding of $\mathrm{GL}(1)$ as scalar matrices. $X_{\mathbb{Q}}(\mathrm{GL}(n))$ is rank 1, with generator given by the determinant homomorphism to $\mathrm{GL}(1)$. In particular, we have

$$\mathrm{GL}(n,\mathbb{R})^1 = \{g \in \mathrm{GL}(n,\mathbb{R}) \mid \log|\det(g)| = 0\}.$$

The condition is equivalent to $|\det(g)| = 1$, which gives the isomorphism

$$\mathrm{GL}(n,\mathbb{R})^1 \cong \mathrm{SL}(n,\mathbb{R}) \times \{\pm 1\}.$$

Pick K_∞ as a maximal compact subgroup of $G(\mathbb{R})^1$. If one now replaces $G(\mathbb{R})$ with $G(\mathbb{R})^1$, one can apply the second half of strong approximation, giving the decomposition

$$G(\mathbb{Q})\backslash G(\mathbb{R})^1 G(\mathbb{A}_{\mathrm{fin}})/K_\infty K_f = \bigsqcup_{i=1}^{k} \Gamma^i \backslash G(\mathbb{R})^1/K_\infty. \tag{1.2.13}$$

This way, we are allowed to work on locally symmetric spaces even in the general reductive case. Having multiple (but finitely many) components is not a problem, as one can restrict to each component separately and glue together the results.

As our primary example, we let $K'(N)$ be the open compact subgroup of $\mathrm{GL}(n,\mathbb{A}_{\mathrm{fin}})$ analogous to $K(N) \subseteq \mathrm{SL}(n,\mathbb{A}_{\mathrm{fin}})$ defined above - simply replace SL with GL everywhere in the definition. One then constructs the space

$$Y(N) := \mathrm{GL}(n,\mathbb{Q})\backslash \mathrm{GL}(n,\mathbb{R})^1 \mathrm{GL}(n,\mathbb{A}_{\mathrm{fin}})/\mathrm{O}(n)K(N).$$

Note that $\mathrm{GL}(n,\mathbb{R})^1/\mathrm{O}(n) = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. Using the decomposition (1.2.13), it follows directly from the definition (1.2.8) and the fact that $K'(N)$ is normal that $Y(N)$ is a disjoint union of $\varphi(N)$ many copies of the symmetric space (1.2.10).

Parabolic subgroups, roots and weights

Continuing from section 1.2, we saw that the Selberg trace formula is not applicable when $\Gamma\backslash G$ is not compact, G here being a Lie group and Γ a discrete subgroup. In the setting of the previous section, for G a connected reductive group over $\mathbb{Q}$, this is the analogous to saying $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is noncompact. Essentially, this defect is caused by parabolic subgroups of G, and hence this is where we shall look to for modifying the formula.

Definition 1.2.7. A *parabolic subgroup* of G is an algebraic subgroup $P \subseteq G$ such that $P(\mathbb{C})\backslash G(\mathbb{C})$ is compact.

For us, we will implicitly assume all parabolic subgroups are defined over $\mathbb{Q}$. It is a standard result that every parabolic subgroup has a *Levi decomposition*, $P = MN$, a semidirect product of a reductive group M of G and a normal unipotent subgroup N of G, both defined over $\mathbb{Q}$. M is called the *Levi component* of P. Furthermore, N is the unipotent radical of P, and in particular it is unique. M is unique up to conjugation by $P(\mathbb{Q})$.

A reductive group G often has many parabolic subgroups conjugate to one another, so to structure them, we fix a *minimal* parabolic subgroup P_0 and a Levi decomposition $P_0 = M_0N_0$, and call any subgroup $P \subseteq G$ containing P_0 a *standard parabolic subgroup* of G. Note that they are automatically parabolic subgroups. A standard parabolic subgroup then has a canonical Levi decomposition, namely $P = MN$ such that $M_0 \subseteq M$. The set of standard parabolic subgroups is finite, and it is a set of representatives of all $G(\mathbb{Q})$-conjugacy classes of parabolic subgroups of G.

Turning to our favourite example, the standard choice of P_0 in $\mathrm{GL}(n)$ is the subgroup of invertible upper triangular matrices, with M_0 the subgroup of diagonal matrices and N_0 the unipotent upper triangular matrices, i.e. those with 1's on the diagonal. There is then a bijection between partitions of n and standard parabolic subgroups of $\mathrm{GL}(n)$, given by associating to a partition $(n_1, \dots, n_k)$ the subgroup of block upper triangular matrices with blocks of sizes $n_1, \dots, n_k$. In particular, the partition $(1, \dots, 1)$ correspond to P_0, while the trivial partition (n) corresponds to all of $\mathrm{GL}(n)$. For the parabolic subgroup corresponding to $(n_1, \dots, n_k)$, its canonical Levi component is the matrices consisting of invertible block submatrices of corresponding sizes, canonically isomorphic to $M = \mathrm{GL}(n_1) \times \cdots \times \mathrm{GL}(n_k)$. The unipotent radical is the block upper triangular matrices with all the block matrices being the identity.

We return to the general setting. As M is reductive, we can apply our theory of reductive groups and lift it to P, essentially by ignoring N, and further to G. By Gram-Schmidt, we have the decomposition $\mathrm{GL}(n, \mathbb{R}) = P_0(\mathbb{R})\mathrm{O}(n)$, and by applying an analogous procedure to the norm $||v|| := \max\{|v_i|_p : i \leq n\}$ on $\mathbb{Q}_p^n$, we have a decomposition $\mathrm{GL}(n, \mathbb{Q}_p) = P_0(\mathbb{Q}_p)\,\mathrm{GL}(n, \mathbb{Z}_p)$ for p prime.

Combining all these gives an adelic decomposition

$$\mathrm{GL}(n, \mathbb{A}) = P_0(\mathbb{A})K,$$

with $K := \mathrm{O}(n) \times \prod_p \mathrm{GL}(n, \mathbb{Z}_p)$. Here K is a compact subgroup of $\mathrm{GL}(n, \mathbb{A})$.

A similar argument works for any reductive group, giving $G(\mathbb{A}) = P_0(\mathbb{A})K$ for an admissible choice of K (see [Art81], §1). Recalling that $P = MN$ the above decomposition becomes $G(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})K$.

A quick remark on measures: Throughout, we will pick the Haar measures on our groups such that K and $N_P(\mathbb{Q})\backslash N_P(\mathbb{A})$ both have measure 1, and they respect the decompositions $G(\mathbb{A}) = P(\mathbb{A})K = M(\mathbb{A})N(\mathbb{A})K = M(\mathbb{A})^1 N(\mathbb{A})A(\mathbb{R})^0 K$.

We will often write $A_P = A_M$ when M is canonical. We consider the function $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$, defined analogously to (1.2.11). Write also $\mathfrak{a}_P = \mathfrak{a}_M$. We can now lift the function to G by defining, for $g = nmk \in G(\mathbb{A}) = N(\mathbb{A})M(\mathbb{A})K$,

$$H_P : G(\mathbb{A}) \to \mathfrak{a}_P,$$
$$H_P(nmk) = H_M(m).$$

Let now Φ_P be the roots of (P, A_P). This is then a (finite) subset of $X(A_P)_\mathbb{Q}$, and we furthermore identify it as a subset of $\mathfrak{a}_P^*$ using the canonical inclusion $X(A_P)_\mathbb{Q} \hookrightarrow X(A_P)_\mathbb{Q} \otimes \mathbb{R} = \mathfrak{a}_P^*$. The last equality holds as the map $X(G)_\mathbb{Q} \to X(A_G)_\mathbb{Q}$ given by restriction is injective and has finite cokernel, in particular their real spans are isomorphic.

Consider two parabolic subgroups P_1, P_2 with $P_1 \subseteq P_2$ and write $P_i = M_i N_i$ for $i = 1, 2$. The inclusion $M_1 \subseteq M_2$ gives a restriction map $X(M_2)_\mathbb{Q} \to X(M_1)_\mathbb{Q}$. This homomorphism is injective, which gives an injection $\mathfrak{a}_{P_2}^* \hookrightarrow \mathfrak{a}_{P_1}^*$, and dually a surjection $\mathfrak{a}_{P_1} \to \mathfrak{a}_{P_2}$. We write $\mathfrak{a}_{P_1}^{P_2}$ for the kernel of the latter map, thus getting the decomposition $\mathfrak{a}_{P_1} \cong \mathfrak{a}_{P_2} \oplus \mathfrak{a}_{P_1}^{P_2}$.

Whenever we would subscript P_0 the minimal parabolic, we instead choose to simply subscript 0. Let Φ_0 be the roots of (P_0, A_0). Then $((\mathfrak{a}_0^G)^*, \Phi_0 \cup (-\Phi_0))$ is a root system, and we pick Δ_0 the simple roots associated to Φ_0, and write further $\Delta_0^\vee$, $\hat{\Delta}_0$, and $\hat{\Delta}_0^\vee$ for the simple coroots, the simple weights, and the simple co-weights respectively. We let W_0 be the Weyl group of the root system. By the above, we may consider $\mathfrak{a}_P$ as a subspace of $\mathfrak{a}_0$, for P any standard parabolic subgroup. In fact, they all have a description as the zero-set of some subset Δ_0^P of Δ_0, i.e.

$$\mathfrak{a}_P = \{B \in \mathfrak{a}_0 \mid \alpha(B) = 0, \ \forall \alpha \in \Delta_0^P\}.$$

This gives a bijective correspondence between standard parabolic subgroups and subsets of Δ_0. Also, Δ_P is bijective to $\Delta_0 - \Delta_0^P$, which justifies the definition

$$\hat{\Delta}_P = \{\omega_\alpha \mid \alpha \in \Delta_0 - \Delta_0^P\},$$

where ω_α is the simple weight associated to $\alpha \in \Delta_0$. This is then also a basis of $(\mathfrak{a}_P^G)^*$. One can again define $\hat{\Delta}_P^\vee$, $\Delta_P^\vee$ as the dual basis of Δ_P, respectively $\hat{\Delta}_P$.

Finally, we consider two hypercones in $\mathfrak{a}_P$ given by

$$\mathfrak{a}_P^+ = \{B \in \mathfrak{a}_P \mid \alpha(B) > 0, \ \forall \alpha \in \Delta_P\}, \tag{1.2.14}$$

$$\hat{\mathfrak{a}}_P^+ = \{B \in \mathfrak{a}_P \mid \alpha(B) > 0, \ \forall \alpha \in \hat{\Delta}_P\}. \tag{1.2.15}$$

We let $\tau_P, \hat{\tau}_P$ be the respective characteristic functions on $\mathfrak{a}_P$ of these subsets. These will be our truncation functions.

The Arthur-Selberg trace formula

One can, in analogy with our presentation of the Selberg trace formula, consider the right regular representation R_P for any parabolic subgroup P of G, defined by

$$R_P = \mathrm{Ind}_{N_P(\mathbb{A})M_P(\mathbb{Q})}^{G(\mathbb{A})}(1_{N_P(\mathbb{A})M_P(\mathbb{Q})}).$$

Here $1_{N_P(\mathbb{A})M_P(\mathbb{Q})}$ denotes the trivial representation of $N_P(\mathbb{A})M_P(\mathbb{Q})$.

Following the recipe of Section 1.2, we pick $f \in C_c^\infty(G(\mathbb{A}))$ and construct the integral operator $R_P(f)$ on $L^2(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A}))$ with kernel

$$K_P(x,y) = \int_{N_P(\mathbb{A})} \sum_{\gamma \in M_P(\mathbb{Q})} f(x^{-1}\gamma n y) dn.$$

For $R(f) = R_G(f)$, this is the adelic version of the kernel we saw in Section 1.2. We will use a truncation of K_P for each proper parabolic subgroup P to modify this kernel to ensure convergence. Let $T \in \mathfrak{a}_0^+$, which we will use as a truncation parameter. We define the modified kernel as

$$k^T(x) = \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_P(\delta x, \delta x)\hat{\tau}_P(H_P(\delta x) - T). \tag{1.2.16}$$

One can then define a family of integrals parametrized by T that will serve as a starting point for the trace formula,

$$J^T(f) := \int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k^T(x) dx.$$

It is not hard to check that k^T is invariant under $G(\mathbb{Q})$ and $A(\mathbb{R})^0$, meaning that it makes sense to consider its integral over this domain. A priori, this is only well defined for sufficiently regular T, a notion we will not get into here (see [Art78]). Here they also prove that for such T, the integral is absolutely convergent. The proof of this fact centers around showing cancellation in the alternating sum over P. It critically uses the approximate fundamental domains for $G(\mathbb{Q})\backslash G(\mathbb{A})$ of Borel and Harish-Chandra ([BHC62], see also [Bor63]), and involves a careful treatment of characteristic functions. See ([Art78]) and ([Art05], §8) for more details.

In [Art81], it is shown that $J^T(f)$ is a polynomial in T, and defined for all $T \in \mathfrak{a}_0$. Furthermore, a canonical point $T_0 \in \mathfrak{a}_0$ is found such that $J^{T_0}(f)$ is not dependent on the choice of minimal parabolic P_0. It is for this value that we define the distribution at the center of the trace formula,

$$J(f) \coloneqq J^{T_0}(f).$$

The next step is to give a spectral interpretation of the kernel $k(x) \coloneqq k^{T_0}(x)$, mirroring that of the geometric expression (1.2.16). As for the Selberg trace formula, one wishes to decompose the regular representation R on the L^2-space of G, in our setting namely $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Unfortunately, when $G(\mathbb{Q})\backslash G(\mathbb{A})$ is noncompact, it does not in general decompose discretely, and one needs to use Eisenstein series to control the continuous part of the spectrum. As mentioned earlier, we will not go into detail with the spectral side of the trace formula, see instead ([Art78]) and ([Art79]). Rather, in the following chapter we will delve deeper into the geometric side and present the coarse and fine geometric expansions.

The geometric expansions of the trace formula

The distribution $J(f)$ for $f \in C_c^\infty(G(\mathbb{A}))$ as constructed in the previous section is a priori not very suitable for applications and computations. To give a more detailed and explicit description, we wish to turn the geometric expression of the kernel (1.2.16) into an expansion of the distribution, called the *coarse geometric expansion*. In essence, we wish to obtain a formula similar to the left-hand side of the Selberg trace formula (1.2.6), summing over conjugacy classes. We will need a set of particular conjugacy classes which we introduce now.

Definition 1.2.8. For G any algebraic group over a perfect field k, we say that $g \in G(k)$ is *semisimple*, respectively *unipotent*, if its image under any (equivalently, every) closed embedding $G \to \mathrm{GL}(n)$ is semisimple, respectively unipotent, as a matrix.

A *Jordan decomposition* of $g \in G(k)$ is an identity $g = g_s g_u = g_u g_s$ with $g_s \in G(k)$ semisimple and $g_u \in G(k)$ unipotent. It is a fact of algebraic groups that a unique Jordan decomposition always exist (see e.g. [Mil18], 7.10). We will refer to g_s as the *semisimple part* of g, and g_u the *unipotent part*.

We return to G being connected reductive. Define an equivalence relation on $G(\mathbb{Q})$ by setting two elements equivalent if their semisimple parts are $G(\mathbb{Q})$-conjugate to each other. We denote by $\mathcal{O}$ the set of such equivalence classes in $G(\mathbb{Q})$. Note that these are in bijection with conjugacy classes of semisimple elements. For any such equivalence class $\mathfrak{o} \in \mathcal{O}$, we may split up the kernel K_P associated to a parabolic subgroup P by defining

$$K_{P,\mathfrak{o}}(x,y) := \int_{N_P(\mathbb{A})} \sum_{\gamma \in M_P(\mathbb{Q}) \cap \mathfrak{o}} f(x^{-1}\gamma n y) dn, \qquad x, y \in G(\mathbb{A})^1, \tag{1.2.17}$$

such that

$$K_P(x,y) = \sum_{\mathfrak{o} \in \mathcal{O}} K_{P,\mathfrak{o}}(x,y).$$

Defining $k_{\mathfrak{o}}^T(x)$ analogously to $k^T(x)$ by replacing $K_P(x,x)$ with $K_{P,\mathfrak{o}}(x,x)$, we also immediately get the decomposition

$$k^T(x) = \sum_{\mathfrak{o} \in \mathcal{O}} k_{\mathfrak{o}}^T(x).$$

Continuing the analogy, define

$$J_{\mathfrak{o}}^T(f) = \int_{G(\mathbb{Q}) \backslash G(\mathbb{A})^1} k_{\mathfrak{o}}^T(x) dx.$$

One shows the absolute convergence of $\sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}^T(f)$ in a similar way as the absolute convergence of $J^T(f)$. The behaviour of $J_{\mathfrak{o}}^T(f)$ in T is akin to that of $J^T(f)$, i.e. polynomial in T, and one picks the same canonical choice for the truncation parameter, $J_{\mathfrak{o}}(f) := J_{\mathfrak{o}}^{T_0}(f)$. By Fubini's theorem, we arrive at the coarse geometric expansion

$$J(f) = J_{\text{geo}}(f) := \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f). \tag{1.2.18}$$

A similar coarse spectral expansion can be completed ([Art78], [Art79]) using the spectral interpretation of the kernel $k^T(x)$, and the equality of these two sides is the "coarse" Arthur-Selberg trace formula, generalizing the Selberg trace formula. However, it is still of little help for applications if we cannot more explicitly describe the distributions $J_{\mathfrak{o}}(f)$. Luckily, this is also provided in Arthur's work, expressing the distributions as orbital integrals. To present this, we will need some notation.

Let P and P' be standard parabolic subgroups. We denote by $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ Langlands' *Weyl set*, the set of linear isomorphisms from $\mathfrak{a}_P$ to $\mathfrak{a}_{P'}$, both thought of as subsets of $\mathfrak{a}_0$, that are obtained as restrictions of the Weyl group action on $\mathfrak{a}_0$. In particular, this set is empty unless $\mathfrak{a}_P$ and $\mathfrak{a}_{P'}$ have the same dimension, but this is not a sufficient criterion. Any Weyl group action is given by conjugation, so for $s \in W_0$, we write $w_s \in G(\mathbb{Q})$ for any element representing this action.

Recall the bijection of $\mathcal{O}$ and semisimple conjugacy classes in $G(\mathbb{Q})$. For a semisimple conjugacy class, we say it is *anisotopic* if it does not intersect $P(\mathbb{Q})$

for any proper parabolic subgroup P of G. We define an *anisotropic rational datum* to be an equivalence class of (P, α) for $P = MN$ a standard parabolic subgroup and α an anisotropic conjugacy class in $M(\mathbb{Q})$, where (P, α) is equivalent to (P', α') if α and α' are $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$-conjugate, i.e. $\alpha' = w_s^{-1} \alpha w_s$. There is a bijection between the set of anisotropic rational data and the set $\mathcal{O}$ by sending the class of (P, α) to the $G(\mathbb{Q})$-conjugacy class of α.

Now we can describe the first part of the explicit formulation of $J_{\mathfrak{o}}(f)$. Let $\mathfrak{o} \in \mathcal{O}$ be anisotropic (i.e. it corresponds to an anisotropic semisimple conjugacy class), and let (P, α) be a representative of the corresponding anisotropic root datum, with $P = MN$ the canonical Levi decomposition of P. Let γ be any element of α. Then

$$J_{\mathfrak{o}}(f) = \mathrm{vol}(M(\mathbb{Q})_\gamma \backslash M(\mathbb{A})^1_\gamma) \int_{G(\mathbb{A})_\gamma \backslash G(\mathbb{A})} f(x^{-1} \gamma x) v_P(x) dx.$$

$v_P(x)$ should be thought of as a weight function; it is the volume of the projection onto $\mathfrak{a}_P^G$ of the convex hull of the set

$$\{-s^{-1} H_{P'}(\tilde{w}_s x) \mid s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'}),\ P' \supset P_0\}.$$

$\tilde{w}_s$ here is a representative of s contained in the maximal compact subgroup K of $G(\mathbb{A})$.

Note that not all classes $\mathfrak{o} \in \mathcal{O}$ are anisotropic; an important counterexample is the set of unipotent elements, corresponding to the semisimple conjugacy set $\{1\}$. This intersects *every* parabolic subgroup, and is in a sense the most difficult class to handle because of this. The solution with this problem is closely tied to the fine geometric expansion, hence we will present the two in tandem in the following section.

Local orbital integrals and the fine geometric expansion

From now on, we will allow the choice of minimal parabolic to vary for more flexibility. Let M be a Levi subgroup of G over $\mathbb{Q}$, i.e. a Levi component of a parabolic subgroup of G. We write $\mathcal{L}(M)$ for the set of Levi subgroups containing M, and $\mathcal{F}(M)$ for the set of parabolic subgroups containing M. Then any $P \in \mathcal{F}(M)$ has a canonical Levi decomposition by picking its Levi component to lie in $\mathcal{L}(M)$. Finally, $\mathcal{P}(M)$ denotes the set of parabolic subgroups for which M is a Levi component. Note that then $\mathcal{F}(M) = \bigsqcup_{L \in \mathcal{L}(M)} \mathcal{P}(L)$. These are all finite sets.

Let S be a finite set of places including ∞, and let M be a Levi subgroup. Choose a $\gamma = \prod_{\nu \in S} \gamma_\nu \in M(\mathbb{Q}_S)$. The centralizer G_{γ_ν} is an algebraic group over $\mathbb{Q}_\nu$, so we define the group scheme over $\mathbb{Q}_S$

$$G_\gamma = \prod_{\nu \in S} G_{\gamma_\nu}.$$

It follows from the short paper of Rao ([Rao72]) that there exists a right-$G(\mathbb{Q}_S)$-invariant Radon measure on the quotient $G_\gamma(\mathbb{Q}_S)\backslash G(\mathbb{Q}_S)$, meaning we have a $G(\mathbb{Q}_S)$-invariant linear form taking $f \in C_c^\infty(G(G_S))$ to

$$\int_{G_\gamma(\mathbb{Q}_S)\backslash G(\mathbb{Q}_S)} f(x^{-1}\gamma x)dx. \tag{1.2.19}$$

We will now assume $G_\gamma = M_\gamma$ to simplify the description. At the end, we explain how to go to the general case. Define the generalized Weyl discriminant,

$$D(\gamma) = \prod_{\nu \in S} D(\gamma_\nu),$$
$$D(\gamma_\nu) = \det(1 - \mathrm{Ad}((\gamma_\nu)_s))_{\mathfrak{g}/\mathfrak{g}_{(\gamma_\nu)_s}},$$

where $(\gamma_\nu)_s$ is the semisimple part of γ_ν, and $\mathfrak{g}_{\sigma_\nu}$ the Lie algebra of G_{σ_ν}. This discriminant is used for normalization purposes. Then we define the weighted orbital integral

$$J_M(f,\gamma) = |D(\gamma)|^{\frac{1}{2}} \int_{G_\gamma(\mathbb{Q}_S)\backslash G(\mathbb{Q}_S)} f(x^{-1}\gamma x) v_M(x) dx. \tag{1.2.20}$$

Here $v_M(x)$ is another weight function, which we define now. Let $P \in \mathcal{P}(M)$ and for $\lambda \in i\mathfrak{a}_M^*$ set

$$\theta_P(\lambda) = \mathrm{vol}(\mathfrak{a}_M^G / \operatorname{span}_{\mathbb{Z}} \Delta_P^\vee)^{-1} \cdot \prod_{\alpha \in \Delta_P} \lambda(\alpha^\vee).$$

Write $x = \prod_{\nu \in S} x_\nu$. It follows directly from the definition that $H_P(x) = \sum_{\nu \in S} H_P(x_\nu)$, when considering x_ν as an element of $G(\mathbb{A})$ through the inclusion $G(\mathbb{Q}_\nu) \to G(\mathbb{A})$, and similarly for $x \in G(\mathbb{Q}_S) \to G(\mathbb{A})$. We define some simpler weight functions

$$v_P(\lambda, x) = e^{-\lambda(H_P(x))}. \tag{1.2.21}$$

By the sum formula for H_P above, it is clear that $v_P(\lambda, x) = \prod_{\nu \in S} v_P(\lambda, x_\nu)$. Then we can construct our desired weight functions as

$$v_M(x) = \lim_{\lambda \to 0} \sum_{P \in \mathcal{P}(M)} v_P(\lambda, x)\theta_P(\lambda)^{-1}. \tag{1.2.22}$$

The fact that the limit exists, along with many more details, are contained in ([Art88a]). It also has an, admittedly simpler, interpretation as the volume of the projection of the convex hull of

$$\{-H_P(x) \mid P \in \mathcal{P}(M)\}.$$

However, the above description will be useful when we wish to decompose our weighted orbital integrals further.

