UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

PhD thesis

Energies of Dilute Bose Gases

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Submitted: August 31, 2025

This thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen

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Abstract

In this thesis, we study the Dilute Bose Gas under various assumptions in two and three dimensions. In three dimensions, we establish a new lower bound for the free energy at low temperature, valid under stronger interactions than previously treated. In two dimensions, we prove the full two dimensional analogue Lee–Huang–Yang formula for the ground state energy, confirming its universality, and derive an upper bound for the corresponding free energy expansion, consistent with Bogoliubov theory despite the absence of Bose–Einstein condensation at positive temperature.

Beyond two-body interactions, we resolve the leading-order ground state energy of a Bose gas with three-body hardcore interactions by providing a sharp upper bound, matching existing lower bounds. Finally, we derive Hartree theory for two dimensional Bose gases with attractive interactions in an almost Gross-Pitaevskii regime, proving stability and convergence to the nonlinear Schrödinger functional.

Resumé

I denne afhandling studerer vi Bose gassen ved lav tæthed under forskellige antagelser i to og tre dimensioner. I tre dimensioner etablerer vi en ny nedre grænse for den frie energi ved lave temperaturer, gyldig for stærkere interaktioner end tidligere behandlet. I to dimensioner beviser vi den fulde to dimensionelle analog til Lee–Huang–Yang-formelen for grundtilstandsenergien, hvilket bekræfter dens universalitet, og vi udleder en øvre grænse for den tilsvarende ekspansion af den frie energi, konsistent med Bogoliubov-teorien trods fraværet af Bose–Einstein-kondensation ved positive temperaturer.

Ud over to legeme interaktioner løser vi den ledende orden af grundtilstandsenergien for en Bose-gas med tre legeme hardcore interaktioner ved at give en skarp øvre grænse, som matcher eksisterende nedre grænser. Endelig udleder vi Hartree teorien for todimensionale Bose gasser med attraktive interaktioner i et næsten Gross-Pitaevskii system, hvor vi beviser stabilitet samt konvergens mod det ikke-lineære Schrödinger-funktional.

Aknowledgements

Now that I am handing in my thesis I wish to address my gratitude towards the people and institutions who have helped me. My first thanks goes to my advisor, Søren Fournais, who introduced me to the topic of dilute Bose gases and guided me through this journey. Working with you has been a pleasure.

I also want to thank my collaborators all the way back from Århus university, Marco, Theo and Leo. It has been amazing spending time with you, both at and away from work. I hope our collaboration and friendship will continue for many years.

I am grateful to Phan Thành Nam for many stimulating discussions during my stay at LMU. I would also like to thank my collaborators at LMU, Francois and Florian, who made the exchange experience both productive and enjoyable.

A warm thank you also goes to all of Q-MATH, for valuable mathematical discussions and for sharing many other interesting conversations. In particular, I am grateful to Tommaso, Gabrielle, and August for their help with the final stages of this thesis.

Finally, nobody deserves my gratitude more than my girlfriend Olivia. You have been there through everything, always supportive—even when I asked you to move away from family and friends twice. I could not have done this without you.

Papers included in the thesis

In the thesis 4 papers are included in full.

1. "The ground state energy of a two dimensional Bose gas" [Fou+24c].
2. "The free energy of dilute Bose gases at low temperatures interaction via strong potentials" [Fou+24a].
3. "Derivation of Hartree theory for two dimensional attractive Bose gases in very dilute regime" [JV25].
4. "Ground state energy of a dilute Bose gas with three-body hard-core interactions.

To each paper all authors have contributed equally. The thesis also includes a review paper "Lower bounds on the energy of the Bose gas" [Fou+24b]. Lastly it includes a draft of the paper "A second order upper bound to the free energy of the two dimensional Bose Gas.". Again all authors have contributed equally.

Contents

Chapter 1

Introduction

The Dilute Bose Gas has drawn much attention for mathematicians and physicists alike. It has the rare property of being descriptive enough to predict measurements of certain well made experiments, yet simple enough for one to provide proof of said predictions. The first prediction was made by Einstein and Bose in [Bos24; Ein24] in 1924 concerning the ideal gas. On ideas of Bose, Einstein proved that past a critical density ρ_c, depending on the temperature, all particles would enter a single quantum state. This phenomenon would be known as Bose Einstein Condensation-"BEC". Although BEC has only been proven in the ideal gas, it is reasonable to assume that it exists in systems where the interaction is weak, which corresponds to a dilute setting. And indeed in 1995 BEC was experimentally verified in the dilute setting through the groundbreaking results of Wieman, Cornell et al. in [And+95] and Ketterle et al. in [Dav+95]. Although theoretically predicted and experimentally verified, a mathematical proof of BEC in the interaction gas still eludes us, and stands as a major open problem in mathematical physics.

Without proving BEC one could assume its existence and use it to compute the ground state energy and excitations. Precisely this was done in the seminal work of Bogoliubov in 1947 [Bog47], where he would provide a prediction for the low lying energy excitation spectrum. Although groundbreaking, it had several mathematical flaws, which meant that some of the predictions, especially concerning the ground state energy, were wrong. Ten years later Lee, Huang and Yang would fix these errors using non-rigorous perturbative arguments, and in the now much celebrated work [LHY57] they estimated the second order expansion of the ground state energy density in the thermodynamic limit at zero temperature and found

$$e^3(\rho) = 4\pi a\rho^2\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3}\right) \tag{1.1}$$

where ρ is the density of the system and a is the scattering length-effective range of the interaction potential. The formula (1.1) is expected to hold in the dilute regime $\rho a^3 \ll 1$ i.e. when the interactions are weak enough to not deplete the BEC of the ideal gas. The first term $4\pi a\rho^2$ was already predicted in 1929 by Lenz [Len29]. Another interesting property of eq. (1.1) is the notion of universality, namely that the potential only appears through its scattering length, and the overall shape of the potential is irrelevant. eq. (1.1) has recently been rigorously proven. The upper bound for the Lenz term by Dyson [Dys57], and corresponding lower bound by Lieb and Yngvason [LY98]. The second order-"Lee-Huang-Yang term" was proved to be correct by Yau and Yin [YY09] for an upper bound and Fournais and Solovej [FS20] for a lower bound. The Lee-Huang-Yang term was also experimentally observed in [Nav+10].

The two dimensional Bose gas is rather different from the three dimensional case. One difference is that the ideal gas in two dimensions does not exhibit BEC for positive temperature, however, for computing the ground state energy at zero temperature this is less relevant. Another difference is the "scattering energy": for two particles in a thermodynamic 3 dimensional box their scattering energy is $4\pi a$, exactly the quantity appearing in eq. (1.1), however, in 2 dimensions that same quantity is 0. This latter fact has caused discrepancies in the physics literature [CS01; HFM78; MC09; Pil+05; Sch71] where multiple predictions for the ground state energy was provided. In 2009 Mora and Castin [MC09] predicted the following formula as a 2D analogue of the Lee-Huang-Yang eq. (1.1)

$$e^2(\rho) = 2\pi\rho\delta\left(1 + \left(\frac{1}{4} + \Gamma + \frac{\log(\pi)}{2}\right)\delta\right), \qquad \delta = \frac{2}{|\log \rho a^2 |\log \rho a^2|^{-1}|} \tag{1.2}$$

where $\Gamma \sim 0.57$ is the Euler-Mascheroni constant. The first term, sometimes referred to as the Schick term [Sch71], was again rigorously proven by Lieb and Yngvason in 2001 [LY01], and the full formula is proven in [Fou+24c] chapter 3 of this thesis. Equation (1.2) exhibits the same form of universality as eq. (1.1) since the potential only appears through its scattering length. Unfortunately eq. (1.2) has yet to be experimentally observed.

After settling eq. (1.1) and eq. (1.2) one can ask whether the other predictions made in Bogoliubov [Bog47] and Lee-Huang-Yang [LHY57] surrounding the low lying excitation spectrum can be rigorously verified. This turns out to be essentially as difficult, as proving BEC. So instead one can focus on the collective behaviour of the excitations and attempt to compute the free energy, which also predicts the energetic behaviour of the gas at positive temperature. For $T \ll T_c$, with T_c being the critical BEC temperature, the following formula was predicted in [LHY57]

$$f^3(\rho, T) = 4\pi a\rho^2\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3}\right) + \frac{T^{\frac{5}{2}}}{(2\pi)^3}\int_{\mathbb{R}^3} \log\left(1 - e^{-\sqrt{p^4 + \frac{16\pi\rho a}{T}p^2}}\right) dp. \tag{1.3}$$

The first term is exactly the ground state energy density and as $T \to 0$ we retrieve eq. (1.1). The second term is a temperature term involving the dispersion relation $D_p = \sqrt{p^4 + 16\pi a\rho p^2}$ predicted by Bogoliubov [Bog47]. Even though a thermodynamic proof of the dispersion relation is out of reach, we have been able to verify eq. (1.3) first done by Haberberger et al. [Hab+24a]-lower bound and Haberberger. et al. [Hab+24b]-upper bound. In Chapter 4 we present a different proof for the lower bound of eq. (1.3), which includes stronger interactions.

One could expect a similar free energy expansion in the 2 dimensional case. The dispersion relation is expected to be the same and thus the predicted formula becomes

$$f^2(\rho, T) = 2\pi\rho\delta\left(1 + \left(\frac{1}{4} + \Gamma + \frac{\log(\pi)}{2}\right)\delta\right) + \frac{T^2}{(2\pi)^2}\int_{\mathbb{R}^2} \log\left(1 - e^{-\sqrt{p^4 + \frac{8\pi\rho\delta}{T}p^2}}\right) dp. \tag{1.4}$$

The above formula holds despite the non existence of BEC at non zero temperature in 2 dimensions. Even though we need BEC to verify eq. (1.4) we only need it at some finite length scales, where BEC turns out to exist. We provide a proof of the upper bound of eq. (1.4) in Chapter 5.

1.1 Content

This thesis consists of eight chapters.

- Chapter 2 sets the historical and methodological framework. In Section 2.2 we present the early physical predictions and methods for the formulae eqs. (1.1) to (1.4), which also serve as motivation and guidelines for the mathematical work in the field. In Section 2.3 we provide brief reviews of the relevant techniques and results in the literature on the dilute Bose gas. We emphasize when techniques or results are directly applied in later chapters, and we state the actual theorems proved there.
- Chapter 3 is a self-contained paper titled "The Ground State Energy of a Two Dimensional Bose gas" [Fou+24c]. Here we provide a complete proof of eq. (1.2) including both upper and lower bound. Section 2.3.1 in Chapter 2 sets the methodological framework for this chapter.
- Chapter 4 is a Review titled "Lower bounds on the energy of the Bose gas" [Fou+24b]. This review is meant to serve as a less technical exposition of [Fou+24c] and [FS20], where many of the technicalities are avoided by considering the Gross-Pitaevskii regime.
- Chapter 5 is a self contained paper titled "The free energy of dilute Bose gases at low temperatures interacting via strong potentials" [Fou+24a]. Here we present a lower bound for formula (1.3). The result apply to hardcore interactions, making it an important extension of [Hab+24a]. At the time of writing, this also provides the simplest proof of eq. (1.1). Section 2.3.2 in Chapter 2 reviews relevant literature for this chapter.
- Chapter 6 is a paper-draft titled "A second order upper bound to the free energy of the two dimensional Bose Gas.". We provide the upper bound for the formula (1.4). Section 2.3.2 in Chapter 2 discusses the relevant literature for this draft.
- Chapter 7 is a short paper titled "Ground state energy of a dilute Bose gas with three-body hard-core interactions"[JV24]. It serves as a simple generalisation of [Vis24] building upon Dyson's work in [Dys57].
- Chapter 8 is a paper titled "Derivation of Hartree theory for two-dimensional attractive Bose gases in very dilute regime" [JV25]. We consider an attractive potential in essentially any scaling regime softer than Gross-Pitaevskii. We provide a proof of convergence to the Hartree energy, from which a proof of BEC follows. Section 2.1.2 and Section 2.3.3 provide the motivation and an introduction to the techniques applied in [JV25].

Chapter 2

Mathematical model and prior results

2.1 The mathematical setting

We introduce here the mathematical definitions and some of the notions described in the introduction. Our aim is to keep the exposition short and concise, while adapting the notation used in later chapters.

2.1.1 The main model

The central model under consideration is the many body Schrödinger operator with pair interactions

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{i<j} v(x_i - x_j). \tag{2.1}$$

Here H_N acts on the Hilbert space $\mathcal{H}_N = L^2_{sym}(\Lambda^N)$ where sym refers to the bosonic subspace of $L^2(\Lambda^N)$ and $\Lambda = [0, L]^d$ denotes a thermodynamic box. The ground state energy is defined as

$$E_N = \inf_{\|\Psi\|=1} \langle \Psi, H_N \Psi \rangle. \tag{2.2}$$

In eq. (2.2) we did not specify boundary conditions, since they play no role here. We define the ground state energy density in the thermodynamic limit as

$$e^d(\rho) = \lim_{\substack{L,N\to\infty \\ \frac{N}{L^d}=\rho}} \frac{E_N}{L^d}, \tag{2.3}$$

which is exactly the quantity appearing on the left hand side of eqs. (1.1) and (1.2). At positive temperature $T > 0$, the minimizing procedure eq. (2.2) changes to instead consider a minimization of density matrices, which we will also call states:

$$S_1(\mathcal{H}_N) = \{\Gamma \in B(\mathcal{H}_N) | \quad \mathrm{Tr}(\Gamma) = 1, \quad \Gamma \geq 0\}. \tag{2.4}$$

The free energy is given by the variational principle,

$$F_N = \inf_{\Gamma \in S_1(\mathcal{H}_N)} \mathrm{Tr}(H\Gamma) + T\,\mathrm{Tr}(\Gamma \log \Gamma), \tag{2.5}$$

where the second term is minus the Von Neumann entropy. As T approaches 0 eq. (2.5) converges to eq. (2.2) as it should. For positive temperature, however, excitations contribute to the energy. By a variational principle, the minimum in eq. (2.5) is attained at

$$\Gamma_0 = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}, \qquad F_N = \mathrm{Tr}(H\Gamma_0) + T\,\mathrm{Tr}(\Gamma_0 \log \Gamma_0) = -T \log \mathrm{Tr}(e^{-\beta H_N}), \tag{2.6}$$

with $\beta = \frac{1}{T}$. Analogously to eq. (2.3), we define the free energy density in the thermodynamic limit

$$f^d(\rho, T) = \lim_{\substack{L,N\to\infty \\ \frac{N}{L^d}=\rho}} \frac{F_N}{L^d}, \tag{2.7}$$

which is the quantity appearing on the left hand side of eqs. (1.3) and (1.4). We will occasionally deviate from the Hamiltonian eq. (2.1) when considering 3-body interactions in Chapter 7

$$H_{N,3-body} = \sum_{i=1}^{N} -\Delta_i + \sum_{i<j<k} v(x_i - x_j, x_i - x_k), \tag{2.8}$$

with ground state energy density as in eq. (2.3).

Assumption 2.1. *Throughout Chapters 2-7 the potential v appearing in eq.* (2.1) *or eq.* (2.8) *will always be assumed positive (repulsive) and radial. For compactly supported potentials the radius of the support will be denoted R.*

One potential of particular interest is the hardcore potential

$$v_{hardcore}(x) = \begin{cases} \infty & |x| \le a, \\ 0 & |x| > a. \end{cases} \tag{2.9}$$

which serves both as a "toy model" for physicists and a genuine challenge for mathematicians. The parameter a is also the scattering length of v_{hardcore}, so no ambiguity arises.

Although assumption 2.1 makes it clear that negative potentials are hard to deal with, one can actually allow for some negativity in softer scaling regimes. Softer scaling regimes implies a connection between the thermodynamic limit in eq. (2.3) and the strength of the interaction potential. To be precise we define

$$H_N^\beta = \sum_{i=1}^{N} -\Delta_i + V^{trap}(x_i) + \frac{N^{d\beta}}{N-1} \sum_{i<j} v(N^\beta(x_i - x_j)), \tag{2.10}$$

where V^{trap} is a trapping potential, ensuring the gas to be essentially confined in a bounded region. The parameter $\beta \ge 0$ determines whether the gas is weak and long range or strong and short range. In 3 dimensions $\beta = 1$ is the well know Gross-Pitaevskii scaling, where the gas changes structure from being described by Hartree theory ($\beta < 1$) to being described by Gross-Piteavskii theory. In 2 dimensions the scaling need to be exponential for the same structure change. See section 2.1.2 for a heuristic explanation.

We also briefly introduce the grand canonical setting and Fock space formalism. The Fock space of $L^2(\Lambda)$ is defined by

$$\mathcal{F}(\Lambda) := \bigoplus_{n=0}^{\infty} \bigotimes_{sym}^{n} L^2(\Lambda) = \bigoplus_{n=0}^{\infty} L^2_{sym}(\Lambda^n). \tag{2.11}$$

The Hamiltonian with chemical potential μ acting on this Fock space is

$$H_\mu = \bigoplus_{n=0}^{\infty} H_n - \mu n = \sum_{p\in\Lambda^*} (p^2-\mu)a_p^* a_p + \frac{1}{2L^d} \sum_{p,k,q\in\Lambda^*} \widehat{v}(k) a_p^* a_q^* a_{p+k} a_{q-k}. \tag{2.12}$$

where a_p is the annihilation operator associated with the plane wave e^{ipx}, and $\Lambda^* = \frac{2\pi}{L}\mathbb{Z}^d$. The definitions of ground state energy eq. (2.2) and ground state energy density eq. (2.3) carry over analogously, except that the limit is only taken in L:

$$f^d_{Fock}(\mu, T) = \lim_{L\to\infty} \frac{-T}{L^d} \operatorname{Tr}(e^{-\frac{1}{T}H_\mu}), \tag{2.13}$$

The quantities eq. (2.13) and eq. (2.7) are related by the Legendre transform see [Rue69] i.e.

$$f^d(\rho, T) = \sup_\mu \{\mu\rho + f^d_{Fock}(\mu, T)\}. \tag{2.14}$$

2.1.2 Scattering length

The quantity a appearing in eqs. (1.1) to (1.3) is the scattering length of the potential. As is evident from these formulae, it plays a crucial role in the analysis of the dilute gas. We present here the necessary definitions and properties, for further details we refer to [Lie+05].

Definition 2.2. *For a radial potential v with support in $B(0,R)$ we define the quantity for any $\tilde{R} \geq R$*

$$E^d(\tilde{R}) = \inf\left\{ \int_{B(0,\tilde{R})} |\nabla\phi|^2 + \frac{1}{2}v\phi^2 dx \quad \middle| \quad \phi \in H^1(B(0,\tilde{R})), \quad \phi_{|\partial B(0,\tilde{R})} = 1 \right\} \tag{2.15}$$

where

$$E^d(\tilde{R}) = \begin{cases} \frac{2\pi}{\log(\frac{\tilde{R}}{a})} & d = 2, \\ \frac{4\pi a}{1-\frac{a}{\tilde{R}}} & d = 3. \end{cases} \tag{2.16}$$

The scattering length $a > 0$ is then the unique length solving eq. (2.16).

It is straightforward to verify that the above definition of a is independent of the normalization length $\tilde{R}$. By a variational principle the minimizers of eq. (2.16) solve the scattering equation

$$-\Delta\varphi + \frac{1}{2}v\varphi = 0. \tag{2.17}$$

Although the normalisation $\tilde{R}$ only plays a trivial role in the two body setting, it still plays the role of an important parameter in the two dimensional many body setting. We therefore briefly explain the choice of this parameter.

The quantity $E^d(\tilde{R})\tilde{R}^{-d}$ is essentially twice the energy of two particles in box of size $\tilde{R}$. From this, the Lenz term in eq. (1.1) can be interpreted as $\frac{N(N-1)}{2}$ particle pairs each carrying the pair energy $2E^d(L)L^{-d}$, i.e

$$4\pi a\rho N \sim \frac{N(N-1)}{2} 2E^3(L)L^{-3}. \tag{2.18}$$

Thus, in three dimensions the "correct" normalization is at the thermodynamic length scale, which effectively means taking $\tilde{R} = \infty$.

In two dimensions we examine eq. (1.2) and desire

$$2\pi\rho\delta N \sim \frac{N(N-1)}{2} 2E^2(\tilde{R})L^{-2} \implies \tilde{R} = ae^{\delta^{-1}}. \tag{2.19}$$

Here the L^{-2} factor is retained because the wave function is expected to be approximately constant between the $\tilde{R}$ and L scales. The above choice of $\tilde{R}$ is the same as in chapters 3, 4 and 6. However the parameter δ in eq. (1.2) deviates from δ_0 in chapters 3 and 4 by a factor of 2, precisely in order to satisfy eq. (2.19). The author acknowledges that this reasoning is somewhat "backwards" and offers limited insight into the true origin of δ, which remains unclear.

Let φ denote the unique minimizer to eq. (2.16) with $\tilde{R}$ chosen as above. From the fundamental solution of the Laplacian and eq. (2.17) we have for $|x| \geq R$

$$\varphi = \begin{cases} \frac{\log(\frac{|x|}{a})}{\log(\frac{\tilde{R}}{a})} & d = 2, \\ 1 - \frac{a}{|x|} & d = 3. \end{cases} \tag{2.20}$$

We further define

$$g = v\varphi, \qquad \omega = 1 - \varphi. \tag{2.21}$$

The integral of g, using eq. (2.17), satisfies

$$\int g dx = \widehat{g}(0) = \begin{cases} 4\pi\delta & d = 2, \\ 8\pi a & d = 3 \end{cases} \tag{2.22}$$

where δ is the quantity introduced in eq. (1.2), which we recall for convenience

$$\delta = \frac{2}{|\log(\rho a^2)| \log(\rho a^2)||}. \tag{2.23}$$

Pertubative expansion of the scattering length

We briefly explore the Born expansion of the scattering length, which can be used to heuristically justify the formulas eqs. (1.1) and (1.2). We will not enter into the mathematically rigorous aspects of this perturbative theory, but instead only present the heuristic arguments. In this section we think of $v \in C_0^\infty(\mathbb{R}^d)$ with small integral.

Starting with the three dimensional setting, let φ be the function from eq. (2.20) and eq. (2.17). Then according to eq. (2.22), the scattering length is determined by the integral of φ against v i.e.

$$E^3 = 4\pi a = \int \frac{1}{2} v\varphi dx. \tag{2.24}$$

The goal is then to approximate φ and, through this relation, approximate a. Using eq. (2.17) we can write

$$(\Delta^{-1}\frac{1}{2}v)\varphi = \varphi. \tag{2.25}$$

This equation yields an iterative pertubative description of φ, namely

$$\varphi = \sum_{n=0}^{\infty} \Psi_n, \qquad \text{where} \qquad \Psi_n = (\Delta^{-1}\frac{1}{2}v)^n \Psi_0, \tag{2.26}$$

and $\Psi_0 \equiv 1$ is the constant function. A straightforward computation shows that the above definition of φ indeed solves the scattering equation (2.17)

$$(-\Delta + \frac{1}{2}v)\sum_{n=0}^{\infty} \Psi_n = -\sum_{n=0}^{\infty} \frac{1}{2}v(-\Delta^{-1}\frac{1}{2}v)^n \Psi_0 + \frac{1}{2}v\sum_{n=0}^{\infty} \Psi_n = 0$$

where we used $-\Delta\Psi_0 = 0$. Moreover $\varphi(x)$ from eq. (2.26) converges to 1 as $|x|$ goes to infinity since $\Psi_n \to 0$ for $n \geq 1$. By uniqueness of solution we may conclude that eq. (2.26) is valid. According to eq. (2.24) an approximation for $4\pi a$ is therefore

$$4\pi a \sim \sum_{k \leq n} \int \frac{1}{2} v \Psi_k dx = \sum_{k \leq n} \int \frac{1}{2}(\Delta^{-1}\frac{1}{2}v)^k v \Psi_0 dx$$

We compute this for $n = 1$, obtaining

$$4\pi a \sim \int \frac{1}{2} v \Psi_0 dx + \int \frac{1}{2} v (\Delta^{-1}\frac{1}{2}v)\Psi_0 dx = \frac{1}{2}\int v dx - \frac{1}{(2\pi)^3}\int \frac{\widehat{v}(p)^2}{4p^2} dp, \tag{2.27}$$

the last equality follows from Plancherel's formula.

We return to the same question in two dimensions. A key observation in the 3 dimensional case was that $\Psi_n(x) \to 0$ for $x \to \infty$ for $n \geq 1$, owing to the fact that (without mention) we used the fundamental solution of the Laplacian that decays at infinity. However, in two dimensions we do not wish to normalize at ∞, as explained in the previous section. Instead, we wish to normalise at some fixed parameter $\tilde{R}$, and thus choose the solution

$$\Psi_1 = \frac{1}{2}(\Delta^{-1}v)(x) = \int v(y)\frac{1}{4\pi}\log\left(\frac{|x-y|}{\tilde{R}}\right) dy, \tag{2.28}$$

which, when $\tilde{R} \gg R$ (the support of v), essentially satisfies the desired normalization. The same approximation as eq. (2.27) in two dimensions then reads

$$\frac{2\pi}{\ln(\frac{\tilde{R}}{a})} \sim \int \frac{1}{2} v \Psi_0 dx + \int \frac{1}{2} v (\Delta^{-1}\frac{1}{2}v)\Psi_0 dx = \frac{1}{2}\int v dx - \frac{1}{(2\pi)^2}\int \frac{\widehat{v}(p)^2}{4p^2} - \frac{\widehat{v}(0)^2}{4p^2} 1_{|p| \leq 2e^{-\Gamma}\tilde{R}^{-1}} dp, \tag{2.29}$$

where the last equality follows from the Plancherel's formula and the Fourier transform of the logarithm (see Chapter 3 Section 3.4).

N scaling in scattering length

In eq. (2.10) and further in Chapter 8 we consider a potential of the form

$$v_N^{\beta}(x) = \frac{N^{d\beta}}{N} v(N^{\beta} x). \tag{2.30}$$

One may ask: what is the scattering length of v_N^β and how does it depend on β? In three dimensions these questions are quickly answered by scaling out some of the N-dependence. Indeed, from eq. (2.16) we have

$$\begin{aligned} 4\pi a_N^\beta &= \inf\left\{\int_{\mathbb{R}^3} |\nabla\phi|^2 + \frac{1}{2} v_N^\beta \phi^2 dx \quad \Big| \quad \lim_{|x|\to\infty} \phi(x) = 1\right\} \\ &= N^{-\beta} \inf\left\{\int_{\mathbb{R}^3} |\nabla\phi|^2 + \frac{1}{2} N^{\beta-1} v \phi^2 dx \quad \Big| \quad \lim_{|x|\to\infty} \phi(x) = 1\right\}. \end{aligned} \tag{2.31}$$

We see that due to the factor $N^{\beta-1}$ in front of the potential, the kinetic energy is dominant for $\beta < 1$, so the minimizing ϕ will be essentially constant and $8\pi a_N^\beta N^\beta \sim \int v$. However, for $\beta > 1$, the potential term dominates and $a_N^\beta N^\beta \sim R$ where R is radius of the support of v. Clearly for $\beta = 1$ the scattering length of the underlying potential v emerges.

In two dimensions we wish to find at what value of β the scaling causes transitions from depending essentially on the integral of v to depending on the range of v. Since we cannot let $\tilde{R} = \infty$, we set, for simplicity, $\tilde{R} = R$ the support of v. From the same scaling as in eq. (2.31) we find

$$\begin{aligned} \frac{2\pi}{\log(\frac{R}{a_N^\beta})} &= \inf\left\{\int_{B(0,R)} |\nabla\phi|^2 + \frac{1}{2} v_N^\beta \phi^2 dx \quad \Big| \quad \phi_{|\partial B(0,R)} = 1\right\} \\ &= \inf\left\{\int_{B(0,RN^\beta)} |\nabla\phi|^2 + \frac{1}{2N} v \phi^2 dx \quad \Big| \quad \phi_{|\partial B(0,RN^\beta)} = 1\right\}. \end{aligned}$$

If the kinetic term of the above is dominant, then

$$\frac{2\pi}{\log(\frac{R}{a_N^\beta})} \sim \frac{1}{2N} \int v.$$

If, however, the potential term dominates we find

$$\frac{2\pi}{\log(\frac{R}{a_N^\beta})} \sim \frac{2\pi}{\log(\frac{RN^\beta}{R})}.$$

Setting the two right hand sides equal we find

$$\frac{1}{2N} \int v = \frac{2\pi}{\log(\frac{RN^\beta}{R})} \implies N^\beta = e^{\frac{4\pi N}{\int v dx}}. \tag{2.32}$$

From this simple heuristic computation, we may draw the following conclusion.

Remark 2.3. In two dimensions the potential $v_N^\beta(x) = N^{2\beta-1} v(N^\beta x)$ for any $\beta \geq 0$ has a scattering length that depends only on the integral of v in the large N limit. The same conclusion holds for any sub exponential scaling.

The above remark provides believability to the result in Chapter 8, whereas eq. (2.32) gives a hint of optimality.

2.2 Early works on Bose gases

We will briefly discuss the seminal works of Bose and Einstein [Bos24; Ein24], Bogoliubov [Bog47] and Lee, Huang and Yang [LHY57]. Although all these papers only address the 3 dimensional setting we will carry out the analysis in both 2 and 3 dimensions.

2.2.1 The Ideal Bose gas

We begin by reviewing some of Einstein's computations, in which he discovered the existence of Bose–Einstein condensation (BEC) in the ideal Bose gas. The ideal Bose gas is modelled by the Hamiltonian eq. (2.1) with $v = 0$, that is we consider

$$H_N^{free} = \sum_{i=1}^{N} -\Delta_i \tag{2.33}$$

acting on $L^2_{sym}(\Lambda^N)$. Although one can proceed in the canonical setting (see [ZUK77]) it is simpler and clearer for our analysis to work in the grand canonical setting. Hence we consider the Fock space Hamiltonian eq. (2.12)

$$H_\mu^{free} = \sum_{n=0}^{\infty} H_n^{free} - \mu n = \sum_{p\in\Lambda^*} (p^2-\mu) a_p^* a_p. \tag{2.34}$$

According to Equation (2.6) the state with lowest free energy at temperature $T = \frac{1}{\beta}$ is

$$\Gamma_\mu = \frac{e^{-\beta H_\mu^{free}}}{\operatorname{Tr} e^{-\beta H_\mu^{free}}} = \frac{\sum_{n=0}^{\infty} e^{\beta\mu n} e^{-\beta H_n^{free}}}{\operatorname{Tr} e^{-\beta H_\mu^{free}}}. \tag{2.35}$$

Our goal is compute the average particle density in this state. The average number of particles is given by

$$\operatorname{Tr}(\mathcal{N}\Gamma_\mu) = \frac{\sum_{n=1}^{\infty} n e^{\beta\mu n} \operatorname{Tr}(e^{-\beta H_n^{free}})}{\sum_{n=0}^{\infty} e^{\beta\mu n} \operatorname{Tr}(e^{-\beta H_n^{free}})} = T\frac{\partial}{\partial\mu} \log \operatorname{Tr} \sum_{n=0}^{\infty} e^{\beta\mu n} e^{-\beta H_n^{free}}.$$

Using the second expression of eq. (2.34), one can use the occupancy numbers basis to compute the trace

$$\operatorname*{Tr}_{\mathcal{F}(L^2(\Lambda))} (e^{-\beta\sum_{p\in\Lambda^*}(p^2-\mu)a_p^* a_p}) = \prod_{p\in\Lambda^*} \sum_{n=0}^{\infty} e^{-\beta n (p^2-\mu)} = \prod_{p\in\Lambda^*} \frac{1}{1-e^{\beta\mu-\beta p^2}}.$$

From this we obtain the average density in the system

$$\rho_\mu = \frac{\operatorname{Tr}(\mathcal{N}\Gamma_\mu)}{L^d} = \frac{T}{L^d}\frac{\partial}{\partial\mu} \sum_{p\in\Lambda^*} -\log(1-e^{\beta\mu-\beta p^2}) = \frac{1}{L^d}\sum_{p\in\Lambda^*} \frac{e^{\beta\mu-\beta p^2}}{1-e^{\beta\mu-\beta p^2}}. \tag{2.36}$$

Naively taking the limit $L \to \infty$ yields the integral

$$\rho_\mu \to \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{\beta\mu-\beta p^2}}{1-e^{\beta\mu-\beta p^2}} dp. \tag{2.37}$$

From this we see that the average density ρ in our thermodynamic gas is bounded by the limit of $\mu \to 0^-$ of ρ_μ which yields the following conclusion

$$\rho \le \rho_c = \lim_{\mu\to\infty} \int_{\mathbb{R}^d} \frac{e^{\beta\mu-\beta p^2}}{1-e^{\beta\mu-\beta p^2}} dp = \begin{cases} \infty & d=2, \\ \zeta(\frac{3}{2})(4\pi\beta)^{-3/2} & d=3. \end{cases} \tag{2.38}$$

For $d=2$ this result is void, but perfectly valid, however, in three dimensions we see the density is bounded by a constant depending only on the temperature. This is the absurdity discovered by Einstein [Ein24]. He goes on to resolve this conjecture, by arguing that a macroscopic number of particles occupy the $0'th$ momentum state. In other words, the limit in eq. (2.37) should be taken jointly in μ and L, producing an additional δ_0 term whose size corresponds to the excess density all of which resides in the 0'th momentum mode. We thus obtain the following theorem:

Theorem 2.4. *The three dimensional ideal Bose gas, described either by the Hamiltonian eq.* (2.33) *or eq.* (2.34), *exhibits Bose Einstein condensation for* $\rho > \rho_c$. *That is,*

$$\frac{\operatorname{Tr}(\mathcal{N}_0\Gamma_\mu)}{L^d} \to \rho - \rho_c \tag{2.39}$$

where $\mathcal{N}_0 = a_0^* a_0$ *and the limit is taken jointly in* μ *and* L, *while keeping the average density fixed at* ρ. ρ_c *is given in eq.* (2.38). *In two dimensions the ideal Bose gas only exhibits Bose Einstein condensation for* $T = 0$.

The above theorem is extended to the non ideal setting as a conjecture in the dilute regime ($\rho a^d \ll 1$), where the interaction v may only play a minor role and the BEC of the ideal gas could persist.

Conjecture 2.5. *For* $v \ge 0$ *the Gibbs state from eq.* (2.6) *exhibits Bose Einstein condensation. I.e. there exist a* $\rho_c(v,T) > 0$ *such that for* $\rho > \rho_c$ *and* $\rho a^d \ll 1$ *we have*

$$\frac{\operatorname{Tr}(\mathcal{N}_0 e^{-\beta H_N})}{\operatorname{Tr}(e^{-\beta H_N})L^d} \to \rho - \rho_c \tag{2.40}$$

where the limit is the thermodynamic limit with fixed density ρ. *In three dimensions* $\rho_c(T,v)$ *may be finite for any temperature, while in 2 dimensions it is expected to be finite only at* $T = 0$.

At present, there are essentially no mathematically rigorous results concerning this conjecture in the thermodynamic limit. However, lower bounds on ρ_c for the occurrence of BEC have been obtained (see [SU09]), and positive results exist in softer scaling regimes such as Gross–Pitaevskii or for the Hamiltonian eq. (2.30) with $\beta \le 1$ (see [Boc+18; CCS21; LS02]). Although we will not present it here, we mention that Penrose and Onsager [PO56] provided some elegant mathematical formulations of BEC via the one-particle density matrix. This is the definition of BEC used in Chapter 8.

2.2.2 Bogoliubov approximation

Turning now to the famous 1947 paper by Bogoliubov [Bog47]. The starting point is the second quantisation of the Hamiltonian eq. (2.1)

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{i<j} v(x_i - x_j) = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{1}{2L^d} \sum_{p,q,k\in\Lambda^*} \widehat{v}(k) a_p^* a_q^* a_{p+k} a_{q-k}. \qquad (2.41)$$

We consider this Hamiltonian at zero temperature and focus on its ground state energy. Motivated by Theorem 2.4 and Conjecture 2.5 one may expect that the ground state of H_N also exhibits Bose-Einstein condensation and thus the a_0's play a distinct role. More precisely we have

$$a_0^* a_0 = N_0, \qquad [a_0, a_0^*] = 1 \qquad (2.42)$$

where we assume $N_0 \sim N$. Since the commutator $[a_0, a_0^*]$ is negligible compared to the size of a_0, a_0^*, we simply assume they commute and replace a_0 and a_0^* with the number $\sqrt{N}_0$. This leads to the approximation

$$\begin{aligned} H_N \sim H_{N_0}^{app} = &\sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{\widehat{v}(0) N_0^2}{2L^d} + \frac{N_0}{L^d} \sum_{p\neq 0} (\widehat{v}(0) + \widehat{v}(p)) a_p^* a_p + \frac{\widehat{v}(p)}{2} (a_p^* a_{-p}^* + a_p a_{-p}) \\ &+ \frac{\sqrt{N_0}}{2L^d} \sum_{p,q,p+q\neq 0} \widehat{v}(k) \left(2a_p^* a_{-k} a_{p+k} + 2a_k^* a_p^* a_{p+k} \right) + \frac{1}{2L^d} \sum_{\substack{p,q\neq 0 \\ p+k,q-k\neq 0}} \widehat{v}(k) a_p^* a_q^* a_{p+k} a_{q-k}. \end{aligned}$$

Although Bogoliubov did not address this, one should note that the above Hamiltonian is no longer particle preserving and must be defined on the Fock space. We emphasize this by removing the N in the notation of the Hamiltonian.

The above terms naturally group into Q_0, Q_2, Q_3 and Q_4 depending on the number of creation/annihilation operators. Bogoliubov argued that Q_3 and Q_4 are of lower order since they contain fewer factors of N_0. Ignoring them yields the quadratic Hamiltonian

$$H_N \sim H_{N_0}^{quadratic} = \frac{\widehat{v}(0) N_0^2}{2L^d} + \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{N_0}{L^d} \sum_{p\in\Lambda^*} (\widehat{v}(0) + \widehat{v}(k)) a_p^* a_p + \frac{1}{2} (a_p^* a_{-p}^* + a_p a_{-p}). \quad (2.43)$$

The terms involving $\widehat{v}(0)$ combine

$$\frac{\widehat{v}(0) N_0^2}{2L^d} + \frac{N_0}{L^d} \sum_{p\neq 0} \widehat{v}(0) a_p^* a_p \sim \frac{N^2 \widehat{v}(0)}{2L^d}. \qquad (2.44)$$

At this point Bogoliubov realised that the quadratic Hamiltonian can be diagonalised via what is now known as a Bogoliubov transformation. In order to implement it we introduce the creation and annihilation operators

$$b_p = \frac{a_p + \alpha_p a_{-p}^*}{\sqrt{1 - \alpha_p^2}}, \qquad \alpha_p = \frac{p^2 + \widehat{v}(p) \rho_0 - D_p^v}{\rho_0 \widehat{v}(p)}, \quad D_p^v = \sqrt{p^4 + 2p^2 \rho_0 \widehat{v}(p)} \qquad (2.45)$$

where $\rho_0 = \frac{N_0}{L^d}$, the operators b_p, b_p^* satisfy the canonical commutation relations, and a straightforward computation shows

$$H_{N_0}^{quadratic} = \frac{\widehat{v}(0)N^2}{2L^d} - \frac{1}{2}\sum_{p\neq 0}\rho_0\widehat{v}(p)\alpha_p + \sum_{p\neq 0} D_p^v b_p^* b_p. \tag{2.46}$$

At the end of [Bog47], Bogoliubov mentions that Landau pointed out the $\widehat{v}(p)$ should be replaced by $\widehat{g}(p)$, see eq. (2.21). This replacement can be partly justified using the first order approximation of the Born series eqs. (2.27) and (2.29). Although not explicitly stated by Bogoliubov the above formulates the following conjecture.

Conjecture 2.6. *In the thermodynamic limit the low excitation spectrum of H_N is described by the Bogoliubov Hamiltonian H^{bog}. More precisely,*

$$H_N \sim H^{bog} = NC(L,\rho,v) + \sum_{p\neq 0} D_p b_p^* b_p \tag{2.47}$$

where C is a constant depending on L, v and ρ. D_p is the dispersion relation

$$D_p = \sqrt{p^4 + 2p^2\rho\widehat{g}(p)}. \tag{2.48}$$

The above conjecture is completely open, and basically unapproachable with current techniques. In fact, we know nothing of the excitation spectrum in the thermodynamic limit. We don't even know how the gap closes. However, in the intermediate Gross-Pitaevskii scaling regimes, Conjecture 2.6 has been established in 3 dimensions [Boc21; Boc+19; BSS22; HST22a; NT23] and in 2 dimensions [CCS23].

Rather than focus on the full spectrum, one may try to compute the constant $C(L,\rho,v)$. Bogoliubov already derived such a constant: from eq. (2.46), replacing v by g and ρ_0 by ρ, his ground state energy estimate becomes

$$C(\rho,v) = N\rho\left(\widehat{g}(0) - \frac{1}{4\rho^2(2\pi)^d}\int p^2 + \widehat{g}(p)\rho - \sqrt{p^4 + 2p^2\rho\widehat{g}(p)}dp\right). \tag{2.49}$$

Using eq. (2.22) to evaluate $\widehat{g}(0)$, the first term coincides with that of the Lee–Huang–Yang formula eq. (1.1). However, the second-order term has the wrong sign and may even diverge—an issue that was later corrected by Lee, Huang and Yang.

2.2.3 The Lee Huang Yang Formula

We now revisit a method of deriving the Lee–Huang–Yang formulas eqs. (1.1) and (1.2) using the Bogoliubov and Born approximations. Starting from the quadratic Hamiltonian in three dimensions after the Bogoliubov approximation eq. (2.46), we add and subtract the term

$$\frac{N\rho}{(2\pi)^3}\int \frac{\widehat{v}(p)^2}{4p^2}, \tag{2.50}$$

which gives

$$H_N^{quadratic} = \frac{\widehat{v}(0)N^2}{2L^3} - \frac{N\rho}{(2\pi)^3}\int \frac{\widehat{v}(p)^2}{4p^2}dp \\ - \frac{1}{4}\sum_{p\neq 0}\rho_0\widehat{v}(p)\alpha_p + \frac{N\rho}{(2\pi)^3}\int \frac{\widehat{v}(p)^2}{4p^2}dp + \sum_{p\neq 0} D_p^v b_p^* b_p.$$

The added and subtracted term is precisely the second term from eq. (2.27). This has two consequences:

1. The first line above becomes a better approximation for $4\pi N\rho a$, since it now matches the two-term Born expansion eq. (2.27).
2. It regularizes the potentially divergent sum, which in the thermodynamic limit is replaced by an integral.

Using the approximation eq. (2.27), we obtain

$$E_N^{quadratic} \sim 4\pi a\rho N - \frac{L^3}{2(2\pi)^3}\int p^2 + \widehat{v}(p)\rho - \sqrt{p^4 + 2p^2\rho\widehat{v}(p)} - \rho^2\frac{\widehat{v}(p)^2}{2p^2}dp$$

where we have replaced ρ_0 with ρ, since they are expected to be close. This integral can be computed in the low-density regime, where it is valid to approximate $\widehat{v}(p)$ by $\widehat{v}(0)$. Performing the computation yields

$$e^{3,quadratic}(\rho) = \lim_{\substack{L,N\to\infty \\ \frac{N}{L^d}=\rho}} \frac{E_N^{quadratic}}{L^3} = 4\pi a\rho^2 - \frac{(\rho\widehat{v}(0))^{\frac{5}{2}}}{2(2\pi)^3}\int p^2 + 1 - \sqrt{p^4 + 2p^2} - \frac{1}{2p^2}dp \\ = 4\pi a\rho^2\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3}\left(\frac{\widehat{v}(0)}{8\pi a}\right)^{\frac{5}{2}}\right).$$

If we further use the first-order Born approximation $\widehat{v}(0) \sim 8\pi a$, we recover exactly eq. (1.1).

Applying the same approximation scheme to the two dimensional setting, now using eq. (2.29) and keeping $\tilde{R}$ as a parameter, we obtain

$$e^{2,quadratic}(\rho) = \frac{2\pi\rho^2}{\log(\frac{\tilde{R}}{a})} + \frac{\rho^2\widehat{v}(0)^2}{8\pi}\log(\sqrt{\rho\widehat{v}(0)}\tilde{R}) \\ + \frac{(\rho\widehat{v}(0))^2}{2(2\pi)^2}\int p^2 + 1 - \sqrt{p^4 + 2p^2} - \frac{1}{2p^2}1_{p\geq 2e^{-\Gamma}}dp. \tag{2.51}$$

Here we set $\tilde{R} = ae^{\delta^{-1}}$ as in eq. (2.19). As in 3 dimensions we also replace $\widehat{v}(0) \sim \widehat{g}(0) = 4\pi\delta$. Combining this yields

$$e^{2,quadratic}(\rho) = 2\pi\rho\delta(1 + (\frac{1}{4} + \Gamma + \frac{\log(\pi)}{2})\delta),$$

which matches eq. (1.2).

It should be noted that this approach provides no clear justification for the choice of δ. In fact, if $\widehat{v}(0)$ could always be replaced by $\widehat{g}(0)$ in eq. (2.51), the optimal choice would be $\delta = 0$. Nevertheless, it still offers a heuristic picture of where the constant originates.

Lee, Huang and Yang did in fact not use this approximation to arrive at eq. (1.1). Instead they considered the pseudo potential

$$v = 8\pi a \delta_0 \frac{\partial}{\partial |x|} |x|.$$

Where $8\pi a\delta_0$ replaces all previous $\widehat{v}(p)$ with $8\pi a$ and $|x|\frac{\partial}{\partial |x|}$ adds the term from eq. (2.50). This in total produces the same type of approximation.

Finally, note that the formulas eqs. (1.3) and (1.4) follow by inserting the constants from eqs. (1.1) and (1.2) into eq. (2.47) and computing the trace as an ideal gas computation

$$\frac{-T}{|\Lambda|} \operatorname*{Tr}_{\mathcal{F}^{\perp}}(e^{-\beta \sum_p D_p b_p^* b_p}) \to \frac{T}{(2\pi)^d} \int \log(1 - e^{-\beta D_p}) dp = \frac{T^{1+\frac{d}{2}}}{(2\pi)^d} \int_{\mathbb{R}^d} \log\left(1 - e^{-\sqrt{p^4 + \frac{2\rho \widehat{g}(0)}{T} p^2}}\right) dp,$$

where $\mathcal{F}^{\perp}$ denotes the Fock space orthogonal to the constant function.

2.3 Rigorous works

In section 2.2 we described the ideas and intuitions for eqs. (1.1) to (1.4). But the lack of rigour is unsatisfactory for a mathematician. In particular, the use of perturbation theory (or equivalently pseudo-potentials) and the neglect of Q_3 and Q_4 are difficult to justify. Here we present some of the results that laid the rigorous foundations on which later chapters are build. This section is not intended as a full review, but rather as a way to set the historical and methodological framework for the papers discussed later in the thesis. For more detailed reviews, see [BCC23; Lie+05; Rou20; Sol25].

2.3.1 Lee-Huang-Yang formula, again

From a rigorous perspective the Lee-Huang-Yang formula eq. (1.1) should be read as two inequalities, that will be proved separately: an upper bound and a lower bound. We start by providing some of the history and methodology of the upper bounds, as the lower bounds usually draws inspiration from the upper.

Upper bounds

The fundamental idea of an upper bound is straightforward: one "guesses" a good trial state and then computes its energy. Both steps, however, can be extremely demanding. The first rigorous upper bound to $e^3(\rho)$ was found by Dyson [Dys57]. He constructed a trial state Ψ_N satisfying

$$\frac{\langle \Psi_N, H_N \Psi_N \rangle}{\|\Psi_N\|^2} \leq 4\pi a \rho N (1 + C(\rho a^3)^{\frac{1}{3}}) \tag{2.52}$$

which by the variational principle immediately yields

$$e^3(\rho) \leq \lim_{\substack{L,N\to\infty \\ \frac{N}{L^3}=\rho}} \frac{\langle \Psi_N, H_N \Psi_N \rangle}{L^3 \|\Psi_N\|^2} \leq 4\pi a \rho^2 (1 + C(\rho a^3)^{\frac{1}{3}}). \tag{2.53}$$

This bound recovers the correct first order (Lenz) term in the dilute expansion. Dyson's trial state was a variant of

$$F = \prod_{i<j} f_{ij}^b \tag{2.54}$$

where f_{ij}^b is the scattering solution to the pair (x_i, x_j) normalised at length b, and extended by 1 outside $B(0,b)$. The strength of this ansatz is that it correctly captures the behaviour near particle collisions. The order of the error $(\rho a^3)^{\frac{1}{3}}$ comes from the minimization of two error terms. The first coming from the truncation of the scattering equation, eq. (2.16)

$$E^3(b) \leq 4\pi a (1 - \frac{2a}{b}). \tag{2.55}$$

The second comes from the use of the inequality

$$F^2 = \prod_{i<j} (f_{ij}^b)^2 \geq \left(1 - \sum_{i=2}^{N} 1 - (f_{1i}^b)^2\right) \prod_{1<i<j} (f_{ij}^b)^2 \tag{2.56}$$

which yields the following lower bound on the denominator

$$\|\Psi_N\|^2 \geq (L^3 - CNab^2) \int \prod_{1<i<j} (f_{ij}^b)^2 dx_2..dx_N.$$

Here L^3 is the main order, and the above generates an error of order $\rho a b^2$, thus the choice of $b = \rho^{-\frac{1}{3}}$ minimizes the sum of the two errors and leads to eq. (2.53).

Recently in [Bas+24], the energy computation of F_N was improved by applying eq. (2.56) to all indices and making a much finer comparison between denominator and numerator. This allowed the choice $b = C^{-1}(\rho a)^{-1/2}$, leading to the bound

$$e^3(\rho) \leq 4\pi a \rho^2 (1 + C(\rho a^3)^{\frac{1}{2}}).$$

The constant C here is unfortunately much larger than the constant in the LHY formula (1.1). Nevertheless, this is currently the best known upper bound on $e^3(\rho)$ for the hardcore potential eq. (2.9), and so eq. (1.1) remains an open problem for the hardcore gas.

In two dimensions the same trial state eq. (2.54) can be used with analogous methods. This was carried out in [LY01], where the following upper bound was obtained

$$\frac{\langle \Psi_N, H_N \Psi_N \rangle}{\|\Psi_N\|^2} \leq \rho N E^2(b) (1 + \rho \int 1 - (f^b)^2 dx) \leq N \frac{2\pi\rho}{\log(\frac{b}{a})} \left(1 + C \frac{\rho b^2}{\log(\frac{b}{a})}\right).$$

A minimization over b gives $b = \frac{1}{\sqrt{\rho}}$ which, using the notation $Y = \frac{1}{|\log(\rho a^2)|}$, yields

$$e^2(\rho) \leq 4\pi \rho^2 Y (1 + CY). \tag{2.57}$$

Replacing $\delta = Y - Y^2 \log Y + O(Y^2)$ shows that this reproduces the correct first-order term of eq. (1.2), but is wrong at order $Y^2 \log Y$, which is larger than the second-order term.

After a gap of more than 50 years, Erdős, Schlein, and Yau [ESY08] improved Dyson's bound by considering minimization over quasi-free states, i.e., states of the form

$$\Psi = W_{N_0} e^{\mathcal{B}} \Omega \in \mathcal{F}(\Lambda). \tag{2.58}$$

where $\mathcal{B}$ is a Bogoliubov transform on the excited Fock space, Ω is the vacuum, and W_{N_0} is a Weyl shift, which effectively produces Bogoliubov first approximation $a_0, a_0^* \mapsto N_0$, eq. (2.43). The N_0 is chosen last such that

$$\langle \mathcal{N} \rangle_\Psi = \rho L^3. \tag{2.59}$$

Evaluating the Grand canonical Hamiltonian in such a state using Wicks rule (see [Sol07] for details) gives

$$\langle \Psi, H\Psi \rangle = \frac{N_0 \rho_0 \widehat{v}(0)}{2} + \sum_{0 \neq p \in \Lambda^*} (p^2 + \rho_0 \widehat{v}(p))\gamma_p + \rho_0 \widehat{v}(p)\alpha_p + \frac{1}{2L^d} \sum_{p,q \in \Lambda^*} (\alpha_p \alpha_q + \gamma_p \gamma_q)\widehat{v}(p+q), \tag{2.60}$$

where

$$\gamma_p = \langle a_p^* a_p \rangle_\Psi, \qquad \alpha_p = \langle a_p^* a_{-p}^* \rangle_\Psi = \langle a_p a_{-p} \rangle_\Psi, \qquad |\alpha_p| \leq \sqrt{\gamma_p (1 + \gamma_p)}.$$

The last inequality is a requirement for $\mathcal{B}$ to be a Bogoliubov transformation. Note also that in eq. (2.60) there is no cubic term, since odd-order terms have zero expectation in a quasi free state. In [ESY08], the authors inserted the ansatz $\alpha_p \sim -\rho \widehat{\omega}(p)$ (see [Sol25] for a nice heuristic argument indicating this ansatz) and then carried out an exact minimization over the quadratic terms in α and γ. This yields

$$e^3(\rho) \leq 4\pi a \rho^2 \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3}\right) + C\rho^2 \sqrt{\rho a^3} (\widehat{v}(0) - \widehat{g}(0)) \tag{2.61}$$

where we also used eq. (2.14) combined with eq. (2.59). In the setting of a "weak" potential whose integral is close to the scattering length, the above indeed finds the Lee-Huang-Yang energy. However, in the hardcore setting the above bound is trivial. Funnily enough the ground state of H^{bog} eq. (2.47) would have achieved the same energy precision, even though the approximation leading to H^{bog} completely neglects Q_4, which now is known to contribute at leading order.

Seeing the success of the quasi-free states, one might hope that a more precise minimisation of eq. (2.60) could yield the Lee–Huang–Yang formula. However, this is not the case, since the cubic term plays a significant role exactly at the Lee-Huang-Yang order. See [NRS18a] and [NRS18b] for a true minimisation of eq. (2.60).

It should be noted that the same quasi-free approach in two dimensions already fails at the leading order. This is due to the fact that $\widehat{v}(0) \gg \widehat{g}(0)$, which makes the setting resemble the hardcore case, something quasi-free states do not handle well. See [Fou+19] for a result in the setting $\widehat{v}(0) \sim \widehat{g}(0)$.

The correct upper bound for eq. (1.1) was obtained by Yau and Yin in [YY09] who realised that Q_3 contained the negative contribution needed to establish the Lee, Huang Yang formula (1.1). They first applied a clever Dirichlet localisation technique from [Rob71] to obtain

$$\inf_{\|\Psi\|=1} \frac{\langle \Psi, H_{L^3\rho}\Psi\rangle}{L^3} \leq \inf_{\|\Psi\|=1} \frac{\langle \Psi, H_{\ell^3\rho}\Psi\rangle}{\ell^3}(1+C\frac{R}{\ell}) + \frac{C\rho L^3}{R\ell} \tag{2.62}$$

for any ℓ satisfying $R \ll \ell \ll L$. In order to capture the Lee–Huang–Yang correction, using this localisation, one needs $\ell \gg (a^2\rho)^{-1}$. After localising, the problem reduces to a finite box. The state they constructed, on this finite box, resembled

$$\Psi = e^{\mathcal{A}+\mathcal{B}} W_{N_0}\Omega, \tag{2.63}$$

with $\mathcal{B}$ a Bogoliubov transformation and

$$\mathcal{A} = \sum_{k\in\Lambda^*} \sum_{v\sim\sqrt{\rho a}} \lambda_{k,v} a^*_{k+v} a^*_{-k+v} a_{2v} - h.c.. \tag{2.64}$$

This transformation introduces what they called soft pairs: two particles with high momentum colliding to produce one particle with zero momentum and one with low momentum. They characterised low momenta to be of order $\sqrt{\rho a}$, and high momenta as of order a^{-1}, this idea has since been used religiously.

Yau and Yin did not quite use the state of eq. (2.63) as they needed to only have each soft pair appear once. So they expanded the exponential and only kept the terms where each soft pair appeared at most once. This led to a lengthy technical computation, later simplified in [BCS21]. There, the authors ensured that each soft pair was created only once by inserting a projection into $\mathcal{A}$, which allowed them to work with the full exponential. This also permitted a significant relaxation of the assumptions on v.

One could attempt to adapt the methods of [BCS21] or [YY09] to the two-dimensional setting, but our own attempts only succeeded in obtaining the second-order term, without determining the constant.

Very recently, in [Bro+25] the authors provided an upper bound including the third order correction. First predicted by Wu in 1959 [HP59; Saw59; Wu59]. They proved

$$e^3(\rho) \leq 4\pi a\rho^2\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + 8(\frac{4\pi}{3} - \sqrt{3})\rho a^3 \log(\rho a^3) + O(\rho a^3)\right). \tag{2.65}$$

The state they considered was also of the form eq. (2.63), though here $\mathcal{A}$ had to include not only soft pairs but essentially all momenta, which greatly increased the complexity of the computations. It should be noted that formula eq. (2.65) includes the last term where universality is expected. This work built on the corresponding Gross–Pitaevskii result in [Car+25], where the associated lower bound was also established.

Lastly, we briefly discuss the upper bound [Bas+22], which had significant influence on the upper bound in [Fou+24c] Chapter 3. There, the authors used a state of the form

$$\Psi = Fe^{\mathcal{B}}\Omega, \tag{2.66}$$

with F the Jastrow factor from eq. (2.54). This is the same type of state we use in Chapter 3. The key point is that F "softens" the potentials, to the extent that the last term in eq. (2.61) is an error. To see this, one applies the IMS localisation formula

$$F\sum_i -\Delta_i F = \sum_i F_{\hat{i}}^2\left(-\Delta_i + |\nabla_i F_i|^2 - \sqrt{1-F_i^2} - \Delta_i\sqrt{1-F_i^2}\right) \tag{2.67}$$

where $F_i = \prod_{j\neq i} f_{ij}$ and $F_i F_{\hat{i}} = F$. For an upper bound we may throw away the last negative term. We can combine the middle term with the potential using integration by parts to get

$$FHF \leq \sum_i F_{\hat{i}}^2(-\Delta_i) + \sum_{i\neq j} F_{\hat{i}}^2\left(-\Delta_i f_{ij}^b + \frac{1}{2}v(x_i - x_j)f_{ij}\right) + \mathcal{R} \tag{2.68}$$

where $\mathcal{R}$ are terms in which ∇_i hits two different factors of F. If one ignores the $F_{\hat{i}}$, the above is a Hamiltonian with the interaction potential $\tilde{v} = -2\Delta f^b + vf^b$. This potential has the same scattering length as v and satisfies

$$\widehat{\tilde{v}}(0) - \widehat{g}(0) \leq \begin{cases} \frac{C\log(\frac{\tilde{R}}{b})}{\log(\frac{\tilde{R}}{a})\log(\frac{b}{a})} & d = 2, \\ \frac{Ca^2}{b} & d = 3. \end{cases} \tag{2.69}$$

One may then apply the result of [ESY08] or [Fou+19] for appropriate b. The reason this method does not work in 3 dimensions stems from the difficulty of estimating the impact from F on the norm of Ψ, however, in two dimensions we made this part work.

Lower bounds

The first Rigorous lower bound to the energy is also due to Dyson [Dys57], where he provided a lower bound for the hardcore potential eq. (2.9). He used what is now commonly referred to as the "Dyson lemma".

Lemma 2.7. *Let v be positive and radial with scattering length a and support R_0 then for any radial U which satisfies*

$$\int_{\mathbb{R}^3} Ur^2 dr \leq 1, \qquad \textit{and} \quad U(r) = 0 \quad \textit{for} \quad r < R_0, \tag{2.70}$$

we have for any starshaped domain $B \subset \mathbb{R}^3$ and $\Psi \in H^1(B)$

$$\int_B |\nabla\Psi|^2 + \frac{1}{2}v|\Psi|^2 \geq a\int_B U|\Psi|^2. \tag{2.71}$$

The same holds in 2 dimensions, with eq. (2.70) replaced by

$$\int U\log(\frac{r}{a})rdr \leq 1, \qquad \textit{and} \quad U(r) = 0 \quad \textit{for} \quad r < R_0,$$

and with a in eq. (2.71) replaced by 1.

The Lemma follows from minimizing along each radial ray (hence star shaped), where the minimizer is shown to solve the 1D scattering equation.

Defining for each i

$$H_N^i = -\Delta_i + \frac{1}{2}\sum_{j\neq i} v(x_i - x_j),$$

then for a fixed configuration of points $(x_1..,x_{i-1},x_{i+1},..,x_N)$ we may apply Dyson's lemma on the convex Voronoi cells

$$B_j = \{x \in \Lambda | \min_k |x_i - x_k| = |x_i - x_j|\}$$

resulting in the lower bound

$$H_N^i \geq aW_i(x), \qquad W_i(x) = U(\min_{j\neq i}|x_i - x_j|).$$

Dyson used this lower bound with the specific choice

$$U(r) = \frac{\max(R - r^3, 0)}{3(\frac{1}{2}R^2 - R_0(R - \frac{1}{2}R_0))} 1_{r\geq R_0}. \tag{2.72}$$

where $R \geq R_0$ is a parameter. Since v is the hardcore potential, one can apply a sphere packing result (Dyson used [Bli29]) and minimize over R, leading to

$$H_N \geq a\sum_{i=1}^N W_i(x) \geq \frac{1}{10}\sqrt{2\pi}N\rho a(1 - CR_0\rho^{\frac{1}{3}}).$$

Although this bound does not give the correct constant, Dyson's method paved the way for Lieb and Yngvason, who proved the correct first-order lower bound in [LY98]. They applied Dyson's lemma to $(1-\varepsilon)$ of the kinetic energy obtaining

$$H_N \geq \varepsilon\sum_{i=1}^N -\Delta_i + (1-\varepsilon)\sum_{i=1}^N aW_i(x). \tag{2.73}$$

Treating the potential term as a perturbation of the kinetic one finds using Temple's inequality

$$H_N \geq \langle H_N\rangle_{\Psi_0} - \frac{\langle H_N^2\rangle_{\Psi_0} - \langle H_N\rangle_{\Psi_0}^2}{E_1 - \langle H_N\rangle_{\Psi_0}} \tag{2.74}$$

where Ψ_0 is the constant function (ground state of kinetic energy) and E_1 is the second eigenvalue of H. This bound is valid when $E_1 > \langle H_N\rangle_{\Psi_0}$. Choosing U similar to eq. (2.72), one can compute $\langle H_N\rangle_{\Psi_0} \sim 4\pi N\rho a$ and we know $E_1 \geq \frac{2\pi\varepsilon}{L^2}$, where L is the side length of the box. Thus for the second term of eq. (2.74) to be an error we require

$$\frac{\epsilon}{L^2} \gg 4\pi\rho a N,$$

which implies

$$L \ll a(\rho a^3)^{-\frac{2}{5}}.$$

Hence the proof can only work in boxes not too large, necessitating a localization. We localize using a Neumann localization to get

$$H_N \geq \sum_{\sum n_j = N} \sum_j H_{n_j}^{B_j} \tag{2.75}$$

where we ignored particle interactions between boxes and allowed jumps in continuity of the wave functions between the boxes. Finally, estimating $H_{n_j}^{B_j}$ using eq. (2.74) and choosing the parameters R, L and ε appropriately yields the result

$$H_N \geq 4\pi\rho a N(1 - C(\rho a^3)^{\frac{1}{17}}). \tag{2.76}$$

In [LY01], Lieb and Yngvason applied the same method to the two dimensional gas and found essentially the same result retrieving the first order (Schick) term of eq. (1.2).

We also mention a condensation bound which follows from the proof of eq. (2.76) [LS02]. Here Seiringer and Lieb observed that most of the kinetic energy was not used in eq. (2.73); indeed, we only needed the kinetic energy $-\Delta_i$ when the nearest neighbour of x_i was closer than R. Keeping this "outer kinetic energy" and using a Poincaré inequality leads to a condensation estimate.

Theorem 2.8. *Let H_N be the Hamiltonian given in eq.* (2.1) *with Neumann boundary conditions, $\Lambda = [0, L]^d$ and $v \geq 0$ with scattering length a. Then for $\rho a^3 \ll 1$ there exist a C such that for any trial state Ψ we have*

$$\langle H_N \rangle_\Psi \geq 4\pi a\rho N(1 - (C\rho a^3)^{\frac{1}{17}}) + N\frac{\langle n_+\rangle_\Psi^2}{L^2} \tag{2.77}$$

where n_+ is the second quantisation of the projection orthogonal to the constant function. Thus if Ψ satisfies $\langle H_N \rangle_\Psi \leq 4\pi a\rho N(1 + (C\rho a^3)^{\frac{1}{17}})$ we conclude

$$\frac{\langle n_+\rangle_\Psi}{N} \leq CL\sqrt{a\rho}(\rho a^3)^{\frac{1}{34}}. \tag{2.78}$$

Equation (2.78) bounds the fraction of non-condensed particles in a wave function of sufficiently low energy. This was used in [LS02] to prove condensation in the Gross-Pitaevskii limit. The condensation estimate is also used as a priori bounds in both [Hab+24b] and Chapter 5. An equivalent result to Theorem 2.8 can, with the same method, also be proven in two dimensions.

The correct Lee-Huang-Yang energy has been obtained in various settings [Boc+18; Boc+19] for the Gross-Pitaevskii regime, and in [GS09] under the assumptions of weak interactions. However, it was the seminal work by Fournais and Solovej [FS20; FS22] that first provided a rigorous lower bound for eq. (1.1) for a general potential, thereby settling the formula definitively. We shall provide a brief overview of the result [FS20] as many of the techniques are reused in Chapters 3 to 5.

The first step is a localisation just as in [LY98], but instead of using a Neumann localisation we would like our resulting Hamiltonian to be periodic. For this purpose we draw inspiration from [BS20] and [BFS20] in which the authors develop the so-called "sliding localisation technique".

Theorem 2.9. *Let Δ denote the Laplacian on $\mathbb{R}^d$ then*

$$-\Delta \geq \int \mathcal{T}_u du := \int_{\mathbb{R}^d} Q_u \chi_u \left[\sqrt{-\Delta} - \frac{1}{2s\ell}\right]_+^2 \chi_u Q_u + b\frac{Q_u}{\ell^2} du \tag{2.79}$$

where $\chi_u \in C_0^M(B_u)$, $B_u = [u - \frac{\ell}{2}, u + \frac{\ell}{2}]^d$ and Q_u is the projection away from the condensate in B_u. Lastly $s > 0$ and $b > 0$ are constants depending only on χ.

While the above theorem is a simplification of what is used in [FS20; FS22], it does convey the message. The high-momentum kinetic energy is essentially unchanged, which is crucial, as high momenta dominate the kinetic energy contribution. The additional term bQ/ℓ^2 is positive and larger than many error terms, allowing one to use positivity arguments rather than delicate estimates. Theorem 2.9 reduces the thermodynamic problem to one on a box B_ℓ with the modified kinetic energy eq. (2.79). To approximate the Bogoliubov sum by an integral, one requires $\ell \gg (\rho a)^{-1/2}$, but ℓ must also be small enough to make use of the spectral gap $\frac{bQ}{\ell^2}$ and to bound the number of excited particles, as in Theorem 2.8.

The second step is a clever algebraic decomposition of the potential v, extracting the relevant contributions and leaving a complicated but positive remainder Q_4^{ren}, which can be discarded for a lower bound,

$$v = Q_0^{ren} + Q_1^{ren} + Q_2^{ren} + Q_2^{ex} + Q_3^{ren} + Q_4^{ren}, \tag{2.80}$$

where the indices indicate the number of non-zero creation and annihilation operators (for Q_4^{ren} this is not true). Full definitions appear in Chapter 5 Lemma 2.2. Importantly, v only appears in Q_4^{ren}; all other terms involve g or $g\omega$ eq. (2.21), allowing momentum-space analysis even for the hardcore potential. By simple computation,

$$Q_0^{ren} = \sum_{i<j} P_i P_j g + g\omega P_i P_j = (\widehat{g}(0) + \widehat{g\omega}(0))\frac{n_0(n_0-1)}{2|\Lambda|}, \qquad Q_1^{ren} = \sum_{i<j} P_i P_j g Q_i P_j \sim 0 \tag{2.81}$$

where $n_0 = a_0^* a_0$. In fact without localisation, Q_1^{ren} would vanish exactly due to momentum conservation. Combining Q_2^{ren} and the kinetic energy while ignoring some technicalities, we have

$$Q_2^{ren} + \mathcal{T} \sim \sum_p \left(p^2 + \frac{\widehat{g}(p)n_0}{|\Lambda|}\right) a_p^* a_p + \frac{1}{2}\widehat{g}(p)(a_p^* a_{-p}^* a_0 a_0 + a_p a_{-p} a_0^* a_0^*) + \widehat{g}(0)\frac{n_0}{|\Lambda|} a_p^* a_p. \tag{2.82}$$

The last term can be combined with the first term of Q_0^{ren}, just as Bogoliubov did eq. (2.44). We carry out the replacement $a_0 \mapsto \sqrt{n_z}$, using c-number substitution, see [Lie+05], which is a rigorous way of using a coherent state quantization in order to do exactly what Bogoliubov did. After this replacement we can diagonalise eq. (2.82) using the Bogoliubov transformation eq. (2.45)

$$\begin{aligned} Q_0^{ren} + Q_2^{ren} + \mathcal{T} \geq{}& \frac{N(N-1)}{2|\Lambda|}\widehat{g}(0) + \widehat{g\omega}(0)\frac{n_z(n_z-1)}{2|\Lambda|} - \frac{1}{|\Lambda|}\sum p^2 + \widehat{g}(p)\frac{n_z}{|\Lambda|} - \sqrt{p^4 + 2p^2\frac{n_z}{|\Lambda|}\widehat{g}(p)} \\ &+ \sum_p D_p b_p^* b_p - errors. \end{aligned}$$

We recognise $\widehat{g\omega}(0)$ as the term that normalises the sum/integral yielding the Lee-Huang-Yang term. For $n_z = N$, we find

$$Q_0^{ren} + Q_2^{ren} + \mathcal{T} \geq 4\pi N\rho a(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3}) - errors + \sum_p D_p b_p^* b_p. \tag{2.83}$$

Finally one combines the last terms and finds

$$\sum_p D_p b_p^* b_p + Q_2^{ex} + Q_3^{ren} \geq 0 - errors.$$

This step is the most technical, both in [FS20; FS22] and in Chapter 3. Here one uses the idea from Yau-Yin's upper bound [YY09]), to discard all momenta not corresponding to a soft pair. After that, one carries out a second Bogoliubov transformation and bounds the errors using a priori condensation estimates.

In [FS20; FS22] and Chapter 3 we use Theorem 2.9 again to go into boxes of size $\ell_s \ll (\rho a)^{-\frac{1}{2}}$ where they improve upon Theorem 2.8, see [Fou20] for review of the method and result. It is, however, possible to make due with the condensation estimates provided by Theorem 2.8, which we do in Chapter 5 significantly reducing technicalities.

In Chapter 3 we prove the following theorem, which is a restatement of Chapter 3 Theorem 2.1.

Theorem 2.10. *For any constants $C_0, \eta_0 > 0$ there exists $C, \eta > 0$ such that the following holds. Let v be a non-negative, measurable and radial potential with scattering length a and satisfying*

$$v(x) \leq \frac{C_0}{|x|^2}\left(\frac{a}{|x|}\right)^{\eta_0}, \qquad |x| \geq C_0 a.$$

Then for $\rho a^2 \leq C^{-1}$

$$\left| e^2(\rho) - 2\pi\rho\delta\left(1 + \left(\frac{1}{4} + \Gamma + \frac{\log(\pi)}{2}\right)\delta\right)\right| \leq C\rho^2\delta^{2+\eta}$$

with δ given in eq. (1.2)*,*

$$\delta = \frac{2}{|\log(\rho a^2 |\log(\rho a^2)|^{-1})|}.$$

2.3.2 Free energy

We briefly review some rigorous results concerning the free energy of the dilute Bose gas eqs. (1.3) and (1.4). Most relevant are [Hab+24a] and [Hab+24b], in which the authors establish eq. (1.3) through lower- and upper bounds respectively.

For the lower bound, one could in principle compute the free energy by establishing Conjecture 2.6, with the right constant, in a sufficiently large box. However, the sliding localisation technique (Theorem 2.9) seems to alter the kinetic energy too much to recover the excitation spectrum. Therefore, in [Hab+24a] the authors instead return to [LY98] and use a Neumann localisation eq. (2.75)

$$H \geq \sum_{boxes} \sum -\Delta_{box}^{\mathcal{N}} + \sum_{i<j} v_{box}(x_i - x_j), \tag{2.84}$$

where the box is of size ℓ. This has the disadvantage that the potential no longer has a nice structure in the Neumann basis. They resolve this by "Neumann symmetrising" the potential, which effectively amounts to adding “mirror particles". If one assumes v decreasing so that $v^{sym} \leq Cv$ then for sufficiently low energy states Ψ, one can show

$$\langle \sum_{i<j} v_{box}^{sym}(x_i - x_j) - v_{box}(x_i - x_j) \rangle_\Psi \leq \frac{R}{\ell} \widehat{v}(0) \frac{N^2}{\ell^3}. \tag{2.85}$$

The above is an error if v is integrable and the box is much larger than $(\rho a)^{-\frac{1}{2}}$. Choosing the box just larger than $(\rho a)^{-\frac{1}{2}}$, essentially puts us in the Gross-Pitaevskii scaling with a nice kinetic and potential operator. In this setting there are many impressive results, which compute the excitation spectrum, some of which we already mentioned. In [Hab+24a] the authors in particular draw inspiration from [HST22b; NT23]. It should be noted that the error in eq. (2.85) is bigger than the size of the first eigenvalue, thus the individual eigenvalues in the excitation spectrum are not deduced in [Hab+24a], but instead only their collective behaviour.

Equation (2.85) highlights a particular difficulty posed by the hardcore potential (eq. (2.9)). In Chapter 5 we avoid the issue of symmetrising v by first using the renormalisation eq. (2.80) and then discarding Q_4^{ren}. Thus we only need to symmetrize g and $g\omega$, which are both integrable.

In [Hab+24b] the authors take inspiration from [BCS21], which we briefly discussed in Section 2.3.1. They use a trial state similar to the one in [BCS21], but instead of having the cubic and quadratic transformations act on the vacuum eq. (2.63), they act on the Gibbs state of the quadratic Bogoliubov Hamiltonian eq. (2.47). I.e the state they use is essentially

$$\Gamma = e^{\mathcal{B}_2} e^{\mathcal{A}} e^{\mathcal{B}_1} W_{N_0} e^{-\sum D_p b_p^* b_p} W_{N_0}^* e^{-\mathcal{B}_1} e^{-\mathcal{A}} e^{-\mathcal{B}_2} \tag{2.86}$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are Bogoliubov transformations and $\mathcal{A}$ is the cubic transformation. Not evaluating on the vacuum complicates matters significantly. One idea they employ is to "open up" $e^{\mathcal{A}}$,

$$e^{\mathcal{A}} = e^{\sum \mathcal{A}_k} \sim \prod_k e^{\mathcal{A}_k}$$

so instead of working with $e^{\mathcal{A}}$ directly, they use the product representation on the right. This allows them to compute only the individual commutator $[\mathcal{A}_k, H]$ which is significantly simpler than the full commutator.

Although the “full” expansions of the free energy eqs. (1.3) and (1.4) have only attracted attention since the resolution of eqs. (1.1) and (1.2), there have been insightful results computing the free energy to leading order, and at much higher temperatures. In [Sei08] and [Yin10] the authors prove a lower and upper bound respectively, to the formula

$$f^3(\rho, \beta) = f_0^3(\rho, \beta) + 4\pi a (2\rho^2 - [\rho - \rho_c(\beta)]_+^2) \tag{2.87}$$

where $\rho_c(\beta)$ is the critical density eq. (2.38) and $f_0(\rho,\beta)$ is the free energy of the ideal gas, which was essentially computed in Section 2.2.1. The above holds in the dilute limit and $\beta \leq C\beta_c(\rho)$ for any large constant C, therefore the critical temperature is included. See also [Bas+25] for a recent improvement on the upper bound of eq. (2.87), allowing for more singular potentials.

In 2 dimensions an equivalent result to eq. (2.87) was recently found in [DMS20] and [MS20] for lower and upper bound respectively:

$$f^2(\rho,\beta) = f_0^2(\beta,\rho-\rho_s) + \frac{4\pi}{|\log(\rho a^2)|}(2\rho^2 - \rho_s^2),$$

where ρ_s is the superfluid fraction

$$\rho_s = \rho[1 - \frac{\beta_c}{\beta}]_+$$

and β_c is the Berezinskii–Kosterlitz–Thouless critical temperature [KT73],

$$\beta_c = \frac{4\pi \log|\log(\rho a^2)|}{\rho}.$$

We remark that the state in [MS20] resembles the one from Chapter 6, except that in Chapter 6 we use the Gibbs state of the Bogoliubov Hamiltonian eq. (2.47) (with a Jastrow factor), whereas [MS20] considers the Gibbs state of the free Hamiltonian.

In Chapter 5 we prove the following theorem, which is a restatement of Theorem 1.1 from the same chapter.

Theorem 2.11. *For any $C_0 > 0$ there exists $C > 0$ such that for $\eta > 0$ small enough the following holds. Let v be a positive radially decreasing potential with scattering length a and compact support $R \leq C_0 a$. Then for $\rho a^3 \leq C^{-1}$, $\nu \in (0, \frac{\eta}{3})$, and $T \leq \rho a(\rho a^3)^{-\nu}$,*

$$f(\rho,T) \geq 4\pi a\rho^2\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3}\right) + \frac{T^{\frac{5}{2}}}{(2\pi)^3}\int_{\mathbb{R}^3} \log\left(1 - e^{-\sqrt{p^4 + \frac{16\pi\rho a}{T}p^2}}\right) dp.$$

In chapter 6 we prove the following theorem, which is a restatement of Theorem 1 from the same chapter.

Theorem 2.12. *For a radial positive potential v with compact support R and scattering length a, there exist a constant c (only depending on the support and scattering length of v) such that for $T \leq cT_c = c\beta_c^{-1}$ and $\rho a^2 \leq c$ we have*

$$f^2(\rho,T) \leq 2\pi\rho^2\delta\left(1 + \left(\Gamma + \frac{1}{4} + \frac{\log(\pi)}{2}\right)\delta\right) + \frac{T^2}{(2\pi)^2}\int_{\mathbb{R}^2} \log\left(1 - e^{-\sqrt{p^4 + \frac{8\pi\rho\delta}{T}p^2}}\right) + c^{-1}\rho^2\delta\left(\delta^2|\log(\delta)| + \frac{T^2}{T_c^2}\right).$$

Here $\Gamma = 0.577\ldots$ is the Euler-Mascheroni constant and

$$\delta = \frac{2}{|\log(\rho a^2)| + |\log\left(|\log(\rho a^2)|\right)|}.$$

2.3.3 Mean field scaling

The mean-field system described by the Hamiltonian eq. (2.10) has also gotten a lot of attention in recent times. Perhaps the most impressive result is that of [BPS21] which provides a full description of the low lying eigenvalues and eigenvectors for $\beta = 0$, see also [BP24] for a nice review on the topic. However, these works do not address the particularly interesting and difficult case of attractive potentials. We will instead briefly review some of the works that tackle the case of negative pair potentials [LNR15; LNR16; LNR14; NR20; Tri18].

In the mean field setting, where the potential is much weaker and longer range, it is reasonable to assume that the ground state is essentially a bosonic product state i.e.

$$\begin{aligned}\inf_{\|\Psi\|=1} \frac{1}{N}\langle \Psi, H_N^\beta \Psi\rangle &\sim \inf_{\|u\|=1} \frac{1}{N}\langle u^{\otimes N}, H_N^\beta u^{\otimes N}\rangle \\ &= \inf_{\|u\|=1} \int |\nabla u|^2 + \frac{1}{2}\int |u|^2 (v_N^\beta * |u|)^2 =: \inf_{\|u\|=1} \mathcal{E}_N^H(u),\end{aligned} \tag{2.88}$$

with

$$v_N^\beta(x) = N^{d\beta} v(N^\beta(x)).$$

The above is obviously an upper bound to the ground state energy. However, proving that it is also a lower bound can be tricky. The aim is therefore to prove

$$\lim_{N\to\infty} \inf_{\|\Psi\|=1} \frac{1}{N}\langle \Psi, H_N^\beta \Psi\rangle \geq \lim_{N\to\infty} \inf_{\|u\|=1} \mathcal{E}_N^H(u). \tag{2.89}$$

In [LNR14] they prove the above inequality under various external potentials and $\beta = 0$. Then in [LNR16] this is extended to 2 dimensions and positive β for an external trapping potential $V^{trap}(x) \geq C|x|^s$, $s > 0$. In particular, they found that eq. (2.89) holds under the condition

$$\beta < \frac{1}{d(1 + \frac{d}{2} + \frac{d}{s})}. \tag{2.90}$$

The range of β in two dimensions would be further increased to $\frac{s+1}{s+2}$ in [LNR15] and to $\beta < 1$ in [NR20]. In three dimensions the best known result is $\beta < \frac{1}{3} + \frac{s}{45+42s}$ in [Tri18]. In chapter 8 we extend the range of β in the two dimensional setting to infinity, we even allow for essentially any sub-exponential scaling.

In order to introduce some of the techniques, we go through some details of the proof found in [LNR16]. For convenience we will henceforth assume

$$v \in L^1 \cap L^\infty(\mathbb{R}^d), \qquad V^{trap}(x) \geq C|x|^s. \tag{2.91}$$

The general strategy of proving eq. (2.89) is to apply some version of a quantitative quantum de Finetti result. The following is due to Christandl, König, Renner and Mitchison [Chr+07].

Theorem 2.13. *Let Ψ be an N-particle bosonic state on a Hilbert space $\mathcal{H}$ of dimension D. Then for all $k \in \mathbb{N}$ there exist a probability measure μ_k on the unit sphere $S\mathcal{H}$ such that*

$$\left\| \Gamma_\Psi^{(k)} - \int_{S\mathcal{H}} |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u) \right\|_1 \leq \frac{kD}{N} \tag{2.92}$$

where $\|\cdot\|_1$ is the trace norm and $\Gamma_\Psi^{(k)} = \mathrm{Tr}_{k+1\to N} |\Psi\rangle\langle\Psi|$ is the reduced particle density matrix.

The point is then to approximate any trial state Ψ to a finite but N dependent dimensional subspace and use Theorem 2.13 on this subspace. To be more precise, we define the wanted subspace P as

$$P = 1_{T \leq f(N)} \quad \text{where} \quad T = -\Delta + V^{trap}, \tag{2.93}$$

where f is a suitable positive function, usually polynomial, and T is the one body operator of our hamiltonian. Note that from eq. (2.91) P becomes finite dimensional, indeed the dimension of P can be bounded by a CLR-type estimate

$$\dim(P) \leq f(N)^{\frac{d}{s}+\frac{d}{2}}. \tag{2.94}$$

Using the language of the reduced density matrices we have

$$\frac{1}{N}\langle \Psi, H_N^\beta \Psi\rangle = \mathrm{Tr}(\Gamma_\Psi^{(2)} H_{N,2}^\beta)$$

where

$$H_{N,2}^\beta = \frac{1}{2}(T_1 + T_2) + \frac{1}{2} v_N^\beta (x_1 - x_2).$$

We then write the identity $1 = P + Q$ four times

$$\mathrm{Tr}(\Gamma_\Psi^{(2)} H_{N,2}^\beta) = \mathrm{Tr}((P+Q)\otimes(P+Q)\Gamma_\Psi^{(2)}(P+Q)\otimes(P+Q)H_{N,2}^\beta).$$

We use an easy extension of eq. (2.92) on the state $P \otimes P\Gamma_\Psi P \otimes P$ to find

$$\begin{aligned} \mathrm{Tr}(\Gamma_\Psi^{(2)} H_{N,2}^\beta) &= \mathrm{Tr}(P\otimes P\Gamma_\Psi^{(2)} P\otimes P) H_{N,2}^\beta \\ &\geq \int \langle u^{\otimes 2}, H_N^\beta u^{\otimes 2}\rangle d\mu(u) - C\|PH_N^\beta P\|_\infty \frac{dim(P)}{N}. \end{aligned} \tag{2.95}$$

The first term on the right hand side is bounded by the Hartree energy (the right hand side of eq. (2.89)), the second term becomes an error if $f(N)$ is not too big depending on β and s. Lastly, the Q-terms can be bounded if $f(N)$ is large enough depending on β and s, one would need

$$Q\otimes Q v_N^\beta Q \otimes Q \leq Q \otimes Q(T_1 + T_2) Q \otimes Q$$

which by assumption (2.91) would follow if $f(N) \geq N^{d\beta}$. Taking $f(N) \sim CN^{d\beta}$ the error in eq. (2.95) becomes

$$C\|PH_N^\beta P\|_\infty \frac{dim(P)}{N} \leq CN^{d\beta} \frac{N^{d\beta(\frac{d}{s}+\frac{d}{2})}}{N} \ll 1 \qquad \text{if} \qquad \beta < \frac{1}{d(1+\frac{d}{s}+\frac{d}{2})},$$

which proves the result of [LNR16]. It is clear that the dimensional factor appearing in eq. (2.92) is largely contributes to the worsening of the bound. Therefore, the following theorem due to Brandao and Harrow [Rou17] enabled noticeable improvements.

Theorem 2.14. *Let Γ_N be a bosonic state on $\mathcal{H}^{\otimes_{sym}^N}$ with $\mathcal{H}$ of dimension D. Then there exist a probability measure μ on $S_1(\mathcal{H}) = \{\gamma \in B(\mathcal{H}) | \gamma \geq 0, \mathrm{Tr}(\gamma) = 1)\}$ such that*

$$\left|\mathrm{Tr}\, A \otimes B \left(\Gamma_N^{(2)} - \int_{S_1(\mathcal{H})} \gamma^{\otimes 2} d\mu(\gamma)\right)\right| \leq 3\|A\|_\infty \|B\|_\infty \sqrt{\frac{\log(D)}{N}} \tag{2.96}$$

where A and B are any self adjoint bounded operators.

The above theorem was applied in [LNR15] with a slightly different technique from that used in [LNR16]. In Chapter 8 we also use Theorem 2.14, but with a different technique: Instead of having one projection P we introduce a number of projections depending on β. In Chapter 8 we prove the following theorem a restatement of Theorem 2 of the same chapter

Theorem 2.15. *Assume $v \in L^1(\mathbb{R}^2) \cap L^{1+\eta}(\mathbb{R}^2)$ to be even and $V^{trap}(x) \geq C^{-1}|x|^{-s} - C$ for some $\eta, C, s > 0$, further assume*

$$\int v^- < a^*$$

where $a^* > 0$ is the optimal constant in the 2D-$H^1 - L^4$ Gagliardo-Nirenberg inequality. Then we have

$$\lim_{N\to\infty} \inf_{\|\Psi\|=1} \frac{1}{N} \langle \Psi, H_N^\beta \Psi\rangle = \lim_{N\to\infty} \inf_{\|u\|=1} \mathcal{E}_N^H(u) = \inf_{\|u\|=1} \mathcal{E}^{nls}(u) > -\infty,$$

where

$$\mathcal{E}^{nls}(u) = \int |\nabla u|^2 + \frac{1}{2} \int v dx \int |u|^4 dx.$$

Moreover we have convergence of states in the following sense: For a sequence Ψ_N of ground states of H_N, there exists a probability measure μ supported on the minimizers of $\mathcal{E}^{nls}$ such that

$$\lim_{N\to\infty} \mathrm{Tr} \left| \Gamma_{\Psi_N}^{(k)} - \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u)\right| = 0, \qquad \forall k \in \mathbb{N}.$$

Chapter 3

Paper: The Ground State Energy of a Two Dimensional Bose gas

This chapter contains the paper [Fou+24c] by Fournais, Girardot, Morin, Olivieri and the author. It proves eq. (1.2) in full for almost all positive pair interaction potentials. The paper is included in its entirety in the published form in Communications in Mathematical Physics, which can be found at https://doi.org/10.1007/s00220-023-04907-2. It can be located within the thesis by the colour ■ at the top of the page.

Commun. Math. Phys. (2024) 405:59
Digital Object Identifier (DOI) https://doi.org/10.1007/s00220-023-04907-2

Communications in Mathematical Physics

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The Ground State Energy of a Two-Dimensional Bose Gas

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Received: 25 April 2023 / Accepted: 13 November 2023

Abstract: We prove the following formula for the ground state energy density of a dilute Bose gas with density ρ in 2 dimensions in the thermodynamic limit

$$e^{2\mathrm{D}}(\rho) = 4\pi\rho^2 Y\Big(1 - Y|\log Y| + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)Y\Big) + o(\rho^2 Y^2),$$

as $\rho a^2 \to 0$. Here $Y = |\log(\rho a^2)|^{-1}$ and a is the scattering length of the two-body potential. This result in 2 dimensions corresponds to the famous Lee–Huang–Yang formula in 3 dimensions. The proof is valid for essentially all positive potentials with finite scattering length, in particular, it covers the crucial case of the hard core potential.

Contents

Published online: 23 February 2024

1. Introduction

The calculation of the ground state energy of a dilute gas of bosons is of fundamental importance and has been the focus of much attention in recent years. This question can be posed in all dimensions of the ambient space, but of course, the most important case from the point of view of Physics is the 3-dimensional situation. However, also 1 and 2 dimensions are experimentally realizable. In this paper we study the 2-dimensional setting and prove an asymptotic formula analogous to the famous Lee–Huang–Yang formula in 3-dimensions.

Let us be more precise about the setting of the result. We consider positive, measurable potentials $v : \mathbb{R}^2 \to [0, +\infty]$ that are radial. Given such a potential, we will let $a = a(v)$

be its scattering length (for details on the scattering length see Sect. 3) and define the Hamiltonian

$$H(N, L) = \sum_{j=1}^{N} -\Delta_j + \sum_{j<k} v(x_j - x_k), \tag{1.1}$$

on $L^2(\Omega^N)$, with $\Omega = [-\frac{L}{2}, \frac{L}{2}]^2$. The ground state energy density in the thermodynamic limit $e^{2\mathrm{D}}(\rho)$ is then defined by

$$e^{2\mathrm{D}}(\rho) := \lim_{\substack{L\to\infty \\ N/L^2\to\rho}} L^{-2} \inf_{\Psi\in C_0^\infty(\Omega^N)} \frac{\langle \Psi, H(N, L)\Psi\rangle}{\|\Psi\|^2}. \tag{1.2}$$

It is a standard result that the limit exists, and actually our analysis of $e^{2\mathrm{D}}(\rho)$ proceeds by giving upper bounds on the lim sup and lower bounds on the lim inf. It is also well-known that the limit is independent of the boundary conditions. The fact that we consider $\Psi \in C_0^\infty$ in the formula above, corresponds to the choice of Dirichlet boundary conditions for concreteness.

Theorem 1.1 (Main result). *For any constants $C_0, \eta_0 > 0$, there exist $C, \eta > 0$ (depending only on C_0 and η_0) such that the following holds. If the (measurable) potential $v : \mathbb{R}^2 \to [0, +\infty]$ is non-negative and radial with scattering length a and $\rho a^2 < C^{-1}$, and, furthermore,*

$$v(x) \leq \frac{C_0}{|x|^2}\Big(\frac{a}{|x|}\Big)^{\eta_0}, \qquad \textit{for all } |x| \geq C_0 a. \tag{1.3}$$

Then

$$\Big|e^{2D}(\rho) - 4\pi\rho^2\delta_0\Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)\delta_0\Big)\Big| \leq C\rho^2\delta_0^{2+\eta}, \tag{1.4}$$

with

$$\delta_0 := |\log(\rho a^2|\log(\rho a^2)|^{-1})|^{-1}, \tag{1.5}$$

where $\Gamma = 0.577\ldots$ is the Euler–Mascheroni constant.

In terms of the simpler parameter $Y = |\log(\rho a^2)|^{-1}$, we get from (1.4), expanding δ_0 in terms of Y, the three-term asymptotics

$$e^{2\mathrm{D}}(\rho) = 4\pi\rho^2 Y\Big(1 - Y|\log Y| + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)Y\Big) + \mathcal{O}(\rho^2 Y^{2+\eta}). \tag{1.6}$$

Here the third term in the asymptotics is analogous to the famous Lee–Huang–Yang term in the 3-dimensional situation.

Notice, in particular, that the decay assumption (1.3) is valid for potentials with compact support. So Theorem 1.1 applies to the very important special case of the hard core potential of radius a:

$$v_{\mathrm{hc}}(x) = \begin{cases} 0, & |x| > a, \\ +\infty, & |x| \leq a. \end{cases} \tag{1.7}$$

For this potential the radius of the support is equal to its scattering length.

The proof of Theorem 1.1 will proceed by establishing upper and lower bounds. In Theorems 2.1 and 2.3 below, we will state more precisely the estimates for the upper and lower bounds, respectively, and the assumptions necessary for each of these. In Sect. 2 below, we will give an outline of the paper as well as these precise statements.

The first term $4\pi\rho^2 Y$ in (1.6) was understood in [1] but a full proof was only given in 2001 in the paper [2]. Calculations beyond leading order were given in [3–6], but have so far not been rigorously proven. The recent papers [7,8] give an analogous expansion of the ground state energy in the setting of the Gross–Pitaevskii regime, giving furthermore information about the excitation spectrum. The constant in the second order term was also found in [9] by restricting to quasi-free states in a special scaling regime.

In the 3-dimensional case, the asymptotic formula for the energy density (with $e^{3\mathrm{D}}(\rho)$ defined analogously to (1.2) and a being here the 3-dimensional scattering length) is

$$e^{3\mathrm{D}}(\rho) = 4\pi a\rho^2\Big(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3}\Big) + o(a\rho^2\sqrt{\rho a^3}). \tag{1.8}$$

This is the famous Lee–Huang–Yang formula. The leading order term goes back to [10], and the second term—the Lee–Huang–Yang (LHY) term—were given in [11,12]. Mathematically rigorous proofs of the leading order term were given in [13] (upper bound) and [14] (matching lower bound). Upper bounds for sufficiently regular potentials to the precision of the LHY-term were given in [15] (correct order only), [16] (first upper bound with correct constant on the LHY-term) with recent improvements in [17]. Lower bounds of second order were given in [18] (potentials in L^1) and [19] (general case including the hard core potential). The upper bound in 3-dimensions in the case of potentials with large L^1-norm, in particular the key example of hard core potentials, is still open.

As can be understood from this overview of results from the analysis of the $3D$ case, it is difficult to prove precise results on the energy when $\int_{\mathbb{R}^3} v$ is much larger than the scattering length $a(v)$, i.e., the hard core case. In 2-dimensions the analogous comparison is between $\int_{\mathbb{R}^2} v$ and δ_0, which always satisfy $\int_{\mathbb{R}^2} v \gg \delta_0$. So in 2-dimensions we face similar challenges as in the 3D hard core case, even for regular potentials. This is one of the reasons why progress on the 2D problem has been slower. It is therefore remarkable that Theorem 1.1 can be established, including both upper and lower bounds, without any extra assumptions on the potentials. Also, the $2D$ case comes with its own challenges due to the logaritmic divergences and changes of the lengthscales. In particular, the small parameter in $3D$ is (ρa^3), i.e. it is a power of the density parameter, whereas in the present $2D$ case, our small parameter is $Y = |\log(\rho a^2)|^{-1}$ which is logarithmic in the density.

Throughout the paper we will use the standard convention that $C > 0$ will denote an arbitrarily large universal constant whose value can change from one line to the other.

Notation We will use the following notation for Fourier transforms,

$$\widehat{f}(p) = \widehat{f}_p = \int e^{-ixp} f(x)\,\mathrm{d}x.$$

In the paper we will use the notation $A \ll B$ in a precise sense given by (H1).

2. Strategies of the Proofs

2.1. Upper bound. As upper bound we prove the following theorem.

Theorem 2.1. *For any constants C_0, $\eta_0 > 0$, there exists C (that depends only on C_0 and η_0) such that the following holds. Let $v : \mathbb{R}^2 \to [0, \infty]$ be a non-negative, measurable and radial potential with scattering length $a < \infty$, and satisfying the following decay property,*

$$v(x) \leq \frac{C_0}{|x|^2}\Big(\frac{a}{|x|}\Big)^{\eta_0} \quad \text{for} \quad |x| \geq C_0 a. \tag{2.1}$$

Then, if $\rho a^2 < C^{-1}$,

$$e^{2D}(\rho) \leq 4\pi\rho^2\delta_0\Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)\delta_0\Big) + C\rho^2\delta_0^3|\log(\delta_0)|,$$

with δ_0 given by (1.5).

In order to prove Theorem 2.1, we will reduce the analysis to the case of compactly supported potentials on a smaller periodic box $\Lambda = \Lambda_\beta = [-\frac{L_\beta}{2}, \frac{L_\beta}{2}]^2$ with length

$$L_\beta = \rho^{-1/2}Y^{-\beta}, \qquad \beta > 0. \tag{2.2}$$

In this box, if the density is ρ, the number of particles is $N = \rho L_\beta^2 = Y^{-2\beta} \gg 1$. Throughout the paper we find conditions on β over which we will optimize. For a potential v with $\operatorname{supp} v \subseteq B(0, \frac{L_\beta}{2})$, we consider the following Hamiltonian acting on the Fock space $\mathscr{F}_s(L^2(\Lambda_\beta))$,

$$\mathcal{H}_v = \bigoplus_{n\geq 0}\Big(\sum_{i=1}^{n} -\Delta_{x_i}^{\text{per}} + \sum_{1\leq i<j\leq n} v^{\text{per}}(x_i - x_j)\Big). \tag{2.3}$$

Here Δ^{per} is the periodic Laplacian, and $v^{\text{per}}(x) = \sum_{m\in\mathbb{Z}^2} v(x + L_\beta m)$ is the periodic version of v. Note that for any $p \in \frac{2\pi}{L_\beta}\mathbb{Z}^2$, the Fourier coefficient of v^{per} is equal to the Fourier transform $\widehat{v}(p)$, because the radius of the support of v is smaller than L_β. In this setting we prove the following result.

Theorem 2.2. *For any $\beta \geq \frac{3}{2}$, there exists $C > 0$, depending only on β such that the following holds. Let $\rho > 0$ and $v : \mathbb{R}^2 \to [0, \infty]$ be a non-negative, measurable and radial potential with scattering length a and $\operatorname{supp} v \subset B(0, R)$ for some $R > 0$. If $\rho R^2 \leq Y^{2\beta+2}$ and $\rho a^2 \leq C^{-1}$, then there exists a normalized trial state $\Psi \in \mathscr{F}_s(L^2(\Lambda_\beta))$, such that,*

$$\langle \mathcal{H}_v\rangle_\Psi \leq 4\pi L_\beta^2\rho^2\delta_0\Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)\delta_0\Big) + CL_\beta^2\rho^2\delta_0^3|\log(\delta_0)|.$$

Moreover Ψ satisfies $\langle\mathcal{N}\rangle_\Psi \geq N(1 - CY^2)$, and $\langle\mathcal{N}^2\rangle_\Psi \leq 9N^2$, where $\mathcal{N}$ is the number operator on $\mathscr{F}_s(L^2(\Lambda_\beta))$ and $N = \rho L_\beta^2 = Y^{-2\beta}$.

2.1.1. Strategy for the upper bound

1. We will show in "Appendix A" how Theorem 2.1 follows from Theorem 2.2. This corresponds to go from the result on the box Λ_β to the thermodynamic limit.
2. The rest of the proof, Sects. 4 and 5, is dedicated to the proof of Theorem 2.2. We first prove in Sect. 4 a weaker upper bound with the assumption that the potential is regular enough. We call it a *soft* potential. Under this assumption, we use a quasi-free trial state Φ built over a Weyl transform W_{N_0} to create the condensate and a unitary T_ν to deal with the excitations. We then minimize over the parameters of this state. This is an adaptation of the method of [15,20–22] to the 2D case. We show in Theorem 4.1 that, with a good choice of Φ to our level of precision, we have
$$\begin{aligned}\langle \mathcal{H}_v \rangle_\Phi \leq 4\pi L_\beta^2 \rho^2 \delta_0 \Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)\delta_0\Big) \\ + C L_\beta^2 \rho^2 \delta_0 (\widehat{v}_0 - \widehat{g}_0) + C L_\beta^2 \rho^2 \delta_0^2 \widehat{v}_0.\end{aligned} \tag{2.4}$$
Here $g = \varphi v$ and φ is the scattering solution associated to v (see Sect. 3 for the precise definition of φ and with parameter δ_0). This provides a first upper bound, but it is not enough to prove Theorem 2.2, unless v admits a Fourier transform and $\widehat{v}_0$ is of order $\widehat{g}_0$.
3. In Sect. 5 we explain how to reduce from any v to a soft potential. To this end, we take care of the influence of the potential on a much shorter length scale by introducing φ_b as the scattering solution normalized at
$$b = \rho^{-\frac{1}{2}} Y^{\beta + \frac{1}{2}} \tag{2.5}$$
and use it to build a *Jastrow function* as follows
$$F_n(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} f(x_i - x_j), \tag{2.6}$$
with $f = \min(1, \varphi_b)$. Then our complete trial state will be the following product state
$$\Psi = \bigoplus_{n \geq 1} F_n \Phi_n, \quad F_n, \Phi_n \in L_s^2(\Lambda_\beta^n), \tag{2.7}$$
where $\Phi = \sum \Phi_n$ is a quasi-free state. When we compute the energy of such a state Ψ we get
$$\langle \mathcal{H}_v \rangle_\Psi \leq \langle \mathcal{H}_{\widetilde{v}} \rangle_\Phi + \langle \mathcal{R} \rangle_\Phi, \tag{2.8}$$
where $\mathcal{R}$ is an error term and $\widetilde{v}$ is the following soft potential,
$$\widetilde{v} = 2 f'(b) \delta_{\{|x| = b\}}. \tag{2.9}$$
The power of Y driven by the parameter β in b is chosen minimal such that $\|\Psi\|^2 = \|\Phi\|^2 + O(Y^2)$, see Lemma 5.3. For the result to apply for the widest range of potentials we will want to choose β as small as possible.

The potential in (2.9) is soft in the sense that it has a decaying Fourier transform and $\widehat{\widetilde{v}}_0 \simeq \widehat{\widetilde{g}}_0$ (see Lemma 3.10 for precise estimates). Then we can take for Φ the optimal quasi-free state satisfying (2.4) for $\widetilde{v}$ and this turns out to be enough to prove Theorem 2.2.

2.1.2. Remarks Since the Jastrow factor (2.6) encodes all 2-particle interactions—at least on short scales—it is a natural trial state for getting upper bounds on the energy. In particular, it has been used to get the correct first order upper bound, both in 3D [13] and 2D [2]. In the product state Ψ, the Jastrow factor deals with short distance correlations between particles (when $|x_i - x_j| \leq b$), while long range effects are dealt with by the quasi-free state Φ. In the case of hard core potentials, the Jastrow factor also imposes the necessary condition that our state vanishes whenever two particles are too close.

We emphasize the following major differences between 2D and 3D. To be able to reduce to the quasi-free state Φ, we need to bound $\mathcal{O}(N^2)$ terms of the form $f(x_i - x_j)$ by 1. The number of particles N in our box is not too large (powers of $|\log(\rho a^2)|$, since the relevant length-scale is $\rho^{-1/2}$ up to logarithmic factors) thus making this error controllable. This is not at all the case in dimension 3, because the number of particles in the box is of order $(\rho a^3)^{-2}$ (since the relevant length-scale in this case is $\frac{1}{\rho a^2}$). However, a similar state as ours was successfully used in the 3D Gross–Pitaevskii regime [22] (length-scale $\frac{1}{\sqrt{\rho a}}$). In this regime the number of particles is $(\rho a^3)^{-\frac{1}{2}}$, which allows the authors, with substantially more work, to get through to a good upper bound. More precisely, they use more accurate bounds on the Jastrow factor compared to our Sect. 5 and obtain the LHY order in the box. See Remark 5.1 for additional information.

Finally, one should notice that Φ is a quasi-free state, and does not include the soft pair interactions that were necessary in [16,17] to get the correct upper bound in 3D. Indeed, for a quasi-free state Φ the second order energy bounds are in terms of $\widehat{v}_0$ and to get the correct constant one needs to change $\widehat{v}_0$'s into $\widehat{g}_0$'s. This is the role of soft pairs. However, our potential $\widetilde{v}$ from (2.9) already satisfies $\widehat{\widetilde{v}}_0 - \widehat{\widetilde{g}}_0 = \mathcal{O}(Y^2|\log Y|)$ (see Lemma 3.10) and this replacement only gives errors of order $\rho^2 Y^3 |\log Y|$. It is possible that we could add the soft pair interactions into Φ to reduce this error at the expense of a much longer and more technical proof.

We conclude this section by proving Theorem 2.1 using Theorem 2.2 and the classical theory of localization to smaller boxes which is added for convenience in "Appendix A".

2.2. Lower bound. In this section we provide the strategy of proof for the theorem below.

Theorem 2.3. *For any constant $\eta_1 > 0$ there exist $C, \eta > 0$ (depending only on η_1) such that the following holds. Let $\rho > 0$ and $v : \mathbb{R}^2 \to [0, +\infty]$ be a non-negative, measurable and radial potential with scattering length $a < \infty$. If $\rho a^2 < C^{-1}$ and*

$$\int_{\{|x| > \rho^{-1/2}\}} v(x) \log\Big(\frac{|x|}{a}\Big)^2 \mathrm{d}x \leq Y^{\eta_1}, \tag{2.10}$$

then

$$e^{2D}(\rho) \geq 4\pi\rho^2\delta_0\Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)\delta_0\Big) + C\rho^2\delta_0^{2+\eta}, \tag{2.11}$$

with δ_0 as defined in (1.5).

We introduce the lengths

$$\ell = \rho^{-1/2} Y^{-1/2-\alpha}, \qquad \ell_{\delta_0} = \frac{1}{2} e^{\Gamma} \rho^{-1/2} Y^{-1/2}, \tag{2.12}$$

for a certain $\alpha \in (0, 1)$, the second of which being called the *healing length*. The proof of Theorem 2.3 will depend on a precise choice of a number of parameters. For convenience these and the relations between them have been collected in "Appendix H".

We work at three different lengthscales:

- the thermodynamical scale, in the box $\Omega = [-L/2, L/2]^2$, where we state the main result in the limit $L \to +\infty$;
- the large box scale $\Lambda = [-\ell/2, \ell/2]^2$, where we prove most of the results and by the sliding localization techniques we integrate over all these boxes to prove the lower bound in the whole thermodynamical box;
- the small box scale $B = [-d\ell/2, d\ell/2]^2$, with $d \ll 1$, where we derive a bound for the number of particles excited out from the condensate, fundamental for the general strategy, obtaining the Bose–Einstein condensation (BEC).

The relations

$$d\ell \ll \ell_{\delta_0} \ll \ell \ll L, \tag{2.13}$$

guarantee that the boxes are in a chain of inclusions.

2.2.1. Strategy for the lower bound The overall strategy for the lower bound has the same structure as in the $3D$ hard core case analyzed in [19]. Therefore, many of the steps below are the same as in that case. We will only indicate when a step differs from its $3D$ counterpart. However, the $2D$ case comes with its own challenges due to the logaritmic divergences and changes of the lengthscales.

1. In Sect. 6.1 we reformulate the problem in a grand canonical setting, adding a chemical potential ρ_μ to the Hamiltonian, in order to control the distribution of particles in later localization steps. The resulting Hamiltonian $\mathcal{H}_{\rho_\mu}$ acts on the symmetric Fock space $\mathcal{F}_s(L^2(\Omega))$. We also reduce the analysis to compactly supported potentials with norm[1] $\|v\|_1 \leq Y^{-1/8}$, using the analysis of the scattering equation from Sect. 3 (the details of this part are different from the $3D$ case). Theorem 2.3 is shown to be a consequence of Theorem 6.1.
2. In order to prove Theorem 6.1, in Sect. 6.2 we use a sliding localization technique to reduce the problem from the thermodynamical box Ω to the large box Λ. The result of this procedure is an inequality of the form (in the quadratic form sense)

$$\mathcal{H}_{\rho_\mu} \geq \int_{\mathbb{R}^2} \mathcal{H}_{\Lambda_u}(\rho_\mu)\, du,$$

   where $\mathcal{H}_{\Lambda_u}(\rho_\mu)$ is a Hamiltonian localized to a box Λ_u which is the translation of the fixed box Λ to be centered at u. The main result is then reduced to the proof of an analogous lower bound for $\mathcal{H}_\Lambda(\rho_\mu)$, namely Theorem 6.7. The next sections focus on this proof.
3. We split the potential energy on the large box in Sect. 7.1 by means of projectors P and Q onto and outside the condensate, respectively, or in other words onto the zero momentum sector and its complement. The splitting produces terms involving from 0 to 4 Q projectors. This is similar to the approach in [11,12].
   By an algebraic identity (see Lemma 7.1), we identify a positive term $\mathcal{Q}_4^{\text{ren}}$ that can be discarded for a lower bound. This procedure also changes the terms with 0 to 3

[1] The power $\frac{1}{8}$ is not optimal but chosen for convenience, in particular to be in agreement with (H24).

Q's. By this procedure, all occurrences of the potential v are replaced by the function g related to the scattering equation and to the parameter δ_0. This idea has its roots in [23] and was a key step in [18,19]. Since $\widehat{g}(0) = 8\pi\delta_0 \ll \widehat{v}(0)$, this can be interpreted as a renormalization procedure.

4. In [16] it was understood how the interaction of the so-called *soft pairs* contributes significantly to the energy. These correspond to two interacting high-momenta producing one 0-momentum and one low-momentum. This is the main contribution of the $3Q$ term. The soft pairs appear after estimating the other parts of the $3Q$ term to be of lower order. This is done in two steps, the first (restriction to low outgoing momentum) is proved in Sect. 7.2 and the second one (high incoming momenta) in Lemma 8.2, the latter being easier treated in second quantization.
5. A key step in both 2 and 3 dimensions is to be able to focus on states where the operator counting the number of excitations satisfies a norm bound. To handle the 3 dimensional hard-core case, in [19] it was realized that such a bound is only possible when restricting to excitations with low momentum. In the 2 dimensional case, we face this difficulty even if the potential v has small integral (i.e., it is soft).
The reason for this difficulty is that the bound on the excitations involves the integral of v, i.e. $\widehat{v}(0)$. This has to be compared to the main term of the energy, where the relevant parameter is $\widehat{g}(0)$, and as previously noticed, in 2 dimensions $\widehat{g}(0) = 8\pi\delta_0 \ll \widehat{v}(0)$. The solution to this problem follows the same general approach as in [19], namely to not bound all excitations but only those with low momentum. This is the result of Theorem 7.7. The analysis for this bound is carried out in Sect. 7.3 and based on estimates on Bose–Einstein Condensation from Theorem 7.6 proven in "Appendix D". Some other important ingredients of the proof are delegated to "Appendix E". Theorem 7.7 and its proof are somewhat simpler and more along the lines of an IMS-localization estimate than the ones in [19].
6. Section 8 contains lower bounds that use a second quantization formalism in momenta space. We first write the Hamiltonian in this formalism in Sect. 8.2. Then we use the c-number substitution in Sect. 8.3, thus reducing to a problem of minimization for particles outside the condensate. The operators related to the condensate act as numbers over the class of coherent states over which we minimize. After this procedure we arrive at an operator containing terms of order up to 3 creation and annihilation operators of non-zero momenta.
7. In Sect. 9 we distinguish the two cases where the density of particles in the condensate ρ_z is far from or close to ρ_μ, the expected density. Since we have Bose–Einstein condensation, we expect on physical grounds to be in the second case, and indeed fairly rough bounds suffice in the first case. These are given in Sect. 9.1. In the second case, $\rho_z \approx \rho_\mu$, a more careful analysis is needed. We use standard techniques, collected in "Appendix B", to diagonalize the main quadratic part of the Hamiltonian the ground state energy of which appears as an integral.
This integral is calculated in "Appendix C", and we show how together with the constant term of the Hamiltonian, we get the energy to the desired precision.
What remains at this point is to show how the remainders, including the localized $3Q$ term, are error terms, and this is the content of the technical Sect. 9.3. There we show how the contribution of the soft pairs is compensated by the remaining quadratic part of the Hamiltonian. Here in particular, the logarithmic divergencies specific to the 2-dimensional situation makes many estimates delicate and require extra localizations in momentum space.

8. Finally, in Sect. 9.4 we use all the previous results to give a proof of Theorem 6.7, with the choices of the parameters in "Appendix H", where all the conditions used to prove the lower bound are collected.
9. In the proof we need two technical estimates, namely (8.12), (8.13) and (E1), which are taken from the 3D case and are independent of dimension. They are only stated and we refer to [19] for the proof.

3. The Scattering Solution in 2 Dimensions

3.1. Basic theory. In this section we establish the notation and results surrounding the two dimensional two body scattering problem. The standard properties of the scattering solutions stated below are well known and can be found in [24, Appendix A]. We will only consider radial and positive potentials $v : \mathbb{R}^2 \to [0, \infty]$, furthermore if v is compactly supported we denote by R the radius of the support of v, i.e., $v(x) = 0$ if $|x| \geq R$.

Definition 3.1. For a compactly supported v its scattering length $a = a(v)$ is defined as

$$\frac{2\pi}{\log(\frac{\widetilde{R}}{a})} = \inf\Big\{ \int_{B(0,\widetilde{R})} \Big(|\nabla u|^2 + \frac{1}{2}v|u|^2\Big)\mathrm{d}x \;\Big|\; u \in H^1(B(0,\widetilde{R})), \quad u|_{\partial B(0,\widetilde{R})} = 1\Big\}, \tag{3.1}$$

where $\widetilde{R} > R$ is arbitrary.

By the positivity of the right hand side we find $a \leq R$. It is also easy to verify that a is an increasing function of v and is independent of $\widetilde{R} > R$. Furthermore for any $\widetilde{R}$ the above functional has a unique minimizer $\varphi_{v,\widetilde{R}} = \log(\frac{\widetilde{R}}{a(v)})^{-1}\varphi_v^{(0)}(x)$, where, for $v \in L^1(\mathbb{R}^2)$, we have

$$-\Delta\varphi_v^{(0)} + \frac{1}{2}v\varphi_v^{(0)} = 0 \quad \text{on } \mathbb{R}^2, \tag{3.2}$$

in the distributional sense. Furthermore,

$$\varphi_v^{(0)}(r) = \log\Big(\frac{r}{a(v)}\Big), \qquad \text{for } r \geq R,$$

and $\varphi_v^{(0)}$ is a monotone, non-decreasing and non-negative, radial function. We will omit the v in the notation of the scattering length if the potential is clear from the context.

The logarithm in the $2D$-scattering solution is clearly unbounded for large values of r. This is a major difference to the $3D$ behaviour (where the scattering solution behaves as $1 - \frac{a}{r}$ at infinity). Therefore the scattering solution normalized to 1 at a certain length $\widetilde{R}$ is of much greater importance. Using the parameter

$$\delta = \frac{1}{2}\log\Big(\frac{\widetilde{R}}{a}\Big)^{-1}, \qquad \text{i.e.} \qquad \widetilde{R} = ae^{\frac{1}{2\delta}}, \tag{3.3}$$

we define on $\mathbb{R}^2$

$$\varphi = \varphi_{v,\delta} = 2\delta\varphi^{(0)}, \qquad \omega = 1 - \varphi, \qquad g = v\varphi = v(1-\omega). \tag{3.4}$$

Clearly,

$$-\Delta\omega = \frac{1}{2}g, \tag{3.5}$$

and, using the divergence theorem,

$$\int g \, \mathrm{d}x = 8\pi\delta. \tag{3.6}$$

We remark here again a difference between the $2D$ and $3D$ case: in $3D$, φ would be normalized to 1 at infinity and (3.6) would have an a instead of δ.

Remark 3.2. (On the parameters δ and $\widetilde{R}$) We clearly have some freedom in the choice of δ, which amounts to determine a normalization lengthscale $\widetilde{R}$ for φ. Throughout the paper, we will need δ to be of the same order as $Y = |\log(\rho a^2)|^{-1}$, namely

$$\frac{Y}{2} \leq \delta \leq 2Y, \qquad \text{or, equivalently,} \quad (\rho a^2)^{-1/4} \leq \frac{\widetilde{R}}{a} \leq (\rho a^2)^{-1}. \tag{3.7}$$

With this condition we can always exchange Y and δ when estimating errors. We thus get upper and lower bounds on the energy depending on the parameter δ. In both cases, it turns out that the optimal choice is given by (1.5), i.e.

$$\delta = \delta_0 = |\log(\rho a^2 |\log(\rho a^2)|^{-1})|^{-1}, \tag{3.8}$$

which corresponds to

$$\frac{\widetilde{R}}{a} = (\rho a^2 Y)^{-1/2}. \tag{3.9}$$

See also Remarks 4.9 and C.4.

3.2. Potentials without compact support.

Definition 3.3. For a potential v without compact support the scattering length is defined as

$$a(v) = \lim_{n\to\infty} a(v\mathbb{1}_{B(0,n)}).$$

Since a is an increasing function of v the limit exits if and only if $\{a(v\mathbb{1}_{B(0,n)})\}_n$ is bounded, which by [25, Lemma 1] is true if and only if there exists a $\widetilde{b} > 0$ such that

$$\int_{\{|x|>\widetilde{b}\}} v(x) \log\left(\frac{|x|}{\widetilde{b}}\right)^2 \mathrm{d}x < \infty.$$

We need to localize our potentials to have compact support. The next result estimates the change this localization induces in the scattering length.

Lemma 3.4. *For a potential v with finite scattering length a and $R > a$, let $v_R = \mathbb{1}_{B(0,R)} v$ and a_R be its associated scattering length. Then,*

$$0 \leq \frac{2\pi}{\log(\frac{R}{a})} - \frac{2\pi}{\log(\frac{R}{a_R})} \leq \frac{1}{2}\int_{\{|x|>R\}} v(x) \frac{\log(\frac{|x|}{a})^2}{\log(\frac{R}{a})^2} \mathrm{d}x. \tag{3.10}$$

Proof. Let φ_1 be the scattering solution for v_R normalized at R, and let

$$\varphi_n(x) := \begin{cases} \varphi_1(x)\frac{\log(\frac{R}{a})}{\log(n)}, & |x| \leq R, \\ \frac{\log(\frac{|x|}{a})}{\log(n)}, & |x| \geq R. \end{cases} \tag{3.11}$$

Notice that φ_n is normalized at $a \cdot n$ and continuous. We use it as a trial function in the variational problem of $v_n = \mathbb{1}_{B(0,a\cdot n)} v$, with $n \cdot a > R$, to get (with $a_n := a(v_n)$)

$$\frac{2\pi}{\log(\frac{a\cdot n}{a_n})} \leq \int_{\{|x|<R\}} \Big(|\nabla\varphi_n|^2 + \frac{1}{2}v_n\varphi_n^2\Big)\mathrm{d}x + \int_{\{R<|x|<a\cdot n\}} \Big(|\nabla\varphi_n|^2 + \frac{1}{2}v_n\varphi_n^2\Big)\mathrm{d}x. \tag{3.12}$$

Since φ_n is just a multiple of the scattering solution of φ_1 inside R the first integral gives

$$\int_{\{|x|<R\}} \Big(|\nabla\varphi_n|^2 + \frac{1}{2}v_n\varphi_n^2\Big)\mathrm{d}x = \frac{2\pi}{\log(\frac{R}{a_R})}\frac{\log(\frac{R}{a})^2}{\log(n)^2}. \tag{3.13}$$

The second term is directly calculated using the explicit formula for φ_n,

$$\begin{aligned} &\int_{\{R<|x|<a\cdot n\}} \Big(|\nabla\varphi_n|^2 + \frac{1}{2}v_n\varphi_n^2\Big)\mathrm{d}x \\ &\quad \leq \frac{2\pi\log(\frac{a\cdot n}{R})}{\log(n)^2} + \frac{1}{2\log(n)^2}\int_{\{|x|>R\}} v\log\Big(\frac{|x|}{a}\Big)^2. \end{aligned} \tag{3.14}$$

By (3.13), multiplying (3.12) through with $\log(n)^2$ and letting $n \to \infty$, whereby $a_n \to a$, yields

$$2\pi\log\Big(\frac{R}{a}\Big) \leq \frac{2\pi\log(\frac{R}{a})^2}{\log(\frac{R}{a_R})} + \frac{1}{2}\int_{\{|x|>R\}} v\log\Big(\frac{|x|}{a}\Big)^2\mathrm{d}x.$$

The result then follows by dividing through with $\log(\frac{R}{a})^2$. □

3.3. Compactly supported potentials with large integrals. We state and prove here in the $2D$ setting a similar approximation result as the one found in [19, Theorem 1.6] for the scattering length in $3D$.

Lemma 3.5. *For a radial, positive $v \in L^1(\mathbb{R}^2)$ with support contained in $B(0, R)$ there exists, for any $T > 0$, a $v_T : \mathbb{R}^2 \to [0, +\infty]$ satisfying*

$$0 \leq v_T(x) \leq v(x), \quad \textit{for all } x \in \mathbb{R}^2, \textit{ and} \quad \int v_T \leq 4\pi T, \tag{3.15}$$

and such that

$$\frac{2\pi}{\log(\frac{R}{a})} - \frac{2\pi}{\log(\frac{R}{a_T})} \leq \frac{2\pi}{\log(\frac{R}{a})^2 T}. \tag{3.16}$$

Proof. Due to the integrability assumption on v we may define

$$R_T = \inf\left\{R' > 0 : \int_{\{|x|\geq R'\}} v\mathrm{d}x < 4\pi T\right\}$$

and

$$v_T := v\mathbb{1}_{\{|x|>R_T\}}. \tag{3.17}$$

Clearly,

$$\int v_T = 4\pi T. \tag{3.18}$$

Also, we may assume $R_T > 0$. Otherwise there is nothing to prove.

Let φ be the scattering solution of v and φ_T the scattering solution of v_T both normalized at $\widetilde{R} > R$. We have from (3.6), using that φ_T is a non-decreasing function,

$$\frac{4\pi}{\log(\frac{\widetilde{R}}{a_T})} = \int v_T\varphi_T\mathrm{d}x \geq \varphi_T(R_T)\int v_T = 4\pi\varphi(R_T)T,$$

and hence

$$\varphi(R_T) \leq \frac{1}{\log(\frac{\widetilde{R}}{a_T})T}. \tag{3.19}$$

Next we define

$$u = \mathbb{1}_{\{|x|>R_T\}}(\varphi_T - \omega_T\varphi_T(R_T)) \qquad \text{where} \quad \omega_T(x) = 1 - \frac{\log(\frac{|x|}{R_T})}{\log(\frac{\widetilde{R}}{R_T})}.$$

Observe that $u(\widetilde{R}) = 1$ and we may therefore apply it as a trial function in the functional for a to get

$$\frac{2\pi}{\log(\frac{\widetilde{R}}{a})} \leq \int_{\{|x|<\widetilde{R}\}}\Big(|\nabla u|^2 + \frac{1}{2}vu^2\Big)\mathrm{d}x := E_1 + E_2 + E_3, \tag{3.20}$$

with

$$E_1 = \int_{\{R_T<|x|<\widetilde{R}\}}\Big(|\nabla\varphi_T|^2 + \frac{1}{2}v_T\varphi_T^2\Big)\mathrm{d}x = \frac{2\pi}{\log(\frac{\widetilde{R}}{a_T})},$$

$$E_2 = -2\varphi_T(R_T)\int_{\{R_T<|x|<\widetilde{R}\}}\Big(\nabla\varphi_T\nabla\omega_T + \frac{1}{2}v_T\varphi_T\omega_T\Big)\mathrm{d}x = 0,$$

$$E_3 = \varphi_T(R_T)^2\int_{\{R_T<|x|<\widetilde{R}\}}\Big(|\nabla\omega_T|^2 + \frac{1}{2}v_T\omega_T^2\Big)\mathrm{d}x.$$

For E_2 we integrated by parts and used that φ_T is harmonic inside $B(0, R_T)$, thus constant, which makes the boundary term vanish. For E_3 we use that $\omega_T \leq 1$ on the given interval, so combining (3.18), (3.19) and (3.20) yields

$$\frac{2\pi}{\log(\frac{\widetilde{R}}{a})} - \frac{2\pi}{\log(\frac{\widetilde{R}}{a_T})} \leq E_3 \leq \frac{2\pi}{\log(\frac{\widetilde{R}}{a_T})^2}\bigg(\frac{1}{\log(\frac{\widetilde{R}}{R_T})T^2} + \frac{1}{T}\bigg). \tag{3.21}$$

Using that $a \geq a_T$ we may replace a_T with a on the right hand side. Secondly, we observe that the function

$$\left(\frac{1}{\log(\frac{\widetilde{R}}{a})} - \frac{1}{\log(\frac{\widetilde{R}}{a_T})}\right) \log\left(\frac{\widetilde{R}}{a}\right)^2,$$

is increasing in $\widetilde{R}$ so we may replace $\widetilde{R}$ with R in the above expression and use (3.21) to get

$$\frac{2\pi}{\log(\frac{R}{a})} - \frac{2\pi}{\log(\frac{R}{a_T})} \leq \frac{2\pi}{\log(\frac{R}{a})^2}\left(\frac{1}{\log(\frac{\widetilde{R}}{R_T})T^2} + \frac{1}{T}\right).$$

Now the result follows by letting $\widetilde{R}$ go to infinity. □

We are ready to prove the main theorem of this section which gives us the ability to deal with a wide range of potentials including, most notably, the hard core.

Theorem 3.6. *For a radial, positive potential $v : \mathbb{R}^2 \to [0, \infty]$ with finite scattering length a there exists, for any $R > a$ and $T, \epsilon > 0$, a potential $v_{T,R,\epsilon}$ such that*

$$\operatorname{supp}(v_{T,R,\epsilon}) \subset B(0, R), \qquad 0 \leq v_{T,R,\epsilon}(x) \leq v(x), \qquad \int v_{T,R,\epsilon} \leq 4\pi T, \tag{3.22}$$

and its scattering length $a_{T,R,\epsilon}$ satisfies

$$\frac{2\pi}{\log(\frac{R}{a})} - \frac{2\pi}{\log(\frac{R}{a_{T,R,\epsilon}})} \leq \frac{1}{\log(\frac{R}{a})^2}\left(\frac{2\pi}{T}(1+\epsilon) + \frac{1}{2}\int_{\{|x|>R\}} v(x)\log\left(\frac{|x|}{a}\right)^2\right)\mathrm{d}x. \tag{3.23}$$

Proof. Lemma 3.5 applied to $v_R^n = \mathbb{1}_{B(0,R)}\min(n, v)$ yields a $v_{R,T}^n$ satisfying all three conditions of (3.22) and

$$\frac{2\pi}{\log(\frac{R}{a_R^n})} - \frac{2\pi}{\log(\frac{R}{a_{T,R}^n})} \leq \frac{2\pi}{\log(\frac{R}{a})^2 T}, \tag{3.24}$$

for all $n \in \mathbb{N}$. In the above we used that $a_{T,R}^n \leq a_R^n \leq a_R \leq a$ (where $a_{T,R}^n, a_R^n, a_R$ are the scattering lengths of $v_{T,R}^n, v_R^n, v_R$, respectively). Choosing n_0 large enough such that $a_R^{n_0}$ is close enough to a_R gives an $a_{T,R,\epsilon} := a_{T,R}^{n_0}$ satisfying

$$\frac{2\pi}{\log(\frac{R}{a_R})} - \frac{2\pi}{\log(\frac{R}{a_{T,R,\epsilon}})} \leq \frac{2\pi}{\log(\frac{R}{a})^2 T}(1+\epsilon). \tag{3.25}$$

We conclude using (3.10) which gives the integral term of (3.23). □

3.4. Fourier analysis on the scattering equation. Due to Theorem 3.6 we may assume our potentials to be compactly supported and L^1, thus making the Fourier transform well defined. The scattering solution φ will be the one defined in (3.4) which is normalized to 1 outside the support of v. In order to discuss the Fourier transform of the scattering solution, we recall some standard results surrounding the Fourier transform of the logarithm. We denote by $\mathcal{S}$ and $\mathcal{S}'$ the Schwartz space and the space of tempered distribution on $\mathbb{R}^2$, respectively.

Lemma 3.7. *For $D > 0$, let L_D denote the tempered distribution given by the function $\log(|x|/D)$ in $\mathbb{R}^2$. The Fourier transform of L_D satisfies for any $h \in \mathcal{S}$*

$$\langle \widehat{L_D}, h\rangle_{\mathcal{S}',\mathcal{S}} = -(2\pi)\int_{\mathbb{R}^2} \frac{h(p) - h(0)\mathbb{1}_{\{|p|\leq 2e^{-\Gamma}D^{-1}\}}}{p^2}\,\mathrm{d}p, \tag{3.26}$$

where Γ denotes the Euler–Mascheroni constant,

$$\Gamma := -\int_0^\infty e^{-x}\log x\,\mathrm{d}x \approx 0.5772. \tag{3.27}$$

The proof is an exercise in distribution theory, with details for instance given in the recent book [26, Theorem 4.73].

It follows from (3.26) that, for any $f \in \mathcal{S}$,

$$\begin{aligned}&\iint_{\mathbb{R}^2\times\mathbb{R}^2} \overline{f(x)}f(y)\log\left(\frac{|x-y|}{D}\right)\mathrm{d}x\mathrm{d}y\\ &\quad = -2\pi\int_{\mathbb{R}^2}\frac{|\widehat{f}(p)|^2 - |\widehat{f}(0)|^2\mathbb{1}_{\{|p|\leq 2e^{-\Gamma}D^{-1}\}}}{p^2}\,\mathrm{d}p.\end{aligned} \tag{3.28}$$

Using the notation from (3.4) and (3.5), we may compute the Fourier transform of ω. In the $3D$ case one gets that $\widehat{\omega}(p) = \frac{\widehat{g}(p)}{2p^2}$, but in $2D$ this formula has to be corrected by a distribution supported at the origin according to Lemma 3.7, see Lemma 3.8 below.

Lemma 3.8. *Let $\widehat{\omega}$ denote the Fourier transform of ω. Then $\widehat{\omega}$ is the tempered distribution given by*

$$\langle \widehat{\omega}, u\rangle_{\mathcal{S}',\mathcal{S}} = \int \frac{\widehat{g}(p)u(p) - \widehat{g}(0)u(0)\mathbb{1}_{\{|p|\leq \ell_\delta^{-1}\}}}{2p^2}\mathrm{d}p, \tag{3.29}$$

for any $u \in \mathcal{S}$ where, recalling the definition of $\widetilde{R}$ in (3.3),

$$\ell_\delta := \frac{ae^\Gamma}{2}e^{\frac{1}{2\delta}} = \frac{1}{2}e^\Gamma\widetilde{R}. \tag{3.30}$$

Notice that if $\delta = \delta_0$ from (1.5), then ℓ_δ coincides with ℓ_{δ_0} introduced in (2.12).

Proof. We first recall the definition (3.4) of ω and write

$$\omega = -\frac{\widehat{g}(0)}{4\pi}\log\left(\frac{r}{\tilde{R}}\right) + \widetilde{\omega}, \tag{3.31}$$

where $\widetilde{\omega}$ is compactly supported, and we recall $\widehat{g}(0) = 8\pi\delta$. Hence, using the Fourier transform of the logarithm as recalled in Lemma 3.7,

$$\langle \widehat{\omega}, u\rangle_{\mathcal{S}',\mathcal{S}} = \widehat{g}(0)\int_{\mathbb{R}^2}\frac{u(p) - u(0)\mathbb{1}_{\{|p|\leq 2e^{-\Gamma}\tilde{R}^{-1}\}}}{2p^2}\,\mathrm{d}p + \int \widehat{\widetilde{\omega}}(p)u(p)\,\mathrm{d}p. \tag{3.32}$$

Using the scattering equation (3.5) we find $\widehat{g}(p) = 2p^2\widehat{\omega}(p) = \widehat{g}(0) + 2p^2\widehat{\widetilde{\omega}}(p)$, where we used that the logarithm is the fundamental solution of the Laplacian. Since $\widehat{\widetilde{\omega}}$ is a smooth function we deduce

$$\widehat{\widetilde{\omega}}(p) = \frac{\widehat{g}(p) - \widehat{g}(0)}{2p^2},$$

and this concludes the proof. □

Thanks to the previous lemma we are able to prove some important properties of $\widehat{g\omega}(0)$ which are going to be key through all the paper.

Lemma 3.9. *The following identity holds*

$$\widehat{g\omega}(0) = \int_{\mathbb{R}^2} \frac{\widehat{g}(k)^2 - \widehat{g}(0)^2 \mathbb{1}_{\{|k| \leq \ell_\delta^{-1}\}}}{2k^2} \mathrm{d}k \tag{3.33}$$

and, furthermore, the following bounds hold

$$|\widehat{g\omega}(0)| \leq C\delta, \tag{3.34}$$

$$\left| \int_{\{|k| \leq \ell_\delta^{-1}\}} \frac{\widehat{g}(k)^2 - \widehat{g}(0)^2}{2k^2} \mathrm{d}k \right| \leq CR^2\delta^2\ell_\delta^{-2}, \tag{3.35}$$

$$\left| \int_{\{|k| \geq \ell_\delta^{-1}\}} \frac{\widehat{g}(k)^2}{2k^2} \mathrm{d}k \right| \leq C\delta. \tag{3.36}$$

Proof. Formula (3.33) is formally given by an application of Lemma 3.8 choosing $u = \widehat{g}$. Since $\widehat{g}$ is not a Schwartz function, we need to apply a regularization argument, by truncating in momentum space. This truncation can then be removed at the end and one arrives at (3.33).

The first bound (3.34) follows because in the support of g, $\omega \leq 1$ and $\widehat{g}_0 = 8\pi\delta$. The last bound (3.36) follows once we have proved the second one. In order to do that, we consider a Taylor expansion to the second order of $|\widehat{g}_k - \widehat{g}_0| \leq C|k|^2\|\widehat{g}_k''\|_\infty$ as a radial function, due to the symmetry of g. We use that v has a compact support R and the definition

$$\widehat{g}_k = \int_{\mathbb{R}^2} v\varphi e^{-ik.x} \mathrm{d}x \tag{3.37}$$

to bound $\widehat{g}_k''$ by $R^2\widehat{g}_0$ to obtain

$$\left| \int_{\{|k| \leq \ell_\delta^{-1}\}} \frac{\widehat{g}_k^2 - \widehat{g}_0^2}{2k^2} \mathrm{d}k \right| \leq CR^2|\widehat{g}_0|^2 \int_{\{|k| \leq \ell_\delta^{-1}\}} \mathrm{d}k = CR^2\delta^2\ell_\delta^{-2}. \tag{3.38}$$

□

3.5. Spherical measure potentials. For the upper bound, we will change the potential in order to ensure small L^1 norm. For a potential v supported in $B(0, R)$ and $b > R$, let

$$f(x) := \min\Big(1, \varphi^{(0)}(x) \log\Big(\frac{b}{a}\Big)^{-1}\Big).$$

Thus, f is the scattering solution in $B(0, b)$ normalized at b and extended by one. The new potential $\widetilde{v}$ will then be described by the deviation of f being the actual scattering solution, i.e.,

$$\widetilde{v} = 2\Big(- \Delta f + \frac{1}{2} v f\Big), \tag{3.39}$$

where the above equality is to be thought of in a distributional sense. The factor 2 is important and should be thought of as the number of particles involved in the scattering process. A quick calculation shows that

$$\widetilde{v} = 2 f'(b) \delta_{\{|x|=b\}} = 2 \frac{1}{b \log(\frac{b}{a})} \delta_{\{|x|=b\}}, \tag{3.40}$$

where $\delta_{\{|x|=b\}}$ is the uniform measure on the circle $\{|x| = b\}$ normalized so that $\int \delta_{\{|x|=b\}} = 2\pi b$, and where $f'(b)$ is to be understood as the radial derivative (from the left) of f at length b. We show in Sect. 5 how we reduce to this potential. The simple, but essential properties of $\widetilde{v}$ are stated in the lemma below.

Lemma 3.10. *Let v and $\widetilde{v}$ be given as above. We use the notation $\widetilde{a} = a(\widetilde{v})$ and $a = a(v)$. Furthermore, let $\widetilde{\varphi}$ be the scattering solution of $\widetilde{v}$ normalized at $\widetilde{R} > b$ and $\widetilde{g} = \widetilde{v}\widetilde{\varphi}$. Then*

1. *The scattering lengths agree, i.e., $\widetilde{a} = a$.*
2. *$\widehat{\widetilde{v}}(p) = 2 f'(b) b J_0(b|p|)$, where J_0 is the zeroth spherical Bessel function. In particular there exists a universal constant $C > 0$ such that*

$$|\widehat{\widetilde{v}}(p)| \leq C \frac{\widehat{\widetilde{v}}(0)}{\sqrt{b|p|}}. \tag{3.41}$$

3. *$\widehat{\widetilde{v}}(0) := \langle v, 1 \rangle = \dfrac{4\pi}{\log(b/a)}$, and $\widehat{\widetilde{g}}(0) = \widehat{\widetilde{v}\widetilde{\varphi}}(0) = \dfrac{4\pi}{\log(\widetilde{R}/a)}$.*

Proof. The potential $\widetilde{v}$ is a spherical measure on the sphere $\{|x| = b\}$ and thus $\widetilde{\varphi}$ is harmonic both inside and outside this sphere. We may therefore conclude from the continuity of $\widetilde{\varphi}$ that

$$\log(\widetilde{R}/\widetilde{a}) \widetilde{\varphi}(r) = \begin{cases} \log(b/\widetilde{a}), & \text{if } r \leq b, \\ \log(r/\widetilde{a}), & \text{if } r > b. \end{cases} \tag{3.42}$$

From the scattering equation

$$-\Delta \widetilde{\varphi} + \frac{1}{2} \widetilde{v} \widetilde{\varphi} = 0,$$

applied to a $u \in C_c^\infty(\mathbb{R}^2)$ we obtain, using Green's formula,

$$- \int_{\{|x|=b\}} u \nabla \widetilde{\varphi} \cdot \mathrm{d}\boldsymbol{n} = f'(b) \widetilde{\varphi}(b) \int_{\{|x|=b\}} u \tag{3.43}$$

and then deduce

$$\widetilde{\varphi}'(b) = f'(b)\widetilde{\varphi}(b) \tag{3.44}$$

where $\widetilde{\varphi}'(b)$ denotes the outgoing radial derivative of $\widetilde{\varphi}$ at length b. Combining (3.42) and (3.44) yields 1. Property 2. is a direct consequence of $\widetilde{v}$ being a uniform measure on the sphere $\{|x|\} = b$ and the behaviour of J_0 at infinity. Finally, the identities in 3.. follow immediately after realizing that

$$g = \widetilde{\varphi}(b)v.$$

□

4. Upper Bound for a Soft Potential

We denote by $\mathscr{M}_c$ the set of potentials of the form $v = v_{\mathrm{reg}} + v_{\mathrm{m}}$, where $v_{\mathrm{reg}} \in L^1(\mathbb{R}^2)$ is radial, positive and has compact support, and where $v_{\mathrm{m}} = C\delta_{\{|x|=r\}}$ for some $C \geq 0$ and $r > 0$. If $v \in \mathscr{M}_c$, it admits a bounded and continuous Fourier transform $\widehat{v}$. The aim of this section is to prove an upper bound on the ground state energy of

$$\mathcal{H}_v = \bigoplus_{n\geq 0}\Big(\sum_{i=1}^{n} -\Delta_{x_i}^{\mathrm{per}} + \sum_{1\leq i<j\leq n} v^{\mathrm{per}}(x_i - x_j)\Big) \tag{4.1}$$

on the box $\Lambda_\beta = [-\frac{L_\beta}{2}, \frac{L_\beta}{2}]^2$ for potentials $v \in \mathscr{M}_c$, under some additional decay assumption on the Fourier transform of v. We recall that $L_\beta = \rho^{-\frac{1}{2}}Y^{-\beta}$.

In this section, we will denote by φ the scattering solution of the given v, normalized at length $\widetilde{R}$, and $g = \varphi v$, see (3.4). Notice here that the theory of Sect. 3 extends to potentials $v \in \mathscr{M}_c$; for this we use in particular, that if $u \in H^1(\mathbb{R}^2)$, then $u|_{\{|x|=r\}} \in L^2$ so the variational problem in Definition 3.1 is well posed. In particular, the scattering equation (3.2) is valid in the distributional sense. We recall that $0 \leq \widehat{g}_0 = 8\pi\delta \leq CY$ by (3.6) and (3.3). We prove the following upper bound, which is very similar in spirit to the upper bound of [15] in the $3D$ case.

Theorem 4.1. *For any given $c_0 > 0$ and $\beta \geq \frac{3}{2}$, there exists $C_\beta > 0$ (only depending on c_0 and β) such that the following holds. Let $\rho > 0$ and $v \in \mathscr{M}_c$ be a radial positive measure with scattering length a and $\operatorname{supp} v \subset B(0, R)$, for some $R > 0$. Let $\mathcal{H}_v$ be as defined in (2.3). Assume that*

$$|\widehat{g}_p| \leq c_0 \frac{\widehat{g}_0}{\sqrt{R|p|}}, \qquad \forall |p| \geq a^{-1}. \tag{4.2}$$

Then, if $\rho R^2 \leq Y$ and $\rho a^2 \leq C_\beta^{-1}$, one can find a normalized trial state $\Phi \in \mathscr{F}_s(L^2(\Lambda_\beta))$ satisfying

$$\langle \mathcal{H}_v\rangle_\Phi \leq 4\pi L_\beta^2\rho^2\delta_0\Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)\delta_0\Big) + CL_\beta^2\rho^2\delta_0(\widehat{v}_0 - \widehat{g}_0) + CL_\beta^2\rho^2\delta_0^2\widehat{v}_0$$

with $\langle \mathcal{N}\rangle_\Phi = N$, and $\langle \mathcal{N}^2\rangle_\Phi \leq 9N^2$, where $N = \rho L_\beta^2 = Y^{-2\beta}$.

Remark 4.2. Note that this result is much weaker than Theorem 2.2. Indeed, the remainders are only of order $\rho^2 L_\beta^2 \delta_0^2$ and $\rho^2 L_\beta^2 \delta_0$ and thus much larger than the 2D-LHY term, unless $\widehat{v}_0 = \widehat{g}_0 + o(\delta_0)$. Moreover, Theorem 4.1 only holds for potentials with finite integral and, in particular, it does not allow for a hard core. However, in the proof of Theorem 2.2 in Sect. 5 we will show how to reduce to such potentials. More precisely, we will apply Theorem 4.1 to a surface potential of the form (3.40) (with the choice of b given in (2.5)).

Remark 4.3. The specific $\delta = \delta_0$ defined in (1.5) is chosen to minimize the upper bound (4.22) up to the LHY precision. This corresponds to fixing the normalisation length of the soft potential $\widetilde{R} = ae^{\frac{1}{2\delta}}$. See also Remarks 3.2, 4.9 and C.4.

The rest of Sect. 4 is dedicated to the proof of Theorem 4.1. We will give an explicit trial state and state several technical calculations as lemmas. In the end we collect the pieces and finish the proof.

4.1. A quasi-free state. We will define our trial state Φ in second quantization formalism. On the bosonic Fock space $\mathscr{F}(L^2(\Lambda_\beta))$, we will denote by $a_p^\dagger$ and a_p the creation and annihilation operators associated to the function $x \mapsto |\Lambda_\beta|^{-\frac{1}{2}} \exp(ipx)$, for $p \in \Lambda_\beta^* = (\frac{2\pi}{L_\beta}\mathbb{Z})^2$. Our quasi-free state is $\Phi = T_\nu W_{N_0} \Omega$ where Ω is the vacuum, W_{N_0} creates the condensate and T_ν the excitations:

$$W_{N_0} = \exp\Big(\sqrt{N_0}(a_0^\dagger - a_0)\Big), \quad T_\nu = \exp\Big(\frac{1}{2}\sum_{p\neq 0} \nu_p (a_p^\dagger a_{-p}^\dagger - a_p a_{-p})\Big), \tag{4.3}$$

for a given $N_0 \leq N$ associated with $\rho_0 := N_0/L_\beta^2$. These operators have the nice properties that

$$W_{N_0}^* a_0 W_{N_0} = a_0 + \sqrt{N_0}, \quad \text{and} \quad T_\nu^* a_p T_\nu = \cosh(\nu_p) a_p + \sinh(\nu_p) a_{-p}^\dagger. \tag{4.4}$$

In particular, for any $p, q \in \Lambda_\beta^*$,

$$\begin{aligned} \langle a_q^\dagger a_p \rangle_\Phi &= \begin{cases} N_0, & \text{if } p = q = 0, \\ 0, & \text{if } p \neq q, \\ \gamma_q, & \text{if } p = q \neq 0, \end{cases} \qquad \text{and} \\ \langle a_q a_p \rangle_\Phi = \langle a_q^\dagger a_p^\dagger \rangle_\Phi &= \begin{cases} N_0, & \text{if } p = q = 0, \\ 0, & \text{if } p \neq -q, \\ \alpha_q, & \text{if } p = -q \neq 0, \end{cases} \end{aligned} \tag{4.5}$$

where $\alpha_p = \cosh(\nu_p)\sinh(\nu_p)$ and $\gamma_p = \sinh(\nu_p)^2$. We choose the coefficient ν_p such that

$$\alpha_p = \frac{-\rho_0 \widehat{g}_p}{2\sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2}}, \qquad \gamma_p = \frac{p^2 + \rho_0 \widehat{g}_p - \sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2}}{2\sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2}} \geq 0, \tag{4.6}$$

this specific choice coming from a minimization of the energy

$$(p^2 + \rho_0 \widehat{g}_p)\gamma_p + \rho_0 \widehat{g}_p \alpha_p$$

obtained in Lemma 4.6 up to changing $\widehat{v}$ into $\widehat{g}$. Note that by $(\cosh(x)^2 - \sinh(x)^2) = 1$ we have $\alpha_p^2 = \gamma_p(\gamma_p + 1)$, making it a possible choice. These coefficients satisfy the following estimates.

Lemma 4.4. *We estimate the sum (over Λ_β^*) of α_p and γ_p:*

$$\sum_{p\neq 0} |\alpha_p| \leq CN, \quad \text{and} \quad \sum_{p\neq 0} \gamma_p \leq CN\delta. \tag{4.7}$$

Proof. We start from the expression of α_p (4.6) and split the sum between $|p| \leq \sqrt{\rho_0 \widehat{g}_0}$ and $|p| \geq \sqrt{\rho_0 \widehat{g}_0}$:

$$\begin{aligned}
\sum_{p\neq 0} |\alpha_p| &\leq C\sqrt{\rho_0 \widehat{g}_0} \sum_{0<|p|\leq \sqrt{\rho_0 \widehat{g}_0}} \frac{1}{|p|} + C\rho_0 \sum_{|p|\geq \sqrt{\rho_0 \widehat{g}_0}} \frac{|\widehat{g}_p|}{|p|^2} \\
&\leq CL_\beta^2 \sqrt{\rho_0 \widehat{g}_0} \int_0^{\sqrt{\rho_0 \widehat{g}_0}} \mathrm{d}u + CL_\beta^2 \rho_0 \widehat{g}_0 \int_{\sqrt{\rho_0 \widehat{g}_0}}^{a^{-1}} \frac{\mathrm{d}u}{u} + CL_\beta^2 \rho_0 \int_{a^{-1}}^{+\infty} \frac{\widehat{g}_0}{R^{1/2} u^{3/2}} \mathrm{d}u \\
&\leq CL_\beta^2 \rho_0 \widehat{g}_0 (1 + |\log(a^2 \rho_0 \widehat{g}_0)|) \leq CN,
\end{aligned}$$

where we used the decay of $\widehat{g}_p$ at infinity (4.2) and the bound $a \leq R$.
For γ_p we also split the sum this way. For $p \leq \sqrt{\rho_0 \widehat{g}_0}$ we obtain that

$$\sum_{|p|\leq \sqrt{\rho_0 \widehat{g}_0}} |\gamma_p| \leq C \sum_{|p|\leq \sqrt{\rho_0 \widehat{g}_0}} \frac{\sqrt{\rho_0 \widehat{g}_0}}{|p|} \leq CL_\beta^2 \rho_0 \widehat{g}_0 \leq CN_0 \delta.$$

For $p \geq \sqrt{\rho_0 \widehat{g}_0}$ we expand the square root and find

$$\sum_{|p|\geq \sqrt{\rho_0 \widehat{g}_0}} |\gamma_p| \leq C \sum_{|p|\geq \sqrt{\rho_0 \widehat{g}_0}} \frac{(\rho_0 \widehat{g}_0)^2}{|p|^4} \leq CL_\beta^2 \rho_0 \widehat{g}_0 \leq CN_0 \delta,$$

which concludes the proof. □

Finally choose N_0 such that

$$\rho L_\beta^2 = N = N_0 + \sum_{p\neq 0} \gamma_p. \tag{4.8}$$

Note that with this choice Φ has the expected average number of particles as stated in the next lemma.

Lemma 4.5. *The state $\Phi = T_\nu W_{N_0} \Omega$ satisfies*

$$\langle \mathcal{N} \rangle_\Phi = N, \qquad \langle \mathcal{N}^2 \rangle_\Phi \leq CN^2,$$

where $\mathcal{N} = \sum_{p\in\Lambda_\beta^*} a_p^\dagger a_p$ is the number operator.

Proof. First we can use the property (4.5) to find

$$\langle \mathcal{N} \rangle_\Phi = N_0 + \sum_{p \neq 0} \gamma_p = N. \tag{4.9}$$

For $\mathcal{N}^2$ we split the sums according to zero and non-zero momenta, and then conjugate by W_{N_0},

$$\begin{aligned}
\langle \mathcal{N}^2 \rangle_\Phi &= \sum_{q,p} \langle a_p^\dagger a_p a_q^\dagger a_q \rangle_\Phi = N_0^2 + N_0 + \sum_{q \neq 0} \langle a_0^\dagger a_0 a_q^\dagger a_q + \text{h.c} \rangle_\Phi + \sum_{q \neq 0, p \neq 0} \langle a_p^\dagger a_p a_q^\dagger a_q \rangle_\Phi \\
&= N_0^2 + N_0 \Big(1 + 2 \sum_{p \neq 0} \gamma_p \Big) + \sum_{q \neq 0, p \neq 0} \langle a_p^\dagger a_p a_q^\dagger a_q \rangle_\Phi.
\end{aligned}$$

Now we use Lemma 4.4 and apply Wick's Theorem [27, Theorem 10.2] to the state $T_\nu \Omega$ to find

$$\begin{aligned}
\langle \mathcal{N}^2 \rangle_\Phi &\leq 4N^2 + \sum_{q \neq 0, p \neq 0} \Big(\langle a_p^\dagger a_p \rangle_\Phi \langle a_q^\dagger a_q \rangle_\Phi + \langle a_p^\dagger a_q^\dagger \rangle_\Phi \langle a_p a_q \rangle_\Phi + \langle a_p^\dagger a_q \rangle_\Phi \langle a_p a_q^\dagger \rangle_\Phi \Big) \\
&\leq 4N^2 + \Big(\sum_{p \neq 0} \gamma_p \Big)^2 + \sum_{p \neq 0} \alpha_p^2 + \sum_{p \neq 0} (\gamma_p^2 + \gamma_p) \leq CN^2
\end{aligned}$$

using Lemma 4.4. □

4.2. Energy of Φ. In order to get an upper bound on the energy of Φ we first introduce the quantity

$$D(A, B) = \frac{1}{(4\pi^2)^2} \int \widehat{v} * A(p) B(p) \mathrm{d}p = \frac{1}{(4\pi^2)^2} \langle B, \widehat{v} * A \rangle, \tag{4.10}$$

and observe that it is symmetric in the entries. Then we prove the following result.