If $G_\gamma \neq M_\gamma$, we have a more complicated formula. For $a \in A_M(\mathbb{Q}_S)$ small in general position, there exists canonical functions $r_M^L(\gamma, a)$ such that the following exists and is well defined:

$$J_M(\gamma, f) = \lim_{a \to 1} \sum_{L \in \mathcal{L}(M)} r_M^L(\gamma, a) J_L(a\gamma, f).$$

This is one of the central theorems of ([Art88b]), see also there for the relevant existence and convergence results.

Considering the obvious injection of $C_c^\infty(G(\mathbb{Q}_S)^1)$ into $C_c^\infty(G(\mathbb{A})^1)$, we may form the distribution $J_{\mathfrak{o}}(f)$ for $f \in C_c^\infty(G(\mathbb{Q}_S)^1)$. We need a finer equivalence relation to index over.

Definition 1.2.9. We say two elements $\gamma_1, \gamma_2 \in M(\mathbb{Q}) \cap \mathfrak{o}$ are (G, S)-*equivalent* if they are $G(\mathbb{Q}_S)$-conjugate. The set of such equivalence classes $(M(\mathbb{Q}) \cap \mathfrak{o})_{G,S}$ is then finite.

Finally, we have the characterization of $J_{\mathfrak{o}}(f)$ in terms of weighted orbital integrals.

Theorem 1.2.10. *Let $f \in C_c^\infty(G(\mathbb{Q}_S)^1)$. There exists a finite set of places $S_{\mathfrak{o}}$, only depending on $\mathfrak{o}$, containing ∞ and such that for any finite S with $S_{\mathfrak{o}} \subseteq S$, we have*

$$J_{\mathfrak{o}}(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1} \sum_{\gamma \in (M(\mathbb{Q}) \cap \mathfrak{o})_{M,S}} a^M(S, \gamma) J_M(\gamma, f),$$

where $a^M(S, u)$ are some uniquely determined coefficients.

Proof. See ([Art86]). □

The coefficients $a^M(S, \gamma)$, sometimes called the *global coefficients*, are difficult to compute in general. In the special case $\gamma = 1$, we have

$$a^M(S, 1) = \mathrm{vol}(M(\mathbb{Q}) \backslash M(\mathbb{A})^1).$$

Furthermore, for $\mathrm{GL}(n)$ defined over a number field F, a bound has been established in [Mat15] for the global coefficients associated to unipotent γ. We state it here only for $F = \mathbb{Q}$.

Proposition 1.2.11 ([Mat15], Theorem 1.1 and Remark 1.2)**.** *There exists a constant $C \geq 0$ such that for any a finite set of places S of $\mathbb{Q}$ containing ∞, for any $M \in \mathcal{L}$ and any $\gamma \in M(\mathbb{Q})$ unipotent, we have that*

$$\left|a^M(S, \gamma)\right| \leq C \sum_{\substack{s_p \in \mathbb{N},\, p \in S_{\text{fin}} \\ \sum s_p \leq \dim \mathfrak{a}_0^M}} \prod_{p \in S_{\text{fin}}} \frac{(\log p)^{s_p}}{p-1}.$$

A simplified version of this bound is used in Chapters 2 and 3.

We have now given a description of the coarse geometric expansion in terms of orbital integrals. As a bonus, note that the definition (1.2.19), and the following general case, gives the definition of a p-adic and real weighted orbital integral by picking S to be a set of one element. These integrals are the simplest to compute, and so we would like to decompose our formulas into these.

This is possible if $f \in C_c^\infty(G(\mathbb{A}))$ can be expressed as a product $f = \prod_\nu f_\nu$ with $f_\nu \in C_c^\infty(G(\mathbb{Q}_\nu))$ and all but finitely many $f_p = 1_{K_p}$. As mentioned, the simple weight functions (1.2.21) satisfy a nice decomposition into local parts, but the general ones (1.2.22) do not. Applying the decomposition of the simple weight functions to the definition of $v_M(x)$ gives a formula involving sums of products of local parts. Handling this in a smart way yields the result below (see [Art81]).

Assume $S = S_1 \sqcup S_2$, and $f = f_1 f_2$ with $f_i \in C_c^\infty(M(\mathbb{Q}_{S_i}))$ along with $\gamma = \gamma_1\gamma_2$ such that $\gamma_i \in M(\mathbb{Q}_{S_i})$. We assume $G_\gamma = M_\gamma$, implying that $G_{\gamma_i} = M_{\gamma_i}$. For $Q_i = MN_i \in \mathcal{P}(M)$ define

$$f_{i,Q_i}(m) = \delta_{Q_i}(m)^{\frac{1}{2}} \int_{K_{S_i}} \int_{N_i(\mathbb{Q}_{S_i})} f_i(k^{-1}mnk)dndk. \tag{1.2.23}$$

Here $\delta_Q(m) = e^{2\rho_Q(H_Q(m))}$, with ρ_Q the half-sum of the roots of (Q, A_Q). Then we have a general decomposition formula given by

$$J_M(\gamma, f) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_M^G(L_1, L_2) J_M^{L_1}(\gamma_1, f_{1,Q_1}) J_M^{L_2}(\gamma_2, f_{2,Q_2}). \tag{1.2.24}$$

Applying this inductively let's us reduce to sets S_i of one element. It is this decomposition, used on the terms $J_{\mathfrak{o}}(f)$ on the geometric side of the Arthur-Selberg trace formula, that I call the *fine geometric expansion*. When $G_\gamma \neq M_\gamma$, the same procedure yields an expansion in limits of sums of the terms above.

A final remark: So far, we have worked exclusively with compactly supported test functions, and for our purposes, this will be too restrictive. Fortunately, the coarse geometric expansion has been extended in [FLM15] to the domain of *adelic Schwartz functions*. This will be presented and applied in Chapter 2.

2 Asymptotics of analytic torsion for congruence quotients of $\mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$

Abstract. In this paper we prove a sharpened asymptotic for the growth of analytic torsion of congruence quotients of $\mathrm{SL}(n, \mathbb{R}) / \mathrm{SO}(n)$ in terms of the volume. The result is based on bounds on the trace of the heat kernel, allowing control of the large time behaviour of certain orbital integrals, as well as a careful analysis of error terms. The result requires the existence of λ-strongly acyclic representations, which we define and show exists in plenitude for any $\lambda > 0$. The motivation is possible applications to torsion in the cohomology of arithmetic groups.

2.1 Introduction

Analytic torsion is an invariant of Riemannian manifolds, defined by Ray and Singer in the 70's ([RS71]). More recently, it was used by Bergeron and Venkatesh to study the torsion in the homology of cocompact arithmetic groups ([BV13]). This made it desirable to extend the definition of analytic torsion to manifolds associated to non-cocompact arithmetic groups. This was first accomplished by Müller–Pfaff ([MP13]) for finite volume hyperbolic manifolds. Later, it was extended further by Matz and Müller, first for congruence subgroups of $\mathrm{SL}(n)$ in [MM17], and later for more general arithmetic groups in [MM23].

One of the reasons for studying torsion in the homology of arithmetic groups is the Ash conjecture ([Ash92]), partially proven by Scholze ([Sch15]). In rough terms, this conjecture links systems of Hecke eigenvalues occurring in the mod p cohomology of arithmetic groups to Galois representations with matching Frobenius eigenvalues. See ([BV13], Section 6) for a discussion on the Ash conjecture and its connection to analytic torsion. As explained in Scholze's paper, by applying the work of Bergeron–Venkatesh and others, in certain situations the Ash conjecture predicts the existence of many Galois representations, more than is predicted by the global Langlands correspondence.

Further control of the behaviour of analytic torsion should translate to better understanding of torsion homology, and conjecturally to existence of Galois representations. With this motivation we give improved asymptotics of analytic torsion in terms of the volume of the associated manifolds. To state our theorem and the connection to homology, we recall the setups of [BV13] and [MM20].

Analytic torsion in the compact setting

Let X be a compact oriented Riemannian manifold of dimension d, and let ρ be a finite-dimensional representation of $\pi_1(X)$. To such a representation, one may associate a flat vector bundle $E_\rho \to X$ over X. Fixing a Hermitian fiber metric on E_ρ, we let $\Delta_p(\rho)$ be the Laplace operator on E_ρ-valued p-forms. For $t > 0$, we define $e^{-t\Delta_p(\rho)}$ to be the heat operator, and we let $k_p(\rho) = \dim\ker\Delta_p(\rho)$. One can associate a zeta function $\zeta_p(s,\rho)$ to $\Delta_p(\rho)$, a meromorphic function on $\mathbb{C}$ defined by

$$\zeta_p(s,\rho) = \frac{1}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}\, e^{-t\Delta_p(\rho)} - k_p(\rho))t^{s-1}dt \tag{2.1.1}$$

for $\Re(s) > \frac{d}{2}$. We then define analytic torsion $T_X(\rho)$ of M by

$$\log T_X(\rho) = \frac{1}{2}\sum_{p=1}^{d}(-1)^p p \frac{d}{ds}\zeta_p(s;\rho)|_{s=0}. \tag{2.1.2}$$

There is a corresponding L^2-invariant, called L^2-torsion denoted $T_X^{(2)}(\rho)$, defined by Lott and Mathai (resp. [Lot92], [Mat92]). For $\tilde{X}$ the universal covering of X, it may be expressed as

$$\log T_X^{(2)}(\rho) = t_{\tilde{X}}^{(2)}(\rho)\,\mathrm{vol}(X), \tag{2.1.3}$$

where $t_{\tilde{X}}^{(2)}(\tau)$ is an invariant associated to $\tilde{X}$ and ρ.

To relate this to arithmetic groups, let G be a semisimple algebraic group with $G(\mathbb{R})$ of non-compact type, let K be a maximal compact subgroup of $G(\mathbb{R})$, and suppose $\Gamma \subseteq G(\mathbb{Q})$ is a torsion-free congruence lattice. Assume further that Γ is cocompact in $G(\mathbb{R})$. Then $\tilde{X} := G(\mathbb{R})/K$ is a Riemannian symmetric space, and $X := \Gamma\backslash\tilde{X}$ is a compact locally symmetric space, and as $\tilde{X}$ is contractible, we have $\pi_1(X) = \Gamma$. Thus, given Γ and a finite-dimensional representation ρ of Γ, we associate to it the analytic torsion $T_X(\rho)$ of the space X as defined above.

By a theorem of independently Cheeger and Müller, later generalized by Bismut–Zhang ([Che79], [Mül78], [Mül93], [BZ94]), there is an equality of analytic torsion and *Reidemeister torsion* on X, the latter being a topological invariant of X computed in terms of homology (see [Rei35], [Fra35]). Since X is a classifying space for Γ, the homology of X computes the homology of Γ, and thus one may study the homology of Γ through its associated analytic torsion.

Consider a descending chain of finite index subgroups $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \ldots$ in $G(\mathbb{Q})$ satisfying $\bigcap_{i=1}^{\infty}\Gamma_i = \{1\}$, and let $X_i := \Gamma_i\backslash\tilde{X}$. Let τ be an irreducible finite-dimensional representation of $G(\mathbb{R})$. By abuse of notation, we set $T_{X_i}(\tau) := T_{X_i}(\tau|_{\Gamma_i})$. Given a non-degeneracy condition on τ, namely requiring that it be *strongly acyclic*, Bergeron and Venkatesh prove ([BV13], Theorem 4.5) that

$$\lim_{i\to\infty} \frac{\log T_{X_i}(\tau)}{\mathrm{vol}(X_i)} = t_{\tilde{X}}^{(2)}(\tau). \tag{2.1.4}$$

Bergeron and Venkatesh show that $t_{\tilde{X}}^{(2)}(\tau)$ is non-zero if and only if the *deficiency* of G, defined $\delta(G) := \operatorname{rank} G(\mathbb{R}) - \operatorname{rank} K$, is 1 ([BV13], Proposition 5.2). The assumption that τ is strongly acyclic also guarantees that the free homology is trivial, so Reidemeister torsion is expressed solely in torsion homology. Using the Cheeger-Müller theorem, this is interpreted as there being a lot of torsion in the homology when $\delta(G) = 1$, and little torsion in all other cases. "Little torsion" should here be understood in a weak sense, as we only know that there is not full exponential growth of analytic torsion in terms of the volume. A natural question is then, also presented in [AGMY20]: what growth rate should we expect when the deficiency is not 1, and can we more precisely describe the behaviour of analytic torsion in terms of the volume? The results of this paper is a first step in answering these questions.

The result (2.1.4) can be thought of as an approximation theorem. In particular, noting that every Γ_i is a finite index subgroup of Γ, we see that $\mathrm{vol}(X_i) = \mathrm{vol}(X)[\Gamma : \Gamma_i]$, where $X \coloneqq X_0$, and thus the result is equivalent to

$$\lim_{i\to\infty} \frac{\log T_{X_i}(\tau)}{[\Gamma : \Gamma_i]} = \log T_X^{(2)}(\tau). \tag{2.1.5}$$

In words, analytic torsion associated to a tower of subgroups can be used to approximate the L^2-torsion associated to the group. Keeping in mind the equality of analytic torsion and Reidemeister torsion, this statement should be thought of as a torsion version of the analogous result on Betti numbers $b_p(X)$ and L^2-Betti numbers $b_p^{(2)}(X)$, proven by W. Lück ([Lüc94]), which we state here in our setting.

$$\lim_{i\to\infty} \frac{b_p(X_i)}{[\Gamma : \Gamma_i]} = b_p^{(2)}(X). \tag{2.1.6}$$

To further the analogy, here there is a criterion based on the deficiency as well: We have that $b_p^{(2)}(X) = 0$ unless $\delta(G) = 0$. For a comprehensive survey on approximation of L^2-invariants, see [Lüc16].

For certain approximation theorems, their rate of convergence has been explored (see e.g. [BLLS12], and [Lüc16] Chapter 5), but is not very developed for analytic and L^2-torsion. This is mostly due to the fact that this approximation result is still conjectural in general, and only proven in very particular cases, such as the setting of this paper. Viewed in this light, Theorem 2.1.1 presented below can be seen as an example of a rate of convergence result for analytic torsion in this aspect.

Many of the most important arithmetic groups in number theory are not cocompact. In general for non-cocompact arithmetic groups in semisimple Lie groups, we do not have a replacement for the Cheeger-Müller theorem, though certain special cases have recently been proven (see [MR20], [MR24]). The non-cocompact setting do, however, have one advantage, as it allows certain terms to contribute (namely the non-identity unipotent part, see section 2.7) which vanish in the cocompact setting, and these terms are very probable candidates for where second order terms should show up. For this reason, the present paper will focus on the non-cocompact setting.

Analytic torsion in the non-compact setting

To state our main theorem, let us recall the setup of Matz and Müller ([MM17], [MM20]). We switch to an adelic framework and focus our attention on $\mathrm{SL}(n)$, $n \geq 2$. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$, with $\mathbb{A}_f$ the finite adeles. Let $X = \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. For $N \geq 3$, let $K(N) \subseteq \mathrm{SL}(n,\mathbb{A}_f)$ be the open compact subgroup given by $K(N) = \prod_p K_p(p^{\nu_p(N)})$, with $K_p(p^e) \coloneqq \ker(\mathrm{SL}(n,\mathbb{Z}_p) \to \mathrm{SL}(n,\mathbb{Z}_p/p^e\mathbb{Z}_p))$. Define

$$X(N) \coloneqq \mathrm{SL}(n,\mathbb{Q})\backslash(\tilde{X} \times \mathrm{SL}(n,\mathbb{A}_f))/K(N)).$$

By strong approximation, we get that $X(N) = \Gamma(N)\backslash\tilde{X}$, with $\Gamma(N) \subseteq \mathrm{SL}(n,\mathbb{Z})$ the standard principal congruence subgroup of level N. Given a finite-dimensional representation τ of $\mathrm{SL}(n,\mathbb{R})$, one can now follow the construction of the Laplace operator $\Delta_p(\tau)$ of the Laplace operator on p-forms with values in E_τ, as above. As $X(N)$ is not compact, however, $\Delta_p(\tau)$ has continuous spectrum, and hence we need a refined definition of the analytic torsion. This is constructed by Matz and Müller in [MM17]. Most strikingly, they define the regularized trace of the heat kernel $\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)})$ as the geometric side of the *Arthur trace formula* for $\mathrm{SL}(n,\mathbb{R})$ applied to the test function $h_t^{\tau,p} \otimes \chi_{K(N)}$, where $h_t^{\tau,p}$ is the trace of the heat kernel and $\chi_{K(N)}$ the normalized characteristic function of $K(N)$. We give the full definition in Section 2.5.4. This also reduces to the standard definition in the cocompact setting. In [MM20], they prove the same limit behaviour as Bergeron-Venkatesh for this setup, namely the approximation formula

$$\lim_{N\to\infty} \frac{\log T_{X(N)}(\tau)}{\mathrm{vol}(X(N))} = t^{(2)}_{\tilde{X}}(\tau). \tag{2.1.7}$$

Formulated in terms of asymptotics of analytic torsion, the above is equivalent to

$$\log T_{X(N)}(\tau) = \log T^{(2)}_{X(N)}(\tau) + o(\mathrm{vol}(X(N))) \quad \text{as} \quad N \to \infty. \tag{2.1.8}$$

Results

As a step towards better understanding the behavior of analytic torsion in terms of the volume, in this paper we prove a stronger version of the asymptotic (2.1.8). In particular, the new result provides an improved upper bound on analytic torsion when the deficiency is not 1, as well as a bound on second order terms when the deficiency is 1. We need a slightly stronger non-degeneracy assumption on τ which we call λ-strongly acyclic. Our result is the following.

Theorem 2.1.1. *Assume τ is a λ-strongly acyclic representation of $\mathrm{SL}(n,\mathbb{R})$, for a certain λ depending only on n. Then there exists some $a > 0$ such that*

$$\log T_{X(N)}(\tau) = \log T^{(2)}_{X(N)}(\tau) + O(\mathrm{vol}(X(N))N^{-(n-1)}\log(N)^a)$$

as N tends to infinity.

Remark 2.1.2. Computing the size of $\mathrm{vol}(X(N))$ in terms of N (see the appendix), the theorem implies a more intrinsic version:

$$\log T_{X(N)}(\tau) = \log T^{(2)}_{X(N)}(\tau) + O(\mathrm{vol}(X(N))^{1-\frac{1}{n+1}}\log(\mathrm{vol}(X(N))^a) \tag{2.1.9}$$

as $N \to \infty$.

To ensure that the technical constraint is justified, we prove the existence of infinitely many λ-strongly acyclic representations for any connected semisimple algebraic group G with $\delta(G) \geq 1$. This result is an extension of the result ([BV13], Section 8.1), stating that strongly acyclic representations always exist for semisimple G with $\delta(G) = 1$.

It seems likely that up to log-terms, the bound in the theorem is strict in the case of deficiency 1, though this is not proven here. When the deficiency is not 1, a better upper bound is expected. A lower bound would be extremely interesting as well, but this is not in the scope of this paper.

Because of technical reasons, we will work with $\mathrm{GL}(n)$ instead of $\mathrm{SL}(n)$. For $Y(N)$ the analogous adelic locally symmetric space of $\mathrm{GL}(n)$, one can define analytic torsion $T_{Y(N)}(\tau)$ in a similar manner (see (2.5.4) for the general definition). Our proof is then that of Theorem 2.5.1, which is the analogous statement of Theorem 2.1.1 for $Y(N)$, and we show the two theorems are equivalent.

The proof is based on the framework of [MM20]. The main contribution of this paper lies in the change of perspective to allow certain parameters, namely the compactification parameter (see Section 2.5) and the truncation parameter T (see (2.7.1)) to vary with the level N, and in the work needed to allow for this variation. In particular, we show the existence of infinitely many λ-strongly acyclic representations of any connected semisimple algebraic group (Proposition 2.3.2), and we adapt to our setting certain bounds on the trace of the heat kernel in the large time aspect (e.g. Proposition 2.4.2). These bounds are used to gain control over large time behaviour of the archimedean orbital integrals showing up in the analysis of the Arthur trace formula.

Organization of the paper

In Section 2.2, we present the setup for the heat kernel on symmetric spaces. In Section 2.3, we prove the existence of infinitely many λ-strongly acyclic representations for any semisimple algebraic group. Section 2.4 gives a proof of large t asymptotics of the trace of the heat kernel. We give a brief introduction to the Arthur trace formula and its geometric expansions in Section 2.5 and define analytic torsion, as well as express the trace formula applied to a compactification of our test function. In Section 2.6, we handle the asymptotics of our local orbital integrals. Finally, we combine all the ingredients in Section 2.7 to prove our main theorem.

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2.2 Preliminaries

Arithmetic manifolds

This section will deal with the general setup of G being a reductive algebraic group defined over $\mathbb{Q}$ with $\mathbb{Q}$-split zenter Z_G. See ([Art05], Section I.2) for a partial reference. Let $K_f \subseteq G(\mathbb{A}_f)$ be any open compact subgroup. We write $G(\mathbb{A}) = G(\mathbb{A})^1 \times A_G$ for A_G the identity component of Z_G, and we set $G(\mathbb{R})^1 := G(\mathbb{A})^1 \cap G(\mathbb{R})$, which is then a semisimple real Lie group. We fix a Cartan involution θ of $G(\mathbb{R})$ and let K denote its fixpoints. We set $\tilde{X} := G(\mathbb{R})^1/K$. The group $G(\mathbb{R})$ has a natural action on the finite double coset space,

$$A_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})K_f.$$

Taking a set of representatives $z_1, \ldots, z_m$ for the double cosets in $G(\mathbb{A}_f)$ and defining

$$\Gamma_j := (G(\mathbb{R}) \times z_j K_f z_j^{-1}) \cap G(\mathbb{Q})$$

for all $1 \leq j \leq m$, the action induces the decomposition

$$A_G(\mathbb{R})^0 G(\mathbb{Q})\backslash G(\mathbb{A})/K_f \cong \bigsqcup_{j=1}^{m} (\Gamma_j \backslash G(\mathbb{R})^1). \tag{2.2.1}$$

Now we define the arithmetic manifold associated to K_f. Set

$$X(K_f) := G(\mathbb{Q})\backslash(\tilde{X} \times G(\mathbb{A}_f))/K_f. \tag{2.2.2}$$

Using (2.2.1), we get

$$X(K_f) \cong \bigsqcup_{j=1}^{m} (\Gamma_j \backslash \tilde{X}). \tag{2.2.3}$$

Here, $\Gamma_j \backslash \tilde{X}$ is a locally symmetric space. We will assume that K_f is neat such that $X(K_f)$ is a locally symmetric manifold of finite volume.

The heat kernel

In the setup above, given a finite-dimensional unitary representation of K, we let $\tilde{E}_\nu$ be the associated homogeneous Hermitian vector bundle over $\tilde{X}$. Using homogeneity, one can push down this vector bundle to give locally homogeneous Hermitian vector bundles over each component $\Gamma_j\backslash\tilde{X}$. Taking the disjoint union of these vector bundles, we get a locally homogeneous vector bundle E_ν over $X(K_f)$.

As it is sufficient to work on each component in (2.2.3), we continue this section with the simplifying assumption that G is a connected semisimple algebraic group. Let $K \subseteq G(\mathbb{R})$ be a maximal compact subgroup, and $\Gamma \subseteq G(\mathbb{R})$ a torsion-free lattice. We let $\tilde{X} = G(\mathbb{R})/K$ and $X = \Gamma\backslash\tilde{X}$. Take a finite-dimensional unitary representation (ν, V_ν) of K with inner product $\langle\cdot,\cdot\rangle_\nu$, and define

$$\tilde{E}_\nu := G(\mathbb{R}) \times_\nu V_\nu$$

as the associated homogenous vector bundle over $\tilde{X}$. The action of K on $G(\mathbb{R})$ is by right multiplication. The inner product $\langle\cdot,\cdot\rangle_\nu$ induces a $G(\mathbb{R})$-invariant metric $\tilde{h}_\nu$ on this vector bundle. Let $E_\nu := \Gamma\backslash\tilde{E}_\nu$ be the associated locally homogenous vector bundle over X, with metric h_ν induced by $\tilde{h}_\nu$ using $G(\mathbb{R})$-invariance. Let $C^\infty(\tilde{X}, \tilde{E}_\nu)$ denote the space of smooth sections of $\tilde{E}_\nu$. Now, set

$$C^\infty(G(\mathbb{R}),\nu) := \{f : G(\mathbb{R}) \to V_\nu \mid f \in C^\infty,$$
$$f(gk) = \nu(k)^{-1}f(g)\ \forall k \in K, g \in G(\mathbb{R})\}.$$

There is an isomorphism ([Mia80], p. 4) interpreting the smooth sections as smooth functions on $G(\mathbb{R})$,

$$C^\infty(\tilde{X}, \tilde{E}_\nu) \xrightarrow{\sim} C^\infty(G(\mathbb{R}),\nu). \tag{2.2.4}$$

This extends to an isometry of corresponding L^2-spaces. Let $\mathcal{C}(G(\mathbb{R}))$ denote Harish-Chandra's Schwartz space, and $\mathcal{C}^q(G(\mathbb{R}))$ Harish-Chandra's L^q-Schwartz space.

We now specify the above to our setting. Let (τ, V_τ) be an irreducible finite-dimensional representation of $G(\mathbb{R})$, $E_\tau := E_{\tau|_K}$, and F_τ the flat vector bundle over X associated to the restriction of τ to Γ. Then we have a canonical isomorphism (see [MM63], Proposition 3.1)

$$E_\tau \cong F_\tau.$$

As K is compact there exists an inner product on V_τ with respect to which $\tau|_K$ is unitary. Fix such an inner product. From this we induce a metric on E_τ, and hence on F_τ as well. Define $\Lambda^p(X, F_\tau) := \Lambda^p T^*(X) \otimes F_\tau$. Under the isomorphism above, this is isomorphic to the vector bundle associated to the representation $\Lambda^p \mathrm{Ad}^* \otimes \tau$ of K on $\Lambda^p \mathfrak{p}^* \otimes V_\tau$. We will denote this representation by $\nu_{\tau,p}$.

Let $\Delta_p(\tau)$ be the Laplace operator on $\Lambda^p(X, F_\tau)$, and let $\tilde{\Delta}_p(\tau)$ be its lift to the universal covering $\tilde{X}$. Let also $\tilde{F}_\tau$ be the pullback of F_τ to $\tilde{X}$. Then $\tilde{\Delta}_p(\tau)$ is an operator on the space $\Lambda^p(\tilde{X}, \tilde{F}_\tau)$ of $\tilde{F}_\tau$-valued p-forms on $\tilde{X}$. By (2.2.4), we get an isomorphism

$$\Lambda^p(\tilde{X}, \tilde{F}_\tau) \cong C^\infty(G(\mathbb{R}), \nu_{\tau,p}). \tag{2.2.5}$$

Let R be the right regular representation of $G(\mathbb{R})$ on $C^\infty(G(\mathbb{R}), \nu_{\tau,p})$, and Ω the Casimir element of $G(\mathbb{R})$. With respect to the above isomorphism, Kuga's lemma implies

$$\tilde{\Delta}_p(\tau) = \tau(\Omega) - R(\Omega). \tag{2.2.6}$$

The operator $\tilde{\Delta}_p(\tau)$ is formally self-adjoint and non-negative. Regarded as an operator with domain the space of compactly supported smooth p-forms, it has a unique self-adjoint extension to $L^2(\tilde{X}, \tilde{F}_\tau)$, the L^2-sections of $\tilde{F}_\tau$, which we by abuse of notation will also denote $\tilde{\Delta}_p(\tau)$. This extension inherits non-negativity. We denote by $e^{-t\tilde{\Delta}_p(\tau)}$, with $t > 0$, the heat semigroup associated to $\tilde{\Delta}_p(\tau)$. Considered as a bounded operator on $L^2(G(\mathbb{R}), \nu_{\tau,p})$ under the extension of the isomorphism (2.2.5), it is a convolution operator, and thus it has a kernel

$$H_t^{\tau,p} : G(\mathbb{R}) \to \mathrm{End}(\Lambda^p \mathfrak{p}^* \otimes V_\tau)$$

called the heat kernel. It satisfies the following covariance property

$$H_t^{\tau,p}(k^{-1}gk') = \nu_{\tau,p}(k)^{-1} \circ H_t^{\tau,p}(g) \circ \nu_{\tau,p}(k'), \quad \forall k, k' \in K, g \in G(\mathbb{R}). \tag{2.2.7}$$

By analogy of the proof of ([BM83], Proposition 2.4), we have that

$$H_t^{\tau,p} \in \mathcal{C}^q(G(\mathbb{R})) \otimes \mathrm{End}(\Lambda^p \mathfrak{p}^* \otimes V_\tau) \tag{2.2.8}$$

for any $q > 0$. We may define

$$h_t^{\tau,p}(g) := \mathrm{tr}\, H_t^{\tau,p}(g), \quad g \in G(\mathbb{R}), \tag{2.2.9}$$

for tr being the trace over $\Lambda^p \mathfrak{p}^* \otimes V_\tau$. By (2.2.7) and (2.2.8), we have that $h_t^{\tau,p}(g) \in \mathcal{C}^q(G(\mathbb{R}))_{K \times K}$ for any $q > 0$, with $\mathcal{C}^q(G(\mathbb{R}))_{K \times K}$ the subspace of $\mathcal{C}^q(G(\mathbb{R}))$ consisting of left and right K-finite functions.

2.3 Strongly acyclic representations

We continue with the setup from Section 2.2. In particular, G is a connected semisimple algebraic group. In the setting that $\Gamma\backslash\tilde{X}$ is compact, ([BV13], §4) defines an irreducible finite-dimensional representation (τ, V_τ) of $G(\mathbb{R})$ to be *strongly acyclic* if there exists some positive constant $\eta > 0$ such that every eigenvalue of $\Delta_p(\tau)$ is $\geq \eta$ for every choice of Γ and every p. Furthermore, they show that for any such τ not fixed by the Cartan involution, then τ must be strongly acyclic. This condition, i.e. $\tau \neq \tau \circ \theta$, is shown to give the necessary bounds in the noncompact setting in ([MM20], Lemma 6.1), and is used throughout that paper.

For our purposes, we need a slightly stronger condition that we define now. This will be the central object of this section.

Definition 2.3.1. Let (τ, V_τ) be a finite-dimensional representation of $G(\mathbb{R})$, and let $\lambda > 0$. We say that τ is λ-*strongly acyclic* if

$$\tau(\Omega) - \pi(\Omega) \geq \lambda$$

for all irreducible unitary representation π satisfying $\mathrm{Hom}_K(\Lambda^p\mathfrak{p} \otimes V_\tau^*, \pi) \neq 0$ for some p.

We see the analogy to the definition of strongly acyclic by comparing to (2.2.6). For τ a λ-strongly acyclic representation, we will sometimes say that λ is the *spectral gap* of τ. We now prove that there are plenty of such representations to choose from.

Proposition 2.3.2. *For any $\lambda > 0$, there exists infinitely many λ-strongly acyclic representations of $G(\mathbb{R})$ if $\delta(G(\mathbb{R})) \geq 1$.*

The proof takes up the remainder of the section. We set $F = V_\tau^*$ to be the dual of the representation space to match the notation of [BV13]. Consider $\mathfrak{g} := \mathrm{Lie}(G(\mathbb{R}))$ with $\mathfrak{g}_\mathbb{C}$ its complexification. Recall that θ is a Cartan involution on $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. This also gives the decomposition $\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C}$. Let $\mathfrak{h}^+$ be a Cartan subalgebra in $\mathfrak{k}$. Denote by $\mathfrak{h}$ the centralizer of $\mathfrak{h}^+$ in $\mathfrak{g}$, which is then a Cartan subalgebra in $\mathfrak{g}$.

The following two lemmas are elementary and certainly well known, but we were not able to find precise references. Hence, we will give the proofs. Similar, but not identical, results are presented in ([Hel62], Section VI.3).

Lemma 2.3.3. *There exists an abelian subalgebra $\mathfrak{h}^- \subseteq \mathfrak{p}$ such that $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}^-$.*

Proof. Take any basis $(K_1, \dots, K_d)$ of $\mathfrak{h}^+$, and extend it to a basis

$$(K_1, \dots, K_d, X_1, \dots, X_s)$$

of $\mathfrak{h}$. Using the Cartan decomposition, write $X_i = M_i + P_i$ for $M_i \in \mathfrak{k}$ and $P_i \in \mathfrak{p}$. Since $[K, K'] \in \mathfrak{k}$ and $[P, K] \in \mathfrak{p}$ for any $\lambda, K' \in \mathfrak{k}$ and $P \in \mathfrak{p}$, and $\mathfrak{k} \cap \mathfrak{p} = 0$, we see that $[M_i, K_j] + [P_i, K_j] = [X_i, K_j] = 0$ for all i, j implies that $[M_i, K_j] = [P_i, K_j] = 0$. In particular, $M_i \in \mathfrak{k}$ commutes with all of $\mathfrak{h}^+$, but as this is a *maximal* abelian subalgebra of $\mathfrak{k}$, we get $M_i \in \mathfrak{h}^+$. Thus, it becomes clear that

$$\mathfrak{h} = \operatorname{span}\{K_1, \dots, K_d, P_1, \dots, P_s\}.$$

□

Let Φ_k be the roots of $\mathfrak{h}^+_{\mathbb{C}}$ in $\mathfrak{k}_{\mathbb{C}}$, and let Φ be the roots of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Pick Φ_k^+ a set of positive roots of Φ_k, and let Φ^+ be a *compatible* choice of positive roots of Φ (see [BW00], II, §6.6). A compatible root system Φ_k^+ is defined as a root system which is closed under the Cartan involution (acting by precomposition), and every element in Φ_k^+ is the restriction of some element in Φ^+ to $\mathfrak{h}^+_{\mathbb{C}}$. Let ρ and ρ_k be the respective half-sum of positive roots. Set W to be the Weyl group of $\mathfrak{g}_{\mathbb{C}}$, and let W^1 be the subset of elements w such that $w\Phi^+$ is again a compatible system of positive roots.

We will be concerned with the weight lattice of $\mathfrak{g}_{\mathbb{C}}$, which lives inside the real span of the roots, $\operatorname{span}_{\mathbb{R}} \Phi$. To utilize our decomposition above, we need the following elementary lemma. We extend $(\mathfrak{h}^+)^*$ to a subspace of $\mathfrak{h}^*$ be setting elements to be identically zero on $\mathfrak{h}^-$, and extend $(\mathfrak{h}^-)^*$ by setting elements identically zero on $\mathfrak{h}^+$. We further identify these as (real) subspaces of $\mathfrak{h}^*_{\mathbb{C}}$ using that $\mathfrak{h}^*_{\mathbb{C}} = \mathfrak{h}^* \oplus i\mathfrak{h}^*$.

Lemma 2.3.4. *We have a decomposition* $\operatorname{span}_{\mathbb{R}} \Phi = i(\mathfrak{h}^+)^* \oplus (\mathfrak{h}^-)^*$*, and this decomposition is orthogonal with respect to the inner product induced by the Killing form.*

Proof. Define $\mathfrak{h}_0^* \coloneqq \operatorname{span}_{\mathbb{R}} \Phi$. Given $H \in \mathfrak{h}$, note that the values $\varphi(H)$ for $\varphi \in \Phi \sqcup \{0\}$ are exactly the eigenvalues of ad_H, by definition. Also, it is an elementary fact that for any element X in a compact subalgebra, all eigenvalues of ad_X are imaginary – one could argue as follows: Its Lie group is compact, thus its action on the Lie algebra by conjugation has eigenvalues of absolute value 1. Now take logarithms.

As $\mathfrak{k}$ is a compact Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$, the above implies that for $K \in \mathfrak{h}^+$ and $\varphi \in \Phi_k$ we have $\varphi(K) \in i\mathbb{R}$. Thus, the roots must take values in $\mathbb{R}$ on $i\mathfrak{h}^+$. Similarly, as $i\mathfrak{p}$ is a subspace of the compact real form $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$, the roots must take values in $i\mathbb{R}$ on $i\mathfrak{p}$, and thus values in $\mathbb{R}$ on $\mathfrak{p}$. This proves the inclusion $\mathfrak{h}_0^* \subseteq i(\mathfrak{h}^+)^* \oplus (\mathfrak{h}^-)^*$, and equality follows from comparing dimensions.

The second claim follows easily from the fact that $\mathfrak{k} \oplus \mathfrak{p}$ is orthogonal with respect to the Killing form, hence so is $i\mathfrak{h}^+ \oplus \mathfrak{h}^-$, and this space is isomorphic to its dual induced by the inner product given by the Killing form.

□

As this may be a nonstandard choice of positive root systems to the reader, let us consider an example.

Example 2.3.5. Consider $G(\mathbb{R}) = \mathrm{SL}(3,\mathbb{R})$ with the usual Cartan involution on its Lie algebra $\theta(X) = -X^t$ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into skew-symmetric and symmetric traceless real 3×3 matrices. Let

$$K = \begin{pmatrix} & 1 & \\ -1 & & \\ & & \end{pmatrix}, \quad P = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}.$$

Then $\mathfrak{h}^+ := \mathrm{span}_{\mathbb{R}}\{K\}$ is a Cartan subalgebra of $\mathfrak{k}$, and its centralizer is $\mathfrak{h} = \mathrm{span}_{\mathbb{R}}\{K, P\}$. Complexifying, we have $\mathfrak{h}_{\mathbb{C}} = \mathrm{span}_{\mathbb{C}}\{K, P\}$. One can check that the adjoint action of K on $\mathfrak{k}_{\mathbb{C}}$ has eigenvalues $0, i, -i$ with respective eigenspaces

$$\mathfrak{h}^+_{\mathbb{C}}, \quad \mathrm{span}_{\mathbb{C}} \begin{pmatrix} & & 1 \\ & & i \\ -1 & -i & \end{pmatrix}, \quad \mathrm{span}_{\mathbb{C}} \begin{pmatrix} & & 1 \\ & & -i \\ -1 & i & \end{pmatrix}.$$

In particular, we may choose a positive root system $\Phi_k^+ := \{\varphi\}$ for $\mathfrak{h}^+_{\mathbb{C}}$ with $\varphi : \mathfrak{h}^+_{\mathbb{C}} \to \mathbb{C}$ given by $\varphi(K) = i$.

Consider now the dual space $\mathfrak{h}^*_{\mathbb{C}}$ and the dual basis $\{K^*, P^*\}$ defined by $K^*(K) = 1$, $K^*(P) = 0$ and analogously for P^*. One can then check that the six roots of $\mathfrak{h}_{\mathbb{C}}$, in terms of this basis, are $\pm(2i, 0), \pm(i, 3)$ and $\pm(i, -3)$. As the Cartan involution acts by precomposition, we see that the dual basis inherits the action of the Cartan involution from the original basis, i.e. $\theta(K^*) = K^*$ and $\theta(P^*) = -P^*$.

In particular, the action of the Cartan involution on a vector written in the dual basis is $(a, b) \mapsto (a, -b)$. We then have exactly one choice of a compatible positive root system for $\mathfrak{h}$: As it must restrict to Φ_k^+, we must pick $(i, 3)$ or $(i, -3)$, and as we have to be closed under the Cartan involution we have to pick both. Thus, by the axioms of positive root systems, we see that $\Phi^+ = \{(i, 3), (i, -3), (2i, 0)\}$.

We return to the general setting. As the adjoint action of $\mathfrak{k}$ preserves the Killing form on $\mathfrak{p}$, which defines $\mathfrak{so}(\mathfrak{p})$, we get a natural map $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$, and thus, we get an induced representation of $\mathfrak{k}$ on $S := \mathrm{Spin}(\mathfrak{p})$. In the proof of ([BV13], Lemma 4.1), it is noted that every highest weight of $F \otimes S$ is of the form $\frac{1}{2}(\mu + \theta\mu) + w\rho - \rho_k$ for some weight μ of F and some $w \in W^1$. We use this to give a more precise statement.

Lemma 2.3.6. *Any highest weight of $F \otimes S$ is always of the form $\frac{1}{2}(\nu + \theta\nu) + w\rho - \rho_k$ for ν the highest weight of F, and some $w \in W^1$.*

Proof. Let $\frac{1}{2}(\mu_0 + \theta\mu_0) + w_0\rho - \rho_k$ be a highest weight of $F \otimes S$, for some μ_0 and $w_0 \in W^1$. We show that this μ_0 is ν. Let $M(\mu) := \frac{1}{2}(\mu + \theta\mu) + w_0\rho - \rho_k$. We claim that if $\mu \succ \mu'$, then $M(\mu) \succ M(\mu')$. Indeed, as the Cartan involution fixes the set of our positive roots, by choice of a compatible root system, it fixes the fundamental Weyl chamber, such that μ is a positive combination of positive roots if and only if the same is true for $\theta(\mu)$. Now we see that

$$M(\mu) - M(\mu') = \frac{1}{2}((\mu - \mu') + \theta(\mu - \mu')),$$

and so, if we assume $(\mu - \mu')$ is a positive combination, so is $\theta(\mu - \mu')$, and thus so is their half-sum, proving the claim.

As every irreducible representation has a unique highest weight, we have a unique highest weight ν for F. By the above, we immediately get $M(\nu) \succeq M(\mu_0)$, with equality iff $\nu = \mu_0$. ☐

We are now ready to prove Proposition 2.3.2. It is shown in the proof of ([BV13], Lemma 4.1) that

$$\tau(\Omega) - \pi(\Omega) \geq \eta(\tau) \tag{2.3.1}$$

for each irreducible unitary representation π of $G(\mathbb{R})$ satisfying $\mathrm{Hom}_K(\Lambda^p\mathfrak{p} \otimes V_\tau^*, \pi) \neq 0$, where $\eta(\tau)$ is given by

$$\eta(\tau) = |\nu + \rho|^2 - \left|\frac{1}{2}(\mu + \theta\mu) + w\rho\right|^2 \tag{2.3.2}$$

for the choice of μ and w given above ([BV13], (4.1.2)), and the norm associated to the inner product induced by the Killing form. By Lemma 2.3.2, this μ must be ν, so we have

$$\eta(\tau) = |\nu + \rho|^2 - \left|\frac{1}{2}(\nu + \theta\nu) + w\rho\right|^2$$

Consider $(K_1^*, \ldots, K_d^*, P_1^*, \ldots, P_s^*)$, an orthonormal basis respecting the decomposition in Lemma 2.3.4. With respect to this basis, we will write any weight μ as $\mu = \mu^+ + \mu^-$, where $\mu^+ = (\mu_1^+, \ldots, \mu_d^+, 0, \ldots, 0)$ and $\mu^- = (0, \ldots, 0, \mu_1^-, \ldots, \mu_s^-)$. Recall the fact that $\mathfrak{k}_\mathbb{C}$, $\mathfrak{p}_\mathbb{C}$ are the eigenspaces of θ with eigenvalues 1 and -1, respectively. As the dual basis inherits the action from the Cartan involution, we get that $\theta(\mu^+ + \mu^-) = \mu^+ - \mu^-$. In particular, $\frac{1}{2}(\mu + \theta\mu) = \mu^+$. Thus, for $\nu = (\nu_1^+, \ldots, \nu_d^+, \nu_1^-, \ldots, \nu_s^-)$ the highest weight of F, expressed in the basis above, we may write

$$\eta(\tau) = |\nu + \rho|^2 - \left|\frac{1}{2}(\nu + \theta\nu) + w\rho\right|^2 = \sum_{i=1}^{s} |\nu_i^-|^2 + \text{linear terms in } \nu. \tag{2.3.3}$$

Note that the second equality above follows from the orthogonality of the basis. By (2.3.1), we need only argue that there exists infinitely many τ such that $\eta(\tau) \geq \lambda$. By the expression above, it is sufficient to be able to find infinitely many representations with highest weight having large $\mathfrak{h}^-$-part. This is possible by the theorem of the highest weight, stating that every dominant integral element is the unique highest weight of an irreducible representation. The dominant integral elements constitute a lattice in the fundamental Weyl chamber associated to Φ^+, which is some cone in the weight space. By the assumption $\delta(G(\mathbb{R})) \geq 1$, we know that $s \geq 1$.

To be precise, we do the following: Fix any half-line in the fundamental Weyl chamber starting at 0 and passing through a lattice point, and not lying in the hyperplane given by $x_{d+1} = x_{d+2} = \cdots = x_{d+s} = 0$. This line is guaranteed to pass through infinitely many lattice points. For any $\lambda > 0$ and any linear polynomial p in $d+s$ variables there exists some $r > 0$ such that at distance at least r from 0, every point on this line will have $\sum_{i=1}^{s} |x_{d+i}|^2 - |p(x_1, \ldots, x_{d+s})|$ larger than λ. Considering the expression (2.3.3), we see that any lattice point lying on the line with distance at least r to 0 must have $\eta(\tau) \geq \lambda$. Thus, the line contains infinitely many lattice points associated to λ-strongly acyclic representations. This finishes the proof of Proposition 2.3.2.

Remark 2.3.7. Instead of the final paragraph using half-lines, let us provide a more visual and geometric argument. We take $G = \mathrm{SL}(3,\mathbb{R})$ for simplicity. Here, the fundamental Weyl chamber is a cone in two-dimensional space. Writing points (x, y) in terms of the basis given in Example 2.3.5, we see that by (2.3.3), the inequality $\eta(\tau) \geq \lambda$ turns into an inequality of the form

$$\lambda \leq y^2 + ay - bx - c,$$

for $a, b, c \in \mathbb{R}$. Geometrically, this means that we are considering points outside some parabola. Below we have visualized the positive Weyl chamber as enclosed by the blue lines, the parabola in red and the area outside it in the cone in yellow.

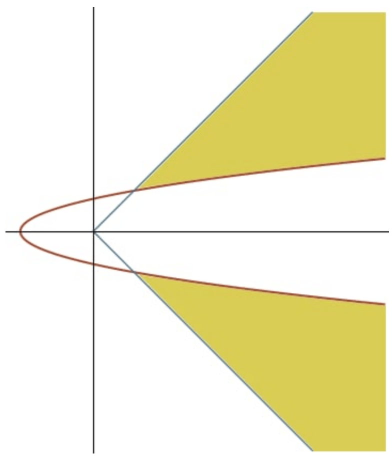

Figure 1. The λ-strongly acyclic region in the fundamental Weyl chamber of $\mathrm{SL}(3,\mathbb{R})$.

From the picture it is clear that the yellow region contains infinitely many lattice points for any lattice in the cone. In fact, it contains *most* of them.

As $\delta(\mathrm{SL}(n,\mathbb{R})) \geq 1$ for $n \geq 3$, we have an immediate corollary to the proposition.

Corollary 2.3.8. *For any $\lambda > 0$ and $n \geq 3$, There exists infinitely many λ-strongly acyclic representations of* $\mathrm{SL}(n,\mathbb{R})$.

2.4 A bound on the heat kernel

We continue with the same notation as the previous section, such that G is a connected semisimple algebraic group. From now on, we assume that τ is an irreducible λ-strongly acyclic representation of $G(\mathbb{R})$, with $\lambda > 0$ to be chosen later. The goal of this section is to present an upper bound on the trace of the heat kernel for large t. This will be used to bound certain orbital integrals in Section 2.6.

Induced operators

Let $(\pi, \mathcal{H}_\pi)$ be a unitary representation of $G(\mathbb{R})$. For any $f \in \mathcal{C}(G(\mathbb{R}))$, the operator

$$\pi(f) := \int_{G(\mathbb{R})} f(g)\pi(g)dg$$

is a trace class operator on $\mathcal{H}_\pi$. If furthermore f is left- and right K-finite, the operator is of finite rank. We consider

$$\pi(H_t^{\tau,p}) := \int_{G(\mathbb{R})} \pi(g) \otimes H_t^{\tau,p}(g)dg$$

as a bounded operator acting on $\mathcal{H}_\pi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau$. Let P be the orthogonal projection to the K-invariant subspace $(\mathcal{H}_\pi \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau)^K$. We note that as π and $\nu_{\tau,p}$ are unitary, so is their tensor product, and thus P has the form

$$P = \int_K \pi(k) \otimes v_{\tau,p}(k)dk.$$

Using the covariance property, one easily checks that

$$P \circ \pi(H_t^{\tau,p}) = \pi(H_t^{\tau,p}) \circ P = \pi(H_t^{\tau,p}).$$

Following the argument of ([BM83], Corollary 2.2), we can then use (2.2.6) to get

$$\pi(H_t^{\tau,p}) = e^{-t(\tau(\Omega)-\pi_{\sigma,i\nu}(\Omega))}P. \tag{2.4.1}$$

We will need the following lemma.