Lemma 4.6. *Under the assumptions of Theorem* 4.1*, there exists a constant* $C > 0$*, independent of* v *and* ρ*, such that*

$$|\Lambda_\beta|^{-1} \langle \mathcal{H}_v \rangle_\Phi \leq \frac{\rho^2}{2} \widehat{v}_0 + \int \Big((p^2 + \rho_0 \widehat{v}_p) \gamma_p + \rho_0 \widehat{v}_p \alpha_p \Big) \frac{\mathrm{d}p}{4\pi^2} + \frac{1}{2} D(\alpha, \alpha) + C \widehat{v}_0 \rho^2 Y^3.$$

Proof. One can write $\mathcal{H}_v$ in second quantization in momentum variable,

$$\mathcal{H}_v = \sum_{p \in \Lambda_\beta^*} p^2 a_p^\dagger a_p + \frac{1}{2|\Lambda_\beta|} \sum_{p,q,r} \widehat{v}_r a_{p+r}^\dagger a_q^\dagger a_{q+r} a_p,$$

and express the energy of Φ in terms of α_p and γ_p as follows. We conjugate by W_{N_0} using (4.4), which amounts to change the a_0's in $\sqrt{N_0}$. Since $\Phi = T_\nu W_{N_0} \Omega$ with no a_0

in T_ν (see (4.3)), when we apply Φ we find

$$\begin{aligned}\langle\mathcal{H}_v\rangle_\Phi &= \sum_{p\neq 0} p^2\langle a_p^\dagger a_p\rangle_\Phi + \frac{N_0^2}{2|\Lambda_\beta|}\widehat{v}_0 + \frac{N_0}{|\Lambda_\beta|}\sum_{p\neq 0}(\widehat{v}_0+\widehat{v}_p)\langle a_p^\dagger a_p\rangle_\Phi \\ &+ \frac{N_0}{|\Lambda_\beta|}\sum_{r\neq 0}\widehat{v}_r\langle a_r a_{-r}\rangle_\Phi + \frac{\sqrt{N_0}}{|\Lambda_\beta|}\sum_{\substack{q,r\neq 0\\ q+r\neq 0}}\widehat{v}_r\langle a_r^\dagger a_q^\dagger a_{q+r}\rangle_\Phi \\ &+ \frac{\sqrt{N_0}}{|\Lambda_\beta|}\sum_{\substack{p,r\neq 0\\ p+r\neq 0}}\widehat{v}_r\langle a_{p+r}^\dagger a_{-r}^\dagger a_p\rangle_\Phi + \frac{1}{2|\Lambda_\beta|}\sum_{\substack{p,q\neq 0\\ p+r,q+r\neq 0}}\widehat{v}_r\langle a_{p+r}^\dagger a_q^\dagger a_{q+r}a_p\rangle_\Phi.\end{aligned} \tag{4.11}$$

We can use Wick's Theorem [27, Theorem 10.2] to the state $T_\nu\Omega$. By definition of α_p and γ_p in (4.5) together with $N^2 = (N_0 + \sum_{p\neq 0}\gamma_p)^2$ we deduce

$$\begin{aligned}\langle\mathcal{H}_v\rangle_\Phi &= \frac{N^2}{2|\Lambda_\beta|}\widehat{v}_0 + \sum_{p\neq 0}p^2\gamma_p + \frac{N_0}{|\Lambda_\beta|}\sum_{p\neq 0}\left(\widehat{v}_p\gamma_p + \widehat{v}_p\alpha_p\right) \\ &+ \frac{1}{2|\Lambda_\beta|}\sum_{\substack{q\neq 0\\ q+r\neq 0}}\widehat{v}_r\alpha_q\alpha_{q+r} + \frac{1}{2|\Lambda_\beta|}\sum_{\substack{q\neq 0\\ q+r\neq 0}}\widehat{v}_r\gamma_q\gamma_{q+r}.\end{aligned} \tag{4.12}$$

We bound the last term in the above using Lemma 4.4. With $\rho = N|\Lambda_\beta|^{-1}$ and $\rho_0 = N_0|\Lambda_\beta|^{-1}$ we deduce

$$\begin{aligned}|\Lambda_\beta|^{-1}\langle\mathcal{H}_v\rangle_\Phi &\leq \frac{1}{2}\rho^2\widehat{v}_0 + \frac{1}{|\Lambda_\beta|}\sum_{p\neq 0}\left((p^2+\rho_0\widehat{v}_p)\gamma_p + \rho_0\widehat{v}_p\alpha_p\right) \\ &+ \frac{1}{2|\Lambda_\beta|^2}\sum_{\substack{q\neq 0\\ q+r\neq 0}}\widehat{v}_r\alpha_q\alpha_{q+r} + C\widehat{v}_0\rho^2Y^2.\end{aligned} \tag{4.13}$$

Up to errors $\mathcal{E} \leq C\widehat{v}_0\rho^2 Y^{\frac{1}{2}+\beta}$, we can approximate these Riemann sums by integrals (see Lemma G.1) and the lemma follows. In fact, the requirement $\beta \geq 3/2$ in Theorem 4.1 comes from here. □

Lemma 4.7. *Under the assumptions of Theorem* 4.1*, there exists a constant* $C > 0$*, independent of* v *and* ρ*, such that*

$$\begin{aligned}&|\Lambda_\beta|^{-1}\langle\mathcal{H}_v\rangle_\Phi \\ &\leq \frac{\rho^2}{2}\widehat{g}_0 + \frac{1}{2}\int\left(\sqrt{p^4+2\rho_0p^2\widehat{g}_p} - p^2 - \rho_0\widehat{g}_p + \rho_0^2\frac{\widehat{g}_p^2 - \widehat{g}_0^2\mathbb{1}_{\{p\leq 2e^{-\Gamma}\widetilde{R}^{-1}\}}}{2p^2}\right)\frac{\mathrm{d}p}{4\pi^2} \\ &+ \frac{1}{2}D(\alpha+\rho_0\widehat{\omega},\alpha+\rho_0\widehat{\omega}) + C\rho^2Y(\widehat{v}_0-\widehat{g}_0) + C\widehat{v}_0\rho^2Y^2.\end{aligned}$$

Proof. We recall the definition (3.4) of ω, and we insert $\rho_0\widehat{\omega}$ into $D(\alpha,\alpha)$,

$$D(\alpha,\alpha) = -\rho_0^2 D(\widehat{\omega},\widehat{\omega}) - 2\rho_0 D(\alpha,\widehat{\omega}) + D(\alpha+\rho_0\widehat{\omega},\alpha+\rho_0\widehat{\omega}).$$

Inserting this into Lemma 4.6 we find

$$\begin{aligned}\frac{\langle\mathcal{H}_v\rangle_\Phi}{|\Lambda_\beta|} \le{}& \frac{\rho^2}{2}\widehat{v}_0 - \frac{\rho_0^2}{2}D(\widehat{\omega},\widehat{\omega}) + \int\Big((p^2+\rho_0\widehat{v}_p)\gamma_p + \rho_0(\widehat{v}_p - \widehat{v\omega}_p)\alpha_p\Big)\frac{\mathrm{d}p}{4\pi^2}\\ &+\frac{1}{2}D(\alpha+\rho_0\widehat{\omega},\alpha+\rho_0\widehat{\omega}) + C\widehat{v}_0\rho^2Y^2. \end{aligned}\tag{4.14}$$

Now note that $\widehat{g}_p = \widehat{v}_p - (\widehat{v}*\widehat{\omega})_p$ and,

$$\begin{aligned}\frac{\rho^2}{2}\widehat{v}_0 &= \frac{\rho^2}{2}\widehat{g}_0 + \frac{\rho_0^2}{2}(\widehat{v\omega})_0 + \frac{\rho^2-\rho_0^2}{2}(\widehat{v\omega})_0\\ &= \frac{\rho^2}{2}\widehat{g}_0 + \frac{\rho_0^2}{2}(\widehat{g\omega})_0 + \frac{\rho_0^2}{2}(\widehat{v\omega^2})_0 + \frac{\rho^2-\rho_0^2}{2}(\widehat{v\omega})_0.\end{aligned}$$

This equality inserted in (4.14), together with $D(\widehat{\omega},\widehat{\omega}) = (\widehat{v\omega^2})_0$ implies

$$\begin{aligned}\frac{\langle\mathcal{H}_v\rangle_\Phi}{|\Lambda|} ={}& \frac{\rho^2}{2}\widehat{g}_0 + \frac{\rho_0^2}{2}(\widehat{g\omega})_0 + \int\Big((p^2+\rho_0\widehat{v}_p)\gamma_p + \rho_0\widehat{g}_p\alpha_p\Big)\frac{\mathrm{d}p}{4\pi^2}\\ &+\frac{1}{2}D(\alpha+\rho_0\widehat{\omega},\alpha+\rho_0\widehat{\omega}) + \frac{\rho^2-\rho_0^2}{2}(\widehat{v\omega})_0 + C\widehat{v}_0\rho^2Y^2.\end{aligned}\tag{4.15}$$

Our choice of γ and α minimizes the integral where we replaced $\widehat{v}_p$ by $\widehat{g}_p$, and by explicit computation using the definition (4.6) of α and γ we find

$$\int (p^2+\rho_0\widehat{g}_p)\gamma_p + \rho_0\widehat{g}_p\alpha_p\frac{\mathrm{d}p}{2\pi^2} = \frac{1}{2}\int\Big(\sqrt{p^4+2\rho_0p^2\widehat{g}_p} - p^2 - \rho_0\widehat{g}_p\Big)\frac{\mathrm{d}p}{4\pi^2}.\tag{4.16}$$

Moreover the formula for $\widehat{g\omega}$ from Lemma 3.9 yields

$$(\widehat{g\omega})_0 = \langle\widehat{\omega},\widehat{g}\rangle = \int\frac{\widehat{g}_p^2 - \widehat{g}_0^2\mathbb{1}_{\{p\le\ell_\delta^{-1}\}}}{2p^2}\frac{\mathrm{d}p}{4\pi^2},$$

where $\ell_\delta = \frac{1}{2}e^\Gamma\widetilde{R}$. Inserting this and (4.16) into (4.15) we find

$$\begin{aligned}\frac{\langle\mathcal{H}_v\rangle_\Phi}{|\Lambda|} ={}& \frac{\rho^2}{2}\widehat{g}_0 + \frac{1}{2}\int\Big(\sqrt{p^4+2\rho_0p^2\widehat{g}_p} - p^2 - \rho_0\widehat{g}_p + \rho_0^2\frac{\widehat{g}_p^2 - \widehat{g}_0^2\mathbb{1}_{\{p\le\ell_\delta^{-1}\}}}{2p^2}\Big)\frac{\mathrm{d}p}{4\pi^2}\\ &+\frac{1}{2}D(\alpha+\rho_0\widehat{\omega},\alpha+\rho_0\widehat{\omega})\\ &+\frac{\rho^2-\rho_0^2}{2}(\widehat{v\omega})_0 + \rho_0\int(\widehat{v}_p - \widehat{g}_p)\gamma_p\frac{\mathrm{d}p}{4\pi^2} + C\widehat{v}_0\rho^2Y^2,\end{aligned}$$

where the last integral comes from the replacement of $\widehat{v}_p$ by $\widehat{g}_p$ in the first term of the integral in (4.15). Since $\rho-\rho_0\le C\rho Y$ (Lemma 4.4 and Lemma G.1) and $|\widehat{v}_p - \widehat{g}_p| \le (\widehat{v\omega})_0 = \widehat{v}_0 - \widehat{g}_0$, we can bound

$$\frac{\rho^2-\rho_0^2}{2}(\widehat{v\omega})_0 + \rho_0\int(\widehat{v}_p - \widehat{g}_p)\gamma_p\frac{\mathrm{d}p}{4\pi^2} \le C\rho^2Y(\widehat{v}_0 - \widehat{g}_0),$$

and the lemma follows. □

In the following lemma we estimate the remainder term from Lemma 4.7.

Lemma 4.8. *There is a $C > 0$ independent of v and ρ such that:*

$$D(\alpha + \rho_0\widehat{\omega}, \alpha + \rho_0\widehat{\omega}) \leq \rho_0^2\delta^2\widehat{v}_0\Big|\frac{1}{\delta} - \frac{1}{Y} + \log\delta\Big| + \rho_0^2\delta^2\widehat{v}_0\Big(\frac{1}{\delta} - \frac{1}{Y} + \log\delta\Big)^2 + C\widehat{v}_0\rho_0^2\widehat{g}_0^2.$$

In particular, with $\delta = \delta_0$ defined in (1.5) *we deduce*

$$D(\alpha + \rho_0\widehat{\omega}, \alpha + \rho_0\widehat{\omega}) \leq C\widehat{v}_0\rho^2\delta_0^2.$$

Proof. We recall the definition of ℓ_δ in (3.30). We first estimate $h_p := \langle \alpha + \rho_0\widehat{\omega}, \widehat{v}_{p-\cdot}\rangle$, using Lemma 3.8 as

$$\begin{aligned} h_p &= \int \Big(\frac{\rho_0\widehat{g}_q\widehat{v}_{p-q} - \rho_0\widehat{g}_0\widehat{v}_p\mathbb{1}_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}}}{2q^2} + \frac{\rho_0\widehat{g}_0\widehat{v}_p\mathbb{1}_{\{\ell_\delta^{-1}<|q|<\sqrt{\rho_0\widehat{g}_0}\}}}{2q^2} \\ &\quad - \frac{\rho_0\widehat{g}_q\widehat{v}_{p-q}}{2\sqrt{q^4 + 2\rho_0q^2\widehat{g}_q}}\Big)\frac{\mathrm{d}q}{4\pi^2} \\ &= h_p^{(1)} + h_p^{(2)} + h_p^{(3)}, \end{aligned} \tag{4.17}$$

with

$$\begin{aligned} |h_p^{(1)}| &= \Big|\int_{\{|q|>\sqrt{\rho_0\widehat{g}_0}\}} \frac{\rho_0\widehat{g}_q\widehat{v}_{p-q}}{2q^2}\Big(1 - \frac{1}{\sqrt{1 + \frac{2\rho_0\widehat{g}_q}{q^2}}}\Big)\frac{\mathrm{d}q}{4\pi^2}\Big| \\ &\leq C\widehat{v}_0\int_{\{|q|>\sqrt{\rho_0\widehat{g}_0}\}} \frac{(\rho_0\widehat{g}_0)^2}{q^4}\mathrm{d}q \leq C\widehat{v}_0\rho_0\widehat{g}_0. \end{aligned} \tag{4.18}$$

We also calculate

$$\begin{aligned} |h_p^{(2)}| &= \Big|\int_{\{\ell_\delta^{-1}<|q|<\sqrt{\rho_0\widehat{g}_0}\}} \frac{\rho_0\widehat{g}_0\widehat{v}_p}{2q^2}\frac{\mathrm{d}q}{4\pi^2}\Big| \\ &\leq C\rho_0\widehat{g}_0\widehat{v}_0\Big|\log\Big(\sqrt{\rho_0a^2}\sqrt{\widehat{g}_0}\frac{e^\Gamma}{2}e^{\frac{1}{2\delta}}\Big)\Big| \\ &= \rho_0\delta\widehat{v}_0\Big|\frac{1}{\delta} - \frac{1}{Y} + \log\delta + C\Big|. \end{aligned} \tag{4.19}$$

In the case where $\ell_\delta^{-1} \geq \sqrt{\rho_0\widehat{g}_0}$, the same estimates hold true. Only the inequalities inside the indicator function in (4.17) change. Using (3.30) we have,

$$\begin{aligned} |h_p^{(3)}| &= \Big|\int_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}} \frac{\rho_0(\widehat{g}_q\widehat{v}_{p-q} - \widehat{g}_0\widehat{v}_p)}{2q^2}\frac{\mathrm{d}q}{4\pi^2} - \int_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}} \frac{\rho_0\widehat{g}_q\widehat{v}_{p-q}}{2\sqrt{q^4 + 2\rho_0q^2\widehat{g}_q}}\frac{\mathrm{d}q}{4\pi^2}\Big| \\ &\leq \int_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}} \frac{\rho_0\widehat{g}_0|\widehat{v}_{p-q} - \widehat{v}_p| + \rho_0|\widehat{g}_q - \widehat{g}_0|\widehat{v}_0}{2q^2}\frac{\mathrm{d}q}{4\pi^2} \\ &\quad + C\widehat{v}_0\sqrt{\rho_0\widehat{g}_0}\int_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}} \frac{1}{|q|}\mathrm{d}q \\ &\leq C\|\nabla\widehat{v}\|_\infty\int_{\{|q|<\sqrt{\rho_0\widehat{g}_0}\}} \frac{\rho_0\widehat{g}_0}{|q|}\mathrm{d}q + C\widehat{v}_0\rho_0\widehat{g}_0 \leq C\widehat{v}_0\rho_0\widehat{g}_0, \end{aligned}$$

where we used $\ell_\delta^{-1} \sim \sqrt{\rho \widehat{g}_0} \leq R$ and $\|\nabla \widehat{v}_0\|_\infty \leq R \widehat{v}_0$. In the end we obtain

$$|h_p| \leq C \widehat{v}_0 \rho_0 \widehat{g}_0 + \rho_0 \delta \widehat{v}_0 \Big(\frac{1}{\delta} - \frac{1}{Y} + \log \delta + C \Big).$$

Similarly we have bounds on the gradient of h, namely

$$|\nabla h_p| \leq C R \widehat{v}_0 \rho_0 \widehat{g}_0. \tag{4.20}$$

Now we turn to

$$\begin{aligned} D(\alpha + \rho_0 \widehat{\omega}, \alpha + \rho_0 \widehat{\omega}) &= \langle \alpha + \rho_0 \widehat{\omega}, h \rangle \\ &= \int \Big(\frac{\rho_0 \widehat{g}_q h_q - \rho_0 \widehat{g}_0 h_0 \mathbb{1}_{\{|q| > \ell_\delta^{-1}\}}}{2q^2} - \frac{\rho_0 \widehat{g}_q h_q}{2\sqrt{q^4 + 2\rho_0 q^2 \widehat{g}_q}} \Big) \frac{\mathrm{d}q}{4\pi^2}, \end{aligned}$$

which we in the same way write as $D_1 + D_2 + D_3$ with

$$\begin{aligned} |D_1| &= \Big| \int_{\{|q| > \sqrt{\rho_0 \widehat{g}_0}\}} \frac{\rho_0 \widehat{g}_q h_q}{2q^2} \Big(1 - \frac{1}{\sqrt{1 + \frac{2\rho_0 \widehat{g}_q}{q^2}}} \Big) \frac{\mathrm{d}q}{4\pi^2} \Big| \\ &\leq \frac{\|h\|_\infty}{8\pi^2} \int_{\{|q| > \sqrt{\rho_0 \widehat{g}_0}\}} \frac{(\rho_0 \widehat{g}_0)^2}{q^4} \mathrm{d}q \\ &\leq C \widehat{v}_0 \rho_0^2 \widehat{g}_0^2 + \rho_0^2 \delta^2 \widehat{v}_0 \Big(\frac{1}{\delta} - \frac{1}{Y} + \log \delta + C \Big), \end{aligned}$$

and using the bounds on h, we find $|D_1| \leq C \widehat{v}_0 \rho_0^2 \widehat{g}_0^2$. The technique to bound D_2 is the same as for $h^{(2)}$ and its provides

$$|D_2| = \Big| \int_{\{\ell_\delta^{-1} < |q| < \sqrt{\rho_0 \widehat{g}_0}\}} \frac{\rho_0 \widehat{g}_0 h_0}{2q^2} \frac{\mathrm{d}q}{4\pi^2} \Big| \leq \widehat{v}_0 \rho_0^2 \delta^2 \Big(\frac{1}{\delta} - \frac{1}{Y} + \log \delta + C \Big)^2.$$

Lastly D_3 is bounded just as $h^{(3)}$,

$$\begin{aligned} |D_3| &= \Big| \int_{\{|q| < \sqrt{\rho_0 \widehat{g}_0}\}} \frac{\rho_0 (\widehat{g}_q h_q - \widehat{g}_0 h_0)}{2q^2} \frac{\mathrm{d}q}{4\pi^2} - \int_{\{|q| < \sqrt{\rho_0 \widehat{g}_0}\}} \frac{\rho_0 \widehat{g}_q h_q}{2\sqrt{q^4 + 2\rho_0 q^2 \widehat{g}_q}} \frac{\mathrm{d}q}{4\pi^2} \Big| \\ &\leq C \widehat{v}_0 \rho_0^2 \widehat{g}_0^2, \end{aligned}$$

from which the first result follows. The second comes from the fact that when $\delta = \delta_0$ we have

$$\begin{aligned} \delta_0^{-1} &= Y^{-1} + |\log Y| + O(Y |\log Y|^2), \\ \log \delta_0 &= \log Y + \log(1 - Y |\log Y|), \end{aligned}$$

providing that

$$|\delta_0^{-1} - Y^{-1} + \log \delta_0| \leq C. \tag{4.21}$$

□

Now we have all necessary ingredients to conclude the proof of Theorem 4.1.

Proof of Theorem 4.1. We take the trial state Φ defined in Sect. 4.1, which has the expected bounds on number of particles from Lemma 4.5. The energy of Φ is bounded by Lemma 4.7 together with Lemma 4.8, and using $\delta_0 \geq \frac{1}{2}Y$ we find

$$\begin{aligned}\frac{\langle \mathcal{H}_v \rangle_\Phi}{|\Lambda_\beta|} \leq \frac{\rho^2}{2}\widehat{g}_0 + \frac{1}{2}\int \Big(\sqrt{p^4 + 2\rho_0 p^2 \widehat{g}_p} - p^2 - \rho_0 \widehat{g}_p + \rho_0^2 \frac{\widehat{g}_p^2 - \widehat{g}_0^2 \mathbb{1}_{\{|p| \leq \ell_\delta^{-1}\}}}{2p^2}\Big)\frac{\mathrm{d}p}{4\pi^2} \\ + \rho_0^2 \delta^2 \widehat{v}_0 \Big(\frac{1}{\delta} - \frac{1}{Y} + \log \delta\Big) + \rho_0^2 \delta^2 \widehat{v}_0 \Big(\frac{1}{\delta} - \frac{1}{Y} + \log \delta\Big)^2 \\ + C\rho^2 \delta(\widehat{v}_0 - \widehat{g}_0) + C\widehat{v}_0 \rho_0^2 \widehat{g}_0^2.\end{aligned}$$

Now this integral can be estimated by Proposition C.3 and using $\rho - \rho_0 \leq C\rho Y$ we find

$$\begin{aligned}\frac{\langle \mathcal{H}_v \rangle_\Phi}{|\Lambda_\beta|} \leq \frac{\rho^2}{2}\widehat{g}_0 + 4\pi\rho^2\delta^2\Big(\frac{1}{\delta} - \frac{1}{Y} + \log\delta + \Big(\frac{1}{2} + 2\Gamma + \log\pi\Big)\Big) \\ + \frac{1}{2}\rho^2\delta^2\widehat{v}_0\Big|\frac{1}{\delta} - \frac{1}{Y} + \log\delta\Big| + \frac{1}{2}\rho^2\delta^2\widehat{v}_0\Big(\frac{1}{\delta} - \frac{1}{Y} + \log\delta\Big)^2 \\ + C\rho^2\delta(\widehat{v}_0 - \widehat{g}_0) + C\widehat{v}_0\rho^2\widehat{g}_0^2. \end{aligned} \tag{4.22}$$

Finally, with $\widehat{g}_0 = 8\pi\delta$ and the specific choice $\delta = \delta_0$ we deduce

$$\frac{\langle \mathcal{H}_v \rangle_\Phi}{|\Lambda_\beta|} \leq 4\pi\rho^2\delta_0\Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)\delta_0\Big) + C\rho^2\delta_0(\widehat{v}_0 - \widehat{g}_0) + C\rho^2\delta_0^2\widehat{v}_0. \tag{4.23}$$

□

Remark 4.9. In the case of the spherical measure potential (3.40) (with the choice of b given in (2.5)), one can see that the upper bound (4.22) is minimized (to the available energy precision) by the choice $\delta = \delta_0$. Indeed, even though the first three terms suggest to choose the smallest δ possible, including the remaining contributions yields a minimizer of the form

$$\delta = Y(1 + cY|\log Y|). \tag{4.24}$$

Notice that first line in (4.22) is independent of the choice of c to our precision. We pick for simplicity $c = -1$ to obtain our δ_0 providing useful cancelations, see (4.21). This also fixes the value of $\widetilde{R} = ae^{\frac{1}{2\delta}}$.

5. General Upper Bound

In this section we prove Theorem 2.2, using the results of Sect. 4. We let $\beta \geq \frac{3}{2}$ be given and we work on the box $\Lambda_\beta = [-\frac{L_\beta}{2}, \frac{L_\beta}{2}]^2$ of size $L_\beta = \rho^{-\frac{1}{2}}Y^{-\beta}$. Moreover, the number of particles at density ρ is $N = Y^{-2\beta}$.

5.1. Trial state. Let v be a non-negative measurable and radial potential with scattering length a and $\mathrm{supp}(v) \subset B(0,R)$, with $\rho R^2 \leq Y^{2\beta+2}$. We consider φ_b the associated scattering solution normalized at length $b = \rho^{-1/2} Y^{\beta+\frac{1}{2}}$. In other words $\varphi_b = 2\delta_\beta \varphi^{(0)}$ with $\delta_\beta = \frac{1}{2}\log(b/a)^{-1}$, see (3.3). Note that $R \ll b$. Let $f = \min(1, \varphi_b)$ be the truncated scattering solution. It satisfies

$$-\Delta f(x) + \frac{1}{2} v(x) f(x) = 0 \quad \text{on } B(0,b), \tag{5.1}$$

and is normalized such that $f(x) = 1$ for $|x| \geq b$. We define a grand canonical trial state as

$$\Psi = \sum_{n\geq 0} \Phi_n F_n \in \mathscr{F}_s\big(L^2\left(\Lambda_\beta\right)\big) \tag{5.2}$$

where $\Phi = \sum_n \Phi_n \in \mathscr{F}_s\big(L^2\left(\Lambda_\beta\right)\big)$ is a quasi-free state defined in (4.3) and F_n is the Jastrow factor

$$F_n(x_1, \ldots, x_n) = \prod_{1\leq i<j\leq n} f(x_i - x_j). \tag{5.3}$$

We will use the notation $f(x_i - x_j) = f_{ij}$ and $\nabla f(x_i - x_j) = \nabla f_{ij}$. Finally note that

$$\nabla_i F_n(x_1, \ldots, x_n) = \sum_{\substack{j=1\\ j\neq i}}^{n} \nabla_i f_{ij} \frac{F_n}{f_{ij}}. \tag{5.4}$$

Remark 5.1. To estimate the energy of Ψ we use the bound

$$1 \geq \prod_{1\leq i<j\leq n} f(x_i - x_j)^2 \geq 1 - \sum_{1\leq i<j\leq n} (1 - f(x_i - x_j)^2). \tag{5.5}$$

A similar trial state is used in [22] in 3 dimensions but there it is necessary to expand the product (5.5) to one order higher to be able to reach the LHY precision. This substantially complicates the estimates in that case.

5.2. Reduction to a soft potential. In this section we prove that the energy $\langle \Psi, \mathcal{H}_v \Psi\rangle$ can be bounded by $\langle \Phi, \mathcal{H}_{\widetilde{v}} \Phi\rangle$ where $\widetilde{v}$ is a nicer potential. This is the effect of the Jastrow factor F_n, and we are thus reduced to optimizing the choice of the quasi-free state Φ according to the potential $\widetilde{v}$.

Lemma 5.2. *Consider the radial potential $\widetilde{v}(x) = 2f'(b)\delta_{\{|x|=b\}}$ (with f' being understood as the radial derivative). Then the state Ψ defined in* (5.2) *satisfies*

$$\langle \Psi, \mathcal{H}_v \Psi\rangle \leq \langle \Phi, \mathcal{H}_{\widetilde{v}} \Phi\rangle - \langle \Phi, \mathcal{R}\Phi\rangle,$$

where $\mathcal{R} = \oplus_n \mathcal{R}_n$ with

$$\mathcal{R}_n = \sum_{\{i,j,k\}} \frac{\nabla f_{ij}}{f_{ij}} \cdot \frac{\nabla f_{ik}}{f_{ik}} F_n^2,$$

where we introduced the notation

$$\{i,j,k\} = \{\textit{set of pairwise distinct indices } i,j,k \textit{ running from } 1 \textit{ to } n\}.$$

Proof. The energy of the n-th sector state is

$$\langle \Psi_n, \mathcal{H}_n \Psi_n \rangle = \sum_{i=1}^{n} \int_{\Lambda^n} \left(F_n^2 |\nabla_i \Phi_n|^2 + |\nabla_i F_n|^2 \Phi_n^2 + 2 F_n \nabla_i F_n \cdot \Phi_n \nabla_i \Phi_n \right) \mathrm{d}x + \sum_{1 \le i < j \le n} \int_{\Lambda^n} v(x_i - x_j) F_n^2 \Phi_n^2 \mathrm{d}x. \tag{5.6}$$

The second term in (5.6) can be written via (5.4) as

$$\sum_{i=1}^{n} \int_{\Lambda^n} |\nabla_i F_n|^2 \Phi_n^2 \mathrm{d}x = \sum_{i \neq j} \int_{\Lambda^n} |\nabla f_{ij}|^2 \frac{F_n^2}{f_{ij}^2} \Phi_n^2 \mathrm{d}x + \sum_{\{i,j,k\}} \int_{\Lambda^n} \frac{\nabla f_{ij}}{f_{ij}} \cdot \frac{\nabla f_{ik}}{f_{ik}} F_n^2 \Phi_n^2 \mathrm{d}x. \tag{5.7}$$

Note that, in the first part of (5.7) the integration in x_i is only supported on the ball $|x_i - x_j| \le b$, because $f_{ij} = 1$ outside this ball. We integrate by parts on this ball to find

$$\begin{aligned} &\sum_{i=1}^{n} \int_{\Lambda^n} |\nabla_i F_n|^2 \Phi_n^2 \mathrm{d}x \\ &= -\sum_{i \neq j} \int_{\{|x_i - x_j| \le b\}} \Delta f_{ij} \frac{F_n^2}{f_{ij}} \Phi_n^2 \mathrm{d}x - \sum_{i \neq j} \int_{\Lambda^n} \nabla f_{ij} \frac{F_n^2}{f_{ij}} \cdot \nabla_i (\Phi_n^2) \mathrm{d}x \\ &\quad - \sum_{\{i,j,k\}} \int_{\Lambda^n} \frac{\nabla f_{ij}}{f_{ij}} \cdot \frac{\nabla f_{ik}}{f_{ik}} F_n^2 \Phi_n^2 \mathrm{d}x + \sum_{i \neq j} \int_{\Lambda^{n-1}} \int_{\{|x_i - x_j| = b\}} \partial_r f(b) F_n^2 \Phi_n^2 \mathrm{d}x_i \mathrm{d}\hat{x}_i, \end{aligned} \tag{5.8}$$

where $\hat{x}_i = (x_1, \ldots x_{i-1}, x_{i+1}, \ldots, x_n)$. The second term in the right hand side of (5.8) is precisely $-2 F_n \nabla_i F_n \cdot \Phi_n \nabla_i \Phi_n$ thanks to (5.4). We use the scattering equation (5.1) to transform

$$\sum_{i \neq j} \int_{\{|x_i - x_j| \le b\}} \Delta f_{ij} \frac{F_n^2}{f_{ij}} \Phi_n^2 \mathrm{d}x = \sum_{1 \le i < j \le n} v(x_i - x_j) F_n^2 \Phi_n^2 \mathrm{d}x, \tag{5.9}$$

in (5.8) (note that there is no half factor because the sum is on $i < j$). Using (5.8) and (5.9) in (5.6) we deduce

$$\langle \Psi_n, \mathcal{H}_n \Psi_n \rangle = \sum_{i=1}^{n} \int_{\Lambda^n} F_n^2 |\nabla_i \Phi_n|^2 + 2 \sum_{i<j} \int_{\Lambda^{n-1}} \int_{\{|x_i - x_j| = b\}} \partial_r f(b) F_n^2 \Phi_n^2 \mathrm{d}x_i \mathrm{d}\hat{x}_i - \sum_{\{i,j,k\}} \int_{\Lambda^n} \frac{\nabla f_{ij}}{f_{ij}} \cdot \frac{\nabla f_{ik}}{f_{ik}} F_n^2 \Phi_n^2 \mathrm{d}x. \tag{5.10}$$

In the first two terms we bound F_n by 1, and the last one we consider as a remainder. Thus,

$$\langle \Psi_n, \mathcal{H}_n \Psi_n \rangle \le \int_{\Lambda^n} |\nabla \Phi_n|^2 + \sum_{i<j} \int_{\Lambda^n} \Phi_n^2 \widetilde{v}(x_i - x_j) \mathrm{d}x - \mathcal{R}_n.$$

□

We comment here how in the proof we used nowhere that Φ is a quasi-free state, therefore the lemma holds true for more general $\Phi \in \mathscr{F}_s(L^2(\Lambda_\beta))$.

5.3. Number of particles in our trial state. Now for Φ we choose the quasi-free state given by Theorem 4.1, applied to the potential $\widetilde{v}$. We recall that $\Phi = W_{N_0} T_\nu \Omega$ is defined in (4.3), and Ψ in (5.2). In this section we prove the following two lemmas, giving estimates on the norm of Ψ and the average number of particles in Ψ. The idea is to use the properties of F_n to derive the bounds on $\Psi_n = F_n \Phi_n$ from the bounds on the quasi-free state Φ.

Lemma 5.3. *There is a $C > 0$, independent of v and ρ, such that*

$$\|\Psi\|^2 \geq \|\Phi\|^2 \Big(1 - CNY^{2\beta+2}\Big).$$

Proof. The norm of our trial state is bounded from below by

$$\begin{aligned}\|\Psi\|^2 &= \sum_{n\geq 0} \int_{\Lambda^n} F_n^2(x)\Phi_n^2(x)\mathrm{d}x \\ &\geq \sum_{n\geq 0} \Big(\int_{\Lambda^n} \Phi_n^2 \mathrm{d}x - \sum_{1\leq i<j\leq n} \int_{\Lambda^n} (1 - f(x_i - x_j)^2)\Phi_n^2(x)\mathrm{d}x \Big),\end{aligned} \tag{5.11}$$

where we used the inequality

$$\prod_{1\leq i<j\leq n} f(x_i - x_j)^2 \geq 1 - \sum_{1\leq i<j\leq n} (1 - f(x_i - x_j)^2). \tag{5.12}$$

The second term is the 2-body interaction potential energy of Φ, thus we can write it as

$$\begin{aligned}\sum_{n\geq 0} \sum_{1\leq i<j\leq n} \int (1 - f(x_i - x_j)^2)\Phi_n^2 \mathrm{d}x &= \frac{1}{2|\Lambda_\beta|} \sum_{p,q,r} \widehat{(1 - f^2)}_r \langle a_q^* a_{p+r}^* a_{q+r} a_p \Phi, \Phi\rangle \\ &\leq \frac{\widehat{(1 - f^2)}_0}{2|\Lambda_\beta|} \sum_{p,q,r} \langle a_q^* a_{p+r}^* a_{q+r} a_p \Phi, \Phi\rangle.\end{aligned} \tag{5.13}$$

Since $\Phi = W_{N_0} T_\nu \Omega$ is a quasi-free state we can estimate this term as already done in (4.11). We first conjugate by W_{N_0} which amounts to change the a_0's into $N_0 \leq N$. Together with Lemma 4.4 and Wick's theorem we deduce

$$\sum_{p,q,r} \langle a_q^* a_{p+r}^* a_{q+r} a_p \rangle_\Phi \leq CN^2 + \sum_{\substack{p\neq 0, q\neq 0, r\neq 0 \\ p+r\neq 0, q+r\neq 0}} \langle a_q^* a_{p+r}^* a_{q+r} a_p \rangle_\Phi. \tag{5.14}$$

Then we use again Wick's Theorem to estimate the remaining sum, which is then bounded by CN^2 by Lemma 4.4. Thus Eq. (5.13) gives

$$\sum_{n\geq 0} \sum_{1\leq i<j\leq n} \int (1 - f(x_i - x_j)^2)\Phi_n^2 \mathrm{d}x \leq C \frac{N^2}{|\Lambda_\beta|} \int_\Lambda (1 - f(x)^2)\mathrm{d}x \, \|\Phi\|^2. \tag{5.15}$$

Using $\frac{\mathrm{d}}{\mathrm{d}r}[r^2 \log\left(\frac{r}{a}\right)^2 - r^2\log\left(\frac{r}{a}\right) + \frac{r^2}{2}] = 2r\log\left(\frac{r}{a}\right)^2$ and $a \le R \le b$ we have that

$$\begin{aligned}\int_\Lambda (1-f(x)^2)\mathrm{d}x &= 2\pi\int_R^b \left(1 - \frac{\log\left(\frac{r}{a}\right)^2}{\log\left(\frac{b}{a}\right)^2}\right) r\mathrm{d}r + 2\pi\int_0^R (1-f(r)^2) r\mathrm{d}r \\ &\le C\frac{b^2}{\log\left(\frac{b}{a}\right)} + CR^2 \le C\rho^{-1}Y^{2\beta+2}, \end{aligned} \tag{5.16}$$

where we used $\rho R^2 \le Y^{2\beta+2}$ and $b^2 = \rho^{-1}Y^{2\beta+1}$. We use this last bound in (5.15) and (5.11) to get

$$\|\Psi\|^2 \ge \|\Phi\|^2\big(1 - CNY^{2\beta+2}\big).$$

□

Lemma 5.4. *There is a $C > 0$ independent of ρ and υ such that,*

$$\langle\Psi, \mathcal{N}\Psi\rangle \ge N(1-CY^2)\|\Psi\|^2, \qquad \langle\Psi, \mathcal{N}^2\Psi\rangle \le CN^2\|\Psi\|^2.$$

Proof. First we have by Lemmas 4.5 and 5.3 that

$$\langle\Psi, \mathcal{N}^2\Psi\rangle = \sum_{n\ge 0} n^2 \int_{\Lambda^n} F_n^2\Phi_n^2\mathrm{d}x \le \sum_{n\ge 0} n^2\|\Phi_n\|^2 = \langle\Phi, \mathcal{N}^2\Phi\rangle \le CN^2\|\Psi\|^2.$$

For the bound on $\langle\mathcal{N}\rangle_\Psi$ we use the same idea as in the proof of Lemma 5.3. From inequality (5.12) we deduce

$$\begin{aligned}\langle\Psi, \mathcal{N}\Psi\rangle &= \sum_{n\ge 0} n\int_{\Lambda^n} F_n^2\Phi_n^2\mathrm{d}x \\ &\ge \sum_{n\ge 0} n\Big(\int_{\Lambda^n}\Phi_n^2\mathrm{d}x - \sum_{i<j}\int_{\Lambda^n}(1-f(x_i-x_j)^2)\Phi_n^2\mathrm{d}x\Big). \end{aligned} \tag{5.17}$$

In the second term we recognize a number operator and a 2-particles interaction energy, which can be rewritten as

$$\sum_{n\ge 0} n\sum_{i<j}\int_{\Lambda^n}(1-f(x_i-x_j)^2)\Phi_n^2\mathrm{d}x = \sum_{k,p,q,r\in\Lambda^*}\frac{\widehat{(1-f^2)}_r}{2L_\beta^2}\langle a_k^* a_k a_{p+r}^* a_q^* a_{q+r} a_p\Phi, \Phi\rangle.$$

We can compute this term using the same techniques as for (5.13), i.e., extract the a_0's and then apply Wick's Theorem yielding many terms of the form $A_1A_2A_3$ with $A_i \in \{\langle a_0^\dagger a_0\rangle_\Phi, \sum_{p\neq 0}\alpha_p, \sum_{p\neq 0}\gamma_p\}$ (see (4.5)). These terms are bounded by N^3 by Lemma 4.4. Thus

$$\sum_{n\ge 0} n\sum_{i<j}\int_{\Lambda^n}(1-f(x_i-x_j)^2)\Phi_n^2\mathrm{d}x \le C\frac{N^3}{L_\beta^2}\int(1-f(x)^2)\mathrm{d}x\|\Phi\|^2.$$

Now we use the inequality (5.16) to bound the right hand side of the quantity above and plug it in (5.17) to obtain

$$\langle\Psi, \mathcal{N}\Psi\rangle \ge (N - CN^2Y^{2\beta+2})\|\Psi\|^2 = N(1-CY^2)\|\Psi\|^2,$$

where in the equality used that $N = Y^{-2\beta}$. □

5.4. Remainder term. Here we prove that the remainder term in Lemma 5.2 is indeed small.

Lemma 5.5. *There is a $C > 0$ independent of v and ρ such that*

$$|\langle\Phi, \mathcal{R}\Phi\rangle| \le CL_\beta^2\rho^2 Y^{2\beta+2}\|\Phi\|^2.$$

Proof. The remainder term can be bounded by

$$|\langle\Phi, \mathcal{R}\Phi\rangle| \le \sum_{n\ge 3}\sum_{\{i,j,k\}}\int_{\Lambda^n} W(x_i - x_k)W(x_i - x_j)\Phi_n^2 \mathrm{d}x,$$

where $W(x) = |f(x)\nabla f(x)|$. This is a three-body interaction potential, which can be rewritten in second quantization as

$$|\langle\Phi, \mathcal{R}\Phi\rangle| \le \frac{1}{|\Lambda_\beta|^2}\sum_{p,q,r,k,\ell\in\Lambda^*}\widehat{W}_k\widehat{W}_\ell\langle a^*_{p+\ell+k}a^*_{q-k}a^*_{r-\ell}a_r a_q a_p\rangle_\Phi\,\|\Phi\|^2.$$

We can again use Wick's Theorem to estimate this part, and since Lemma 4.4 provides

$$\max\Big\{\sum_{p\ne 0}\alpha_p, \sum_{p\ne 0}\gamma_p\Big\} \le N,$$

we find

$$|\langle\Phi, \mathcal{R}\Phi\rangle| \le C\frac{N^3}{|\Lambda_\beta|^2}\widehat{W}_0^2\,\|\Phi\|^2. \tag{5.18}$$

Now since $f(x) = \log\left(\frac{b}{a}\right)^{-1}\log\left(\frac{|x|}{a}\right)$ outside the support of v and is radially increasing, we have

$$\begin{aligned}\widehat{W}_0 &\le 2\pi\int_R^b \frac{\log\left(\frac{r}{a}\right)}{\log\left(\frac{b}{a}\right)^2}\mathrm{d}r + 2\pi\int_0^R f(r)f'(r)r\mathrm{d}r\\ &\le C\frac{b}{\log\left(\frac{b}{a}\right)} + CR\left(\frac{\log\left(\frac{R}{a}\right)}{\log\left(\frac{b}{a}\right)}\right)^2 \le C\rho^{-1/2}Y^{\beta+1},\end{aligned}$$

where we used $|\log\left(\frac{b}{a}\right)|^{-1} \le Y$ and $|\log\left(\frac{R}{a}\right)| \le |\log\left(\frac{b}{a}\right)|$. Inserting this bound in (5.18) we get the result. □

5.5. Conclusion: Proof of Theorem 2.2. Using Lemmas 5.2, 5.3 and 5.5 we know that our trial state Ψ satisfies

$$\langle\mathcal{H}_v\rangle_\Psi \le \left(\langle\mathcal{H}_{\widetilde{v}}\rangle_\Phi + CL_\beta^2\rho^2Y^{2\beta+2}\right)\left(1 + CNY^{2\beta+2}\right). \tag{5.19}$$

For Φ we choose the quasi-free state given by Theorem 4.1 applied to the soft potential $\widetilde{v}$. Recall the definition of $\widetilde{g}$ from Lemma 3.10. We deduce that

$$\frac{1}{|\Lambda_\beta|}\langle\mathcal{H}_{\widetilde{v}}\rangle_\Phi \le 4\pi\rho^2\delta_0\left(1 + \left(2\Gamma + \frac{1}{2} + \log(\pi)\right)\delta_0\right) + C\rho^2\delta_0\left(\widehat{\widetilde{v}}(0) - \widehat{\widetilde{g}}(0)\right) + C\rho^2\delta_0^2\widehat{\widetilde{v}}(0). \tag{5.20}$$

From Lemma 3.10 we have

$$\widehat{\widetilde{v}}(0) = \frac{4\pi}{\log b/a} \quad \text{and} \quad \widehat{\widetilde{g}}(0) = \frac{4\pi}{\log \widetilde{R}/a} = 8\pi\delta_0,$$

where we recall from (3.3) that $\widetilde{R} = ae^{\frac{1}{2\delta_0}}$. Therefore, remembering the choices $b = \rho^{-1/2}Y^{1/2+\beta}$ and $\beta \geq 3/2$, we can estimate $(\widehat{\widetilde{v}}(0) - \widehat{\widetilde{g}}(0)) \leq CY^2 \log Y$ and we get

$$\frac{1}{|\Lambda_\beta|}\langle \mathcal{H}_{\widetilde{v}} \rangle_\Phi \leq 4\pi\rho^2\delta_0\Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)\delta_0\Big) + \beta C\rho^2\delta_0^3|\log(\delta_0)| + C\rho^2\delta_0^3. \tag{5.21}$$

We insert this into (5.19) together with $N = \rho L_\beta^2 = Y^{-2\beta}$ and $Y \leq 2\delta_0$, which concludes the proof of Theorem 2.2. □

6. Localization to Large Boxes for the Lower Bound

In this section we reduce the proof of Theorem 2.3 to an analogous statement localized to a box of size ℓ defined in (6.6), namely Theorem 6.7.

6.1. Grand canonical ensemble. We rewrite the Hamiltonian in a grand canonical setting to approach the problem in the Fock space description. To emphasize the fact that the density parameter appears through a chemical potential in this setting, we introduce the notation $\rho_\mu > 0$ as new parameter. The corresponding Y will be $Y = |\log(\rho_\mu a^2)|^{-1}$ and we fix δ to be

$$\delta = \delta_\mu, \qquad \delta_\mu := \frac{1}{|\log(\rho_\mu a^2 |\log(\rho_\mu a^2)|^{-1})|}. \tag{6.1}$$

This corresponds to normalizing the scattering solution at length $\widetilde{R} = (\rho_\mu Y)^{-1/2}$ in (3.3). With this choice we recall the definition (3.4) of g. This definition is analogous to the one of δ_0 (1.5) but with ρ_μ in place of ρ. We are going to choose, a posteriori, $\rho_\mu = \rho$ which implies $\delta_\mu = \delta_0$.

That this choice of δ is optimal follows by an evaluation of the relevant integral giving the constant in the correction term in (1.4). Please see Remark C.4 for the evaluation of this integral and the discussion of the optimal choice.

We consider the operator $\mathcal{H}_{\rho_\mu}$ acting on the symmetric Fock space $\mathscr{F}_s(L^2(\Omega))$ and commuting with the number operator, whose action on the N-bosons space is

$$\begin{aligned}\mathcal{H}_{\rho_\mu,N} &= H(N,L) - 8\pi\delta\rho_\mu N = \sum_{j=1}^N -\Delta_j + \sum_{i<j} v(x_i - x_j) - 8\pi\delta\rho_\mu N \\ &= \sum_{j=1}^N \Big(-\Delta_j - \rho_\mu \int_{\mathbb{R}^2} g(x_j - y)\,\mathrm{d}y\Big) + \sum_{i<j} v(x_i - x_j).\end{aligned} \tag{6.2}$$

We define the ground state energy density of $\mathcal{H}_{\rho_\mu}$:

$$e_0(\rho_\mu) := \lim_{|\Omega|\to+\infty} \frac{1}{|\Omega|} \inf_{\Psi\in\mathscr{F}_s(L^2(\Omega))\setminus\{0\}} \frac{\langle \Psi | \mathcal{H}_{\rho_\mu} | \Psi \rangle}{\|\Psi\|^2}. \tag{6.3}$$

In the rest of the paper we prove the following lower bound on $e_0(\rho_\mu)$.

Theorem 6.1. *There exists $C, \eta > 0$ such that the following holds. Let $\rho_\mu > 0$ and $v \in L^1(\Omega)$ be a positive, spherically symmetric potential with scattering length a and* $\mathrm{supp}(v) \subset B(0, R)$ *such that $\|v\|_1 \leq Y^{-1/8}$ and $R \leq \rho_\mu^{-1/2}$. Then, if $\rho_\mu a^2 \leq C^{-1}$, we have, for any $\rho_\mu > 0$,*

$$e_0(\rho_\mu) \geq -4\pi\rho_\mu^2\delta\Big(1 - \Big(2\Gamma + \frac{1}{2} + \log\pi\Big)\delta\Big) - C\rho_\mu^2\delta^{2+\eta}. \tag{6.4}$$

We now show that Theorem 6.1 implies the main lower bound Theorem 2.3.

Proof of Theorem 2.3. We start by reducing the problem to a potential which is L^1 and compactly supported. For a given v satisfying the assumptions of Theorem 2.3, we apply Theorem 3.6 with $T = (4\pi Y)^{-1/8}$, $R = \rho^{-1/2}$ and $\varepsilon = 1$. This provides us with a potential $\widetilde{v} = v_{T,R,\epsilon}$ to which we can apply Theorem 6.1. Then for this new potential we use the ground state of $\mathcal{H}_N$ as a trial function for $\mathcal{H}_{\rho_\mu}$ and get

$$\begin{aligned} e^{2\mathrm{D}}(\rho, \widetilde{v}) &\geq e_0(\rho_\mu, \widetilde{v}) + 8\pi\widetilde{\delta}\rho\rho_\mu \\ &\geq -4\pi\rho_\mu^2\widetilde{\delta}\Big(1 - \Big(2\Gamma + \frac{1}{2} + \log\pi\Big)\widetilde{\delta}\Big) - C\rho_\mu^2\widetilde{\delta}^{2+\eta} + 8\pi\widetilde{\delta}\rho\rho_\mu, \end{aligned}$$

where $\widetilde{\delta} = \frac{1}{|\log(\rho\widetilde{a}^2|\log(\rho\widetilde{a}^2)|^{-1})|}$ and $\widetilde{a}$ is the scattering length of $\widetilde{v}$. Since $\widetilde{v} \leq v$ we have $e^{2\mathrm{D}}(\rho, v) \geq e^{2\mathrm{D}}(\rho, \widetilde{v})$. Moreover, by Eq. (3.23) we can change $\widetilde{\delta}$ into δ up to an error of order

$$\frac{1}{\log\left(\frac{R}{a}\right)^2 T} + \frac{1}{\log\left(\frac{R}{a}\right)^2}\int_{\{|x|>R\}} v(x)\log\left(\frac{|x|}{a}\right)^2 \mathrm{d}x \leq C\delta^{2+\min\left(\frac{1}{8},\eta_1\right)}. \tag{6.5}$$

Choosing $\rho_\mu = \rho$ concludes the proof. □

6.2. Reduction to large boxes. We now make use of the sliding localization technique developed in [28] to reduce the proof of Theorem 2.3 to a localized problem in a large box $\Lambda \subset \Omega$. We introduce the length scale

$$\ell := K_\ell\,\rho_\mu^{-1/2}Y^{-\frac{1}{2}}, \tag{6.6}$$

where $K_\ell \gg 1$ is a parameter fixed in "Appendix H", and we carry out the analysis in the large box

$$\Lambda := \Big[-\frac{\ell}{2}, \frac{\ell}{2}\Big]^2. \tag{6.7}$$

For any $u \in \mathbb{R}^2$, we denote by

$$\Lambda_u := \ell u + \Lambda \tag{6.8}$$

the translated large box. Let us introduce the localization functions: the sharp characteristic function

$$\theta_u := \mathbb{1}_{\Lambda_u} \tag{6.9}$$

and the regular one: let $\chi \in C_0^M(\mathbb{R}^2)$, for $M \in \mathbb{N}$ with $\operatorname{supp}\chi = [-\frac{1}{2}, \frac{1}{2}]^2$ be the spherically symmetric function defined in "Appendix F", and

$$\chi_\Lambda(x) := \chi\left(\frac{x}{\ell}\right), \qquad \chi_u(x) := \chi_\Lambda(x - \ell u). \tag{6.10}$$

The parameter M is fixed in "Appendix H". Define the following projections on $L^2(\Lambda)$,

$$P := \ell^{-2}|\mathbb{1}_\Lambda\rangle\langle\mathbb{1}_\Lambda|, \qquad Q := \mathbb{1} - P, \tag{6.11}$$

i.e. P is the orthogonal projection in $L^2(\Lambda)$ onto the constant functions and Q is the orthogonal projection to the complement. Using these definitions, we define the following operators on $\mathscr{F}_s(L^2(\Lambda))$ through their action on any N-particles sector:

$$n_0 := \sum_{j=1}^N P_j, \qquad n_+ := \sum_{j=1}^N Q_j = N - n_0. \tag{6.12}$$

The definition is based on the idea that low energy eigenstates of the system should concentrate in the constant function. Thus, n_0 counts the number of particles in the condensate and n_+ the number of particles excited out of the condensate.

We start by stating the result for the kinetic energy.

Lemma 6.2 (Kinetic energy localization). *Let $-\Delta_u^{\mathcal{N}}$ denote the Neumann Laplacian in Λ_u and $-\Delta$ the Laplacian on $\mathbb{R}^2$. If the regularity of χ is $M > 5$ and the positive parameters $\varepsilon_N, \varepsilon_T, d, s, b$ are smaller than some universal constant, then for all $\ell > 0$ we have*

$$-\Delta \geq \int_{\mathbb{R}^2} \mathcal{T}_u \,\mathrm{d}u, \tag{6.13}$$

in terms of quadratic forms in $H^1(\mathbb{R}^2)$, where

$$\mathcal{T}_u := \varepsilon_N(-\Delta_u^{\mathcal{N}}) + (1 - \varepsilon_N)(\mathcal{T}_u^{Neu,s} + \mathcal{T}_u^{Neu,l} + \mathcal{T}_u^{gap} + \mathcal{T}_u^{kin}), \tag{6.14}$$

with

$$\mathcal{T}_u^{Neu,s} := \frac{\varepsilon_T}{2(d\ell)^2}\frac{-\Delta_u^{\mathcal{N}}}{-\Delta_u^{\mathcal{N}} + (d\ell)^{-2}}, \tag{6.15}$$

$$\mathcal{T}_u^{Neu,l} := \frac{b}{\ell^2}Q_u, \tag{6.16}$$

$$\mathcal{T}_u^{gap} := b\frac{\varepsilon_T}{(d\ell)^2}Q_u\mathbb{1}_{(d^{-2}\ell^{-1},+\infty)}(\sqrt{-\Delta})Q_u, \tag{6.17}$$

$$\mathcal{T}_u^{kin} := Q_u\chi_u\left\{(1-\varepsilon_T)\left[\sqrt{-\Delta} - \frac{1}{2s\ell}\right]_+^2 + \varepsilon_T\left[\sqrt{-\Delta} - \frac{1}{2ds\ell}\right]_+^2\right\}\chi_u Q_u. \tag{6.18}$$

Proof. The proof is identical to the one of [28, Lemma 3.7] and its adaptation to our context in [19, Lemma 6.4], which are independent of dimension. □

Remark 6.3. The kinetic energy is composed of several terms which have to remedy some problems related to the main kinetic energy term and play the following roles:

- $\mathcal{T}_u^{\mathrm{kin}}$ is the main kinetic energy term;

- $-\Delta^{\mathcal{N}}$ is the Neumann Laplacian and compensates the loss of ellipticity at the boundary caused by the localization function χ in $\mathcal{T}_u^{\mathrm{kin}}$;
- $\mathcal{T}_u^{\mathrm{Neu,s}}$ is the Neumann gap in the small box. Worth to remark is that, for large momenta, it behaves like a gap, while for small momenta its action is like a Neumann Laplacian;
- $\mathcal{T}_u^{\mathrm{Neu,l}}$ is a fraction b of the Neumann gap in the large box. We don't think of b as a parameter but as a fixed small constant. In the remaining we then choose and fix the value of b.
- $\mathcal{T}_u^{\mathrm{gap}}$ is another spectral gap which we need in order to control the number of excitations with large momenta.

The localization of the potential energy relies on a direct calculation of the integral which can be found in [28, Proposition 3.1]. Assuming that $R\ell^{-1}$ is sufficiently small, we can introduce the following localized potentials

$$W(x) := \frac{v(x)}{\chi * \chi(x/\ell)}, \qquad w(x,y) := \chi_\Lambda(x) W(x-y)\chi_\Lambda(y), \tag{6.19}$$

$$W_1(x) := \frac{g(x)}{\chi * \chi(x/\ell)}, \qquad w_1(x,y) := \chi_\Lambda(x) W_1(x-y)\chi_\Lambda(y), \tag{6.20}$$

$$W_2(x) := \frac{g(x)+g(x)\omega(x)}{\chi * \chi(x/\ell)}, \qquad w_2(x,y) := \chi_\Lambda(x) W_2(x-y)\chi_\Lambda(y), \tag{6.21}$$

where we observe that W, W_1, W_2 and w, w_1, w_2 are localized versions of v, g, $(1+\omega)g$, respectively, defined in (3.4).

Furthermore, we introduce the translated versions for $u \in \Lambda$

$$w_{1,u}(x,y) = w_1(x-\ell u, y-\ell u) \tag{6.22}$$

and similarly for $w_{2,u}$ and w_u. We are going to make use of the following approximation result. We recall the defintion of the lengthscale ℓ_δ from (3.30), which, with our choice $\delta = \delta_\mu$ from (6.1) becomes

$$\ell_\delta = \frac{e^\Gamma}{2}\rho_\mu^{-1/2} Y^{-1/2}, \tag{6.23}$$

and corresponds to the so-called healing length.

Lemma 6.4. *There exists a universal constant $C > 0$ such that, if $R\ell^{-1} < C^{-1}$, we have*

- *W_1 can be approximated by g up to the following error*

$$0 \le W_1(x) - g(x) \le C g(x) \frac{\min\{|x|^2, R^2\}}{\ell^2}, \tag{6.24}$$

and in particular $\|W_1\|_{L^1} \le 8\pi\delta(1 + CR^2\ell^{-2})$ due to (3.6).
- *For any $h \in L^1(\mathbb{R}^2)$ such that $h(x) = h(-x)$ and $\operatorname{supp} h \subseteq B(0,R)$,*

$$\left| h * \chi_\Lambda(x) - \chi_\Lambda(x)\int_{\mathbb{R}^2} \mathrm{d}x\ h(x)\right| \le C \max_{i,j} \|\partial_i\partial_j\chi\|_\infty \frac{R^2}{\ell^2}\|h\|_{L^1}. \tag{6.25}$$

• *It also holds*

$$\left|\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} dk\,\frac{\widehat{W}_1(k)^2-\widehat{W}_1^2(0)\mathbb{1}_{\{|k|\le \ell_\delta^{-1}\}}}{2k^2}-\widehat{g\omega}(0)\right|\le C\frac{R^2}{\ell^2}\delta. \tag{6.26}$$

• *It holds*

$$\left|\int_{\mathbb{R}^2}\frac{(\widehat{W}_1(k)-\widehat{g}(k))^2-(\widehat{W}_1(0)-\widehat{g}(0))^2\mathbb{1}_{\{|k|\le \ell_\delta^{-1}\}}}{2k^2}\mathrm{d}k\right|\le C\frac{R^4}{\ell^4}\widehat{g\omega}(0). \tag{6.27}$$

Proof. For (6.24) we use that the support of g is contained in the set $\{|x|<R\}$, therefore it is enough to give here our estimate. Using the symmetries of χ, the normalization $\|\chi\|_2=1$ (Appendix F) and a Taylor expansion we see that

$$\begin{aligned}\left|1-\frac{1}{\chi*\chi(x/\ell)}\right| &\le \frac{1}{|\chi*\chi(x/\ell)|}\left|\int_{\mathbb{R}^2}\chi(y)[\chi(y)-\chi(x/\ell-y)]\right|\mathrm{d}y\\ &\le C\frac{|x|^2}{\ell^2}\max_{i,j}\|\partial_i\partial_j\chi\|_\infty,\end{aligned}$$

which implies the first bound. (6.25) is proved similarly. For the bound (6.26), by the Lemma 3.8 we know that

$$(\widehat{g\omega})_0=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\frac{\widehat{g}_k^2-\widehat{g}_0^2\mathbb{1}_{\{|k|\le \ell_\delta^{-1}\}}}{2k^2}\,\mathrm{d}k, \tag{6.28}$$

and using (3.28) for both the expressions of W and g we get

$$\begin{aligned}\frac{1}{(2\pi)^2}&\left|\int_{\mathbb{R}^2}\frac{\widehat{g}_k^2-\widehat{g}_0^2\mathbb{1}_{\{|k|\le \ell_\delta^{-1}\}}-\widehat{W}_1^2(k)+\widehat{W}_1^2(0)\mathbb{1}_{\{|k|\le \ell_\delta^{-1}\}}}{2k^2}\mathrm{d}k\right|\\ &\le -C\iint |g(x)g(y)-W_1(x)W_1(y)|\log\Big(\frac{x-y}{\ell_\delta}\Big)\mathrm{d}x\mathrm{d}y\\ &\le -\frac{C}{\ell^2}\iint |x|^2 g(x)g(y)\log\Big(\frac{x-y}{\ell_\delta}\Big)\mathrm{d}x\mathrm{d}y\\ &=\frac{C}{\ell^2}\int |x|^2 g(x)\omega(x)\mathrm{d}x\\ &\le C\frac{R^2}{\ell^2}\delta,\end{aligned}$$

where we used first (6.24), then the fact that in 2 dimension the log term produces a convolution with the Green's function of the Laplacian and finally formulas (3.5) and (3.6) (together with the bounds $\omega\le 1$ in the support of g and $2R<\ell_\delta$). The last inequality has a similar proof and is omitted. □

We now give a result of localization to large boxes for the potential part in the Hamiltonian (6.2).

Lemma 6.5 (Localization of the potential). *The following identity holds*

$$-\rho_\mu \sum_{j=1}^{N} \int_{\mathbb{R}^2} g(x_j - y)\mathrm{d}y + \sum_{i<j} v(x_i - x_j)$$
$$= \int_{\mathbb{R}^2} \Big[-\rho_\mu \sum_{j=1}^{N} \int_{\mathbb{R}^2} w_{1,u}(x_j, y)\mathrm{d}y + \sum_{i<j} w_u(x_i, x_j) \Big] \mathrm{d}u. \tag{6.29}$$

Proof. It is proven by direct calculation following the same lines as [28, Proposition 3.1]. □

Therefore, joining the results from Lemmas 6.2, 6.5 and introducing the large box Hamiltonian acting on $\mathscr{F}_s(L^2(\Lambda_u))$ as

$$\mathcal{H}_{\Lambda_u}(\rho_\mu)_N := \sum_{j=1}^{N} \mathcal{T}_u^{(j)} - \rho_\mu \sum_{j=1}^{N} \int_{\mathbb{R}^2} w_{1,u}(x_j, y)\mathrm{d}y + \sum_{i<j} w_u(x_i, x_j), \tag{6.30}$$

where $\mathcal{T}_u^{(j)}$ is (6.14) for the x_j variable, and the ground state energy and its density

$$E_\Lambda(\rho_\mu) := \inf \mathrm{Spec}(\mathcal{H}_\Lambda(\rho_\mu)), \qquad e_\Lambda(\rho_\mu) := \frac{1}{\ell^2} E_\Lambda(\rho_\mu), \tag{6.31}$$

we are able to prove the following. Recall that e_0 is defined in (6.3).

Lemma 6.6. *Under the assumptions of Lemma 6.2,*

$$e_0(\rho_\mu) \geq e_\Lambda(\rho_\mu). \tag{6.32}$$

Proof. By direct application of Lemma 6.2 and Lemma 6.5 we have

$$\mathcal{H}_{\rho_\mu,N}(\rho_\mu) \geq \int_{\ell^{-1}(\Omega+B(0,\ell/2))} \mathcal{H}_{\Lambda_u}(\rho_\mu)_N \mathrm{d}u \geq \ell^{-2}|\Omega + B(0,\ell/2)| E_\Lambda(\rho_\mu), \tag{6.33}$$

where the last inequality is guaranteed by the unitary equivalence $\mathcal{H}_{\Lambda_u} \cong \mathcal{H}_{\Lambda_{u'}}$ via the relation

$$w_{u'}(x, y) = w_u(x - \ell(u' - u), y - \ell(u' - u)). \tag{6.34}$$

The proof is concluded taking the infimum of the spectrum of the left-hand side and dividing by $|\Omega|$ observing that $\frac{|\Omega+B(0,\ell/2)|}{|\Omega|} \xrightarrow[|\Omega|\to+\infty]{} 1$. □

Therefore, Lemma 6.6 shows that in order to prove our main result Theorem 6.1, it is enough to give an analogous estimate on the Hamiltonian on the large box, and it is the content of the next theorem.

Theorem 6.7. *There exist $C, \eta > 0$ such that the following holds. Let $\rho_\mu > 0$ and $v \in L^1(\Omega)$ be a positive, spherically symmetric potential with scattering length a and* $\mathrm{supp}(v) \subset B(0, R)$ *such that $\|v\|_1 \leq Y^{-1/8}$ and $R \leq \rho_\mu^{-1/2}$. Then, if $\rho_\mu a^2 \leq C^{-1}$, and the parameters are chosen as in "Appendix H", we have*

$$E_\Lambda(\rho_\mu) \geq -4\pi \ell^2 \rho_\mu^2 \delta \Big(1 - \Big(2\Gamma + \frac{1}{2} + \log \pi\Big)\delta\Big) - C\ell^2 \rho_\mu^2 \delta^{2+\eta}. \tag{6.35}$$

The proof of Theorem 6.7 is given in the remaining sections of the article.

7. Lower Bounds in Position Space

7.1. Splitting of the potential. By the definitions (6.11) of the projectors P and Q, we see that we can split the potential in a way presented in the lemma below.

Lemma 7.1. *We have, recalling the definitions in* (6.19)*, that*

$$-\rho_\mu \sum_{j=1}^{N} \int_{\mathbb{R}^2} w_1(x_j, y)\mathrm{d}y + \frac{1}{2}\sum_{i\neq j} w(x_i, x_j) = \sum_{j=0}^{4} \mathcal{Q}_j^{ren} \tag{7.1}$$

with

$$0 \leq \mathcal{Q}_4^{ren} := \frac{1}{2}\sum_{i\neq j}\Big[Q_iQ_j + \big(P_iP_j + P_iQ_j + Q_iP_j\big)\,\omega(x_i - x_j)\Big]w(x_i, x_j) \times\Big[Q_jQ_i + \omega(x_i - x_j)\,\big(P_jP_i + P_jQ_i + Q_jP_i\big)\Big], \tag{7.2}$$

$$\mathcal{Q}_3^{ren} := \sum_{i\neq j} P_iQ_jw_1(x_i, x_j)Q_iQ_j + h.c., \tag{7.3}$$

as well as

$$\mathcal{Q}_2^{ren} := \sum_{i\neq j} P_iQ_jw_2(x_i, x_j)Q_iP_j + \sum_{i\neq j} P_iQ_jw_2(x_i, x_j)P_iQ_j \tag{7.4}$$

$$+\frac{1}{2}\sum_{i\neq j} P_iP_jw_1(x_i, x_j)Q_iQ_j + h.c. - \rho_\mu \sum_{j=1}^{N} Q_i \int_{\mathbb{R}^2} w_1(x_i, y)\mathrm{d}y\; Q_i, \tag{7.5}$$

$$\mathcal{Q}_1^{ren} := \sum_{i,j} Q_iP_jw_2(x_i, x_j)P_iP_j - \rho_\mu \sum_{i=1}^{N} Q_i \int_{\mathbb{R}^2} w_1(x_i, y)\mathrm{d}y\; P_i + h.c., \tag{7.6}$$

and

$$\mathcal{Q}_0^{ren} := \frac{1}{2}\sum_{i\neq j} P_iP_jw_2(x_i, x_j)P_iP_j - \rho_\mu \sum_{j=1}^{N} P_j \int_{\mathbb{R}^2} w_1(x_j, y)\mathrm{d}y\; P_j. \tag{7.7}$$

Proof. It follows from an elementary calculation, using that $P + Q = \mathbb{1}$ on $L^2(\Lambda)$ and, where needed, the identity

$$w_1 = w_2 - w\omega + w\omega^2. \tag{7.8}$$

□

We rewrite now some of the previous Q terms in the lemma below.

Lemma 7.2. *With the notation* $\rho_0 = \frac{n_0}{\ell^2}$ *we have*

$$\mathcal{Q}_0^{ren} = \frac{\rho_0(n_0-1)}{2}(\widehat{g}(0) + \widehat{g\omega}(0)) - \rho_\mu n_0 \widehat{g}(0), \tag{7.9}$$

$$\mathcal{Q}_1^{ren} = (\rho_0 - \rho_\mu)\sum_{i=1}^N Q_i \chi_\Lambda(x_i) W_1 * \chi_\Lambda(x_i) P_i + h.c.$$
$$+ \rho_0 \sum_{i=1}^N Q_i \chi_\Lambda(x_i)((W_1\omega) * \chi_\Lambda)(x_i) P_i + h.c., \tag{7.10}$$

and

$$\mathcal{Q}_2^{ren} \geq \sum_{i\neq j} P_i Q_j w_2(x_i, x_j) Q_i P_j + \frac{1}{2}\sum_{i\neq j}(P_i P_j w_1(x_i, x_j) Q_i Q_j + h.c.)$$
$$+ ((\rho_0 - \rho_\mu)\widehat{W}_1(0) + \rho_0 \widehat{W_1\omega}(0)) \sum_{j=1}^N Q_j \chi_\Lambda(x_j)^2 Q_j - C(\rho_\mu + \rho_0)\delta\left(\frac{R}{\ell}\right)^2 n_+. \tag{7.11}$$

Proof. The first two identities are straightforward after having observed that

$$\sum_{j=1}^N P_j w_1(x_i, x_j) P_j = \frac{1}{\ell^2}\sum_{j=1}^N P_j \int_\Lambda w_1(x_i, y)\mathrm{d}y = \rho_0 \int_\Lambda w_1(x_i, y)\mathrm{d}y, \tag{7.12}$$

and

$$\int_\Lambda w_1(x_i, y) dy = \chi_\Lambda(x_i)(W_1 * \chi_\Lambda)(x_i). \tag{7.13}$$

For the $\mathcal{Q}_2^{ren}$ term, the only parts which require a different approach are

$$(\rho_0 - \rho_\mu)\sum_{j=1}^N Q_j \chi_\Lambda(x_j) W_1 * \chi_\Lambda(x_j) Q_j + \rho_0 \sum_{j=1}^N Q_j \chi_\Lambda(x_j)((W_1\omega) * \chi_\Lambda)(x_j) Q_j. \tag{7.14}$$

Using Eq. (6.25) of Lemma 6.4 we can bound

$$\sum_{j=1}^N Q_j \chi_\Lambda(x_i) W_1 * \chi_\Lambda(x_i) Q_j \geq \|W_1\|_{L^1} \sum_{j=1}^N Q_j \chi_\Lambda(x_j)^2 Q_j$$
$$- C \max_{i,j} \|\partial_i \partial_j \chi\|_\infty \frac{R^2}{\ell^2} \|W_1\|_{L^1} \|\chi\|_\infty n_+. \tag{7.15}$$

Recalling that $\|W_1\|_{L^1} \leq C\delta$ (Lemma 6.4) and acting similarly for the other term, this concludes the proof. □

As a direct consequence of the lemma above, we can derive the following first lower bound for the large box Hamiltonian.

Corollary 7.3. *The following bound holds for the Hamiltonian in the large box*

$$\mathcal{H}_\Lambda(\rho_\mu)\big|_N \geq \sum_{j=1}^N \mathcal{T}^{(j)} + \frac{\rho_0(n_0-1)}{2}(\widehat{g}(0)+\widehat{g\omega}(0)) - \rho_\mu n_0 \widehat{g}_0 \tag{7.16}$$

$$+ \Big(\rho_0 - \rho_\mu\Big)\sum_{i=1}^N Q_i \chi_\Lambda(x_i) W_1 * \chi_\Lambda(x_i) P_i + h.c. \tag{7.17}$$

$$+ \rho_0 \sum_{i=1}^N Q_i \chi_\Lambda(x_i)((W_1\omega)*\chi_\Lambda)(x_i) P_i + h.c. \tag{7.18}$$

$$+ \sum_{i\neq j} P_i Q_j w_2(x_i,x_j) Q_i P_j + \frac{1}{2}\sum_{i\neq j}(P_i P_j w_1(x_i,x_j) Q_i Q_j + h.c.) \tag{7.19}$$

$$+ ((\rho_0-\rho_\mu)\widehat{W}_1(0) + \rho_0 \widehat{W_1\omega}(0)) \sum_{j=1}^N Q_j \chi_\Lambda(x_j)^2 Q_j \tag{7.20}$$

$$- C(\rho_\mu + \rho_0)\delta\Big(\frac{R}{\ell}\Big)^2 n_+ + \mathcal{Q}_3^{ren} + \mathcal{Q}_4^{ren}. \tag{7.21}$$

In the lemma below we prove an estimate which is going to be useful in Sect. 7.2 to localize the $\mathcal{Q}_3^{\mathrm{ren}}$ term.

Lemma 7.4. *Let Q' be a possibly non self-adjoint operator on $L^2(\Lambda)$ such that $QQ' = Q'$ and $\|Q'\| \leq 1$. Then for all $c \in (0,1)$ there is a $C > 0$ such that, if $R \leq \ell$,*

$$\sum_{i\neq j}(P_i Q'_j w_1(x_i,x_j) Q_i Q_j + h.c.)$$

$$\geq -\frac{1}{4}\mathcal{Q}_4^{ren} - \sum_{i\neq j}(P_i Q'_j w_1 \omega P_i P_j + h.c.) - \delta n_0\Big(c K_\ell^{-2}\frac{n_+}{\ell^2} + C\frac{K_\ell^2}{\ell^2}\sum_{j=1}^N Q'_j(Q'_j)^\dagger\Big).$$

Proof. The idea is to reobtain the Q_4 term in the inequalities.

$$\sum_{i\neq j}(P_i Q'_j w_1 Q_i Q_j + h.c.) = \sum_{i\neq j} P_i Q'_j w_1 \left[Q_i Q_j + \omega(P_i P_j + P_i Q_j + Q_i P_j)\right] + h.c.$$
$$- \sum_{i\neq j} P_i Q'_j w_1 \omega (P_i P_j + P_i Q_j + Q_i P_j) + h.c. \tag{7.22}$$

We use Cauchy–Schwarz inequality on both the terms on the right-hand side. The first line of (7.22), using that $w_1 \leq w$, is controlled by

$$C\sum_{i\neq j} P_i Q'_j w_1 (P_i Q'_j)^\dagger + \frac{1}{4}\mathcal{Q}_4^{\mathrm{ren}} = C\frac{n_0}{\ell^2}\sum_{j=1}^N Q'_j \chi_\Lambda(x_j)(W_1 * \chi_\Lambda)(x_j)(Q'_j)^\dagger + \frac{1}{4}\mathcal{Q}_4^{\mathrm{ren}}$$
$$\leq C\frac{n_0}{\ell^2}\|\chi_\Lambda\|_\infty^2 \delta \sum_{j=1}^N Q'_j(Q'_j)^\dagger + \frac{1}{4}\mathcal{Q}_4^{\mathrm{ren}},$$

where we used (7.12), (7.13), the bound $\|W_1\|_{L^1} \leq C\delta(1+R^2\ell^{-2})$ and $R \leq \ell$. For the second line of (7.22) we keep the PP contribution and treat the other terms separately. They can be estimated as above. For instance,

$$\sum_{i\neq j}(P_iQ'_jw_1\omega P_iQ_j + h.c.) \leq \varepsilon^{-1}\sum_{i\neq j}P_iQ'_jw_1\omega(P_iQ'_j)^\dagger + \varepsilon\sum_{i\neq j}P_iQ_jw_1\omega P_iQ_j$$
$$\leq C\delta\frac{n_0}{\ell^2}\Big(\varepsilon^{-1}\sum_{j=1}^N Q'_j(Q'_j)^\dagger + \varepsilon n_+\Big), \tag{7.23}$$

where we used the Cauchy–Schwarz inequality with weight $\varepsilon > 0$. Choosing $\varepsilon = cC^{-1}K_\ell^{-2}$ with $c \in (0,1)$, we get

$$\sum_{i\neq j}P_iQ'_jw_1\omega P_iQ_j \leq c^{-1}C^2\delta\frac{n_0d^2K_\ell^2}{(d\ell)^2}\sum_{j=1}^N Q'_j(Q'_j)^\dagger + c\delta n_0K_\ell^{-2}\frac{n_+}{\ell^2}, \tag{7.24}$$

and the lemma follows. □

7.2. Localization of $3Q$ *term.* In this section we show how we can restrict the action of one of the Q projectors in the $\mathcal{Q}_3^{\text{ren}}$ term to low momenta. More precisely we define the following two sets of low and high momenta respectively,

$$\mathcal{P}_L := \{p \in \mathbb{R}^2 \mid |p| \leq d^{-2}\ell^{-1}\}, \qquad \mathcal{P}_H := \{p \in \mathbb{R}^2 \mid |p| \geq K_H\ell^{-1}\}. \tag{7.25}$$

We choose the parameters d and K_H satisfying (H6) so that the two sets are disjoint. We will localize the Q projector using the following cutoff function,

$$f_L(r) := f(d^2\ell r), \qquad f(r) := \begin{cases} 1, & \text{if } r \leq 1, \\ 0, & \text{if } r \geq 2, \end{cases} \tag{7.26}$$

where $f \in C^\infty(\mathbb{R})$ is a non-increasing function. The localized projectors are

$$Q_L := Qf_L(\sqrt{-\Delta}), \qquad \overline{Q}_L := Q - Q_L, \tag{7.27}$$

and the localized version of $\mathcal{Q}_3^{\text{ren}}$ is

$$\mathcal{Q}_3^{\text{low}} := \sum_{i\neq j}(P_iQ_{L,j}w_1(x_i,x_j)Q_iQ_j + h.c.). \tag{7.28}$$

The number of high excitations, namely the number of bosons outside from the condensate and with momenta not in $\mathcal{P}_L$ is

$$n_+^H := \sum_{j=1}^N Q_j\,\mathbb{1}_{(d^{-2}\ell^{-1},\infty)}(\sqrt{-\Delta_j})Q_j. \tag{7.29}$$

It is easy to see that

$$\sum_{j=1}^N \overline{Q}_{L,j}\overline{Q}_{L,j}^\dagger \leq n_+^H. \tag{7.30}$$

The next lemma shows how the $\mathcal{Q}_3^{\text{ren}}$ term added to a small contribution from $\mathcal{Q}_4^{\text{ren}}$ and to the spectral gap from the kinetic energy (see (6.14)), can be bounded above by $\mathcal{Q}_3^{\text{low}}$.