Lemma 2.4.1. *Let $(\pi, \mathcal{H}_\pi)$ be an admissible unitary representation of $G(\mathbb{R})$, and let $A : \mathcal{H}_\pi \to \mathcal{H}_\pi$ be a bounded operator. Then $(A \otimes 1) \circ \pi(H_t^{\tau,p})$ is of trace class, and*

$$\mathrm{Tr}\left((A \otimes 1) \circ \pi(H_t^{\tau,p})\right) = \mathrm{Tr}\left(A \circ \pi(h_t^{\tau,p})\right),$$

Here, the Tr *on the left-hand side is the trace over $\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau$, while on the right-hand side it is the trace over $\mathcal{H}_\pi$.*

Proof. By (2.4.1), we may restrict computing the trace of $(A \otimes 1) \circ \pi(H_t^{\tau,p})$ to computing it over $(\mathcal{H}_\pi \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K$, and as π was assumed admissible, this is finite-dimensional, thus the trace is well defined and finite. The equality now follows by arguing as in the proof of ([BM83], Lemma 5.1). □

Application of the Plancherel formula

Let $P = MAN$ be a real standard parabolic subgroup of $G(\mathbb{R})$. Let $\mathfrak{a}$ be the Lie algebra of A. We denote by $\langle \cdot, \cdot \rangle$ the inner product on the real vector space $\mathfrak{a}^*$ induced by the Killing form, and $|| \cdot ||$ its associated norm. Fix a restricted positive root system of $\mathfrak{a}$ and let $\rho_\mathfrak{a}$ denote their half sum. Let (σ, W_σ) be a discrete series representation of M, i.e. an irreducible unitary subrepresentation of the left regular representation of M on $L^2(M)$, and let $i\nu \in i\mathfrak{a}^*$. We denote by $(\pi_{\sigma,i\nu}, \mathcal{H}^{\sigma,i\nu})$ the induced *principal series representation* of G, defined by

$$\begin{aligned}\mathcal{H}^{\sigma,i\nu} &= \{f : G \to W_\sigma \mid f(gman) = a^{-(\nu+\rho_\mathfrak{a})}\sigma(m)^{-1}f(g) \\ &\qquad \forall g \in G(\mathbb{R}), man \in MAN,\ \ f|_K \in L^2(K, W_\sigma)\}, \\ (\pi_{\sigma,\nu}(g)f)(x) &= f(g^{-1}x).\end{aligned}$$

This is an irreducible unitary representation of $G(\mathbb{R})$, in particular admissible, and thus by Lemma 2.4.1, we have

$$\mathrm{Tr}(\pi_{\sigma,\nu}(H_t^{\tau,p})\pi_{\sigma,\nu}(g)) = \mathrm{Tr}(\pi_{\sigma,\nu}(h_t^{\tau,p})\pi_{\sigma,\nu}(g)). \tag{2.4.2}$$

As in the lemma, the traces are over the appropriate spaces. Applying (2.4.1) to the above, we get that

$$\mathrm{Tr}(\pi_{\sigma,\nu}(h_t^{\tau,p})\pi_{\sigma,\nu}(g)) = e^{-t(\tau(\Omega)-\pi_{\sigma,i\nu}(\Omega))}\,\mathrm{Tr}(\pi^K_{\sigma,i\nu}(g)), \tag{2.4.3}$$

where we by $\pi^K_{\sigma,i\nu}(g)$ denote the operator $P(\pi_{\sigma,i\nu}(g) \otimes \mathrm{Id})P$ on $\mathcal{H}^{\sigma,i\nu} \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau$, with P the projection onto the K-fixed vectors. We consider this as an operator on $(\mathcal{H}^{\sigma,i\nu} \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau)^K$, as this subspace contains its image and it is 0 elsewhere. This subspace is a finite-dimensional space, and by Frobenius reciprocity we have that

$$\dim(\mathcal{H}^{\sigma,i\nu} \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau)^K = \dim(W_\sigma \otimes \Lambda^p\mathfrak{p}^* \otimes V_\tau)^{K_M},$$

with $K_M := K \cap M$. By virtue of being unitary, we have the inequality

$$|\operatorname{Tr}(\pi^K_{\sigma,i\nu}(g))| \leq \dim(W_\sigma \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^{K_M}. \tag{2.4.4}$$

With this bound established, we turn to the scalar appearing in (2.4.3). Define a constant $c(\sigma)$ by

$$c(\sigma) := \sigma(\Omega_M) - ||\rho_{\mathfrak{a}}||.$$

Then we have that (see [Kna01], Proposition 8.22)

$$\pi_{\sigma,i\nu}(\Omega) = c(\sigma) - ||\nu||^2.$$

Assume that $\dim(\mathcal{H}^{\sigma,i\nu} \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^K \neq 0$. By the assumption that τ is λ-strongly acyclic, we immediately get that

$$\tau(\Omega) - c(\sigma) \geq \lambda - ||\nu||^2,$$

and as this is true for all $\nu \in \mathfrak{a}^*$, we get

$$\tau(\Omega) - c(\sigma) \geq \lambda. \tag{2.4.5}$$

Now we are ready to prove a large t asymptotic for $h_t^{\tau,p}$. Using the Harish-Chandra Plancherel formula (see [Olb02], Theorem 2.2) and (2.4.2) we get that

$$\begin{aligned} h_t^{\tau,p}(g) &= \sum_P \sum_{\sigma \in \hat{M}_d} \int_{\mathfrak{a}^*} \operatorname{Tr}(\pi_{\sigma,i\nu}(h_t^{\tau,p})\pi(g^{-1}))p_\sigma(i\nu)d\nu \\ &= e^{-t(\tau(\Omega)-c(\sigma))} \sum_P \sum_{\sigma \in \hat{M}_d} \int_{\mathfrak{a}^*} e^{-t||\nu||^2} \operatorname{Tr}(\pi^K_{\sigma,i\nu}(g))p_\sigma(i\nu)d\nu, \end{aligned}$$

where $p_\sigma : i\mathfrak{a}^* \to [0,\infty)$ is the Plancherel density, an analytic function of polynomial growth. The first sum runs over $P = MAN$ real standard parabolic subgroups, and the second over discrete series representations of M the Levi subgroup of P. Since the trace inside the integral on the right-hand side is bounded absolutely by $\dim(W_\sigma \otimes \Lambda^p \mathfrak{p}^* \otimes V_\tau)^{K_M}$, and this is non-zero only for finitely many pairs (P,σ) (see [Olb02], Corollary 2.3), we may take the double sum to be finite. Furthermore, which pairs contribute is governed solely by τ and p.

The latter integral is convergent and vanishing for $t \to \infty$, and so for $t \geq 1$ it may be bounded independently of t. Putting these observations together and applying (2.4.5), we get the following result.

Proposition 2.4.2. *Assume $t \geq 1$. Then there exists a constant $C > 0$ only depending on τ and p such that for any $g \in G(\mathbb{R})$, we have*

$$|h_t^{\tau,p}(g)| \leq C\, e^{-\lambda t}.$$

Later, we will also need a vanishing behaviour of $h_t^{\tau,p}(g)$ in terms of g. This has been explored in [LM10] for the heat kernel associated to Laplace operators on standard p-forms. By following the proof of their Theorem 3.1, adapted to our setting, we get a refined bound on the heat kernel, which turns into the following bound on its trace. Let $r(g) = d(gK, K)$ be the geodesic distance from gK to K on $G(\mathbb{R})/K$. Then there exists constants $A, c, C > 0$ such that

$$|h_t^{\tau,p}(g)| \leq C\, e^{-\lambda t}\, e^{-c\frac{r(g)^2}{t}} \tag{2.4.6}$$

for all $t > 1$ and $g \in G(\mathbb{R})$ satisfying $r(g) > A$.

2.5 Analysis of the trace formula

Review of the geometric side of the trace formula

Let G be a reductive algebraic group over $\mathbb{Q}$ with $G(\mathbb{R})$ noncompact. Fix a minimal parabolic subgroup P_0 of G with a Levi decomposition $P_0 = M_0N_0$. For $\mathrm{GL}(n)$, we pick as minimal parabolic subgroup the subgroup of upper triangular matrices, with M_0 the diagonal matrices in G. We set $\mathcal{F}$ to be the set of parabolic subgroups of G defined over $\mathbb{Q}$ containing M_0. We let $\mathcal{L}$ denote the set of subgroups in G containing M_0 that are also Levi components of some group in $\mathcal{F}$. Furthermore, any $L \in \mathcal{L}$ is a reductive group, and for $M \in \mathcal{L}$ a Levi subgroup we shall denote by $\mathcal{L}^L(M)$ the set of Levi subgroups in L containing M. Finally we will write $\mathcal{P}^L(M)$ for the set of parabolic subgroups of L for which M is a Levi component. If $L = G$, we drop the superscript and write $\mathcal{L}(M)$ and $\mathcal{P}(M)$.

Let $\mathbb{A}_f$ be the finite adeles over $\mathbb{Q}$. Given $K_f \subseteq G(\mathbb{A}_f)$ an open compact subgroup, we can define the adelic Schwartz space $\mathcal{C}(G(\mathbb{A})^1, K_f)$ as the space of smooth right K_f-invariant functions on $G(\mathbb{A})^1$ all of whose derivatives lies in $L^1(G(\mathbb{A})^1)$. We denote by $\mathcal{C}(G(\mathbb{A})^1)$ the union of $\mathcal{C}(G(\mathbb{A})^1, K_f)$ over all such K_f.

For $f \in C_c^\infty(G(\mathbb{A})^1)$, let $J_{\mathrm{geo}}(f)$ be the geometric side of the Arthur trace formula (see [Art78]). This is a distribution with test function f. We give a very brief sketch of its construction (see [Art05] for an excellent introduction). In essence, we wish to integrate the function

$$K(x, y) = \sum_{\gamma \in G(\mathbb{Q})} f(x^{-1}\gamma y), \qquad x \in G(\mathbb{A})^1 \tag{2.5.1}$$

over $G(\mathbb{Q})\backslash G(\mathbb{A})^1$. In our non-compact case however, this function is often not integrable, and lacks some of our desired properties. Correction terms must be added, and it turns out to be a good idea to add one for each parabolic subgroup. Let $P \in \mathcal{F}$ with canonical Levi decomposition $P = M_PN_P$, i.e.

such that $M_P \in \mathcal{L}(M_0)$. We then define

$$K_P(x,y) := \int_{N_P(\mathbb{A})} \sum_{\gamma \in M_P(\mathbb{Q})} f(x^{-1}\gamma n y) dn, \qquad x, y \in G(\mathbb{A})^1.$$

From this, one constructs a kernel function $k^T(x,f)$ as a sum over standard parabolic subgroups and with a truncation parameter T which serves as a replacement for the function (2.5.1), see ([Art78]) for a precise definition. One then defines $J^T(f)$ as the $G(\mathbb{Q})\backslash G(\mathbb{A})^1$-integral of $k^T(f,x)$. Finally, one picks a particular truncation parameter $T = T_0$ and defines $J(f) := J^{T_0}(f)$. In the case of $G = \mathrm{GL}(n)$, the canonical choice is $T_0 = 0$.

Recall that any element of an algebraic group has a Jordan decomposition, i.e. for any $g \in G(k)$ with k a perfect field, we have a decomposition $g = g_s g_u = g_u g_s$ with g_s semisimple and g_u unipotent. The terms *semisimple* and *unipotent* simply mean that their image has this property under some (equivalently, any) closed embedding $G \to \mathrm{GL}(n)$. Now, define an equivalence relation on $G(\mathbb{Q})$: Say that two elements are equivalent if their semisimple parts are $G(\mathbb{Q})$-conjugate to each other. Let $\mathcal{O}$ be the set of equivalence classes in $G(\mathbb{Q})$. Then for $\mathfrak{o} \subseteq \mathcal{O}$, one could also consider the function

$$K_{P,\mathfrak{o}}(x,y) := \int_{N_P(\mathbb{A})} \sum_{\gamma \in M_P(\mathbb{Q}) \cap \mathfrak{o}} f(x^{-1}\gamma n y) dn, \qquad x, y \in G(\mathbb{A})^1. \tag{2.5.2}$$

From this, one can again construct a modified kernel function $k_{\mathfrak{o}}(x,f)$ as above, and define another distribution with test function f analogously that we will denote $J_{\mathfrak{o}}(f)$ (see *loc. sit.*). We see that we have an equality

$$K_P(x,y) = \sum_{\mathfrak{o} \in \mathcal{O}} K_{P,\mathfrak{o}}(x,y).$$

This equality directly extends to the identity on kernel functions $k^T(f,x) = \sum_{\mathfrak{o} \in \mathcal{O}} k^T_{\mathfrak{o}}(f,x)$ once proper definitions are given. The content of the *coarse geometric expansion* is the fact that the analogous identity holds true for the associated distributions. Precisely, it is the equality

$$J_{\mathrm{geo}}(f) = \sum_{\mathfrak{o} \in \mathcal{O}} J_{\mathfrak{o}}(f). \tag{2.5.3}$$

The coarse geometric expansion has been extended to the domain $\mathcal{C}(G(\mathbb{A})^1, K_f)$ in ([FL16]), which will be important for our applications.

We note that the distributions $J_{\mathfrak{o}}(f)$ have a formulation in terms of integrals of $f(x^{-1}\gamma x)$, for $\gamma \in \mathfrak{o}$ and $x \in G(\mathbb{A})$ (e.g. [Art79], Theorem 8.1). In particular, $J_{\mathfrak{o}}(f)$ is non-zero only if the union of $G(\mathbb{A})$-conjugacy classes of $\mathfrak{o}$ intersects the support of f. This will be used in Section 2.5 to show that in some cases, only one particular class contribute.

Analytic torsion

We assume that G is $\mathrm{GL}(n)$ or $\mathrm{SL}(n)$, and let K_f and $X(K_f)$ be defined as in Section 2.2, with K_f neat. We denote by $h_t^{\tau,p}$ the trace of the heat kernel as defined in (2.2.9). Set 1_{K_f} as the indicator function of K_f on $G(\mathbb{A}_f)$, and define

$$\chi_{K_f} := \frac{1_{K_f}}{\mathrm{vol}(K_f)}.$$

Then $h_t^{\tau,p} \otimes \chi_{K_f} \in \mathcal{C}(G(\mathbb{A})^1, K_f)$. Following ([MM17]), we define the regularized trace of the heat operator as

$$\mathrm{Tr}_{\mathrm{reg}}\left(e^{-t\Delta_p(\tau)}\right) := J_{\mathrm{geo}}\left(h_t^{\tau,p} \otimes \chi_{K_f}\right).$$

If $X(K_f)$ is compact, the heat operator is of trace class, and the regularized trace defined above is then equal to the usual trace. We define the associated spectral zeta functions $\zeta_p(s,\tau)$ as in the compact case (2.1.1), replacing the usual trace with the regularized version. Absolute convergence and existence of meromorphic continuation was shown in ([MM17]). However, unlike the compact case, the zeta function might have a pole at $s=0$, and we have to be slightly more careful. For a meromorphic function $f(s)$ on $\mathbb{C}$ and $z \in \mathbb{C}$, we define $\mathrm{FP}_{s=a} f$ as the zeroth coefficient of the Laurent expansion of $f(s)$ at $s=a$. Analytic torsion of $X(K_f)$ is then defined by

$$\log T_{X(K_f)}(\tau) = \frac{1}{2}\sum_{p=0}^{d}(-1)^p\, p\ \mathrm{FP}_{s=0}\left(\frac{\zeta_p(s,\tau)}{s}\right). \qquad (2.5.4)$$

Congruence quotients of $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$

We stay in the same setup as the previous subsection. Recall that if K_f is neat, $X(K_f)$ is a locally symmetric manifold of finite volume. This assumption is true in the primary setting of this paper that we now explain. Let $N \in \mathbb{N}$, $N \geq 3$ and define

$$K(N) := \prod_p K_p\left(p^{\nu_p(N)}\right) \subseteq \mathrm{GL}(n,\mathbb{A}_f), \qquad (2.5.5)$$

where $K_p(p^e)$ is the kernel of the canonical map $\mathrm{GL}(n,\mathbb{Z}_p) \to \mathrm{GL}(n,\mathbb{Z}/p^e\mathbb{Z})$. Then $K(N)$ is an open compact subgroup of $\mathrm{GL}(n,\mathbb{A}_f)$. We further define $K'(N)$ to be the completely analogous subgroup for $\mathrm{SL}(n)$. Pick the maximal compact subgroups of $\mathrm{GL}(n,\mathbb{R})^1$ and $\mathrm{SL}(n,\mathbb{R})$ to be $\mathrm{O}(n)$, respectively $\mathrm{SO}(n)$. We now set $Y(N)$ to be the adelic symmetric space for $\mathrm{GL}(n)$ associated to $K(N)$ as in (2.2.2), and similarly let $X(N)$ be the space for $\mathrm{SL}(n)$ associated to $K'(N)$.

It follows from Section 2.2 that $X(N) \cong \Gamma(N)\backslash \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ with $\Gamma(N)$ the standard principal congruence subgroup of $\mathrm{SL}(n,\mathbb{Z})$ of level N, and also that $Y(N)$ is a disjoint union of $\varphi(N)$ many copies of $X(N)$. Thus, it is reasonable to also call N the level of the subgroup $K(N)$. In particular, we get

$$\mathrm{vol}(Y(N)) = \varphi(N)\,\mathrm{vol}(X(N)). \tag{2.5.6}$$

We can consider the associated analytic torsion as defined in (2.5.4) in both of these settings. As one would hope for, by ([MM20], (11.5)) we have

$$\log T_{Y(N)}(\tau) = \varphi(N) \log T_{X(N)}(\tau). \tag{2.5.7}$$

Combining (2.5.6), (2.5.7) and (2.1.3), we see that Theorem 2.1.1 is equivalent to the following.

Theorem 2.5.1. *Assume τ is a λ-strongly acyclic representation of $\mathrm{GL}(n,\mathbb{R})^1$, for a certain λ depending only on n. Then there exists some $a > 0$ such that*

$$\log T_{Y(N)}(\tau) = \log T^{(2)}_{Y(N)}(\tau) + O(\mathrm{vol}(Y(N))N^{-(n-1)}\log(N)^a)$$

as N tends to infinity.

The remainder of the paper is essentially a proof of this theorem.

Compactification of the test function

We continue with the same setup, and assume that $G = \mathrm{GL}(n)$ and $K_f = K(N)$, $N \geq 3$. Let $r(g) = d(K, gK)$ be the geodesic distance of $g \in G(\mathbb{R})^1$ from the identity on $\tilde{X} = G(\mathbb{R})^1/K$. We will often write $d(I, g)$ for the same expression. Let $\varphi_R : G(\mathbb{R})^1 \to [0,1]$ be a smooth function identically 1 on $B(R) = \{g \in G(\mathbb{R})^1 \mid r(g) < R\}$ and identically 0 outside $B(R+\varepsilon)$ for some small $\varepsilon > 0$. We now define

$$h^{\tau,p}_{t,R}(g) := \varphi_R(g) h^{\tau,p}_t(g).$$

In a moment, we will need this small lemma on this distance function.

Lemma 2.5.2. *Let $||\cdot||$ denote the Frobenius norm on $M_{n\times n}(\mathbb{R})$. Then*

$$r(g) \geq \log ||g||, \quad \forall g \in G(\mathbb{R})^1.$$

Proof. By ([BH99], II. Corollary 10.42(2)), we have that $d(I, e^X) = ||X||$ for X a symmetric matrix. Further, by the basic inequality $e^{||X||} \geq ||e^X||$, we see that

$$d(I, e^X) \geq \log ||e^X||.$$

Since both the metric and the Frobenius norm are invariant under multiplication by orthogonal matrices, this inequality holds when replacing e^X with any $g \in G(\mathbb{R})^1$, using the Cartan decomposition $G(\mathbb{R})^1 = KAK$ with A the set of diagonal matrices in $G(\mathbb{R})^1$. ☐

Let J_{unip} denote the distribution defined in Section 2.5 associated to the equivalence class of elements with semisimple part being the identity, i.e. the unipotent elements. Replacing $h_t^{\tau,p}$ with $h_{t,R}^{\tau,p}$ allows us to reduce the geometric side of the Arthur trace formula to only the unipotent contribution, if we keep R small relative to the level N. This is the content of the following theorem.

Proposition 2.5.3. *For N large enough and $R \leq C_n \log N$, the constant $C_n > 0$ only depending on n, we have that*

$$J_{\text{geo}}(h_{t,R}^{\tau,p} \otimes \chi_{X(N)}) = J_{\text{unip}}(h_{t,R}^{\tau,p} \otimes \chi_{X(N)}).$$

Proof. In the coarse geometric expansion of the Arthur trace formula (2.5.3), we are summing distributions indexed over equivalence classes in $G(\mathbb{Q})$. For our test function f, we wish to pick R such that $J_{\mathfrak{o}}(f) = 0$ for any $\mathfrak{o}$ not the unipotent class. As explained at the end of Section 2.5, for this it is sufficient to show that the $G(\mathbb{A})$-conjugacy classes of $\mathfrak{o}$ do not intersect the support of the test function.

Our test function $f = h_{t,R}^{\tau,d} \otimes \chi_{K(N)}$ has its support inside $B_R K(N) \subseteq G(\mathbb{A})$. We will pick R such that for any γ with semisimiple part $\gamma_{ss} \neq I$ in $G(\mathbb{Q})$, every conjugate lies outside the support. Take such a γ, and let $g = x^{-1}\gamma x$ for some $x \in G(\mathbb{A})$. Write $g = g_\infty \prod_p g_p$ and assume $\prod_p g_p \in K(N)$. We will then pick R such that $g_\infty \notin B_R$.

Let $q(x) \in \mathbb{Q}[x]$ be the characteristic polynomial of $\gamma - I$. Note that by conjugation invariance, this is also the characteristic polynomial of $g_\nu - I$ for all places ν. As γ is not unipotent by assumption, this polynomial has a non-leading, non-zero coefficient, say for the degree k term, $0 \leq k \leq n-1$. Recall that this polynomial is $q(x) = \det(xI - (g_\nu - I))$. Then for every prime p this coefficient, call it a_k, satisfies

$$\nu_p(a_k) \geq (n-k) \cdot \nu_p(N),$$

since it is a sum of products of $n-k$ elements of $p^{\nu_p(N)}\mathbb{Z}_p$, as g_p lies in $K_p(p^{\nu_p(N)})$. In particular, it is integral, and as it is non-zero we get that $|a_k| \geq N^{n-k}$. This implies that at least one of the entries of $g_\infty - I$ has norm greater than $c_n N$, for some constant $c_n > 0$ only depending on n, and this in turn implies the same lower bound on the Frobenius norm of $g_\infty - I$. Applying Lemma 2.5.2, we get that there exists some $C_n > 0$ such that

$$r(g_\infty) = d(I, g_\infty) \geq \log ||g_\infty|| \geq C_n \log N$$

for N large enough, only depending on n. The last inequality is just using the reverse triangle inequality on our lower bound on $||g_\infty - I||$. Thus, it is clear that picking $R \leq C_n \log N$, we have that $g_\infty \notin B_R$ as desired. □

We must ensure that in replacing $h_t^{\tau,p}$ with its compactification, we can control the change in the trace formula. This was shown in [MM20]:

Proposition 2.5.4 ([MM20], Proposition 7.2). *There exists constants $C_1, C_2, C_3 > 0$ such that*

$$\left|J_{\text{spec}}(h_t^{\tau,p} \otimes \chi_{X_N}) - J_{\text{spec}}(h_{t,R}^{\tau,p} \otimes \chi_{X_N})\right| \leq C_3 e^{-C_1 R^2/t + C_2 t} \operatorname{vol}(Y(N))$$

for all $t > 0$, $R \geq 1$ and $N \in \mathbb{N}$.

It is in controlling this error term that we need to vary R with N. The specifics will be discussed in Section 2.7. Importantly, we may pick C_1, C_2 independent of the representation τ.

The fine geometric expansion

By the coarse geometric expansion (2.5.3), Proposition 2.5.3 and Proposition 2.5.4, when analyzing $\operatorname{Tr}_{\text{reg}}(e^{-t\Delta_p(\tau)})$ we may restrict our attention to $J_{\text{unip}}(h_{t,R}^{\tau,p} \otimes \chi_{K(N)})$. This distribution can be expressed as a finite sum of weighted orbital integrals with certain coefficients, a result known as the fine geometric expansion ([Art85], Corollary 8.3). First we need a bit of notation.

Let S be a finite set of primes containing ∞, and set $\mathbb{Q}_S = \prod_{\nu \in S} \mathbb{Q}_\nu$ and $\mathbb{Q}^S = \prod'_{\nu \notin S} \mathbb{Q}_\nu$, with $\prod'$ the usual restricted product. We define $G(\mathbb{Q}_S)^1 := G(\mathbb{Q}_S) \cap G(\mathbb{A})^1$, and write $C_c^\infty(G(\mathbb{Q}_S)^1)$ for the space of functions $C_c^\infty(G(\mathbb{Q}_S))$ restricted to $G(\mathbb{Q}_S)^1$.