Lemma 7.5. *Assume $R \leq \ell$ and the relation* (H26) *between the parameters. Then there exists $C > 0$ such that, for any n-particles state $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ with $n \leq 2\rho_\mu \ell^2$,*

$$\langle \mathcal{Q}_3^{ren} \rangle_\Psi + \frac{1}{4}\langle \mathcal{Q}_4^{ren} \rangle_\Psi + \frac{b}{100}\Big(\frac{\langle n_+ \rangle_\Psi}{\ell^2} + \varepsilon_T \frac{\langle n_+^H \rangle_\Psi}{(d\ell)^2}\Big) \geq \langle \mathcal{Q}_3^{low} \rangle_\Psi - C\delta \frac{n^2}{\ell^2}(d^{2M-2} + R^2\ell^{-2})$$

where the fixed number b was introduced in Lemma 6.2.

Proof. By definition

$$\mathcal{Q}_3^{\text{ren}} - \mathcal{Q}_3^{\text{low}} = \sum_{i \neq j} (P_i \overline{Q}_{L,j} w_1(x_i, x_j) Q_i Q_j + h.c.). \tag{7.31}$$

We use now Lemma 7.4 with $Q' = \overline{Q}_L$ and the estimate (7.30) to get

$$\begin{aligned} \mathcal{Q}_3^{\text{ren}} - \mathcal{Q}_3^{\text{low}} \geq & -\frac{1}{4}\mathcal{Q}_4^{\text{ren}} - \sum_{i \neq j}(P_i \overline{Q}_{L,j} w_1 \omega P_i P_j + h.c.) \\ & - \delta n_0 \Big(c K_\ell^{-2} \frac{n_+}{\ell^2} + C \frac{d^2 K_\ell^2}{(d\ell)^2} n_+^H \Big). \end{aligned} \tag{7.32}$$

By (7.12) we have

$$\begin{aligned} & \sum_{i \neq j}(P_i \overline{Q}_{L,j} w_1 \omega P_i P_j + h.c.) \\ & = \frac{n_0}{\ell^2}\Big(\sum_{j=1}^N \overline{Q}_{L,j} \chi_\Lambda(x_j) \big(\|W_1\omega\|_{L^1} \chi_\Lambda(x_j) + \varepsilon(x_j) \big) P_j + h.c. \Big), \end{aligned} \tag{7.33}$$

with $\varepsilon(x_j) = W_1\omega * \chi_\Lambda(x_j) - \|W_1\omega\|_{L^1}\chi_\Lambda(x_j)$. The $\varepsilon(x_j)$-term can be bounded using a Cauchy–Schwarz inequality and (6.25),

$$\begin{aligned} \frac{n_0}{\ell^2}\Big(\sum_{j=1}^N \overline{Q}_{L,j} \chi_\Lambda(x_j) \varepsilon(x_j) P_j + h.c. \Big) & \leq C \frac{n_0}{\ell^2} \sum_{j=1}^N (\overline{Q}_{L,j} \chi_\Lambda \varepsilon \overline{Q}_{L,j}^\dagger + P_j \chi_\Lambda \varepsilon P_j) \\ & \leq C \frac{n_0 R^2}{\ell^4} \delta (n_+^H + n_0). \end{aligned} \tag{7.34}$$

For the other term we take $M - 1 \leq 2\widetilde{M} \leq M$ and using the notation $D_M := (\ell^{-2} - \Delta_j)^{\widetilde{M}}$, we write

$$\overline{Q}_{L,j} \chi_\Lambda(x_j)^2 P_j + h.c. = \overline{Q}_{L,j} D_M^{-1} [D_M \chi_\Lambda(x_j)^2] P_j + h.c. \tag{7.35}$$

Therefore, by Cauchy–Schwarz inequality with weight $\varepsilon_0 > 0$,

$$\overline{Q}_{L,j} \chi_\Lambda(x_j)^2 P_j + h.c. \leq \varepsilon_0 P_j + \varepsilon_0^{-1} \|D_M \chi_\Lambda^2\|_\infty^2 \overline{Q}_{L,j} D_M^{-2} (\overline{Q}_{L,j})^\dagger.$$

Now using that $\|D_M \chi_\Lambda^2\| \leq C\ell^{-2\widetilde{M}}$ and that $\overline{Q}_L$ cut momenta lower than $d^{-2}\ell^{-1}$ we obtain

$$\begin{aligned} \overline{Q}_{L,j} \chi_\Lambda(x_j)^2 P_j + h.c. & \leq \varepsilon_0 P_j + \varepsilon_0^{-1} C \ell^{-4\widetilde{M}} \overline{Q}_{L,j} (\ell^{-2} - \Delta_j)^{-2\widetilde{M}} (\overline{Q}_{L,j})^\dagger \\ & \leq \varepsilon_0 P_j + \varepsilon_0^{-1} C d^{8\widetilde{M}} \overline{Q}_{L,j} (\overline{Q}_{L,j})^\dagger. \end{aligned} \tag{7.36}$$

Therefore choosing $\varepsilon_0 = d^{4\widetilde{M}}$, we have

$$\frac{n_0}{\ell^2}\left(\sum_{j=1}^{N}\overline{Q}_{L,j}\chi_\Lambda(x_j)^2\|W_1\omega\|_{L^1}P_j + h.c.\right) \leq C\delta d^{2M-2}\frac{n_0}{\ell^2}(n_+^H + n_0). \tag{7.37}$$

Inserting (7.34) and (7.37) into (7.33) we find

$$\sum_{i\neq j}(P_i\overline{Q}_{L,j}w_1\omega P_iP_j + h.c.) \leq C\delta\frac{n_0}{\ell^2}(n_+^H + n_0)(d^{2M-2} + R^2\ell^{-2}). \tag{7.38}$$

We use this last bound in (7.32) and apply it to the state Ψ,

$$\begin{aligned}
&\langle \mathcal{Q}_3^{\text{ren}}\rangle_\Psi - \langle \mathcal{Q}_3^{\text{low}}\rangle_\Psi \\
&\geq -\frac{1}{4}\langle \mathcal{Q}_4^{\text{ren}}\rangle_\Psi - C\delta\frac{n}{\ell^2}(\langle n_+^H\rangle_\Psi + n)(d^{2M-2} + R^2\ell^{-2}) \\
&\quad - c\delta n K_\ell^{-2}\frac{\langle n_+\rangle_\Psi}{\ell^2} - C\delta n d^2 K_\ell^2\frac{\langle n_+^H\rangle_\Psi}{(d\ell)^2} \\
&\geq -\frac{1}{4}\langle \mathcal{Q}_4^{\text{ren}}\rangle_\Psi - C\delta\frac{n^2}{\ell^2}(d^{2M-2} + R^2\ell^{-2}) - c\frac{\langle n_+\rangle_\Psi}{\ell^2} - Cd^2K_\ell^4\frac{\langle n_+^H\rangle_\Psi}{(d\ell)^2},
\end{aligned} \tag{7.39}$$

where we used $n \leq 2\rho_\mu\ell^2$ and $\ell^2 = K_\ell^2\rho_\mu^{-1}Y^{-1}$. We conclude by choosing $c = \frac{b}{100}$ and using the relation (H26) between the parameters. □

7.3. A priori bounds and localization of the number of excitations. The purpose of this section is to get bounds on the number of excitations of the system. First of all, Theorem 7.6 gives a priori bounds on n_+.

Theorem 7.6. *There exists a universal constant $C > 0$ such that, if $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ is a normalized n-bosons vector which satisfies*

$$\langle \mathcal{H}_\Lambda(\rho_\mu)\rangle_\Psi \leq -4\pi\rho_\mu^2\ell^2Y\left(1 - CK_B^2Y|\log Y|\right), \tag{7.40}$$

with K_B fixed in "Appendix H", then

$$\langle n_+\rangle_\Psi \leq CK_B^2K_\ell^2\rho_\mu\ell^2Y|\log Y|, \tag{7.41}$$

$$\langle \mathcal{Q}_4^{ren}\rangle_\Psi \leq CK_B^2K_\ell^2\rho_\mu^2\ell^2Y^2|\log Y|, \tag{7.42}$$

$$\left|\rho_\mu - \frac{n}{\ell^2}\right| \leq CK_BK_\ell\rho_\mu Y^{1/2}|\log Y|^{1/2}. \tag{7.43}$$

Proof. It is proved in "Appendix D", using a second localization to "small boxes" of size $\ll \ell_\delta$. □

We also need to bound the number of low excitations, defined in terms of our modified kinetic energy $\mathcal{T}$. More precisely we define, for a certain $\widetilde{K}_H \gg 1$ fixed in "Appendix H", the projectors

$$\overline{Q}_H = \mathbb{1}_{(0,\widetilde{K}_H^2\ell^{-2})}(\mathcal{T}), \qquad Q_H = Q - \overline{Q}_H, \tag{7.44}$$

which satisfy

$$P + \overline{Q}_H + Q_H = 1_\Lambda. \tag{7.45}$$

We will consider the operators

$$n_+^L := \sum_j \overline{Q}_{H,j}, \qquad \widetilde{n}_+^H := \sum_j Q_{H,j}, \tag{7.46}$$

for which we prove the following result.

Theorem 7.7 (Restriction on n_+^L). *There exist $C, \eta > 0$ such that the following holds. Let $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ be a normalized n-particle vector which satisfies (7.40). Assume that the potential v is such that $\|v\|_1 \leq Y^{-1/8}$. Then, for $\mathcal{M} \gg 1$ satisfying condition (H24) there exists a sequence $\{\Psi^m\}_{m\in\mathbb{Z}} \subseteq \mathscr{F}_s(L^2(\Lambda))$ such that $\sum_m \|\Psi^m\|^2 = 1$ and*

$$\Psi^m = \mathbb{1}_{[0,\frac{\mathcal{M}}{2}+m]}(n_+^L)\Psi^m, \tag{7.47}$$

and such that the following lower bound holds true

$$\begin{aligned}\langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi\rangle \geq &\sum_{2|m|\leq \mathcal{M}} \langle \Psi^m, \mathcal{H}_\Lambda(\rho_\mu)\Psi^m\rangle - C\rho_\mu^2\ell^2 Y^{2+\eta}\\ &-4\pi\rho_\mu^2\ell^2 Y\Big(1 - CK_B^2 Y|\log Y|\Big)\sum_{2|m|>\mathcal{M}} \|\Psi^m\|^2.\end{aligned}$$

Notice that, if such a state Ψ does not exist, then our lower bound is already proven (see when we apply Theorem 7.7 in (9.80)). From this result we see that, in order to prove Theorem 6.7, we only need to bound the energy of states satisfying the bound $n_+^L \leq \mathcal{M}$. In the remainder of this section, we prove Theorem 7.7. The following lemma states that for a lower bound we can restrict to states with finitely many excitations n_+, up to small enough errors. The proof of this lemma is inspired by the localization of large matrices result in [29]. It is also really similar to the bounds in [30, Proposition 21]. It could be interpreted as an analogue of the standard IMS localization formula. The error produced is written in terms of the following quantities d_1^L and d_2^L:

$$\begin{aligned}d_1^L := &-\rho_\mu \sum_i (P_i + Q_{H,i})\int w_1(x_i, y)\mathrm{d}y\, \overline{Q}_{H,i} + h.c.\\ &+\sum_{i\neq j}(P_i + Q_{H,i})\overline{Q}_{H,j} w(x_i, x_j)\overline{Q}_{H,i}\overline{Q}_{H,j} + h.c.\\ &+\sum_{i\neq j}\overline{Q}_{H,i}(P_j + Q_{H,j})w(x_i, x_j)(P_i + Q_{H,i})(P_j + Q_{H,j}) + h.c.\end{aligned} \tag{7.48}$$

and

$$d_2^L := \sum_{i\neq j}(P_i + Q_{H,i})(P_j + Q_{H,j})w(x_i, x_j)\overline{Q}_{H,j}\overline{Q}_{H,i} + h.c. \tag{7.49}$$

Lemma 7.8. *Let $\theta : \mathbb{R} \to [0,1]$ be any compactly supported Lipschitz function such that $\theta(s) = 1$ for $|s| < \frac{1}{8}$ and $\theta(s) = 0$ for $|s| > \frac{1}{4}$. For any $\mathcal{M} > 0$, define $c_{\mathcal{M}} > 0$ and $\theta_{\mathcal{M}}$ such that*

$$\theta_{\mathcal{M}}(s) = c_{\mathcal{M}}\theta\Big(\frac{s}{\mathcal{M}}\Big), \qquad \sum_{s\in\mathbb{Z}} \theta_{\mathcal{M}}(s)^2 = 1.$$

Then there exists a $C > 0$ depending only on θ such that, for any normalized state $\Psi \in \mathscr{F}_s(L^2(\Lambda))$,

$$\langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi\rangle \geq \sum_{m\in\mathbb{Z}} \langle \Psi^m, \mathcal{H}_\Lambda(\rho_\mu)\Psi^m\rangle - \frac{C}{\mathcal{M}^2}\Big(|\langle d_1^L\rangle_\Psi| + |\langle d_2^L\rangle_\Psi|\Big), \quad (7.50)$$

where $\Psi^m = \theta_{\mathcal{M}}(n_+^L - m)\Psi$.

Proof. Notice that $\mathcal{H}_\Lambda$ only contains terms that change n_+^L by $0, \pm 1$ or ± 2. Therefore, we write our operator as $\mathcal{H}_\Lambda(\rho_\mu) = \sum_{|k|\leq 2}\mathcal{H}_k$, with $\mathcal{H}_k n_+^L = (n_+^L + k)\mathcal{H}_k$. Moreover, $\mathcal{H}_k + \mathcal{H}_{-k} = d_k^L$ for $k = 1, 2$. We use this decomposition to estimate the localized energy,

$$\begin{aligned}\sum_{m\in\mathbb{Z}}\langle \Psi^m, \mathcal{H}_\Lambda \Psi^m\rangle &= \sum_{m\in\mathbb{Z}}\sum_{|k|\leq 2}\langle \theta_{\mathcal{M}}(n_+^L - m)\theta_{\mathcal{M}}(n_+^L - m + k)\Psi, \mathcal{H}_k\Psi\rangle \\ &= \sum_{m,s\in\mathbb{Z}}\sum_{|k|\leq 2}\langle \theta_{\mathcal{M}}(s - m)\theta_{\mathcal{M}}(s - m + k)\mathbb{1}_{\{n_+^L = s\}}\Psi, \mathcal{H}_k\Psi\rangle \\ &= \sum_{m,s\in\mathbb{Z}}\sum_{|k|\leq 2}\theta_{\mathcal{M}}(m)\theta_{\mathcal{M}}(m+k)\langle \mathbb{1}_{\{n_+^L = s\}}\Psi, \mathcal{H}_k\Psi\rangle,\end{aligned}$$

where in the last line we changed the index m into $s - m$. We can sum on s to recognize

$$\sum_{m\in\mathbb{Z}}\langle \Psi^m, \mathcal{H}_\Lambda \Psi^m\rangle = \sum_{m\in\mathbb{Z}}\sum_{|k|\leq 2}\theta_{\mathcal{M}}(m)\theta_{\mathcal{M}}(m+k)\langle \Psi, \mathcal{H}_k\Psi\rangle. \quad (7.51)$$

Furthermore the energy of Ψ can be rewritten as

$$\langle \Psi, \mathcal{H}_\Lambda\Psi\rangle = \sum_{|k|\leq 2}\langle \Psi, \mathcal{H}_k\Psi\rangle = \sum_{m\in\mathbb{Z}}\sum_{|k|\leq 2}\theta_{\mathcal{M}}(m)^2\langle \Psi, \mathcal{H}_k\Psi\rangle, \quad (7.52)$$

by definition of $\theta_{\mathcal{M}}$. Thus, the localization error is

$$\sum_{m\in\mathbb{Z}}\langle \Psi^m, \mathcal{H}_\Lambda \Psi^m\rangle - \langle \Psi, \mathcal{H}_\Lambda\Psi\rangle = \sum_{|k|\leq 2}\delta_k\langle \Psi, \mathcal{H}_k\Psi\rangle, \quad (7.53)$$

with

$$\delta_k = \sum_{m\in\mathbb{Z}}\big(\theta_{\mathcal{M}}(m)\theta_{\mathcal{M}}(m+k) - \theta_{\mathcal{M}}(m)^2\big) = -\frac{1}{2}\sum_m\big(\theta_{\mathcal{M}}(m) - \theta_{\mathcal{M}}(m+k)\big)^2. \quad (7.54)$$

Since $\delta_0 = 0$, $\delta_k = \delta_{-k}$ and $d_k^L = \mathcal{H}_k + \mathcal{H}_{-k}$ we find

$$\sum_{m\in\mathbb{Z}}\langle \Psi^m, \mathcal{H}_\Lambda \Psi^m\rangle - \langle \Psi, \mathcal{H}_\Lambda\Psi\rangle = \delta_1\langle d_1^L\rangle_\Psi + \delta_2\langle d_2^L\rangle_\Psi, \quad (7.55)$$

and only remains to prove that $|\delta_k| \leq C\mathcal{M}^{-2}$. Using (7.54) and the definition of $\theta_{\mathcal{M}}$,

$$|\delta_k| = \frac{c_{\mathcal{M}}^2}{2} \sum_{m\in\mathbb{Z}} \Big[\theta\Big(\frac{m}{\mathcal{M}}\Big) - \theta\Big(\frac{m+k}{\mathcal{M}}\Big)\Big]^2. \tag{7.56}$$

We can restrict the sum to $m \in \big[-\frac{\mathcal{M}}{2}, \frac{\mathcal{M}}{2}\big]$, since the other terms vanish due to θ being a cutoff function. This sum contains $\mathcal{M}+1$ terms which we can bound using the Lipschitz constant L of θ,

$$|\delta_k| \leq c_{\mathcal{M}}^2 \frac{\mathcal{M}+1}{2} \frac{L^2k^2}{\mathcal{M}^2} \leq \frac{2L^2k^2}{\mathcal{M}^2}, \tag{7.57}$$

where in the last inequality we used

$$c_{\mathcal{M}}^2 = \Big(\sum_{s\in\mathbb{Z}} \theta\Big(\frac{s}{\mathcal{M}}\Big)^2\Big)^{-1} \leq \frac{1}{\mathcal{M}/4+1}. \tag{7.58}$$

□

To estimate the error in (7.50), we need the following bounds on d_1^L and d_2^L.

Lemma 7.9. *Let* $\widetilde{\mathcal{M}} > 0$ *and* $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ *be a normalized* n*-bosons vector satisfying*

$$\Psi = \mathbb{1}_{[0,\widetilde{\mathcal{M}}]}(n_+^L)\Psi.$$

Then, assuming the choices of parameters in "Appendix H" we have

$$\begin{aligned} &|\langle d_1^L\rangle_\Psi| + |\langle d_2^L\rangle_\Psi| \\ &\leq \rho_\mu^2\ell^2\|v\|_1\Big(\frac{\langle n_+\rangle_\Psi^{1/2}}{n^{1/2}} + \frac{\widetilde{\mathcal{M}}^{1/2}\langle n_+\rangle_\Psi^{1/2}}{n}\varepsilon_N^{-1/4}\widetilde{K}_H + \frac{\widetilde{\mathcal{M}}\langle n_+\rangle_\Psi}{n^2}\varepsilon_N^{-1/2}\widetilde{K}_H^2\Big) + C\langle \mathcal{Q}_4^{ren}\rangle_\Psi. \end{aligned} \tag{7.59}$$

Proof. We give the proof in "Appendix E". □

Now we can combine Lemmas 7.8, 7.9 and Theorem 7.6 to prove Theorem 7.7.

Proof of Theorem 7.7. Given a n-sector state $\Psi \in L^2(\Lambda^n)$ satisfying (7.40), we can apply Lemma 7.8 and write $\Psi^m = \theta_{\mathcal{M}}(n_+^L - m)\Psi$. In (7.50) we split the sum into two. The first part, for $|m| < \frac{1}{2}\mathcal{M}$, we keep. For $|m| > \frac{1}{2}\mathcal{M}$, Ψ_m satisfies

$$\langle n_+\rangle_{\Psi^m} \geq \langle n_+^L\rangle_{\Psi^m} \geq \frac{\mathcal{M}}{4}\|\Psi^m\|^2, \tag{7.60}$$

due to the cutoff $\theta_{\mathcal{M}}(n_+^L - m)$. Thanks to condition (H24) on $\mathcal{M}$, this is a larger bound than (7.41), and thus the assumption of Theorem 7.6 cannot be satisfied for Ψ^m and we must have the lower bound

$$\langle \Psi^m, \mathcal{H}_\Lambda(\rho_\mu)\Psi^m\rangle \geq -4\pi\rho_\mu^2\ell^2 Y\Big(1 - CK_B^2Y|\log Y|\Big)\|\Psi^m\|^2. \tag{7.61}$$

We finally bound the last term in (7.50), using Lemma 7.9 with $\widetilde{\mathcal{M}} = n$,

$$|\langle d_1^L\rangle_\Psi| + |\langle d_2^L\rangle_\Psi| \leq \rho_\mu^2\ell^2\|v\|_1^2\Big(\frac{1+\varepsilon_N^{-1/4}\widetilde{K}_H}{n^{1/2}}\langle n_+\rangle_\Psi + \frac{\varepsilon_N^{-1/2}\widetilde{K}_H^2}{n}\langle n_+\rangle_\Psi\Big) + C\langle \mathcal{Q}_4^{\mathrm{ren}}\rangle_\Psi.$$

Now we use the condensation estimate (7.41) and the bound (7.42) on Q_4^{ren} to obtain

$$|\langle d_1^L\rangle_\Psi| + |\langle d_2^L\rangle_\Psi| \le \rho_\mu^2 \ell^2 \|v\|_1 \Big(Y^{1/2}|\log Y|^{1/2} K_\ell K_B \widetilde{K}_H \varepsilon_N^{-1/4} + Y|\log Y| K_\ell^2 K_B^2 \widetilde{K}_H^2 \varepsilon_N^{-1/2} \Big). \tag{7.62}$$

The relation (H13) between the parameters implies that the largest term in (7.62) is the first one. Using the conditions (H11) and (H24) on ε_N and $\mathcal{M}$ respectively, and the assumptions on $\|v\|_1$ we find

$$\frac{|\langle d_1^L\rangle_\Psi| + |\langle d_2^L\rangle_\Psi|}{\mathcal{M}^2} \le \rho_\mu^2 \ell^2 Y^{2+\eta}. \tag{7.63}$$

Using the estimates (7.61) for $m > \frac{1}{2}\mathcal{M}$ and (7.63) in formula (7.50) we conclude the proof. □

8. Lower Bounds in Second Quantization

8.1. Second quantization formalism. We rewrite the Hamiltonian in the second quantization formalism. Let us introduce the operators, where # can be nothing or † for the annihilation or creation operators on the space $\mathscr{F}_s(L^2(\Lambda))$, respectively,

$$a_0^\# := \frac{1}{\ell} a^\#(\theta), \qquad \text{and} \qquad [a_0, a_0^\dagger] = 1, \tag{8.1}$$

being the creation and annihilation operators for bosons with zero momentum, where θ is the sharp localization function on Λ (see (6.9)). For $k \in \mathbb{R}^2 \setminus \{0\}$ we also define

$$\widetilde{a}_k^\# := \frac{1}{\ell} a^\#(Q e^{ikx}\theta), \tag{8.2}$$

the creation and annihilation operators for bosons with non-zero momentum with Q defined in (6.11), and their regular analogous

$$a_k^\# := \frac{1}{\ell} a^\#(Q e^{ikx}\chi_\Lambda), \tag{8.3}$$

where χ_Λ is the regular localization function defined in “Appendix F”. We have the usual commutation relations, for $k, h \in \mathbb{R}^2 \setminus \{0\}$

$$[\widetilde{a}_k, \widetilde{a}_h] = [a_k, a_h] = 0, \qquad \text{and} \qquad [\widetilde{a}_k, \widetilde{a}_h^\dagger] = \frac{1}{\ell^2}\langle Q e^{ikx}, Q e^{ihx}\rangle. \tag{8.4}$$

Using that $P = \mathbb{1} - Q$ and $\widehat{\chi}_\Lambda(k) = \ell^2 \widehat{\chi}(k\ell)$,

$$[a_k, a_h^\dagger] = \frac{1}{\ell^2}\langle Q e^{ikx}\chi_\Lambda,\ Q e^{ihx}\chi_\Lambda\rangle = \widehat{\chi^2}((k-h)\ell) - \widehat{\chi}(k\ell)\overline{\widehat{\chi}}(h\ell), \tag{8.5}$$

and

$$[a_k, a_h^\dagger] \le 1. \tag{8.6}$$

Let us observe, first of all, that

$$n_0 = a_0^\dagger a_0, \qquad n_+ = \frac{\ell^2}{(2\pi)^2} \int \widetilde{a}_k^\dagger \widetilde{a}_k \mathrm{d}k. \tag{8.7}$$

Let us introduce, for $k \in \mathbb{R}^2$, the kinetic Fourier multiplier

$$\tau(k) := (1-\varepsilon_T)\Big[|k| - \frac{1}{2s\ell}\Big]_+^2 + \varepsilon_T\Big[|k| - \frac{1}{2ds\ell}\Big]_+^2. \tag{8.8}$$

We will need the following technical lemma to control the number operators.

Lemma 8.1. *Assume the relation* (H27) *between the parameters. Let* $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ *be a normalized state satisfying*

$$\mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Psi = \Psi, \qquad \mathbb{1}_{[0,2\rho_\mu \ell^2]}(n_+)\Psi = \Psi, \tag{8.9}$$

then the following bounds hold

$$\Big\langle \ell^2 \int_{\{|k|\le 2K_H\ell^{-1}\}} (a_k^\dagger a_k + \widetilde{a}_k^\dagger \widetilde{a}_k)\mathrm{d}k \Big\rangle_\Psi \le C\mathcal{M}, \tag{8.10}$$

$$\Big\langle \ell^2 \int_{\mathbb{R}^2} (a_k^\dagger a_k + \widetilde{a}_k^\dagger \widetilde{a}_k)\mathrm{d}k \Big\rangle_\Psi \le C\mathcal{M} + C\langle n_+^H\rangle_\Psi. \tag{8.11}$$

Proof. The proof is analogous for both the addends, therefore we give the proof only for the $a_k^\#$. We want to compare localization in terms of kinetic energy with localization in momenta. We use [19, Lemma 5.2] adapted to dimension 2:

$$Q\chi_\Lambda \mathbb{1}_{\{|p|\le K_H\ell^{-1}\}}\chi_\Lambda Q \le C\overline{Q}_H + C\Big(\Big(\frac{K_H}{\widetilde{K}_H}\Big)^M + \varepsilon_N^{3/2}\Big), \tag{8.12}$$

$$Q\mathbb{1}_{\{|p|\le K_H\ell^{-1}\}}Q \le C\overline{Q}_H + C\Big(\Big(\frac{K_H}{\widetilde{K}_H}\Big)^M + \varepsilon_N^{3/2}\Big), \tag{8.13}$$

where we recall the definition (7.44) of $\overline{Q}_H$. Using (8.12) we have the following inequality in the N-th Fock sector

$$\begin{aligned}\frac{\ell^2}{(2\pi)^2}\int_{\{|k|\le 2K_H\ell^{-1}\}} a_k^\dagger a_k \mathrm{d}k\Big|_N &= \sum_{j=1}^N Q_j\chi_\Lambda(x_j)\mathbb{1}_{(0,2K_H\ell^{-1}]}(\sqrt{-\Delta_j})\chi_\Lambda(x_j)Q_j \\ &\le Cn_+^L + \Big(\Big(\frac{K_H}{\widetilde{K}_H}\Big)^M + \varepsilon_N^{3/2}\Big)n_+.\end{aligned}$$

Using the bounds from (8.9) and the relation (H27) we deduce

$$\Big\langle \ell^2\int_{\{|k|\le 2K_H\ell^{-1}\}} a_k^\dagger a_k \mathrm{d}k\Big\rangle_\Psi \le C\mathcal{M} + C\Big(\Big(\frac{K_H}{\widetilde{K}_H}\Big)^M + \varepsilon_N^{3/2}\Big)\rho_\mu\ell^2 \le C\mathcal{M}, \tag{8.14}$$

thus proving (8.10). In order to obtain (8.11) it is enough to estimate the integral on the complementary subset. We have, again on the N-th sector,

$$\ell^2\int_{\{|k|\ge 2K_H\ell^{-1}\}} a_k^\dagger a_k \mathrm{d}k\Big|_N = \sum_{j=1}^N Q_j\chi_\Lambda(x_j)\mathbb{1}_{\{|k|\ge 2K_H\ell^{-1}\}}(\sqrt{-\Delta_j})\chi_\Lambda(x_j)Q_j. \tag{8.15}$$

We insert $1 = \mathbb{1}_{\mathcal{P}_L} + \mathbb{1}_{\mathcal{P}_L^c}$ and use the Cauchy–Schwarz inequality to estimate the right-hand side,

$$
\begin{aligned}
&Q\chi_\Lambda \mathbb{1}_{[2K_H\ell^{-1},+\infty)}(\sqrt{-\Delta})\chi_\Lambda Q \\
&\quad \leq 2Q\mathbb{1}_{\mathcal{P}_L^c}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{[2K_H\ell^{-1},+\infty)}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{\mathcal{P}_L^c}(\sqrt{-\Delta})Q. \\
&\qquad + 2Q\mathbb{1}_{\mathcal{P}_L}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{[2K_H\ell^{-1},+\infty)}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{\mathcal{P}_L}(\sqrt{-\Delta})Q.
\end{aligned}
$$

On $\mathcal{P}_L^c$ we can use the bound

$$
Q\mathbb{1}_{\mathcal{P}_L^c}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{[2K_H\ell^{-1},+\infty)}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{\mathcal{P}_L^c}(\sqrt{-\Delta})Q \leq \|\chi_\Lambda\|_\infty^2 Q\mathbb{1}_{\mathcal{P}_L^c}(\sqrt{-\Delta})Q.
$$

On $\mathcal{P}_L$ we bound the operator norm, multiplying and dividing by an M power of the Laplacian and using that χ has M bounded derivatives,

$$
\begin{aligned}
&\|\mathbb{1}_{\mathcal{P}_L}(\sqrt{-\Delta})\chi_\Lambda \mathbb{1}_{[2K_H\ell^{-1},+\infty)}\| \\
&\quad \leq \|\mathbb{1}_{\mathcal{P}_L}(\sqrt{-\Delta})\chi_\Lambda(-\Delta)^{M/2}\|\|(-\Delta)^{-M/2}\mathbb{1}_{[2K_H\ell^{-1},+\infty)}\| \leq C(d^2K_H)^{-M}.
\end{aligned}
$$

We deduce

$$
\ell^2 \int_{\{|k|\geq 2K_H\ell^{-1}\}} a_k^\dagger a_k \mathrm{d}k \leq Cn_+^H + C(d^2K_H)^{-2M}n_+, \tag{8.16}
$$

and we conclude using (H27) and the assumptions on Ψ. □

8.2. Second quantized Hamiltonian. We can rewrite the $\mathcal{Q}_3^{\text{low}}$ term (7.28) in second quantized formalism

$$
\mathcal{Q}_3^{\text{low}} = \frac{\ell^2}{(2\pi)^4}\int_{\mathbb{R}^2\times\mathbb{R}^2} f_L(p)\widehat{W}_1(k)a_0^\dagger \widetilde{a}_p^\dagger a_{p-k}a_k \mathrm{d}k\mathrm{d}p + h.c. \tag{8.17}
$$

An important consideration is that we can restrict the contributions in $\mathcal{Q}_3^{\text{low}}$ to the high momenta. This is the content of the next lemma.

Lemma 8.2 (Localization of $\mathcal{Q}_3^{low}$ to high momenta). *Assume $R \leq \ell$ and the relations (H16), (H17), (H27) between the parameters. If $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ is a n-particle state satisfying* (7.40) *and $\mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Psi = \Psi$ then we have*

$$
\langle \Psi \mid \mathcal{Q}_3^{low}\Psi\rangle \geq \langle \Psi \mid \mathcal{Q}_3^{high}\Psi\rangle - \frac{b}{100\ell^2}\langle n_+\rangle_\Psi, \tag{8.18}
$$

where

$$
\mathcal{Q}_3^{high} = \frac{\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2} f_L(p)\widehat{W}_1(k)a_0^\dagger \widetilde{a}_p^\dagger a_{p-k}a_k \mathrm{d}k\mathrm{d}p + h.c., \tag{8.19}
$$

with $\mathcal{P}_H$ defined in (7.25).

Proof. First note that

$$\langle \Psi \mid (\mathcal{Q}_3^{\mathrm{low}} - \mathcal{Q}_3^{\mathrm{high}})\Psi\rangle = \frac{\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H^c\times\mathbb{R}^2} f_L(p)\widehat{W}_1(k)\langle \Psi \mid a_0^\dagger \widetilde{a}_p^\dagger a_{p-k}a_k\Psi\rangle \mathrm{d}k\mathrm{d}p + h.c. \tag{8.20}$$

For any $\varepsilon > 0$, using Cauchy–Schwarz on the creation and annihilation operators,

$$\begin{aligned}&\langle \Psi \mid (\mathcal{Q}_3^{\mathrm{low}} - \mathcal{Q}_3^{\mathrm{high}})\Psi\rangle \\ &\geq -C\delta\ell^2\int_{\mathcal{P}_H^c\times\mathbb{R}^2} f_L(p)\Big(\varepsilon\langle\Psi \mid \widetilde{a}_p^\dagger a_0^\dagger a_0\widetilde{a}_p\Psi\rangle + \varepsilon^{-1}\langle \Psi \mid a_k^\dagger a_{p-k}^\dagger a_{p-k}a_k\Psi\rangle\Big)\mathrm{d}k\mathrm{d}p,\end{aligned} \tag{8.21}$$

where we used the fact that $\|\widehat{W}_1\|_\infty \leq \|W_1\|_1 \leq C\delta$ (from Lemma 6.4). We now use the following inequalities, obtained by Lemma 8.1 and bounding f_L by 1,

$$\ell^2\int_{\mathcal{P}_H^c\times\mathbb{R}^2} f_L(p)\langle\Psi \mid \widetilde{a}_p^\dagger a_0^\dagger a_0\widetilde{a}_p\Psi\rangle\mathrm{d}k\mathrm{d}p \leq n\langle n_+\rangle_\Psi\int_{\mathcal{P}_H^c}\mathrm{d}k = \frac{n\langle n_+\rangle_\Psi}{\ell^2}K_H^2, \tag{8.22}$$

$$\ell^4\int_{\mathcal{P}_H^c\times\mathbb{R}^2} f_L(p)\langle\Psi \mid a_k^\dagger a_{p-k}^\dagger a_{p-k}a_k\Psi\rangle\mathrm{d}k\mathrm{d}p \leq C\mathcal{M}\langle n_+\rangle_\Psi. \tag{8.23}$$

Therefore, applying to (8.21) we obtain

$$\langle \Psi \mid (\mathcal{Q}_3^{\mathrm{low}} - \mathcal{Q}_3^{\mathrm{high}})\Psi\rangle \geq -C\delta\frac{\langle n_+\rangle_\Psi}{\ell^2}n\Big(\varepsilon K_H^2 + \varepsilon^{-1}\frac{\mathcal{M}}{n}\Big). \tag{8.24}$$

Choosing $\varepsilon = K_H^{-1}\frac{\mathcal{M}^{1/2}}{n^{1/2}}$, we obtain

$$\langle \Psi \mid (\mathcal{Q}_3^{\mathrm{low}} - \mathcal{Q}_3^{\mathrm{high}})\Psi\rangle \geq -C\delta\frac{\langle n_+\rangle_\Psi}{\ell^2}n\frac{K_H\mathcal{M}^{1/2}}{n^{1/2}}. \tag{8.25}$$

We use Theorem 7.6 and (H17) to bound $n^{1/2}$ by $2\rho_\mu^{1/2}\ell$ and get

$$\langle \Psi \mid (\mathcal{Q}_3^{\mathrm{low}} - \mathcal{Q}_3^{\mathrm{high}})\Psi\rangle \geq -C\delta\rho_\mu^{1/2}\ell K_H\mathcal{M}^{1/2}\frac{\langle n_+\rangle_\Psi}{\ell^2}.$$

By the assumption (H16) the error can be absorbed in a small fraction of the spectral gap. □

We are ready to state a bound for the second quantized Hamiltonian.

Proposition 8.3. *Assume $R \ll (\rho_\mu\delta)^{-1/2}$ and the relations of "Appendix H" between the parameters. Let Ψ be a normalized n-particle state satisfying (7.40) and $\Psi = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Psi$. Then*

$$\langle \Psi \mid \mathcal{H}_\Lambda(\rho_\mu)\Psi\rangle \geq \langle \Psi \mid \mathcal{H}_\Lambda^{2nd}(\rho_\mu)\Psi\rangle - C\ell^2\rho_\mu^2\delta\Big(d^{2M-2} + R^2\ell^{-2}\Big), \tag{8.26}$$

where

$$\mathcal{H}^{2nd}_{\Lambda} := \frac{\ell^2}{(2\pi)^2}\int_{\mathbb{R}^2}(1-\varepsilon_N)\tau(k)a_k^\dagger a_k \mathrm{d}k + \frac{b}{2\ell^2}n_+ + b\frac{\varepsilon_T}{8d^2\ell^2}n_+^H + b\frac{\varepsilon_T n_0 n_+^H}{16d^2\ell^2(\rho_\mu \ell^2)} \tag{8.27}$$

$$+\frac{1}{2\ell^2}a_0^\dagger a_0^\dagger a_0 a_0(\widehat{g}_0+\widehat{g\omega}(0)) - \rho_\mu a_0^\dagger a_0 \widehat{g}_0 \tag{8.28}$$

$$+\Big(\Big(\frac{1}{\ell^2}a_0^\dagger a_0 - \rho_\mu\Big)\widehat{W}_1(0)\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{\chi}_\Lambda(k)a_k^\dagger a_0 \mathrm{d}k + h.c.\Big) \tag{8.29}$$

$$+\Big(\frac{1}{\ell^2}a_0^\dagger a_0\widehat{\omega W_1}(0)\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{\chi}_\Lambda(k)a_k^\dagger a_0 \mathrm{d}k + h.c.\Big) \tag{8.30}$$

$$+\mathcal{Q}_2^{rest} + \mathcal{Q}_3^{high} \tag{8.31}$$

$$+\Big(\Big(\frac{1}{\ell^2}a_0^\dagger a_0 - \rho_\mu\Big)\widehat{W}_1(0) + \frac{1}{\ell^2}a_0^\dagger a_0\widehat{W_1\omega}(0)\Big)\frac{\ell^2}{(2\pi)^2}\int_{\mathbb{R}^2}a_k^\dagger a_k \mathrm{d}k, \tag{8.32}$$

where $\tau(k)$ *is defined in* (8.8) *and with*

$$\mathcal{Q}_2^{rest} = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}(\widehat{W}_1(k)+\widehat{(W_1\omega)}(k))a_0^\dagger a_k^\dagger a_k a_0 \mathrm{d}k + \frac{1}{2}\int_{\mathbb{R}^2}\widehat{W}_1(k)\Big(a_0^\dagger a_0^\dagger a_k a_{-k} + a_k^\dagger a_{-k}^\dagger a_0 a_0\Big)\mathrm{d}k.$$

Proof. We use the lower bound for $\mathcal{H}_\Lambda(\rho_\mu)$ from Corollary 7.3. First of all, in the kinetic energy expression (6.14) we remove the positive parts depending on the Neumann Laplacian, namely $\varepsilon_N(-\Delta_\mathcal{N})$ and $\mathcal{T}^{\mathrm{Neu,s}}$. Using the quantization, we obtain from (6.14) the expressions in (8.27) with the main kinetic energy term and the spectral gaps. We bounded part of the spectral gap to get the last term in (8.27) using $n_0 \leq 2\rho_\mu\ell^2$ (which follows from (7.43) and (H17)). This term will be useful later (in particular in the proof of Lemma 9.2).

The expressions (8.28), (8.29), (8.30), $\mathcal{Q}_2^{\mathrm{rest}}$ and (8.32) are obtained from (7.16), (7.17), (7.18), (7.19) and (7.20) respectively, via a straightforward application of the quantization rules. Note that in (8.29) and (8.30) we have changed a $\widehat{W}_1(k)$ (resp. $\widehat{\omega W_1}(k)$) into $\widehat{W}_1(0)$ (resp. $\widehat{\omega W_1}(0)$). This can be justified by using (6.25) in (7.17) and (7.18), the error being of order $R^2\rho_\mu^2\delta$. We can reabsorb the term

$$-C(\rho_\mu+\rho_0)\delta R^2\frac{n_+}{\ell^2},$$

in a fraction of the spectral gap because $R \ll (\rho_\mu\delta)^{-1/2}$. Let us observe that thanks to Lemma 7.5 we can replace $\mathcal{Q}_3^{\mathrm{ren}}+\frac{1}{4}\mathcal{Q}_4^{\mathrm{ren}}$ by $\mathcal{Q}_3^{\mathrm{low}}$ in $\mathcal{H}_\Lambda(\rho_\mu)$. Part of the error is absorbed in the spectral gap, other part appears in (8.26). Finally we change $\mathcal{Q}_3^{\mathrm{low}}$ into $\mathcal{Q}_3^{\mathrm{high}}$ using Lemma 8.2, the error being absorbed in a fraction of the spectral gap again. □

8.3. c-number substitution. In this section we show how the energy can be bounded if we minimize over a specific class of coherent states, which are eigenvectors for the annihilation operator of the condensate. In this way we can turn the action of the condensate operators in the form of multiplication per complex numbers. Let us define

$$|z\rangle = e^{-\left(\frac{|z|^2}{2}+za_0^\dagger\right)}\,\Omega, \tag{8.33}$$

for any $z \in \mathbb{C}$. As anticipated, we have

$$a_0|z\rangle = z\,|z\rangle. \tag{8.34}$$

Given any state Ψ we define the z-dependent state

$$\Phi(z) := \langle z \,|\, \Psi\rangle, \tag{8.35}$$

obtained by the partial inner product in $\mathscr{F}_s(\operatorname{Ran} P)$. One can verify that these states generate the space $\mathscr{F}_s(\operatorname{Ran} Q)$. Moreover,

$$1 = \frac{1}{\pi}\int_{\mathbb{C}} |z\rangle\langle z|\,\mathrm{d}z. \tag{8.36}$$

We define the following z-dependent density,

$$\rho_z := \frac{|z|^2}{\ell^2}, \tag{8.37}$$

and z-dependent Hamiltonian,

$$\mathcal{K}(z) = \frac{1}{2}\rho_z^2\ell^2(\widehat{g}_0 + \widehat{g\omega}(0)) - \rho_\mu\rho_z\widehat{g}_0\ell^2 \tag{8.38}$$

$$+ \mathcal{K}^{\mathrm{Bog}} + \frac{b}{2\ell^2}n_+ + \frac{\varepsilon_T b}{8d^2\ell^2}n_+^H + b\frac{\varepsilon_T|z|^2 n_+^H}{16d^2\ell^2(\rho_\mu\ell^2)} + \varepsilon_R(\rho_\mu - \rho_z)^2\delta\ell^2 \tag{8.39}$$

$$+ (\rho_z - \rho_\mu)\widehat{W}_1(0)\frac{\ell^2}{(2\pi)^2}\int_{\mathbb{R}^2} a_k^\dagger a_k \mathrm{d}k + \mathcal{Q}_1^{\mathrm{ex}}(z) + \mathcal{Q}_2^{\mathrm{ex}}(z) + \mathcal{Q}_3(z), \tag{8.40}$$

where $\varepsilon_R \ll 1$ is fixed in "Appendix H", and

$$\mathcal{K}^{\mathrm{Bog}} := \frac{\ell^2}{2(2\pi)^2}\int_{\mathbb{R}^2}\Big(\mathcal{A}(k)(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \mathcal{B}(k)(a_k a_{-k} + a_k^\dagger a_{-k}^\dagger) + \mathcal{C}(k)(a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k})\Big)\mathrm{d}k, \tag{8.41}$$

with

$$\mathcal{A}(k) := (1-\varepsilon_N)\tau(k) + \mathcal{B}(k), \qquad \mathcal{B}(k) := \rho_z\widehat{W}_1(k),$$
$$\mathcal{C}(k) := \frac{(\rho_z - \rho_\mu)}{\ell^2}\widehat{W}_1(0)\widehat{\chi}_\Lambda(k)z,$$
$$\mathcal{Q}_1^{\mathrm{ex}}(z) := \rho_z\widehat{(\omega W_1)}(0)\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}\widehat{\chi}_\Lambda(k)a_k^\dagger z\mathrm{d}k + h.c.,$$
$$\mathcal{Q}_2^{\mathrm{ex}}(z) := \frac{\ell^2}{(2\pi)^2}\rho_z\int_{\mathbb{R}^2}(\widehat{\omega W_1}(0) + \widehat{\omega W_1}(k))a_k^\dagger a_k\mathrm{d}k,$$
$$\mathcal{Q}_3(z) := \frac{\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2} f_L(p)\widehat{W}_1(k)\Big(\overline{z}\widetilde{a}_p^\dagger a_{p-k}a_k + h.c.\Big)\mathrm{d}k\mathrm{d}p \tag{8.42}$$

and $\tau(k)$ defined in (8.8). With these notations, the following theorem holds. Recall that $\mathcal{H}_\Lambda^{\mathrm{2nd}}$ is given by Proposition 8.3.

Theorem 8.4. *Assume $R \leq \ell$ and* (H17). *For any normalized n-particle state Ψ satisfying $\Psi = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Psi$ and* (7.40) *we have*

$$\langle \Psi \,|\, \mathcal{H}_\Lambda^{2nd}\Psi\rangle \geq \inf_{z\in\mathbb{R}_+} \inf_{\Phi} \langle \Phi \,|\, \mathcal{K}(z)\Phi\rangle - C\rho_\mu\delta(1+\varepsilon_R K_\ell^4 K_B^2 |\log Y|), \tag{8.43}$$

where the second infimum is over all the normalized states in $\mathscr{F}(\mathrm{Ran}\, Q)$ such that

$$\Phi = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Phi, \qquad and \qquad \Phi = \mathbb{1}_{[0,2\rho_\mu\ell^2]}(n_+)\Phi. \tag{8.44}$$

Proof. The theorem is proven via a standard technique of calculating the actions of creation and annihilation operators for the condensate on the coherent state and using its eigenvector properties, for details see [19, Theorem 8.5]. Practically speaking it consists in the formal substitutions

$$a_0 \mapsto z, \qquad a_0^\dagger \mapsto \overline{z}, \qquad a_0^\dagger a_0 \mapsto |z|^2 - 1, \tag{8.45}$$

and getting rid of the lower order terms in $|z|$ because they produce errors of the form

$$\rho_\mu\delta = \rho_\mu^2\ell^2\delta^2 K_\ell^{-2}. \tag{8.46}$$

In order to make the last term in (8.39) appear, we add and subtract $\varepsilon_R(\rho_\mu - n_0\ell^{-2})^2\delta\ell^2$ to $\mathcal{H}_\Lambda^{\text{2nd}}$ and estimate the negative contribution, recalling the estimates in Theorem 7.7 and that $n_+^2 \leq nn_+$ we get

$$\begin{aligned} -\varepsilon_R\Big(\rho_\mu - \frac{n_0}{\ell^2}\Big)^2\delta\ell^2 &\geq -2\varepsilon_R\delta\ell^{-2}((\rho_\mu\ell^2 - n)^2 + n_+ n) \\ &\geq -C\varepsilon_R\frac{\delta}{\ell^2}n^2 K_B^2 Y|\log Y|K_\ell^2 = -C\varepsilon_R\rho_\mu\delta K_B^2|\log Y|K_\ell^4, \end{aligned}$$

which is coherent with the error terms. □

9. Lower Bounds for the Hamiltonian $\mathcal{K}$

9.1. Estimate of $\mathcal{K}$ for ρ_z far from ρ_μ. The purpose of this section is to show that for values of ρ_z far from the density ρ_μ it is possible to prove a rough estimate on the energy and eliminate these values from the analysis. This is the content of the proposition below. We recall that $\mathcal{K}(z)$ is defined in (8.38), and we use the notations $\varepsilon_{\mathcal{M}} = \frac{\mathcal{M}}{\rho_\mu\ell^2}$ and

$$\delta_1 = \frac{\varepsilon_T^2\varepsilon_{\mathcal{M}}}{d^8K_\ell^4}\Big(1+\frac{K_\ell^2}{K_H^2}\Big), \quad \delta_2 = \varepsilon_{\mathcal{M}}^{1/2}, \quad \delta_3 = \delta|\log(dsK_\ell)| + \frac{(dK_\ell)^4}{\varepsilon_T^2}. \tag{9.1}$$

Proposition 9.1. *Assume the relations between the parameters in "Appendix H". There exists a $C > 0$, such that if we have $\rho_\mu a^2 \leq C^{-1}$ and*

$$|\rho_\mu - \rho_z| \geq C\rho_\mu \max\Big((\delta_1+\delta_2+\delta_3)^{1/2}, \delta^{1/2}\Big), \tag{9.2}$$

then for any state $\Phi \in \mathscr{F}(\mathrm{Ran}\, Q)$ satisfying (8.44)*, we have*

$$\langle \Phi \,|\, \mathcal{K}(z)\Phi\rangle \geq -4\pi\rho_\mu^2\ell^2\delta + 8\pi\Big(\frac{1}{2}+2\Gamma+\log\pi\Big)\rho_\mu^2\ell^2\delta^2. \tag{9.3}$$

Notice that the second order term in (9.3) is larger than the one aimed for in Theorem 6.7. So the statement of the proposition is that the energy is too large unless $|\rho_\mu - \rho_z|$ is small. The proof of the proposition relies on the technical estimate given by the following lemma.

Lemma 9.2. *Assume the relations between the parameters in "Appendix H". For any normalized* $\Phi \in \mathscr{F}(\mathrm{Ran}\, Q)$ *such that* (8.44) *holds,*

$$\langle \Phi \,|\, \mathcal{K}(z)\Phi \rangle \geq -4\pi \rho_\mu^2 \ell^2 \delta + 4\pi \ell^2 (\rho_\mu - \rho_z)^2 \delta - C\rho_z \rho_\mu \ell^2 \delta \delta_1 - C\rho_\mu^{1/2}(\rho_\mu + \rho_z)^{3/2} \ell^2 \delta \delta_2 - C\rho_z^2 \ell^2 \delta \delta_3 - C\rho_\mu \delta^2 K_\ell^{-2} (ds)^{-4}. \quad (9.4)$$

Proof of Lemma 9.2. We start by estimating the Q_1 terms. We have for any $\varepsilon > 0$

$$\begin{aligned} &\int_{\mathbb{R}^2} \widehat{\chi}_\Lambda(k)(a_k^\dagger z + a_k \bar{z})\mathrm{d}k \\ &\leq \int_{\mathbb{R}^2} |\widehat{\chi}_\Lambda(k)|(\varepsilon |z|^2 + \varepsilon^{-1} a_k^\dagger a_k)\mathrm{d}k \\ &\leq C\Big(\varepsilon |z|^2 + \varepsilon^{-1} |\widehat{\chi}_\Lambda(0)| \int_{k \in \mathcal{P}_H^c} a_k^\dagger a_k \mathrm{d}k + \varepsilon^{-1} \int_{k \in \mathcal{P}_H} |\widehat{\chi}_\Lambda(k)| a_k^\dagger a_k \mathrm{d}k\Big). \end{aligned}$$

Considering a Φ like in the assumption we have, using $|\widehat{\chi}_\Lambda(0)| = \ell^2 \|\chi\|_1$ together with Lemma 8.1,

$$\begin{aligned} &\Big\langle |\widehat{\chi}_\Lambda(0)| \int_{k \in \mathcal{P}_H^c} a_k^\dagger a_k \mathrm{d}k \;+ \int_{k \in \mathcal{P}_H} |\widehat{\chi}_\Lambda(k)| a_k^\dagger a_k \mathrm{d}k \Big\rangle_\Phi \\ &\leq C\Big(\mathcal{M} + \rho_\mu \ell^2 \sup_{k \in \mathcal{P}_H} (\ell^{-2} |\widehat{\chi}_\Lambda(k)|)\Big). \end{aligned} \quad (9.5)$$

Now, using (F4) and optimizing with $\varepsilon = \sqrt{\mathcal{M}/|z|^2}$,

$$\begin{aligned} &\langle -\frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{C}(k)(a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k})\mathrm{d}k + \mathcal{Q}_1^{\mathrm{ex}}(z) \rangle_\Phi \\ &\geq -C\delta \sqrt{\mathcal{M}} |z| (|\rho_z - \rho_\mu| + \rho_z) \\ &\geq -C\Big(\frac{\mathcal{M}}{\rho_\mu \ell^2}\Big)^{1/2} \rho_\mu^{1/2} \ell^2 \delta (\rho_\mu + \rho_z)^{3/2}. \end{aligned} \quad (9.6)$$

For the terms that are quadratic in the field operators, we use the estimate

$$\Big| \Big\langle \ell^2 \int_{\mathbb{R}^2} \widehat{W}_1(k) a_k^\dagger a_k \mathrm{d}k \Big\rangle_\Phi \Big| \leq C\delta (\mathcal{M} + \langle n_+^H \rangle_\Phi), \quad (9.7)$$

from Lemma 8.1 to obtain that

$$\begin{aligned} &\Big\langle \mathcal{Q}_2^{\mathrm{ex}} + (\rho_z - \rho_\mu)\widehat{W}_1(0) \frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} a_k^\dagger a_k \mathrm{d}k + \frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} \mathcal{B}_k a_k^\dagger a_k \mathrm{d}k \Big\rangle_\Phi \\ &\geq -C(\rho_z + \rho_\mu)\big(\rho_\mu \ell^2 \delta \varepsilon_{\mathcal{M}} + \delta \langle n_+^H \rangle_\Phi\big), \end{aligned} \quad (9.8)$$

where the $\mathcal{B}_k$ has been extracted from the expression of the $\mathcal{A}_k$. The first term is coherent with the error in the result and the last one can be reabsorbed in a fraction of the spectral gap because of relation (H8).

For the remaining part of $\mathcal{A}_k$ involving τ_k we add and subtract $-\rho_z\delta\varepsilon^{-1/2}+\varepsilon\tau_k$, with $\varepsilon \geq \varepsilon_N$ and estimate

$$(1-\varepsilon_N)\tau_k \geq \widetilde{\mathcal{A}}_k - \rho_z\delta\varepsilon^{-1/2} + \varepsilon\tau_k, \tag{9.9}$$

with

$$\widetilde{\mathcal{A}}_k = (1-2\varepsilon)\Big[|k| - \frac{1}{2ds\ell}\Big]_+^2 + \rho_z\delta\varepsilon^{-1/2}. \tag{9.10}$$

We treat the terms in (9.9) separately, adding them to the remaining parts of the Hamiltonian. The simplest one is

$$-\frac{\ell^2}{(2\pi)^2}\rho_z\varepsilon^{-1/2}\delta\Big\langle \int_{\mathbb{R}^2} a_k^\dagger a_k \mathrm{d}k\Big\rangle_\Phi \geq -C\varepsilon^{-1/2}\rho_z\delta(\mathcal{M}+\langle n_+^H\rangle_\Phi), \tag{9.11}$$

where we used Lemma 8.1. We use this estimate to fix the choice of ε in order to absorb the last term in the fraction of the spectral gap represented by the second to last term in (8.39). This yields

$$\varepsilon = C^{-1}\varepsilon_T^{-2}(dK_\ell)^4, \tag{9.12}$$

for some sufficiently large constant C and the relations (H23), (H8) ensure that $\varepsilon_N \leq \varepsilon \ll 1$. For the $\widetilde{\mathcal{A}}$ term plus the B terms in the Hamiltonian we use the Bogoliubov diagonalization procedure stated in Theorem B.1 to obtain

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathbb{R}^2} \widetilde{\mathcal{A}}_k a_k^\dagger a_k + \frac{\mathcal{B}_k}{2}(a_k^\dagger a_{-k}^\dagger + a_k a_{-k})\,\mathrm{d}k \geq -\frac{\ell^2}{2(2\pi)^2}\int_{\mathbb{R}^2} \widetilde{\mathcal{A}}_k - \sqrt{\widetilde{\mathcal{A}}_k^2 - \mathcal{B}_k^2}\,\mathrm{d}k, \tag{9.13}$$

and then we use Lemma C.5 and its proof choosing the parameters $K_1 = \rho_z\varepsilon^{-1/2}/2$, $K_2 = 2\rho_z$, $K = (2ds\ell)^{-1}$ and $\kappa = (1-2\varepsilon)$ to derive that

$$\begin{aligned}(9.13) \geq -\frac{\ell^2}{2(2\pi)^2}\Big(&\rho_z^2\frac{1+\varepsilon}{1-2\varepsilon}\int_{\mathbb{R}^2}\mathrm{d}k\frac{\widehat{W}_1^2(k)-\widehat{W}_1^2(0)\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2|k|^2} + C\rho_z\varepsilon^{1/2}\delta(ds\ell)^{-2}\\ &+\frac{C}{(1-2\varepsilon)}\rho_z^2\delta^2(1+R^2\ell_\delta^{-2}) + \frac{C\rho_z^2}{1-2\varepsilon}\delta^2|\log((ds\ell)^{-1}\ell_\delta)|\Big).\end{aligned} \tag{9.14}$$

Using now Cauchy–Schwarz on the second term, Lemma 6.4, writing only the dominant terms due to the relations between the parameters and recalling the definition (6.23) of ℓ_δ we obtain

$$(9.13) \geq -\frac{1}{2}\rho_z^2\ell^2\widehat{g\omega}(0) - C\rho_z^2\ell^2\delta\big(\varepsilon + \delta^2\rho_\mu R^2 + \delta|\log(dsK_\ell)|\big) - C\delta\ell^2(ds\ell)^{-4}. \tag{9.15}$$

Due to relation (H3) the second term gives δ_3, while the third one gives the last term in (9.4).

We continue considering the third term in (9.9) and adding it to the $\mathcal{Q}_3$. The latter is an integral for $k \in \mathcal{P}_H$, and dropping the part of the τ_k for $k \in \mathcal{P}_H^c$ and using that for $k \in \mathcal{P}_H$ then $\tau_k \geq |k|^2/2$, we have to estimate

$$\frac{\ell^2}{(2\pi)^2}\int_{k\in\mathcal{P}_H}\Big(\frac{\varepsilon}{2}k^2 a_k^\dagger a_k + \frac{1}{(2\pi)^2}\int f_L(p)\widehat{W}_1(k)(\bar{z}\widetilde{a}_p^\dagger a_{p-k}a_k + a_k^\dagger a_{p-k}^\dagger \widetilde{a}_p z)\Big)\mathrm{d}p\mathrm{d}k. \tag{9.16}$$

We complete the square in the previous expression, introducing the operators

$$\sigma_k := a_k + \frac{2}{(2\pi)^2}\int f_L(p)\frac{\widehat{W}_1(k)}{\varepsilon|k|^2} z a_{p-k}^\dagger \widetilde{a}_p \mathrm{d}p, \tag{9.17}$$

so that

$$\begin{aligned}
(9.16) &= \frac{\ell^2}{(2\pi)^2}\int_{k\in\mathcal{P}_H}\Big(\frac{\varepsilon}{2}k^2\sigma_k^\dagger\sigma_k \\
&\quad - \frac{2|z|^2}{\varepsilon(2\pi)^4}\iint f_L(p)f_L(s)\frac{\widehat{W}_1(k)^2}{k^2}\widetilde{a}_s^\dagger a_{s-k}a_{p-k}^\dagger\widetilde{a}_p\Big)\mathrm{d}p\mathrm{d}s\mathrm{d}k \\
&\geq -\frac{2|z|^2\ell^2}{\varepsilon(2\pi)^6}\int_{k\in\mathcal{P}_H}\frac{\widehat{W}_1(k)^2}{k^2} \\
&\quad \iint f_L(p)f_L(s)\widetilde{a}_s^\dagger(a_{p-k}^\dagger a_{s-k} + [a_{s-k}, a_{p-k}^\dagger])\widetilde{a}_p\mathrm{d}p\mathrm{d}s\mathrm{d}k.
\end{aligned}$$

For the term without commutator, estimated on a state Φ which satisfies (8.44) and using Cauchy–Schwarz

$$\widetilde{a}_s^\dagger a_{p-k}^\dagger a_{s-k}\widetilde{a}_p \leq C(\widetilde{a}_s^\dagger a_{p-k}^\dagger a_{p-k}\widetilde{a}_s + \widetilde{a}_p^\dagger a_{s-k}^\dagger a_{s-k}\widetilde{a}_p), \tag{9.18}$$

we have

$$\begin{aligned}
&\frac{2|z|^2\ell^2}{\varepsilon(2\pi)^6}\Big\langle\int_{k\in\mathcal{P}_H} dk\,\frac{\widehat{W}_1(k)^2}{k^2}\iint f_L(p)f_L(s)\widetilde{a}_s^\dagger a_{p-k}^\dagger a_{p-k}\widetilde{a}_s\mathrm{d}p\mathrm{d}s\Big\rangle_\Phi \\
&\quad \leq C|z|^2\varepsilon^{-1}\frac{\ell^4\delta^2}{K_H^2}\Big\langle\int_{k\in\mathcal{P}_H}\int f_L(s)\widetilde{a}_s^\dagger a_k^\dagger a_k\widetilde{a}_s\mathrm{d}s\mathrm{d}k\Big\rangle_\Phi\int_{p\in\mathcal{P}_L}\mathrm{d}p \\
&\quad \leq C\varepsilon^{-1}\frac{\delta^2}{K_H^2}d^{-4}\mathcal{M}\rho_\mu\rho_z\ell^2,
\end{aligned} \tag{9.19}$$

where we used Lemma 8.1 since the support of f_L is included in the complement of $\mathcal{P}_H$, and the estimate, for $k \in \mathcal{P}_H$,

$$\frac{\widehat{W}_1(k)^2}{2k^2} \leq CK_H^{-2}\delta^2\ell^2. \tag{9.20}$$

For the commutator part we use the estimate (8.6), the Cauchy–Schwarz inequality

$$\widetilde{a}_s^\dagger[a_{s-k}, a_{p-k}^\dagger]\widetilde{a}_p \leq C\widetilde{a}_s^\dagger\widetilde{a}_s + C\widetilde{a}_p^\dagger\widetilde{a}_p, \tag{9.21}$$

and Lemma 3.9 applied to $\widehat{W}_1$ instead of $\widehat{g}$ paying a small error, we get

$$-\frac{2|z|^2\ell^2}{\varepsilon(2\pi)^6}\left\langle\int_{k\in\mathcal{P}_H}\frac{\widehat{W}_1(k)^2}{k^2}\iint f_L(p)f_L(s)\widetilde{a}_s^\dagger[a_{s-k},a_{p-k}^\dagger]\widetilde{a}_p\mathrm{d}p\mathrm{d}s\mathrm{d}k\right\rangle_\Phi$$
$$\geq -C\frac{|z|^2\ell^2}{\varepsilon}\delta\left\langle\iint f_L(p)f_L(s)\widetilde{a}_p^\dagger\widetilde{a}_p\mathrm{d}p\mathrm{d}s\right\rangle_\Phi \geq -C\varepsilon^{-1}\rho_z\delta\mathcal{M}d^{-4}, \quad (9.22)$$

where in the last inequality we used Lemma 8.1.

Collecting formulas (9.6), (9.8), (9.15), (9.19) and (9.22) and observing that

$$\frac{1}{2}\rho_z^2\ell^2\widehat{g}_0 - \rho_z\rho_\mu\ell^2\widehat{g}_0 = \frac{1}{2}(\rho_z-\rho_\mu)^2\ell^2\widehat{g}_0 - \frac{1}{2}\rho_\mu^2\ell^2\widehat{g}_0, \quad (9.23)$$

we obtain the result. □

Proof of Proposition 9.1. We observe that, thanks to the relations (H6), (H8), (H22), we have $\delta_j \ll 1$ for $j = 1, 2, 3$. Each coefficient of the δ_j in formula (9.4) can be bounded by

$$C\delta(\rho_\mu-\rho_z)^2\ell^2 + C\rho_\mu^2\ell^2\delta. \quad (9.24)$$

Therefore, Lemma 9.2 and $\widehat{g}_0 = 8\pi\delta$ implies the bound

$$\begin{aligned}\langle\mathcal{K}(z)\rangle_\Phi \geq & -\frac{1}{2}\rho_\mu^2\ell^2\widehat{g}_0 + \frac{1}{2}(\rho_\mu-\rho_z)^2\ell^2\widehat{g}_0(1-C(\delta_1+\delta_2+\delta_3)) \\ & - C\rho_\mu^2\ell^2\delta(\delta_1+\delta_2+\delta_3+\delta^2(K_\ell ds)^{-4}) \\ \geq & -\frac{1}{2}\rho_\mu^2\ell^2\widehat{g}_0 + \frac{1}{4}\ell^2\widehat{g}_0(\rho_\mu-\rho_z)^2 - C\rho_\mu^2\ell^2\delta(\delta_1+\delta_2+\delta_3+\delta^2(K_\ell ds)^{-4}).\end{aligned}$$

Note that $\delta^2(K_\ell ds)^{-4} \ll \delta$ due to (H12) and (H17). By the assumption on $(\rho_\mu-\rho_z)^2$ the second term is of higher order both of the δ_j errors and of the desired quantity in the statement of the Proposition. □

9.2. Estimate of $\mathcal{K}$ for $\rho_z \simeq \rho_\mu$. We study here the main case, that is when ρ_z is close to ρ_μ. More precisely, we consider the complementary situation to (9.2), when

$$|\rho_\mu-\rho_z| \leq K_\ell^{-2}\rho_\mu, \quad (9.25)$$

where we used that, thanks to the choices of the parameters (H8), (H17) and (H22), we have

$$K_\ell^2\max\left((\delta_1+\delta_2+\delta_3)^{1/2},\delta^{1/2}\right) \leq C^{-1}. \quad (9.26)$$

Using again (9.23) and reabsorbing the term $(\rho_z-\rho_\mu)\widehat{W}_1(0)\frac{\ell^2}{(2\pi)^2}\int a_k^\dagger a_k\mathrm{d}k$ in part of the spectral gap of n_+, we have the estimate of $\mathcal{K}(z)$ from (8.38),

$$\begin{aligned}\mathcal{K}(z) \geq & -\frac{1}{2}\rho_\mu^2\ell^2\widehat{g}_0 + \frac{1}{2}\rho_z^2\ell^2\widehat{g\omega}(0) + \frac{1}{2}(\rho_z-\rho_\mu)^2\ell^2\widehat{g}_0 \\ & + \mathcal{K}^{\mathrm{Bog}} + \frac{b}{4\ell^2}n_+ + b\frac{\varepsilon_T}{8d^2\ell^2}n_+^H + b\frac{\varepsilon_T|z|^2n_+^H}{16d^2\ell^2(\rho_\mu\ell^2)} + \varepsilon_R(\rho_\mu-\rho_z)^2\delta\ell^2 \\ & + \mathcal{Q}_1^{\mathrm{ex}}(z) + \mathcal{Q}_2^{\mathrm{ex}}(z) + \mathcal{Q}_3(z),\end{aligned} \quad (9.27)$$

and in the following we want to give a lower bound for the expression above using a diagonalization method for the Bogoliubov Hamiltonian. In order to do that, let us introduce a couple of new creation and annihilation operators

$$b_k := \frac{1}{\sqrt{1-\alpha_k^2}}(a_k + \alpha_k a_{-k}^\dagger + c_k), \tag{9.28}$$

where

$$\alpha_k := \mathcal{B}(k)^{-1}\Big(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\Big),$$
$$c_k := \frac{2\mathcal{C}(k)}{\mathcal{A}(k) + \mathcal{B}(k) + \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}} \mathbb{1}_{\{|k| \le \frac{1}{2} K_H \ell^{-1}\}},$$

with $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are defined in (8.41) and the diagonalized Bogoliubov Hamiltonian

$$\mathcal{K}_H^{\text{Diag}} := \frac{\ell^2}{(2\pi)^2} \int_{\{|k| \ge \frac{1}{2} K_H \ell^{-1}\}} \mathcal{D}(k) b_k^\dagger b_k \mathrm{d}k, \tag{9.29}$$

where

$$\mathcal{D}(k) := \frac{1}{2}\big(\mathcal{A}(k) + \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\big). \tag{9.30}$$

Theorem 9.3. *Assume the relations between the parameters in "Appendix H". For any state* $\Phi \in \mathscr{F}_s(L^2(\Lambda))$ *such that* (8.44) *holds and* $\frac{9}{10}\rho_\mu \le \rho_z \le \frac{11}{10}\rho_\mu$ *we have*

$$\begin{aligned}
&\langle \mathcal{K}^{Bog} \rangle_\Phi + \frac{1}{2}\rho_z^2 \ell^2 (\widehat{g\omega})_0 + \frac{1}{2}(\rho_z - \rho_\mu)^2 \ell^2 \widehat{g}_0 \\
&\quad \ge (1 - \varepsilon_K)\Big\langle \mathcal{K}_H^{Diag} \Big\rangle_\Phi + 4\pi\Big(2\Gamma + \frac{1}{2} + \log\pi\Big)\rho_z^2 \ell^2 \delta^2 \\
&\quad\quad - C(\rho_\mu - \rho_z)^2 \ell^2 \delta^2 \rho_\mu R^2 - C\rho_\mu^2 \ell^2 \delta (K_H^{4-M} K_\ell \delta^{-1/2}) + Cr(\rho_\mu)\ell^2,
\end{aligned}$$

where the error term is given by

$$r(\rho_\mu) := \rho_\mu^2 \delta^2 \big(\delta|\log(\delta)| R^2 \rho_\mu + \delta|\log(\delta)| + d + \varepsilon_T |\log\delta| + (sK_\ell)^{-1} + \varepsilon_N \delta^{-1}\big).$$

In the proof of Theorem 9.3 we are going to use the following formulas and estimates for the commutators of the operators, recalling that $\widehat{\chi}_\Lambda$ is even,

$$[b_k, b_h] = \frac{\alpha_k - \alpha_h}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_h^2}}\Big(\widehat{(\chi^2)}((k+h)\ell) - \widehat{\chi}(k\ell)\widehat{\chi}(h\ell)\Big), \tag{9.31}$$

$$[b_k, b_h^\dagger] = \frac{1 - \alpha_k \alpha_h}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_h^2}}\Big(\widehat{(\chi^2)}((k-h)\ell) - \widehat{\chi}(k\ell)\widehat{\chi}(h\ell)\Big), \tag{9.32}$$

$$[\widetilde{a}_p^\dagger, b_k^\dagger] = \frac{\alpha_k}{\sqrt{1-\alpha_k^2}}[\widetilde{a}_p^\dagger, a_k] = \frac{\alpha_k}{\sqrt{1-\alpha_k^2}}\ell^{-2}\langle e^{ipx}, Q\chi_\Lambda e^{ikx}\rangle, \tag{9.33}$$

$$[\widetilde{a}_p, b_{-k}^\dagger] = \frac{1}{\sqrt{1-\alpha_k^2}}[\widetilde{a}_p, a_{-k}^\dagger] = \frac{1}{\sqrt{1-\alpha_k^2}}(\widehat{\chi}((p+k)\ell) - \widehat{\theta}(p\ell)\widehat{\chi}(k\ell)). \tag{9.34}$$

Proof. Let us start by showing that the contribution coming from the $\mathcal{C}(k)$ gives an error term for $|k| > \frac{1}{2}K_H\ell^{-1}$.

By Cauchy–Schwarz we have $a_k^\dagger + a_k \leq a_k^\dagger a_k + 1$ and then we recognize n_+ (8.7),

$$\begin{aligned}&\frac{\ell^2}{2(2\pi)^2}\int_{\{|k|>\frac{1}{2}K_H\ell^{-1}\}}\mathcal{C}(k)(a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k})\,\mathrm{d}k\\ &\quad\geq -C|\rho_\mu - \rho_z||\widehat{W}_1(0)||z|\int_{\{|k|>\frac{1}{2}K_H\ell^{-1}\}}|\widehat{\chi_\Lambda}(k)|(a_k^\dagger a_k + 1)\,\mathrm{d}k\\ &\quad\geq -C\rho_\mu\delta|z|(n_+ + 1)K_H^{4-M},\end{aligned}$$

where we use the assumption on ρ_z and that by Lemma F.1,

$$\ell^{-2}\sup_{|k|>\frac{1}{2}K_H\ell^{-1}}(1+(k\ell)^2)^2|\widehat{\chi_\Lambda}(k)| \leq CK_H^{4-M}. \tag{9.35}$$

When we apply to Φ we have $n_+ \leq 2\rho_\mu\ell^2$ and

$$\frac{\ell^2}{2(2\pi)^2}\int_{\{|k|>\frac{1}{2}K_H\ell^{-1}\}}\mathcal{C}(k)\langle a_k^\dagger + a_{-k}^\dagger + a_k + a_{-k}\rangle_\Phi\,\mathrm{d}k \geq -C\rho_\mu^2\ell^2\delta(K_H^{4-M}\sqrt{\rho_\mu}\ell). \tag{9.36}$$

Therefore

$$\mathcal{K}^{\mathrm{Bog}} \geq \widetilde{\mathcal{K}}^{\mathrm{Bog}} - C\rho_\mu^2\ell^2\delta(K_H^{4-M}K_\ell\delta^{-1/2}), \tag{9.37}$$

where $\widetilde{\mathcal{K}}^{\mathrm{Bog}}$ is the same as $\mathcal{K}^{\mathrm{Bog}}$ but with $\mathcal{C}(k)$ substituted by

$$\widetilde{\mathcal{C}}(k) := \mathcal{C}(k)\mathbb{1}_{\{|k|\leq\frac{1}{2}K_H\ell^{-1}\}}. \tag{9.38}$$

The bound on the commutator (8.6) allows us to use Theorem B.1 to diagonalize the Bogoliubov Hamiltonian

$$\begin{aligned}\widetilde{\mathcal{K}}^{\mathrm{Bog}} \geq \widetilde{\mathcal{K}}^{\mathrm{Diag}} &- \frac{\ell^2}{2(2\pi)^2}\int_{\mathbb{R}^2}\Big(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\Big)\mathrm{d}k\\ &- (\rho_z - \rho_\mu)^2\widehat{W}_1(0)^2\frac{z^2}{(2\pi)^2\ell^2}\int_{\{|k|\leq\frac{1}{2}K_H\ell^{-1}\}}\frac{|\widehat{\chi_\Lambda}(k)|^2}{\mathcal{A}(k)+\mathcal{B}(k)}\mathrm{d}k,\end{aligned}$$

where

$$\widetilde{\mathcal{K}}^{\mathrm{Diag}} = \frac{\ell^2}{(2\pi)^2}\int(1-\alpha_k^2)\mathcal{D}_k b_k^\dagger b_k\mathrm{d}k \geq \frac{\ell^2}{(2\pi)^2}\int_{\{|k|>\frac{1}{2}K_H\ell^{-1}\}}(1-\alpha_k^2)\mathcal{D}_k b_k^\dagger b_k\mathrm{d}k. \tag{9.39}$$

Using the inequality $|\alpha_k| \leq C\rho_z\delta k^{-2} \leq CK_\ell^2K_H^{-2}$ we find

$$\widetilde{\mathcal{K}}^{\mathrm{Diag}} \geq \mathcal{K}_H^{\mathrm{Diag}}(1 - CK_\ell^4K_H^{-4}). \tag{9.40}$$

The calculation of the Bogoliubov integral is given in “Appendix C”. Combining the results of Lemma C.1, Lemma C.2 and Proposition C.3 and multiplying everything by ℓ^2 we find

$$-\frac{\ell^2}{2(2\pi)^2}\int_{\mathbb{R}^2}\Big(\mathcal{A}(k)-\sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2}\Big)\mathrm{d}k+\frac{1}{2}\widehat{g\omega}(0)\rho_z^2\ell^2$$
$$\geq 4\pi\Big(2\Gamma+\frac{1}{2}+\log\pi\Big)\rho_z^2\ell^2\delta^2+r(\rho_\mu)\ell^2, \tag{9.41}$$

where $r(\rho_\mu)$ is defined in the statement of the theorem. For the remaining term we use the estimate

$$\mathcal{A}(k)+\mathcal{B}(k)\geq 2\rho_z\widehat{W}_1(k)\geq 2\rho_z\widehat{W}_1(0)(1-C\delta(kR)^2), \tag{9.42}$$

where we used a Taylor expansion and the fact that W_1 is even. By this last estimate, together with Lemma F.1 and (6.24) we obtain

$$-(\rho_z-\rho_\mu)^2\widehat{W}_1^2(0)\frac{z^2}{(2\pi)^2\ell^2}\int_{\{|k|\leq\frac{1}{2}K_H\ell^{-1}\}}\frac{|\widehat{\chi_\Lambda}(k)|^2}{\mathcal{A}(k)+\mathcal{B}(k)}$$
$$\geq-(\rho_z-\rho_\mu)^2\frac{\widehat{W}_1(0)}{2}\ell^2\big(1+C\rho_\mu\delta^2R^2K_H^2K_\ell^{-2}\big)$$
$$\geq-(\rho_z-\rho_\mu)^2\frac{\widehat{g}(0)}{2}\ell^2\big(1+C\rho_\mu\delta R^2\big),$$

where in the last line we used $K_H\ll\delta^{-1/2}$ from (H16). □

9.3. Contribution of $\mathcal{Q}_3$. The aim of this section is to bound the $3Q$ term from below, namely

$$\mathcal{Q}_3(z)=\frac{\bar{z}\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\widehat{W}_1(k)f_L(p)(\widetilde{a}_p^\dagger a_{p-k}a_k+\text{h.c.})\mathrm{d}k\mathrm{d}p,$$

which turns out to be controlled by the quadratic Hamiltonian $\mathcal{K}_H^{\text{Diag}}$ defined in (9.29), absorbing $\mathcal{Q}_2^{\text{ex}}$ and $\mathcal{Q}_1^{\text{ex}}$. More precisely we prove

Theorem 9.4. *Assume the relations between the parameters in “Appendix H” to be satisfied. Then there exists a universal constant $C>0$ such that for any state Φ satisfying* (8.44) *we have*

$$\Big\langle(1-\varepsilon_K)\mathcal{K}_H^{Diag}+\mathcal{Q}_3(z)+\mathcal{Q}_2^{ex}+\mathcal{Q}_1^{ex}+\frac{b}{100}\frac{n_+}{\ell^2}+\frac{\varepsilon_T b}{100}\frac{n_+^H}{(d\ell)^2}\Big\rangle_\Phi$$
$$\geq-C\rho_z^2\ell^2\delta^2\Big(\delta K_H^{-8}K_\ell^{10}d^{-4}+\varepsilon_K^{-1}K_H^{-12}K_\ell^{10}d^{-8}+d^{-8}K_\ell^2K_H^{-4}$$
$$+\varepsilon_K^{-1}K_H^{-2M-8}K_\ell^6d^{-8}+\delta K_\ell^2|\log\delta|^2+\delta^{-1}K_\ell^2d^{8M-2}\varepsilon_T^{-1}$$
$$+\varepsilon_{\mathcal{M}}^{1/2}(K_\ell^4K_H^{-4}+\delta^{-1}K_H^{-M}d^{-2})\Big).$$

Remark 9.5. Note that we control $\mathcal{Q}_3+\mathcal{Q}_2^{\mathrm{ex}}$ using a large fraction of $\mathcal{K}_H^{\mathrm{Diag}}$. It is important to remember that $\mathcal{K}_H^{\mathrm{Diag}}$ is not the kinetic energy, but the Hamiltonian arising from the Bogoliubov diagonalization—sometimes $\mathcal{K}^{\mathrm{Diag}}$ is called the excitation Hamiltonian. The kinetic energy is already contributing to main order in the energy, and we use it to obtain the LHY term (Theorem 9.3). The operator $\mathcal{K}_H^{\mathrm{Diag}}$ is much smaller than the kinetic energy, and this is why we can use all of it to control $\mathcal{Q}_3 + \mathcal{Q}_2^{\mathrm{ex}}$.

In order to prove this theorem, we start by rewriting $\mathcal{Q}_3(z)$ in terms of the b_k's defined in (9.28). Notice that $c_k = c_{p-k} = 0$ if $k \in \mathcal{P}_H$ and $p \in \mathcal{P}_L$, and

$$a_k = \frac{b_k - \alpha_k b_{-k}^\dagger}{\sqrt{1-\alpha_k^2}}, \qquad a_{p-k} = \frac{b_{p-k} - \alpha_{p-k} b_{k-p}^\dagger}{\sqrt{1-\alpha_{p-k}^2}}. \tag{9.43}$$

Therefore,

$$a_{p-k}a_k = \frac{b_{p-k}b_k - \alpha_k b_{p-k}b_{-k}^\dagger - \alpha_{p-k}b_{k-p}^\dagger b_k + \alpha_{p-k}\alpha_k b_{k-p}^\dagger b_{-k}^\dagger}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}},$$

and $\mathcal{Q}_3(z) = \mathcal{Q}_3^{(1)} + \mathcal{Q}_3^{(2)} + \mathcal{Q}_3^{(3)} + \mathcal{Q}_3^{(4)}$ where

$$\mathcal{Q}_3^{(1)} = \frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\big(\widetilde{a}_p^\dagger b_{p-k}b_k + \alpha_k\alpha_{p-k}\widetilde{a}_p^\dagger b_{k-p}^\dagger b_{-k}^\dagger + h.c.\big), \tag{9.44}$$

$$\mathcal{Q}_3^{(2)} = -\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)\alpha_k}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\big(\widetilde{a}_p^\dagger b_{-k}^\dagger b_{p-k} + b_{p-k}^\dagger b_{-k}\widetilde{a}_p\big), \tag{9.45}$$

$$\mathcal{Q}_3^{(3)} = -\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)\alpha_{p-k}}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\big(\widetilde{a}_p^\dagger b_{k-p}^\dagger b_k + b_k^\dagger b_{k-p}\widetilde{a}_p\big), \tag{9.46}$$

$$\mathcal{Q}_3^{(4)} = -\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\alpha_k[b_{p-k}, b_{-k}^\dagger](\widetilde{a}_p^\dagger + \widetilde{a}_p). \tag{9.47}$$

In the remaining of this section, we get lower bounds on those four terms (Lemmas 9.7, 9.9 and 9.11 below) hence proving Theorem 9.4.

We collect here some important technical estimates which are going to be useful in the following.

Lemma 9.6. *The following bounds hold:*

$$|\alpha_k| \le C\rho_z\delta|k|^{-2} \le CK_\ell^2 K_H^{-2}, \quad \text{for } |k| \ge \frac{1}{2}K_H\ell^{-1}, \tag{9.48}$$

$$\mathcal{D}_k \ge \frac{1}{2}|k|^2 \ge \frac{1}{8}K_H^2\ell^{-2}, \quad \text{for } |k| \ge \frac{1}{2}K_H\ell^{-1}, \tag{9.49}$$

$$\left|\rho_z\widehat{(\omega W_1)}(0) - \frac{1}{(2\pi)^2}\int_{\mathcal{P}_H}\widehat{W}_1(k)\alpha_k \mathrm{d}k\right| \le C\rho_z\delta^2|\log\delta|, \tag{9.50}$$

$$\left|\widehat{(\omega W_1)}(0) - \frac{1}{(2\pi)^2}\int_{\mathcal{P}_H}\frac{\widehat{W}_1(k)^2}{2\mathcal{D}_k}\mathrm{d}k\right| \le C\delta^2|\log\delta|, \tag{9.51}$$

and

$$\rho_z \frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{(W_1\omega)}(k) a_k^\dagger a_k \mathrm{d}k \geq \rho_z \widehat{(W_1\omega)}(0) \frac{\ell^2}{(2\pi)^2} \int_{\mathbb{R}^2} a_k^\dagger a_k \mathrm{d}k - 4\rho_z \delta n_+^H - C\rho_z \delta d^{-2} \frac{R}{\ell} n_+. \tag{9.52}$$

Proof. The first two inequalities are straightforward from the definitions of the terms. For the third one we split the difference in the following way,

$$\begin{aligned} &\left|\rho_z \widehat{(\omega W_1)}(0) - \frac{1}{(2\pi)^2} \int_{k\in\mathcal{P}_H} \widehat{W}_1(k)\alpha_k \mathrm{d}k\right| \\ &\leq C\left|\rho_z \int_{k\notin\mathcal{P}_H} \frac{\widehat{W}_1(k)\widehat{g}_k - \widehat{W}_1(0)\widehat{g}_0 \mathbb{1}_{\{|k|\leq \ell_\delta^{-1}\}}}{2k^2} \mathrm{d}k\right| + C\left|\int_{k\in\mathcal{P}_H} \widehat{W}_1(k)\Big(\alpha_k - \rho_z \frac{\widehat{g}_k}{2k^2}\Big) dk\right| \\ &=: (I) + (II). \end{aligned} \tag{9.53}$$

For the first integral we do a further splitting of the domain of integration, considering $(I) \leq (I, <) + (I, >)$ for $|k| \leq \ell_\delta^{-1}$ or otherwise, respectively. For $(I, <)$ we consider a Taylor expansion of the numerator and we get, recalling the symmetry of g which in the integration drops the first order,

$$(I, <) \leq C\rho_z R^2 \delta^2 \int_{\{|k|\leq \ell_\delta^{-1}\}} \leq C\rho_z R^2 \delta^2 \ell_\delta^{-2}. \tag{9.54}$$

For the $(I, >)$ we proceed by a direct calculation and obtain

$$(I, >) \leq C\rho_z \delta^2 \log(K_H \ell^{-1} \ell_\delta). \tag{9.55}$$

Let us analyze the second integral. We have that $|\mathcal{B}_k/\mathcal{A}_k| \leq 1/2$ and therefore we can expand in the following way

$$\widehat{W}_1(k)\alpha_k = \rho_z^{-1}\mathcal{A}_k\left(1 - \sqrt{1 - \frac{\mathcal{B}_k^2}{\mathcal{A}_k^2}}\right) \simeq \rho_z \frac{\widehat{W}_1(k)^2}{2\mathcal{A}_k} + C\rho_z^3 \frac{\widehat{W}_1(k)^4}{\mathcal{A}_k^3}. \tag{9.56}$$

We deduce

$$\begin{aligned} (II) &\leq C\left|\int_{k\in\mathcal{P}_H} \Big(\widehat{W}_1(k)\alpha_k - \rho_z \frac{\widehat{W}_1(k)^2}{2\mathcal{A}_k}\Big)\mathrm{d}k\right| + C\rho_z\left|\int_{k\in\mathcal{P}_H} \widehat{W}_1(k)\Big(\frac{\widehat{W}_1(k)}{2\mathcal{A}_k} - \frac{\widehat{g}_k}{2k^2}\Big)\mathrm{d}k\right| \\ &\leq C\rho_z^3 \int_{k\in\mathcal{P}_H} \frac{\widehat{W}_1(k)^4}{\mathcal{A}_k^3}\mathrm{d}k + C\rho_z\left|\int_{k\in\mathcal{P}_H} \widehat{W}_1(k)\Big(\frac{\widehat{W}_1(k)}{2\mathcal{A}_k} - \frac{\widehat{g}_k}{2k^2}\Big)\mathrm{d}k\right| \\ &\leq C\rho_z^3 \ell^4 \delta^4 K_H^{-4} + C\rho_z\left|\int_{k\in\mathcal{P}_H} \widehat{W}_1(k)^2\Big(\frac{1}{2\mathcal{A}_k} - \frac{1}{|k|^2}\Big)\mathrm{d}k\right| \\ &\quad + C\rho_z\left|\int_{k\in\mathcal{P}_H} \Big(\widehat{W}_1(k)\frac{\widehat{W}_1(k) - \widehat{g}_k}{2k^2}\Big)\mathrm{d}k\right|, \end{aligned}$$

where we used that $\mathcal{A}_k \geq \frac{1}{2}|k|^2$ for $k \in \mathcal{P}_H$. For the remaining terms, we use that in $\mathcal{P}_H$ we have $0 < k^2 - \tau_k \leq 2|k|(ds\ell)^{-1}$,

$$C\rho_z\left|\int_{k\in\mathcal{P}_H}\frac{\widehat{W}_1(k)^2}{k^2}\left(\frac{k^2-\mathcal{A}_k}{\mathcal{A}_k}\right)\mathrm{d}k\right| \leq C\rho_z\int_{k\in\mathcal{P}_H}\frac{\widehat{W}_1(k)^2}{k^2}\left(\frac{2|k|(ds\ell)^{-1}}{k^2}+\rho_z\frac{\widehat{W}_1(k)}{k^2}\right)$$
$$\leq C\rho_z\delta^2(ds)^{-1}K_H^{-1} + C\rho_z^2\ell^2\delta^3K_H^{-2}.$$

By Cauchy–Schwarz inequality we get for the last term

$$\rho_z\left|\int_{k\in\mathcal{P}_H}\left(\widehat{W}_1(k)\frac{\widehat{W}_1(k)-\widehat{g}_k}{2k^2}\right)\mathrm{d}k\right|$$
$$\leq C\rho_z\delta\int_{k\in\mathcal{P}_H}\frac{\widehat{W}_1(k)^2}{2k^2}\mathrm{d}k + C\rho_z\delta^{-1}\int_{k\in\mathcal{P}_H}\frac{(\widehat{W}_1(k)-\widehat{g}_k)^2}{2k^2}\mathrm{d}k.$$

We complete the domain of the integrals: by Lemma 6.4 we get

$$\rho_z\delta\int_{k\in\mathcal{P}_H}\frac{\widehat{W}_1(k)^2}{2k^2}\mathrm{d}k \leq C\rho_z\delta\widehat{g\omega}(0) + C\rho_z\delta\int_{k\notin\mathcal{P}_H}\frac{\widehat{W}_1(k)^2-\widehat{W}_1(0)^2\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2k^2}\mathrm{d}k$$
$$\leq C\rho_z\delta^2 + C\rho_z\delta^3(R^2\ell_\delta^{-2}+\log(K_H\ell^{-1}\ell_\delta)),$$

and

$$\rho_z\delta^{-1}\int_{k\in\mathcal{P}_H}\frac{(\widehat{W}_1(k)-\widehat{g}_k)^2}{2k^2}\mathrm{d}k$$
$$\leq C\rho_z\delta^{-1}\frac{R^4}{\ell^4}\widehat{g\omega}(0) + C\rho_z\delta^{-1}\left|\int_{k\notin\mathcal{P}_H}\frac{(\widehat{W}_1(k)-\widehat{g}_k)^2-(\widehat{W}_1(0)-\widehat{g}_0)^2\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2k^2}\mathrm{d}k\right|$$
$$\leq C\rho_z\frac{R^4}{\ell^4} + C\rho_z\delta\left(R^2\ell_\delta^{-2}+\frac{R^2}{\ell^2}\log(K_H\ell^{-1}\ell_\delta)\right).$$

We conclude the proof of (9.50) by collecting all the previous estimates and exploiting the relations between the parameters so that $\rho_z\delta^2|\log\delta|$ is the dominant term.

For the inequality (9.51), we can derive it from (9.50) and the control on the first term of (II) above using that, for $k \in \mathcal{P}_H$, $\left|1-\frac{\mathcal{A}_k}{\mathcal{D}_k}\right| \leq \frac{\mathcal{B}_k^2}{\mathcal{A}_k} \leq C\rho_z^2\delta^2|k|^{-4}$.

For the last inequality, we estimate the difference, splitting the integral for $|k| \leq \xi\ell^{-1}$ or otherwise,

$$\rho_z\frac{\ell^2}{(2\pi)^2}\int_{\mathbb{R}^2}((\widehat{W_1\omega})(k)-(\widehat{W_1\omega})(0))a_k^\dagger a_k\mathrm{d}k$$
$$\geq -C\rho_z\xi^2\delta\frac{R^2}{\ell^2}n_+ - \frac{2\ell^2}{(2\pi)^2}\rho_z\delta\int_{\mathbb{R}^2}a_k^\dagger\mathbb{1}_{\{|k|\geq\xi\ell^{-1}\}}a_k\mathrm{d}k$$

where we used a Taylor expansion and estimated the integral for $|k| \leq \xi\ell^{-1}$. For the second term we exploit the second quantization in a N-bosons sector and we insert symmetrically the sum of projectors $1 = \mathbb{1}_{\{\sqrt{-\Delta}\in\mathcal{P}_L\}} + \mathbb{1}_{\{\sqrt{-\Delta}\in\mathcal{P}_L^c\}}$

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathbb{R}^2}a_k^\dagger\mathbb{1}_{\{|k|\geq\xi\ell^{-1}\}}a_k\mathrm{d}k\Big|_N = \sum_{j=1}^N Q_j\chi_\Lambda(x_j)\mathbb{1}_{\{\sqrt{-\Delta_j}\geq\xi\ell^{-1}\}}\chi_\Lambda(x_j)Q_j$$
$$\geq 2n_+^H + 2\mathcal{N}n_+$$

where we estimated by a Cauchy–Schwarz the cross terms $(\mathcal{P}_L, \mathcal{P}_L^c)$ to make them comparable to the diagonal terms and denoted by

$$\mathcal{N} := \|\mathbb{1}_{\{\sqrt{-\Delta}\in\mathcal{P}_L\}}\chi_\Lambda(x)\mathbb{1}_{\{\sqrt{-\Delta}\geq\xi\ell^{-1}\}}\|^2 \leq C\xi^{-2}d^{-4}. \tag{9.57}$$

Here we used the regularity properties of χ_Λ dividing and multiplying by $-\Delta$. We conclude optimizing ξ by the choice $\xi^2 = d^{-2}\frac{\ell}{R}$. □

9.3.1. Estimates on $\mathcal{Q}_3^{(1)}$ The first part $\mathcal{Q}_3^{(1)}$ will absorb $\mathcal{Q}_2^{\text{ex}}$ using $\mathcal{K}_H^{\text{Diag}}$.

Lemma 9.7 (Estimates on $\mathcal{Q}_3^{(1)}$). *For any state* Φ *satisfying* (8.44) *we have*

$$\begin{aligned}&\left\langle \mathcal{Q}_3^{(1)} + \mathcal{Q}_2^{ex} + \big(1-2\varepsilon_K\big)\mathcal{K}_H^{Diag} + \frac{b}{100}\frac{n_+}{\ell^2} + \frac{b}{100}\frac{\varepsilon_T n_+^H}{(d\ell)^2}\right\rangle_\Phi \\ &\quad \geq -C\rho_z^2\ell^2\delta^2\Big(\delta K_H^{-8}K_\ell^{10}d^{-4} + \varepsilon_K^{-1}K_H^{-12}K_\ell^{10}d^{-8} + d^{-8}K_\ell^2K_H^{-4}\Big).\end{aligned}$$

Proof. We first reorder the creation an annihilation operators, applying a change of variables $k \mapsto -k$, $p \mapsto -p$ in the α terms,

$$\begin{aligned}\mathcal{Q}_3^{(1)} &= \frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}} \\ &\quad\times\big(\widetilde{a}_p^\dagger b_{p-k}b_k + \alpha_k\alpha_{p-k}\widetilde{a}_{-p}^\dagger b_{p-k}^\dagger b_k^\dagger + b_k^\dagger b_{p-k}^\dagger\widetilde{a}_p + \alpha_k\alpha_{p-k}b_kb_{p-k}\widetilde{a}_{-p}\big)\mathrm{d}k\mathrm{d}p \\ &= \frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\Big(\big(\widetilde{a}_p^\dagger b_{p-k} + \alpha_k\alpha_{p-k}b_{p-k}\widetilde{a}_{-p}\big)b_k \\ &\quad + b_k^\dagger\big(b_{p-k}^\dagger\widetilde{a}_p + \alpha_k\alpha_{p-k}\widetilde{a}_{-p}^\dagger b_{p-k}^\dagger\big) \\ &\quad + \alpha_k\alpha_{p-k}\big(\big[b_k, b_{p-k}\widetilde{a}_{-p}\big] + \big[\widetilde{a}_{-p}^\dagger b_{p-k}^\dagger, b_k^\dagger\big]\big)\Big)\mathrm{d}k\mathrm{d}p.\end{aligned}$$

We can complete the square to get, for $\varepsilon_K \ll 1$ fixed in "Appendix H",

$$\begin{aligned}\mathcal{Q}_3^{(1)} + (1-3\varepsilon_K)\mathcal{K}_H^{\text{Diag}} &= (1-3\varepsilon_K)\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{D}_k\tilde{b}_k^\dagger\tilde{b}_k\mathrm{d}k \\ &\quad + \frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\big(\mathcal{T}_1(k) + \mathcal{T}_2(k)\big)\mathrm{d}k,\end{aligned} \tag{9.58}$$

where we mantained a small portion of $\mathcal{K}_H^{\mathrm{Diag}}$ in order to bound other error terms and we defined

$$\tilde{b}_k := b_k + \frac{z}{\mathcal{D}_k(1-3\varepsilon_K)(2\pi)^2} \int \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}} \Big(b_{p-k}^\dagger \widetilde{a}_p + \alpha_k\alpha_{p-k}\widetilde{a}_{-p}^\dagger b_{p-k}^\dagger\Big)\mathrm{d}p, \tag{9.59}$$

$$\mathcal{T}_1(k) := \frac{z}{(2\pi)^2}\int_{\mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\alpha_k\alpha_{p-k}\big([b_k^\dagger, \widetilde{a}_{-p}^\dagger b_{p-k}^\dagger] + h.c.\big)\mathrm{d}p, \tag{9.60}$$

$$\mathcal{T}_2(k) := -\frac{|z|^2\widehat{W}_1(k)^2}{(1-3\varepsilon_K)\mathcal{D}_k(1-\alpha_k^2)(2\pi)^4}\int \frac{f_L(p)f_L(s)}{\sqrt{1-\alpha_{s-k}^2}\sqrt{1-\alpha_{p-k}^2}} \times \big(\widetilde{a}_p^\dagger b_{p-k} + \alpha_k\alpha_{p-k}b_{p-k}\widetilde{a}_{-p}\big)\big(b_{s-k}^\dagger\widetilde{a}_s + \alpha_k\alpha_{s-k}\widetilde{a}_{-s}^\dagger b_{s-k}^\dagger\big)\mathrm{d}p\mathrm{d}s. \tag{9.61}$$

• Let us estimate the error term $\mathcal{T}_1(k)$. We use $\big[b_k^\dagger, \widetilde{a}_{-p}^\dagger b_{p-k}^\dagger\big] = \widetilde{a}_{-p}^\dagger\big[b_k^\dagger, b_{p-k}^\dagger\big] + \big[b_k^\dagger, \widetilde{a}_{-p}^\dagger\big]b_{p-k}^\dagger$ and the Cauchy–Schwarz inequality with weights $\varepsilon_1, \varepsilon_2 > 0$,

$$\mathcal{T}_1(k) \geq -Cz\int_{\mathbb{R}^2} \frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}|\alpha_k\alpha_{p-k}| \Big(\big(\varepsilon_1\widetilde{a}_{-p}^\dagger\widetilde{a}_{-p} + \varepsilon_1^{-1}\big)|[b_k^\dagger, b_{p-k}^\dagger]| + |[b_k^\dagger, \widetilde{a}_{-p}^\dagger]|\big(\varepsilon_2 b_{p-k}^\dagger b_{p-k} + \varepsilon_2^{-1}\big)\Big)\mathrm{d}p.$$

By (9.33) and (9.31) we have $|[b_k^\dagger, \widetilde{a}_{-p}^\dagger]| \leq C|\alpha_k|$ and $|[b_k^\dagger, b_{p-k}^\dagger]| \leq C|\alpha_k|$. Therefore using (9.48),

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{T}_1(k)\mathrm{d}k \geq -C\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\frac{|f_L(p)|\rho_z^3\delta^4}{k^6} \times\Big((\varepsilon_1\widetilde{a}_{-p}^\dagger\widetilde{a}_{-p} + \varepsilon_1^{-1}) + (\varepsilon_2 b_{p-k}^\dagger b_{p-k} + \varepsilon_2^{-1})\Big)\mathrm{d}k\mathrm{d}p.$$

Due to the presence of the cutoff f_L on low momenta and the bounds

$$\int_{\mathcal{P}_L}(\varepsilon_1\widetilde{a}_{-p}^\dagger\widetilde{a}_{-p} + \varepsilon_1^{-1})\mathrm{d}p \leq C\frac{\varepsilon_1 n_+}{\ell^2} + \varepsilon_1^{-1}\frac{d^{-4}}{\ell^2}, \tag{9.62}$$

$$\int_{\mathcal{P}_L}(\varepsilon_2 b_k^\dagger b_k + \varepsilon_2^{-1})\mathrm{d}p \leq C\frac{d^{-4}}{\ell^2}(\varepsilon_2 b_k^\dagger b_k + \varepsilon_2^{-1}), \tag{9.63}$$

where we changed the k variable, we find,

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{T}_1(k)\mathrm{d}k \geq -Cz\rho_z^3\delta^4\int_{\mathcal{P}_H}\frac{1}{k^6}\big((\varepsilon_1 n_+ + \varepsilon_1^{-1}d^{-4}) + d^{-4}(\varepsilon_2 b_k^\dagger b_k + \varepsilon_2^{-1})\big)\mathrm{d}k.$$

We insert $\mathcal{D}_k \geq C^{-1}k^2$ in front of $b_k^\dagger b_k$ and get the bound

$$\begin{aligned}
&\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{T}_1(k)\mathrm{d}k\\
&\geq -Cz\rho_z^3\delta^4\ell^6 K_H^{-4}\varepsilon_1\frac{n_+}{\ell^2} - Cz\varepsilon_1^{-1}d^{-4}\rho_z^3\delta^4\ell^4 K_H^{-4}\\
&\quad - C\varepsilon_2\ell^6 z\rho_z^3\delta^4 K_H^{-8}d^{-4}\ell^2\int_{\mathcal{P}_H}\mathcal{D}_k b_k^\dagger b_k\mathrm{d}k - C\varepsilon_2^{-1}\ell^4 z\rho_z^3\delta^4 d^{-4}K_H^{-4}.
\end{aligned}$$

One can choose ε_1, ε_2 such that the first and third terms are absorbed in the positive $\frac{b}{100}\frac{n_+}{\ell^2}$ and $\varepsilon_K\mathcal{K}_H^{\mathrm{Diag}}$ respectively. With this choice the second and fourth terms are errors of respective sizes

$$C\ell^2\rho_z^2\delta^2(\delta K_\ell^{10}K_H^{-8}d^{-4}) \quad \text{and} \quad C\ell^2\rho_z^2\delta^2(\delta K_\ell^{10}K_H^{-12}d^{-4}\varepsilon_K^{-1}).$$

• Let us now focus on the square term $\mathcal{T}_2(k)$ in (9.61). One can write, in normal order,

$$\widetilde{a}_p^\dagger b_{p-k} + \alpha_k\alpha_{p-k}b_{p-k}\widetilde{a}_{-p} = \widetilde{a}_p^\dagger b_{p-k} + \alpha_k\alpha_{p-k}\widetilde{a}_{-p}b_{p-k} + \alpha_k\alpha_{p-k}[b_{p-k},\widetilde{a}_{-p}],$$

and use the Cauchy–Schwarz inequality with weight ε_K on the cross terms to find

$$\mathcal{T}_2(k) \geq (1+\varepsilon_K)\mathcal{T}_2'(k) + (1+\varepsilon_K^{-1})\mathcal{T}_2''(k), \tag{9.64}$$

with

$$\begin{aligned}
\mathcal{T}_2'(k) &= -\frac{|z|^2\widehat{W}_1(k)^2}{(1-3\varepsilon_K)\mathcal{D}_k(1-\alpha_k^2)(2\pi)^4}\int\frac{f_L(p)f_L(s)}{\sqrt{1-\alpha_{p-k}^2}\sqrt{1-\alpha_{s-k}^2}}\\
&\quad\times(\widetilde{a}_p^\dagger + \alpha_k\alpha_{p-k}\widetilde{a}_{-p})b_{p-k}b_{s-k}^\dagger(\widetilde{a}_s + \alpha_k\alpha_{s-k}\widetilde{a}_{-s}^\dagger)\mathrm{d}p\mathrm{d}s,\\
\mathcal{T}_2''(k) &= -\frac{|z|^2\widehat{W}_1(k)^2}{(1-3\varepsilon_K)\mathcal{D}_k(1-\alpha_k^2)(2\pi)^4}\int\frac{f_L(p)f_L(s)}{\sqrt{1-\alpha_{s-k}^2}\sqrt{1-\alpha_{p-k}^2}}\\
&\quad\times\alpha_k^2\alpha_{p-k}\alpha_{s-k}|[b_{p-k},\widetilde{a}_{-p}]||[\widetilde{a}_{-s}^\dagger, b_{s-k}^\dagger]|\mathrm{d}p\mathrm{d}s.
\end{aligned}$$

$\mathcal{T}_2''$ we can estimate (for $k\in\mathcal{P}_H$),

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{T}_2''(k)\mathrm{d}k \geq -C\rho_z\ell^4\Big(\int_{\mathcal{P}_H}\frac{\widehat{W}_1(k)^2}{(1-3\varepsilon_K)\mathcal{D}_k}|\alpha_k|^4\mathrm{d}k\Big)d^{-8}\ell^{-4}\sup|[b_{p-k},\widetilde{a}_{-s}]|^2,$$

and by (9.51), (9.33) and (9.48) we get

$$(1+\varepsilon_K^{-1})\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{T}_2''(k)\mathrm{d}k \geq -C\rho_z^2\ell^2\delta^2(\varepsilon_K^{-1}K_H^{-12}K_\ell^{10}d^{-8}). \tag{9.65}$$

Now we use a commutator to write $\mathcal{T}_2' = \mathcal{T}_{2,\mathrm{op}}' + \mathcal{T}_{2,\mathrm{com}}'$ in normal order for the b_k, with

$$\mathcal{T}_{2,\mathrm{op}}'(k) = -\frac{|z|^2 \widehat{W}_1(k)^2}{(2\pi)^4(1-3\varepsilon_K)\mathcal{D}_k(1-\alpha_k^2)} \int \frac{f_L(p) f_L(s)}{\sqrt{1-\alpha_{p-k}^2}\sqrt{1-\alpha_{s-k}^2}} \times (\widetilde{a}_p^\dagger + \alpha_k \alpha_{p-k} \widetilde{a}_{-p}) b_{s-k}^\dagger b_{p-k} (\widetilde{a}_s + \alpha_k \alpha_{s-k} \widetilde{a}_{-s}^\dagger) \mathrm{d}p \mathrm{d}s,$$
$$\mathcal{T}_{2,\mathrm{com}}'(k) = -\frac{|z|^2 \widehat{W}_1(k)^2}{(2\pi)^4(1-3\varepsilon_K)\mathcal{D}_k(1-\alpha_k^2)} \int \frac{f_L(p) f_L(s)}{\sqrt{1-\alpha_{p-k}^2}\sqrt{1-\alpha_{s-k}^2}} \times (\widetilde{a}_p^\dagger + \alpha_k \alpha_{p-k} \widetilde{a}_{-p}) [b_{p-k}, b_{s-k}^\dagger] (\widetilde{a}_s + \alpha_k \alpha_{s-k} \widetilde{a}_{-s}^\dagger) \mathrm{d}p \mathrm{d}s. \tag{9.66}$$

• In order to estimate the error term $\mathcal{T}_{2,\mathrm{op}}'$, we introduce

$$\tau_s := \widetilde{a}_s + \alpha_k \alpha_{s-k} \widetilde{a}_{-s}^\dagger \quad \text{and} \quad \mathcal{C} := \sup_{p,s \in \mathcal{P}_L, k \in \mathcal{P}_H} \| [b_{p-k}, \tau_s] \|. \tag{9.67}$$

In $\mathcal{T}_{2,\mathrm{op}}'$ we commute the b's trough the a's,

$$\tau_p^\dagger b_{s-k}^\dagger b_{p-k} \tau_s = b_{s-k}^\dagger \tau_p^\dagger \tau_s b_{p-k} + [\tau_p^\dagger, b_{s-k}^\dagger] \tau_s b_{p-k} + b_{s-k}^\dagger \tau_p^\dagger [b_{p-k}, \tau_s] + [\tau_p^\dagger, b_{s-k}^\dagger][b_{p-k}, \tau_s],$$

and use the Cauchy–Schwarz inequality

$$\tau_p^\dagger b_{s-k}^\dagger b_{p-k} \tau_s \leq C (b_{s-k}^\dagger \tau_p^\dagger \tau_p b_{s-k} + b_{p-k}^\dagger \tau_s^\dagger \tau_s b_{p-k} + \mathcal{C}^2).$$

Inserting it in $\mathcal{T}_{2,\mathrm{op}}'$, bounding $(1-3\varepsilon_K)(1-\alpha_k) \geq 1/2$ and noticing that we can exchange s and p in the integral, we find

$$\mathcal{T}_{2,\mathrm{op}}'(k) \geq -C \frac{|z|^2 \widehat{W}_1(k)^2}{\mathcal{D}_k} \int \frac{f_L(p) f_L(s)}{\sqrt{1-\alpha_{p-k}^2}\sqrt{1-\alpha_{s-k}^2}} (b_{s-k}^\dagger \tau_p^\dagger \tau_p b_{s-k} + \mathcal{C}^2) \mathrm{d}p \mathrm{d}s.$$

When we apply this operator to the state Φ which satisfies $\mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Phi = \Phi$ we can apply Lemma 8.1 for the vector $b_{s-k}\Phi$ to get the estimate

$$\langle \mathcal{T}_{2,\mathrm{op}}'(k) \rangle_\Phi \geq -C \frac{|z|^2 \widehat{W}_1(k)^2}{\mathcal{D}_k} \Big(\ell^{-2} \mathcal{M} \int f_L(s) \langle b_{s-k}^\dagger b_{s-k} \rangle_\Phi \mathrm{d}s + d^{-8} \ell^{-4} \mathcal{C}^2 \Big),$$

and finally, using again (9.51) and (9.49), and the fact that $\mathcal{C} \leq C K_\ell^2 K_H^{-2}$ by (9.33) and Lemma F.1,

$$\frac{\ell^2}{(2\pi)^2} \int_{\mathcal{P}_H} \langle \mathcal{T}_{2,\mathrm{op}}'(k) \rangle_\Phi \mathrm{d}k \geq -C\rho_z \ell^2 \delta^2 K_H^{-4} d^{-4} \mathcal{M} \langle \mathcal{K}_H^{\mathrm{Diag}} \rangle_\Phi - C\rho_z \delta d^{-8} K_\ell^4 K_H^{-4} \geq -C K_\ell^4 K_H^{-4} d^{-4} \varepsilon_{\mathcal{M}} \mathcal{K}_H^{\mathrm{Diag}} - C\rho_z^2 \ell^2 \delta^2 d^{-8} K_\ell^2 K_H^{-4}. \tag{9.68}$$

The first part can be absorbed in the positive $\varepsilon_K \mathcal{K}_H^{\mathrm{Diag}}$, as long as the relation (H21) holds, and the second part contributes to the error term.

• We now turn to $\mathcal{T}'_{2,\mathrm{com}}$ given in (9.66). This term will absorb $\mathcal{Q}_2^{ex}$. We first use Lemma F.1, (9.32) and (9.48) to estimate the commutator,

$$
\begin{aligned}
\left|[\, b_{p-k}, b^\dagger_{s-k}] - \widehat{\chi^2}((p-s)\ell)\right| &= \left|\alpha_{p-k}\alpha_{s-k}\widehat{\chi^2}((p-s)\ell)\right| \\
&\quad + \left|(1-\alpha_{p-k}\alpha_{s-k})\widehat{\chi}((p-k)\ell)\widehat{\chi}((s-k)\ell)\right| \\
&\leq C K_\ell^4 K_H^{-4},
\end{aligned}
$$

and bounding then by a Cauchy–Schwarz inequality

$$
\begin{aligned}
&(\widetilde{a}_p^\dagger + \alpha_k\alpha_{p-k}\widetilde{a}_{-p})(\widehat{(\chi^2)}((p-s)\ell) + C K_\ell^4 K_H^{-4})(\widetilde{a}_s + \alpha_k\alpha_{s-k}\widetilde{a}^\dagger_{-s}) \\
&\quad \leq \widetilde{a}_p^\dagger\widehat{\chi^2}((p-s)\ell)\widetilde{a}_s + C(\widetilde{a}^\dagger_{-p}\widetilde{a}_{-p} + \widetilde{a}^\dagger_s\widetilde{a}_s)K_\ell^4 K_H^{-4}.
\end{aligned}
$$

We get, by using Lemma 8.1

$$
\begin{aligned}
\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{T}'_{2,\mathrm{com}}(k)\mathrm{d}k \geq &-\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\frac{|z|^2\widehat{W}_1(k)^2}{(2\pi)^4(1-3\varepsilon_K)\mathcal{D}_k(1-\alpha_k^2)} \\
&\times\int\frac{f_L(p)f_L(s)}{\sqrt{1-\alpha_{p-k}^2}\sqrt{1-\alpha_{s-k}^2}}\widetilde{a}_p^\dagger\widehat{\chi^2}((p-s)\ell)\widetilde{a}_s\mathrm{d}p\mathrm{d}s\mathrm{d}k \\
&-C\Big(\int_{\mathcal{P}_H}\frac{|z|^2\widehat{W}_1(k)^2}{\mathcal{D}_k}\mathrm{d}k\Big)d^{-4}K_\ell^4K_H^{-4}\frac{n_+}{\ell^2}.
\end{aligned}
$$

Using (9.49) the last part is of order $K_\ell^6 K_H^{-4} d^{-4}\frac{n_+}{\ell^2}$ and can be absorbed in a fraction of the positive $\frac{b}{100}\frac{n_+}{\ell^2}$ by (H8). For the first term we use the following formula valid in a Fock sector with N bosons

$$
\frac{\ell^4}{(2\pi)^4}\int f_L(p)f_L(s)\widetilde{a}_p^\dagger\widehat{\chi^2}((p-s)\ell)a_s\mathrm{d}s\mathrm{d}p|_N = \sum_{j=1}^N Q^\dagger_{L,j}\chi^2_\Lambda(x_j)Q_{L,j}, \tag{9.69}
$$

to rewrite, due to (H8) and by (9.49),

$$
\begin{aligned}
&\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H}\mathcal{T}'_{2,\mathrm{com}}(k)\mathrm{d}k \\
&\quad\geq -\frac{(1+C\varepsilon_K)}{(2\pi)^2}\int\frac{\rho_z\widehat{W}_1(k)^2}{\mathcal{D}_k(1-\alpha_k^2)}\mathrm{d}k\sum_{j=1}^N Q^\dagger_{L,j}\chi^2_\Lambda(x_j)Q_{L,j} - \frac{b}{200}\frac{n_+}{\ell^2} \\
&\quad\geq -(1+C\varepsilon_K + CK_\ell^4K_H^{-4}) \\
&\qquad\times\Big(2\rho_z\widehat{(\omega W_1)}(0) + C\rho_z\delta^2|\log\delta|\Big)\sum_{j=1}^N Q^\dagger_{L,j}\chi^2_\Lambda Q_{L,j} - \frac{b}{200}\frac{n_+}{\ell^2}.
\end{aligned}
$$

In this last expression we want to replace $Q_{L,j}$ by Q_j. Using Cauchy–Schwarz with weight ε_0 we find

$$
Q^\dagger_{L,j}\chi^2_\Lambda Q_{L,j} \leq (1+\varepsilon_0)Q_j\chi^2_\Lambda Q_j + (1+\varepsilon_0^{-1})Q_j(f_L-1)\chi_j^2(f_L-1)Q_j, \tag{9.70}
$$

and since f_L localizes on low momenta we can bound the second term by n_+^H, and the term $\varepsilon_0 Q_j \chi_j^2 Q_j$ by $C\varepsilon_0 n_+$,

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H} \mathcal{T}'_{2,\mathrm{com}}(k)\mathrm{d}k \geq -2\rho_z\widehat{(\omega W_1)}(0)\sum_{j=1}^{N}\left(Q_j\chi_j^2 Q_j + C\varepsilon_0^{-1}n_+^H + C\varepsilon_0 n_+\right)$$
$$- C\Big(\rho_z\ell^2\delta^2|\log\delta| + \rho_z\ell^2\delta\varepsilon_K + \rho_z\ell^2\delta K_\ell^4 K_H^{-4} + \frac{b}{200}\Big)\frac{n_+}{\ell^2}.$$

The n_+^H-part can be absorbed by the positive $\frac{b}{100}\frac{\varepsilon_T n_+^H}{(d\ell)^2}$ if we choose $\varepsilon_0 \simeq \frac{\rho_z\delta d^2\ell^2}{\varepsilon_T} \simeq \frac{d^2K_\ell^2}{\varepsilon_T}$. With this choice the n_+ terms are of order

$$\Big(\frac{d^2K_\ell^4}{\varepsilon_T} + \delta K_\ell^2|\log\delta| + K_\ell^2\varepsilon_K + K_\ell^6 K_H^{-4} + \frac{b}{200}\Big)\frac{n_+}{\ell^2}. \tag{9.71}$$

Those terms are absorbed in a fraction of the positive $\frac{b}{100}\frac{n_+}{\ell^2}$, as long as we have the relations (H8), (H10), (H17) and (H20). We deduce

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H} \mathcal{T}'_{2,\mathrm{com}}(k)\mathrm{d}k \geq -2\rho_z\widehat{(\omega W_1)}(0)\sum_{j=1}^{N} Q_j\chi_j^2 Q_j - \frac{b}{150}\frac{n_+}{\ell^2} - \frac{b}{100}\frac{\varepsilon_T n_+^H}{(d\ell)^2}.$$

To compare the remaining part with $\mathcal{Q}_2^{\mathrm{ex}}$ we use (9.52) to find

$$\begin{aligned}
\mathcal{Q}_2^{\mathrm{ex}} &= \rho_z\frac{\ell^2}{(2\pi)^2}\int\Big(\widehat{W\omega}(k) + \widehat{W\omega}(0)\Big)a_k^\dagger a_k\mathrm{d}k \\
&\geq 2\rho_z\frac{\ell^2}{(2\pi)^2}\widehat{W_1\omega}(0)\int a_k^\dagger a_k \mathrm{d}k - C\rho_z\delta(d^{-2}R\ell^{-1}n_+ + Cn_+^H) \\
&= 2\rho_z\widehat{W_1\omega}(0)\sum_j Q_j\chi_\Lambda^2 Q_j - C\rho_z\delta d^{-2}R\ell^{-1}n_+ + C\rho_z\delta n_+^H.
\end{aligned}$$

Using that $\rho_z \simeq \rho_\mu$, the remaining parts are absorbed by the spectral gaps and then we get

$$\frac{\ell^2}{(2\pi)^2}\int_{\mathcal{P}_H} \mathcal{T}'_{2,\mathrm{com}}(k)\mathrm{d}k + \mathcal{Q}_2^{\mathrm{ex}} \geq -\frac{b}{100}\frac{n_+}{\ell^2} - \frac{b}{100}\frac{\varepsilon_T n_+^H}{(d\ell)^2}.$$

This last estimate, together with (9.58), (9.64), (9.65) and (9.68) concludes the proof. □

Remark 9.8. It was necessary to replace a_k's by b_k's before estimating $\mathcal{Q}_3(z) + \mathcal{Q}_2^{ex}$, otherwise we would need a fraction of the kinetic energy instead of $\mathcal{K}_H^{\mathrm{diag}}$ in Lemma 9.7, and this we cannot allow (see Remark 9.5). In other words, it is important that the positive term in (9.58) is given in terms of $\tilde{b}_k$ (Eq. (9.59)) whose main part is b_k.

9.3.2. Estimates on $\mathcal{Q}_3^{(2)}$ and $\mathcal{Q}_3^{(3)}$

Lemma 9.9 (Estimates on $\mathcal{Q}_3^{(2)}$ and $\mathcal{Q}_3^{(3)}$). *For any normalized state* Φ *satisfying* (8.44) *we have*

$$\Big\langle \mathcal{Q}_3^{(2)} + \mathcal{Q}_3^{(3)} + \frac{\varepsilon_K}{100}\mathcal{K}_H^{Diag}\Big\rangle_\Phi \geq -C\rho_z^2\ell^2\delta^2\varepsilon_K^{-1}K_H^{-2M-8}K_\ell^6 d^{-8}.$$

Proof. We focus on $\mathcal{Q}_3^{(3)}$ (the estimates on $\mathcal{Q}_3^{(2)}$ are similar), and decompose it into $\mathcal{Q}_3^{(3)} = I + II$, where

$$I = -\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\frac{f_L(p)\widehat{W}_1(k)\alpha_{p-k}}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\big(b_{k-p}^\dagger\widetilde{a}_p^\dagger b_k + b_k^\dagger\widetilde{a}_p b_{k-p}\big)\mathrm{d}k\mathrm{d}p, \tag{9.72}$$

and

$$II = -\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\frac{f_L(p)\widehat{W}_1(k)\alpha_{p-k}}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\big([\widetilde{a}_p^\dagger, b_{k-p}^\dagger]b_k + b_k^\dagger[b_{k-p},\widetilde{a}_p]\big)\mathrm{d}k\mathrm{d}p. \tag{9.73}$$

The first part we estimate using Cauchy–Schwarz with weight ε, and by (9.48)

$$\begin{aligned} I &\geq -\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\frac{f_L(p)\widehat{W}_1(k)\alpha_{p-k}}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}(\varepsilon b_{k-p}^\dagger\widetilde{a}_p^\dagger\widetilde{a}_p b_{k-p} + \varepsilon^{-1}b_k^\dagger b_k)\mathrm{d}k\mathrm{d}p \\ &\geq -Cz\ell^2\delta K_\ell^2 K_H^{-2}\int_{\mathcal{P}_H\times\mathbb{R}^2} f_L(p)(\varepsilon b_{k-p}^\dagger\widetilde{a}_p^\dagger\widetilde{a}_p b_{k-p} + \varepsilon^{-1}b_k^\dagger b_k)\mathrm{d}k\mathrm{d}p, \end{aligned}$$

and using Lemma 8.1,

$$\langle\Phi, \int f_L(p)b_{k-p}^\dagger\widetilde{a}_p^\dagger\widetilde{a}_p b_{k-p}\mathrm{d}p\Phi\rangle \leq C\ell^{-2}\mathcal{M}\langle\Phi, b_k^\dagger b_k\Phi\rangle. \tag{9.74}$$

We choose $\varepsilon = \sqrt{d^{-4}/\mathcal{M}}$, and insert $\mathcal{D}_k \geq K_H^2\ell^{-2}$,

$$\begin{aligned} \langle I\rangle_\Phi &\geq -Cz\delta K_\ell^2 K_H^{-2}(\varepsilon\mathcal{M} + \varepsilon^{-1}d^{-4})\int_{k\in\mathcal{P}_H}\langle b_k^\dagger b_k\rangle_\Phi\mathrm{d}k \\ &\geq -C(\rho_z^{1/2}\ell\delta K_\ell^2 K_H^{-4}\mathcal{M}^{1/2}d^{-2})\ell^2\int_{k\in\mathcal{P}_H}\mathcal{D}_k\langle b_k^\dagger b_k\rangle_\Phi\mathrm{d}k. \end{aligned} \tag{9.75}$$

Thanks to condition (H21), I can be absorbed in the positive $\frac{\varepsilon_K}{100}\mathcal{K}_H^{\mathrm{Diag}}$ term. Now we return to the commutator term, which can be estimated using a Cauchy–Schwarz inequality with new weight ε,

$$II \geq -2\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}|[b_{k-p},\widetilde{a}_p]||f_L(p)|\widehat{W}_1(k)\alpha_k|(\varepsilon b_k^\dagger b_k + \varepsilon^{-1})\mathrm{d}k\mathrm{d}p.$$

We use the commutator bound $|[b_{k-p},\widetilde{a}_p]| \leq C\alpha_{k-p}\sup_{k\in\mathcal{P}_H}\widehat{\chi}(k\ell)$ from (9.33),

$$II \geq -CzK_\ell^2K_H^{-2}\Big(\sup_{k\in\mathcal{P}_H}\widehat{\chi}(k\ell)\Big)d^{-4}\Big(\varepsilon K_\ell^2K_H^{-4}\delta\ell^2\int_{\mathcal{P}_H}\mathcal{D}_k b_k^\dagger b_k\mathrm{d}k + \varepsilon^{-1}\int_{\mathcal{P}_H}\widehat{W}_1(k)\alpha_k\mathrm{d}k\Big).$$

With $\varepsilon^{-1} \simeq \varepsilon_K^{-1} z d^{-4} K_\ell^4 K_H^{-6} \delta\big(\sup \widehat{\chi}\big)$ and our choice of parameters, the first part is absorbed in the positive $\varepsilon_K \mathcal{K}^{\mathrm{Diag}}$ term. We estimate the last part using (9.50) and Lemma F.1, and then II contributes with an error of order $\varepsilon_K^{-1} \rho_z^2 \ell^2 \delta^2 K_H^{-2M-8} K_\ell^6 d^{-8}$. □

9.3.3. Estimates on $\mathcal{Q}_3^{(4)}$ First we rewrite $\mathcal{Q}_1^{\mathrm{ex}}$ as a term appearing in $\mathcal{Q}_3^{(4)}$.

Lemma 9.10. *Assume that Assumptions of "Appendix H" are satisfied. Then there exists a universal constant $C > 0$ such that*

$$\mathcal{Q}_1^{ex} + \frac{b}{100}\frac{n_+}{\ell^2} + \frac{b}{100}\frac{\varepsilon_T n_+^H}{(d\ell)^2} \geq \frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \widehat{W}_1(k) \alpha_k \widehat{\chi^2}(p\ell) f_L(p) (\widetilde{a}_p^\dagger + \widetilde{a}_p) \mathrm{d}k \mathrm{d}p$$
$$- C\rho_z^2 \ell^2 \delta^3 K_\ell^2 |\log\delta|^2 - \rho_z^2 \ell^2 \delta K_\ell^2 d^{8M-2} \varepsilon_T^{-1}.$$

Proof. First we can rewrite $\mathcal{Q}_1^{\mathrm{ex}}$ in terms of the $\widetilde{a}_p$'s,

$$\mathcal{Q}_1^{\mathrm{ex}} = z\rho_z \widehat{\omega W_1}(0) \frac{\ell^2}{(2\pi)^2} \int \widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger + \widetilde{a}_p)\mathrm{d}p,$$

and then we use (9.50) to compare $\widehat{\omega W_1}(0)$ with an integral in k, and using the bound $K_\ell^2 K_H^{-2} \ll 1$,

$$\mathcal{Q}_1^{\mathrm{ex}} \geq \frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \widehat{W}_1(k) \alpha_k \widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger + \widetilde{a}_p)\mathrm{d}k\mathrm{d}p$$
$$- C\rho_z \delta^2 |\log\delta| z\ell^2 \int_{\mathbb{R}^2} \widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger + \widetilde{a}_p)\mathrm{d}p. \tag{9.76}$$

The second integral can be estimated using a Cauchy–Schwarz inequality with weight ε,

$$\rho_z \ell^2 \delta^2 z \int_{\mathbb{R}^2} \widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger + \widetilde{a}_p)\mathrm{d}p$$
$$\leq \varepsilon \rho_z \ell^2 \delta^2 z \int |\widehat{\chi^2}(p\ell)| + C\varepsilon^{-1} \rho_z \ell^2 \delta^2 z \int |\widehat{\chi^2}(p\ell)| \widetilde{a}_p^\dagger \widetilde{a}_p \mathrm{d}p$$
$$\leq C\varepsilon\rho_z\delta^2 z + C\varepsilon^{-1}\rho_z\delta^2 z n_+. \tag{9.77}$$

where we used Lemma (F.1). With $\varepsilon \simeq z\delta K_\ell^2 |\log\delta|$, the second part is absorbed by the positive fraction of $\frac{n_+}{\ell^2}$, and the first term is of order $\rho_z^2 \ell^2 \delta^3 K_\ell^2 |\log\delta|$. Hence,

$$\mathcal{Q}_1^{\mathrm{ex}} \geq \frac{z\ell^2}{(2\pi)^4} \int_{\mathcal{P}_H \times \mathbb{R}^2} \widehat{W}_1(k) \alpha_k \widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger + \widetilde{a}_p)\mathrm{d}k\mathrm{d}p$$
$$- C\rho_z^2 \ell^2 \delta^3 K_\ell^2 |\log\delta|^2 - \frac{b}{100}\frac{n_+}{\ell^2}. \tag{9.78}$$

Finally we want to insert the cutoff $f_L(p)$ inside the integral. The error we make is estimated similarly,

$$\begin{aligned}
&\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\widehat{W}_1(k)\alpha_k\widehat{\chi^2}(p\ell)(1-f_L(p))(\widetilde{a}_p^\dagger+\widetilde{a}_p)\mathrm{d}k\mathrm{d}p\\
&\geq -Cz\ell^2\rho_z\delta\int_{\mathcal{P}_L^c}\widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger+\widetilde{a}_p)\mathrm{d}p\\
&\geq -C\varepsilon z\rho_z\delta\ell^2\int_{\mathcal{P}_L^c}|\widehat{\chi^2}(p\ell)|\mathrm{d}p - C\varepsilon^{-1}z\rho_z\delta\ell^2\int_{\mathcal{P}_L^c}|\widehat{\chi^2}(p\ell)|\widetilde{a}_p^\dagger\widetilde{a}_p\mathrm{d}p\\
&\geq -C\varepsilon z\rho_z\delta d^{4M-4} - C\varepsilon^{-1}z\rho_z\delta d^{4M}n_+^H,
\end{aligned}$$

where we used $\sup_{p\in\mathcal{P}_L^c}|\widehat{\chi^2}(p\ell)|\leq Cd^{4M}$. With $\varepsilon\simeq zK_\ell^2d^{4M+2}\varepsilon_T^{-1}$ the first part is of order $\rho_z^2\ell^2\delta K_\ell^2d^{8M-2}\varepsilon_T^{-1}$ and the second is absorbed in a fraction of $\frac{\varepsilon_T n_+^H}{(d\ell)^2}$. □

Now we have all we need to estimate $\mathcal{Q}_3^{(4)}$.

Lemma 9.11 (Estimates on $\mathcal{Q}_3^{(4)}$). *For any state* Φ *satisfying* (8.44) *we have*

$$\begin{aligned}
\left\langle \mathcal{Q}_3^{(4)}+\mathcal{Q}_1^{ex}+\frac{b}{100}\frac{n_+}{\ell^2}+\frac{b}{100}\frac{\varepsilon_T n_+^H}{(d\ell)^2}\right\rangle_\Phi \geq &-C\rho_z^2\ell^2\delta^3K_\ell^2|\log\delta|^2 - C\rho_z^2\ell^2\delta K_\ell^2 d^{8M-2}\varepsilon_T^{-1}\\
&-C\rho_z^2\ell^2\delta^2\varepsilon_{\mathcal{M}}^{1/2}(K_\ell^4K_H^{-4}+\delta^{-1}K_H^{-M}d^{-2}).
\end{aligned}$$

Proof. We use the commutator formula

$$[b_{p-k},b_{-k}^\dagger] = (1-\alpha_k\alpha_{p-k})\Big(\widehat{\chi^2}(p\ell)-\widehat{\chi}(k\ell)\widehat{\chi}((p-k)\ell)\Big),$$

and split into $\mathcal{Q}_3^{(4)} = I + II$, with

$$I = -\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}\frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\alpha_k(1-\alpha_k\alpha_{p-k})\widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger+\widetilde{a}_p)\mathrm{d}k\mathrm{d}p,$$

and

$$\begin{aligned}
II = -\frac{z\ell^2}{(2\pi)^4}\int_{\mathcal{P}_H\times\mathbb{R}^2}&\frac{f_L(p)\widehat{W}_1(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\\
&\times\alpha_k(1-\alpha_k\alpha_{p-k})\widehat{\chi}(k\ell)\widehat{\chi}((p-k)\ell)(\widetilde{a}_p^\dagger+\widetilde{a}_p)\mathrm{d}k\mathrm{d}p.
\end{aligned}$$

In I we recognize the lower bound on $\mathcal{Q}_1^{\mathrm{ex}}$ given by Lemma 9.10 with opposite sign, up to an error term:

$$\begin{aligned}
&I+\mathcal{Q}_1^{\mathrm{ex}}+\frac{b}{100}\frac{n_+}{\ell^2}+\frac{b}{100}\frac{\varepsilon_T n_+^H}{(d\ell)^2}+C\rho_z^2\ell^2\delta^3K_\ell^2|\log\delta|^2+\rho_z^2\ell^2\delta K_\ell^2d^{8M-2}\varepsilon_T^{-1}\\
&\geq -Cz\ell^2\int_{\mathcal{P}_H\times\mathbb{R}^2}f_L(p)\widehat{W}_1(k)\alpha_k^3\widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger+\widetilde{a}_p)\mathrm{d}k\mathrm{d}p.
\end{aligned}$$

This remaining integral can be estimated, by (9.48), as

$$\begin{aligned}&\left|z\ell^2\int_{\mathcal{P}_H\times\mathbb{R}^2} f_L(p)\widehat{W}_1(k)\alpha_k^3\widehat{\chi^2}(p\ell)(\widetilde{a}_p^\dagger+\widetilde{a}_p)\mathrm{d}k\mathrm{d}p\right|\\&\leq C|z|\ell^2\rho_z^3\delta^4\int_{\mathcal{P}_H}k^{-6}\mathrm{d}k\int_{\mathcal{P}_L}|\widehat{\chi^2}(p\ell)|(\widetilde{a}_p^\dagger+\widetilde{a}_p)\mathrm{d}p,\end{aligned}$$

and after applying to the state Φ we use a Cauchy–Schwarz inequality with weight $\sqrt{\mathcal{M}}$,

$$\begin{aligned}&\left|z\ell^2\int_{\mathcal{P}_H\times\mathbb{R}^2} f_L(p)\widehat{W}_1(k)\alpha_k^3\widehat{\chi^2}(p\ell)\langle\widetilde{a}_p^\dagger+\widetilde{a}_p\rangle_\Phi\mathrm{d}k\mathrm{d}p\right|\\&\leq Cz\rho_z^3\ell^6\delta^4K_H^{-4}\Big(\sqrt{\mathcal{M}}\int_{\mathcal{P}_L}|\widehat{\chi^2}(p\ell)|\mathrm{d}p+\frac{1}{\sqrt{\mathcal{M}}}\int_{\mathcal{P}_L}|\widehat{\chi^2}(p\ell)|\langle\widetilde{a}_p^\dagger\widetilde{a}_p\rangle_\Phi\mathrm{d}p\Big)\\&\leq Cz\rho_z^3\ell^4\delta^4K_H^{-4}\sqrt{\mathcal{M}}\leq C\rho_z^2\ell^2\delta^2\big(K_\ell^4K_H^{-4}\sqrt{\varepsilon_{\mathcal{M}}}\big).\end{aligned}$$

Finally we bound, by (F4) and (9.50),

$$\begin{aligned}|\langle II\rangle_\Phi|&\leq z\ell^2\sup_{h\in\mathcal{P}_H}|\widehat{\chi}(h\ell)|\int_{\mathcal{P}_H}|\widehat{W}_1(k)|\alpha_k|\widehat{\chi}(k\ell)|\mathrm{d}k\int_{\mathcal{P}_L}\langle\widetilde{a}_p^\dagger+\widetilde{a}_p\rangle_\Phi\mathrm{d}p\\&\leq Cz\rho_z\ell^2\delta K_H^{-M}\Big(d^2\mathcal{M}^{1/2}\int_{\mathcal{P}_L}\mathrm{d}p+d^{-2}\mathcal{M}^{-1/2}\int_{\mathcal{P}_L}\langle\widetilde{a}_p^\dagger\widetilde{a}_p\rangle_\Phi\mathrm{d}p\Big),\end{aligned}$$

where we used a Cauchy–Schwarz inequality with weight $d^2\sqrt{\mathcal{M}}$. Thus,

$$|\langle II\rangle_\Phi|\leq C\rho_z^2\ell^2\delta K_H^{-M}d^{-2}\varepsilon_{\mathcal{M}}^{1/2}.$$

□

9.4. Conclusion: Proof of Theorem 6.7. In Sect. 6 we showed how the proof of Theorem 2.3 is reduced to the proof of Theorem 6.7, which we give here.

Proof of Theorem 6.7. Recall the choices of the parameters in "Appendix H". Let us consider a normalized n-particle state $\Psi\in\mathscr{F}_s(L^2(\Lambda))$ which satisfies (7.40) for a certain large constant $C_0>0$,

$$\langle\mathcal{H}_\Lambda(\rho_\mu)\rangle_\Psi\leq-4\pi\rho_\mu^2\ell^2Y\big(1-C_0K_B^2Y|\log Y|\big).\tag{9.79}$$

If such a state does not exists, our desired lower bound follows, because

$$-4\pi\rho_\mu^2\ell^2Y\big(1-C_0K_B^2Y|\log Y|\big)\geq-4\pi\rho_\mu^2\ell^2\delta\Big(1-\Big(2\Gamma+\frac{1}{2}+\log\pi\Big)\delta\Big).\tag{9.80}$$

So we can assume the existence of Ψ. By Theorem 7.7 there exists a sequence of n-particle states $\{\Psi^m\}_{m\in\mathbb{Z}}\subseteq\mathscr{F}_s(L^2(\Lambda))$ and $C_1,\eta_1>0$ such that

$$\begin{aligned}\langle\Psi,\mathcal{H}_\Lambda(\rho_\mu)\Psi\rangle\geq&\sum_{2|m|\leq\mathcal{M}}\langle\Psi^{(m)},\mathcal{H}_\Lambda(\rho_\mu)\Psi^m\rangle-C_1\rho_\mu^2\ell^2\delta^{2+\eta_1}\\&-4\pi\rho_\mu^2\ell^2Y\big(1-C_1K_B^2Y|\log Y|\big)\sum_{2|m|>\mathcal{M}}\|\Psi^m\|^2.\end{aligned}$$

For $|m| \leq \frac{\mathcal{M}}{2}$, we have that $\Psi^m = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Psi^m$. If we prove the lower bound for all Ψ^m such that $|m| \leq \frac{\mathcal{M}}{2}$ then we would get (using (9.80) with C_0 replaced by C_1)

$$\langle \Psi, \mathcal{H}_\Lambda(\rho_\mu)\Psi\rangle \geq -4\pi\rho_\mu^2\ell^2\delta\Big(1 - \Big(2\Gamma + \frac{1}{2} + \log\pi\Big)\delta\Big)\sum_m \|\Psi^{(m)}\|^2 - C_1\rho_\mu^2\ell^2\delta^{2+\eta_1},$$

Therefore, the theorem is proven if we derive the corresponding lower bound for any n-particle, normalized state $\widetilde{\Psi} \in \mathscr{F}_s(L^2(\Lambda))$ such that

$$\widetilde{\Psi} = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\widetilde{\Psi}. \tag{9.81}$$

By Proposition 8.3, for such a state there exists a constant $C_2 > 0$ such that

$$\langle\widetilde{\Psi}, \mathcal{H}_\Lambda(\rho_\mu)\widetilde{\Psi}\rangle \geq \langle\widetilde{\Psi}, \mathcal{H}_\Lambda^{\mathrm{2nd}}(\rho_\mu)\widetilde{\Psi}\rangle - C_2\ell^2\rho_\mu^2\delta\big(d^{2M-2} + R^2\ell^{-2}\big), \tag{9.82}$$

where the last term is an error term of order $\rho_\mu^2\ell^2\delta^{2+\eta_2}$, for some $\eta_2 > 0$, thanks to relations (H25) and (H3). Then, by Theorem 8.4, there exists a constant $C_3 > 0$ such that

$$\langle\widetilde{\Psi}, \mathcal{H}_\Lambda^{\mathrm{2nd}}\widetilde{\Psi}\rangle \geq \inf_{z\in\mathbb{R}_+}\inf_{\Phi}\langle\Phi, \mathcal{K}(z)\Phi\rangle - C_3\rho_\mu\delta(1 + \varepsilon_R K_\ell^4 K_B^2|\log Y|), \tag{9.83}$$

where the infimum is over the Φ's which satisfy (8.44). The last term is an error term of order $\rho_\mu^2\ell^2\delta^{2+\eta_3}$ for some $\eta_3 > 0$, thanks to relation (H19). The proof is reduced now to getting a lower bound for $\mathcal{K}(z)$. We have two cases, according to different values of z:

- If $|\rho_z - \rho_\mu| \geq C\rho_\mu \max((\delta_1 + \delta_2 + \delta_3)^{1/2}, \delta^{1/2})$ then Proposition 9.1 implies the bound

$$\langle\mathcal{K}(z)\rangle_\Phi \geq -\frac{1}{2}\rho_\mu^2\ell^2\widehat{g}_0 + 8\pi\Big(2\Gamma + \frac{1}{2} + \log\pi\Big)\rho_\mu^2\ell^2\delta^2, \tag{9.84}$$

and the second term is twice the LHY-term and positive, therefore there is nothing more to prove;
- Otherwise $|\rho_z - \rho_\mu| \leq \rho_\mu K_\ell^{-2}$ (see Sect. 9.2). In this case we can use (9.27) and Theorem 9.3 to obtain $C_4, \eta_4 > 0$, such that

$$\begin{aligned}\langle\mathcal{K}(z)\rangle_\Phi \geq{}& -\frac{1}{2}\rho_\mu^2\ell^2\widehat{g}_0 + (1-\varepsilon_K)\langle\mathcal{K}_H^{\mathrm{Diag}}\rangle_\Phi + 4\pi\Big(2\Gamma + \frac{1}{2} + \log\pi\Big)\rho_z^2\ell^2\delta^2\\ &+\Big\langle b\frac{n_+}{4\ell^2} + b\frac{\varepsilon_T n_+^H}{8d^2\ell^2} + \mathcal{Q}_1^{\mathrm{ex}}(z) + \mathcal{Q}_2^{\mathrm{ex}} + \mathcal{Q}_3(z)\Big\rangle_\Phi - C_4\rho_\mu^2\ell^2\delta^{2+\eta_4},\end{aligned} \tag{9.85}$$

where we used that $C\rho_\mu^2\ell^2\delta(K_H^{4-M}K_\ell\delta^{-1/2}) + |r(\rho_\mu)|\ell^2 \leq C\rho_\mu^2\ell^2\delta^{2+\eta_4}$, thanks to the relations (H7), (H8) and that $M > 4$. We conclude observing that, thanks to Theorem 9.4, we have the existence of $C_5, \eta_5 > 0$ such that

$$\Big\langle(1-\varepsilon_K)\mathcal{K}_H^{\mathrm{Diag}} + \mathcal{Q}_3(z) + \mathcal{Q}_2^{\mathrm{ex}} + \mathcal{Q}_1^{\mathrm{ex}} + \frac{b}{100}\frac{n_+}{\ell^2} + \frac{b\varepsilon_T}{100}\frac{n_+^H}{(d\ell)^2}\Big\rangle_\Phi \geq -C_5\rho_z^2\ell^2\delta^{2+\eta_5}, \tag{9.86}$$

where the error has been obtained using relations (H10), (H18), (H21), (H26) and (H27). Thanks to the assumptions on ρ_z and ρ_μ, there exist $C_6, \eta_6 > 0$ such that

$$|\rho_z^2\ell^2\delta^2 - \rho_\mu^2\ell^2\delta^2| \leq C_6\rho_\mu^2\ell^2\delta^2K_\ell^{-2} = C_6\rho_\mu^2\ell^2\delta^{2+\eta_6}, \tag{9.87}$$

so that, plugging (9.86) into (9.85) and substituting the ρ_z by the ρ_μ using (9.87) gives the desired lower bound and the right order for the error terms.

We choose $C = \sum_{j=1}^{6} C_j$ and $\eta = \min_{j=1,\dots,6} \eta_j$. We conclude using that $\widehat{g}_0 = 8\pi\delta$ to get that

$$\inf_{z\in\mathbb{R}_+} \inf_{\Phi} \langle \Phi, \mathcal{K}(z)\Phi\rangle \geq -4\pi\ell^2\rho_\mu^2\delta\Big(1 - \Big(2\Gamma + \frac{1}{2} + \log\pi\Big)\delta\Big) - C\rho_\mu^2\ell^2\delta^{2+\eta}.$$

This finishes the proof of Theorem 6.7. □

Funding Open access funding provided by Copenhagen University. This research was partially supported by the grant 0135-00166B from the Independent Research Fund Denmark. SF and LM were partially funded by the European Union. Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors are not aware of any conflict of interest regarding this manuscript.

Appendix A: Reduction to Smaller Boxes for the Upper Bound

We provide here the necessary tools to go from a fixed box with compactly support potentials in the grand canonical setting, Theorem 2.2, to the thermodynamic limit with potentials allowing a tail, Theorem 2.1. The same techniques can be found in [17] with only minor deviations surrounding the non-compactness of the potential.

Given a potential v, we define

$$e(\rho) := \lim_{L\to\infty} e_L(\rho) = \lim_{L\to\infty} \inf_{\psi\in H_0^1(\Lambda_L^{\rho L^2})} \frac{\langle\psi, \mathcal{H}_v^{\rho L^2}\psi\rangle}{L^2},$$

where the limit is taken such that $\rho L^2 = N \in \mathbb{N}$ and

$$\mathcal{H}_v^N = \sum_{i=1}^{N} -\Delta_i + \sum_{i<j}^{N} v(x_i - x_j).$$

We write $v = v\mathbb{1}_{B(0,R)} + v\mathbb{1}_{B(0,R)^c} = v_R + v_{tail}$ where the v_{tail} will always be treated as an error term. Let $v_R^{per}(x) = \sum_{k\in\mathbb{Z}^2} v_R(x + kL)$. In order for this to be finite we understand

R to be smaller than L. We omit the N in the hamiltonian when it is operating on the Fock space.

The result below evaluates the error when going from periodic boundary conditions to Dirichlet boundary conditions.

Lemma A.1. *There exists a universal $C > 0$, such that given $R_0 > 0$ and a periodic, bosonic trial function $\Psi_L \in \mathscr{F}(\Lambda_L)$, there exists a Dirichlet trial function $\Psi^D_{L+2R_0} \in \mathscr{F}(L^2(\Lambda_{L+2R_0}))$ satisfying, for $j \in \mathbb{N}_0$,*

$$\langle \Psi^D_{L+2R_0}, \mathcal{N}^j \Psi^D_{L+2R_0} \rangle = \langle \Psi_L, \mathcal{N}^j \Psi_L \rangle, \tag{A1}$$

and

$$\langle \Psi^D_{L+2R_0}, \mathcal{H}_{v_R} \Psi^D_{L+2R_0} \rangle \leq \langle \Psi_L, \mathcal{H}_{v_R^{per}} \Psi_L \rangle + \frac{C}{LR_0} \langle \Psi_L, \mathcal{N} \Psi_L \rangle. \tag{A2}$$

Proof. The result is independent of dimension, see [31, Lemma 2.1.3] or [17, Lemma A.1] for a proof in the 3D case. □

Next step is to glue the Dirichlet boxes together in order to construct a trial function on a thermodynamic box.

Theorem A.2. *Let $\Psi^D_{L+2R_0} \in \mathscr{F}_s(L^2(\Lambda_{L+2R_0}))$ be a trial function with Diriclet boundary conditions and extend it to $\mathbb{R}^2$ by 0. Then for $L_k = k(L + 2R_0 + R)$, $k \in \mathbb{N}$, we define $\Psi_{L_k} \in \mathscr{F}_s(L^2(\Lambda_{L_k}))$ by*

$$\Psi^{(m)}_{L_k}(x_1, \ldots, x_m) = \frac{1}{\|(\Psi^D_{L+2R_0})^{(n)}\|^{k^2-1}} \prod_{i=1}^{k^2} (\Psi^D_{L+2R_0})^{(n)}(x_{1+n(i-1)} - c_i, \ldots, x_{in} - c_i), \tag{A3}$$

if $m = nk^2$, and $\Psi^{(m)}_{L_k} = 0$ otherwise. Here c_i defines an enumeration of the lattice points on $\mathbb{Z}^2(L + 2R_0)$. Then Ψ_{L_k} satisfies

$$\langle \Psi_{L_k}, \mathcal{N}^j \Psi_{L_k} \rangle = k^{2j} \langle \Psi^D_{L+2R_0}, \mathcal{N}^j \Psi^D_{L+2R_0} \rangle, \qquad j \in \mathbb{N}_0. \tag{A4}$$

Furthermore if v satisfies the decay condition (1.3) of Theorem 1.1, then there exits a constant C only depending on η_0 and C_0 such that

$$\langle \Psi_{L_k}, \mathcal{H}_v \Psi_{L_k} \rangle \leq k^2 \langle \Psi^D_{L+2R_0}, \mathcal{H}_{v_R} \Psi^D_{L+2R_0} \rangle + k^2 \langle \Psi^D_{L+2R_0}, \mathcal{N}^2 \Psi^D_{L+2R_0} \rangle \frac{Ca^{\eta_0}}{R^{2+\eta_0}}. \tag{A5}$$

Proof. The expectation of $\mathcal{N}^j$ can be computed using that

$$\|\Psi^{(m)}_{L_k}\|^2 = \begin{cases} \|\Psi^{(n)}_{L+2R_0}\|^2 & \text{if} \quad m = k^2 n, \\ 0 & \text{otherwise.} \end{cases}$$

However for the potential energy we need to estimate the interaction between the boxes and the long range interaction inside the box. We observe that

$$\langle \Psi_{L_k}, \mathcal{H}_v \Psi_{L_k} \rangle - k^2 \langle \Psi^D_{L+2R_0}, \mathcal{H}_{v_R} \Psi^D_{L+2R_0} \rangle = \textstyle\sum_{n \geq 0} \sum_{i<j}^{k^2 n} \int |\Psi^{(k^2 n)}_{L_k}|^2 v_{tail}(x_i - x_j) \mathrm{d}x, \tag{A6}$$

where we used that the kinetic energy of the two terms are equal and only the tail of the potential survives due to the corridors between the boxes. We further estimate

$$\sum_{n\geq 0}\sum_{i<j}^{k^2 n}\int |\Psi_{L_k}^{(k^2 n)}|^2 v_{tail}(x_i - x_j)\mathrm{d}x \leq \sum_{n\geq 0} k^2 n \sum_{j=2}^{k^2 n}\int |\Psi_{L_k}^{(k^2 n)}|^2 v_{tail}(x_1 - x_j)\mathrm{d}x$$
$$\leq \sum_{n\geq 0} k^2 n \sum_{j=2}^{k^2 n}\int |\Psi_{L_k}^{(k^2 n)}|^2 \frac{Ca^{\eta_0}}{|x_1 - x_j|^{2+\eta_0}}\mathrm{d}x, \tag{A7}$$

where we used (1.3). If $s \in \mathbb{N}$ denotes the number of aligned boxes separating x_1 from x_j, then $|x_1 - x_j| \geq (s-1)L + R$ and there are $4(s+1)+1$ of such possible boxes. Summing on s we get

$$(A7) \leq \sum_{n\geq 0} k^2 n \sum_{s=1}^{k} \frac{C_0 a^{\eta_0} n(4(s+1)+1)}{((s-1)L+R)^{2+\eta_0}} \|\Psi_{L_k}^{(k^2 n)}\|^2$$
$$\leq k^2 \langle \Psi^D_{L+2R_0}, \mathcal{N}^2 \Psi^D_{L+2R_0}\rangle C_0 \Big(\frac{9a^{\eta_0}}{R^{2+\eta_0}} + \frac{a^{\eta_0}}{L^{2+\eta_0}}\sum_{s=1}^{\infty}\frac{4}{s^{1+\eta_0}} + \frac{9}{s^{2+\eta_0}}\Big).$$

In fact the largest term is the contribution of v_{tail} inside the box and its 8 neighbours which here is represented by the term $\frac{9a^{\eta_0}}{R^{2+\delta}}$. □

We have thus far constructed a sequence of grand canonical trial functions on larger and larger boxes, where we control the energy and the expected number of particles. The last part will be to relate this sequence to $e(\rho)$. For this we will use the continuity and convexity of $e(\rho)$ see [32, Thm. 3.5.8 and 3.5.11] together with the Legendre transformation being an involution on such functions.