Given $M \in \mathcal{L}$, denote by $\mathcal{U}_M(\mathbb{Q})$ the finite set of conjugacy classes of unipotent elements of $M(\mathbb{Q})$. Assume $f = f_S \otimes 1_{K^S}$, with $f_S \in C_c^\infty(G(\mathbb{Q}_S)^1)$ and 1_{K^S} the characteristic function of the standard compact subgroup in $G(\mathbb{Q}^S)$. In the case of $\mathrm{GL}(n)$, as all orbits are stable, the fine geometric expansion can then be expressed as

$$J_{\text{unip}}(f) = \sum_{M \in \mathcal{L}} \sum_{\mathfrak{u} \in \mathcal{U}_M(\mathbb{Q})} a^M(S, \mathfrak{u}) J_M(f_S, \mathfrak{u}). \tag{2.5.8}$$

Here $J_M(f, \mathfrak{u})$ is the weighted orbital integral associated to $(M, \mathfrak{u})$ of f, and $a^M(S, \mathfrak{u})$ are the *global coefficients*. One finds the general definition of weighted orbital integrals in [Art88b] - we will express them in our specific situation in a moment. We will apply this to our test function $h_{t,R}^{\tau,p} \otimes \chi_{K(N)}$. We will by abuse of notation write $\chi_{K(N)}$ both for the normalised characteristic function of $K(N)$ in $G(\mathbb{A}_f)$ and in $G(\mathbb{Q}_{S(N)})$, where $S(N) = \{p \text{ prime} : p \mid N\}$. In this case, $a^M(S(N), \mathfrak{u})$ depend on N only by its prime divisors, and does not grow too quickly, as seen from the following lemma.

Lemma 2.5.5 ([Mat15]). *There exists $b, c > 0$ such that for all N, M and $\mathfrak{u}$ we have*

$$|a^M(S(N), \mathfrak{u})| \leq c(1 + \log N)^b.$$

To describe the weighted orbital integrals, we split them into archimedean and non-archimedean parts (see [Art81]). Assume that $f = f_\infty \otimes f_{\text{fin}}$. For $L \in \mathcal{L}(M)$ and $Q = LV \in \mathcal{P}(L)$, define

$$f_{\infty,Q}(m) := \delta_Q(m)^{\frac{1}{2}} \int_{K_\infty} \int_{V(\mathbb{R})} f_\infty(k^{-1}mvk)dvdk, \quad m \in M(\mathbb{R}). \tag{2.5.9}$$

Define $f_{\text{fin},Q}$ analogously. Then we have the expression

$$J_M(f,\mathfrak{u}) = \sum_{L_1,L_2\in\mathcal{L}(M)} d_M^G(L_1,L_2) J_M^{L_1}(f_{\infty,Q_1},\mathfrak{u}_\infty) J_M^{L_2}(f_{\text{fin},Q_2},\mathfrak{u}_{\text{fin}}). \tag{2.5.10}$$

Here Q_i is some parabolic in $\mathcal{P}(L_i)$, and $\mathfrak{u}_{\text{fin}} = (\mathfrak{u}_p)_p$, where $\mathfrak{u}_p$ the $M(\mathbb{Q}_p)$-conjugacy class of $\mathfrak{u}$. Denote $\mathfrak{u}_\infty$ analogously. The archimedean part is an integral of the form (see [Art88b])

$$J_M^L(f_{\infty,Q},\mathfrak{u}_\infty) = \int_{U(\mathbb{R})} f_{\infty,Q}(u)\omega(u)du \tag{2.5.11}$$

for a certain weight function ω depending on the class $\mathfrak{u}$, as well as on M and L. Here, $U = U_{Q_1}$ is the unipotent radical of a semistandard parabolic subgroup $S = M_{Q_1}U_{Q_1}$ in M such that Q_1 is a Richardson parabolic for $\mathfrak{u}$ in M. By splitting up the finite part further into local parts and computing explicitly, Matz and Müller showed the following lemma.

Lemma 2.5.6 ([MM20],(9.4)). *There exist $c,d > 0$ only depending on the group G such that*

$$\left|J_M^L(\chi_{K(N),Q},\mathfrak{u}_{fin})\right| \leq cN^{-\dim \mathrm{Ind}_M^G \mathfrak{u}/2}(\log N)^d \operatorname{vol}(K(N))^{-1}.$$

Remark 2.5.7. Unless $(M,\mathfrak{u}) = (G,\{1\})$, we have that $\frac{\dim \mathrm{Ind}_M^G \mathfrak{u}}{2} \geq n-1$, and this is where the power saving in our main result comes from.

As the non-archimedean part does not see the variable t, nor does it see the representation τ or the radius R, we may focus on the integrals

$$J_M^L(((h_{t,R}^{\tau,p})_Q,\mathfrak{u}_\infty) = \int_{U(\mathbb{R})} (h_{t,R}^{\tau,p})_Q(u)\omega(u)du. \tag{2.5.12}$$

Recalling the definition of f_Q, we may use that $h_{t,R}^{\tau,p}$ is bi-K_∞-invariant, and letting $MV' := M_{Q_1}U_{Q_1}V$ such that V' is the unipotent radical of a Richardson parabolic for $\mathrm{Ind}_M^G(\mathfrak{u})$ in G, we may rewrite this as

$$J_M^L(((h_{t,R}^{\tau,p})_Q,\mathfrak{u}_\infty) = \int_{V'(\mathbb{R})} h_{t,R}^{\tau,p}(v)\omega(v)dv. \tag{2.5.13}$$

This description will be used in the following section.

2.6 Asymptotic time behaviour

We continue in the setting of Section 2.5 with $G = \mathrm{GL}(n)$ and $K_f = K(N)$. It is clear from the definition that to understand analytic torsion, we must analyze the terms

$$\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^\infty \mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)})t^{s-1}dt\right), \tag{2.6.1}$$

where we by the integral in fact means its meromorphic continuation to all of $\mathbb{C}$ as a function of s. As $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1, we see that $\frac{1}{s\Gamma(s)}$ is holomorphic at $s = 0$ with value 1. Thus, we are reduced to examining the meromorphic continuation of the Mellin transform of $\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)})$.

From the standard theory of Mellin transforms (see e.g. [Zag06]), we can understand the meromorphic continuation of the Mellin transform of a function f by giving asymptotics of $f(t)$ for $t \to 0$ and $t \to \infty$. More precisely, it is sufficient to establish the following:

$$f(t) = O(e^{-ct}) \qquad \text{as } t \to \infty \tag{2.6.2}$$

$$f(t) = \sum_{i=0}^{B}\sum_{j=0}^{r_i} c_{ij}t^{\alpha_i}(\log t)^j + O(t^{\alpha_{B+1}}) \qquad \text{as } t \to 0 \tag{2.6.3}$$

for $(\alpha_i)_{i\in\mathbb{N}_0}$ a sequence of real, possibly negative numbers with $\alpha_i < \alpha_{i+1}$ and tending to $+\infty$, and $c > 0$. Then we know that the Mellin transform of f converges in some half plane and has a meromorphic continuation to all of $\mathbb{C}$. Furthermore, its residues at its poles can be described in terms of the coefficients $c_{i,j}$. In the following, we present such asymptotics in the different settings we will need for our proof.

Large t asymptotic of the spectral side

During the proof of the main theorem, we will need to control large t behaviour of the entire trace formula at once to control the error term incurred from truncating the the Mellin transform. This was done in ([MM20], Corollary 6.7), using the fine spectral expansion of the spectral side of the Arthur trace formula and results on logarithmic derivatives of intertwining operators. The result is

$$\left|J_{\mathrm{spec}}(h_t^{\tau,p}\otimes\chi_{K(N)})\right| \le Ce^{-ct}\,\mathrm{vol}(Y(N)) \tag{2.6.4}$$

for some $C, c > 0$, for all $t \ge 1$, $p = 0, \ldots, n$ and $N \in \mathbb{N}$.

Remark 2.6.1. Going carefully through the proof of the above result in [MM20], one sees that we may pick $c = \lambda(1-\varepsilon)$ for any $0 < \varepsilon < 1$, where λ is the spectral gap guaranteed by assuming τ is λ-strongly acyclic. Indeed, in ([MM20], (6.20)) they pick $c = \frac{\lambda}{2}$, but the proof works for any multiple of λ with a factor less than 1.

Small t asymptotics of orbital integrals

By the definition of the regularized trace of the heat operator, along with Proposition 2.5.3 and the fine geometric expansion (2.5.8), once we have switched to a compactified test function it is sufficient to establish asymptotics for the weighted orbital integrals. By the decomposition into archimedean and non-archimedean part (2.5.10), as the non-archimedean part does not depend on the variable t in our case, we are reduced to analyzing the archimedean parts.

The desired small-t asymptotics (2.6.3) for our archimedean weighted orbital integrals given in (2.5.13) were shown in [MM17]. Combining ([MM17], Proposition 12.3) and ([MM17], (13.14)) we get

Proposition 2.6.2. *Let $M \in \mathcal{L}$ and $\mathfrak{u} \in \mathcal{U}_M(\mathbb{Q})$ with $(M, \mathfrak{u}) \neq (G, \{1\})$. For every $N \geq 3$, there is an expansion*

$$J_M^G((h_{t,R}^{\tau,p})_Q, \mathfrak{u}) = t^{-(d-k)/2} \sum_{j=0}^{N} \sum_{i=0}^{r_j} c_{ij}(\tau,p) t^{j/2} (\log t)^i + O(t^{(N-d+k+1)/2})$$

as $t \to 0^+$.

Here k is the dimension of the Lie algebra of $V(\mathbb{R})$, and $c_{ij}(\tau,p)$ are certain coefficients depending only on i, j, τ, p. Note that the results ([MM17], Proposition 12.3) is stated for $M \neq G$, but their proof holds in the more general case above without modification. Furthermore, it is sufficient for our purposes to state the result for $L = G$ as above, since every Levi subgroup L of $\mathrm{GL}(n)$ is canonically isomorphic to a finite direct product of $\mathrm{GL}(m)$'s, $m \leq n$, and the orbital integral splits accordingly.

Remark 2.6.3. The proof of ([MM17], Proposition 12.3) does not utilize the fact that the support of the test function is compactified, i.e. their proof holds when replacing $h_{t,R}^{\tau,p}$ by $h_t^{\tau,p}$ - this can be seen in the first inequality of Section 12.2. In particular, neither the coefficients nor the implied constants in the error term depend on the radius of compactification R.

Large t asymptotics of orbital integrals

We now show exponential decay as t goes to infinity of the orbital integrals appearing in (2.5.13). Throughout, we will assume $t > 1$. Let $A, c, C > 0$ be given as in (2.4.6). Recall that $B(k)$ are the elements g of $G(\mathbb{R})^1$ with $r(g) < k$. We now decompose $V(\mathbb{R})$ into disjoint subsets of growing radius:

$$V(\mathbb{R}) = (V(\mathbb{R}) \cap B(A)) \cup \bigcup_{k=1}^{\infty} V(\mathbb{R}) \cap (B(A+k) \setminus B(A+k-1)).$$

To simplify notation, for $k \geq 1$ write

$$D(k) := V(\mathbb{R}) \cap (B(A+k) \setminus B(A+k-1))$$

and set $D(0) := (V(\mathbb{R}) \cap B(A))$. The point is that for $v \in D(k)$ we then have the bounds $A+k-1 \leq r(v) < A+k$. The orbital integral can be decomposed

$$\int_{V(\mathbb{R})} h_{t,R}^{\tau,p}(v)\omega(v)dv = \int_{D(0)} h_{t,R}^{\tau,p}(v)\omega(v)dv + \sum_{k=1}^{\infty} \int_{D(k)} h_{t,R}^{\tau,p}(v)\omega(v)dv. \quad (2.6.5)$$

As a consequence of log-homogeneity ([MM17], Proposition 7.1), the weight function $\omega(v)$ is bounded by a polynomial of uniformly bounded degree in powers of $\log\|v\|$, and hence by a polynomial in $r(v)$ by Lemma 2.5.2. This will be used in the following.

The integral over $D(0)$ can be handled with the use of Proposition 2.4.2. Indeed, we get the bound

$$\begin{aligned}\left|\int_{D(0)} h_{t,R}^{\tau,p}(v)\omega(v)dv\right| &\leq Ce^{-\lambda t}\int_{D(0)}(1+r(v))^k dv \\ &\leq Ce^{-\lambda t}(1+A)^k \operatorname{vol}(D(0)).\end{aligned}$$

As $D(0)$ is a compact domain, thus of finite volume, only depending on A and V, and we consider only finitely many such groups V, we see that the above gives a bound of the form $C'e^{-\lambda t}$ for some constant C' only depending on G.

For the integral over $D(k)$, $k \geq 1$, we note that (2.4.6) is applicable. Hence we get

$$\left|\int_{D(k)} h_{t,R}^{\tau,p}(v)\omega(v)dv\right| \leq Ce^{-\lambda t}\int_{D(k)} e^{-c\frac{r(v)^2}{t}}(1+r(v))^b dv. \quad (2.6.6)$$

Pick any small $\varepsilon > 0$ and a constant $C'' > 0$ such that $(1+r(v))^b \leq C''e^{\varepsilon r(v)}$. The volume of $D(k)$ is bounded by the volume of $V(\mathbb{R}) \cap B(A+k)$, which is again bounded by the volume of

$$\{v \in V(\mathbb{R}) \mid \|v\| \leq e^{A+k}\} \quad (2.6.7)$$

by Lemma 2.5.2. since the Haar measure on $V(\mathbb{R})$ is compatible with the Lebesgue measure on the Euclidean space $\mathbb{R}^{\dim V}$, the volume of the ball (2.6.7) is bounded by a polynomial of degree $\dim V$ in the radius. Hence, for some constant C_A depending on A, we get

$$\operatorname{vol}(D(k)) \leq C_A e^{\dim V \cdot k}.$$

Let $c_1 = \dim V + \varepsilon$. Then we can bound the integral on the right hand side of (2.6.6) by

$$\begin{aligned}\int_{D(k)} e^{-c\frac{r(v)^2}{t}}(1+r(v))^k dv &\leq Ce^{-c\frac{(k+A-1)^2}{t}}e^{\varepsilon k}\operatorname{vol}(D(k)) \\ &\leq C'_A e^{-c\frac{k^2}{t}}e^{c_1 k},\end{aligned}$$

for some other constant C'_A depending only on A. This bound guarantees the absolute convergence of the sum in (2.6.5), meaning we can bound it by

$$C'_A \sum_{k=1}^{\infty} e^{-c\frac{k^2}{t}} e^{c_1 k}.$$

As the function $e^{-c\frac{x^2}{t}} e^{c_1 x}$ for $x \geq 0$ is increasing for $2c\frac{x}{t} - c_1 \leq 0$, and decreasing elsewhere, the above sum can be bounded by the corresponding integral up to an absolute constant, and this integral is computable:

$$\int_0^{\infty} e^{-c\frac{x^2}{t}} e^{c_1 x} dx = \sqrt{\frac{\pi t}{4c}} e^{c_1^2 t/c} \left(1 - \operatorname{erf}\left(c_1 \sqrt{\frac{t}{c}}\right)\right).$$

Here erf is the *error function* $\operatorname{erf}(z) = \frac{\sqrt{\pi}}{2} \int_0^z e^{-t^2} dt$, satisfying $0 < \operatorname{erf}(z) < 1$ for all $z > 0$. Set $c' = \frac{c_1^2}{c}$. Putting everything together, we have proven the following result.

Proposition 2.6.4. *Assume $t > 1$. Then there exist constants $C_{\tau,p} > 0$ only depending on τ and p and $c' > 0$ only depending on G such that*

$$\left| \int_{V(\mathbb{R})} h_{t,R}^{\tau,p}(v) \omega(v) dv \right| \leq C_{\tau,p}\, e^{-(\lambda - c')t}.$$

This gives us the desired large t asymptotics (2.6.2) if one assumes a large enough λ. We will put this to work in the following section.

2.7 Proof of the main theorem

We continue to assume $G = \mathrm{GL}(n)$, $n \geq 3$ and τ is λ-strongly acyclic. To prove Theorem 2.5.1, we first turn to an analysis of the terms (2.6.1). We truncate the integral in this expression with respect to a parameter $T > 0$, leaving a remainder term.

$$\int_0^{\infty} \mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)}) t^{s-1} dt = \int_0^{T} \mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)}) t^{s-1} dt \tag{2.7.1}$$

$$+ \int_T^{\infty} \mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)}) t^{s-1} dt. \tag{2.7.2}$$

We focus first on the second integral on the right hand side, what we call the remainder. It is convergent for all s, in particular it is holomorphic at $s = 0$. For any meromorphic function f holomorphic at 0 it holds that $\mathrm{FP}_{s=0}(f(s)) = f(0)$, thus by linearity, we need only analyze

$$E_0(0,T) := \int_T^{\infty} \mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)}) t^{-1} dt.$$

By applying our large t asymptotic for the spectral side (2.6.4) and its following remark, and using the trace formula, i.e. the equality of the geometric side and spectral side, we get the following lemma.

Lemma 2.7.1. *There exists a $C > 0$ only depending on G such that for any $\varepsilon > 0$,*

$$|E_0(0,T)| \le Ce^{-\lambda(1-\varepsilon)T}\operatorname{vol}(Y(N)).$$

Restricting to unipotent contribution

We return to the first integral on the right-hand side of (2.7.1). We want to substitute the test function with its compactified version. By Proposition 2.5.4, we may write

$$\int_0^T J_{\text{geo}}(h_t^{\tau,p} \otimes \chi_{K(N)})t^{s-1}dt = \int_0^T J_{\text{geo}}(h_{t,R}^{\tau,p} \otimes \chi_{K(N)})t^{s-1}dt + E_1(s,R,T),$$

with the error term $E_1(s,R,T)$ given by an integral convergent for all $s \in \mathbb{C}$ and satisfying

$$|E_1(0,R,T)| \le C_3 \int_0^T e^{-C_1R^2/t + C_2t}t^{-1}dt\operatorname{vol}(Y(N)) \tag{2.7.3}$$

$$\le C_3e^{-C_4R^2/T + C_2T}\int_0^{T/R^2} e^{-C_4/t}t^{-1}dt\operatorname{vol}(Y(N)). \tag{2.7.4}$$

Again, $C_2, C_4 > 0$ both only depend on the group G. We will return to this estimate in a moment. The reward for compactifying our test function is that by Proposition 2.5.3, only the unipotent part of the coarse geometric expansion (2.5.3) contribute if we keep the radius of the support small enough. Assume $R \le C_n \log N$. Then

$$\int_0^T J_{\text{geo}}(h_{t,R}^{\tau,p} \otimes \chi_{K(N)})t^{s-1}dt = \int_0^T J_{\text{unip}}(h_{t,R}^{\tau,p} \otimes \chi_{K(N)})t^{s-1}dt$$

Applying the fine geometric expansion

To deal with the truncated Mellin transform of J_{unip}, we recall the fine geometric expansion (2.5.8). We isolate the term for $(M,\mathfrak{u}) = (G,\{1\})$, which is exactly $h_{t,R}^{\tau,p}(1)\operatorname{vol}(Y(N))$, and denote the remaining sum by $J_{\text{unip}-1}$. This allows us to write

$$\int_0^T J_{\text{unip}}(h_{t,R}^{\tau,p} \otimes \chi_{K(N)})t^{s-1}dt = \int_0^T h_{t,R}^{\tau,p}(1)t^{s-1}dt\operatorname{vol}(Y(N)) \tag{2.7.5}$$

$$+ \int_0^T J_{\text{unip}-1}(h_{t,R}^{\tau,p} \otimes \chi_{K(N)})t^{s-1}dt. \tag{2.7.6}$$

The first integral on the right-hand side is the truncated Mellin transform of $h_{t,R}^{\tau,p}(1)$. As we are evaluating at $1 \in G(\mathbb{R})$, the compactification has no effect, and hence we can ignore R. By ([MM17], (5.11)) we have an asymptotic expansion

$$h_t^{\tau,p}(1) \sim \sum_{i=0}^{\infty} a_i t^{-d/2+i}, \quad t \to 0.$$

Furthermore, as a special case of Theorem 2.4.2 we have for $t \geq 1$,

$$|h_t^{\tau,p}(1)| \leq C_{\tau,p} e^{-\lambda t}.$$

Thus, the integral is convergent for s in some half-plane and has a meromorphic extension to all of $\mathbb{C}$. We may write

$$\int_0^T h_{t,R}^{\tau,p}(1) t^{s-1} dt \operatorname{vol}(Y(N)) = \int_0^\infty h_t^{\tau,p}(1) t^{s-1} dt \operatorname{vol}(Y(N)) + E_2(s,T),$$

with the error term $E_2(s,T)$ given by an integral convergent for all $s \in \mathbb{C}$ and satisfying

$$|E_2(0,T)| \leq C \int_T^\infty e^{-\lambda t} t^{-1} dt \operatorname{vol}(Y(N)) \tag{2.7.7}$$

$$\leq C' e^{-\lambda T} \operatorname{vol}(Y(N)) \tag{2.7.8}$$

for $T \geq 1$. We return to the second integral on the right-hand side of (2.7.5). Using the splitting of the orbital integrals into local parts, i.e. equation (2.5.10), we may express this integral as

$$\sum_{(M,\mathfrak{u}) \neq (G,\{1\})} a^M(S(N),\mathfrak{u}) \sum_{L_1,L_2 \in \mathcal{L}(M)} d_M(L_1,L_2) A_M^{L_1}(h_{t,R}^{\tau,p},\mathfrak{u}_\infty,T) J_M^{L_2}(\chi_{K(N)},\mathfrak{u}_{\mathrm{fin}}), \tag{2.7.9}$$

with the archimedean part defined as

$$A_M^L(h_{t,R}^{\tau,p},\mathfrak{u}_\infty,T) = \int_0^T J_M^L((h_{t,R}^{\tau,p})_Q,\mathfrak{u}_\infty) t^{s-1} dt.$$

We will treat this term like we did the integral of $h_{t,R}^{\tau,p}(1)$ above. By the asymptotics given in Proposition 2.6.2, this is convergent for s in some half plane. By Proposition 2.6.4, we may write

$$A_M^L(h_{t,R}^{\tau,p},\mathfrak{u}_\infty,T) = \int_0^\infty J_M^L((h_{t,R}^{\tau,p})_Q,\mathfrak{u}_\infty) t^{s-1} dt + E_3(s,T), \tag{2.7.10}$$

with $E_3(s,T)$ some error term, given by an integral convergent for all $s \in \mathbb{C}$, analogously to E_2 satisfying

$$|E_3(0,T)| \leq C'' e^{-(\lambda - c')T} \tag{2.7.11}$$

for $T > 1$ and some constants $c', C'' > 0$. We will write $A_M^L(h_{t,R}^{\tau,p}, \mathfrak{u}_\infty)$ for the integral on the right-hand side of (2.7.10), which is convergent in some half plane with a meromorphic extension to all of $s \in \mathbb{C}$. This means that

$$\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)} A_M^L(h_{t,R}^{\tau,p}, \mathfrak{u}_\infty)\right)$$

is well defined.

Lemma 2.7.2. *Assume $\lambda > c'$. Then there exists constants $C > 0$ only depending on τ, p and λ and $b > 0$ only depending on the group such that for $T > 1$, we have*

$$\begin{aligned}&\left|\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J_{\mathrm{unip}-1}(h_{t,R}^{\tau,p} \otimes \chi_{K(N)}) t^{s-1} dt\right)\right| \\ \leq\ & CN^{-(n-1)}(1+\log N)^b \operatorname{vol}(Y(N)).\end{aligned}$$

Proof. Considering the expression (2.7.9) and the discussion above, it suffices to appropriately bound the three following terms:

$$\begin{aligned}&a^M(S(N), \mathfrak{u}),\\ &J_M^{L_2}(\chi_{K(N)}, \mathfrak{u}_{\mathrm{fin}}) \text{ for } (M, \mathfrak{u}) \neq (G, \{1\}),\\ &\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\left(A_M^L(h_{t,R}^{\tau,p}, \mathfrak{u}_\infty) + E_3(s,T)\right)\right).\end{aligned}$$

The global coefficients $a^M(S(N), \mathfrak{u})$ were bounded in Lemma 2.5.5, and the local orbital integrals $J_M^{L_2}(\chi_{K(N)}, \mathfrak{u}_{\mathrm{fin}})$ were bounded in Lemma 2.5.6 and its remark. Note that $\operatorname{vol}(Y(N)) = c \operatorname{vol}(K(N))^{-1}$ for some constant c depending only on normalisations of measures (see the appendix).