Theorem A.3. *Let $\Psi_{L_k} \in \mathscr{F}(L^2(\Lambda_{L_k}))$ be a sequence with Dirichlet boundary conditions such that $L_k \to \infty$ as $k \to \infty$. Assume that there exist $C, c > 0$ such that, for all $k \in \mathbb{N}$,*

$$\langle \Psi_{L_k}, \mathcal{N}\Psi_{L_k}\rangle \geq \rho(1+c\rho)L_k^2, \qquad \langle \Psi_{L_k}, \mathcal{N}^2\Psi_{L_k}\rangle \leq C(\rho L_k^2)^2,$$

then

$$e(\rho) \leq \lim_{k\to\infty}\frac{\langle \Psi_{L_k}, \mathcal{H}_v \Psi_{L_k}\rangle}{L_k^2}.$$

Proof. We insert a chemical potential μ, and find that, using the positivity of $\mathcal{H}_v$ and $\mathcal{N}$, for any $\mu \geq 0$ and $M > 0$ we have

$$\frac{\langle \Psi_{L_k}, \mathcal{H}_v \Psi_{L_k}\rangle}{L_k^2}$$
$$\geq \frac{\langle \Psi_{L_k}, (\mathcal{H}_v - \mu\mathcal{N})\chi(\mathcal{N} \leq ML_k^2)\Psi_{L_k}\rangle}{L_k^2} + \frac{\mu}{L_k^2}\big(\langle \Psi_{L_k}, \mathcal{N}\Psi_{L_k}\rangle - \langle \Psi_{L_k}, \mathcal{N}\chi(\mathcal{N} \geq ML_k^2)\Psi_{L_k}\rangle\big)$$
$$\geq \sum_{m=0}^{ML_k^2}\Big(e_{L_k}\Big(\frac{m}{L_k^2}\Big) - \mu\frac{m}{L_k^2}\Big)\|\Psi_{L_k}^{(m)}\|^2 + \frac{\mu}{L_k^2}\Big(\langle \Psi_{L_k}, \mathcal{N}\Psi_{L_k}\rangle - \frac{1}{ML_k^2}\langle \Psi_{L_k}, \mathcal{N}^2\Psi_{L_k}\rangle\Big).$$

Fixing M large enough in terms of C and c then gives

$$\frac{\langle \Psi_{L_k}, \mathcal{H}\Psi_{L_k}\rangle}{L_k^2} \geq \sum_{m=0}^{ML^2} \Big(e_{L_k}\Big(\frac{m}{L_k^2}\Big) - \mu \frac{m}{L_k^2}\Big) \|\Psi_{L_k}^{(m)}\|^2 + \mu\rho. \tag{A8}$$

As in Theorem A.2, we glue several copies of a minimizer of $e_{L_k}(\rho)$, each copy living on a different box. We leave corridors of size $L_k^{1-\epsilon}$ between the boxes and this has the consequence of changing the density to $\rho(1+L_k^{-\epsilon})^{-2}$. Assuming further that v satisfies the conditions of Theorem 1.1 we estimate the ignored interactions to find

$$e(\rho(1+L_k^{-\epsilon})^{-2}) \leq e_{L_k}(\rho)(1+L_k^{-\epsilon})^{-2} + \frac{C(\rho L_k^2)^2}{L_k^{4+\eta_0-(2+\eta_0)\epsilon}}. \tag{A9}$$

Using (A9) in (A8) yields

$$\begin{aligned} L_k^{-2}\langle \Psi_{L_k}, \mathcal{H}\Psi_{L_k}\rangle &\geq \mu\rho + (1+L_k^{-\epsilon})^2 \sum_{m=0}^{ML_k^2} \Big(e\Big(\frac{m}{L_k^2}(1+L_k^{-\epsilon})^{-2}\Big) \\ &\quad - (1+L_k^{-\epsilon})^{-2}\mu\frac{m}{L_k^2} - \frac{Cm^2}{L_k^{4+\eta_0-(2+\eta_0)\epsilon}}\Big)\|\Psi_{L_k}^{(m)}\|^2 \\ &\geq \mu\rho - (1+L_k^{-\epsilon})^2 e^*(\mu) - C\rho^2 L_k^{-\eta_0+(2+\eta_0)\epsilon}, \end{aligned}$$

where * defines the Legendre transformation with respect to the interval $[0, M]$. Choosing $\epsilon > 0$ small enough and letting k go to infinity yields

$$\lim_{k\to\infty} \frac{\langle \Psi_{L_k}, \mathcal{H}\Psi_{L_k}\rangle}{L_k^2} \geq \sup_{\mu\in[0,\infty)} (\mu\rho - e^*(\mu)) = \sup_{\mu\in\mathbb{R}} (\mu\rho - e^*(\mu)) = e(\rho), \tag{A10}$$

where we used that $e^*(\mu) \geq 0$ for all $\mu \in \mathbb{R}$ and that the Legendre transformation is an involution. □

We end the section by giving the proof of the final upper bound Theorem 2.1 using the result of Theorem 2.2.

Proof of Theorem 2.1. We first cut our potential in order to apply Theorem 2.2. We write

$$v = v\mathbb{1}_{B(0,R)} + v\mathbb{1}_{B(0,R)^c} = v_R + v_{tail},$$

where $R = \rho^{-\frac{1}{2}} Y^{\beta+2}$. We denote by a_R the scattering length of v_R. To get estimates on the energy density $e(\rho)$ we use the standard theory developed in "Appendix A". The idea is to extend L_β with R_0 and force the trial function to have Dirichlet boundary conditions on the box of sidelength $L_\beta + R_0$. Thereafter one glues together these small Dirichlet boxes, separated by corridors of size R. Since this process will slightly change the density, we choose for a given $\rho > 0$, the larger density $\widetilde{\rho}$ satisfying

$$\rho = \widetilde{\rho}(1 - 2C\widetilde{Y}^2)\Big(1 + \frac{2R_0}{L_\beta} + \frac{R}{L_\beta}\Big)^{-2},$$

where C is the same as in Theorem 2.2, and $R_0 = \rho^{-\frac{1}{2}} Y^{-\frac{1}{2}}$. This choice of R_0 is in fact optimal as one can see from the error term $C\frac{\rho}{L_\beta R_0}$ coming from the glueing process in

(A17). Here we use the notation $\widetilde{Y} = |\log(\widetilde{\rho} a_R^2)|^{-1}$ and $\widetilde{\delta_0} = |\log(\widetilde{\rho} a_R^2 \widetilde{Y})|^{-1}$. If ρa^2 is small enough then $\widetilde{\rho} a_R^2 \leq C^{-1}$, and we may use Theorem 2.2 to find a periodic trial state Ψ for the density $\widetilde{\rho}$ and potential v_R satisfying

$$\langle \mathcal{H}_{v_R} \rangle_\Psi \leq 4\pi L_\beta^2 \widetilde{\rho}^2 \widetilde{\delta_0} \Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)\widetilde{\delta_0}\Big) + C L_\beta^2 \widetilde{\rho}^2 \widetilde{\delta_0}^3 |\log(\widetilde{\delta_0})|, \quad \text{(A11)}$$

with $\langle \mathcal{N} \rangle_\Psi \geq \widetilde{\rho} L_\beta^2 (1 - C\widetilde{Y}^2)$, and $\langle \mathcal{N}^2 \rangle_\Psi \leq C \widetilde{\rho}^2 L_\beta^4$.

Since $\frac{R}{L_\beta} \ll \frac{R_0}{L_\beta} = Y^{-\frac{1}{2}+\beta} \ll 1$, we have $|\rho - \widetilde{\rho}| \leq C\rho Y^{-\frac{1}{2}+\beta}$, and we can change $\widetilde{\rho}$ into ρ in (A11) up to smaller errors if

$$\beta \geq \frac{5}{2}. \quad \text{(A12)}$$

One can show that the C appearing in (A11) only increases in β (see (5.21)). Thus we find $\beta = 5/2$ to be optimal. We can also change a_R into a because the right-hand side of (A11) is an increasing function of the scattering length and $a_R \leq a$. Thus,

$$\langle \mathcal{H}_{v_R} \rangle_\Psi \leq 4\pi L_\beta^2 \rho^2 \delta_0 \Big(1 + \Big(2\Gamma + \frac{1}{2} + \log(\pi)\Big)\delta_0\Big) + C L_\beta^2 \rho^2 \delta_0^3 |\log(\delta_0)|, \quad \text{(A13)}$$

and the bounds on the number of particles become

$$\langle \mathcal{N} \rangle_\Psi \geq (\rho + c\rho^2)(L_\beta + 2R_0 + R)^2, \quad \text{and} \quad \langle \mathcal{N}^2 \rangle_\Psi \leq C(\rho L_\beta^2)^2, \quad \text{(A14)}$$

for some $c > 0$.

Now we can use Lemma A.1 and Theorem A.2 to glue small boxes together. We get a sequence $\Psi_{k(L_\beta+2R_0+R)} \in \mathscr{F}_s\big(L^2\big(\Lambda_{k(L_\beta+2R_0+R)}\big)\big)$ with Dirichlet boundary conditions, for $k \in \mathbb{N}$, on the box

$$\Lambda_{k(L_\beta+2R_0+R)} = \Big[-\frac{1}{2}k(L_\beta + 2R_0 + R), \frac{1}{2}k(L_\beta + 2R_0 + R)\Big]^2,$$

satisfying

$$\langle \mathcal{N} \rangle_{\Psi_{k(L_\beta+2R_0+R)}} = k^2 \langle \mathcal{N} \rangle_\Psi, \qquad \langle \mathcal{N}^2 \rangle_{\Psi_{k(L_\beta+2R_0+R)}} = k^4 \langle \mathcal{N}^2 \rangle_\Psi, \quad \text{(A15)}$$

and

$$\langle \mathcal{H}_v \rangle_{\Psi_{k(L_\beta+2R_0+R)}} \leq k^2 \Big(\langle \mathcal{H}_{v_R} \rangle_\Psi + \langle \mathcal{N} \rangle_\Psi \frac{C}{L_\beta R_0} + \langle \mathcal{N}^2 \rangle_\Psi \frac{C a^{\eta_0}}{R^{2+\eta_0}}\Big), \quad \text{(A16)}$$

where C only depends on η_0 and C_0. Note that we have the original potential in the left-hand side of (A16) because by (A5), v_{tail} only produces an error term. By construction this sequence satisfies the conditions on the number of particles for Theorem A.3, and we conclude

$$e^{2D}(\rho) \leq \lim_{k\to\infty} \frac{\langle \mathcal{H}_v \rangle_{\Psi_{k(L_\beta+2R_0+R)}}}{k^2 L_\beta^2} \leq \frac{\langle \mathcal{H}_{v_R} \rangle_\Psi}{L_\beta^2} + C \frac{\rho}{L_\beta R_0} + C a^{\eta_0} \frac{\rho^2 L_\beta^2}{R^{2+\eta_0}}, \quad \text{(A17)}$$

where in the last inequality we used (A16), (A14) and that $\langle \mathcal{N} \rangle_\Psi^2 \leq \langle \mathcal{N}^2 \rangle_\Psi$. With our choice of parameters including (A12), the two last terms in (A17) are errors. Theorem 2.1 follows from (A17) and (A13). □

Appendix B: Bogoliubov Diagonalization

Theorem B.1. *Let $a_\pm$ be operators on a Hilbert space satisfying $\big[a_+, a_-\big] = 0$. For $\mathcal{A} > 0$, $\mathcal{B} \in \mathbb{R}$ satisfying either $|\mathcal{B}| < \mathcal{A}$ or $\mathcal{B} = \mathcal{A}$ and arbitrary $\kappa \in \mathbb{C}$, we have the operator identity*

$$\begin{aligned}&\mathcal{A}(a_+^\dagger a_+ + a_-^\dagger a_-) + \mathcal{B}(a_+^\dagger a_-^\dagger + a_+ a_-) + \kappa(a_+^\dagger + a_-) + \overline{\kappa}(a_+ + a_-^\dagger)\\&\quad = (1-\alpha^2)\mathcal{D}(b_+^\dagger b_+ + b_-^\dagger b_-) - \frac{1}{2}(\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2})([a_+, a_+^\dagger] + [a_-, a_-^\dagger]) - \frac{2|\kappa|^2}{\mathcal{A}+\mathcal{B}},\end{aligned}$$

where $\mathcal{D} = \frac{1}{2}\big(\mathcal{A} + \sqrt{\mathcal{A}^2 - \mathcal{B}^2}\big)$ and

$$b_+ = \frac{1}{\sqrt{1-\alpha^2}}\big(a_+ + \alpha a_-^\dagger + \bar{c}_0\big), \quad b_- = \frac{1}{\sqrt{1-\alpha^2}}\big(a_- + \alpha a_+^\dagger + c_0\big), \tag{B1}$$

with

$$\alpha = \mathcal{B}^{-1}\big(\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2}\big), \quad c_0 = \frac{2\bar{\kappa}}{\mathcal{A} + \mathcal{B} + \sqrt{\mathcal{A}^2 - \mathcal{B}^2}}. \tag{B2}$$

Remark B.2. Note that the normalization of $b_\pm$ is chosen such that

$$[b_+, b_+^\dagger] = \frac{[a_+, a_+^\dagger] - \alpha^2[a_-, a_-^\dagger]}{1-\alpha^2}, \tag{B3}$$

and we recover the canonical commutation relations $[b_+, b_+^\dagger] = 1$ when a_+ and a_- satisfies them as well.

Proof. This follows directly from algebraic computations. □

Appendix C: Calculation of the Bogoliubov Integral

For functions α, β, and parameter $\varepsilon \geq 0$, we define

$$\begin{aligned}I_\varepsilon(\alpha, \beta) := \frac{1}{2(2\pi)^2}\int_{\mathbb{R}^2}\Big(&\sqrt{(1-\varepsilon)^2\alpha^2(k) + 2(1-\varepsilon)\rho\alpha(k)\beta(k)} - (1-\varepsilon)\alpha(k) - \rho\beta(k)\\&+ \rho^2\frac{\widehat{g}_k^2 - \widehat{g}_0^2\mathbb{1}_{\{|k|\leq \ell_\delta^{-1}\}}}{k^2}\Big)\mathrm{d}k.\end{aligned} \tag{C1}$$

We recall that $\widehat{g}_0 = 8\pi\delta$, where δ satisfies $\frac{1}{2}Y \leq \delta \leq 2Y$. We are mainly interested into two special cases, namely $I_0(k^2, \widehat{g})$ and $I_{\varepsilon_N}(\tau, \widehat{W}_1)$. In this section we estimate these integrals.

Lemma C.1. *We can replace τ_k by k^2 up to the following error,*

$$|I_{\varepsilon_N}(\tau, \widehat{W}_1) - I_{\varepsilon_N}(k^2, \widehat{W}_1)| \leq C\rho^2\delta^2\big(d + \varepsilon_T|\log Y| + (sK_\ell)^{-1}\big).$$

Proof. We recall the definition (8.8) of τ_k,

$$\tau_k = (1-\varepsilon_T)\Big(|k| - \frac{1}{2s\ell}\Big)_+^2 + \varepsilon_T\Big(|k| - \frac{1}{2ds\ell}\Big)_+^2,$$

from which we deduce the bounds

$$|\tau_k - k^2| \leq \begin{cases} \frac{1}{2s\ell}|k| + \frac{1}{2(s\ell)^2}, & \text{if } \quad |k| > \frac{1}{2ds\ell}, \\ \varepsilon_T |k|^2 + \frac{3}{2s\ell}|k|, & \text{if } \quad \frac{1}{2s\ell} < |k| < \frac{1}{2ds\ell}. \end{cases} \tag{C2}$$

We write the integral as

$$I_{\varepsilon_N}(\tau, \widehat{W}_1) = \frac{1}{2(2\pi)^2}\int_{\mathbb{R}^2} F_k(\tau_k, \widehat{W}_1(k))\mathrm{d}k, \tag{C3}$$

with

$$F_k(\tau, w) = \sqrt{(1-\varepsilon_N)^2\tau^2 + 2(1-\varepsilon_N)\rho w\tau} \\ - (1-\varepsilon_N)\tau - \rho w + \rho^2\frac{\widehat{g}^2(k) - \widehat{g}^2(0)\mathbb{1}_{\{|k|\leq \ell_\delta^{-1}\}}}{2k^2}.$$

We first consider separately the small k's. Indeed, $\tau_k = 0$ for $|k| \leq \frac{1}{2s\ell}$ and thus

$$|F_k(\tau_k, \widehat{W}_1(k)) - F_k(k^2, \widehat{W}_1(k))| = \Big|\sqrt{(1-\varepsilon_N)^2k^4 + 2(1-\varepsilon_N)\rho\hat{W}_1(k)k^2} - (1-\varepsilon)k^2\Big| \\ \leq C\sqrt{\rho\delta}|k|,$$

(recall that $(sK_\ell)^{-1} \ll 1$) and

$$\frac{1}{2(2\pi)^2}\int_{|k|\leq(2s\ell)^{-1}} |F_k(\tau_k, \widehat{W}_1(k)) - F_k(k^2, \widehat{W}_1(k))|\mathrm{d}k \leq C\rho^2\delta^2(sK_\ell)^{-3}. \tag{C4}$$

The part with larger k we bound using the derivatives of F and deduce

$$|I_{\varepsilon_N}(\tau, \widehat{W}_1) - I_{\varepsilon_N}(k^2, \widehat{W}_1)| \\ \leq \frac{1}{2(2\pi)^2}\int_{\{|k|>(2s\ell)^{-1}\}} \sup_{\tau\in[\tau_k,k^2]} |\partial_\tau F_k(\tau, \widehat{W}_1(k))| \cdot |\tau_k - k^2|\mathrm{d}k + C\rho^2\delta^2(sK_\ell)^{-3}. \tag{C5}$$

The derivative of F is given by

$$\partial_\tau F(\tau, w) = \frac{(1-\varepsilon_N)^2\tau + (1-\varepsilon_N)\rho w}{\sqrt{(1-\varepsilon_N)^2\tau^2 + 2(1-\varepsilon_N)\rho w\tau}} - (1-\varepsilon_N) \tag{C6}$$

and can be estimated for $\tau \in [\tau_k, k^2]$ as

$$|\partial_\tau F_k(\tau, \widehat{W}_1(k))| \leq \begin{cases} C\dfrac{\sqrt{\rho\delta}}{|k| - (2s\ell)^{-1}}, & \text{if } (2s\ell)^{-1} < |k| < \sqrt{\rho\delta}, \\ C\dfrac{\rho^2\delta^2}{k^4}, & \text{if } |k| > \sqrt{\rho\delta}. \end{cases} \tag{C7}$$

Indeed, for $|k| < \sqrt{\rho\delta}$, we just need to bound individualy each term in (C6), whereas for $|k| > \sqrt{\rho\delta}$, we have $\tau_k > C\rho\widehat{W}_1(k)$ and we use a Taylor expansion of the square root to get

$$|\partial_\tau F_k(\tau, \widehat{W}_1(k))| \leq C\frac{\rho^2\widehat{W}_1(k)^2}{(1-\varepsilon_N)\tau^2} \leq C\frac{\rho^2\delta^2}{k^4}.$$

Now we split the integral in (C5) into 3 parts I_1, I_2, I_3, corresponding to the integration on the domains $\{(2s\ell)^{-1} < k < \sqrt{\rho\delta}\}$, $\{\sqrt{\rho\delta} < k < (2ds\ell)^{-1}\}$ and $\{k > (2ds\ell)^{-1}\}$, respectively, and we use (C7), (C2) to bound it and find:

$$\begin{aligned} I_1 &\leq C\rho^2\delta^2\Big(\varepsilon_T + \frac{1}{sK_\ell}\Big), \\ I_2 &\leq C\rho^2\delta^2\Big(\varepsilon_T|\log Y| + \frac{1}{sK_\ell}\Big), \\ I_3 &\leq C\rho^2\delta^2 d. \end{aligned}$$

□

Lemma C.2. *We can replace $\widehat{W}_1(k)$ by $\widehat{g}_k$ up to the following error*

$$|I_{\varepsilon_N}(k^2, \widehat{W}_1) - I_{\varepsilon_N}(k^2, \widehat{g})| \leq C\rho^2\delta^2 K_\ell^{-1} + C\rho^2\delta\varepsilon_N.$$

Proof. Recall that $I_{\varepsilon_N}(k^2, \widehat{W}_1)$ is given by (C3). We first use (6.26) and (3.34) to replace the last part,

$$\rho^2\int_{\mathbb{R}^2}\frac{\widehat{g}_k^2 - \widehat{g}_0^2\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2k^2}\mathrm{d}k = \rho^2\int_{\mathbb{R}^2}\frac{\widehat{W}_1(k)^2 - \widehat{W}_1(0)^2\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2(1-\varepsilon_N)k^2}\mathrm{d}k + \mathcal{O}(\rho^2\delta\varepsilon_N + \rho^2\delta^2K_\ell^{-2}),$$

so that

$$I_{\varepsilon_N}(k^2, \widehat{W}_1) = J(\widehat{W}_1) + \mathcal{O}(\rho^2\delta\varepsilon_N + \rho^2\delta^2K_\ell^{-2}) \quad \text{and} \quad I_{\varepsilon_N}(k^2, \widehat{g}) = J(\widehat{g}), \tag{C8}$$

with

$$J(w) = \frac{1}{2(2\pi)^2}\int_{\mathbb{R}^2} G_k(w_k, w_0)\mathrm{d}k, \tag{C9}$$

and

$$G_k(w, w_0) = \sqrt{(1-\varepsilon_N)^2 k^4 + 2(1-\varepsilon_N)\rho w k^2} \\ - (1-\varepsilon_N)k^2 - \rho w + \rho^2 \frac{w^2 - w_0^2 \mathbb{1}_{\{|k| \le \ell_\delta^{-1}\}}}{2(1-\varepsilon_N)k^2}.$$

Note that G_k is independent of w_0 for $|k| > \ell_\delta^{-1}$. Then we split $J(w)$ into two parts,

$$J(w) = \frac{1}{2(2\pi)^2} \int_{\{|k|<\ell_\delta^{-1}\}} G_k(w_k, w_0)\mathrm{d}k + \frac{1}{2(2\pi)^2} \int_{\{|k|>\ell_\delta^{-1}\}} G_k(w_k)\mathrm{d}k \\ =: J_<(w) + J_>(w). \tag{C10}$$

For $k > \ell_\delta^{-1}$ we use

$$|J_>(\widehat{W}_1) - J_>(\widehat{g})| \le \frac{1}{2(2\pi)^2} \int_{\{|k|>\ell_\delta^{-1}\}} \sup_{w \in [\widehat{g}_k, \widehat{W}_1(k)]} |\partial_w G_k(w)| \cdot |\widehat{W}_1(k) - \widehat{g}_k|\mathrm{d}k, \tag{C11}$$

with

$$\partial_w G = \frac{\rho}{\sqrt{1 + \frac{2\rho w}{(1-\varepsilon_N)k^2}}} - \rho + \frac{\rho^2 w}{(1-\varepsilon_N)k^2}. \tag{C12}$$

We use a Taylor expansion of the square root to get

$$|J_>(\widehat{W}_1) - J_>(\widehat{g})| \le C\rho^3 \int_{|k|>\ell_\delta^{-1}} \frac{\widehat{g}_k^2}{k^4} |\widehat{W}_1(k) - \widehat{g}_k|\mathrm{d}k.$$

Since $|\widehat{W}_1(k) - \widehat{g}_k| \le C\delta^2 K_\ell^{-1}$ (by (6.24)) and $\int \widehat{g}_k^2 k^{-2}\mathrm{d}k < C\delta$ (see (3.34)) we deduce

$$|J_>(\widehat{W}_1) - J_>(\widehat{g})| \le C\rho^2\delta^2 K_\ell^{-1}. \tag{C13}$$

For $k < \ell_\delta^{-1}$ we start by focusing on the first part of G_k,

$$F_k(w) = \sqrt{(1-\varepsilon_N)^2 k^4 + 2(1-\varepsilon_N)\rho w k^2} - (1-\varepsilon_N)k^2 - \rho w. \tag{C14}$$

Since $|\partial_w F_k| \le C\rho$, we have

$$\Big| \int_{\{|k|<\ell_\delta^{-1}\}} F_k(\widehat{W}_1(k)) - F_k(\widehat{g}_k)\mathrm{d}k \Big| \le C\rho \int_{\{|k|<\ell_\delta^{-1}\}} |\widehat{W}_1(k) - \widehat{g}_k|\mathrm{d}k \le C\rho^2\delta^3 K_\ell^{-1}. \tag{C15}$$

Now

$$\begin{aligned}|J_<(\widehat{W}_1) - J_<(\widehat{g})| &\leq C\Big|\int_{\{|k|<\ell_\delta^{-1}\}} F_k(\widehat{W}_1(k)) - F_k(\widehat{g}_k)\mathrm{d}k\Big| \\ &\quad + C\Big|\int_{\{|k|<\ell_\delta^{-1}\}} \rho^2 \frac{\widehat{W}_1(k)^2 - \widehat{W}_1(0)^2}{2(1-\varepsilon_T)k^2}\mathrm{d}k\Big| \\ &\quad + C\Big|\int_{\{|k|<\ell_\delta^{-1}\}} \rho^2 \frac{\widehat{g}_k^2 - \widehat{g}_0^2}{2(1-\varepsilon_T)k^2}\mathrm{d}k\Big| \\ &\leq C\rho^2\delta^3 K_\ell^{-1} + C\rho^2 R^2\delta^2\ell_\delta^{-2} \leq C\rho^2\delta^2 K_\ell^{-1},\end{aligned}$$

where we used (3.35). Combining this with (C8) and (C13) the lemma is proved. □

Proposition C.3. *There exists a universal constant $C > 0$ such that, for any $\varepsilon \in [0, 1)$,*

$$\begin{aligned}&\Big|I_\varepsilon(k^2, \widehat{g}_k) - 4\pi\rho^2\delta\Big(1 - \frac{\delta}{Y} + \delta\log\delta + \Big(\frac{1}{2} + 2\Gamma + \log(\pi)\Big)\delta\Big)\Big| \\ &\quad \leq C\rho^2\delta^3\big(|\log(\delta)|R^2\rho + 1\big) + C\rho^2\delta\varepsilon,\end{aligned}$$

where I_ε is defined in (C1). *In particular when $\delta = \delta_0$ we deduce*

$$\Big|I_\varepsilon(k^2, \widehat{g}_k) - 4\pi\rho^2\delta_0^2\Big(\frac{1}{2} + 2\Gamma + \log(\pi)\Big)\Big| \leq C\rho^2\delta^3\big(|\log(\delta)|R^2\rho + 1\big) + C\rho^2\delta\varepsilon.$$

Proof. At first we want to replace $\widehat{g}_k$ by $\widehat{g}_0$ in the integral I:

$$|I_\varepsilon(k^2, \widehat{g}_k) - I_\varepsilon(k^2, \widehat{g}_0)| \leq \int_{\mathbb{R}^2} |F(k^2, \widehat{g}_k) - F(k^2, \widehat{g}_0)|\mathrm{d}k \tag{C16}$$

$$\leq \int_{\mathbb{R}^2} \sup_{g\in[\widehat{g}_k, \widehat{g}_0]} \Big|\partial_g F(k^2, g)\Big| \, |\widehat{g}_k - \widehat{g}_0| \,\mathrm{d}k =: I_\leqslant + I_\geqslant, \tag{C17}$$

where we split for values of $|k|$ under or above $(\rho\delta)^{1/2}$. Notice that

$$\partial_g F(k^2, \widehat{g}_k) = \frac{\rho k^2}{\sqrt{k^4 + 2\rho\widehat{g}_k k^2}} - \rho + \frac{\rho^2\widehat{g}_k}{k^2}. \tag{C18}$$

By a Taylor expansion we can prove that

$$I_\leqslant \leq C\int_{\{|k|\leq(\rho\delta)^{1/2}\}} \Big(R^2(\rho\widehat{g}_0)^{1/2}k^3 + \rho\widehat{g}_0 k^2 + R^2\big(\rho\widehat{g}_0\big)^2\Big)\mathrm{d}k \leq CR^2(\rho\delta)^3. \tag{C19}$$

In the other case we have, by Taylor expansion of the square root in (C18),

$$\begin{aligned}I_\geqslant &\leq C\rho\int_{\{|k|\geq(\rho\delta)^{1/2}\}} \frac{(\rho\widehat{g}_0)^2}{k^4}|\widehat{g}_k - \widehat{g}_0|\mathrm{d}k \\ &\leq C\,(\rho\widehat{g}_0)^3\Big(\int_{\{(\rho\delta)^{1/2}\leq|k|\leq(\rho\delta)^{1/2}\widehat{g}_0^{-1/2}\}} \frac{R^2}{k^2}\mathrm{d}k + \int_{\{|k|\geq(\rho\delta)^{1/2}\widehat{g}_0^{-1/2}\}} \frac{\mathrm{d}k}{k^4}\Big) \\ &\leq C\,(\rho\delta)^3\,R^2|\log\delta| + C\,(\rho\delta)^2\,\delta.\end{aligned}$$

We deduce that $|I_\varepsilon(k^2,\widehat{g}_k) - I_\varepsilon(k^2,\widehat{g}_0)| \leq C\rho^2\delta^3(1+R^2\rho\log(\delta))$. Now remains to compute $I_\varepsilon(k^2,\widehat{g}_0)$. In this integral we use the new variable $q = k(\rho\widehat{g}_0)^{-\frac12}(1-\varepsilon)^{\frac12}$,

$$I_\varepsilon(k^2,\widehat{g}_0) = \frac{(\rho\widehat{g}_0)^2}{2(2\pi)^2(1-\varepsilon)}\int_{\mathbb{R}^2}\Big(\sqrt{q^4+2q^2} - q^2 - 1 + \frac{\mathbb{1}_{\{|q|>(1-\varepsilon)^{\frac12}\ell_\delta^{-1}(\rho\widehat{g}_0)^{-\frac12}\}}}{2q^2}\Big)\mathrm{d}q. \tag{C20}$$

In term of $c_0 = (1-\varepsilon)^{\frac12}\ell_\delta^{-1}(\rho\widehat{g}_0)^{-\frac12}$, this integral is explicitly computable and equal to

$$I_\varepsilon(k^2,\widehat{g}_0) = \frac{(\rho\widehat{g}_0)^2}{4\pi(1-\varepsilon)}\Big(\frac18 - \frac{\log 2}{4} + \frac12\log(c_0^{-1})\Big). \tag{C21}$$

With $\widehat{g}_0 = 8\pi\delta$ and $c_0 = 2e^{-\Gamma}e^{-\frac{1}{2\delta}}(1-\varepsilon)^{\frac12}\widehat{g}_0^{-\frac12}(\rho a^2)^{-\frac12}$ (see (3.30)), we find

$$I_\varepsilon(k^2,\widehat{g}_0) = 4\pi\rho^2\delta^2\Big(\frac1\delta - \frac1Y + \log\delta + \frac12 + 2\Gamma + \log(\pi)\Big)(1+\mathcal{O}(\varepsilon)). \tag{C22}$$

□

Remark C.4. With the arbitrary parameter δ (within the range $\frac12 Y \leq \delta \leq 2Y$), one can deduce from Proposition C.3 that our lower bound on the energy is

$$e^{2\mathrm{D}}(\rho) \geq 4\pi\rho^2\delta\Big(2 - \frac{\delta}{Y} + \delta\log\delta + \Big(\frac12 + 2\Gamma + \log(\pi)\Big)\delta\Big) - C\rho^2Y^{2+\eta}. \tag{C23}$$

However, this lower bound is maximized by $\delta = Y(1 - Y|\log Y| + o(Y|\log Y|))$, thus leading to the optimal choice $\delta = \delta_0$.

We conclude this section by a general bound on Bogoliubov integrals that is used several times throughout the paper.

Lemma C.5. *For two functions $\mathcal{A},\mathcal{B}:\mathbb{R}^2\to\mathbb{R}$ such that*

$$\mathcal{A}(k) \geq \kappa[|k|-K]_+^2 + 2K_1\delta, \qquad |\mathcal{B}(k)| \leq K_2\delta, \qquad |\mathcal{B}(k)-\mathcal{B}(0)| \leq K_2R^2\delta|k|^2, \tag{C24}$$

for constants $\kappa > 0$, $0 < K_2 \leq K_1$, $\ell_\delta^{-1} < K$, then there exists $C > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^2}\big(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2-\mathcal{B}(k)^2}\big)\mathrm{d}k &\\ \leq \kappa^{-1}\int_{\mathbb{R}^2}\frac{\mathcal{B}^2(k)-\mathcal{B}^2(0)\mathbb{1}_{\{|k|\leq\ell_\delta^{-1}\}}}{2|k|^2}\mathrm{d}k &\\ + C\frac{K_2^2}{K_1}\delta K^2 + C\kappa^{-1}K_2^2\delta^2(1+R^2\ell_\delta^{-2}) + C\kappa^{-1}K_2^2\delta^2|\log(2K\ell_\delta)| &\\ + C\min\Big(K_2^4\delta^4\kappa^{-3}K^{-4}, C\frac{K_2^2}{K_1^2}\kappa^{-1}\int_{\mathbb{R}^2}\frac{\mathcal{B}(k)^2-\mathcal{B}(0)^2\mathbb{1}_{\{|k|<\ell_\delta^{-1}\}}}{|k|^2}\mathrm{d}k\Big).&\end{aligned} \tag{C25}$$

Proof. We show that the difference

$$\int_{\mathbb{R}^2} \Big(\big(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\big) - \kappa^{-1} \frac{\mathcal{B}^2(k) - \mathcal{B}^2(0)\mathbb{1}_{\{|k| \le \ell_\delta^{-1}\}}}{2|k|^2} \Big) \mathrm{d}k, \tag{C26}$$

is bounded by the desired error terms.

For $|k| \le 2K$ we have that

$$\begin{aligned} \int_{|k|\le 2K} \big(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\big) \mathrm{d}k &\le C \int_{\{|k|\le 2K\}} \frac{\mathcal{B}^2(k)}{\mathcal{A}(k)} \mathrm{d}k \\ &\le C \frac{K_2^2}{K_1} \delta \int_{\{|k|\le 2K\}} \mathrm{d}k \; = C \frac{K_2^2}{K_1} \delta K^2, \end{aligned}$$

while for the $\mathcal{B}$ part, using the assumption on $|\mathcal{B}(k) - \mathcal{B}(0)|$,

$$\kappa^{-1} \int_{\{|k| \le \ell_\delta^{-1}\}} \frac{|\mathcal{B}^2(k) - \mathcal{B}^2(0)|}{2|k|^2} \mathrm{d}k \le C\kappa^{-1} K_2^2 R^2 \delta^2 \int_{\{|k| \le \ell_\delta^{-1}\}} \mathrm{d}k = C\kappa^{-1} K_2^2 R^2 \delta^2 \ell_\delta^{-2}, \tag{C27}$$

and

$$\kappa^{-1} \int_{\{\ell_\delta^{-1} \le |k| \le 2K\}} d\, \frac{\mathcal{B}_k^2}{2|k|^2} \mathrm{d}k \le C\kappa^{-1} K_2^2 \delta^2 |\log(2K\ell_\delta)|. \tag{C28}$$

For $|k| \ge 2K$ we have, by a Taylor expansion,

$$\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2} \le \frac{1}{2} \frac{\mathcal{B}(k)^2}{\mathcal{A}(k)} + C \frac{\mathcal{B}(k)^4}{\mathcal{A}(k)^3}. \tag{C29}$$

For the first term we observe that

$$\frac{\mathcal{B}(k)^2}{\mathcal{A}(k)} \le \kappa^{-1} \frac{\mathcal{B}(k)^2}{(|k| - K)^2} \le \kappa^{-1} \frac{\mathcal{B}(k)^2}{|k|^2} \Big(1 + \frac{K}{|k|}\Big), \tag{C30}$$

giving

$$\int_{\{|k|\ge 2K\}} \Big(\frac{\mathcal{B}(k)^2}{\mathcal{A}(k)} - \kappa^{-1} \frac{\mathcal{B}(k)^2}{2|k|^2} \Big) \mathrm{d}k \le CK\kappa^{-1} \int_{\{|k|\ge 2K\}} \frac{\mathcal{B}(k)^2}{|k|^3} \mathrm{d}k \le C\kappa^{-1} K_2^2 \delta^2,$$

while for the second one we can bound either

$$\int_{\{|k|\ge 2K\}} \frac{\mathcal{B}(k)^4}{\mathcal{A}(k)^3} \mathrm{d}k \le CK_2^4 \delta^4 \kappa^{-3} \int_{\{|k|\ge 2K\}} \frac{\mathrm{d}k}{|k|^6} \le CK_2^4 \delta^4 \kappa^{-3} K^{-4}, \tag{C31}$$

or as in the following,

$$\int_{\{|k|\ge 2K\}} \frac{\mathcal{B}(k)^4}{\mathcal{A}(k)^3} \mathrm{d}k \le \frac{K_2^2}{K_1^2} \kappa^{-1} \int_{\{|k|\ge 2K\}} \frac{\mathcal{B}(k)^2}{\mathcal{A}(k)} \mathrm{d}k \le C \frac{K_2^2}{K_1^2} \kappa^{-1} \int_{\{|k|\ge 2K\}} \frac{\mathcal{B}(k)^2}{|k|^2} \mathrm{d}k,$$

adding and subtracting the term $C \frac{K_2^2}{K_1^2} \kappa^{-1} \int_{\{|k|<2K\}} \frac{\mathcal{B}(k)^2 - \mathcal{B}(0)^2 \mathbb{1}_{\{|k|<\ell_\delta^{-1}\}}}{|k|^2}$ we have

$$\int_{\{|k|\ge 2K\}} \frac{\mathcal{B}(k)^4}{\mathcal{A}(k)^3} \mathrm{d}k \le C \frac{K_2^2}{K_1^2} \kappa^{-1} \int_{\mathbb{R}^2} \frac{\mathcal{B}(k)^2 - \mathcal{B}(0)^2 \mathbb{1}_{\{|k|<\ell_\delta^{-1}\}}}{|k|^2} \mathrm{d}k. \tag{C32}$$

This finishes the proof of Lemma C.5. □

Appendix D: A Priori Bounds

In this section, we prove Theorem 7.6. We study a localized problem on a shorter length scale $d\ell$ such that

$$d\ell \ll \ell_\delta \ll \ell. \tag{D1}$$

where we recall that ℓ_δ is the healing length. We are able, in this section, to prove Bose–Einstein condensation in boxes with length scale smaller than the healing length. A key point is that, at this scale, we can use a larger Neumann gap to reabsorb the errors. We will show how the proof of Theorem 7.6 reduces to this localized problem.

We introduce the small box centered at $u \in \mathbb{R}^2$ to be

$$B_u = \Lambda \cap \left\{ d\ell u + \left[-\frac{d\ell}{2}, \frac{d\ell}{2} \right]^2 \right\}. \tag{D2}$$

The associated localization functions are

$$\chi_{B_u}(x) := \chi\left(\frac{x}{\ell}\right) \chi\left(\frac{x}{d\ell} - u\right), \tag{D3}$$

where we highlight that

$$\iint |\chi_{B_u}|^2 \mathrm{d}x \mathrm{d}u = \ell^2. \tag{D4}$$

In order to construct the small box Hamiltonian, we introduce the localized potentials

$$W^{\mathrm{s}}(x) := \frac{W(x)}{\chi * \chi(x/d\ell)}, \qquad w_{B_u}(x, y) := \chi_{B_u}(x) W^s(x - y) \chi_{B_u}(y), \tag{D5}$$

$$W_1^{\mathrm{s}}(x) := \frac{W_1(x)}{\chi * \chi(x/d\ell)}, \qquad w_{1,B_u}(x, y) := \chi_{B_u}(x) W_1^s(x - y) \chi_{B_u}(y), \tag{D6}$$

$$W_2^{\mathrm{s}}(x) := \frac{W_2(x)}{\chi * \chi(x/d\ell)}, \qquad w_{2,B_u}(x, y) := \chi_{B_u}(x) W_2^s(x - y) \chi_{B_u}(y), \tag{D7}$$

where we recall that W, W_1, W_2 are localized versions of $v, g, (1 + \omega)g$, respectively (see formulas (6.19) and (3.4)). Since v has support in $B(0, R)$, we see that W^{s} is well-defined as $d\ell$ is required to be larger then R. Clearly W^{s} depends on $d\ell$ and thus ρ_μ, but we will not reflect this in our notation.

Similarly to Lemma 6.4, W_1^{s} satisfies the following inequalities which can be proven in analogous ways considering the length scale $d\ell$ in place of ℓ

$$\int W_2^{\mathrm{s}} \leq 2 \int W_1^{\mathrm{s}} \leq C\delta, \tag{D8}$$

$$0 \leq W_1^{\mathrm{s}}(x) - g(x) \leq C g(x) \frac{|x|^2}{(d\ell)^2}, \tag{D9}$$

$$\left| \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\widehat{W}_1^{\mathrm{s}}(k)^2 - \widehat{W}_1^{\mathrm{s}}(0)^2 \mathbb{1}_{\{|k| \leq \ell_\delta^{-1}\}}}{2k^2} \mathrm{d}k - \widehat{g\omega}(0) \right| \leq C \frac{R^2 \delta}{(d\ell)^2}. \tag{D10}$$

We define furthermore, as operators on $L^2(B_u)$,

$$P_{B_u} := \frac{1}{|B_u|} |\mathbb{1}_{B_u}\rangle\langle \mathbb{1}_{B_u}|, \qquad Q_{B_u} := \mathbb{1}_{B_u} - P_{B_u}, \tag{D11}$$

i.e., P_{B_u} is the orthogonal projection in $L^2(B_u)$ onto the constant functions and Q_{B_u} is the projection to the orthogonal complement. We can therefore introduce the number operators as well

$$n_{B_u} := \sum_{j=1}^{N} \mathbb{1}_{B_u,j}, \quad n_{B_u,0} := \sum_{j=1}^{N} P_{B_u,j}, \quad n_{B_u,+} := \sum_{j=1}^{N} Q_{B_u,j}, \tag{D12}$$

and the small-box kinetic energy

$$\mathcal{T}_{B_u} := Q_{B_u}\Big(\chi_{B_u}\Big[\sqrt{-\Delta} - \frac{1}{ds\ell}\Big]_+^2 \chi_{B_u} + \frac{\varepsilon_T}{2}(1+\pi^{-2})\frac{1}{(d\ell)^2}\Big)Q_{B_u}. \tag{D13}$$

We are now ready to define the localized Hamiltonian $\mathcal{H}_{B_u}$ which acts on the symmetric Fock space $\mathcal{F}_s(L^2(B_u))$. It preserves particle number and is given as

$$\mathcal{H}_{B_u}(\rho_\mu)_N := \sum_{i=1}^{N}(1-\varepsilon_N)\mathcal{T}_{B_u}^{(i)} - \rho_\mu \sum_{i=1}^{N} \int w_{1,B_u}(x_i, y)\,\mathrm{d}y + \frac{1}{2}\sum_{i\neq j} w_{B_u}(x_i, x_j), \tag{D14}$$

on the N-particle sector where ε_N was introduced in Lemma 6.2.

An adaptation to dimension 2 of [28, Theorem 3.10] allows us relate $\mathcal{H}_B(\rho_\mu)$ to the original Hamiltonian in the large box, using the condition (H28). This gives the lower bound

$$\mathcal{H}_\Lambda(\rho_\mu) \geq (1-\varepsilon_N)\frac{b}{\ell^2}\sum_{j=1}^{N} Q_{\Lambda,j} + \int_{\mathbb{R}^2} \mathcal{H}_{B_u}(\rho_\mu)\mathrm{d}u. \tag{D15}$$

We would like to restrict the previous integral to boxes that are not too small. Therefore, we identify the following sets of integration, for $\xi \in [0,1]$,

$$\Lambda_\xi := \Big\{u \in \mathbb{R}^2 \,\Big|\, |\ell d u|_\infty - \frac{\ell}{2}(d+1) \leq -\xi d\ell\Big\}, \tag{D16}$$

underlying the property

$$\Lambda_{\xi_1} \subseteq \Lambda_{\xi_2} \quad \text{if} \quad \xi_1 \geq \xi_2, \tag{D17}$$

and we observe that integration outside Λ_0 is zero because there is no more intersection between the small box and Λ. The following Lemma guarantees that we can restrict the integration for the potential over set $\Lambda_{1/10}$ (where $1/10$ is chosen arbitrarily) and estimate the remaining part by a frame inside $\Lambda_{1/10}$.

Lemma D.1. *For all $x \in \Lambda$ we have the estimate*

$$\begin{aligned} -\rho_\mu &\iint w_{1,B_u}(x,y)\mathrm{d}y\mathrm{d}u \\ &\geq -\rho_\mu \int_{\Lambda_{\frac{1}{10}}} \int w_{1,B_u}(x,y)\mathrm{d}y\mathrm{d}u - 3\rho_\mu \int_{\Lambda_{\frac{1}{10}}\setminus\Lambda_{\frac{1}{5}}} \int w_{1,B_u}(x,y)\mathrm{d}y\mathrm{d}u. \end{aligned} \tag{D18}$$

Proof. The proof follows the same lines as in [19, Lemma E.1]. We split the domain of integration $\Lambda_{1/10}$ and $\Lambda_0 - \Lambda_{1/10}$ and we estimate the integral over the latter. By the definition of $w_{1,B}$ we have simply to estimate the quantity

$$\int_{\Lambda_0\setminus\Lambda_{1/10}} \chi\left(\frac{x}{\ell d} - u\right)\chi\left(\frac{y}{\ell d} - u\right)\mathrm{d}u. \tag{D19}$$

We use that χ is a product of decreasing functions in the variables u_1, u_2 and observe that

$$\begin{aligned}&\max_{\frac{1}{2}(d^{-1}+1)-\frac{(\ell d)^{-1}}{10}\le|u_1|\le\frac{1}{2}(d^{-1}+1)} \chi\left(\frac{x}{\ell d} - u\right)\chi\left(\frac{y}{\ell d} - u\right)\\ &\le \min_{\frac{1}{2}(d^{-1}+1)-\frac{2(d\ell)^{-1}}{10}\le|u_1|\le\frac{1}{2d}+\frac{1}{2}-\frac{(d\ell)^{-1}}{10}} \chi\left(\frac{x}{\ell d} - u\right)\chi\left(\frac{y}{\ell d} - u\right),\end{aligned} \tag{D20}$$

so that we can estimate the integral over the frame pointwise, getting a factor of 3 due to the presence of the corners. □

Thanks to Lemma D.1 we can write

$$\mathcal{H}_\Lambda(\rho_\mu) \ge (1-\varepsilon_N)\frac{b}{\ell^2}\sum_{j=1}^N Q_{\Lambda,j} + \int_{\Lambda_{\frac{1}{5}}} \mathcal{H}_{B_u}(\rho_\mu)\mathrm{d}u + \int_{\Lambda_{\frac{1}{10}}\setminus\Lambda_{\frac{1}{5}}} \mathcal{H}_{B_u}(4\rho_\mu)\mathrm{d}u, \tag{D21}$$

where we dropped the positive part of $\mathcal{H}_{B_u}$ in $\Lambda_0\setminus\Lambda_{\frac{1}{10}}$. We are now ready to give lower bounds for kinetic and potential energies in terms of the number of particles. From this the lower bound for the small box Hamiltonian is going to follow in Corollary D.6 below.

In order to prove Theorem 7.6, we provide a lower bound on $\mathcal{H}_{B_u}(\rho_\mu)$. For notational simplicity we will remove the index u. Lemmas D.2 and D.3 below give first lower bounds on the potential and kinetic energy respectively.

Lemma D.2. *There exists a constant $C > 0$ depending only on χ such that*

$$\begin{aligned}&-\rho_\mu\int\sum_{j=1}^N w_{1,B}(x,y)\mathrm{d}y + \frac{1}{2}\sum_{i\neq j} w_B(x_i,x_j)\\ &\quad\ge A_0 + A_2 + \frac{1}{2}Q_4^{ren,\,s} - C\delta\Big(\rho_\mu + \frac{n_{0,B}}{|B|}\Big)n_{+,B},\end{aligned}$$

with

$$\begin{aligned}A_0 &:= \frac{n_{0,B}(n_{0,B}+1)}{2|B|^2}\iint w_{2,B}(x,y)\mathrm{d}x\mathrm{d}y\\ &\quad-\Big(\frac{\rho_\mu n_{0,B}}{|B|} + \frac{1}{4}\big(\rho_\mu - \frac{n_{0,B}-1}{|B|}\big)^2\Big)\iint w_{1,B}(x,y)\mathrm{d}x\mathrm{d}y,\\ A_2 &:= \frac{1}{2}\sum_{i\neq j} P_iP_j w_{1,B} Q_iQ_j + h.c.,\end{aligned}$$

and $Q_4^{ren,s}$ is the analogue of (7.2), *but for the small box B.*

Proof. The proof follows from an analogous potential splitting like in Lemma 7.1 and Lemma 7.2 and by the same lines as [19, Lemma E.7]. □

Lemma D.3. *For the kinetic energy on the small box in the N'th sector we have*

$$Q_B\chi_B\Big[\sqrt{-\Delta}-\frac{1}{ds\ell}\Big]_+^2\chi_B Q_B + A_2$$
$$\geq -\frac{1}{2}\widehat{g\omega}(0)\frac{N(N+1)}{|B|^2}\int\chi_B^2+\mathcal{E}_2(N)+\mathcal{E}_4(N)-C\delta\frac{N+1}{|B|}n_{+,B},$$

where

$$\mathcal{E}_2(N):=-C\delta\Big(\frac{R^2}{(d\ell)^2}+\delta|\log(dsK_\ell)|+\delta^2\Big)\frac{N(N+1)}{|B|^2}\int\chi_B^2, \tag{D22}$$
$$\mathcal{E}_4(N):=-C\Big(\delta^4(ds\ell)^4\Big(\frac{N+1}{|B|}\Big)^3+\delta(ds\ell)^{-2}\Big)\frac{N}{|B|}\int\chi_B^2. \tag{D23}$$

Proof. Let us introduce the operators

$$d_p^\dagger:=\frac{1}{|B|^{1/2}}a^\dagger(Q_B\chi_Be^{-ipx})a_0, \tag{D24}$$

where $a_0=\frac{1}{\ell}a(1)$ and $a, a^\dagger$ are the annihilation and creation operators on $\mathscr{F}_s(L^2(\Lambda))$. Further we introduce

$$A_1:=\frac{\widehat{W}_1^s(0)}{(2\pi)^2}\int_{\mathbb{R}^2}(d_p^\dagger d_p+d_{-p}^\dagger d_{-p})\mathrm{d}p. \tag{D25}$$

Now using that on the N'th sector we have

$$\int\Big[|p|-\frac{1}{ds\ell}\Big]_+^2 d_p^\dagger d_p\mathrm{d}p=\frac{(n_0+1)}{|B|}\sum_{j=1}^N Q_{B,j}\chi_B(x_j)\Big[\sqrt{-\Delta}-\frac{1}{ds\ell}\Big]_+^2\chi_B(x_j)Q_{B,j}, \tag{D26}$$

and that $n_0\leq N$, we get, adding A_1 and A_2 to the kinetic energy

$$\sum_{j=1}^N Q_{B,j}\chi_{B,j}\Big[\sqrt{-\Delta}-\frac{1}{ds\ell}\Big]_+^2\chi_{B,j}Q_{B,j}+A_1+A_2\geq\frac{1}{2(2\pi)^2}\int h_p\mathrm{d}p, \tag{D27}$$

where

$$h_p:=\mathcal{A}_p(d_p^\dagger d_p+d_{-p}^\dagger d_{-p})+\mathcal{B}_p(d_p^\dagger d_{-p}^\dagger+d_{-p}d_p), \tag{D28}$$

with

$$\mathcal{A}_p:=(1-\varepsilon_N)\frac{|B|}{N+1}\Big[|p|-\frac{1}{ds\ell}\Big]_+^2+2\widehat{W}_1^s(0),\qquad \mathcal{B}_p:=\widehat{W}_1^s(p). \tag{D29}$$

The additional A_1 term is estimated, thanks to (D8), by

$$A_1\leq C\delta\frac{n_0+1}{|B|}n_{+,B}, \tag{D30}$$

which contributes to the last term in the result of the lemma. By an application of Theorem B.1 we get the bound

$$\frac{1}{2(2\pi)^2}\int h_p \mathrm{d}p \geq -\frac{1}{2(2\pi)^2}\frac{N}{|B|}\int \chi_B^2 \int \Big(\mathcal{A}_p - \sqrt{\mathcal{A}_p^2 - \mathcal{B}_p^2}\Big)\mathrm{d}p, \tag{D31}$$

and therefore we want to bound the latter. We observe that, thanks to (D9), $\widehat{W}_1^s(0) \geq C\delta$ for a certain $C < 8\pi$. Choosing the parameters

$$K = (ds\ell)^{-1}, \quad \kappa = (1-\varepsilon_N)\frac{|B|}{N+1}, \quad K_1 = 1, \quad K_2 = C, \tag{D32}$$

we can apply Lemma C.5 to obtain

$$\begin{aligned}
&-\frac{1}{2(2\pi)^2}\int (\mathcal{A}_p - \sqrt{\mathcal{A}_p^2 - \mathcal{B}_p^2})\mathrm{d}p \\
&\geq -\frac{N+1}{|B|(1-\varepsilon_N)}\widehat{g\omega}(0) - \frac{N+1}{|B|}\frac{R^2}{(d\ell)^2}\delta + C\delta(ds\ell)^{-2} \\
&\quad - C\frac{N+1}{|B|}\delta^2\Big(1+\Big(\frac{R}{\ell_\delta}\Big)^2\Big) - C\frac{N+1}{|B|}\delta^2|\log(2dsK_\ell)| - C\delta^4\frac{(N+1)^3}{|B|^3}(ds\ell)^4,
\end{aligned}$$

where we used that $\varepsilon_N \leq 1/2$ and (D10) to approximate the leading term by $\widehat{g\omega}(0)$ getting the second term as an error. Plugging the last estimate in (D31) we get the result with the error terms coherent with the definitions of $\mathcal{E}_2(n)$ and $\mathcal{E}_4(n)$. □

We will also need the following estimates.

Lemma D.4. *Let ℓ_{min} denote the shortest length of the box B, then there exists a constant $C > 0$ such that*

$$\Big|\iint w_{1,B}(x,y)\mathrm{d}x\mathrm{d}y - 8\pi\delta\int \chi_B^2\Big| \leq C\delta\Big(\frac{R}{\ell_{min}}\Big)^2\int \chi_B^2, \tag{D33}$$

and

$$\iint w_{2,B}(x,y)\mathrm{d}x\mathrm{d}y \geq \iint w_{1,B}(x,y)\mathrm{d}x\mathrm{d}y + \widehat{g\omega}(0)\int \chi_B^2 - C\frac{R\delta^2}{\ell_{\min}^2}\int \chi_B^2. \tag{D34}$$

Proof. Since $8\pi\delta = \int g$, and thanks to (D9), we can write the inequality

$$\Big|\iint (W_1^s(x) - g(x))\chi_B^2(y)\mathrm{d}x\mathrm{d}y\Big| \leq C\delta\Big(\frac{R}{\ell_{\min}}\Big)^2\int \chi_B^2, \tag{D35}$$

where we used $\ell_{\min} \leq d\ell$. By a Taylor expansion for the localization function and the fact that W is spherically symmetric and (D8), we have, on the other hand,

$$\begin{aligned}
\Big|\iint w_{1,B}(x,y)\mathrm{d}x\mathrm{d}y - \int W_1^s(x)\mathrm{d}x\int \chi_B^2\Big| &\leq CR^2\|\nabla^2\chi_B\|_\infty\int W_1^s(x)\mathrm{d}x\int \chi_B \\
&\leq C\Big(\frac{R}{\ell_{\min}}\Big)^2\delta\int \chi_B^2,
\end{aligned} \tag{D36}$$

and where we used that $|B|^{-1}(\int \chi_B)^2 \leq \int \chi_B^2$ and the bound (F6).

Then inequality (D33) follows by (D35) and (D36). The inequality (D34) follows from a very similar argument. □

Combining the results of Lemma D.2 and D.3, we deduce that the Hamiltonian on the small box has the following lower bound, which is coherent with the main order of the energy expansion.

Theorem D.5. *Assume the conditions from "Appendix H", then for any box B we have the following bound on the N'th sector*

$$\mathcal{H}_B(\rho_\mu)|_N \geq \Big(\frac{1}{4}\Big(\rho_\mu - \frac{N}{|B|}\Big)^2 - \frac{1}{2}\rho_\mu^2\Big) \iint w_{1,B} + \frac{1}{2}\mathcal{Q}_4^{ren,s} + \mathcal{E}_2(N) + \mathcal{E}_4(N), \tag{D37}$$

with $\mathcal{E}_2$ and $\mathcal{E}_4$ defined in (D22).

Proof. The combination of Lemmas D.2, D.3 gives

$$\begin{aligned}\mathcal{H}_B(\rho_\mu) \geq& \sum_{j=1}^{n} Q_{B,j}\Big(\frac{\varepsilon_T}{2}(1+\pi^{-2})\frac{1}{(d\ell)^2}\Big)Q_{B,j} + A_0 - \frac{1}{2}\widehat{g\omega}(0)\frac{N(N+1)}{|B|^2}\int \chi_B^2 \\ &+ \frac{1}{2}\mathcal{Q}_4^{\text{ren,s}} + \mathcal{E}_2(N) + \mathcal{E}_4(N) - C\delta\rho_\mu n_{+,B}.\end{aligned}$$

We observe that we can choose a constant $C' > 0$ such that

$$\sum_{j=1}^{N} Q_{B,j}\Big(\frac{\varepsilon_T}{2}\frac{1}{(d\ell)^2}\Big) \geq C'\rho_\mu\delta n_{+,B}, \tag{D38}$$

where we used (H8) and, choosing the right C', we can cancel the last term with $n_{+,B}$ for a lower bound. The same can be said for the errors produced by replacing $n_0 = N - n_+$ by N. By Lemma D.4 we have

$$\begin{aligned}&A_0 - \frac{1}{2}\widehat{g\omega}(0)\frac{N(N+1)}{|B|}\int \chi_B^2 \\ &\geq \Big(-\frac{N^2}{2|B|^2} - \Big(\rho_\mu\frac{N}{|B|} + \frac{1}{4}\Big(\rho_\mu - \frac{N}{|B|}\Big)^2\Big)\Big) \iint w_{1,B} - C\frac{N^2}{|B|^2}\frac{R\delta^2}{\ell_{\text{min}}^2}\int \chi_B^2 \\ &\geq \Big(\frac{1}{4}\Big(\rho_\mu - \frac{N}{|B|}\Big)^2 - \frac{1}{2}\rho_\mu^2\Big) \iint w_{1,B} - C\frac{N^2}{|B|^2}\frac{R\delta^2}{\ell_{\text{min}}^2}\int \chi_B^2,\end{aligned}$$

and this gives the result since the last term can be reabsorbed in the $\mathcal{E}_2$ term. □

We deduce the following corollary.

Corollary D.6. *Assume B is a small box with shortest side length $\ell_{min} \geq \frac{d\ell}{10}$ and that the conditions of "Appendix H" hold true. Then we have the following lower bound*

$$\mathcal{H}_B(\rho_\mu) \geq -\frac{1}{2}\rho_\mu^2 \iint w_{1,B}(x,y)\mathrm{d}x\mathrm{d}y - C\rho_\mu^2\delta^2(dsK_\ell)^{-2}\int \chi_B^2 - C\rho_\mu\delta\frac{1}{|B|}\int \chi_B^2.$$

Proof. We split the particles in m subsets of n' particles and a remaining group of n'', with $n'' < n' < n$. If we ignore the positive interactions between the subsets, and denoting by $e_B(n, \rho_\mu)$ the ground state energy of $\mathcal{H}_B(\rho_\mu)$ restricted to states with n particles in the box B, then

$$e_B(n, \rho_\mu) \geq m e_B(n', \rho_\mu) + e_B(n'', \rho_\mu). \tag{D39}$$

From formula (D37) in Theorem D.5 applied for n' in place of n and, choosing $n' = 3\rho_\mu |B|$, we see that the first term becomes, thanks to Lemma D.4

$$\frac{1}{2}\rho_\mu^2 \iint w_{1,B} \geq 4\pi \rho_\mu^2 \delta\Big(1 - C\Big(\frac{R}{\ell_{\min}}\Big)^2\Big) \int \chi_B^2. \tag{D40}$$

From the following controls on the error terms

$$\mathcal{E}_2(n') \leq C\rho_\mu^2 \delta^2 (dK_\ell)^{-2} \int \chi_B^2, \tag{D41}$$

$$\mathcal{E}_4(n') \leq C\rho_\mu^2 \delta^2 (dsK_\ell)^{-2} \int \chi_B^2, \tag{D42}$$

$$C\rho_\mu^2 \delta\Big(\frac{R}{\ell_{\min}}\Big)^2 \int \chi_B^2 \leq C\rho_\mu^2 \delta \frac{R^2}{(d\ell)^2} \int \chi_B^2 \leq C\rho_\mu^2 \delta^2 (dK_\ell)^{-2} \int \chi_B^2, \tag{D43}$$

we see that the first term is the leading term of the energy. Since it is clearly positive, we obtain that with the aforementioned choice of n', we have $e_B(n', \rho_\mu) \geq 0$ and, then, using the previous equality, we can state that

$$e_B(n, \rho_\mu) \geq e_B(n'', \rho_\mu). \tag{D44}$$

The Corollary follows using again Theorem D.5 with n'' in place of n to obtain the lower bound and using (D41) and (D42) for n'' to control the errors, using that $s^{-1} \gg 1$ to obtain one of the error terms in the statement. A further error term of order

$$C\rho_\mu \delta \frac{1}{|B|} \int \chi_B^2,$$

is created by the substitutions of the terms $n'' \pm 1$ by n''. □

We are finally ready to use the lower bound on the small box Hamiltonian to obtain a bound on the number of excited particles in the large box, result stated in the Theorem below. By an abuse notation, from now on, the operators n, n_+, n_0 start again to denote the number operators in the large box.

Theorem D.7. *We have the following lower bound for the large box Hamiltonian*

$$\mathcal{H}_\Lambda(\rho_\mu) \geq -4\pi \rho_\mu^2 \ell^2 Y\Big(1 - \frac{1}{2} Y |\log Y|\Big) + \frac{b}{2\ell^2} n_+, \tag{D45}$$

and if there exists a normalized $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ *with* n *particles in* Λ *such that (7.40) holds:*

$$\langle \mathcal{H}_\Lambda(\rho_\mu)\rangle_\Psi \leq -4\pi \rho_\mu^2 \ell^2 Y(1 - CK_B^2 Y |\log Y|), \tag{D46}$$

then the bound (7.41) *for the number of excitations holds:*

$$\langle n_+ \rangle_\Psi \leq C n K_B^2 K_\ell^2 Y |\log Y|. \tag{D47}$$

Proof. We study the integration over $\Lambda_{1/10} \setminus \Lambda_{1/5}$ from formula (D21). By [18, (C.6)] we have $|\chi_{B_u}| \leq C(\ell_{\min}\ell^{-1})^M$ and then

$$\int_{\Lambda_{1/10}-\Lambda_{1/5}} \int \chi_{B_u}(x)^2 \mathrm{d}x\mathrm{d}u \leq C\Big(\frac{\ell_{\min}}{\ell}\Big)^{2M} (\ell d)^2 d^{-2} \leq C\Big(\frac{\ell_{\min}}{\ell}\Big)^{2M} \ell^2. \quad \text{(D48)}$$

By the joint action of Corollary D.6 and Lemma D.4 we get

$$\mathcal{H}_{B_u}(4\rho_\mu) \geq -C\rho_\mu^2 \delta \int \chi_{B_u}^2 - C\rho_\mu\delta\Big(\rho_\mu\delta(dsK_\ell)^{-2} + \frac{1}{|B_u|}\Big) \int \chi_{B_u}^2, \quad \text{(D49)}$$

and therefore, using (D48) and that $|B| = d^2\ell^2$ we have

$$\int_{\Lambda_{1/10}\setminus\Lambda_{1/5}} \mathcal{H}_{B_u}(4\rho_\mu)\mathrm{d}u \geq -C\rho_\mu^2\ell^2\delta(1+\delta(dsK_\ell)^{-2})\Big(\frac{\ell_{\min}}{\ell}\Big)^{2M} - C\rho_\mu\delta d^{-2}, \text{(D50)}$$

Using the definition of $\ell_{\min} = d\ell/10$ and the relations between the parameters (H12) we get

$$\Big(\frac{\ell_{\min}}{\ell}\Big)^{2M} \leq d^{2M} \leq \delta, \qquad \rho_\mu\delta d^{-2} \leq \rho_\mu^2\ell^2\delta^2(K_\ell d)^{-2} \leq \rho_\mu^2\ell^2\delta^2K_B^2, \quad \text{(D51)}$$

which makes the integral coherent with the statement of the Theorem using the expansion of δ,

$$\delta \simeq Y - Y^2|\log Y| + \mathcal{O}(Y^3|\log Y|^2). \quad \text{(D52)}$$

For the remaining integral in formula (D21) we use Corollary D.6 and Lemma D.4 to get

$$\begin{aligned}
&\int_{\Lambda_{1/10}} \mathcal{H}_{B_u}(\rho_\mu)\mathrm{d}u \\
&\geq -\int_{\Lambda_{1/10}} \Big[\iint \mathrm{d}x\mathrm{d}y\, \frac{1}{2}\rho_\mu^2 w_{1,B_u}(x,y) + C\rho_\mu\delta\Big(\rho_\mu\delta(dsK_\ell)^{-2} + \frac{1}{|B_u|}\Big)\int \chi_{B_u}^2\Big]\mathrm{d}u \\
&\geq -4\pi\rho_\mu^2\ell^2\delta - C\rho_\mu^2\ell^2\delta^2K_B^2,
\end{aligned}$$

where we used (H3), (H12) and

$$\iiint w_{1,B_u}(x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}u = 8\pi\delta\ell^2, \qquad \iint \chi_{B_u}(x)^2\mathrm{d}u\mathrm{d}x = \ell^2.$$

Collecting the previous estimates, together with (D21) and the fact that $\varepsilon_N \leq \frac{1}{2}$, we finally get (D45), using the expansion (D52) of δ.

The proof of the bound on n_+ is proven noting that, joining together the a priori bound (7.40) with the obtained lower bound we get

$$\frac{b}{2\ell^2}\langle n_+\rangle_\Psi \leq CK_B^2\rho_\mu\ell^2Y^2|\log Y|, \quad \text{(D53)}$$

and conclude recalling that $\ell = \rho_\mu^{-1/2}Y^{-1/2}K_\ell$. □

We follow now a similar strategy to obtain a lower bound for the large box Hamiltonian and get an a priori bound on the number of particles and a control on $\mathcal{Q}_4^{\text{ren}}$ in the large box.

Corollary D.8. *If there exists a n-particles state* $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ *such that* (7.40) *holds, then the a priori bounds on n and* $\mathcal{Q}_4^{ren}$ *hold:*

$$\left|\rho_\mu - \frac{n}{\ell^2}\right| \leq C K_B K_\ell \rho_\mu Y^{1/2} |\log Y|^{1/2}, \qquad \langle \mathcal{Q}_4^{ren} \rangle_\Psi \leq C K_B^2 K_\ell^2 \rho_\mu^2 \ell^2 Y^2 |\log Y|. \tag{D54}$$

Proof. We observe that we have the following lower bound, reproducing analogous estimates for potential and kinetic energies from Lemmas D.2 and D.3 but adapted to the large box Λ (for details, see [19, Appendix E.2]), where we estimate the n_+ contributions thanks to Theorem D.7,

$$\langle \mathcal{H}_\Lambda(\rho_\mu) \rangle_\Psi \geq \frac{1}{2} \langle \mathcal{Q}_4^{\text{ren}} \rangle_\Psi - 4\pi \rho_\mu^2 \ell^2 \delta + 2\pi \left(\rho_\mu - \frac{n}{\ell^2}\right)^2 \ell^2 \delta - C K_B^2 K_\ell^2 \rho_\mu^2 \ell^2 \delta^2. \tag{D55}$$

By the assumption, the expansion of δ in terms of Y and (D55) we get

$$\left(\frac{n}{\ell^2} - \rho_\mu\right)^2 \ell^2 \delta + \langle \mathcal{Q}_4^{\text{ren}} \rangle_\Psi \leq C K_B^2 K_\ell^2 \rho_\mu^2 \ell^2 Y^2 |\log Y|, \tag{D56}$$

which implies the desired bounds. □

Appendix E: Technical Estimates for Off-Diagonal Excitation Terms

We give here a proof of Lemma 7.9, bounding the terms d_1^L and d_2^L defined in (7.48) and (7.49). We are going to use the following dimension independent estimates which are proven in [19, Corollary F.6] in order to prove the technical lemma below. There exists $C > 0$, such that, for any $\varphi \in \operatorname{Ran} \overline{Q}_H$,

$$\|\Delta(\chi_\Lambda \varphi)\| \leq C \varepsilon_N^{-1/2} \frac{\widetilde{K}_H^2}{\ell^2}, \qquad \|\Delta^{\mathcal{N}} \varphi\| \leq C \varepsilon_N^{-1} \frac{\widetilde{K}_H^2}{\ell^2}. \tag{E1}$$

Lemma E.1. *If we assume the relations between the parameters in "Appendix H", then there exists* $C > 0$ *such that*

$$\|\overline{Q}_{H,x} w(x,y) \overline{Q}_{H,x}\| \leq C \varepsilon_N^{-1/2} \frac{\widetilde{K}_H^2}{\ell^2} \|v\|_1. \tag{E2}$$

Proof. The proof is an adaptation to 2 dimensions of [19, Lemma 5.3]. Let $\varphi \in \operatorname{Ran} \overline{Q}_{H,x}$ with $\|\varphi\|_2 = 1$, then

$$\|\overline{Q}_{H,x} w(x,y) \overline{Q}_{H,x} \varphi\| \leq I_1 + I_2, \tag{E3}$$

where

$$I_1 = \int_{\mathbb{R}^2} \chi_\Lambda(x)^2 |\varphi(x)|^2 v(x-y) \mathrm{d}x,$$

$$I_2 = \int_{\mathbb{R}^2} \chi_\Lambda(x)\, |\chi_\Lambda(x) - \chi_\Lambda(y)|\, |\varphi(x)|^2 v(x-y) \mathrm{d}x.$$

We use the technical Lemma E.2 below to get

$$|I_1| \leq \|\chi_\Lambda\varphi\|_\infty^2 \|v\|_1 \leq C\|v\|_1 \|\chi_\Lambda\varphi\| \|\Delta\chi_\Lambda\varphi\| \leq C\frac{\widetilde{K}_H^2}{\ell^2}\varepsilon_N^{-1/2}\|v\|_1, \tag{E4}$$

by (E1) and (E6) and

$$\begin{aligned}|I_2| &\leq C\frac{R}{\ell}\|\chi_\Lambda\varphi\|_\infty\|\varphi\|_\infty\|v\|_1 \leq C\frac{R}{\ell}\|\Delta(\chi_\Lambda\varphi)\|^{1/2}\|\Delta^{\mathcal{N}}\varphi\|^{1/2}_{L^2\left(\left[-\frac{L}{2},\frac{L}{2}\right]^2\right)}\|v\|_1 \\ &\leq C\varepsilon_N^{-1/2}\Big(\frac{R}{\ell}\varepsilon_N^{-1/4}\Big)\frac{\widetilde{K}_H^2}{\ell^2}\|v\|_1 \leq C\varepsilon_N^{-1/2}\frac{\widetilde{K}_H^2}{\ell^2}\|v\|_1,\end{aligned}$$

by a Taylor expansion for the localization function, (E5) and (E6) for φ and $\chi_\Lambda\varphi$, respectively, (E1) and the choice of the parameters in (H11), (H3), (H14) and this concludes the proof. □

In the proof of Lemma E.1 we used the following result.

Lemma E.2. *Let $-\Delta^{\mathcal{N}}$ denote the Neumann Laplacian on $[-\frac{L}{2},\frac{L}{2}]^2$. There exists $C > 0$ such that, for all $f \in \mathcal{D}(-\Delta^{\mathcal{N}})$ such that $\int_{[-\frac{L}{2},\frac{L}{2}]^2} f(x)\mathrm{d}x = 0$, we have*

$$\|f\|_\infty \leq C\|f\|^{1/2}_{L^2([-\frac{L}{2},\frac{L}{2}]^2)}\|-\Delta^{\mathcal{N}}f\|^{1/2}_{L^2([-\frac{L}{2},\frac{L}{2}]^2)}. \tag{E5}$$

Also, for all $f \in H^2(\mathbb{R}^2)$,

$$\|f\|_\infty \leq C\|f\|^{1/2}\|\Delta f\|^{1/2}. \tag{E6}$$

Proof. Let us prove the last inequality, the first one being proven by an adaptation for the box (essentially the only difference is to replace sums by integrals). We use a scaling argument defining $f_\lambda(x) := f(\lambda x)$, for $x \in \mathbb{R}^2$. Given an $f \in H^2(\mathbb{R}^2)$, it is clearly possible to choose λ such that $\|f_\lambda\| = \|\Delta f_\lambda\|$. Now, for the given λ, we have

$$\|f_\lambda\|_\infty^2 \leq \frac{1}{(2\pi)^2}\Big(\int_{\mathbb{R}^2}|\hat{f}_\lambda(p)|\mathrm{d}p\Big)^2 \leq C\int_{\mathbb{R}^2}(1+|p|^4)|\hat{f}_\lambda(p)|^2\mathrm{d}p = C\|\Delta f_\lambda\|^2,$$

where we multiplied and divided by $(1+|p|^4)^{1/2}$ and used the Cauchy–Schwarz inequality and the choice of λ. Applied to f_λ with the λ chosen above the previous inequality becomes

$$\|f\|_\infty^2 = \|f_\lambda\|_\infty^2 \leq C\|\Delta f_\lambda\|^2 = C\|f_\lambda\|\|\Delta f_\lambda\| = C\|f\|\|\Delta f\|, \tag{E7}$$

where in the last equality we used the scaling properties of the dilatation in λ w.r.t. the L^2 norm. □

Proof of Lemma 7.9. Let $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ be satisfying the assumptions of Lemma 7.9. Our goal is to prove the following estimate

$$\begin{aligned}&\langle(d_1^L + d_2^L)\rangle_{\widetilde{\Psi}} \\ &\leq \rho_\mu\|v\|_1\Big(\langle n_+\rangle_{\widetilde{\Psi}} + n^{1/2}\langle n_+^L\rangle_{\widetilde{\Psi}}^{1/2} + \langle(n_+^L)^2\rangle_{\widetilde{\Psi}}^{1/2}\varepsilon_N^{-1/4}\widetilde{K}_H + \langle n_+^L\widetilde{n}_+^H\rangle_{\widetilde{\Psi}}\varepsilon_N^{-1/4}\widetilde{K}_H \\ &+ \big(\langle\widetilde{n}_+^H n_+^L\rangle_{\widetilde{\Psi}}^{1/2}\langle(n_+^L)^2\rangle_{\widetilde{\Psi}}^{1/2} + \langle\widetilde{n}_+^H n_+^L\rangle_{\widetilde{\Psi}}\big)n^{-1}\varepsilon_N^{-1/2}\widetilde{K}_H^2\Big) + C\langle\mathcal{Q}_4^{\mathrm{ren}}\rangle_{\widetilde{\Psi}}.\end{aligned} \tag{E8}$$

We split the d_j^L in several terms multiplying out the parentheses in (7.48) and (7.49). All these terms we treat individually using Cauchy–Schwarz inequalities. Similar bounds have been carried out in [19]. Here we just bound some representative examples to illustrate the procedure and the role played by Lemma E.1.

Let us start using the Cauchy–Schwarz inequality for any $\varepsilon > 0$ to get

$$\Big|\Big\langle -\rho_\mu \sum_i P_i \int \mathrm{d}y\, w_1(x_i,y)\, \overline{Q}_{H,i} + h.c.\Big\rangle_\Psi\Big| \leq \frac{n}{\ell^2}\|w_1\|_1\big(\varepsilon n + \varepsilon^{-1}\langle n_+^L\rangle_\Psi\big),$$

observing that $\|w_1\|_1 \leq C\delta$ and choosing $\varepsilon = \langle n_+^L\rangle_\Psi^{1/2} n^{-1/2}$, we obtain the desired quantity.