By Proposition 2.6.2 and its remark, alongside Proposition 2.6.4, the term

$$\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)} A_M^L(h_{t,R}^{\tau,p}, \mathfrak{u}_\infty)\right)$$

is uniformly bounded by a constant, following the theory of Mellin transforms. The bound can be chosen to be uniform over all $(L, M, \mathfrak{u}_\infty)$, as there are only finitely many such triples. Finally, the term

$$\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)} E_3(s,T)\right) = E_3(0,T)$$

was bounded in (2.7.11), and we see that it is vanishing in T if $\lambda > c'$, in particular bounded by a constant. Collecting all these bounds and applying to (2.7.9), we arrive at the desired bound. □

Asymptotics of analytic torsion

Recall the definition of analytic torsion (2.5.4). By collecting our initial analysis in the previous subsections, we may write

$$\log T_{Y(N)}(\tau) = \mathrm{FP}_{s=0}\left(\frac{1}{2}\frac{1}{s\Gamma(s)}\int_0^\infty \sum_{p=1}^{d}(-1)^p p\, h_t^{\tau,p}(1)t^{s-1}dt\right)\mathrm{vol}(Y(N)) \tag{2.7.12}$$

$$+\,\mathrm{FP}_{s=0}\left(\frac{1}{2}\sum_{p=1}^{d}(-1)^p p\frac{1}{s\Gamma(s)}\int_0^T J_{\mathrm{unip}\,-1}(h_{t,R}^{\tau,p}\otimes\chi_{K(N)})t^{s-1}dt\right) \tag{2.7.13}$$

$$+\,\frac{1}{2}\sum_{p=1}^{d}(-1)^p p\,\left(E_0(0,T)+E_1(0,R,T)+E_2(0,T)\right) \tag{2.7.14}$$

The first term on the right-hand side is the L^2-torsion of $Y(N)$ (see [Lot92]). We set

$$t_{\tilde{X}}^{(2)}(\tau) := \frac{1}{2}\frac{d}{ds}\left(\frac{1}{\Gamma(s)}\int_0^\infty \sum_{p=1}^{d}(-1)^p p\, h_t^{\tau,p}(1)t^{s-1}dt\right)\Bigg|_{s=0}.$$

Note that although $\frac{1}{\Gamma(s)}\int_0^\infty h_t^{\tau,p}(1)t^{s-1}dt$ may not be holomorphic at $s=0$, swapping $h_t^{\tau,p}(1)$ with this alternating sum over p seen above turns the meromorphic extension holomorphic at $s=0$, from which the above definition is well defined (see [BV13], §4.4). Then the first term is exactly $\log T_{Y(N)}^{(2)} = t_{\tilde{X}}^{(2)}(\tau)\,\mathrm{vol}(Y(N))$. All that is left to prove Theorem 2.5.1 is to show that the second and third term above both have the form $O(\mathrm{vol}(Y(N))N^{-(n-1)}\log(N)^b)$.

This was accomplished for the second term in Lemma 2.7.2 when $\lambda > c'$, as we assume $R \leq C_n \log N$. Note that the sum over p contributes at most a constant multiple, as the bound was independent of p. To arrive at our desired bound for the third and final term, we wish to pick our parameters R and T such that the error terms E_0, E_1, E_2 are all as small as possible. This is done in the following section.

A dance of error terms

We pick $T=\beta\log N$ and $R=C_n\log N$, with C_n from Proposition 2.5.3 and $\beta>0$ still to be determined. We see that the integral $\int_0^{T/R^2} e^{-C_4/t}t^{s-1}dt$ is then vanishing in N, in particular bounded by a constant for $N\geq 3$. Thus, by Lemma 2.7.1, (2.7.3), and (2.7.7) respectively we have

$$\begin{aligned} E_0(0, \beta \log N) &= O(N^{-\lambda(1-\varepsilon)\beta} \operatorname{vol}(Y(N))), \\ E_1(0, C_n \log N, \beta \log N) &= O(N^{-C_4 C_n^2/\beta + C_2 \beta} \operatorname{vol}(Y(N))), \\ E_2(0, \beta \log N) &= O(N^{-\lambda\beta} \operatorname{vol}(Y(N))). \end{aligned}$$

The implied constants may depend on τ. We now ensure that the error terms above are of the size

$$O(\operatorname{vol}(Y(N) N^{-(n-1)} (\log N)^a) \tag{2.7.15}$$

for some $a > 0$ and N large enough. Here is where the independence of the constants C_2, C_4 on τ becomes important.

Pick β such that

$$C_4 C_n/\beta - C_2 \beta > n - 1.$$

This choice of β then only depends on n, i.e. on the group G. With β fixed, let λ be chosen such that

$$\begin{aligned} \lambda(1-\varepsilon)\beta > n-1, \quad \text{and} \\ \lambda > c'. \end{aligned}$$

The latter inequality is to ensure we can use Lemma 2.7.2. As ε can be chosen arbitrarily close to 0, for the first inequality it is in fact sufficient to satisfy $\lambda > \frac{n-1}{\beta}$. Note that this choice of λ also only depends on C_2 and C_4, which only depend on G. This implies that for τ being λ-strongly acyclic for any λ satisfying these inequalities, all the error terms are of the form (2.7.15) as desired.

We have now proven Theorem 2.5.1, from which Theorem 2.1.1 follows.

2.8 Appendix

For the convenience of the reader, in this section we prove the well known formula of the volume of the group $K(N)$ and show its inverse relation to the volume of its associated adelic locally symmetric space. This is summarized in the following proposition.

Proposition 2.8.1. *Let $K(N)$ be the open compact subgroup of $\mathrm{GL}(n,\mathbb{A}_f)$ given by $K(N)=\prod_p K_p(p^{v_p(N)})$, where $K_p(p^k)=\ker(\mathrm{GL}(n,\mathbb{Z}_p)\to\mathrm{GL}(n,\mathbb{Z}/p^k\mathbb{Z}))$, and let $Y(N)$ be the associated adelic locally symmetric space*

$$Y(N)=\mathrm{GL}(n,\mathbb{Q})\backslash(\mathrm{GL}(n,\mathbb{A})/\mathrm{SO}(n)K(N)).$$

Similarly, let $K'(N)$ be the analogous open compact subgroup of $\mathrm{SL}(n,\mathbb{A}_f)$, with adelic locally symmetric space $X(N)$. Then the following formulas hold:

$$\mathrm{vol}(K(N))=\left(N^{n^2}\prod_{p|N}\prod_{k=1}^{n}\left(1-\frac{1}{p^k}\right)\right)^{-1} \tag{2.8.1}$$

$$\mathrm{vol}(K(N))=\varphi(N)^{-1}\,\mathrm{vol}(K'(N)). \tag{2.8.2}$$

Here φ is Euler's totient function. Furthermore, let

$$c(n):=\mathrm{vol}(\mathrm{SL}(n,\mathbb{Z})\backslash(\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)))^{-1}.$$

This constant depends only on normalizations of measures. We have the relations

$$\mathrm{vol}(K(N))=c(n)\,\mathrm{vol}(Y(N))^{-1} \tag{2.8.3}$$

$$\mathrm{vol}(K'(N))=c(n)\,\mathrm{vol}(X(N))^{-1}. \tag{2.8.4}$$

Proof. As $K(N)=\prod_p K_p(p^{v_p(N)})$, and the measures respect this decomposition, it is sufficient to compute $\mathrm{vol}(K_p(p^{v_p(N)}))$. Fix p, and let $m=v_p(N)$. $K_p(p^m)$ is the kernel of the surjective map $\mathrm{GL}(n,\mathbb{Z}_p)\to\mathrm{GL}(n,\mathbb{Z}_p/p^m\mathbb{Z}_p)\cong\mathrm{GL}(n,\mathbb{Z}/p^m\mathbb{Z})$. Thus,

$$\mathrm{GL}(n,\mathbb{Z}_p)=\bigsqcup_{a\in\mathrm{GL}(n,\mathbb{Z}/p^m\mathbb{Z})}(a\cdot K_p(p^m))$$

where we by abuse of notation let $a\in\mathrm{GL}(n,\mathbb{Z}/p^m\mathbb{Z})$ also denote any lift to $\mathrm{GL}(n,\mathbb{Z}_p)$. Since the cosets are disjoint, and they are homeomorphic through multiplication, they all have the same measure, $\mathrm{vol}(K_p(p^m))$. Since the left-hand side has measure 1 in our normalization, we get that

$$\mathrm{vol}(K_p(p^m))=\frac{1}{|\mathrm{GL}(n,\mathbb{Z}/p^m\mathbb{Z})|}.$$

To determine $|\operatorname{GL}(n,\mathbb{Z}/p^m\mathbb{Z})|$, we use the following short exact sequence of finite groups

$$0 \to \{I + pA \mid A \in M_n(\mathbb{Z}/p^m\mathbb{Z})\} \to \operatorname{GL}(n,\mathbb{Z}/p^m\mathbb{Z}) \to \operatorname{GL}(n,\mathbb{Z}/p\mathbb{Z}) \to 0.$$

The rightmost group has cardinality $\prod_{k=0}^{n-1}(p^n - p^k)$ by counting number of bases, using that $\mathbb{Z}/p\mathbb{Z}$ is a field. The leftmost group is bijective to $\{pA \mid A \in M_n(\mathbb{Z}/p^m\mathbb{Z})\}$, which is the kernel of $M_n(\mathbb{Z}/p^m\mathbb{Z}) \to M_n(\mathbb{Z}/p\mathbb{Z})$, induced by the obvious quotient map. These have cardinality $p^{n^2 m}$ and p^{n^2} respectively, so their kernel has cardinality $p^{m^2(n-1)}$. Plugging everything in, we get that

$$\begin{aligned} |\operatorname{GL}(n,\mathbb{Z}/p^m\mathbb{Z})| &= |\operatorname{GL}(n,\mathbb{Z}/p\mathbb{Z})| \cdot |\{I + pA \mid A \in M_n(\mathbb{Z}/p^m\mathbb{Z})\}| && (2.8.5) \\ &= p^{n^2(m-1)} \prod_{k=0}^{n-1}(p^n - p^k) = (p^m)^{n^2} \prod_{k=0}^{n-1}\left(1 - \frac{1}{p^{n-k}}\right). && (2.8.6) \end{aligned}$$

Reindexing the product, this establishes formula (2.8.1). For the next formula, running through the same argument for $K'(N)$, we get that

$$\operatorname{vol}(K_p'(p^m)) = \frac{1}{|\operatorname{SL}(n,\mathbb{Z}/p^m\mathbb{Z})|}. \quad (2.8.7)$$

As the number of units of $\mathbb{Z}/p^m\mathbb{Z}$ is exactly $\varphi(p^m)$, we get that

$$|\operatorname{GL}(n,\mathbb{Z}/p^m\mathbb{Z})| = \varphi(p^m)|\operatorname{SL}(n,\mathbb{Z}/p^m\mathbb{Z})|,$$

and this proves (2.8.2).

Now, let us relate $\operatorname{vol}(Y(N))$ to $\operatorname{vol}(K(N))$. By combining (2.5.6) and (2.8.2), we see that the two latter statements of the proposition are equivalent, and so it is sufficient to prove (2.8.4). By strong approximation, $X(N) = \Gamma(N)\backslash \operatorname{SL}(n,\mathbb{R})/\operatorname{SO}(n)$, with $\Gamma(N)$ the classical principal congruence subgroup of level N. The fundamental domain of $X(N)$ can be viewed as $[\operatorname{SL}(n,\mathbb{Z}) : \Gamma(N)]$ copies of a fundamental domain of $\operatorname{SL}(n,\mathbb{Z})\backslash \operatorname{SL}(n,\mathbb{R})/\operatorname{SO}(n)$, and hence

$$\operatorname{vol}(X(N)) = \operatorname{vol}(\operatorname{SL}(n,\mathbb{Z})\backslash \operatorname{SL}(n,\mathbb{R})/\operatorname{SO}(n))[\operatorname{SL}(n,\mathbb{Z}) : \Gamma(N)].$$

Note that $[\operatorname{SL}(n,\mathbb{Z}) : \Gamma(N)] = |\operatorname{SL}(n,\mathbb{Z}/N\mathbb{Z})|$ essentially by definition of $\Gamma(N)$. By the Chinese remainder theorem and (2.8.7), we have that

$$|\operatorname{SL}(n,\mathbb{Z}/N\mathbb{Z})| = \prod_{p|N} |\operatorname{SL}(n,\mathbb{Z}/p^{v_p(N)}\mathbb{Z})| = \operatorname{vol}(K'(N))^{-1}.$$

This concludes the proof. □

The proposition immediately implies $\operatorname{vol}(X(N)) = O(N^{n^2-1})$. In particular, we get the inequality

$$N^{-(n-1)} = O\left(\operatorname{vol}(X(N))^{-\frac{1}{n+1}}\right),$$

with which we can rewrite the main theorem into its invariant form in Remark 2.1.2.

3 A generalization

In this chapter, we present work in progress on how to generalize the main theorem of Chapter 2 (Theorem 2.1.1) to congruence subgroups in other reductive groups, and give applications.

The motivation for this generalization, other than the inherent interest in analytic torsion, is that for certain families of non-cocompact arithmetic groups, Cheeger-Müller formulas has been established, while no such formula is available for arithmetic subgroups in $\mathrm{SL}(n, \mathbb{R})$, $n \geq 3$, as of yet. As mentioned in the introduction of the paper, the work of Müller-Rochon ([MR20], [MR21], [MR24]) establishes Cheeger-Müller formulas for locally symmetric spaces associated to certain arithmetic groups of $\mathbb{Q}$-rank 1. They combine this with the main result of [MM23], namely the approximation theorem of analytic torsion analogous to (2.1.7) for a large class of arithmetic groups. Combined, this shows that there is exponential growth of torsion in the cohomology of the arithmetic groups when the deficiency is 1. This is discussed further in Section 3.3.

The strategy is to follow the recipe of Chapter 2, and generalize the intermediate results for $\mathrm{GL}(n)$ to nice connected reductive groups. Fortunately, many results were already shown in this generality in Chapter 2. Furthermore, several of the results for $\mathrm{GL}(n)$ from ([MM17], [MM20]) that our proof relied on were generalized appropriately in [MM23]. This leaves only a manageable amount lemmas that needs generalizing. In Section 3.1, we sketch what intermediate results are missing, then go on to prove these results. Then, in Section 3.2, we state and prove a generalization of Theorem 2.1.1, namely Theorem 3.2.2. In the final section, we provide applications to growth of torsion cohomology when G is $\mathrm{SL}(2)$ defined over a number field.

3.1 The ingredients

The recipe

Let us recall the ingredients in the proof of the main theorem of Chapter 2, so we know what to generalize. Here is a very rough sketch:

1. One expresses the relevant adelic quotient as a finite disjoint union of locally symmetric spaces and associates a vector bundle to a representation of the group G, such that one can describe the p-form Laplace operators using the Casimir element (see (2.2.6)). Assuming we have a spectral gap λ, one then shows a bound in the large t-aspect on the trace of the associated heat kernel (Proposition 2.4.2).

2. This assumption is justified by proving the existence of infinitely many λ-strongly acyclic representations of G (Proposition 2.3.2).

3. To allow for truncation, (a) exponential decay for large t of the spectral side of the trace formula for our test function is established (see (2.6.4)). Using the coarse geometric expansion, one shows that (b) for the compactified test function with compactification parameter R and level N, the geometric side of the trace formula receives contribution only from the unipotent part as long as $R \ll \log N$ (Proposition 2.5.3). One furthermore (c) bounds the difference between the spectral side and its compactified analogue (Proposition 2.5.4).

4. By the fine geometric expansion it suffices to analyze certain local weighted orbital integrals and global coefficients. For the archimedean integrals, one gives (a) an upper bound in the large t-aspect (Proposition 2.6.4), and (b) an asymptotic expansion as $t \to 0$ (Proposition 2.6.2). For the non-archimedean parts, one provides (c) an upper bound with a factor some negative power of the level (Lemma 2.5.6). Finally, (d) a bound is given on the global coefficients (Lemma 2.5.5).

One then follows the strategy of Section 2.7 to obtain Theorem 2.1.1. In essence, if one can do all these steps for a connected reductive group, one would obtain the improved asymptotics for analytic torsion for this group. Some of the steps have already been done in the desired generality. In particular, steps (1) and (2) were completed in Chapter 2 for any connected reductive group G and K_f any neat open compact subgroup of $G(\mathbb{A}_{\text{fin}})$.

The remaining results were presented in the setting of $G = \mathrm{GL}(n)$ and $K_f = K(N)$, meriting at least some remark as to how to generalize. Fortunately, several of those have been generalized for us in [MM23]. Assume G is a reductive group satisfying two conditions called (TWN) and (BD), see [FLM15], Definition 5.2 and Definition 5.9, respectively. $K_f \subseteq G(\mathbb{A}_f)$ is any open compact subgroup in some fixed open compact K_0.

- The lemma in ($3a$), i.e. the bound (2.6.4), is generalized in ([MM23], Proposition 12.1) to the setting above.
- Regarding ($3c$), Proposition 2.5.4 is suitably generalized in ([MM23], Proposition 12.3).
- The same is true for ($4b$), where Proposition 2.6.2 may be replaced by ([MM23], Proposition 7.2 combined with (11.21)).

This leaves ($3b$), ($4a$), ($4c$), and ($4d$). In the following, we will provide proofs for generalizations of the steps ($3b$), ($4a$), and ($4c$). The step ($4d$), i.e. the log-bound on global coefficients, is certainly conjectured to hold in general, but only known for specific groups (for $\mathrm{GL}(n)$ see [Mat15]). Therefore, we will need to either assume this conjecture or make restrictions to which levels N we may vary over. This is discussed in detail in Section 3.2.

Restricting to the unipotent contribution

In this subsection, we prove a version of Proposition 2.5.3 for connected reductive groups. In fact, by embedding correctly into $\mathrm{GL}(n)$, we can reduce the general statement to the one for $\mathrm{GL}(n)$. We state the desired result in a moment. For now, we provide the setup. Let G be a connected reductive group defined over $\mathbb{Q}$ with neat open compact subgroup $K_f = \prod_p K_p \subseteq G(\mathbb{A}_f)$. Take a faithful $\mathbb{Q}$-rational representation $\rho : G \to \mathrm{GL}(V)$ with a lattice $\Lambda \subseteq V$ such that K_f is the stabilizer in $G(\mathbb{A}_f)$ of $\hat{\Lambda} := \hat{\mathbb{Z}} \otimes \Lambda \subseteq \mathbb{A}_f \otimes V$. For $N \in \mathbb{N}$ we define

$$K(N) = \{g \in G(\mathbb{A}_f) \mid \forall v \in \mathbb{A}_f \otimes V : \ \rho(g)v \equiv v \mod N\hat{\Lambda}\}. \tag{3.1.1}$$

Then $K(N) \subseteq K_f$ is a factorizable open compact subgroup in $G(\mathbb{A}_f)$. We see that these generalize the open compact subgroups in the case of $G = \mathrm{GL}(n)$ as defined in the paper (see 2.5.5), by picking ρ the standard representation and Λ the $\mathbb{Z}$-span of the standard basis. For convenience, we denote these by $K_{\mathrm{GL}(n)}(N)$. Furthermore, given $\Lambda \subseteq V$ we may identify $\mathrm{GL}(V)$ with $\mathrm{GL}(n)$ by choosing a $\mathbb{Z}$-basis of Λ as a basis of V, and under this identification, $K(N)$ maps into $K_{\mathrm{GL}(n)}(N)$ under the embedding ρ.

Fix a Cartan involution $\theta : G(\mathbb{R}) \to G(\mathbb{R})$ and let K be the connected component of the identity of the θ-fixpoints. Recall that we presented the coarse geometric expansion of the Arthur-Selberg trace formula in Section 1.2. We let $h_t^{\tau,p} : G(\mathbb{R})^1 \to \mathbb{R}$ be the trace of the heat kernel as defined in (2.2.9) and $\chi(N)$ the normalized characteristic function on $G(\mathbb{A}_f)$ of $K(N)$, i.e.

$$\chi(N) := \frac{1_{K(N)}}{\mathrm{vol}(K(N))}.$$

We take $f = h_t^{\tau,p} \otimes \chi(N) \in \mathcal{C}(G(\mathbb{A})^1, K(N))$ as our test function. Then the geometric side of the Arthur-Selberg trace formula is well-defined for this function, and we have a coarse geometric expansion by ([FL16]). We write J_{unip} for the term associated to the equivalence class of unipotent elements. Set $r(g) = d(K, gK)$ to be the geodesic distance on $G(\mathbb{R})^1/K$ of $g \in G(\mathbb{R})^1$ from the identity, and let $\varphi_R : G(\mathbb{R})^1 \to [0,1]$ be any smooth function identically 1 on $B(R) = \{g \in G(\mathbb{R})^1 \mid r(g) < R\}$ and identically 0 outside $B(R+\varepsilon)$ for some $\varepsilon > 0$ small. We then define the compactification of $h_t^{\tau,p}$ with parameter R as

$$h_{t,R}^{\tau,p}(g) = \varphi_R(g) h_t^{\tau,p}(g), \quad g \in G(\mathbb{R})^1. \tag{3.1.2}$$

Consider the embedding $G(\mathbb{R})^1 \to \mathrm{GL}(n,\mathbb{R})^1$ induced by ρ as above. By possibly composing with an inner automorphism of $\mathrm{GL}(n,\mathbb{R})^1$, we can ensure the following properties (see also [MM23], p. 24): $\rho(K) \subseteq \mathrm{O}(n)$, and for $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of $G(\mathbb{R})^1$, then $\mathfrak{k}$ (resp. $\mathfrak{p}$) is embedded into the skew-symmetric (resp. symmetric) matrices of $\mathfrak{gl}(n,\mathbb{R})$ under the induced map. Finally, the Frobenius norm on the embedding of $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{n},\mathbb{R})$ coincides with the norm induced by the Killing form. By the choice of embedding, we have

$$d(gK, K) = d_{\mathrm{GL}(n)}(\rho(g)\mathrm{O}(n), \mathrm{O}(n)) \tag{3.1.3}$$

where $d_{\mathrm{GL}(n)}$ is the geodesic distance on $\mathrm{GL}(n,\mathbb{R})^1/\mathrm{O}(n)$.

Proposition 3.1.1. *For N large enough and $R \leq C_n \log N$, we have that*

$$J_{\text{geo}}(h_{t,R}^{\tau,p} \otimes \chi(N)) = J_{\text{unip}}(h_{t,R}^{\tau,p} \otimes \chi(N)).$$

Proof. Let $\mathfrak{o} \in \mathcal{O}$ be an equivalence class with representative γ as defined just above (1.2.17), and let $f \in \mathcal{C}(G(\mathbb{A})^1, K(N))$. Considering the formulations of the distributions $J_{\mathfrak{o}}$ in Theorem 1.2.10 and the expression of the orbital integrals in (1.2.20), we see that $J_{\mathfrak{o}}(f) = 0$ if no conjugate of γ in $G(\mathbb{A})$ intersects the support of f.

Let now $\mathfrak{o} \neq \mathfrak{o}_{\text{unip}}$ and assume $R \leq C_n \log N$ as in Proposition 2.5.3. The orbit of γ under $G(\mathbb{A})$ maps into the orbit of $\rho(\gamma)$ in $\mathrm{GL}(n,\mathbb{A})$. Furthermore, under ρ, the ball $B(R) \subseteq G(\mathbb{R})^1$ of radius R maps into $B_{\mathrm{GL}(n)}(R)$ the ball of radius R under geodesic distance in $\mathrm{GL}(n,\mathbb{R})^1$ by (3.1.3), and $K(N)$ maps into $K_{\mathrm{GL}(n)}(N)$ by construction as detailed above.

It was shown in Proposition 2.5.3 that the $\mathrm{GL}(n,\mathbb{A})$-orbit of any such $\rho(\gamma)$ does not intersect $B_{\mathrm{GL}(n)}(R)K_{\mathrm{GL}(n)}(N)$ for N large enough and the chosen R. By the above, this implies that the $G(\mathbb{A})$-orbit of γ does not intersect $B(R)K(N)$, which is the support of $h_{t,R}^{\tau,p} \otimes \chi(N)$. In particular, $J_{\mathfrak{o}}(h_{t,R}^{\tau,p} \otimes \chi(N)) = 0$. By the coarse geometric expansion (1.2.18), this completes the proof. □

Jacobson-Morozov parabolic subgroups

We present some background on Jacobson-Morozov parabolic subgroups. Classical references are Jacobson ([Jac79], Cpt. III, §11) and Collingwood-McGovern ([CM93], Cpt. 3), see also ([Art88b]).

Theorem 3.1.2 (Jacobson-Morozov)**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic* 0*. If $X \in \mathfrak{g}$ is a non-zero nilpotent element, there exists an $\mathfrak{sl}_2$-triple for $\mathfrak{g}$ whose nilpositive element is X.*

We will also use the name *standard triples* for $\mathfrak{sl}_2$-triples. Note that the result immediately extends to reductive Lie algebras, as any nilpotent element of a reductive Lie algebra $\mathfrak{g}$ lies in $[\mathfrak{g}, \mathfrak{g}]$, which is semisimple.

Definition 3.1.3. Let $\mathfrak{g}$ be a reductive Lie algebra. A *parabolic subalgebra* is any subalgebra containing a Borel subalgebra, the latter defined as a maximal solvable subalgebra. For $\mathfrak{h}$ a Cartan subalgebra with roots Φ and a choice of positive roots Φ^+, the associated Borel subalgebra is $\mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$.

To a subset $\Theta \subseteq \Delta$, with Δ the simple roots of Φ^+, set $\langle \Theta \rangle \subseteq \Phi$ the subroot system generated by Θ, and set $\langle \Theta \rangle^+ := \langle \Theta \rangle \cap \Phi^+$. One can associate a parabolic subalgebra as follows:

$$\mathfrak{p}_\Theta := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+ \cup \langle \Theta \rangle} \mathfrak{g}_\alpha.$$

Lemma 3.1.4. *In the notation above, $\mathfrak{p}$ is a parabolic subalgebra containing $\mathfrak{b}$ if and only if it is of the form $\mathfrak{p}_\Theta$ for some $\Theta \subseteq \Delta$. Every parabolic subalgebra is conjugate to $\mathfrak{p}_\Theta$ for some Θ. There is a decomposition $\mathfrak{p}_\Theta = \mathfrak{l} \oplus \mathfrak{n}$, where*

$$\mathfrak{l} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \langle \Theta \rangle} \mathfrak{g}_\alpha, \quad \mathfrak{n} = \sum_{\alpha \in \Phi^+ \setminus \langle \Theta \rangle^+} \mathfrak{g}_\alpha,$$

where $\mathfrak{l}$, called the Levi subalgebra, is reductive, and $\mathfrak{n}$ is the nilradical of $\mathfrak{p}_\Theta$.

Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and take any nonzero nilpotent $X \in \mathfrak{g}$. Then by Jacobson-Morozov, we have standard triple $\{H, X, Y\} \subseteq \mathfrak{g}$. Let $\mathfrak{a} \cong \mathfrak{sl}(2)$ be the subalgebra of $\mathfrak{g}$ spanned by $\{X, Y, Z\}$. Acting by the adjoint representation, $\mathfrak{g}$ decomposes as a direct sum of irreducible submodules. By the representation theory of $\mathfrak{sl}(2)$, we have integral eigenvalues on the irreducible submodules, thus on all of $\mathfrak{g}$. We write $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ for this decomposition, and define

$$\mathfrak{q} = \bigoplus_{i \geq 0} \mathfrak{g}_i, \quad \mathfrak{l} = \mathfrak{g}_0, \quad \mathfrak{u} = \bigoplus_{i \geq 1} \mathfrak{g}_i.$$

Then $\mathfrak{q}$ is a parabolic subalgebra, sometimes denoted $\mathfrak{q}_X$ and called the *Jacobson-Morozov parabolic* of X, and it has Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Sometimes we will use the same name to refer to the parabolic subgroup with $\mathfrak{q}$ as its Lie algebra. Although it is constructed using $\{H, X, Y\}$, it is uniquely defined by X.

Explicit local weighted orbital integrals

In this section we present explicit formulas for local weighted orbital integrals attached to unipotent conjugacy classes, essentially elaborating on Section 1.2. It contrasts the explicit orbital integrals in Section 2.5 (e.g. (2.5.11)) as we do not in general have access to Richardson parabolics, and instead must use Jacobson-Morozov parabolics. The explicit formulas will be used both in the archimedean and non-archimedean setting in the following sections.

We will assume the notation of Section 1.2 unless otherwise mentioned. In particular, G will be a connected reductive group over $\mathbb{Q}$ and $\nu \in \{p, \infty\}$ a place of $\mathbb{Q}$. We let K_ν be an admissible maximal compact subgroup of $G(\mathbb{Q}_\nu)$ as defined in ([Art81], §1). Let M be a Levi subgroup of G with Lie algebra $\mathfrak{m}$, and take $\gamma \in M(\mathbb{Q}_\nu)$ a representative of a unipotent conjugacy class, i.e. $\gamma_{ss} = 1$. For $P \in \mathcal{F}(M)$ we write $P = M_P N_P$ for its Levi decomposition with $M \subseteq M_P$.

By the Jacobson-Morozov theorem there exists an $\mathfrak{sl}_2$-triple (X, Y, H) in $M(\mathbb{Q}_\nu)$ such that $\gamma = \exp X$. Let $\mathfrak{m}(\mathbb{Q}_\nu) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{m}_i$ be the associated eigenspace decomposition, and define

$$Z_\nu := \left\{ p^{-1}\gamma p \mid p \text{ in the normalizer in } M(\mathbb{Q}_\nu) \text{ of } \bigoplus_{i\geq 0} \mathfrak{m}_i \right\}.$$

It is known that Z_ν is an open subset of $\exp \mathfrak{u}_2$, for $\mathfrak{u}_k := \bigoplus_{i\geq k} \mathfrak{m}_i$ (see [Art88b], p. 246). Fix $Q \in \mathcal{F}(M)$ with Levi decomposition $Q = LN_Q$, and take some $R \in \mathcal{P}(M)$ contained in L. Define $\Pi_\nu^Q = Z_\nu N_R(\mathbb{Q}_\nu)$. With Haar measures on K_ν, $N_P(\mathbb{Q}_\nu)$, and the product of Haar measures on Π_ν^Q, the proof of ([Art88b], Corollary 6.2) gives the local weighted orbital integrals as

$$J_M(f,\gamma) = \sum_{Q\in\mathcal{F}(M)} c(Q,\gamma) \int_{K_\nu} \int_{N_Q(\mathbb{Q}_\nu)} \int_{\Pi_\nu^Q} f(k^{-1}\pi n k) v_M^Q(1,\pi) |J_\nu^Q(\pi)|_\nu^{\frac{1}{2}} d\pi dn dk.$$

Our integral is significantly simpler than that of [Art88b] due to the fact that we assume γ is unipotent. In the above, J_ν^Q is a polynomial on $\mathfrak{u}_2(\mathbb{Q}_\nu)$ of $\deg J_\nu^Q = \dim \mathfrak{m}_1$ defined over $\mathbb{Q}_\nu$ (see [Rao72]), and $v_M^Q(1,\pi)$ is a weight function defined in ([Art88b], §5), similar to that of the preliminary section (see (1.2.22)). In particular, the weight function satisfies log-homogeneity, i.e. it is bounded by some polynomial in the norm of their entries.

To simplify the integral, we will instead integrate over the associated Lie algebras. Let $\mathfrak{n}_Q$ be the Lie algebra of $N_Q(\mathbb{Q}_\nu)$, and similarly for $\mathfrak{n}_R$. Let $\mathfrak{v}_Q = \mathfrak{n}_Q \oplus \mathfrak{u}_2 \oplus \mathfrak{n}_R$, which is a nilpotent Lie algebra. Using that Z_ν is dense in $\exp \mathfrak{u}_2$, we can extend both the weight function and the polynomial J_ν to $\exp(\mathfrak{n}_{\mathbb{Q}} \oplus \mathfrak{u}_2 \oplus \mathfrak{n}_R)$ by projecting to the relevant subspace.

$$J_M(f,\gamma) = \sum_{Q\in\mathcal{F}(M)} c(Q,\gamma) \int_{K_\nu} \int_{\mathfrak{v}_Q} f(k^{-1}e^X k) v_M^Q(1,e^X) |J_\nu^Q(X)|_\nu^{\frac{1}{2}} dX dk. \tag{3.1.4}$$

We can relate the dimensions of these Lie algebras with the dimensions of orbits of X. Let $\mathcal{O}_X$ be the orbit of X in H. Then ([CM93], Lemma 4.1.3) yields

$$\dim \mathcal{O}_X = \dim \mathfrak{m} - \dim \mathfrak{m}_0 - \dim \mathfrak{m}_1 = 2 \dim \mathfrak{u}_2 + \dim \mathfrak{m}_1,$$

where we have used that $\dim \mathfrak{m}_k = \dim \mathfrak{m}_{-k}$ for every $k \in \mathbb{Z}$. Furthermore, by ([CM93], Theorem 7.1.1), applied twice using that $\operatorname{Ind}_M^G \mathcal{O}_X = \operatorname{Ind}_L^G \operatorname{Ind}_M^L \mathcal{O}_X$, we have that

$$\begin{aligned} \dim \operatorname{Ind}_M^G \mathcal{O}_X &= \dim \operatorname{Ind}_M^L \mathcal{O}_X + 2 \dim \mathfrak{n}_Q \\ &= \dim \mathcal{O}_X + 2 \dim \mathfrak{n}_R + 2 \dim \mathfrak{n}_Q \\ &= 2(\dim \mathfrak{u}_2 + \dim \mathfrak{n}_R + \dim \mathfrak{n}_Q) + \dim \mathfrak{m}_1 \\ &= 2 \dim \mathfrak{v}_Q + \dim \mathfrak{m}_1. \end{aligned}$$

Let us end this section by giving an explicit description of the weight functions.

Lemma 3.1.5 ([MM23], Lemma 5.4)**.** *There exists constants $r, t \geq 0$ and polynomials $q_1, \ldots, q_r : \mathfrak{v}_Q \to \mathbb{Q}_\nu$ and complex polynomials $R_1, \ldots, R_t$ in r many variables such that for any $X \in \mathfrak{v}_Q$ and $s > 0$, there exists some $C > 0$ such that*

$$v_M^Q(1, I + sX) = \sum_{i=0}^{t} \log(s)^i R_i(\log|q_1(X)|_\nu, \ldots, \log|q_r(X)|_\nu).$$

Note that the proof of ([MM23], Lemma 5.4) was shown in the case of $\nu = \infty$, but it holds in general with only minor modifications. It follows immediately that we have an upper bound on the weight functions. There exists some $k > 0$ such that

$$|v_M^Q(1, e^X)| \leq C(1 + \log\|e^X\|)^k. \tag{3.1.5}$$

The constant $C > 0$ can be chosen to be independent of M and Q.

Large t asymptotics of archimedean orbital integrals

We adopt the setup of the previous subsection. In particular G is a reductive group over $\mathbb{Q}$. We will further assume that it is quasi-split, with $\mathbb{Q}$-split center Z_G. Set r to be the split semisimple rank of G, i.e. $r = \dim \mathfrak{a}_0^G$. Throughout this subsection, we will focus on the infinite place $\nu = \infty$.

In this section, we prove a generalization of Proposition 2.6.4. Recall that it is essentially an application of Proposition 2.4.2 and Proposition 2.4.6, and these will be the main input in the generalization as well. Armed with the explicit formula for local orbital integrals from the previous subsection, one only needs to ensure that the compactification behaves nicely. This is tackled as follows.

we choose an embedding of $G(\mathbb{R})^1$ in $\mathrm{GL}(n,\mathbb{R})^1$ as in the paragraph before (3.1.3). It follows that the geodesic distance function on $\mathrm{GL}(n,\mathbb{R})^1/\mathrm{O}(n)$ agrees with the geodesic distance function on $G(\mathbb{R})^1/K$.

Proposition 3.1.6. *Let $\gamma \in M(\mathbb{R})$ be unipotent, and assume $t > 1$. Then there exists some $C, c' > 0$ such that*

$$|J_M^G(h_{t,R}^{\tau,p}, \gamma)| \leq Ce^{-(\lambda - c')t}. \tag{3.1.6}$$

The proof of this result follows the proof of Proposition 2.6.4 closely.

Proof. By the formula in (3.1.4) and by using the bi-K-invariance of $h_{t,R}^{\tau,p}$, we are reduced to considering

$$J_M^G(h_{t,R}^{\tau,p}, \gamma) = \sum_{Q \in \mathcal{F}(M)} c(Q,\gamma) \int_{\mathfrak{v}_Q} h_{t,R}^{\tau,p}(e^X) v_M^Q(1, e^X) |J_\infty^Q(X)|^{\frac{1}{2}} dX. \tag{3.1.7}$$

Let $A, c, C > 0$ be as in Proposition 2.4.6, and write

$$\mathfrak{v}_Q = \bigcup_{k=0}^{\infty} D(k),$$

where

$$D(0) := \{X \in \mathfrak{v}_Q \mid r(e^X) < A\},$$
$$D(k) := \{X \in \mathfrak{v}_Q \mid A + k \leq r(e^X) < A + k + 1\}, \quad k \geq 1.$$

We will decompose the orbital integral (3.1.7) into a sum of integrals using this disjoint union. Throughout the following, we use the bound (3.1.5), which implies the bound

$$|v_M^Q(1, e^X)| \leq C \left(1 + r(e^X)\right)^b.$$

We will also use that $J_\infty^Q(X)$ is bounded by some polynomial in $\|X\|$, and as $\|X\|$ is bounded by $C(1 + \|e^X\|)$ for some constant $C > 0$ for all unipotent matrices (see e.g. [MM23], Lemma 6.2), it is furthermore bounded by a polynomial in $e^{r(e^X)}$. The integral over $D(0)$ is then handled using and Proposition 2.4.2:

$$\left|\int_{D(0)} h_{t,R}^{\tau,p}(e^X) v_M^Q(1,e^X)|J_\infty^Q(X)|^{\frac{1}{2}} dX\right| \leq C' e^{-\lambda t} \int_{D(0)} \left(1+r(e^X)\right)^b \left(1+e^{r(e^X)}\right)^{\frac{1}{2}} dX$$
$$\leq C' e^{-\lambda t} \left(1+A\right)^b \left(1+e^A\right)^{\frac{1}{2}} \operatorname{vol}(D(0)).$$

As $D(0)$ is compact, its volume only depending on A, we see that the above is $C_A e^{-\lambda t}$, for some constant $C_A > 0$ only depending on A.

By the bound on $\|X\|$ in terms of $\|e^X\|$ above, we may bound the volume of $D(k)$ by the volume of

$$\{X \in \mathfrak{v}_Q \mid \|X\| \leq C(1+e^k)\}$$

As this is a ball in Euclidean space, we get that

$$\operatorname{vol}(D(k)) \leq C(1+e^k)^{\dim \mathfrak{v}_Q}.$$

We may now estimate the integral over $D(k)$. First, use (2.4.6) to write

$$\left|\int_{D(k)} h_{t,R}^{\tau,p}(e^X) v_M^Q(1,e^X)|J_\infty^Q(X)|^{\frac{1}{2}} dX\right| \leq C' e^{-\lambda t} \left|\int_{D(k)} e^{-c\frac{r(e^X)^2}{t}} v_M^Q(1,e^X)|J_\infty^Q(X)|^{\frac{1}{2}} dX\right|.$$

By the above, the latter integral can then be bounded by

$$C' e^{-c\frac{(A+k-1)^2}{t}} \left(1+(A+k)\right)^b \left(1+e^{A+k}\right)^d \operatorname{vol}(D(k))$$
$$\leq C'_A e^{-c\frac{k^2}{t}} e^{c_1 k},$$

where $C'_A, d > 0$ are some constants and $c_1 = \dim \mathfrak{v}_Q + d + \varepsilon$ for some (any) small $\varepsilon > 0$. It now follows by the end of the proof of Proposition 2.6.4 that the sum of the integrals over $D(k)$, $k \geq 1$, converges and is bounded by

$$C_2 e^{-(\lambda - c')t}$$

for some constants $C_2, c' > 0$, with c' only depending on the group G. This finishes the proof. □

Explicit bounds on non-archimedean orbital integrals

Our setup is the following. We will stick to the notation of $K(N)$ and $\chi(N)$ as defined in (3.1.1). Write $K(N) = \prod_p K_p(p^{\nu_p(N)})$ for the factorization of $K(N)$ such that $K_p(p^{\nu_p(N)})$ is mapped into the kernel of the quotient map $\mathrm{GL}(n, \mathbb{Z}_p) \to \mathrm{GL}(n, \mathbb{Z}_p/p^{\nu_p(N)}\mathbb{Z}_p)$ under the embedding $G \to \mathrm{GL}(n)$. We let M be a Levi subgroup of G, and $Q \in \mathcal{F}(M)$ with Levi decomposition $P_1 = LN_Q$. Let $\gamma = \prod_\nu \gamma_\nu$ be a representative of a unipotent conjugacy class $\mathcal{O}_\gamma$ in $M(\mathbb{Q})$.

We wish to generalize Lemma 2.5.6. This lemma is proven in ([MM20], §9) and we follow their method. In particular, using the fine geometric expansion we write the non-archimedean weighted orbital integrals as a product of p-adic orbital integrals, hence it is sufficient to prove that

$$|J_M^L((1_{K_p(p^{\nu_p(N)})})_{P_1}, \gamma_p)| \leq Cp^{-\frac{\nu_p(N)\dim \mathcal{O}_\gamma^G}{2}}(1 + \log(p^{\nu_p(N)}))^r$$

for some $r, C > 0$ depending only on G. From now on we fix p and write $s = \nu_p(N)$. Using the explicit formula for local orbital integrals (3.1.4) on the reductive group L, unfolding the definition of f_Q as given in (1.2.23), and using the fact that $K(N)$ is a normal subgroup, we may write

$$J_M^L((1_{K_p(p^s)})_{P_1}, \gamma) = \sum_{Q \in \mathcal{F}^L(M)} c(Q, \gamma) \int_{\mathfrak{v}_Q} 1_{K_p(p^s)}(e^X) v_M^Q(1, e^X) |J_p^Q(X)|_\nu^{\frac{1}{2}} \, dX. \tag{3.1.8}$$

Considering the characteristic function, we may restrict to computing this integral over $\{X \in v_Q \mid e^X \in K_p(p^s)\}$, which is isomorphic to $p^s\mathbb{Z}_p^{\dim \mathfrak{v}_Q}$. Using the bound on weight functions (3.1.5) and that J_p^Q is homogeneous of degree $\dim \mathfrak{m}_1$, we may estimate the integral on the right hand side above as

$$\begin{aligned}
\left| \int_{\mathfrak{v}_Q} 1_{K_p(p^s)}(e^X) v_M^Q(1, e^X) |J_p^Q(X)|_p^{\frac{1}{2}} \, dX \right| &\leq C \int_{p^s\mathbb{Z}_p^{\dim \mathfrak{v}_Q}} (1 + \log\|e^X\|)^k \|X\|^{\frac{\dim \mathfrak{m}_1}{2}} \, dX \\
&\leq C' p^{-s\dim \mathfrak{v}_Q} (1 + \log(p^{-s}))^k p^{-\frac{s\dim \mathfrak{m}_1}{2}} \\
&\leq C' p^{-\frac{s\dim \mathrm{Ind}_M^G \mathcal{O}_\gamma}{2}} (1 + \log(p^{-s}))^k.
\end{aligned}$$

Here we also used the description of the dimension of orbits provided above Lemma 3.1.5. In the case that $L = M$, the weight function is identically 1, which gives a bound

$$\left| \int_{\mathfrak{v}_Q} 1_{K_p(p^s)}(e^X) v_M^Q(1, e^X) |J_p^Q(X)|_p^{\frac{1}{2}} \, dX \right| \leq C_p^J p^{-\frac{s\dim \mathrm{Ind}_M^G \mathcal{O}_\gamma}{2}}.$$

Here C_p^J is the p-adic valuation of some constant depending only on J_p^Q. In fact, this polynomial may be globally defined: Note that γ_p is just $\gamma \in M(\mathbb{Q})$ embedded into $M(\mathbb{Q}_p)$ through the inclusion $\mathbb{Q} \to \mathbb{Q}_p$. Thus, if we let $\mathfrak{m}(\mathbb{Q})$ be the $\mathbb{Q}$-points of the Lie algebra of M, and let $X \in \mathfrak{m}(\mathbb{Q})$ such that e^X represents the same orbit as γ, the Jacobson-Morozov theorem tells us that X is part of an $\mathfrak{sl}_2$-triple, and we have a full decomposition of $\mathfrak{m}(\mathbb{Q})$ into eigenspaces as in the local case. Picking a basis for the eigenspaces $\mathfrak{m}_1$ and $\mathfrak{m}_2$ globally, which is then also a basis in the local cases, we see from the definition in ([Rao72], Lemma 2) that the polynomial J_p^Q is actually defined algebraically over $\mathbb{Q}$, and is thus global. This implies that the mentioned constant is a rational number, and hence one may pick $C_p^J = 1$ for all but finitely many p. Importantly, those p do not depend on N.

With these inequalities established, the bound on the entire non-archimedean part now follows: Let $S(N)$ be the collection of primes dividing N. By iterative use of the decomposition formula (1.2.24), we get that

$$J_M(1_{K(N)}, \gamma_f) = \sum_{\underline{L} \in \mathcal{L}(M)^{|S(N)|}} d_M(\underline{L}) \prod_{p \in S(N)} J_M^{L_p}((1_{K(N)})_{P_p}, \gamma_p),$$

where $\underline{L} = (L_p)_{p \in S(N)}$ runs over all such tuples of Levi subgroups, with $P_p = L_p N_p$ a certain parabolic subgroup of G chosen as in ([Art05], §§17-18). Here $d_M(\underline{L})$ are certain constants which are non-zero only if at most $\dim \mathfrak{a}_M$-many elements of $\underline{L}$ is not equal to M (see [MM20], Lemma 8.2). This implies that the number of such $\underline{L}$ that actually contributes is bounded by $|S(N)|^{\dim \mathfrak{a}_M}$. As $|S(N)| \leq 2 \log N$, collecting the identities and bounds in this section amounts to the following proposition.

Proposition 3.1.7. *There exists constants $c, m > 0$ independent of N such that*

$$\left|J_M(1_{K(N)}, \gamma_f)\right| \leq c N^{-\frac{\dim \operatorname{Ind}_M^G \mathcal{O}_\gamma}{2}} (1 + \log N)^m. \tag{3.1.9}$$

Remark 3.1.8. Later, we will isolate the orbital integral related to the trivial orbit $\{1\}$ in G, while needing to bound the remaining terms. As there are only finitely many unipotent orbits $\mathfrak{u} \in \mathcal{U}_M(\mathbb{Q})$ for every M, and only finitely many standard Levi subgroups M, one can make the bound above independent of M by picking the exponent to be half the minimal dimension of the non-trivial unipotent orbits. Hence, as ad-hoc notation, we set

$$k(G) \coloneqq \frac{1}{2} \min_{(M,\mathfrak{u}) \neq (G,\{1\})} \dim \operatorname{Ind}_M^G \mathfrak{u}. \tag{3.1.10}$$

This constant is strictly positive, and it is computable in terms of root systems, see e.g. ([CM93], §4.3). For $G = \mathrm{SL}(n)$ and $G = \mathrm{GL}(n)$ over $\mathbb{Q}$, we have $k(G) = n - 1$.

3.2 The proof

In this section, we will prove a generalization of Theorem 2.1.1. It is a sharpening of Theorem 1.5 of [MM23] under certain additional assumptions. Let us recall the setup.

Let G be a connected reductive quasi-split group with $G(\mathbb{R})$ noncompact, and let $K_f \subseteq G(\mathbb{A}_f)$ be a neat open compact subgroup. Define $K(N) \subseteq K_f$ as in (3.1.1) for $N \in \mathbb{N}$, $N \geq 3$. Fix also a maximal compact subgroup K of $G(\mathbb{R})^1$. We let $\tilde{X}$ be the globally symmetric space associated to G, and $X(N)$ the adelic locally symmetric space associated to $K(N)$ in G as defined in (2.2.2). Write $d = \dim \tilde{X}$. Let (τ, V_τ) be a finite-dimensional complex representation of $G(\mathbb{R})^1$. We define the Laplace operator on E_τ-valued p-forms $\Delta_{p,N}(\tau)$ as in Section 2.2, and $\tilde{\Delta}_p(\tau)$ its lift to the universal covering $\tilde{X}$. Let $h_t^{\tau,p} \in \mathcal{C}^q(G(\mathbb{R})^1)$ be the trace of the associated heat kernel as in (2.2.9).