For the following term, for any $\varepsilon > 0$, we use the Cauchy–Schwarz inequality and Lemma E.1,

$$\begin{aligned}\Big|\Big\langle \sum_{i,j} P_i \overline{Q}_{H,j} w \overline{Q}_{H,i} \overline{Q}_{H,j}\Big\rangle_\Psi\Big| &\leq \varepsilon \frac{n}{\ell^2}\|w\|_1 \langle n_+^L\rangle_\Psi + \varepsilon^{-1}\|\overline{Q}_H w \overline{Q}_H\| \sum_{i\neq j}\langle \overline{Q}_{H,i}\overline{Q}_{H,j}\rangle_\Psi \\ &\leq \frac{\|w\|_1}{\ell^2}(\varepsilon n^2 + \varepsilon^{-1}\varepsilon_N^{-1/2}\widetilde{K}_H^2\langle (n_+^L)^2\rangle_\Psi)\end{aligned}$$

where we used that $n_+^L \leq n_+$. Choosing $\varepsilon = \varepsilon_N^{-1/4}\widetilde{K}_H \langle (n_+^L)^2\rangle_\Psi^{1/2} n^{-1}$, we obtain

$$\Big|\Big\langle \sum_{i,j} P_i \overline{Q}_{H,j} w \overline{Q}_{H,i} \overline{Q}_{H,j}\Big\rangle_\Psi\Big| \leq n\frac{\langle (n_+^L)^2\rangle_\Psi^{1/2}}{\ell^2}\varepsilon_N^{-1/4}\widetilde{K}_H\|w\|_1. \tag{E9}$$

For the next term we want to apply a Cauchy–Schwarz inequality to reobtain a Q_4^{ren} term. In order to do that we are going to complete the Q_H to a $Q = Q_H + \overline{Q}_H$.

$$\begin{aligned}&\Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i} P_j w Q_{H,i} Q_{H,j} + h.c.\Big\rangle_\Psi\Big| \\ &\leq \Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i} P_j w Q_i Q_j\Big\rangle_\Psi + h.c.\Big| \\ &\quad + \Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i} P_j w (Q_{H,i}\overline{Q}_j + \overline{Q}_{H,i} Q_{H,j})\Big\rangle_\Psi + h.c.\Big| \\ &\quad + \Big|\Big\langle \sum_{i,j} P_i \overline{Q}_{H,j} w \overline{Q}_{H,i}\overline{Q}_{H,j}\Big\rangle_\Psi\Big|.\end{aligned}$$

The second term and the third terms can be estimated in the same manner as above, so let us focus on completing the first term in order to obtain the Q_4.

$$\Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i} P_j w Q_i Q_j\Big\rangle_\Psi + h.c.\Big| \tag{E10}$$

$$\leq \Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i} P_j w (Q_i Q_j + \omega(P_i P_j + P_i Q_j + Q_i P_j))\Big\rangle_\Psi + h.c.\Big| \tag{E11}$$

$$+ \Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i} P_j w\omega (P_i Q_j + Q_i P_j))\Big\rangle_\Psi + h.c.\Big| \tag{E12}$$

$$+ \Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i} P_j w\omega P_i P_j)\Big\rangle_\Psi + h.c.\Big|. \tag{E13}$$

The second and the third terms are treated as above, using that $0 \leq \omega \leq 1$ on the support of w. By a Cauchy–Schwarz inequality on the first term we get

$$(E11) \leq \langle \mathcal{Q}_4^{\text{ren}} \rangle_\Psi + C\frac{n}{\ell^2}\|w\|_1 \langle n_+ \rangle_\Psi. \tag{E14}$$

Collecting the previous estimates including the ones not explicitly treated, we obtain (E8).

Bounding $n_+^L \leq \widetilde{\mathcal{M}}$ in (E8) where it appears for higher moments than 1, using that $\widetilde{n}_+^H \leq n$ and that $\varepsilon_N^{-1/4} \widetilde{K}_H \geq 1$ by (H15) gives the result. This finishes the proof of Lemma 7.9. □

Appendix F: Properties of the Localization Function

We collect here the definition and some important properties of the localization function that are used throughout the paper.

We define

$$\chi(x) := C_M(\zeta_1(x_1)\zeta_2(x_2))^{M+2}, \tag{F1}$$

where

$$\zeta(y) := \begin{cases} \cos(\pi y), & |y| \leq 1/2, \\ 0, & |y| > 1/2, \end{cases} \tag{F2}$$

where $M \in \mathbb{N}$ is chosen even and large enough. The normalization constant $C_M > 0$ is chosen in order to obtain $\|\chi\|_2 = 1$. We have $0 \leq \chi \in C^M(\mathbb{R}^2)$. We also define $\chi_\Lambda(x) = \chi(x/\ell)$.

Lemma F.1. *Let χ be the localization fuction defined above and let $M \in 2\mathbb{N}$. Then, for all $k \in \mathbb{R}^2$,*

$$|\widehat{\chi}(k)| \leq \frac{C_\chi}{(1+|k|^2)^{M/2}}, \tag{F3}$$

where $C_\chi = \int |(1-\Delta)^{M/2}\chi|$. If, furthermore, $|k| \geq \frac{1}{2}K_K\ell^{-1}$,

$$|\widehat{\chi_\Lambda}(k)| = \ell^2|\widehat{\chi}(k\ell)| \leq C\ell^2 K_H^{-M}. \tag{F4}$$

An important property for the localization function χ_{B_u}, $u \in \mathbb{R}^2$, on the small boxes, namely

$$\chi_{B_u}(x) := \chi_\Lambda(x)\chi\Big(\frac{x}{d\ell} - u\Big), \tag{F5}$$

which is used in "Appendix D", is the following bound

$$\|\nabla^2 \chi_{B_u}\|_\infty \leq C_M \frac{1}{|B_u|\ell_{\min}^2}\int \chi_{B_u}, \tag{F6}$$

which is taken from [18, Appendix C]. Here it is key the fact that we do not consider a smooth function but we require χ to have a finite degree of regularity measured by the parameter M.

Appendix G: Comparing Riemann Sums and Integrals

We will show in this section that we could approximate integrals on $\mathbb{R}^2$ by Riemann sums when it was needed in (4.13) to prove the upper bound. Recall that the assumptions of Theorem 4.1 were

$$R \leq \rho^{-1/2} Y^{1/2}, \qquad L_\beta = \rho^{-1/2} Y^{-\beta}. \tag{G1}$$

We divide $\mathbb{R}^2$ into small squares $\square_p$ of size $\frac{2\pi}{L}$ centered at $p \in \Lambda_L^* = \frac{2\pi}{L}\mathbb{Z}^2$. Then, clearly

$$\Big| \frac{4\pi^2}{L^2} \sum_{p \in \Lambda_L^*} f(p) - \int_{\mathbb{R}^2} f(k) \mathrm{d}k \Big| \leq \frac{C}{L^3} \sum_{p \in \Lambda_L^*} \sup_{\square_p} |\nabla f|. \tag{G2}$$

We consider the functions present in the two sums of (4.13). With α_p and γ_p given in (4.6) the first term is

$$\begin{aligned} f(p) &= p^2 + \rho_0 \widehat{g}_p - \sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2} + \rho_0 \big(\widehat{v}_p - \widehat{g}_p\big)\big(\gamma_p + \alpha_p\big) \\ &= p^2 + \rho_0 \widehat{g}_p - \sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2} + \rho_0 (\widehat{v}_p - \widehat{g}_p)\Big(\frac{p^2}{2\sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2}} - \frac{1}{2}\Big), \end{aligned} \tag{G3}$$

and the second term

$$d(p, r) = \widehat{v}_r \alpha_{p+r} \alpha_p. \tag{G4}$$

We then have the following estimates

Lemma G.1. *Let f, d be as in* (G3) *and* (G4). *Then,*

$$\Big| \frac{1}{|\Lambda_\beta|} \sum_{p \in \Lambda_\beta^*} f(p) - \int_{\mathbb{R}^2} f(k) \frac{\mathrm{d}k}{4\pi^2} \Big| \leq C \rho^2 Y^{1/2+\beta} \widehat{v}_0, \tag{G5}$$

and

$$\Big| \frac{1}{|\Lambda_\beta|^2} \sum_{p,r \neq 0} d(p,r) - \int_{\mathbb{R}^4} d(p,r) \frac{\mathrm{d}p \mathrm{d}r}{(4\pi^2)^2} \Big| \leq C \rho^2 Y^{1/2+\beta} \widehat{v}_0. \tag{G6}$$

Proof. In order to apply (G2), we start by calculating the gradient

$$\begin{aligned} \partial_p f &= 2p + \rho_0 \partial_p \widehat{g}_p - 2p \frac{\Big(1 + \frac{\rho_0 \widehat{g}_p}{p^2} + \frac{\rho_0 \partial_p \widehat{g}_p}{2p}\Big)}{\sqrt{1 + \frac{2\rho_0 \widehat{g}_p}{p^2}}} \\ &\quad + \rho_0 \big(\partial_p \widehat{v}_p - \partial_p \widehat{g}_p\big)\Big(\frac{p^2}{2\sqrt{p^4 + 2\rho_0 \widehat{g}_p p^2}} - \frac{1}{2}\Big) \\ &\quad + \rho_0^2 \left(\widehat{v}_p - \widehat{g}_p\right) \frac{\widehat{g}_p p^3 - \frac{1}{2} \partial_p \widehat{g}_p p^4}{\big(p^4 + 2\rho_0 \widehat{g}_p p^2\big)^{\frac{3}{2}}} \\ &:= A_p + B_p + C_p. \end{aligned} \tag{G7}$$

We will now systematically omit the constants and study separately the cases $p \leq \sqrt{2\rho_0 \widehat{g}_0}$ (case 1 referring to A^1_p) and $p \geqslant \sqrt{2\rho_0 \widehat{g}_0}$ (case 2 referring to A^2_p). We then get by elementary inequalities

$$\left|A^1_p\right| \leq (\rho \widehat{g}_0)^{1/2}, \qquad \left|A^2_p\right| \leq \frac{\rho^2 R \widehat{g}_0 \widehat{g}_p}{p^2} + \frac{(\rho \widehat{g}_p)^2}{p^3},$$

$$\left|B^1_p\right| \leq \rho R(\widehat{v}_0 - \widehat{g}_0), \qquad \left|B^2_p\right| \leq \frac{R\rho^2(\widehat{v}_0 - \widehat{g}_0)\widehat{g}_p}{p^2},$$

$$\left|C^1_p\right| \leq \frac{\rho^{1/2}(\widehat{v}_0 - \widehat{g}_0)}{\widehat{g}_0^{1/2}}, \qquad \left|C^2_p\right| \leq (\widehat{v}_0 - \widehat{g}_0)\Big(\frac{\rho^2 \widehat{g}_0}{p^3} + \frac{R\rho^2 \widehat{g}_p}{p^2}\Big),$$

where we used $|\widehat{g}_0 - \widehat{g}_p| \leq |\widehat{g}_0^{3/2}|$ for $p \leq (\rho \widehat{g}_0)^{1/2}$. This way we can use inequality (G2) and the decay of $\widehat{g}_p$ (3.41) to get

$$\begin{aligned} \frac{1}{L^3_\beta} \sum_{p \in \Lambda^*} \left|\partial_p f\right| \mathrm{d}p &\leq \frac{C}{L_\beta}\big((\rho \widehat{g}_0)^{3/2} + \rho^{3/2} \widehat{g}_0^{1/2}(\widehat{v}_0 - \widehat{g}_0) + \rho^2 \widehat{g}_0 R + R\rho^2(\widehat{v}_0 - \widehat{g}_0)\big) \\ &\leq C \widehat{v}_0 \rho^2 Y^{1/2+\beta}, \end{aligned} \tag{G8}$$

where we used (G1), and $\widehat{g}_0 \leq \widehat{v}_0$. We use the same method to prove (G6). We have

$$|\alpha_p| \leq \begin{cases} \frac{\sqrt{\rho \widehat{g}_0}}{p}, & \text{for } p \leq \sqrt{\rho \widehat{g}_0}, \\ \frac{\rho |\widehat{g}_p|}{p^2}, & \text{for } p \geq \sqrt{\rho \widehat{g}_0}. \end{cases} \tag{G9}$$

We have to calculate

$$\partial_p \alpha_p = -\frac{\rho \partial_p \widehat{g}_p}{2\sqrt{p^4 + 2\rho \widehat{g}_p p^2}} - \frac{\rho \widehat{g}_p \left(4p^3 + 4\rho \widehat{g}_p p + 2\rho \partial_p \widehat{g}_p p^2\right)}{2(p^4 + 2\rho \widehat{g}_p p^2)^{3/2}},$$

yielding

$$|\partial_p \alpha_p| \leq \begin{cases} \frac{\sqrt{\rho \widehat{g}_0} R}{p} + (\rho \widehat{g}_0)^{-1/2} + \frac{\sqrt{\rho \widehat{g}_0}}{p^2} + \frac{\sqrt{\rho \widehat{g}_0} R}{p}, & \text{for } p \leq \sqrt{\rho \widehat{g}_0}, \\ \frac{\rho \widehat{g}_p R}{p^2} + \frac{\rho \widehat{g}_p}{p^3} + \frac{(\rho \widehat{g}_p)^2}{p^5} + \frac{(\rho \widehat{g}_p)^2 R}{p^4}, & \text{for } p \geq \sqrt{\rho \widehat{g}_0}. \end{cases} \tag{G10}$$

The divergence in $p \to 0$ implies to remove a little box around the point 0

$$\begin{aligned} &\Big|\frac{16\pi^4}{L^4_\beta} \sum_{p,r \neq 0} d(p,r) - \int_{\mathbb{R}^4} d(p,r) \mathrm{d}p\mathrm{d}r\Big| \\ &\leq \Big|\frac{16\pi^4}{L^4_\beta} \sum_{p,r \neq 0} d(p,r) - \int_{\left(\mathbb{R}^2 \setminus [-\frac{1}{L}, \frac{1}{L}]^2\right)^2} d(p,r) \mathrm{d}p\mathrm{d}r\Big| \\ &\quad + \Big|\int_{\mathbb{R}^2 \times [-\frac{1}{L}, \frac{1}{L}]^2} d(p,r) \mathrm{d}p\mathrm{d}r\Big| + \Big|\int_{[-\frac{1}{L}, \frac{1}{L}]^2 \times \mathbb{R}^2} d(p,r) \mathrm{d}p\mathrm{d}r\Big|. \end{aligned}$$

where the last two terms in the above can be bounded by $\rho^2 Y^{1/2+\beta}\widehat{v}_0$. Finally a direct computation using the decay of $\widehat{g}_p$, the bounds (4.2), (G9), (G10), and (G2) yields

$$\begin{aligned}
&\Big|\frac{16\pi^4}{L_\beta^4}\sum_{p,r\neq 0} d(p,r) - \int_{\left(\mathbb{R}^2\setminus[-\frac{1}{L},\frac{1}{L}]^2\right)^2} d(p,r)\mathrm{d}p\mathrm{d}r\Big| \\
&\leq \frac{1}{L_\beta^5}\sum_{p,r\neq 0}\sup_{\square_p\times\square_r}\big|\nabla_{p,r} d(p,r)\big| \\
&\leq C\frac{\widehat{v}_0}{L_\beta^5}\sum_{p\neq 0}|\partial_p\alpha_p|\sum_{r\neq 0}|\alpha_r| + C\frac{R\widehat{v}_0}{L_\beta^5}\Big(\sum_r|\alpha_r|\Big)^2 \\
&\leq C\widehat{v}_0\rho^2 Y^{1/2+\beta},
\end{aligned} \tag{G11}$$

where we used the estimates of Lemma 4.4. This concludes the proof. □

Appendix H: Fixing Parameters for the Lower Bound

Here we collect all the relations and dependencies of the several parameters involved in the lower bound for the convenience of the reader. Furthermore, we end the section by making an explicit choice that satisfies all the relations. Recall that we have the small parameter

$$Y = Y_\mu = |\log(\rho_\mu a^2)|^{-1}.$$

We use the following notation throughout the article

$$A \ll B \quad \text{if and only if there exist } C,\varepsilon > 0 \text{ s.t. } A \leq C Y^\varepsilon B. \tag{H1}$$

In the proof of the lower bound, a number of positive parameters are needed. These are the following

$$d, s, \varepsilon_T, \varepsilon_K, \varepsilon_N, \varepsilon_{\mathcal{M}} \ll 1 \ll \mathcal{M}, K_\ell, K_H, \widetilde{K}_H, K_N, K_B. \tag{H2}$$

These will be chosen below.

Furthermore, there are length scales ℓ_δ and R. These will be chosen to satisfy

$$R \leq \rho_\mu^{-1/2}, \qquad \text{Condition on the radius of the support ,} \tag{H3}$$

$$\ell_\delta = \frac{e^\Gamma}{2}\rho_\mu^{-1/2} Y^{-1/2}, \qquad \text{healing length condition.} \tag{H4}$$

Some first relations between the parameters are

$$d \ll 1 \ll K_\ell, \qquad \text{sep. of small and large boxes,} \tag{H5}$$

$$d^{-2} \ll K_H \ll \widetilde{K}_H, \qquad \text{sep. of low and high momenta,} \tag{H6}$$

$$d \ll (sK_\ell)^{-1} \ll 1, \qquad \text{condition for Bog. integral,} \tag{H7}$$

$$d^2 K_\ell^4 \ll \varepsilon_T \ll d s K_\ell, \qquad \text{spectral gap condition,} \tag{H8}$$

$$d s^{-1} \leq C, \qquad \text{localization to small boxes.} \tag{H9}$$

The combination of (H6) and (H8) implies the following relations:

$$K_\ell \ll K_\ell^2 \ll sd^{-1} \ll d^{-1} \ll d^{-2} \ll K_H. \tag{H10}$$

Defining

$$\varepsilon_N := K_N^{-1} Y, \qquad \varepsilon_{\mathcal{M}} := \frac{\mathcal{M}}{\rho_\mu \ell^2}, \tag{H11}$$

we give the following conditions which control the magnitude of the large parameters in terms of Y:

$$(dsK_\ell)^{-1} \ll K_B, \qquad \text{condition errors in small box,} \tag{H12}$$

$$K_B K_\ell \widetilde{K}_H K_N^{1/4} \ll Y^{-1/4}, \qquad \text{small error in large matrices,} \tag{H13}$$

$$K_\ell^{-1} K_N^{1/4} \ll Y^{-1/2}, \qquad \text{technical estimate in large matrices,} \tag{H14}$$

$$\widetilde{K}_H K_N^{-1/4} \gg Y^{1/4}, \qquad \text{technical estimate in large matrices,} \tag{H15}$$

$$K_\ell^2 K_H^2 \mathcal{M} \ll Y^{-1}, \qquad \text{second localization of 3Q term,} \tag{H16}$$

$$K_B^2 K_\ell^2 \ll Y^{-1/4}, \qquad \text{number for high momenta,} \tag{H17}$$

$$K_\ell^{10} K_H^{-8} d^{-4} \ll Y^{-1}, \qquad \text{condition error in } \mathcal{T}_1. \tag{H18}$$

Here the magnitude of the small parameters:

$$\varepsilon_R \ll K_B^{-2} K_\ell^{-2} |\log Y|^{-1}, \qquad \text{Condition on } \varepsilon_R, \tag{H19}$$

$$\varepsilon_K \ll K_\ell^{-2}, \qquad \text{error in } \mathcal{T}'_{2,com}, \tag{H20}$$

$$\varepsilon_K \gg K_\ell^4 K_H^{-4} (d^{-2} \varepsilon_{\mathcal{M}}^{1/2} + d^{-4} \varepsilon_{\mathcal{M}}), \qquad \text{condition error in } \mathcal{T}_1 \text{ and } \mathcal{T}_2, \tag{H21}$$

$$\varepsilon_{\mathcal{M}} \ll d^8 K_\ell^4 \varepsilon_T^{-2}, \qquad \text{condition for error } \delta_1. \tag{H22}$$

$$\varepsilon_N \le \varepsilon_T^{-2} d^4 K_\ell^4, \qquad \text{bound from Lemma 9.2.} \tag{H23}$$

We use the fundamental property of the system that the number of excitations of our state is relatively small compared to the number of particles (expressed by the condition $\varepsilon_{\mathcal{M}} \ll 1$) but still larger that a certain threshold. This property is expressed by the following condition:

$$\mathcal{M} \gg Y^{-7/8} |\log Y|^{1/4} K_B^{1/2} K_\ell^{1/2} K_N^{1/8} \widetilde{K}_H^{1/2} \|v\|_1^{1/2}. \tag{H24}$$

The following are conditions that impose constraints on the size of M, the degree of regularity of the localization function χ:

$$d^{2M-2} \ll Y, \qquad \text{error in localization 3Q,} \tag{H25}$$

$$d^2 K_\ell^4 \ll \varepsilon_T, \qquad \text{error in localization 3Q,} \tag{H26}$$

$$\varepsilon_N^{3/2} + \left(\frac{K_H}{\widetilde{K}_H}\right)^M + (d^2 K_H)^{-2M} \le \varepsilon_{\mathcal{M}}, \qquad \text{number for high momenta,} \tag{H27}$$

$$(s^{-2} + d^{-2})(sd)^{-2} s^M \le C, \qquad \text{localization to small boxes.} \tag{H28}$$

A choice of parameters, non-optimal in the size of the error produced, fitting the previous conditions, is the following,

$$
\begin{aligned}
M &= 258, & \mathcal{M} &= Y^{-\frac{31}{32}}, & \varepsilon_T &= Y^{\frac{1}{512}-\frac{1}{8192}}, \\
K_\ell &= Y^{-\frac{1}{2048}}, & K_H &= Y^{-\frac{1}{128}}, & \widetilde{K}_H &= Y^{-\frac{1}{64}}, \\
d &= Y^{\frac{1}{512}}, & K_N &= Y^{-\frac{1}{512}}, & s &= Y^{\frac{1}{4096}}, \\
K_B &= Y^{-\frac{1}{512}}, & \varepsilon_K &= Y^{\frac{1}{512}}, & \varepsilon_{\mathcal{M}} &= Y^{\frac{1}{32}+\frac{1}{1024}}.
\end{aligned} \tag{H29}
$$

This choice is not made with any particular view towards optimality.

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Communicated by R. Seiringer

Chapter 4

Review: Lower bounds on the energy of the Bose gas

This chapter contains the review article [Fou+24c] by Fournais, Girardot, Morin, Olivieri and the author. We provide short proofs of the formulae eqs. (1.1) and (1.2) in Gross-Pitaevskii regime. The review is included in its entirety in the published form, which can be found at https://doi.org/10.1142/S0129055X23600048. It can be located within the thesis by the colour ■ at the top of the page.

Reviews in Mathematical Physics

Lower bounds on the energy of the Bose gas

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Received (Day Month Year)
Revised (Day Month Year)

We present an overview of the approach to establish a lower bound to the ground state energy for the dilute, interacting Bose gas in a periodic box. In this paper the size of the box is larger than the Gross-Pitaevski length scale. The presentation includes both the 2 and 3 dimensional cases, and catches the second order correction, i.e. the Lee-Huang-Yang term. The calculation on a box of this length scale is the main step to calculate the energy in the thermodynamic limit. However, the periodic boundary condition simplifies many steps of the argument considerably compared to the localized problem coming from the thermodynamic case.

Keywords: many-body quantum mechanics; dilute Bose gases; Bogoliubov theory; Lee-Huang-Yang formula.

Mathematics Subject Classification: 81V73, 81V70

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1. Introduction and Main Results

1.1. *Introduction*

The understanding of the ground state of a Bose gas is of major interest in many-body quantum theory, especially since the first experimental observation of Bose-Einstein condensates [1]. It is a very challenging problem to find properties of this ground state, and the mathematical proof of condensation in the thermodynamic limit is still out of reach. In this paper, we focus on the asymptotic behaviour of the ground state *energy* in the dilute limit, both in dimensions $d = 2$ and 3.

To state the results, we consider a gas of N bosons in a box Ω, in the thermodynamic limit $|\Omega| \to \infty$, with fixed density $\rho = N/|\Omega|$. The first terms of the expansion of the ground state energy density of such a gas depend only on the scattering length a of the inter-particle potential (as defined in Section 1.2 below) and the density ρ. In the 3-dimensional case, the ground state energy density has the following expansion in dilute limit $\rho a^3 \to 0$,

$$e^{3\mathrm{D}}(\rho) = 4\pi\rho^2 a\Big(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3}\Big) + o\big(\rho^2 a\sqrt{\rho a^3}\big). \tag{1.1}$$

The leading term of this asymptotic formula was first derived in [2], and the second term, the Lee-Huang-Yang term, was given in [3,4]. Mathematical proofs of the leading order term were given in [5] for the upper bound and in [6] for the lower bound. The first upper bound to LHY precision was given in [7], and the correct constant in [8] with recent improvements in [9], for sufficiently regular potentials. The matching lower bounds were given in [10,11] including the crucial case of hard core potentials. The upper bound in the case of potentials with large L^1-norm, such as the hard core interactions, is still an open problem. However, the reader may find recent improvements in [12].

In the 2-dimensional case, the asymptotic formula is

$$e^{2\mathrm{D}}(\rho) = 4\pi\rho^2\delta\Big(1 + \Big[2\Gamma + \frac{1}{2} + \log(\pi)\Big]\delta\Big) + o(\rho^2\delta^2), \tag{1.2}$$

where $\Gamma \simeq 0.57$ is the Euler-Mascheroni constant and δ is a small logarithmic parameter given by

$$\delta := \frac{1}{|\log(\rho a^2 |\log(\rho a^2)|^{-1})|}. \tag{1.3}$$

This formula was first given in [13,14,15,16], and the leading order was first proven in [17]. Both upper and lower bounds to second order precision were recently proved in [18], and they include the case of hard core interactions. We refer to [19,20] for overview articles. Similar expansions for Bose gases in $2D$ were obtained, in the Gross-Pitaevskii regime in [21] or in different regimes, see [22].

The case of interacting Fermi gases is equally interesting and has seen major progress in recent years, see for instance [23,24,25,26,27,28,29,30,31,32,33].

The purpose of the present paper is to explain the proof of lower bounds in [10] and [18] for the $3D$ and the $2D$ case, respectively, which are similar in many aspects. The very first step, both in $2D$ and $3D$, is to reduce the problem to length-scales ℓ which are much smaller than the thermodynamic length L but larger than the Gross-Pitaevski length scale. This localization procedure is now quite standard [34], but gives rise to technical complications. Mainly, the kinetic energy is inconveniently modified, including localization functions which affect the algebra of calculations and require more involved estimates. For this reason, we decide here to directly consider a gas of bosons on a periodic box of the right ρ-dependent length scale and to carry out all the analysis in this setting omitting the localization step. Since many terms are simpler and many errors vanish, this should help the interested reader understand the general strategy of lower bounds for Bose gases.

Before introducing the energy and the associated result we need to recall some basic facts about the scattering equation.

1.2. *Scattering length*

An important difference between 2 and 3 dimensions concerns the properties of scattering solutions, which can be found in [35, Appendix A]. We recall here the main definitions, and fix notations.

In this paper we will only consider radial, compactly supported and positive potentials $v : \mathbb{R}^d \to [0, \infty]$, with $R > 0$ such that $\mathrm{supp}(v) \subseteq B^d(0, R)$, where we denote by $B^d(y, r)$ the ball of radius r centered in y in $\mathbb{R}^d$.

Let us consider the minimization problem, for an arbitrary $\widetilde{R} > R$,

$$\mathrm{E}_d(v, \widetilde{R}) = \inf_{\varphi} \int_{B^d(0,\widetilde{R})} \Big(|\nabla \varphi|^2 + \frac{1}{2} v \varphi^2 \Big) \mathrm{d}x, \tag{1.4}$$

where the infimum is taken over $\varphi \in H^1(B^d(0, \widetilde{R}))$ such that $\varphi|_{\partial B^d(0,\widetilde{R})} = 1$. We define the scattering length $a = a(v)$ by

$$\mathrm{E}_2(v, \widetilde{R}) = \frac{2\pi}{\log(\frac{\widetilde{R}}{a})}, \qquad \text{and} \qquad \mathrm{E}_3(v, \widetilde{R}) = \frac{4\pi a}{1 - a/\widetilde{R}}. \tag{1.5}$$

It is a well-known result that a is independent of $\widetilde{R} > R$. The associated minimizers are of the form

$$\varphi_{\mathbb{R}^d} = \begin{cases} \dfrac{1}{\log(\widetilde{R}/a)} \varphi^0_{\mathbb{R}^d}, & \text{if } d = 2, \\[2mm] \dfrac{1}{1 - a/\widetilde{R}} \varphi^0_{\mathbb{R}^d}, & \text{if } d = 3, \end{cases} \tag{1.6}$$

where $\varphi^0_{\mathbb{R}^d}$ solves the scattering equation

$$-\Delta \varphi^0_{\mathbb{R}^d} + \frac{1}{2} v \varphi^0_{\mathbb{R}^d} = 0, \tag{1.7}$$

in a distributional sense. The solution is such that, for $|x| \geq R$, we have the explicit form

$$\varphi^0_{\mathbb{R}^2}(x) = \log\Big(\frac{|x|}{a}\Big), \quad \text{and} \quad \varphi^0_{\mathbb{R}^3}(x) := 1 - \frac{a}{|x|}. \tag{1.8}$$

If $d = 3$, we choose $\widetilde{R} = \infty$ so that $\varphi^0_{\mathbb{R}^3} = \varphi_{\mathbb{R}^3}$. The logarithm in the 2D-scattering solution is clearly unbounded for large values of $|x|$. This is a major difference to the 3D behaviour. Therefore the length $\widetilde{R}$ is of much greater importance. In this paper, when $d = 2$, we choose

$$\widetilde{R} = ae^{\frac{1}{2\delta}}, \qquad \text{i.e.} \qquad \delta = \frac{1}{2}\log\Big(\frac{\widetilde{R}}{a}\Big)^{-1}, \tag{1.9}$$

so that

$$\varphi_{\mathbb{R}^2} := 2\delta\varphi^0_{\mathbb{R}^2} \tag{1.10}$$

is then normalized to 1 at distance $\widetilde{R}$, with δ given in (1.3).

1.3. *Main result*

We consider N interacting bosons on the torus of unit cell $\Lambda = \big[-\frac{\ell}{2}, \frac{\ell}{2}\big]^d$. We define the associated Hamiltonian with periodic boundary conditions

$$\mathcal{H}_N = \sum_{j=1}^{N} -\Delta_j + \sum_{1\leq i<j\leq N} v(x_i - x_j), \tag{1.11}$$

acting on the space of symmetric square integrable functions $L^2_{\rm sym}(\Lambda^N)$, where $-\Delta$ is the periodic Laplacian on Λ and the potential depends on $(x_i - x_j)^*$, the distance between particle i and j on the torus. More precisely, we define $x^* \in \mathbb{R}$ by

$$x^* = \min_{z\in\mathbb{Z}^d} |x - z\ell|, \tag{1.12}$$

and

$$v(x) = v_{\mathbb{R}^d}(x^*), \qquad \text{with} \qquad v_{\mathbb{R}^d} : \mathbb{R}^d \to \mathbb{R}_+. \tag{1.13}$$

We assume $v_{\mathbb{R}^d}$ to be a *positive, radially symmetric interaction with support in the ball of radius* $R \leq \ell/4$. This condition on the support will be made precise later, all we need for now is the support of v to fit in the box. We have here committed a mild abuse of notation using that $v_{\mathbb{R}^d}$ is radially symmetric. Using the positivity of the potential it is standard that $\mathcal{H}_N$ defines a self-adjoint operator. If $\varphi_{\mathbb{R}^d}$ is the scattering solution associated to $v_{\mathbb{R}^d}$, we define

$$\omega_{\mathbb{R}^d} := 1 - \varphi_{\mathbb{R}^d}, \qquad g_{\mathbb{R}^d} := v_{\mathbb{R}^d}(1 - \omega_{\mathbb{R}^d}) = v_{\mathbb{R}^d}\varphi_{\mathbb{R}^d}, \tag{1.14}$$

and their periodic versions

$$\omega(x) := \omega_{\mathbb{R}^d}(x^*), \qquad g(x) := g(x^*), \qquad x \in \Lambda. \tag{1.15}$$

Note that we dropped the dependence on d in the notation. The function g has a specific role in the analysis, and its Fourier transform satisfies, through a manipulation of the scattering equation (1.7), the relation

$$\widehat{g}(0) = \begin{cases} 8\pi\delta, & \text{if } d = 2, \\ 8\pi a, & \text{if } d = 3. \end{cases} \tag{1.16}$$

Notice that since $R \leq \ell/4$, we have that the Fourier transforms and Fourier coefficients agree at zero, i.e. $\widehat{g}(0) = \widehat{g}_{\mathbb{R}^d}(0)$. We scale the system in the following way: for a given density ρ we define

$$\ell := \frac{K_\ell}{\sqrt{\rho \widehat{g}(0)}} \tag{1.17}$$

where $K_\ell \gg 1$ is a large ρ-dependent parameter chosen in (F.13). This scaling has to be understood under the dilute regime assumption, that is $\rho a^d \leq C^{-1}$ for a large enough constant C. The regime $K_\ell = 1$ corresponds to the well-known Gross-Pitaevskii regime. In this paper, the particular choice $K_\ell \gg 1$ is needed to control the errors obtained at the different steps of the proof, as the c-number substitution of Section 3 and to go from sums to integrals at a negligible cost, in particular to get the correct LHY constant.

The number N of particles in the box is defined through

$$N = \rho \ell^d.$$

We can observe using (1.7) that the Fourier transform $\widehat{g\omega}(0)$ can be written by means of an auxiliary function

$$\widehat{g\omega}(0) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} G_d(k) \mathrm{d}k, \qquad G_d(k) = \frac{\widehat{g}_{\mathbb{R}^d}(k)^2 - \widehat{g}_{\mathbb{R}^d}(0)^2 \mathbb{1}_d(\ell_\delta k)}{2k^2}, \tag{1.18}$$

where we introduced the cut-off

$$\mathbb{1}_d(t) := \delta_{d,2} \mathbb{1}_{\{|t| \leq 1\}}(t), \tag{1.19}$$

with $\delta_{i,j}$ being the Kronecker delta, to deal with the 2D case where the Fourier transform of a logarithm involves a renormalization around zero. This renormalization is done at the scale

$$\ell_\delta = \frac{a}{2} e^{\frac{1}{2\delta}} e^{\Gamma} = \frac{1}{2\sqrt{\rho\delta}} e^{\Gamma}(1 + o(1)), \tag{1.20}$$

where we recall that Γ is the Euler-Mascheroni constant. We define the *Lee-Huang-Yang energy* in dimension d as

$$E_d^{\mathrm{LHY}}(\rho, \Lambda) := \frac{\rho^2}{2} |\Lambda| \widehat{g}(0) \lambda_d^{\mathrm{LHY}} I_d^{\mathrm{Bog}}, \tag{1.21}$$

where

$$\lambda_d^{\mathrm{LHY}} = \begin{cases} \sqrt{\rho a^3}, & \text{if } d = 3, \\ \delta, & \text{if } d = 2, \end{cases} \tag{1.22}$$

is the Lee-Huang-Yang correction order, and

$$I_d^{\mathrm{Bog}} := \left(\frac{2}{\pi}\right)^{d/2} \int_{\mathbb{R}^d} \sqrt{(t^2+1)^2-1} - t^2 - 1 + \frac{1}{2t^2}\big(1 + \mathbb{1}_d(\sqrt{2\pi}e^{\Gamma} t)\big)\,\mathrm{d}t, \tag{1.23}$$

is the Bogoliubov integral of dimension d.

We also define the LHY error in dimension d denoted o_d^{LHY} as a quantity of smaller order than the LHY precision in term of the small parameter of the dilute regime ρa^d. For any error term $\mathcal{E}$ we write $\mathcal{E} = o_d^{\mathrm{LHY}}$ if there exist constants $C > 0$ and $\eta > 0$ such that

$$|\mathcal{E}| \leq \begin{cases} C\rho^2|\Lambda|\delta^{2+\eta}, & \text{if } d = 2, \\ C\rho^2|\Lambda|a\big(\rho a^3\big)^{\frac{1}{2}+\eta}, & \text{if } d = 3. \end{cases} \tag{1.24}$$

Let us recall the expressions of $\mathcal{H}_N$ and Λ below, for reader's convenience:

$$\mathcal{H}_N = \sum_{j=1}^N -\Delta_j + \sum_{1\leq i<j\leq N} v(x_i - x_j),$$
$$\Lambda = \Big[-\frac{\ell}{2}, \frac{\ell}{2}\Big]^d, \qquad \ell = \frac{K_\ell}{\sqrt{\rho\widehat{g}(0)}}.$$

We can now state the main theorem of the paper.

Theorem 1.1. *There exists $C > 0$, such that, if $v \in L^2(\Lambda)$ is a positive, spherically symmetric, compactly supported potential with scattering length $a > 0$ and if $\rho > 0$ is such that $\rho a^d \leq C^{-1}$, then for any bosonic, normalized state Ψ in the domain of $\mathcal{H}_N$ we have*

$$\langle \Psi, \mathcal{H}_N \Psi\rangle \geq \frac{1}{2}\rho^2|\Lambda|\widehat{g}(0) + E_d^{\mathrm{LHY}}(\rho,\Lambda) + o_d^{\mathrm{LHY}}.$$

I.e. inserting the values of $\widehat{g}(0)$, E_d^{LHY} and I_d^{Bog},

$$\inf\mathrm{Spec}(\mathcal{H}_N) \geq \begin{cases} 4\pi\rho^2|\Lambda|a\Big(1 + \dfrac{128}{15\sqrt{\pi}}\sqrt{\rho a^3}\Big) + o\big(\rho^2|\Lambda|a\big), & \textit{if } d = 3, \\ 4\pi\rho^2|\Lambda|\delta\Big(1 + \Big[2\Gamma + \dfrac{1}{2} + \log(\pi)\Big]\delta\Big) + o\big(\rho^2|\Lambda|\delta\big), & \textit{if } d = 2. \end{cases} \tag{1.25}$$

Remark 1.1 (Bogoliubov integral). The integral (1.23) can be explicitly calculated and provides the expected coefficients for the LHY corrections

$$I_d^{\mathrm{Bog}} = \begin{cases} 2\Gamma + \dfrac{1}{2} + \log\pi, & \text{if } d = 2, \\ \dfrac{128}{15\sqrt{\pi}}, & \text{if } d = 3. \end{cases} \tag{1.26}$$

Notice furthermore, that the whole second order term E_d^{LHY} of the energy comes from the calculation of the integral

$$\frac{|\Lambda|}{2(2\pi)^d}\int_{\mathbb{R}^d}\Big(\sqrt{k^4 + 2k^2\rho\widehat{g}(k)} - k^2 - \rho\widehat{g}(k) + \rho^2 G_d(k)\Big)\mathrm{d}k, \tag{1.27}$$

from which we recover (1.21) thanks to a change of variables $k \mapsto \sqrt{\rho \widehat{g}(0)}k$, and a passage to the limit $\rho a^d \to 0$.

Remark 1.2 (Assumptions on the potential). The L^2 assumption on v in Theorem 1.1 is technical and not needed in the actual papers dealing with the thermodynamic limit [11,18], where L^1 suffices. In the present paper we need this assumption in the comparison between the discrete sums over the dual lattice and the corresponding continuous integrals (see (A.1) and the proof of Proposition 4.1). Actually, for this point the assumption $v \in L^p(\Lambda)$ for any $p > 6/5$ would suffice.

These L^p-assumptions on the potential v exclude the hard core case. These assumptions are actually also not necessary. Indeed, the inequalities of the proof in the thermodynamic setting allow for a large L^1-norm. This is enough to extend the result to the hard core case approximating it through a sequence of growing L^1-potentials. See [10, Theorem 1.6] and [18, Section 3.3] for the $3D$ and $2D$-case respectively.

The compact support assumption on the potential v can also be relaxed in the thermodynamic regime. We can allow for a tail under a proper decay assumption provided that, avoiding the contribution from the tail does not affect the scattering length too much. See [10, Theorem 1.6] and [18, Section 3.2].

Remark 1.3. The present article reviews, in the simpler setting of the periodic box, results stated in [10, Theorem 1.3] and [18, Theorem 2.3] for the $3D$, $2D$-case respectively, neglecting the complications derived from the double localization for the thermodynamic limit. Nevertheless we included an original bound on the number of high momentum excitations (E.3). Similar results in three dimension were proven in [36] with different methods.

Remark 1.4. As already mentioned, the purpose of the present paper is mainly expository. The main ideas of [10,11,18] are clearest in the periodic setting, which is the setting of this paper. To prove the analogous lower bound in the thermodynamic setting one would first need to localize in such periodic boxes, but it is not clear how to make such a localization with the right precision. Indeed, in [10,11,18], the localization is done by a sliding technique which produces a much more complicated kinetic energy in the boxes.

In the papers [37,38] the corresponding localization procedure is done by imposing Neumann bounday conditions which also introduces substantial technical difficulties compared to the periodic case.

1.4. *Strategy of the proof*

(1) **Splitting of the potential and renormalization.** We expect the ground state of our operator to exhibit condensation, meaning that most particles should have zero momentum. This is why we start by decomposing the potential energy according to creation or annihilation of bosons with zero and non-zero

momenta. We define the following operators on $L^2(\Lambda)$, denoting by $|1\rangle$ the function which has constant value 1 on Λ,

$$P = |\Lambda|^{-1}|1\rangle\langle 1|, \qquad Q = \mathbb{1} - P = \mathbb{1}_{(0,\infty)}(\sqrt{-\Delta}),$$

projecting on the condensate and on excitations respectively. We recall that here $-\Delta$ is the periodic Laplacian on Λ. With this notation, the number of particles in the condensate n_0, and the number of excited particles n_+ are given by

$$n_0 := \sum_{j=1}^{N} P_j, \qquad n_+ := \sum_{j=1}^{N} Q_j = N - n_0,$$

where P_j and Q_j denotes P and Q acting on the j-th variable. We insert these projections in the potential energy,

$$\sum_{i<j} v(x_i - x_j) = \sum_{k=0}^{4} \mathcal{Q}_k, \tag{1.28}$$

where $\mathcal{Q}_k$ contains precisely k occurences of Q's. For instance,

$$\mathcal{Q}_0 = \sum_{i<j} P_i P_j v(x_i - x_j) P_j P_i. \tag{1.29}$$

One should also note that $\mathcal{Q}_1 = 0$ by momentum conservation.

We need terms to depend on g instead of v in order for the scattering length to appear. We are able to overcome this problem modifying each $\mathcal{Q}_j$ into $\mathcal{Q}_j^{\rm ren}$ and collecting in the last term $\mathcal{Q}_4^{\rm ren}$, which is positive, all the error terms produced by the renormalization. For a lower bound $\mathcal{Q}_4^{\rm ren}$ can be discarded.

(2) **c-number substitution.** From this point on, we work in momentum space and second quantization; the operator can be rewritten in terms of creation and annihilation operators of plane waves $a_k^\dagger$, a_k (Proposition 2.1). The next step is a rigorous justification of the so-called *c-number substitution*, which is given by expanding the operator on projectors on coherent states living in the 0-momentum space. This allows us to replace a_0 and $a_0^\dagger$ by their actions as multiplication by complex numbers z on the coherent states (Proposition 3.1). This amounts to consider the condensate of 0-momenta particles having fixed density $\rho_z = |z|^2|\Lambda|^{-1}$ and to only work on the remaining degrees of freedom in the space of excitations.

(3) **Bogoliubov diagonalization.** We first focus on $\mathcal{Q}_0^{\rm ren}$ and the quadratic excitation operator $\mathcal{Q}_2^{\rm ren}$. The sum of these with the kinetic energy produces a $\mathcal{K}(z)$ that can be diagonalized, as in the standard Bogoliubov theory. This procedure gives rise to the Bogoliubov integral $I_d^{\rm Bog}$, times the LHY order, which is the second order term of the energy, together with a positive diagonal operator $\mathcal{K}^{\rm diag}$ (Proposition 4.1). The remaining quadratic terms have to be bounded by the contribution given by the soft-pairs in $\mathcal{Q}_3^{\rm ren}$, introduced in the next step.

(4) **Localization of 3Q terms.** One of the major difficulties is to deal with the 3Q terms $\mathcal{Q}_3^{\rm ren}$. These terms can be interpreted as the energy generated by one pair of excited momenta, interacting to give one zero and one excited momentum or the other way around. The upper bound calculations of [8] show that such pairs are crucial to find the correct energy to LHY precision, and especially the *soft pairs.* Those pairs have high momentum, and interact to create one zero momentum and one low momentum. In fact, we show in Proposition 5.1 that $\mathcal{Q}_3^{\rm ren}$ gives almost the same contribution to the energy as the analogue soft pairs operator $\mathcal{Q}_3^{\rm soft}$.

(5) **The energy of soft pairs.** Section 6 is dedicated to the bounds on $\mathcal{Q}_3^{\rm soft}$. It absorbs the remaining part of the quadratic energy $\mathcal{Q}_2^{\rm ex}$, using the high momenta part of $\mathcal{K}^{\rm diag}$. The precise understanding of the $\mathcal{Q}_3^{\rm soft}$ is a key calculation in our approach.

(6) **Bounds on the number of excitations.** Most of our bounds require estimates on the number of excited particles n_+, the number of high-momenta excited particles n_+^H and the number of low momenta excited particles n_+^L. In Appendix B, we use the technique called *localization of large matrices* to show that we can restrict to states having bounded n_+^L. In Appendix E, we directly get bounds on n_+ and n_+^H, i.e., *condensation estimates* on Λ.

(7) **Conclusion.** In the final Section 7 we combine all the estimates to finish the proof of Theorem 1.1.

The proof depends on several parameters that have to be suitably tuned. These parameters and their relations are collected in Appendix F.

2. Splitting of the Potential Energy and Renormalization

By means of the projectors onto and outside the condensate, we split the potential in a sum of operators by expanding

$$v(x_i - x_j) = (P_i + Q_i)(P_j + Q_j)v(x_i - x_j)(P_j + Q_j)(P_i + Q_i)$$

and reorganize it as a sum of $\mathcal{Q}_j$, where in each $\mathcal{Q}_j$, the projector Q is present j times. An idea similar to this already appeared in the early work [39]. We then renormalize the $\mathcal{Q}_j$ to obtain $\mathcal{Q}_j^{\rm ren}$ where v has been replaced by g. More precisely we have

Lemma 2.1. *The following algebraic identity holds*

$$\frac{1}{2}\sum_{i\neq j} v(x_i - x_j) = \sum_{j=0}^{4} \mathcal{Q}_j^{\rm ren}, \tag{2.1}$$

where

$$0 \le \mathcal{Q}_4^{\mathrm{ren}} := \frac{1}{2}\sum_{i\neq j}\Big[Q_iQ_j + (P_iP_j + P_iQ_j + Q_iP_j)\,\omega(x_i-x_j)\Big]v(x_i-x_j) \times \Big[Q_jQ_i + \omega(x_i-x_j)\,(P_jP_i + P_jQ_i + Q_jP_i)\Big], \tag{2.2}$$

$$\mathcal{Q}_3^{\mathrm{ren}} := \sum_{i\neq j} P_iQ_jg(x_i-x_j)Q_jQ_i + h.c., \tag{2.3}$$

$$\mathcal{Q}_2^{\mathrm{ren}} := \sum_{i\neq j} P_iQ_j(g+g\omega)(x_i-x_j)P_jQ_i + \sum_{i\neq j} P_iQ_j(g+g\omega)(x_i-x_j)Q_jP_i + \frac{1}{2}\sum_{i\neq j} P_iP_jg(x_i-x_j)Q_jQ_i + h.c., \tag{2.4}$$

$$\mathcal{Q}_1^{\mathrm{ren}} := \sum_{i,j}\big(Q_iP_j(g+g\omega)(x_i-x_j)P_jP_i + h.c.\big) = 0, \tag{2.5}$$

and

$$\mathcal{Q}_0^{\mathrm{ren}} := \frac{1}{2}\sum_{i\neq j} P_iP_j(g+g\omega)(x_i-x_j)P_jP_i. \tag{2.6}$$

Proof. The lemma is proven by algebraic computations using that $g = v(1-\omega)$, and $\mathcal{Q}_1^{\mathrm{ren}}$ is zero because, for any $f \in L^1(\Lambda)$,

$$Q_iP_jf(x_i-x_j)P_jP_i = \frac{1}{|\Lambda|}\|f\|_{L^1}Q_iP_i = 0. \qquad \square$$

We continue our analysis in momentum space considering the second quantization of the Hamiltonian. Let us introduce

$$a_k^\dagger := \frac{1}{|\Lambda|^{1/2}}a^\dagger(e^{ikx}), \qquad a_k := \frac{1}{|\Lambda|^{1/2}}a(e^{ikx}), \tag{2.7}$$

i.e. the usual bosonic creation and annihilation operators of bosons with momentum $k \in \Lambda^* = \frac{2\pi}{\ell}\mathbb{Z}^d$. Note that for zero momentum, $a_0^\dagger$ creates the function 1, the *condensate* in Λ. The operator $\mathcal{H}_N$ can be written, by abuse of notation, as the action on the $N-$boson space of a second quantized Hamiltonian acting on the Fock space $\mathscr{F}_s(L^2(\Lambda)) = \bigoplus_{N=0}^\infty L^2_s(\Lambda^N)$ involving a_k and $a_k^\dagger$. We can write the number operators as

$$n_0 = a_0^\dagger a_0, \qquad n_+ = \sum_{k\in\Lambda^*} a_k^\dagger a_k. \tag{2.8}$$

Proposition 2.1. *The Hamiltonian $\mathcal{H}_N$ acts on $L^2_s(\Lambda^N)$ as*

$$\begin{aligned}\mathcal{H}_N = &\sum_{k\in\Lambda^*} k^2 a_k^\dagger a_k + \frac{1}{2|\Lambda|}\big(\widehat{g}(0)+\widehat{g\omega}(0)\big)a_0^\dagger a_0^\dagger a_0 a_0 \\ &+ \frac{1}{|\Lambda|}\sum_{k\in\Lambda^*}\Big((\widehat{g}(k)+\widehat{g\omega}(k))a_0^\dagger a_k^\dagger a_k a_0 + \frac{1}{2}\widehat{g}(k)(a_0^\dagger a_0^\dagger a_k a_{-k} + h.c.)\Big) \\ &+ \Big(\widehat{g}(0)+\widehat{g\omega}(0)\Big)\frac{n_0 n_+}{|\Lambda|} + \mathcal{Q}_3^{\rm ren} + \mathcal{Q}_4^{\rm ren}. \end{aligned} \tag{2.9}$$

Proof. The first term of (2.9) is obtained by a simple application of the second quantization to the Laplacian. The other terms require some manipulations with the $\mathcal{Q}_j^{\rm ren}$. We observe that

$$\sum_{j=1}^n P_j g(x_i-x_j)P_j = \frac{1}{|\Lambda|}\sum_{j=1}^n P_j\int_\Lambda g(x_i-y)dy = \frac{n_0}{|\Lambda|}\widehat{g}(0). \tag{2.10}$$

In particular $\mathcal{Q}_0^{\rm ren}$ is

$$\mathcal{Q}_0^{\rm ren} = \frac{n_0(n_0-1)}{2|\Lambda|}(\widehat{g}(0)+\widehat{g\omega}(0)), \tag{2.11}$$

and by the second quantization we get the second term in (2.9). For $\mathcal{Q}_2^{\rm ren}$, we use (2.10) for

$$\sum_{i\neq j} P_i Q_j (g+g\omega)(x_i-x_j)Q_j P_i = \Big(\widehat{g}(0)+\widehat{g\omega}(0)\Big)\frac{n_0 n_+}{|\Lambda|}. \tag{2.12}$$

The second quantization of the whole $\mathcal{Q}_2^{\rm ren}$ is obtained by a standard calculation which provides the third and fourth terms of (2.9). We only provide here an example of this calculation for the term

$$\mathcal{Q}_2^1 := \sum_{i\neq j} P_i Q_j g(x_i-x_j)P_j Q_j. \tag{2.13}$$

We denote the basis elements $e_p(x) = \frac{e^{ipx}}{\sqrt{|\Lambda|}}$ and write a $\Psi\in L^2(\Lambda^N)$ as

$$\Psi = \sum_{p,k} c_{pk} e_p(x_j)e_k(x_i) \quad \text{with} \quad c_{pk} = \frac{1}{\sqrt{N(N-1)}} a_p a_k \Psi.$$

We can then compute

$$\begin{aligned}\mathcal{Q}_2^1\Psi &= \frac{1}{|\Lambda|}\sum_{k\neq 0}\widehat{g}(k)\sum_{i\neq j} e_k(x_j)e_0(x_i)a_0 a_k\Psi && (2.14)\\ &= \frac{1}{|\Lambda|}\sum_{k\neq 0}\widehat{g}(k)a_k^\dagger a_0^\dagger a_0 a_k\Psi. && (2.15)\end{aligned}$$

□

3. c-Number Substitution

Now that the operator is written in second quantization, as stated in Proposition 2.1, we proceed to the c-number substitution. Thanks to this procedure, we can turn the action of the a_0's into multiplication by complex numbers z. It amounts to consider the condensate of 0-momentum particles as having a fixed density $\rho_z = |z|^2|\Lambda|^{-1}$, and only deal with excitations. This is done by diagonalizing a_0 in the following way. The decomposition $L^2(\Lambda) = \mathrm{Ran}P \oplus \mathrm{Ran}Q$ leads to the splitting of the bosonic Fock space $\mathscr{F}_s(L^2(\Lambda)) = \mathscr{F}_s(\mathrm{Ran}P) \otimes \mathscr{F}_s(\mathrm{Ran}Q)$. Denoting by Ω the vacuum vector, we introduce the class of coherent states in $\mathscr{F}_s(\mathrm{Ran}P)$, labeled by $z \in \mathbb{C}$,

$$|z\rangle = e^{-\left(\frac{|z|^2}{2}+za_0^\dagger\right)}\,\Omega, \tag{3.1}$$

which are eigenvectors for the annihilation operator of the condensate. It is simple to show that

$$a_0|z\rangle = z\,|z\rangle \quad \text{and} \quad 1 = \frac{1}{\pi}\int_{\mathbb{C}} |z\rangle\langle z|\,\mathrm{d}z. \tag{3.2}$$

Here $\langle z|$ is the partial trace along $\mathscr{F}_s(\mathrm{Ran}P)$. Thus, for any $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ the state $\Phi(z) = \langle z|\Psi\rangle$ is in $\mathscr{F}_s(\mathrm{Ran}Q)$.

Proposition 3.1. *For $z \in \mathbb{C}$, set $\rho_z = |z|^2|\Lambda|^{-1}$. The Hamiltonian $\mathcal{H}_N$ acts on $L^2_{\mathrm{sym}}(\Lambda^N)$ as*

$$\mathcal{H} = \frac{1}{\pi}\int_{\mathbb{C}} \mathcal{K}(z)|z\rangle\langle z|\mathrm{d}z + \mathcal{Q}_3^{\mathrm{ren}} + \mathcal{Q}_4^{\mathrm{ren}} + \mathcal{R}_0, \tag{3.3}$$

where the $z-$dependent Hamiltonian is

$$\mathcal{K}(z) := \mathcal{Q}(z) + \mathcal{Q}_2^{\mathrm{ex}}(z) + (\rho_z - \rho)n_+\widehat{g}(0) - \rho\rho_z|\Lambda|\widehat{g}(0) + \rho^2|\Lambda|\widehat{g}(0), \tag{3.4}$$

with

$$\mathcal{Q}(z) := \frac{1}{2}\rho_z^2|\Lambda|(\widehat{g}(0) + \widehat{g\omega}(0)) + \mathcal{K}^{\mathrm{Bog}}, \tag{3.5}$$

where $\mathcal{K}^{\mathrm{Bog}}$ is a quadratic Hamiltonian in creation and annihilation operators that we call the Bogoliubov Hamiltonian:

$$\mathcal{K}^{\mathrm{Bog}} = \frac{1}{2}\sum_{k\neq 0}\mathcal{A}_k\left(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}\right) + \frac{1}{2}\sum_{k\neq 0}\mathcal{B}_k\left(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}\right), \tag{3.6}$$

with

$$\mathcal{A}_k := k^2 + \rho_z\widehat{g}(k), \qquad \mathcal{B}_k := \rho_z\widehat{g}(k). \tag{3.7}$$

The remaining 2Q term is

$$\mathcal{Q}_2^{\mathrm{ex}}(z) = \rho_z\sum_{k\neq 0}\left(\widehat{g\omega}(k) + \widehat{g\omega}(0)\right)a_k^\dagger a_k. \tag{3.8}$$

Moreover, there exists a universal constant $C > 0$ such that the error term satisfies

$$|\langle \mathcal{R}_0 \rangle_\Psi| \leq CN|\Lambda|^{-1}\widehat{g}(0), \qquad \forall \Psi \in L^2_{\rm sym}(\Lambda^N) \quad \textit{normalized}. \tag{3.9}$$

Proof. As a first step, we add and subtract in the Hamiltonian the term $\rho^2|\Lambda|\widehat{g}(0)$, exploiting the identity on $L^2_{\rm sym}(\Lambda^N)$

$$\rho^2|\Lambda|\widehat{g}(0) = \rho(n_0 + n_+)\widehat{g}(0). \tag{3.10}$$

We focus then on $\mathcal{H} - \rho n_0 \widehat{g}(0)$, and apply to this term the c-number substitution, briefly described below. The expansion on coherent states allows to perform, for instance, the following formal substitutions in (2.9)

$$a_0^\dagger a_0^\dagger a_0 a_0 \mapsto |z|^4 - 4|z|^2 + 2, \qquad a_0^\dagger a_0 \mapsto |z|^2 - 1. \tag{3.11}$$

We give an example of the rigorous derivation of the second term in (3.11) as follows. For any $f, g \in \mathscr{F}_s(L^2(\Lambda))$, using (3.2),

$$\langle f | a_0^\dagger a_0 g\rangle = \langle f | a_0 a_0^\dagger g\rangle - \langle f|g\rangle = \frac{1}{\pi}\int_{\mathbb{C}} |z|^2 \langle f|z\rangle\langle z|g\rangle \mathrm{d}z - \frac{1}{\pi}\int_{\mathbb{C}} \langle f|z\rangle\langle z|g\rangle \mathrm{d}z, \tag{3.12}$$

yielding

$$a_0^\dagger a_0 = \frac{1}{\pi}\int_{\mathbb{C}} (|z|^2 - 1)|z\rangle\langle z| \mathrm{d}z, \tag{3.13}$$

and the other terms can be treated in a similar manner. We now prove how low order terms produced in the aforementioned substitution are actually errors. For instance, focusing again on the $|z|^2$ in the first term of (3.11), we have that

$$\frac{\widehat{g}(0)}{2\pi|\Lambda|}\int_{\mathbb{C}} |z|^2 |z\rangle\langle z| \mathrm{d}z = \frac{\widehat{g}(0)}{2|\Lambda|} a_0 a_0^\dagger \geq -C\frac{n_0+1}{|\Lambda|}\widehat{g}(0). \tag{3.14}$$

The substitution step leads to the result, with

$$\mathcal{R}_0 = -\frac{1}{2|\Lambda|}\big(\widehat{g}(0) + \widehat{g\omega}(0)\big)\big(4n_0 - 2\big) - \widehat{g\omega}(0)\frac{n_+}{|\Lambda|} \tag{3.15}$$

$$- \frac{1}{|\Lambda|}\sum_{k\in\Lambda^*}\Big((\widehat{g}(k) + \widehat{g\omega}(k))a_k^\dagger a_k + \frac{1}{2}\widehat{g}_k(a_k a_{-k} + h.c.)\Big), \tag{3.16}$$

and bound the error term using a Cauchy-Schwarz on the $a_k a_{-k}$ terms to get

$$|\mathcal{R}_0| \leq C\frac{n_0 + n_+}{|\Lambda|}\widehat{g}(0) \leq C\frac{N}{|\Lambda|}\widehat{g}(0).$$

Notice that the substitutions of $a_0 a_0$ and $a_0^\dagger a_0^\dagger$ should give a z^2 and a $\overline{z}^2$ in the definition of $\mathcal{B}_k := |z|^2|\Lambda|^{-1}\widehat{g}(k)$. To circumvent this issue we write $z = |z|e^{i\phi}$ and absorb the phase in the a_k's. This does not affect the later computations which only involve commutations of such a_k's. □

By Proposition 3.1 we are reduced to study a Hamiltonian dependent on the free parameter $z \in \mathbb{C}$. The density ρ_z describes the particles in the condensate, but we have no restriction on it. We expect to have full condensation, i.e. $\rho_z \simeq \rho$. In this regime we need to make very precise estimates which are established in the main part of the paper. The regime where ρ_z is far from ρ seems less physical and, in fact, there rougher bounds suffice.

We define the threshold magnitude for the densities

$$\varepsilon_+ := \max\{K_\ell^2 K_L^{-1}, (\lambda_d^{\mathrm{LHY}})^{1/2}\}, \tag{3.17}$$

with K_L being introduced in (5.2) below (and fixed in Appendix F). In the following sections, we will study the regime

$$|\rho_z - \rho| < \rho\varepsilon_+, \tag{3.18}$$

while we deal with the regime $|\rho_z - \rho| \geq \rho\varepsilon_+$ in Appendix D.

4. Estimates for ρ_z close to ρ

4.1. *Diagonalization*

We apply the diagonalization procedure to the operator

$$\mathcal{Q}(z) = \frac{\rho_z^2}{2}|\Lambda|(\widehat{g}(0) + \widehat{g\omega}(0)) + \mathcal{K}^{\mathrm{Bog}} \tag{4.1}$$

defined in (3.5) and containing the LHY integral and a positive operator, diagonal in creation and annihilation of excitations.

Proposition 4.1. *Let ε_+ be as in* (3.17) *and assume the relations between the parameters in Appendix F. For any $z \in \mathbb{C}$ such that $|\rho - \rho_z| \leq \rho\varepsilon_+$, the following equality holds:*

$$\mathcal{Q}(z) = \frac{\rho_z^2}{2}|\Lambda|\widehat{g}(0) + E_d^{\mathrm{LHY}}(\rho_z) + \mathcal{K}^{\mathrm{diag}} + \mathcal{R}_1^{(d)},$$

where we define the diagonalized Bogoliubov Hamiltonian *as*

$$\mathcal{K}^{\mathrm{diag}} = \sum_{k\neq 0} \mathcal{D}_k b_k^\dagger b_k, \qquad \mathcal{D}_k = \sqrt{k^4 + 2k^2\rho_z\widehat{g}(k)}, \tag{4.2}$$

where

$$b_k = \frac{1}{\sqrt{1-\alpha_k^2}}\Big(a_k + \alpha_k a_{-k}^\dagger\Big), \qquad \alpha_k = \frac{k^2 + \rho_z\widehat{g}(k) - \sqrt{k^4 + 2k^2\rho_z\widehat{g}(k)}}{\rho_z\widehat{g}(k)}, \tag{4.3}$$

and where the error term $\mathcal{R}_1^{(d)}(\rho_z)$ satisfies

$$|\mathcal{R}_1^{(d)}(\rho_z)| \leq \begin{cases} C|\Lambda|\rho_z^2\delta^2 K_\ell^{-1}, & \text{if } d = 2, \\ C|\Lambda|\rho_z^2 a\big(\rho_z a^3\big)^{\frac{1}{2}}\log(\rho_z)K_\ell^{-1}, & \text{if } d = 3. \end{cases} \tag{4.4}$$

The constant in (4.4) depends on L^p-norms of the potential.

Proof. Applying Theorem Appendix C.1 with $\mathcal{A}_k = k^2 + \rho_z \widehat{g}(k)$ and $\mathcal{B}_k = \rho_z \widehat{g}(k)$, we get $\mathcal{D}_k$ and α_k from (4.2) and (4.3), and we can write, for all $k \neq 0$,

$$\begin{aligned}\mathcal{A}_k(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}) + \mathcal{B}_k(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}) = \\ \mathcal{D}_k(b_k^\dagger b_k + b_{-k}^\dagger b_{-k}) + \sqrt{k^4 + 2k^2\rho_z\widehat{g}(k)} - k^2 - \rho_z\widehat{g}(k).\end{aligned} \tag{4.5}$$

Then, using that $\mathcal{A}_k$ and $\mathcal{B}_k$ are even functions of k, we deduce

$$\begin{aligned}\mathcal{Q}(z) = \frac{1}{2}\rho_z^2|\Lambda|\widehat{g}(0) + \frac{1}{2}\sum_{k\neq 0}\Big(\sqrt{k^4 + 2k^2\rho_z\widehat{g}(k)} - k^2 - \rho_z\widehat{g}(k)\Big) \\ + \frac{1}{2}\rho_z^2|\Lambda|\widehat{g\omega}(0) + \mathcal{K}^{\mathrm{diag}}.\end{aligned} \tag{4.6}$$

Changing the sum in (4.6) into the integral

$$\frac{|\Lambda|}{2(2\pi)^d}\int\Big(\sqrt{k^4 + 2k^2\rho_z\widehat{g}(k)} - k^2 - \rho_z\widehat{g}(k)\Big)\mathrm{d}k, \tag{4.7}$$

can be done up to an error term $\mathcal{R}_1^{(d)}(\rho_z)$ which can be estimated as in (4.4). The constant in (4.4) depends on L^p-properties of the potential, since we need some decay of $\widehat{g}(k)$ to control the decay of the summand. This is easily achieved through an expansion of the square root and a Hölder inequality on the sum.

We recall here that $\widehat{g\omega}(0)$ defined in (1.18) can be written as an integral,

$$\frac{\rho_z^2}{2}|\Lambda|\widehat{g\omega}(0) = \rho_z^2|\Lambda|\int\frac{\widehat{g}_{\mathbb{R}^d}^2(k) - \widehat{g}_{\mathbb{R}^d}^2(0)\mathbb{1}_d(\ell_\delta k)}{4k^2}\frac{\mathrm{d}k}{(2\pi)^d}. \tag{4.8}$$

The proposition follows then using Lemma Appendix C.2 to calculate the value of the integral. □

5. Localization of 3Q terms

In this section we focus on the effect of the $3Q$-term, namely

$$\mathcal{Q}_3^{\mathrm{ren}} = \sum_{i\neq j} P_i Q_j g(x_i - x_j) Q_i Q_j + h.c. \tag{5.1}$$

Since $3Q$'s appear in this term, we can interpret it as the energy produced when 2 non-zero incoming momenta create 1 non-zero momentum and 1 zero momentum (or vice versa). We prove below that we can restrict this interaction to soft pairs, i.e., when two "high" momenta and one "low" momentum are involved in this process. More precisely, let us define the sets of low and high momenta by

$$\mathcal{P}_L = \{p \in \Lambda^*, \quad 0 < |p| \leq K_L\ell^{-1}\}, \qquad \mathcal{P}_H = \{k \in \Lambda^*, \quad |k| \geq K_H\ell^{-1}\}, \tag{5.2}$$

where the parameters K_L, K_H are fixed in Appendix F. The condition $K_L \ll K_H$, which is part of (F.3), will ensure that these sets are disjoint. We define the localized projectors by

$$Q_L := \mathbb{1}_{\mathcal{P}_L}(\sqrt{-\Delta}), \qquad \overline{Q}_L := Q - Q_L = \mathbb{1}_{(K_L\ell^{-1},\infty)}(\sqrt{-\Delta}), \tag{5.3}$$

$$Q_H := \mathbb{1}_{\mathcal{P}_H}(\sqrt{-\Delta}), \qquad \overline{Q}_H := Q - Q_H = \mathbb{1}_{(0,K_H\ell^{-1})}(\sqrt{-\Delta}). \tag{5.4}$$

The number of high excitations, namely the number of bosons outside the condensate and with momenta not in $\mathcal{P}_L$, is

$$n_+^H := \sum_{j=1}^n \overline{Q}_{L,j}, \tag{5.5}$$

acting on $L^2_{\mathrm{sym}}(\Lambda^n)$ for any n. Similarly, we define the number of low excitations by

$$n_+^L := \sum_{j=1}^n \overline{Q}_{H,j}. \tag{5.6}$$

Notice that $n_+^L + n_+^H \geq n_+$, due to the overlap of the regions in momentum space.

The reduction to soft pairs is then given by the following proposition.

Proposition 5.1. *Assuming the relations between the parameters in Appendix F, there exists a universal constant $C > 0$ such that, for all N-particle states $\Psi \in L^2_{\mathrm{sym}}(\Lambda^N)$ satisfying $\Psi = \mathbb{1}_{[0,2\mathcal{M}]}(n_+^L)\Psi$ and assumption* (E.1)*, we have*

$$|\langle \mathcal{Q}_3^{\mathrm{ren}} \rangle_\Psi - \langle \mathcal{Q}_3^{\mathrm{soft}} \rangle_\Psi| \leq \frac{1}{4}\langle \mathcal{Q}_4^{\mathrm{ren}} \rangle_\Psi + o_d^{LHY}.$$

where

$$\mathcal{Q}_3^{\mathrm{soft}} = \frac{1}{|\Lambda|} \sum_{\substack{k\in\mathcal{P}_H,\\ p\in\mathcal{P}_L}} \widehat{g}(k)\big(a_0^\dagger a_p^\dagger a_{p-k} a_k + h.c.\big). \tag{5.7}$$

The proof of Proposition 5.1 will follow from the Lemmas 5.1 and 5.2 below.

Lemma 5.1. *There exists a universal constant $C > 0$ such that, for all $\varepsilon_1 > 0$ and all N-particle states $\Psi \in L^2_{\mathrm{sym}}(\Lambda^N)$, we have*

$$|\langle \mathcal{Q}_3^{\mathrm{ren}} \rangle_\Psi - \langle \mathcal{Q}_3^{\mathrm{low}} \rangle_\Psi| \leq \frac{1}{4}\langle \mathcal{Q}_4^{\mathrm{ren}} \rangle_\Psi + \rho\widehat{g}(0)\Big(C\varepsilon_1 \langle n_+\rangle_\Psi + (C+\varepsilon_1^{-1})\langle n_+^H\rangle_\Psi\Big), \tag{5.8}$$

where

$$\mathcal{Q}_3^{\mathrm{low}} := \sum_{i\neq j} (P_i Q_{L,j} g(x_i - x_j) Q_i Q_j + h.c.). \tag{5.9}$$

Proof. From the definitions we have

$$\mathcal{Q}_3^{\mathrm{ren}} - \mathcal{Q}_3^{\mathrm{low}} = \sum_{i\neq j} (P_i \overline{Q}_{L,j} g(x_i - x_j) Q_i Q_j + h.c.). \tag{5.10}$$

In the right-hand side we aim to reconstruct the $\mathcal{Q}_4^{\mathrm{ren}}$ terms as

$$\begin{aligned}\sum_{i\neq j} (P_i \overline{Q}_{L,j} g Q_i Q_j + h.c.) &= \sum_{i\neq j} P_i \overline{Q}_{L,j} g\,[Q_i Q_j + \omega(P_i P_j + P_i Q_j + Q_i P_j)] + h.c. \\ &\quad - \sum_{i\neq j} P_i \overline{Q}_{L,j} g \omega (P_i P_j + P_i Q_j + Q_i P_j) + h.c. \end{aligned} \tag{5.11}$$

We use Cauchy-Schwarz inequality on both terms. Using that $g \leq v$ in the support of v, the first line of (5.11) is controlled by

$$C \sum_{i \neq j} P_i \overline{Q}_{L,j} g (P_i \overline{Q}_{L,j})^\dagger + \frac{1}{4} \mathcal{Q}_4^{\mathrm{ren}} = C \widehat{g}(0) \frac{n_0 n_+^H}{|\Lambda|} + \frac{1}{4} \mathcal{Q}_4^{\mathrm{ren}}.$$

In the second line of (5.11), the $P_i P_j$ term vanishes because $\overline{Q}_{L,j} P_j = 0$. The two other terms can be estimated as above. For instance, for any $\varepsilon_1 > 0$,

$$\begin{aligned} \sum_{i \neq j} (P_i \overline{Q}_{L,j} g \omega P_i Q_j + h.c.) &\leq \varepsilon_1^{-1} \sum_{i \neq j} P_i \overline{Q}_{L,j} g \omega (P_i \overline{Q}_{L,j})^\dagger + \varepsilon_1 \sum_{i \neq j} P_i Q_j g \omega P_i Q_j \\ &\leq \widehat{g}(0) \frac{n_0}{|\Lambda|} \Big(\varepsilon_1^{-1} n_+^H + \varepsilon_1 n_+ \Big), \end{aligned} \tag{5.12}$$

and conclude observing that $n_0 \leq N$ when applied to Ψ. □

Lemma 5.2. *There exists a universal constant $C > 0$ such that, for all $\varepsilon_2 > 0$ and all N-particles state $\Psi \in L^2_{\mathrm{sym}}(\Lambda^N)$ we have*

$$|\langle \mathcal{Q}_3^{\mathrm{low}} \rangle_\Psi - \langle \mathcal{Q}_3^{\mathrm{soft}} \rangle_\Psi| \leq C \rho \widehat{g}(0) \Big(\varepsilon_2 K_H^d \langle n_+ \rangle_\Psi + \varepsilon_2^{-1} \frac{\langle n_+ n_+^L \rangle_\Psi}{N} \Big). \tag{5.13}$$

Proof. First of all, we can rewrite (5.9) in second quantization,

$$\mathcal{Q}_3^{\mathrm{low}} = \frac{1}{|\Lambda|} \sum_{p \in \mathcal{P}_L, k \neq 0} \widehat{g}(k) \big(a_0^\dagger a_p^\dagger a_{p-k} a_k + h.c. \big). \tag{5.14}$$

From the definition (5.7) of $\mathcal{Q}_3^{\mathrm{soft}}$ we deduce

$$\mathcal{Q}_3^{\mathrm{low}} - \mathcal{Q}_3^{\mathrm{soft}} = \frac{1}{|\Lambda|} \sum_{\substack{k \in \mathcal{P}_H^c, k \neq 0 \\ p \in \mathcal{P}_L}} \widehat{g}(k) \big(a_0^\dagger a_p^\dagger a_{p-k} a_k + h.c. \big). \tag{5.15}$$

When applying to Ψ, we can use the Cauchy-Schwarz inequality with weight $\varepsilon_2 > 0$ and deduce

$$|\langle \mathcal{Q}_3^{\mathrm{low}} \rangle_\Psi - \langle \mathcal{Q}_3^{\mathrm{soft}} \rangle_\Psi| \leq C \frac{\widehat{g}(0)}{|\Lambda|} \sum_{\substack{k \in \mathcal{P}_H^c, k \neq 0 \\ p \in \mathcal{P}_L}} \big(\varepsilon_2 \langle a_0^\dagger a_p^\dagger a_p a_0 \rangle_\Psi + \varepsilon_2^{-1} \langle a_k^\dagger a_{p-k}^\dagger a_{p-k} a_k \rangle_\Psi \big). \tag{5.16}$$

In the first term of (5.16) we recognize n_+ and a volume of $\mathcal{P}_H^c$. Similarly in the second term, the p-sum gives n_+ and the k-sum gives n_+^L (and the remaining commutator is controlled by the other terms). Thus,

$$|\langle \mathcal{Q}_3^{\mathrm{low}} \rangle_\Psi - \langle \mathcal{Q}_3^{\mathrm{soft}} \rangle_\Psi| \leq C \widehat{g}(0) \Big(\varepsilon_2 K_H^d \frac{N \langle n_+ \rangle_\Psi}{|\Lambda|} + \varepsilon_2^{-1} \frac{\langle n_+ n_+^L \rangle_\Psi}{|\Lambda|} \Big), \tag{5.17}$$

and this concludes the proof. □

We are now ready to prove Proposition 5.1.

Proof. [Proof of Proposition 5.1] Joining together Lemma 5.1 and Lemma 5.2, we get that the error made approximating $\mathcal{Q}_3^{\rm ren}$ by $\mathcal{Q}_3^{\rm soft}$, testing on a state Ψ as in the assumptions such that $n_+^L \le \mathcal{M}$, is bounded by

$$\frac{1}{4}\langle \mathcal{Q}_4^{\rm ren}\rangle_\Psi + C\rho\widehat{g}(0)\Big(K_\ell^{-2} + K_H^{d/2}\Big(\frac{\mathcal{M}}{N}\Big)^{1/2}\Big)\langle n_+\rangle_\Psi + C\rho\widehat{g}(0)K_\ell^2\langle n_+^H\rangle_\Psi \tag{5.18}$$

where we chose $\varepsilon_1 = K_\ell^{-2}$ and $\varepsilon_2 = \Big(\frac{\mathcal{M}}{NK_H^d}\Big)^{1/2}$. Let us focus on the n_+ terms. We use (E.2) of Theorem Appendix E.1 to bound $\langle n_+\rangle_\Psi$ and (F.7) to conclude that the expression is of an order smaller than LHY. For the n_+^H terms, we use (E.3) instead and (F.3). □

6. Bounds on $\mathcal{Q}_3$ when $\rho_z \simeq \rho$: the effect of Soft Pairs

In this section we explain the effects of soft pairs on the energy in the case when ρ_z is close to ρ. In the remaining part of this section, we only assume that $|\rho_z - \rho| \le \frac{1}{2}\rho$, so that we can replace ρ_z by ρ in error estimates.