We recall the definition of analytic torsion from [MM23]. It is a direct generalization of the one we gave in (2.5.4), which was stated for $G = \mathrm{GL}(n)$ or $G = \mathrm{SL}(n)$. First, we defined the regularized trace of the heat operator as

$$\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_{p,N}(\tau)}) := J_{\mathrm{geo}}(h_t^{\tau,p} \otimes \chi(N)).$$

From this, one defines the spectral zeta functions

$$\zeta_p(s,\tau) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_{p,N}(\tau)}) t^{s-1} dt.$$

Due to the small t expansion and the large t bound of $\mathrm{Tr}_{\mathrm{reg}}(e^{-t\Delta_p(\tau)})$ given in ([MM23], Theorem 1.1, resp. Theorem 1.2), this zeta function converges absolutely in some halfplane $\Re(s) \gg 0$ and admits a meromorphic continuation to all of $\mathbb{C}$. Let $\mathrm{FP}_{s=a}(f)$ be the zeroth Laurent coefficient at $a \in \mathbb{C}$ of the meromorphic function f on $\mathbb{C}$. Thus, the analytic torsion $T_{X(N)}(\tau)$ of $X(N)$ may be defined as

$$\log T_{X(N)}(\tau) = \frac{1}{2} \sum_{p=0}^{d} (-1)^p \, p \; \mathrm{FP}_{s=0} \left(\frac{\zeta_p(s,\tau)}{s} \right).$$

In constrast to [MM23], in the proof of our theorem we need to handle the global coefficients $a^M(S_N, \mathfrak{u})$ arising in the fine geometric expansion. This causes difficulty as no general bound is known for them. They depend on the primes dividing N, and so might grow, possibly rapidly, as N goes to infinity. They are believed to grow logarithmically, as already established for $\mathrm{GL}(n)$ ([Mat15]). We state a version of the conjecture here. Let S_N be the set of primes dividing N together with ∞.

Conjecture 3.2.1. *There exists constants $b, c > 0$ such that for all N, M and $\mathfrak{u} \in \mathcal{U}_M(\mathbb{Q})$ we have*

$$\left| a^M(S_N, \mathfrak{u}) \right| \leq c(1 + \log N)^b.$$

For the main theorem, we either have to assume the conjecture, or assume that N only varies over a set of positive integers with all prime divisors lying in some finite fixed set of primes, rendering the global coefficients bounded by a constant. As non-standard notation, let us call such a set *prime-fixed*. A good example of a prime-fixed set is $\{m^k \mid k \in \mathbb{N}\}$ given any $m \in \mathbb{N}$. It is interesting to compare this to the property $K_j \underset{S}{\to} 1$ defined in ([MM23], (1.13)).

We can now state our theorem. Define $T^{(2)}_{X(N)}(\tau)$ to be the L^2-torsion of $X(N)$ as in [Lot92].

Theorem 3.2.2. *Let τ be an irreducible and λ-strongly acyclic representation of $G(\mathbb{R})$*[1]*, for some $\lambda > 0$ only depending on G. Assume that G satisfies the properties (TWN) and (BD) (see [FLM15]). Assume either Conjecture 3.2.1 or that N varies over a prime-fixed set. Then there exists some $a > 0$ such that*

$$\log T_{X(N)}(\tau) = \log T^{(2)}_{X(N)}(\tau) + O(\mathrm{vol}(X(N))N^{-k(G)}(\log N)^a)$$

as N tends to infinity.

The rest of this section is a proof of this theorem. It is essentially step by step the same argument as the proof of Theorem 2.5.1 given in Section 2.7, now using the generalized results of Section 3.1.

The first reduction

Assume the setup of the theorem. Firstly, it follows from ([MM23], Proposition 12.1) that there exists $C > 0$ such that for any $\varepsilon > 0$ and $t \geq 1$ we have

$$|J_{\mathrm{spec}}(h_t^{\tau,p} \otimes \chi(N))| \leq Ce^{-\lambda(1-\varepsilon)t}\,\mathrm{vol}(X(N)) \tag{3.2.1}$$

The fact that their constant c can be replaced by $\lambda(1-\varepsilon)$ follows as in Remark 2.6.1, since the proof of ([MM23], Proposition 12.1) relies on the proof of ([MM20], Lemma 6.3). With this we start the analysis of

$$\mathrm{FP}_{s=0}\left(\frac{\zeta_p(s,\tau)}{s}\right) = \mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^\infty J_{\mathrm{geo}}(h_t^{\tau,p} \otimes \chi(N))t^{s-1}dt\right). \tag{3.2.2}$$

Note that $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1, hence the fraction is holomorphic at 0. Let $T \geq 1$. We split up the integral, and use that $\mathrm{FP}_{s=a}(f) = f(a)$ if f is holomorphic at $s = a$, to express the above as

$$\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J_{\mathrm{geo}}(h_t^{\tau,p} \otimes \chi(N))t^{s-1}dt\right) + \int_T^\infty J_{\mathrm{geo}}(h_t^{\tau,p} \otimes \chi(N))t^{-1}dt. \tag{3.2.3}$$

By the Arthur-Selberg trace formula, any bound on J_{spec} also applies to J_{geo}. In particular, applying (3.2.1) to the second term above yields the following lemma.

Lemma 3.2.3. *There exists $C > 0$ such that for any $\varepsilon > 0$, we have*

$$\left| \int_T^\infty J_{\text{geo}}(h_t^{\tau,p} \otimes \chi(N)) t^{-1} dt \right| \leq C e^{-\lambda(1-\varepsilon)T} \operatorname{vol}(X(N)).$$

That is the second term of (3.2.3) handled, and we return to the first term. We focus on the integral. We want to replace the test function with its compactified version defined in (3.1.2). This is possible due to the following bound.

Lemma 3.2.4 ([MM23], Proposition 12.3)**.** *There exists constants $C_1, C_2, C_3 > 0$ such that*

$$\left| J_{\text{spec}}(h_t^{\tau,p} \otimes \chi(N)) - J_{\text{spec}}(h_{t,R}^{\tau,p} \otimes \chi(N)) \right| \leq C_3 e^{-C_1 R^2/t + C_2 t}.$$

The constants C_1 and C_2 may be chosen independently of τ. Combining this lemma with Proposition 3.1.1, we get that for N sufficiently large and $R \leq C_n \log N$,

$$\int_0^T J_{\text{geo}}(h_t^{\tau,p} \otimes \chi(N)) t^{s-1} dt = \int_0^T J_{\text{unip}}(h_{t,R}^{\tau,p} \otimes \chi(N)) t^{s-1} dt + E_1(s, R, T), \tag{3.2.4}$$

where $E_1(s, R, T)$ is an error term given by an integral convergent for all $s \in \mathbb{C}$, and for $s = 0$ bounded by

$$|E_1(0, R, T)| \leq C_3 e^{-C_4 R^2/T + C_2 T} \int_0^{T/R^2} e^{-C_4/t} t^{-1} dt \ \operatorname{vol}(X(N)). \tag{3.2.5}$$

for some C_4 only depending on G.

Separating the leading term

Having dealt with the latter term of the right hand side of (3.2.4), we use the geometric expansions of the trace formula to analyze the former. As before, let $J_{\text{unip}-1}$ denote the unipotent contribution subtracted the term associated to $(M, \mathfrak{u}) = (G, \{1\})$. By Theorem 1.2.10 specified to $\mathfrak{o} = \mathfrak{o}_{\text{unip}}$ (see also 2.5.8), we may write

$$\int_0^T J_{\text{unip}}(h_{t,R}^{\tau,p} \otimes \chi(N)) t^{s-1} dt = \int_0^T h_{t,R}^{\tau,p}(1) t^{s-1} dt \operatorname{vol}(X(N)) \tag{3.2.6}$$

$$+ \int_0^T J_{\text{unip}-1}(h_{t,R}^{\tau,p} \otimes \chi(N)) t^{s-1} dt \tag{3.2.7}$$

Note that $h_{t,R}^{\tau,p}(1) = h_t^{\tau,p}(1)$, i.e. the compactification does not matter when evaluating at 1. By ([MP13], (5.11)) this has a asymptotic expansion for small t

$$h_t^{\tau,p}(1) \sim \sum_{i=0}^{\infty} a_i t^{i-d/2} \tag{3.2.8}$$

as $t \to 0$. Furthermore, as a special case of Proposition 2.4.2, we have a large t bound for some $C > 0$ and $t \geq 1$

$$|h_t^{\tau,p}(1)| \leq Ce^{-\lambda t}.$$

Together, this implies the absolute convergence in the complex half-plane $\Re(s) > \frac{d}{2}$ and the existence of a meromorphic continuation of its Mellin transform. Furthermore, the upper bound allows us to bound the error term arising in the expression

$$\int_0^T h_{t,R}^{\tau,p}(1) t^{s-1} dt = \int_0^\infty h_t^{\tau,p}(1) t^{s-1} dt + E_2(s,T). \tag{3.2.9}$$

Here $E_2(s,T)$ is also an error term, holomorphic in s and given by an integral which is absolutely convergent everywhere satisfying

$$|E_2(0,T)| \leq Ce^{-\lambda T} \tag{3.2.10}$$

for $t \geq 1$ and some constant $C > 0$.

Handling the non-leading terms

We turn our attention to the truncated Mellin transform of $J_{\text{unip}-1}(h_{t,R}^{\tau,p} \otimes \chi(N))$ in (3.2.6). We let $S = S(N) \sqcup \{\infty\}$. Using the fine geometric expansion, we write

$$J_{\text{unip}-1}(h_{t,R}^{\tau,p} \otimes \chi(N)) = \sum_{(M,\mathfrak{u}) \neq (G,\{1\})} a^M(S,\mathfrak{u}) J_M(h_{t,R}^{\tau,p} \otimes \chi(N), \mathfrak{u}). \tag{3.2.11}$$

By (1.2.24), each of the orbital integrals can be decomposed into local constituents

$$J_M(h_{t,R}^{\tau,p} \otimes \chi(N), \mathfrak{u}) = \sum_{L_1,L_2 \in \mathcal{L}(M)} d_M(L_1,L_2) J_M^{L_1}(h_{t,R}^{\tau,p}, \mathfrak{u}_\infty) J_M^{L_2}(\chi(N), \mathfrak{u}_f). \tag{3.2.12}$$

Since only the archimedean orbital integral sees the variable t, we may write the truncated Mellin transform of $J_M(h_{t,R}^{\tau,p} \otimes \chi(N), \mathfrak{u})$ as

$$\sum_{L_1,L_2 \in \mathcal{L}(M)} d_M(L_1,L_2) \left(\int_0^T J_M^{L_1}(h_{t,R}^{\tau,p}, \mathfrak{u}_\infty) t^{s-1} dt \right) J_M^{L_2}(\chi(N), \mathfrak{u}_f).$$

The local orbital integrals were analyzed in the previous section. Also, an asymptotic expansion of $J_M^{L_1}(h_{t,R}^{\tau,p}, \mathfrak{u}_\infty)$ as $t \to 0$ similar to (3.2.8) is obtained by combining Proposition 7.2 and (11.21) of [MM23]. By Proposition 3.1.6, when $\lambda > c'$, we then see that the truncated Mellin transform $\int_0^T J_M^{L_1}(h_{t,R}^{\tau,p}, \mathfrak{u}_\infty) t^{s-1} dt$ enjoys the same properties as the one of $h_t^{\tau,p}(1)$. In particular, not only is it absolutely convergent in some half-plane and has a meromorphic continuation to all of $\mathbb{C}$, but it is also equal to the full Mellin transform (cf. (3.2.9)) up to some error term $E_3(s,T)$ satisfying

$$|E_3(0,T)| \le C e^{-(\lambda - c')T} \tag{3.2.13}$$

for some constant $C > 0$. Then we see that

$$\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J_M^{L_1}(h_{t,R}^{\tau,p}, \mathfrak{u}_\infty) t^{s-1} dt\right) = \mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^\infty J_M^{L_1}(h_{t,R}^{\tau,p}, \mathfrak{u}_\infty) t^{s-1} dt\right) + E_3(0,T)$$

may be bounded by a uniform constant as T grows to infinity.

The non-archimedean orbital integrals were appropriately bounded in Proposition 3.1.7. Combining these bounds with the decompositions in the current section, and making the technical assumption previously discussed to handle global coefficients, we get the following.

Proposition 3.2.5. *Assume Conjecture 3.2.1 or that N varies over a prime-fixed set, and assume $\lambda > c'$. Then for N sufficiently large and $R \le C_n \log N$, there exists constants $C, b > 0$ with b only depending on G such that*

$$\left|\mathrm{FP}_{s=0}\left(\frac{1}{s\Gamma(s)}\int_0^T J_{\mathrm{unip}-1}(h_{t,R}^{\tau,p} \otimes \chi(N)) t^{s-1} dt\right)\right| \le C N^{-k(G)} (\log N)^b \operatorname{vol}(X(N)). \tag{3.2.14}$$

Approximating L^2-torsion

Collecting the results of Subsection 3.2, we may insert into the definition of analytic torsion and see

$$\begin{aligned}
\log T_{X(N)}(\tau) = {} & \mathrm{FP}_{s=0}\left(\frac{1}{2}\frac{1}{s\Gamma(s)}\int_0^\infty \sum_{p=1}^d (-1)^p p\, h_t^{\tau,p}(1) t^{s-1} dt\right) \operatorname{vol}(X(N)) \\
& + \mathrm{FP}_{s=0}\left(\frac{1}{2}\sum_{p=1}^d (-1)^p p \frac{1}{s\Gamma(s)}\int_0^T J_{\mathrm{unip}-1}(h_{t,R}^{\tau,p} \otimes \chi(N)) t^{s-1} dt\right) \\
& + \frac{1}{2}\sum_{p=1}^d (-1)^p p \left(\int_T^\infty J_{\mathrm{geo}}(h_t^{\tau,p} \otimes \chi(N)) t^{-1} dt + E_1(0,R,T) + E_2(0,T)\right)
\end{aligned}$$

The first term on the right hand side is the L^2-torsion, as we show below. The second term was just handled in the previous subsection, Proposition 3.2.5. The error terms will be dealt with in a moment.

We recall the definition of $t^{(2)}_{\tilde{X}}(\tau)$ given in [Lot92]:

$$t^{(2)}_{\tilde{X}}(\tau) := \frac{1}{2}\frac{d}{ds}\left(\frac{1}{2}\frac{1}{\Gamma(s)}\int_0^\infty \sum_{p=1}^{d}(-1)^p p\, h_t^{\tau,p}(1)t^{s-1}dt\right)\Bigg|_{s=0}.$$

We recall further that $T^{(2)}_{X(N)}(\tau) = t^{(2)}_{\tilde{X}}(\tau)\,\mathrm{vol}(X(N))$. This is equal to the first term above as the meromorphic extension of $\frac{1}{\Gamma(s)}\int_0^\infty \sum_{p=1}^{d}(-1)^p p\, h_t^{\tau,p}(1)t^{s-1}dt$ is holomorphic at $s=0$ (see [BV13], §4.4).

By Proposition 3.2.5, the second term above is of the form

$$O(N^{-k(G)}(\log N)^b\,\mathrm{vol}(X(N)))$$

for N large enough and $b>0$ some constant. This is the correct size for the asymptotic in Theorem 3.2.2. Hence, the only missing step in proving the theorem is to establish the same bound for the error terms.

Correct asymptotics of error terms

To ensure that the error term E_2 vanishes sufficiently quickly, we need to pick $T \sim \log N$. Thus, let $T = \beta \log N$, with the constant $\beta > 0$ to be determined later. We also pick the maximal allowed value for R, namely $R = C_n \log N$. These choices mean that our error terms take the form

$$\begin{aligned}\int_T^\infty J_{\mathrm{geo}}(h_t^{\tau,p}\otimes\chi(N))t^{-1}dt &= O\left(N^{-\lambda(1-\varepsilon)\beta}\,\mathrm{vol}(X(N))\right)\\ E_1(0, C_n\log N, \beta\log N) &= O\left((N^{-C_4C_n^2/\beta + C_2\beta}\,\mathrm{vol}(X(N))\right)\\ E_2(0,\beta\log N) &= O\left(N^{-\lambda\beta}\,\mathrm{vol}(X(N))\right).\end{aligned}$$

This follows from Lemma 3.2.3, (3.2.5), and (3.2.10), respectively. Recall that $C_2, C_4, C_n > 0$ are all fixed constants, only depending on G.

Choose β small such that $-C_4C_n^2/\beta + C_2\beta \le -k(G)$. This ensures the correct asymptotic for E_1. Now, pick the spectral gap λ such that $-\lambda(1-\varepsilon)\beta < -k(G)$ and $\lambda > c'$ as well. Note that this choice is only dependent on G. This ensures that the remaining error terms have the correct asymptotic. We have now finished the proof of Theorem 3.2.2.

3.3 Applications

In this section, we will present some of the applications of Theorem 3.2.2. The main obstruction to applying this result is the lack of generalized Cheeger-Müller formulas, i.e. identities relating analytic torsion to Reidemeister torsion. In fact, Reidemeister torsion is a priori not defined in general for noncompact Riemannian manifolds. In the following, we will instead use the identification of analytic torsion with the Reidemeister torsion of the Borel-Serre compactification which has been established in certain low-rank cases.

We will focus on applications in the setting of SL(2) defined over a number field F, where a Cheeger-Müller formula has been established in [MR24]. The setup is the following.

Let $G_0 = \mathrm{SL}(2)/F$, and set $G = \mathrm{Res}_{F/\mathbb{Q}}(G_0)$ its restriction of scalars to $\mathbb{Q}$. Then $G(\mathbb{Q}) = \mathrm{SL}(2, F)$, and we can continue working over $\mathbb{Q}$. Let r_1 and r_2 be respectively the number of real, respectively pairs of complex, embeddings of F, such that

$$G(\mathbb{R}) = \mathrm{SL}(2, \mathbb{R})^{r_1} \times \mathrm{SL}(2, \mathbb{C})^{r_2}.$$

Let $K = \mathrm{SO}(2)^{r_1} \times \mathrm{SU}(2)^{r_2}$. Then the associated symmetric space $\tilde{X} := G(\mathbb{R})/K$ is

$$\tilde{X} = (\mathbb{H}^2)^{r_1} \times (\mathbb{H}^3)^{r_2}.$$

We embed $\mathrm{SL}(2, \mathcal{O}_F)$ as a discrete subgroup of $G(\mathbb{R})$ induced by the embedding of F in $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. Now let $\Gamma \subseteq \mathrm{SL}(2, \mathcal{O}_F)$ be a torsion free finite index subgroup, also discretely embedded through the inclusion. Then $X := \Gamma\backslash\tilde{X}$ is a (noncompact) locally symmetric manifold of finite volume. With the parametrizations

$$\mathbb{H}^2 = \{x + iy \mid x \in \mathbb{R}, y > 0\}, \qquad \mathbb{H}^3 = \{z + is \mid z \in \mathbb{C}, s > 0\},$$

we give an invariant metric on $\tilde{X}$ by

$$\tilde{g} = \sum_{i=1}^{r_1} \frac{dx_i^2 + dy_i^2}{y_i^2} + 2\sum_{j=1}^{r_2} \frac{|dz_j|^2 + ds_j^2}{s_j^2}.$$

We let g denote the metric on X induced by $\tilde{g}$. Let (τ, V_τ) be a finite dimensional irreducible representation of $G(\mathbb{R})$, and let E be the flat vector bundle associated to $\tau|_\Gamma$. As in Section 2.2, one may equip this bundle with a canonical bundle metric h defined by an admissible inner product on V_τ. Let $T(X, E, g, h)$ denote the associated analytic torsion as defined in [ARS12].

X admits a natural compactification as a manifold with boundary $\overline{X}$ (see [MR24], §3). The Cheeger-Müller formula in this setting is the following. Let $\tau(\overline{X}, E, \mu_X)$ denote the Reidemeister torsion of $(\overline{X}, E)$ with respect to μ_X, a particular basis of $H^*(\overline{X}, E)$ as chosen in ([MR24], (5.2)).

Theorem 3.3.1 ([MR24], Theorem 1.1)**.** *In the notation above, assume that r_2 is odd, $r_1 > 0$ and $r_1 + r_2 > 2$. Then*

$$T(X, E, g, h) = \tau(\overline{X}, E, \mu_X).$$

Their full result also allows for $r_1 = 0$ in exchange for further assumptions on the representation.

Take a faithful $\mathbb{Q}$-rational representation $\rho : G \to \mathrm{GL}(V)$ and a lattice $\Lambda \subseteq V$ stabilized by $\mathrm{SL}(2, \mathcal{O}_F)$. Let $K_f \subseteq G(\mathbb{A}_f)$ denote the stabilizer of $\hat{\mathbb{Z}} \otimes \Lambda$ and subgroups $K(N)$ as defined in (3.1.1). Set

$$\Gamma(N) := G(\mathbb{Q}) \cap (G(\mathbb{R}) \times K(N))$$

where $K(N)$ was defined in (3.1.1). If ρ and Λ are chosen correctly, these are exactly the principal congruence subgroups of level N in $\mathrm{SL}(2, \mathcal{O}_F)$. As we are working with $\mathrm{SL}(2)$, strong approximation guarantees us that $X(N) = \Gamma(N)\backslash\tilde{X}$.

The definition of analytic torsion given in [ARS12] matches the one given in Section 3.2 in this setup (see the discussion in [MR24], §7). Let ρ_∞ be the representation of $G(\mathbb{R})$ induced by ρ. From now on, we assume that it decomposes into a sum of λ-strongly acyclic representations. One can now combine this with our Theorem 3.2.2 to achieve new asymptotics on cohomology.

Let L_ρ be the local system of free $\mathbb{Z}$-modules over $X(N)$ associated to Λ. As being λ-strongly acyclic implies being acyclic, we have that $H^*(X(N), L_\rho)$ is purely torsion. Hence, there is no need for a choice of basis for the free part. Then by ([Che79], (1.4)) we have

$$\tau(\overline{X(N)}, E_{\rho_\infty})^2 = \prod_{q=0}^{d} |H^q(\overline{X(N)}, L_\rho)|^{(-1)^{q+1}}. \tag{3.3.1}$$

Theorem 3.3.2. *Assume that r_2 is odd, $r_1 > 0$ and $r_1 + r_2 > 2$. Assume further that ρ_∞ decomposes into a sum of λ-strongly acyclic representations, with $\lambda > 0$ chosen as in Theorem 3.2.2. Let $k(G)$ be defined as in (3.1.10). Then there exists a contant $a > 0$ such that*

$$\sum_{q=0}^{d}(-1)^{q+1} \log |H^q(\overline{X(N)}, L_\rho)| = 2 \cdot T^{(2)}_{X(N)}(\rho_\infty) + O(\mathrm{vol}(X(N))N^{-k(G)} \log(N)^a)$$

as N tends to infinity.

Proof. If one assumes Conjecture 3.2.1, this is simply the application of Theorem 3.2.2 to Theorem 3.3.1, phrased in terms of cohomology using (3.3.1). The conjecture is proven for $\mathrm{GL}(n)$ over any number field in [Mat15], and one may relate the trace formula of $\mathrm{SL}(n)$ to the one of $\mathrm{GL}(n)$ as in ([MM20], §11). Then, applying Theorem 3.2.2 to $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}(n)/F)$, we get the unconditional result for $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{SL}(n)/F)$. □

This result gives second order terms to the asymptotic growth of cohomology when the deficiency is 1, i.e. when $r_2 = 1$ (cf. [MR24], Theorem 1.3). However, the main motivation for seeking the asymptotics of Theorem 3.2.2 was in its applications when the deficiency is strictly greater than 1.

Corollary 3.3.3. *Assume further that $r_2 > 1$, such that $\delta(G) > 1$. Then $T^{(2)}_{X(N)}(\rho_\infty) = 0$, and hence,*

$$\sum_{q=0}^{d}(-1)^{q+1}\frac{\log|H^q(\overline{X(N)}, L_\rho)|}{[\Gamma(1):\Gamma(N)]} = O(N^{-k(G)}(\log N)^a).$$

as N tends to infinity.

We are not able to guarantee that there is not significant cancellation happening. However, it is interesting to compare this result to similar vanishing results. As an example, essentially the same asymptotic has been established for degrees $i \leq d-2$ of integral homology of principal congruence subgroups of $\mathrm{SL}(d,\mathbb{Z})$ by Abert–Bergeron–Fraczyk–Gaboriau, see ([ABFG25], Theorem B). Heuristically, it seems promising to combine such results with asymptotics on the alternating sums as above to isolate even more degrees of homology, especially in lower ranks.

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