We will see how $\mathcal{Q}_3^{\rm soft}$, $\mathcal{Q}_2^{\rm ex}$ and $\mathcal{K}^{\rm diag}$ can be combined together, as stated in Proposition 6.1 below. Actually, only the high momenta in $\mathcal{K}^{\rm diag}$ are needed, namely

$$\mathcal{K}_H^{\rm diag} = \sum_{k\in\mathcal{P}_H} \mathcal{D}_k b_k^\dagger b_k. \tag{6.1}$$

Note that we can use c-number substitution to rewrite $\mathcal{Q}_3^{\rm soft}$ as

$$\mathcal{Q}_3^{\rm soft} = \int_{\mathbb{C}} \mathcal{Q}_3^{\rm soft}(z)|z\rangle\langle z|{\rm d}z, \tag{6.2}$$

with

$$\mathcal{Q}_3^{\rm soft}(z) = \frac{1}{|\Lambda|}\sum_{k\in\mathcal{P}_H, p\in\mathcal{P}_L} \widehat{g}(k)\big(\bar{z}a_p^\dagger a_{p-k}a_k + h.c.\big). \tag{6.3}$$

With this notation, we prove the following result.

Proposition 6.1. *There exists a universal constant $C > 0$ such that the following holds. Let $\rho a^d \le C^{-1}$ and $z \in \mathbb{C}$ be such that $|\rho_z - \rho| \le \frac{1}{2}\rho$. Then for any normalized state $\Phi \in \mathscr{F}_s(\mathrm{RanQ})$ satisfying*

$$\Phi = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Phi,$$

we have, for a small fraction ε_{gap} of the spectral gap, suitably chosen in Appendix F with the other parameters,

$$\langle \mathcal{Q}_3^{\rm soft}(z) + \mathcal{K}_H^{\rm diag} + \mathcal{Q}_2^{\rm ex}(z)\rangle_\Phi \ge -\varepsilon_{\rm gap}\frac{\langle n_+\rangle_\Phi}{\ell^2} - K_\ell^2\frac{\langle n_+^H\rangle_\Phi}{\ell^2} + o_d^{\rm LHY}. \tag{6.4}$$

In order to prove Proposition 6.1, we start by rewriting $\mathcal{Q}_3^{\rm soft}(z)$ in terms of the b_k's defined in (4.3). Note that

$$a_k = \frac{b_k - \alpha_k b_{-k}^\dagger}{\sqrt{1-\alpha_k^2}}, \qquad a_{p-k} = \frac{b_{p-k} - \alpha_{p-k}b_{k-p}^\dagger}{\sqrt{1-\alpha_{p-k}^2}}. \tag{6.5}$$

Therefore,

$$a_{p-k}a_k = \frac{\Big(b_{p-k}b_k - \alpha_k b_{p-k}b_{-k}^\dagger - \alpha_{p-k}b_{k-p}^\dagger b_k + \alpha_{p-k}\alpha_k b_{k-p}^\dagger b_{-k}^\dagger\Big)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}},$$

and $\mathcal{Q}_3^{\rm soft}(z) = \mathcal{Q}_3^{(1)} + \mathcal{Q}_3^{(2)} + \mathcal{Q}_3^{(3)} + \mathcal{Q}_3^{(4)}$ where

$$\mathcal{Q}_3^{(1)} = \frac{1}{|\Lambda|}\sum_{\substack{k\in\mathcal{P}_H,\\ p\in\mathcal{P}_L}} \frac{\widehat{g}(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\big(\bar{z}a_p^\dagger b_{p-k}b_k + \alpha_k\alpha_{p-k}\bar{z}a_p^\dagger b_{k-p}^\dagger b_{-k}^\dagger + h.c.\big), \tag{6.6}$$

$$\mathcal{Q}_3^{(2)} = -\frac{1}{|\Lambda|}\sum_{\substack{k\in\mathcal{P}_H,\\ p\in\mathcal{P}_L}} \frac{\widehat{g}(k)\alpha_k}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\big(\bar{z}a_p^\dagger b_{-k}^\dagger b_{p-k} + z b_{p-k}^\dagger b_{-k}a_p\big), \tag{6.7}$$

$$\mathcal{Q}_3^{(3)} = -\frac{1}{|\Lambda|}\sum_{\substack{k\in\mathcal{P}_H,\\ p\in\mathcal{P}_L}} \frac{\widehat{g}(k)\alpha_{p-k}}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\big(\bar{z}a_p^\dagger b_{k-p}^\dagger b_k + z b_k^\dagger b_{k-p}a_p\big), \tag{6.8}$$

$$\mathcal{Q}_3^{(4)} = -\frac{1}{|\Lambda|}\sum_{\substack{k\in\mathcal{P}_H,\\ p\in\mathcal{P}_L}} \frac{\widehat{g}(k)\alpha_k}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}[b_{p-k}, b_{-k}^\dagger](\bar{z}a_p^\dagger + z a_p) = 0. \tag{6.9}$$

Notice that $\mathcal{Q}_3^{(4)}$ cancels due to the commutation relation $[b_{p-k}, b_{-k}^\dagger] = \delta_{-k,p-k}$. In Lemmas 6.1 and 6.2 below, we get bounds on $\mathcal{Q}_3^{(1)}$, $\mathcal{Q}_3^{(2)}$, and $\mathcal{Q}_3^{(3)}$, thus proving Proposition 6.1.

6.1. *Estimates on* $\mathcal{Q}_3^{(1)}$

The first part $\mathcal{Q}_3^{(1)}$ absorbs $\mathcal{Q}_2^{\rm ex}$ using $(1-\varepsilon_K)K_H^{\rm diag}$ for some parameter ε_K chosen in Appendix F. The remaining fraction $\varepsilon_K K_H^{\rm diag}$ will be later in the proof to control other terms.

Lemma 6.1. *There exists a universal constant $C > 0$ such that the following holds. If $\rho a^d \le C^{-1}$, $|\rho_z - \rho| \le \frac{1}{2}\rho$, and if the parameters $\varepsilon_K, \varepsilon_{\rm gap} \ll 1$ and $\mathcal{M} > 0$, satisfy the relations in Appendix F, then for any normalized state $\Phi \in \mathscr{F}_s(\mathrm{RanQ})$ satisfying*

$$\Phi = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Phi,$$

we have

$$\big\langle \mathcal{Q}_3^{(1)} + \mathcal{Q}_2^{\rm ex} + \big(1-\varepsilon_K\big)\mathcal{K}_H^{\rm diag}\big\rangle_\Phi \ge -\varepsilon_{\rm gap}\frac{\langle n_+\rangle_\Phi}{\ell^2} - K_\ell^2\frac{\langle n_+^H\rangle_\Phi}{\ell^2} + o_d^{\rm LHY}.$$

Proof. We first reorder the creation and annihilation operators, applying a change of variables $k \mapsto -k, p \mapsto -p$ in the α terms,

$$
\begin{aligned}
\mathcal{Q}_3^{(1)} &= \frac{1}{|\Lambda|} \sum_{k \in \mathcal{P}_H, p \in \mathcal{P}_L} \frac{\widehat{g}(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}} \\
&\quad \times \left(\bar{z} a_p^\dagger b_{p-k} b_k + \alpha_k \alpha_{p-k} \bar{z} a_{-p}^\dagger b_{p-k}^\dagger b_k^\dagger + z b_k^\dagger b_{p-k}^\dagger a_p + \alpha_k \alpha_{p-k} z b_k b_{p-k} a_{-p} \right) \\
&= \frac{1}{|\Lambda|} \sum_{k \in \mathcal{P}_H, p \in \mathcal{P}_L} \frac{\widehat{g}(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}} \Big(\left(\bar{z} a_p^\dagger b_{p-k} + \alpha_k \alpha_{p-k} z b_{p-k} a_{-p} \right) b_k \\
&\quad + b_k^\dagger \left(z b_{p-k}^\dagger a_p + \alpha_k \alpha_{p-k} \bar{z} a_{-p}^\dagger b_{p-k}^\dagger \right) + \alpha_k \alpha_{p-k} \left(z \left[b_k, b_{p-k} a_{-p} \right] + \bar{z} \left[a_{-p}^\dagger b_{p-k}^\dagger, b_k^\dagger \right] \right) \Big).
\end{aligned}
$$

Note that the two last commutators vanish. Thus, we can complete the square to get,

$$
\mathcal{Q}_3^{(1)} + (1-\varepsilon_K) \mathcal{K}_H^{\mathrm{diag}} = (1-\varepsilon_K) \sum_{k \in \mathcal{P}_H} \mathcal{D}_k c_k^\dagger c_k + \sum_{k \in \mathcal{P}_H} \mathcal{T}(k), \tag{6.10}
$$

where we keep a small portion of $\mathcal{K}_H^{\mathrm{diag}}$ in order to bound other error terms, and we define

$$
c_k = b_k + \frac{1}{\mathcal{D}_k (1-\varepsilon_K)|\Lambda|} \sum_{p \in \mathcal{P}_L} \frac{\widehat{g}(k)}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}} \left(z b_{p-k}^\dagger a_p + \alpha_k \alpha_{p-k} \bar{z} a_{-p}^\dagger b_{p-k}^\dagger \right), \tag{6.11}
$$

$$
\begin{aligned}
\mathcal{T}(k) = -&\frac{\widehat{g}(k)^2}{(1-\varepsilon_K)\mathcal{D}_k (1-\alpha_k^2)|\Lambda|^2} \sum_{p,s \in \mathcal{P}_L} \frac{1}{\sqrt{1-\alpha_{s-k}^2}\sqrt{1-\alpha_{p-k}^2}} \\
&\times \left(\bar{z} a_p^\dagger b_{p-k} + \alpha_k \alpha_{p-k} z b_{p-k} a_{-p} \right) \left(z b_{s-k}^\dagger a_s + \alpha_k \alpha_{s-k} \bar{z} a_{-s}^\dagger b_{s-k}^\dagger \right).
\end{aligned} \tag{6.12}
$$

The positive $c_k^\dagger c_k$ term in (6.10) can be dropped for a lower bound, and we can focus on the remaining term $\mathcal{T}(k)$. One can write

$$
\bar{z} a_p^\dagger b_{p-k} + \alpha_k \alpha_{p-k} z b_{p-k} a_{-p} = \bar{z} a_p^\dagger b_{p-k} + \alpha_k \alpha_{p-k} z a_{-p} b_{p-k} + \alpha_k \alpha_{p-k} z [b_{p-k}, a_{-p}],
$$

and the last commutator vanishes. Therefore

$$
\begin{aligned}
\mathcal{T}(k) = -&\frac{\widehat{g}(k)^2}{(1-\varepsilon_K)\mathcal{D}_k (1-\alpha_k^2)|\Lambda|^2} \sum_{p,s \in \mathcal{P}_L} \frac{1}{\sqrt{1-\alpha_{p-k}^2}\sqrt{1-\alpha_{s-k}^2}} \\
&\times (\bar{z} a_p^\dagger + \alpha_k \alpha_{p-k} z a_{-p}) b_{p-k} b_{s-k}^\dagger (z a_s + \alpha_k \alpha_{s-k} \bar{z} a_{-s}^\dagger).
\end{aligned}
$$

Now we use a commutator to write $\mathcal{T} = \mathcal{T}_{\mathrm{op}} + \mathcal{T}_{\mathrm{com}}$ in normal order for the b_k.

Since $[b_{p-k}, b^\dagger_{s-k}] = \delta_{s,p}$ we get

$$\mathcal{T}_{\rm op}(k) = -\frac{\widehat{g}(k)^2}{(1-\varepsilon_K)\mathcal{D}_k(1-\alpha_k^2)|\Lambda|^2}\sum_{p,s\in\mathcal{P}_L}\frac{1}{\sqrt{1-\alpha_{p-k}^2}\sqrt{1-\alpha_{s-k}^2}}$$
$$\times(\bar{z}a^\dagger_p + \alpha_k\alpha_{p-k}za_{-p})b^\dagger_{s-k}b_{p-k}(za_s + \alpha_k\alpha_{s-k}\bar{z}a^\dagger_{-s}), \tag{6.13}$$
$$\mathcal{T}_{\rm com}(k) = -\frac{\widehat{g}(k)^2}{(1-\varepsilon_K)\mathcal{D}_k(1-\alpha_k^2)|\Lambda|^2}\sum_{p\in\mathcal{P}_L}\frac{|z|^2}{1-\alpha_{p-k}^2}$$
$$\times(a^\dagger_p + \alpha_k\alpha_{p-k}a_{-p})(a_p + \alpha_k\alpha_{p-k}a^\dagger_{-p}). \tag{6.14}$$

• In order to estimate the error term $\mathcal{T}_{\rm op}$, we introduce

$$\tau_s := za_s + \alpha_k\alpha_{s-k}\bar{z}a^\dagger_{-s}. \tag{6.15}$$

In $\mathcal{T}_{\rm op}$ we commute the b's through the a's, $\tau^\dagger_p b^\dagger_{s-k}b_{p-k}\tau_s = b^\dagger_{s-k}\tau^\dagger_p\tau_s b_{p-k}$, since the commutators vanish in our range of indices. We use the Cauchy-Schwarz inequality

$$\tau^\dagger_p b^\dagger_{s-k}b_{p-k}\tau_s \le \frac{1}{2}(b^\dagger_{s-k}\tau^\dagger_p\tau_p b_{s-k} + b^\dagger_{p-k}\tau^\dagger_s\tau_s b_{p-k}).$$

Inserting this in $\mathcal{T}_{\rm op}$, bounding $(1-\varepsilon_K)(1-\alpha_k) \ge 1/2$ for $k\in\mathcal{P}_H$ (by Lemma Appendix A.2), and noticing that we can exchange s and p in the sum, we find

$$|\langle\mathcal{T}_{\rm op}(k)\rangle_\Phi| \le C\frac{\widehat{g}(k)^2}{\mathcal{D}_k|\Lambda|^2}\sum_{p,s\in\mathcal{P}_L}\frac{1}{\sqrt{1-\alpha_{p-k}^2}\sqrt{1-\alpha_{s-k}^2}}|\langle b^\dagger_{s-k}\tau^\dagger_p\tau_p b_{s-k}\rangle_\Phi|.$$

For states Φ satisfying $\mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Phi = \Phi$ we get, bounding each $\tau^\dagger_s\tau_s$ by $C|z|^2a^\dagger_s a_s$ directly or by a means of Cauchy-Schwarz inequality and a change of variables, by

$$|\langle\mathcal{T}_{\rm op}(k)\rangle_\Phi| \le C\frac{\widehat{g}(k)^2}{\mathcal{D}_k|\Lambda|^2}|z|^2\mathcal{M}\sum_{s\in\mathcal{P}_L}\langle b^\dagger_{s-k}b_{s-k}\rangle_\Phi.$$

Finally, using (A.3),

$$\sum_{k\in\mathcal{P}_H}|\langle\mathcal{T}_{\rm op}(k)\rangle_\Phi| \le C\rho_z\ell^{4-d}\widehat{g}(0)^2K_H^{-2}K_L^d\mathcal{M}\frac{\langle n_+^H\rangle_\Phi}{\ell^2}. \tag{6.16}$$

This term can be absorbed in $K_\ell^2\ell^{-2}n_+^H$, as long as the relation (F.9) holds.

• We now turn to $\mathcal{T}_{\rm com}$ given in (6.14). This term will absorb $\mathcal{Q}_2^{\rm ex}$. Using the Cauchy-Schwarz inequality we have

$$|\langle(a^\dagger_p + \alpha_k\alpha_{p-k}a_{-p})(a_p + \alpha_k\alpha_{p-k}a^\dagger_{-p})\rangle_\Phi - \langle a^\dagger_p a_p\rangle_\Phi|$$
$$\le C|\alpha_k\alpha_{p-k}|\langle a^\dagger_{-p}a_{-p} + a^\dagger_p a_p\rangle_\Phi + |\alpha_k\alpha_{p-k}|^2.$$

We deduce that

$$\sum_{k\in\mathcal{P}_H}\mathcal{T}_{\rm com}(k) = -\frac{1}{|\Lambda|^2}\sum_{k\in\mathcal{P}_H, p\in\mathcal{P}_L}\frac{|z|^2\widehat{g}(k)^2}{(1-\varepsilon_K)\mathcal{D}_k}a^\dagger_p a_p + \mathcal{E}, \tag{6.17}$$

where (using in particular Lemma Appendix A.2)

$$\begin{aligned}|\langle\mathcal{E}\rangle_\Phi| &\le \frac{C}{|\Lambda|^2}\sum_{k\in\mathcal{P}_H,p\in\mathcal{P}_L}\frac{|z|^2\widehat{g}(k)^2}{\mathcal{D}_k}\big(|\alpha_k\alpha_{p-k}|\langle a_p^\dagger a_p\rangle_\Phi+|\alpha_k\alpha_{p-k}|^2\big)\\ &\le C\rho_z^3\widehat{g}(0)^4\ell^{6-d}K_H^{d-6}\langle n_+\rangle_\Phi+\widehat{g}(0)|\Lambda|^{-1}K_L^dK_\ell^{10}K_H^{d-10}. \qquad (6.18)\end{aligned}$$

The first term in (6.18) can be absorbed in a fraction of the spectral gap if $\rho_z^3\widehat{g}(0)^4\ell^{8-d}K_H^{d-6}\ll\varepsilon_{\text{gap}}$ using Appendix F, the second term is smaller than LHY by (F.3). For the main term in (6.17) we do several approximations. First,

$$\sum_{k\in\mathcal{P}_H}\mathcal{T}_{\text{com}}(k)=-\big(1+\mathcal{O}(\varepsilon_K+\ell^2\rho\widehat{g}(0)K_H^{-2})\big)\frac{\rho_z}{|\Lambda|}\sum_{k\in\mathcal{P}_H}\frac{\widehat{g}(k)^2}{k^2}\sum_{p\in\mathcal{P}_L}a_p^\dagger a_p+\mathcal{E}, \qquad (6.19)$$

where we used (A.3). Second, the k-sum is an approximation of $2|\Lambda|\widehat{g\omega}(0)$ by Lemma Appendix A.1, and thus

$$\sum_{k\in\mathcal{P}_H}\mathcal{T}_{\text{com}}(k)=-2\rho_z\widehat{g\omega}(0)\sum_{p\in\mathcal{P}_L}a_p^\dagger a_p+\mathcal{E}'+\mathcal{E}, \qquad (6.20)$$

with $|\mathcal{E}'|\le C(\varepsilon_K\widehat{g}(0)+\ell^2\rho_z\widehat{g}(0)K_H^{-2}+\widehat{g}(0)^2K_H^{-1}+\mathcal{E}_d)\rho_z n_+$. This error is absorbed in the spectral gap $\varepsilon_{\text{gap}}n_+\ell^{-2}$ using (F.11). Then, for $p\in\mathcal{P}_L$, we can replace $\widehat{g\omega}(0)$ by $\widehat{g\omega}(p)$,

$$\sum_{k\in\mathcal{P}_H}\mathcal{T}_{\text{com}}(k)=-\rho_z\sum_{p\in\mathcal{P}_L}\big(\widehat{g\omega}(0)+\widehat{g\omega}(p)\big)a_p^\dagger a_p+\mathcal{E}''+\mathcal{E}'+\mathcal{E}, \qquad (6.21)$$

with error $|\mathcal{E}''|\le CR^2\ell^{-2}K_L^2\rho_z\widehat{g}(0)n_+$, absorbed in the spectral gap again by (F.12). Finally, if we add $\mathcal{Q}_2^{\text{ex}}$ defined in (3.8), we get a sum on $\mathcal{P}_L^c$ which can be bounded by n_+^H,

$$\Big|\sum_{k\in\mathcal{P}_H}\langle\mathcal{T}_{\text{com}}(k)\rangle_\Phi+\langle\mathcal{Q}_2^{\text{ex}}\rangle_\Phi\Big|\le C\rho_z\widehat{g}(0)\langle n_+^H\rangle_\Phi+|\langle\mathcal{E}+\mathcal{E}'+\mathcal{E}''\rangle_\Phi|, \qquad (6.22)$$

and this concludes the proof of Lemma 6.1. □

6.2. *Estimates on $\mathcal{Q}_3^{(2)}$ and $\mathcal{Q}_3^{(3)}$*

Here we show the remaining $\varepsilon_K K_H^{\text{diag}}$ can control $\mathcal{Q}_3^{(2)}$ and $\mathcal{Q}_3^{(3)}$.

Lemma 6.2. *There exists a universal constant $C>0$ such that the following holds. If $\rho a^d\le C^{-1}$, $|\rho_z-\rho|\le\frac{1}{2}\rho$, and if the parameters satisfy the relations in Appendix F, then for all normalized states $\Phi\in\mathscr{F}_s(\text{RanQ})$ satisfying*

$$\Phi=\mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Phi, \qquad (6.23)$$

we have

$$\Big|\Big\langle\mathcal{Q}_3^{(2)}+\mathcal{Q}_3^{(3)}\Big\rangle_\Phi\Big|\le\varepsilon_K\langle\mathcal{K}_H^{\text{diag}}\rangle_\Phi$$

Proof. Notice that $\mathcal{Q}_3^{(2)}$ and $\mathcal{Q}_3^{(3)}$ are identical except for the substitution of $-k$ by $k-p$, so we can focus on $\mathcal{Q}_3^{(3)}$. We can commute the creation operators to write this term as

$$\mathcal{Q}_3^{(3)} = -\frac{1}{|\Lambda|}\sum_{k\in\mathcal{P}_H,p\in\mathcal{P}_L}\frac{\widehat{g}(k)\alpha_{p-k}}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\big(\bar{z}b_{k-p}^\dagger a_p^\dagger b_k + z b_k^\dagger a_p b_{k-p}\big), \tag{6.24}$$

We use the Cauchy-Schwarz inequality with weight $\varepsilon > 0$, and by (A.3),

$$\begin{aligned}|\langle\mathcal{Q}_3^{(3)}\rangle_\Phi| &\leq \frac{|z|}{|\Lambda|}\sum_{k\in\mathcal{P}_H,p\in\mathcal{P}_L}\frac{|\widehat{g}(k)\alpha_{p-k}|}{\sqrt{1-\alpha_k^2}\sqrt{1-\alpha_{p-k}^2}}\langle\varepsilon b_{k-p}^\dagger a_p^\dagger a_p b_{k-p} + \varepsilon^{-1}b_k^\dagger b_k\rangle_\Phi \\ &\leq C\frac{|z|}{|\Lambda|}\ell^2\rho_z\widehat{g}(0)^2K_H^{-2}\sum_{k\in\mathcal{P}_H,p\in\mathcal{P}_L}\langle\varepsilon b_{k-p}^\dagger a_p^\dagger a_p b_{k-p} + \varepsilon^{-1}b_k^\dagger b_k\rangle_\Phi,\end{aligned}$$

and using (6.23),

$$\sum_{p\in\mathcal{P}_L}\langle b_{k-p}^\dagger a_p^\dagger a_p b_{k-p}\rangle_\Phi \leq C\mathcal{M}\langle b_k^\dagger b_k\rangle_\Phi. \tag{6.25}$$

We choose $\varepsilon = \sqrt{K_L^d/\mathcal{M}}$, and insert $\mathcal{D}_k \geq K_H^2\ell^{-2}$, obtaining

$$\begin{aligned}|\langle\mathcal{Q}_3^{(3)}\rangle_\Phi| &\leq C|z|\ell^{2-d}\rho_z\widehat{g}(0)^2K_H^{-2}(\varepsilon\mathcal{M}+\varepsilon^{-1}K_L^d)\sum_{k\in\mathcal{P}_H}\langle b_k^\dagger b_k\rangle_\Phi \\ &\leq C|z|\ell^{4-d}\rho_z\widehat{g}(0)^2K_H^{-4}K_L^{d/2}\sqrt{\mathcal{M}}\sum_{k\in\mathcal{P}_H}\mathcal{D}_k\langle b_k^\dagger b_k\rangle_\Phi.\end{aligned} \tag{6.26}$$

Thanks to condition (F.8), $\mathcal{Q}_3^{(3)}$ can be absorbed in the positive $\varepsilon_K\mathcal{K}_H^{\text{diag}}$ term. □

7. Conclusion

In all this section, we assume that all our parameters satisfy the relations in Appendix F, and prove Theorem 1.1 by combining as follows all the previous estimates.

Let us first fix $C_B \geq 2I_d^{\text{Bog}}$, and assume that there exists a normalized N-particle state $\Psi \in L^2_{\text{sym}}(\Lambda^N)$ with energy

$$\langle\mathcal{H}\rangle_\Psi \leq \frac{1}{2}\rho^2|\Lambda|\widehat{g}(0)(1+C_B\lambda_d^{\text{LHY}}). \tag{7.1}$$

If Ψ does not exist we are clearly done. For such a state Ψ we use the localization of large matrices Lemma Appendix B.1 to decompose Ψ into Ψ^m's satisfying that,

$$\Psi^m = \mathbb{1}_{\{n_+^L\leq\frac{\mathcal{M}}{2}+m\}}\Psi^m, \qquad \text{and} \qquad \sum_m\|\Psi^m\|^2 = 1 \tag{7.2}$$

with

$$\langle\Psi,\mathcal{H}\Psi\rangle \geq \sum_{2|m|\leq\mathcal{M}}\langle\Psi^m,\mathcal{H}\Psi^m\rangle + \frac{|\Lambda|}{2}\rho^2\widehat{g}(0)\Big(1+2C_B\lambda_d^{\text{LHY}}\Big)\sum_{2|m|>\mathcal{M}}\|\Psi^m\|^2 + o_d^{\text{LHY}}. \tag{7.3}$$

The next goal is then to prove our lower bound for each term of the first sum of above to reconstruct $\sum_m \|\Psi^m\|^2 = 1$. Hence we only have to prove the desired lower bound for states $\Psi \in L^2_{\mathrm{sym}}(\Lambda^N)$ satisfying

$$\Psi = \mathbb{1}_{\{n_+^L \leq \mathcal{M}\}}\Psi. \tag{7.4}$$

For such a Ψ, we use the second quantization from Proposition 2.1, the c-number substitution from Proposition 3.1 and the localization of the $3Q$ term in Proposition 5.1 to deduce

$$\langle \mathcal{H} \rangle_\Psi \geq \frac{1}{\pi} \int_{\mathbb{C}} \langle \mathcal{K}(z) + \mathcal{Q}_3^{\mathrm{soft}}(z) \rangle_{\Phi(z)} \mathrm{d}z + o_d^{\mathrm{LHY}}, \tag{7.5}$$

where $\Phi(z) = \langle \Psi | z \rangle \in \mathscr{F}_s(\mathrm{RanQ})$ was introduced in Section 3. Note that we dropped the remaining part of $\mathcal{Q}_4^{\mathrm{ren}} > 0$, and that the error terms are estimated using Theorem Appendix E.1. Now we split the integral according to the values of ρ_z. We recall that $\varepsilon_+^2 = \max\{K_\ell^4 K_L^{-2}, \lambda_d^{\mathrm{LHY}}\}$ and consider the two following cases.

- If $|\rho_z - \rho| \geq \rho\varepsilon_+$, we can apply Theorem Appendix D.1 to get a lower bound larger than the LHY term, since $E_d^{\mathrm{LHY}} > 0$, i.e.

$$\begin{aligned} &\langle \mathcal{K}(z) + \mathcal{Q}_3^{\mathrm{soft}}(z) \rangle_{\Phi(z)} \\ &\qquad \geq \Big(\frac{1}{2}\rho^2 |\Lambda| \widehat{g}(0) + 2E_d^{\mathrm{LHY}} + o_d^{\mathrm{LHY}} \Big) \|\Phi(z)\|^2 - C\rho\widehat{g}(0) \langle n_+^H \rangle_{\Phi(z)}. \end{aligned} \tag{7.6}$$

  The integral of the last term over $\{z \in \mathbb{C} : |\rho_z - \rho| \geq \varepsilon_+ \rho\}$ can be bounded by the integral over all of $\mathbb{C}$, giving $C\rho\widehat{g}(0)\langle n_+^H \rangle_\Psi$ that, thanks to (E.3), is of order o_d^{LHY}.
- Now we want to prove the desired lower bound for $|\rho_z - \rho| \leq \rho\varepsilon_+$. Recall that $\mathcal{K}(z)$ is given by

$$\mathcal{K}(z) = \mathcal{Q}(z) + \mathcal{Q}_2^{\mathrm{ex}}(z) + (\rho_z - \rho) n_+ \widehat{g}(0) - \rho\rho_z |\Lambda| \widehat{g}(0) + \rho^2 |\Lambda| \widehat{g}(0) + o_d^{\mathrm{LHY}},$$

  where we have omitted the error term $\mathcal{R}_1^{(d)}(\rho_z)$, which is lower order when $\rho_z \approx \rho$. We diagonalize $\mathcal{Q}(z)$ with Proposition 4.1 to get

$$\begin{aligned} &\mathcal{K}(z) - o_d^{\mathrm{LHY}} \\ &\geq \frac{|\Lambda|}{2}\rho_z^2 \widehat{g}(0) + E_d^{\mathrm{LHY}}(\rho_z) + \mathcal{K}^{\mathrm{diag}} + \mathcal{Q}_2^{\mathrm{ex}} + (\rho_z - \rho) n_+ \widehat{g}(0) - \rho\rho_z |\Lambda| \widehat{g}(0) + \rho^2 |\Lambda| \widehat{g}(0) \\ &= \frac{|\Lambda|}{2}\rho^2 \widehat{g}(0) + E_d^{\mathrm{LHY}}(\rho_z) + \mathcal{K}^{\mathrm{diag}} + \mathcal{Q}_2^{\mathrm{ex}} + \frac{1}{2}(\rho - \rho_z)^2 |\Lambda| \widehat{g}(0) + (\rho_z - \rho) n_+ \widehat{g}(0). \end{aligned} \tag{7.7}$$

  The last term we can bound by integrating and using (E.2),

$$\int_{\{|\rho_z - \rho| \leq \rho\varepsilon_+\}} (\rho_z - \rho) \langle n_+ \rangle_{\Phi(z)} \widehat{g}(0) \, \mathrm{d}z \leq C\rho\varepsilon_+ \widehat{g}(0) \langle n_+ \rangle_\Psi = o_d^{\mathrm{LHY}}, \tag{7.8}$$

  thanks to the choice of ε_+. Therefore, we deduce

$$\begin{aligned} &\int_{\{|\rho_z - \rho| \leq \rho\varepsilon_+\}} \langle \mathcal{K}(z) \rangle_{\Phi(z)} \mathrm{d}z \geq \\ &\int_{\{|\rho_z - \rho| \leq \rho\varepsilon_+\}} \Big(\frac{|\Lambda|}{2}\rho^2 \widehat{g}(0) + E_d^{\mathrm{LHY}}(\rho_z) \Big) \|\Phi(z)\|^2 + \langle \mathcal{K}^{\mathrm{diag}} + \mathcal{Q}_2^{\mathrm{ex}}(z) \rangle_{\Phi(z)} \, \mathrm{d}z + o_d^{\mathrm{LHY}}. \end{aligned} \tag{7.9}$$

The contributions of $\mathcal{Q}_3^{\rm soft}$, $\mathcal{Q}_2^{\rm ex}$, and $\mathcal{K}^{\rm diag}$ are combined using Proposition 6.1. Bounding the remaining positive terms by 0 and estimating the errors with the relations from Appendix F, we deduce

$$\begin{aligned}\int_{\{|\rho_z-\rho|\leq\rho\varepsilon_+\}} &\langle\mathcal{K}(z)\rangle_{\Phi(z)}\,dz\\ &\geq \int_{\{|\rho_z-\rho|\leq\rho\varepsilon_+\}}\Big(\frac{1}{2}\rho^2|\Lambda|\widehat{g}(0)+E_d^{\rm LHY}(\rho_z)\Big)\|\Phi(z)\|^2{\rm d}z+o_d^{\rm LHY}. \quad (7.10)\end{aligned}$$

Finally, in this case we can replace ρ_z by ρ up to errors of order $o_d^{\rm LHY}$. Hence we have a lower bound for all z, and we deduce from (7.5), from the contributions of the integrals in (7.6) and (7.10) on the domains $\{z\in\mathbb{C}:|\rho_z-\rho|\geq\varepsilon_+\rho\}$ and $\{z\in\mathbb{C}:|\rho_z-\rho|<\varepsilon_+\rho\}$, respectively, that

$$\langle\mathcal{H}\rangle_\Psi\geq\frac{\rho^2}{2}|\Lambda|\widehat{g}(0)+E_d^{\rm LHY}(\rho)+o_d^{\rm LHY}, \quad (7.11)$$

which concludes the proof of Theorem 1.1.

Appendix

Appendix A. Miscellaneous Estimates

Lemma Appendix A.1. *There exists a constant $C>0$ such that the following estimate holds*

$$\Big|\widehat{g\omega}(0)-\frac{1}{|\Lambda|}\sum_{k\in\mathcal{P}_H}\frac{\widehat{g}(k)^2}{2k^2}\Big|\leq C\widehat{g}(0)K_H^{-1}+\mathcal{E}_d,$$

where

$$\mathcal{E}_d\leq\begin{cases}CR^2\ell_\delta^{-2}\widehat{g}(0)^2+C\widehat{g}(0)^2|\log K_H\ell_\delta\ell^{-1}|, & \text{if } d=2,\\ C\widehat{g}(0)^2K_H\ell^{-1}, & \text{if } d=3.\end{cases}$$

The constant C in the error bounds depends on L^p-properties of the potential, $p>1$.

Proof. First of all, one can replace the sum by an integral,

$$\Big|\frac{1}{|\Lambda|}\sum_{k\in\mathcal{P}_H}\frac{\widehat{g}(k)^2}{2k^2}-\int_{|k|\geq K_H\ell^{-1}}\frac{\widehat{g}(k)^2}{2k^2}\frac{{\rm d}k}{(2\pi)^d}\Big|\leq C\widehat{g}(0)K_H^{-1}. \quad (A.1)$$

This can be proven by bounding the derivatives of the integrand on small boxes of size $(2\pi)\ell^{-1}$, but depends on L^p-properties of the potential, since we need some decay of $\widehat{g}(k)$ to control the decay of the summand. The estimate is obtained through a Hölder inequality on the sum.

Now we can compare the integral with $\widehat{g\omega}(0)$ (in $d=3$ for instance),

$$\begin{aligned}\Big|\widehat{g\omega}(0)-\int_{|k|\geq K_H\ell^{-1}}\frac{\widehat{g}(k)^2}{2k^2}\frac{{\rm d}k}{(2\pi)^d}\Big| &\leq\Big|\int_{|k|\leq K_H\ell^{-1}}\frac{\widehat{g}(k)^2}{2k^2}\frac{{\rm d}k}{(2\pi)^d}\Big|\\ &\leq C\widehat{g}(0)^2K_H\ell^{-1}. \quad (A.2)\end{aligned}$$

The estimate is similar in $d=2$, except we must bound $|\widehat{g}_k - \widehat{g}(0)| \leq R^2\widehat{g}(0)k^2$ for small k's, to have integrability. □

We end this section by stating, without proof, the following simple bounds, which will be useful for further estimates.

Lemma Appendix A.2. *If $|\rho_z - \rho| \leq \frac{1}{2}\rho$ and $|k| \geq K_H \ell^{-1}$. Then*

$$|\alpha_k| \leq C\frac{|\rho_z \widehat{g}(k)|}{k^2}, \quad \text{and} \quad |\mathcal{D}_k - k^2| \leq C\ell^2\rho\widehat{g}_0 K_H^{-2}k^2. \tag{A.3}$$

Appendix B. Localization of Large Matrices: restrictions of n_+^L

Some of our errors depend on n_+^L. Thus, we need a priori bounds on this excitation number, for low energy states. We explain how we can reduce the analysis to states with bounded number of low excitations, $n_+^L \leq \mathcal{M}$, in Proposition Appendix B.1.

Proposition Appendix B.1. *There exist C, $\eta > 0$ such that the following holds. Let $\Psi \in L^2_{\rm sym}(\Lambda^N)$ be a normalized N-particle state which satisfies*

$$\langle \mathcal{H} \rangle_\Psi \leq \frac{1}{2}\rho^2|\Lambda|\widehat{g}(0) + C_B\rho^2|\Lambda|\widehat{g}(0)\lambda_d^{\rm LHY} \tag{B.1}$$

for some $C_B > 0$. Assume that $\mathcal{M}$ and $\|v\|_1$ satisfy (F.4). Then, there exists a sequence $\{\Psi^m\}_{m\in\mathbb{Z}} \subseteq L^2_{\rm sym}(\Lambda^N)$ such that $\sum_m \|\Psi^m\|^2 = 1$ and

$$\Psi^m = \mathbb{1}_{[0,\frac{\mathcal{M}}{2}+m]}(n_+^L)\Psi^m, \tag{B.2}$$

and such that the following lower bound holds true

$$\langle \Psi, \mathcal{H}\Psi\rangle \geq \sum_{2|m|\leq \mathcal{M}} \langle \Psi^m, \mathcal{H}\Psi^m\rangle + \frac{|\Lambda|}{2}\rho^2\widehat{g}(0)\Big(1 + 2C_B\lambda_d^{\rm LHY}\Big)\sum_{2|m|>\mathcal{M}} \|\Psi^m\|^2 + o_d^{\rm LHY}.$$

The proof of Proposition Appendix B.1 will follow from the Lemmas Appendix B.1 and Appendix B.2 below. The proof of Lemma Appendix B.1 is inspired by the localization of large matrices result in [40]. It is also similar to the bounds in [41, Proposition 21]. It can be interpreted as an analogue of the standard IMS localization formula. The error produced is written in terms of the following quantities d_1^L and d_2^L (i.e the terms in the Hamiltonian that change n_+^L by 1 or 2).

$$\begin{aligned} d_1^L := &\sum_{i\neq j}(P_i + Q_{H,i})\overline{Q}_{H,j}v(x_i - x_j)\overline{Q}_{H,i}\overline{Q}_{H,j} + h.c. \\ &+ \sum_{i\neq j}\overline{Q}_{H,i}(P_j + Q_{H,j})v(x_i - x_j)(P_i + Q_{H,i})(P_j + Q_{H,j}) + h.c. \end{aligned} \tag{B.3}$$

and

$$d_2^L := \sum_{i\neq j}(P_i + Q_{H,i})(P_j + Q_{H,j})v(x_i - x_j)\overline{Q}_{H,j}\overline{Q}_{H,i} + h.c. \tag{B.4}$$

where $Q_{H,j}$ is defined in (5.4). These error terms are estimated in Lemma Appendix B.2.

Lemma Appendix B.1. *Let $\theta : \mathbb{R} \to [0,1]$ be any compactly supported Lipschitz function such that $\theta(s) = 1$ for $|s| < \frac{1}{8}$ and $\theta(s) = 0$ for $|s| > \frac{1}{4}$. For any $\mathcal{M} > 0$, define $c_{\mathcal{M}} > 0$ and $\theta_{\mathcal{M}}$ such that*

$$\theta_{\mathcal{M}}(s) = c_{\mathcal{M}}\theta\Big(\frac{s}{\mathcal{M}}\Big), \qquad \sum_{s\in\mathbb{Z}} \theta_{\mathcal{M}}(s)^2 = 1.$$

Then there exists a $C > 0$ depending only on θ such that, for any normalized state $\Psi \in L^2_{\text{sym}}(\Lambda^N)$,

$$\langle \Psi, \mathcal{H}\Psi\rangle \geq \sum_{m\in\mathbb{Z}} \langle \Psi^m, \mathcal{H}\Psi^m\rangle - \frac{C}{\mathcal{M}^2}\left(|\langle d_1^L\rangle_\Psi| + |\langle d_2^L\rangle_\Psi|\right), \tag{B.5}$$

where $\Psi^m = \theta_{\mathcal{M}}(n_+^L - m)\Psi$.

Proof. Notice that $\mathcal{H}$ only contains terms that change n_+^L by $0, \pm 1$ or ± 2. Therefore, we write our operator as $\mathcal{H} = \sum_{|k|\leq 2} \mathcal{H}^{(k)}$, with $\mathcal{H}^{(k)} n_+^L = (n_+^L + k)\mathcal{H}^{(k)}$. Moreover, $\mathcal{H}^{(k)} + \mathcal{H}^{(-k)} = d_k^L$ for $k = 1, 2$. We use this decomposition to estimate the localized energy,

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \langle \Psi^m, \mathcal{H}\Psi^m\rangle &= \sum_{m\in\mathbb{Z}} \sum_{|k|\leq 2} \langle \theta_{\mathcal{M}}(n_+^L - m)\theta_{\mathcal{M}}(n_+^L - m + k)\Psi, \mathcal{H}^{(k)}\Psi\rangle \\
&= \sum_{m,s\in\mathbb{Z}} \sum_{|k|\leq 2} \langle \theta_{\mathcal{M}}(s - m)\theta_{\mathcal{M}}(s - m + k)\mathbb{1}_{\{n_+^L = s\}}\Psi, \mathcal{H}^{(k)}\Psi\rangle \\
&= \sum_{m,s\in\mathbb{Z}} \sum_{|k|\leq 2} \theta_{\mathcal{M}}(m)\theta_{\mathcal{M}}(m + k)\langle \mathbb{1}_{\{n_+^L = s\}}\Psi, \mathcal{H}^{(k)}\Psi\rangle,
\end{aligned}$$

where in the last line we changed the index m into $s - m$. We can sum on s to recognize

$$\sum_{m\in\mathbb{Z}} \langle \Psi^m, \mathcal{H}\Psi^m\rangle = \sum_{m\in\mathbb{Z}} \sum_{|k|\leq 2} \theta_{\mathcal{M}}(m)\theta_{\mathcal{M}}(m + k)\langle \Psi, \mathcal{H}^{(k)}\Psi\rangle. \tag{B.6}$$

Furthermore the energy of Ψ can be rewritten as

$$\langle \Psi, \mathcal{H}\Psi\rangle = \sum_{|k|\leq 2} \langle \Psi, \mathcal{H}^{(k)}\Psi\rangle = \sum_{m\in\mathbb{Z}} \sum_{|k|\leq 2} \theta_{\mathcal{M}}(m)^2 \langle \Psi, \mathcal{H}^{(k)}\Psi\rangle, \tag{B.7}$$

by definition of $\theta_{\mathcal{M}}$. Thus, the localization error is

$$\sum_{m\in\mathbb{Z}} \langle \Psi^m, \mathcal{H}\Psi^m\rangle - \langle \Psi, \mathcal{H}\Psi\rangle = \sum_{|k|\leq 2} \delta_k \langle \Psi, \mathcal{H}^{(k)}\Psi\rangle, \tag{B.8}$$

with

$$\delta_k = \sum_{m\in\mathbb{Z}} \left(\theta_{\mathcal{M}}(m)\theta_{\mathcal{M}}(m+k) - \theta_{\mathcal{M}}(m)^2\right) = -\frac{1}{2}\sum_{m} \left(\theta_{\mathcal{M}}(m) - \theta_{\mathcal{M}}(m+k)\right)^2. \tag{B.9}$$

Since $\delta_0 = 0$, $\delta_k = \delta_{-k}$ and $d_k^L = \mathcal{H}^{(k)} + \mathcal{H}^{(-k)}$ we find

$$\sum_{m\in\mathbb{Z}} \langle \Psi^m, \mathcal{H}\Psi^m\rangle - \langle \Psi, \mathcal{H}\Psi\rangle = \delta_1 \langle d_1^L\rangle_\Psi + \delta_2\langle d_2^L\rangle_\Psi, \tag{B.10}$$

and only remains to prove that $|\delta_k| \leq C\mathcal{M}^{-2}$. This follows from (B.9) using that θ is Lipschitz and restricting the sum to $m \in \big[-\frac{\mathcal{M}}{2}, \frac{\mathcal{M}}{2}\big]$. □

To estimate the error in (B.5), we need the following bounds on d_1^L and d_2^L.

Lemma Appendix B.2. *There exists a universal constant $C > 0$ such that, for any $\Psi \in L^2_{\rm sym}(\Lambda^N)$, with our choices of parameters we have*

$$|\langle d_1^L\rangle_\Psi| + |\langle d_2^L\rangle_\Psi| \leq C\|v\|_1 \rho K_H \langle n_+\rangle_\Psi + C\langle \mathcal{Q}_4^{\rm ren}\rangle_\Psi. \tag{B.11}$$

Proof. First note that we have the following bound on the operator norm

$$\|\overline{Q}_{H,x} v(x-y)\overline{Q}_{H,x}\| \leq CK_H^2 \ell^{-d}\|v\|_1. \tag{B.12}$$

Indeed, for all $\varphi \in \mathrm{Ran}\overline{Q}_{H,x}$,

$$\langle \overline{Q}_{H,x} v(x-y)\overline{Q}_{H,x}\varphi,\varphi\rangle \leq \int_\Lambda |\varphi(x)|^2 v(x-y)\mathrm{d}x \leq \|\varphi\|_\infty^2 \|v\|_1 \leq C\ell^{2-d}\|\Delta\varphi\|\|\varphi\|\|v\|_1, \tag{B.13}$$

by Sobolev inequality. Moreover such φ's satisfy $\|\Delta\varphi\| \leq K_H^2\ell^{-2}\|\varphi\|$ by definition of $\overline{Q}_H$, and (B.12) follows.

We split d_1^L, d_2^L in several terms multiplying out the parentheses in (B.3) and (B.4). Here we just bound some representative examples to illustrate the procedure.

For instance, we can use the Cauchy-Schwarz inequality with weight K_H and equation (B.12) to find,

$$\begin{aligned}\Big|\Big\langle \sum_{i,j} P_i \overline{Q}_{H,j} v \overline{Q}_{H,i}\overline{Q}_{H,j}\Big\rangle_\Psi\Big| &\leq K_H \frac{N}{|\Lambda|}\|v\|_1 \langle n_+^L\rangle_\Psi + K_H^{-1}\|\overline{Q}_H v \overline{Q}_H\| N\langle n_+^L\rangle_\Psi,\\ &\leq C\|v\|_1 K_H \rho\langle n_+\rangle_\Psi\end{aligned}$$

where we used $n_+^L \leq n_+$.

We also estimate a term where the need for $\mathcal{Q}_4^{\rm ren}$ becomes clear. In order to do that we complete the Q_H to a $Q = Q_H + \overline{Q}_H$,

$$\begin{aligned}&\Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i} P_j v Q_{H,i} Q_{H,j} + h.c.\Big\rangle_\Psi\Big|\\ &\quad\leq \Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i} P_j v (Q_{H,i}\overline{Q}_{H,j} + \overline{Q}_{H,i} Q_{H,j})\Big\rangle_\Psi + h.c.\Big|\\ &\qquad + \Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i} P_j v Q_i Q_j\Big\rangle_\Psi + h.c.\Big| + \Big|\Big\langle \sum_{i,j} P_i \overline{Q}_{H,j} v \overline{Q}_{H,i}\overline{Q}_{H,j}\Big\rangle_\Psi\Big|.\end{aligned}$$

The first and the third terms can be estimated in the same manner as above, so let us focus on completing the second term in order to obtain $4Q$ terms.

$$\Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i}P_j v Q_i Q_j \Big\rangle_\Psi + h.c.\Big| \tag{B.14}$$

$$\leq \Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i}P_j v (Q_iQ_j + \omega(P_iP_j + P_iQ_j + Q_iP_j))\Big\rangle_\Psi + h.c.\Big| \tag{B.15}$$

$$+ \Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i}P_j v\omega (P_iQ_j + Q_iP_j))\Big\rangle_\Psi + h.c.\Big| \tag{B.16}$$

$$+ \Big|\Big\langle \sum_{i\neq j} \overline{Q}_{H,i}P_j v\omega P_iP_j)\Big\rangle_\Psi + h.c.\Big|. \tag{B.17}$$

The second and the third terms are treated as above, using that $0 \leq \omega \leq 1$ on the support of v. By a Cauchy-Schwarz inequality on the first term we get

$$\text{(B.15)} \leq \langle \mathcal{Q}_4^{\text{ren}}\rangle_\Psi + C\frac{N}{|\Lambda|}\|v\|_1\langle n_+\rangle_\Psi. \qquad \square$$

Now we can combine Lemmas Appendix B.1 and Appendix B.2 to prove Proposition Appendix B.1.

Proof. [Proposition Appendix B.1] Given $\Psi \in L^2_{\text{sym}}(\Lambda^N)$ satisfying (B.1), we can apply Lemma Appendix B.1 and write $\Psi^m = \theta_{\mathcal{M}}(n_+^L - m)\Psi$. In (B.5) we split the sum into two. The first part, for $|m| < \frac{1}{2}\mathcal{M}$, we keep. For $|m| > \frac{1}{2}\mathcal{M}$, Ψ_m satisfies

$$\langle n_+\rangle_{\Psi^m} \geq \langle n_+^L\rangle_{\Psi^m} \geq \frac{\mathcal{M}}{4}\|\Psi^m\|^2, \tag{B.18}$$

due to the cutoff $\theta_{\mathcal{M}}(n_+^L - m)$. Since we have from (F.5) that $\mathcal{M} \gg \rho^2\ell^2|\Lambda|\widehat{g}(0)\lambda_d^{\text{LHY}}$, this is a larger bound than (E.2), and thus the assumption of Theorem Appendix E.1 cannot be satisfied for Ψ^m and we must have the lower bound

$$\langle \Psi^m, \mathcal{H}\Psi^m\rangle \geq \rho^2|\Lambda|\widehat{g}(0)\Big(\frac{1}{2} + C_B\lambda_d^{\text{LHY}}\Big)\|\Psi^m\|^2. \tag{B.19}$$

We finally bound the last term in (B.5), using Lemma Appendix B.2. We use the condensation estimate (E.2) and the bound (E.4) on $\mathcal{Q}_4^{\text{ren}}$ to obtain

$$\begin{aligned}\mathcal{M}^{-2}\big(|\langle d_1^L\rangle_\Psi| + |\langle d_2^L\rangle_\Psi|\big) &\leq C\mathcal{M}^{-2}\Big(\rho K_H\|v\|_1\ell^2 + 1\Big)|\Lambda|\rho^2\widehat{g}(0)\lambda_d^{\text{LHY}} \\ &= o_d^{\text{LHY}},\end{aligned} \tag{B.20}$$

for $\mathcal{M}$ and $\|v\|_1$ satisfying (F.4). Using the estimates (B.19) for $m > \frac{1}{2}\mathcal{M}$ and (B.20) in formula (B.5) we conclude the proof. $\square$

Appendix C. Rigorous Bogoliubov Theory for Quadratic Hamiltonians

C.1. *Diagonalization of quadratic Hamiltonians*

In the next proposition we show a simple consequence of the Bogoliubov method, see [42, Theorem 6.3] and [34], that we use to diagonalize the quadratic term $\mathcal{Q}(z)$ of Proposition 3.1.

Theorem Appendix C.1. *Let $a_\pm$ be operators on a Hilbert space satisfying $[a_+, a_-] = 0$. For $\mathcal{A} > 0$, $\mathcal{B} \in \mathbb{R}$ satisfying $|\mathcal{B}| < \mathcal{A}$ and arbitrary $\kappa \in \mathbb{C}$, we have the operator identity*

$$\begin{aligned}&\mathcal{A}(a_+^\dagger a_+ + a_-^\dagger a_-) + \mathcal{B}(a_+^\dagger a_-^\dagger + a_+ a_-) + \kappa(a_+^\dagger + a_-) + \overline{\kappa}(a_+ + a_-^\dagger)\\&\quad = \mathcal{D}(b_+^\dagger b_+ + b_-^\dagger b_-) - \frac{1}{2}\alpha\mathcal{B}([a_+, a_+^\dagger] + [a_-, a_-^\dagger]) - \frac{2|\kappa|^2}{\mathcal{A}+\mathcal{B}},\end{aligned}$$

where $\mathcal{D} = \frac{1}{2}\Big(\mathcal{A} + \sqrt{\mathcal{A}^2 - \mathcal{B}^2}\Big)$, and

$$b_+ = \frac{1}{\sqrt{1-\alpha^2}}\big(a_+ + \alpha a_-^\dagger + \bar{c}_0\big), \qquad b_- = \frac{1}{\sqrt{1-\alpha^2}}\big(a_- + \alpha a_+^\dagger + c_0\big), \tag{C.1}$$

with

$$\alpha = \mathcal{B}^{-1}\big(\mathcal{A} - \sqrt{\mathcal{A}^2 - \mathcal{B}^2}\big), \qquad c_0 = \frac{2\bar{\kappa}}{\mathcal{A} + \mathcal{B} + \sqrt{\mathcal{A}^2 - \mathcal{B}^2}}. \tag{C.2}$$

Remark Appendix C.1. Note that the normalization of $b_\pm$ is chosen such that

$$[b_+, b_+^\dagger] = \frac{[a_+, a_+^\dagger] - \alpha^2 [a_-, a_-^\dagger]}{1-\alpha^2}, \tag{C.3}$$

and we recover the canonical commutation relations $[b_+, b_+^\dagger] = 1$ when a_+ and a_- satisfies them as well.

Proof. This follows directly from algebraic computations. □

C.2. *Evaluation of the Bogoliubov integral*

In this section we report two lemmas for the calculation of the Bogoliubov integral. The first one, under weak assumptions, gives a bound for general Bogoliubov-type integrals, expressing the dependence on the parameters involved in the spectral gaps. The second one is a more precise calculation which lets us obtain the exact value of the Lee-Huang-Yang constant. Let us recall the definition of G_d in (1.18):

$$G_d(k) := \frac{\widehat{g}_{\mathbb{R}^d}(k)^2 - \widehat{g}_{\mathbb{R}^d}(0)^2 \mathbb{1}_d(\ell_\delta k)}{2k^2}. \tag{C.4}$$

Lemma Appendix C.1. *Let $\mathcal{A}, \mathcal{B} : \mathbb{R}^d \to \mathbb{R}$ be two functions such that, for parameters satisfying $\kappa > 0$, $0 < K_2 \leq K_1$, $\ell_\delta^{-1} \leq K < a^{-1}$,*

$$\begin{aligned}\mathcal{A}(k) \geq \kappa[|k| - K]_+^2 + 2K_1\widehat{g}(0), \qquad |\mathcal{B}(k)| \leq 2K_2\widehat{g}(0),\\ |\mathcal{B}(k) - \mathcal{B}(0)| \leq K_2 R^2 \widehat{g}(0)|k|^2,\end{aligned} \tag{C.5}$$

and let us introduce the integral, recalling (1.18),

$$I(d) = \int_{\mathbb{R}^d} \Big(\mathcal{A}(k) - \sqrt{\mathcal{A}(k)^2 - \mathcal{B}(k)^2}\Big)\mathrm{d}k - \frac{K_2^2}{\kappa}\int_{\mathbb{R}^d} G_d(k)\mathrm{d}k, \tag{C.6}$$

then there exists a constant $C > 0$ *such that*

- *For* $d = 3$,

$$I(3) \leq C\frac{KK_2^2 a}{\kappa}\widehat{g\omega}(0) + C\widehat{g}(0)K_2^2\big(K_1^{-1}K^3 + \kappa^{-1}\widehat{g}(0)K\log((aK)^{-1})\big) + \min\Big(\kappa^{-3}\widehat{g}(0)^4\frac{K_2^4}{K^3}, \frac{K_2^4}{K_1^2}\widehat{g\omega}(0)\Big).$$

- *For* $d = 2$,

$$I(2) \leq C\widehat{g}(0)K_2^2\Big(\widehat{g}(0)(\rho K_1^{-1} + \kappa^{-1}R^2\ell_\delta^{-2}) + \kappa^{-1}\widehat{g}(0)|\log(2K\ell_\delta)| + \kappa^{-1}\widehat{g}(0)\Big) + \min\Big(\kappa^{-3}\widehat{g}(0)^4\frac{K_2^4}{K^4}, \frac{K_2^4}{K_1^2}\widehat{g\omega}(0)\Big).$$

Proof. The proof of the 3D and 2D cases can be found in [11, Lemma C.1] and [18, Lemma C.5], respectively. □

Lemma Appendix C.2. *There exists a* $C > 0$ *such that*

$$\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}\Big(\sqrt{k^4 - 2k^2\rho\widehat{g}(k)} - k^2\rho\widehat{g}(k) - \rho^2 G_d(k)\Big)\mathrm{d}k = \frac{\rho^2}{2}I_d^{Bog}\widehat{g}(0)\lambda_d^{LHY} + \mathcal{E}_d^{int}(\rho), \tag{C.7}$$

where

$$|\mathcal{E}_d^{int}(\rho)| \leq \begin{cases} C\rho^2\widehat{g}(0)^3\rho R^2\log(\widehat{g}(0)), & \text{if } d = 2,\\ C\rho^2\widehat{g}(0)^3\rho R^2\sqrt{\rho\widehat{g}(0)^3}, & \text{if } d = 3.\end{cases} \tag{C.8}$$

Proof. The idea of the proof is to estimate the error made approximating $\widehat{g}(k)$ with $\widehat{g}(0)$ and then changing variables $k \mapsto \sqrt{\rho\widehat{g}(0)}k$ to reduce to I_d^{Bog}. The details can be found in [18, Proposition C.3] and [11, Lemma C.2] for dimension 2 and 3, respectively. □

Appendix D. When ρ_z is far from ρ

Before establishing the lower bound when $|\rho - \rho_z| \geq \rho\varepsilon_+$, we first need the following intermediate lemma, which states that the elements corresponding to the soft pairs interaction in $\mathcal{Q}_3^{\mathrm{ren}}$ can be bounded at the price of a small part of the kinetic energy. We recall the definition of $\mathcal{Q}_3^{\mathrm{soft}}$ in (5.7) and the definition of the momenta spaces $\mathcal{P}_L$ and $\mathcal{P}_H$ in (5.2).

Lemma Appendix D.1. *There exists a universal constant $C>0$ such that, for any $z\in\mathbb{C}$, any $\varepsilon>0$, and any $\Phi\in\mathcal{F}_s(\mathrm{RanQ})$ satisfying*

$$\langle n_+\rangle_\Phi \le \rho|\Lambda|, \tag{D.1}$$

we have

$$\Big\langle \frac{\varepsilon}{2}\sum_{k\in\mathcal{P}_H} k^2 a_k^\dagger a_k + \mathcal{Q}_3^{\mathrm{soft}}(z)\Big\rangle_\Phi \ge -C|\Lambda|\varepsilon^{-1}\rho\rho_z\widehat{g}(0)\frac{K_\ell^2}{K_H^2}\frac{\langle n_+^L\rangle_\Phi}{N}K_L^d. \tag{D.2}$$

Proof. Introducing the operators

$$b_k := a_k + \frac{2}{\varepsilon|\Lambda|}\sum_{p\in\mathcal{P}_L}\frac{\widehat{g}(k)}{k^2} z a_{p-k}^\dagger a_p, \tag{D.3}$$

and

$$\mathcal{K}_\varepsilon^{\mathrm{diag}} = \frac{\varepsilon}{2}\sum_{k\in\mathcal{P}_H} k^2 a_k^\dagger a_k \tag{D.4}$$

we can complete the square in the following expression, obtaining

$$\begin{aligned}\mathcal{K}_\varepsilon^{\mathrm{diag}} + \mathcal{Q}_3^{\mathrm{soft}} &= \sum_{k\in\mathcal{P}_H}\Big(\frac{\varepsilon}{2}k^2 b_k^\dagger b_k - \frac{2|z|^2}{\varepsilon|\Lambda|^2}\sum_{p,s\in\mathcal{P}_L}\frac{\widehat{g}(k)^2}{k^2}a_s^\dagger a_{s-k}a_{p-k}^\dagger a_p\Big)\\ &\ge -\frac{2|z|^2}{\varepsilon|\Lambda|^2}\sum_{k\in\mathcal{P}_H}\frac{\widehat{g}(k)^2}{k^2}\sum_{p,s\in\mathcal{P}_L}a_s^\dagger\big(a_{p-k}^\dagger a_{s-k} + [a_{s-k},a_{p-k}^\dagger]\big)a_p.\end{aligned}$$

For the term without commutator, estimated on a state Φ which satisfies (D.1) and using the Cauchy-Schwarz inequality

$$a_s^\dagger a_{p-k}^\dagger a_{s-k}a_p \le C(a_s^\dagger a_{p-k}^\dagger a_{p-k}a_s + a_p^\dagger a_{s-k}^\dagger a_{s-k}a_p) \tag{D.5}$$

we have

$$\begin{aligned}&\frac{2|z|^2}{\varepsilon|\Lambda|^2}\Big\langle\sum_{k\in\mathcal{P}_H}\frac{\widehat{g}(k)^2}{k^2}\sum_{p,s\in\mathcal{P}_L}a_s^\dagger a_{p-k}^\dagger a_{p-k}a_s\Big\rangle_\Phi\\ &\le C\varepsilon^{-1}\frac{\rho_z\widehat{g}(0)^2}{|\Lambda|}\sum_{k\in\frac{1}{2}\mathcal{P}_H}\sum_{s\in\mathcal{P}_L}\frac{1}{k^2}\langle a_s^\dagger a_k^\dagger a_k a_s\rangle_\Phi\Big(\sum_{p\in\mathcal{P}_L}1\Big)\\ &\le C\varepsilon^{-1}\rho\rho_z\ell^2\widehat{g}(0)^2\frac{\langle n_+^L\rangle_\Phi K_L^d}{K_H^2},\end{aligned} \tag{D.6}$$

where in the last line we used that the sum over $\frac{1}{2}\mathcal{P}_H$ of $a_k^\dagger a_k$ can be bounded by the number of bosons $N=\rho|\Lambda|$, while the sum over $\mathcal{P}_L$ of the $a_s^\dagger a_s$ can be bounded by $C\langle n_+^L\rangle_\Phi$ thanks to the assumptions on Φ.

On the other hand, the commutator satisfies $a_s^\dagger[a_{s-k}, a_{p-k}^\dagger]a_p = \delta_{s=p}a_p^\dagger a_p$, so we get

$$\frac{2|z|^2}{\varepsilon|\Lambda|^2}\left\langle \sum_{k\in\mathcal{P}_H}\frac{\widehat{g}(k)^2}{k^2}\sum_{p,s\in\mathcal{P}_L} a_s^\dagger[a_{s-k}, a_{p-k}^\dagger]a_p\right\rangle_\Phi \\ \leq C\frac{|z|^2}{\varepsilon|\Lambda|^2}\sum_{k\in\mathcal{P}_H, p\in\mathcal{P}_L}\frac{\widehat{g}(k)^2}{k^2}\langle a_p^\dagger a_p\rangle_\Phi \leq C\varepsilon^{-1}\rho_z\widehat{g}(0)\langle n_+^L\rangle_\Phi, \quad \text{(D.7)}$$

where we used Lemma Appendix A.1, and we obtain a term which is smaller than the error stated in the lemma provided $\frac{K_\ell^2 K_L^d}{K_H^2} \leq 1$.

Combining the inequalities from (D.6) and (D.7) we get the estimate of the lemma. □

We are now ready to state the theorem which gives a lower bound for the expression (3.3) when $|\rho - \rho_z| \geq \rho\varepsilon_+$. We use the notation

$$\Phi(z) := \langle z|\Psi\rangle, \qquad z \in \mathbb{C}, \quad \text{(D.8)}$$

where $|z\rangle$ belongs to the family of coherent states of the form (3.1), so that, from the c-number substitution, we can write

$$\langle\Psi, \mathcal{H}\Psi\rangle = \frac{1}{\pi}\int_{\mathbb{C}}\langle\Phi(z), (\mathcal{K}(z) + \mathcal{Q}_3^{\text{ren}} + \mathcal{Q}_4^{\text{ren}} + \mathcal{R}_0)\Phi(z)\rangle \mathrm{d}z. \quad \text{(D.9)}$$

We further observe that, since $\Psi = \mathbb{1}_{[0,\mathcal{M}]}(n_+^L)\Psi$, we have

$$\langle n_+^L\rangle_{\Phi(z)} \leq \mathcal{M}\|\Phi(z)\|^2, \quad \text{(D.10)}$$

and the simpler

$$\langle n_+\rangle_{\Phi(z)} \leq N\|\Phi(z)\|^2. \quad \text{(D.11)}$$

Theorem Appendix D.1. *Assume $|\rho - \rho_z| \geq \rho\varepsilon_+$ and that the relations between the parameters in Appendix F hold true. If there exists a $C > 0$ such that $\rho a^d \leq C^{-1}$, then for any normalized, $N-$particle state $\Psi \in \mathscr{F}_s(L^2(\Lambda))$ satisfying (E.1) and $\Psi = \mathbb{1}_{[0,2\mathcal{M}]}(n_+^L)\Psi$, the following lower bound holds,*

$$\langle\mathcal{K}(z) + \mathcal{Q}_3^{\text{soft}}(z) + \mathcal{R}_0\rangle_{\Phi(z)} \geq \left(\frac{1}{2}\rho^2|\Lambda|\widehat{g}(0) + 2E_d^{\text{LHY}}\right)\|\Phi(z)\|^2 - C\rho\widehat{g}(0)\langle n_+^H\rangle_{\Phi(z)}.$$

Proof. We start by proving the following lower bound

$$\langle\mathcal{K}(z) + \mathcal{Q}_3^{\text{soft}}\rangle_{\Phi(z)} \\ \geq |\Lambda|\widehat{g}(0)\left(\frac{1}{2}\rho_z^2 + \rho^2 - \rho\rho_z - CK_\ell^2 K_L^{-1}(\rho\rho_z + \rho_z^2 + \rho^2) - C\rho^2\lambda_d^{\text{LHY}}\right)\|\Phi(z)\|^2 \\ - C\rho\widehat{g}(0)\langle n_+^H\rangle_{\Phi(z)}. \quad \text{(D.12)}$$

We use Lemma Appendix D.1. Subtracting a small part of the kinetic energy from $\mathcal{K}(z)$, we get a bound on $\mathcal{Q}_3^{\rm soft}(z)$,

$$\begin{aligned}\frac{\varepsilon}{2\pi}\Big\langle \sum_{k\in\mathcal{P}_H} k^2 a_k^\dagger a_k + \mathcal{Q}_3^{\rm soft}(z)\Big\rangle_{\Phi(z)} &\geq -C|\Lambda|\varepsilon^{-1}\rho\rho_z\widehat{g}(0)\frac{K_\ell^2}{K_H^2}\frac{\langle n_+^L\rangle_{\Phi(z)}}{N}K_L^d \\ &\geq -C|\Lambda|\varepsilon^{-1}\rho\rho_z\widehat{g}(0)\frac{K_\ell^6}{K_L^3}\|\Phi(z)\|^2, \end{aligned} \tag{D.13}$$

where we used (D.10) and the assumption on Ψ to have $\langle n_+^L\rangle_{\Phi(z)} \leq C\mathcal{M}\|\Phi(z)\|^2$ and the relations between the parameters. Choosing

$$\varepsilon = \frac{K_\ell^4}{K_L^2} \ll 1, \tag{D.14}$$

this term can be absorbed in the $K_\ell^2 K_L^{-1}$ term in (D.12).

Subtracting $\varepsilon/2\sum k^2 a_k^\dagger a_k$ from $\mathcal{K}^{\rm Bog}$, for $\varepsilon \ll 1$, this is turned into

$$\widetilde{\mathcal{K}}^{\rm Bog} = \frac{1}{2}\sum_{k\neq 0}\widetilde{\mathcal{A}}_k\big(a_k^\dagger a_k + a_{-k}^\dagger a_{-k}\big) + \frac{1}{2}\sum_{k\neq 0}\mathcal{B}_k\big(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}\big), \tag{D.15}$$

where

$$\widetilde{\mathcal{A}}_k := (1-\varepsilon)k^2 + \rho_z\widehat{g}_k. \tag{D.16}$$

The diagonalization procedure in Proposition 4.1 can be adapted with the modified kinetic energy, and we find

$$\begin{aligned}\widetilde{\mathcal{K}}^{\rm Bog} &\geq -\frac{1}{2}\sum_{k\neq 0}\Big(\widetilde{\mathcal{A}}_k - \sqrt{\widetilde{\mathcal{A}}_k^2 - \mathcal{B}_k^2}\Big) \\ &\geq -\frac{|\Lambda|}{2(2\pi)^2}\int_{\mathbb{R}^d}\Big(\widetilde{\mathcal{A}}_k - \sqrt{\widetilde{\mathcal{A}}_k^2 - \mathcal{B}_k^2}\Big){\rm d}k + o_d^{\rm LHY}, \end{aligned} \tag{D.17}$$

where we approximated the series by the integral obtaining a small error absorbed in the last term. Since

$$\widetilde{\mathcal{A}}_k \geq (1-\varepsilon)\Big[|k| - \sqrt{\rho\widehat{g}(0)}\Big]_+^2 + \frac{1}{2}\rho_z\widehat{g}(0), \tag{D.18}$$

we satisfy the assumptions of Lemma Appendix C.1, with $\kappa = (1-\varepsilon), K = \sqrt{\rho\widehat{g}(0)}$, $K_1 = \frac{1}{2}\rho_z$, $K_2 = \rho_z$, and therefore we get the estimate

$$\begin{aligned}&\frac{1}{2}\rho_z^2|\Lambda|\widehat{g\omega}(0) - \frac{|\Lambda|}{2(2\pi)^2}\int_{\mathbb{R}^d}\Big(\widetilde{\mathcal{A}}_k - \sqrt{\widetilde{\mathcal{A}}_k^2 - \mathcal{B}_k^2}\Big){\rm d}k \\ &\geq -C\varepsilon\rho_z^2|\Lambda|\widehat{g\omega}(0) - C\rho\rho_z|\Lambda|\widehat{g}(0)\lambda_d^{\rm LHY} \\ &\quad - C\rho_z^2|\Lambda|\widehat{g}(0)(1-\varepsilon)^{-1}(\lambda_d^{\rm LHY} + R^2\ell_\delta^{-2}\mathbb{1}_{d,2}) + o_d^{\rm LHY} \\ &\geq -C\rho_z^2|\Lambda|\widehat{g}(0)(\varepsilon + R^2\ell_\delta^{-2}\mathbb{1}_{d,2} + \lambda_d^{\rm LHY}) - C\rho^2|\Lambda|\widehat{g}(0)\lambda_d^{\rm LHY}, \end{aligned} \tag{D.19}$$

where we reconstructed $\widehat{g\omega}(0)$ obtaining an error reabsorbed in the first term of the third line, and we used a Cauchy-Schwarz inequality on the second term in the second line. Thanks to the choice of ε made in (D.14) and the relations between the parameters, we have that ε is the dominant term in the first addend, and it can be reabsorbed in the $K_\ell^2 K_L^{-1}$ term in (D.12), while the second addend is dominated by error term in (D.12).

We bound by zero the positive terms in the quadratic elements in creation and annihilation operators

$$\begin{aligned}\langle(\rho_z-\rho)n_+\widehat{g}(0)+\mathcal{Q}_2^{\mathrm{ex}}(z)\rangle_{\Phi(z)} &\geq -\rho\widehat{g}(0)\langle n_+\rangle_{\Phi(z)}\\ &\geq -C\rho\widehat{g}(0)\big(\mathcal{M}\|\Phi(z)\|^2+\langle n_+^H\rangle_{\Phi(z)}\big), \end{aligned}\tag{D.20}$$

where we used the simple bound $n_+ \leq C(n_+^L + n_+^H)$ and (D.10). The first term, thanks to (F.6), contributes to the $K_\ell^2 K_L^{-1}$ terms in (D.12), and the last term to the relative n_+^H term in (D.12).

Collecting the inequalities (D.13), (D.19) and (D.20), we deduce the lower bound in (D.12).

By the simple algebraic equivalence

$$\frac{1}{2}\rho_z^2+\rho^2-\rho\rho_z=\frac{1}{2}(\rho-\rho_z)^2+\frac{1}{2}\rho^2,\tag{D.21}$$

and using that the coefficients of the $K_\ell^2 K_L^{-1}$ in (D.12) can be bounded by

$$C(\rho-\rho_z)^2\widehat{g}(0)|\Lambda|+C\rho^2\widehat{g}(0)|\Lambda|,\tag{D.22}$$

we get the bound

$$\begin{aligned}(\text{D.12}) \geq{}& \Big(\frac{1}{2}\rho^2|\Lambda|\widehat{g}(0)+\frac{1}{2}(\rho-\rho_z)^2|\Lambda|\widehat{g}(0)(1-CK_\ell^2K_L^{-1})\\ &-C\rho^2|\Lambda|\widehat{g}(0)K_\ell^2K_L^{-1}-C\rho^2|\Lambda|\widehat{g}(0)\lambda_d^{\mathrm{LHY}}\Big)\|\Phi(z)\|^2-C\rho\widehat{g}(0)\langle n_+^H\rangle_{\Phi(z)}\\ \geq{}& \Big(\frac{1}{2}\rho^2|\Lambda|\widehat{g}(0)+\frac{1}{4}(\rho-\rho_z)^2|\Lambda|\widehat{g}(0)-C\rho^2|\Lambda|\widehat{g}(0)K_\ell^2K_L^{-1}\\ &-C\rho^2|\Lambda|\widehat{g}(0)\lambda_d^{\mathrm{LHY}}\Big)\|\Phi(z)\|^2-C\rho\widehat{g}(0)\langle n_+^H\rangle_{\Phi(z)},\end{aligned}\tag{D.23}$$

and we can conclude using the assumption $|\rho-\rho_z| \geq \rho\varepsilon_+$, where ε_+ is chosen in order to dominate the $K_\ell^2 K_L^{-1}$terms and the error and to have that the second term in the previous expression positive and bigger than the Lee-Huang-Yang precision, to obtain the desired bound. □

Appendix E. A priori Bounds for the Number of Excited Bosons

In this section we bound the number of excitations for states of suitably low energy.

Theorem Appendix E.1. *Assume the relations between the parameters in Appendix F and that ρa^d is small enough. There exists a $C_B > 0$ such that, if $\Psi \in L^2_{sym}(\Lambda^N)$ is a normalized state satisfying*

$$\langle\mathcal{H}\rangle_\Psi \leq \frac{1}{2}\rho^2|\Lambda|\widehat{g}(0)(1+C_B\lambda_d^{\mathrm{LHY}}),\tag{E.1}$$

then there exists a $C > 0$ *such that*

$$\langle n_+\rangle_\Psi \le C \begin{cases} C_B N K_\ell^2 \widehat{g}(0), & d=2,\\ C_B N K_\ell^2 \sqrt{\rho a^3}, & d=3. \end{cases} \tag{E.2}$$

$$\langle n_+^H\rangle_\Psi \le C \begin{cases} C_B N\, K_L^{-2} K_\ell^2 \widehat{g}(0), & d=2,\\ C_B N\, K_L^{-2} K_\ell^2 \sqrt{\rho a^3}, & d=3. \end{cases} \tag{E.3}$$

$$\langle \mathcal{Q}_4^{\rm ren}\rangle_\Psi \le C_B \rho^2 |\Lambda| \widehat{g}(0) \lambda_d^{\rm LHY}. \tag{E.4}$$

In order to prove the Theorem Appendix E.1, we need to prove a lower bound on $\mathcal{H}$ localizing on boxes B with Gross-Pitaevskii length scale $\ell_{\rm GP} \ll \ell$, where

$$\ell_{\rm GP} := \rho^{-1/2} \widehat{g}(0)^{-1/2}. \tag{E.5}$$

We introduce the small box centered at $u \in \Lambda$ to be

$$B_u = u + \Big[-\frac{\ell_{\rm GP}}{2}, \frac{\ell_{\rm GP}}{2} \Big]^d. \tag{E.6}$$

The associated localization functions are

$$\chi_{B_u}(x) := \chi\left(\frac{x-u}{\ell_{\rm GP}} \right), \tag{E.7}$$

where $\chi \in C^\infty(\mathbb{R}^d)$, $0 \le \chi$, $\operatorname{supp}\chi \subseteq B_{\frac12}(0)$, $\|\chi\|_{L^2} = 1$. We emphasize that

$$\int_\Lambda \int_{B_u} |\chi_{B_u}|^2 {\rm d}x{\rm d}u = |\Lambda|. \tag{E.8}$$

We also introduce the projectors on the condensate in the small boxes P_{B_u} and their complements Q_{B_u},

$$P_{B_u} := \frac{1}{|B_u|} |\mathbb{1}_{B_u}\rangle\langle \mathbb{1}_{B_u}|, \qquad Q_{B_u} := \mathbb{1}_{B_u} - P_{B_u}. \tag{E.9}$$

In order to construct the small box Hamiltonian, we introduce the localized potentials

$$v^B(x) := \frac{v(x)}{\chi * \chi(x/\ell_{\rm GP})}, \qquad w_{B_u}(x,y) := \chi_{B_u}(x) v^B(x-y) \chi_{B_u}(y), \tag{E.10}$$

$$v_1^B(x) := \frac{g(x)}{\chi * \chi(x/\ell_{\rm GP})}, \qquad w_{1,B_u}(x,y) := \chi_{B_u}(x) v_1^B(x-y) \chi_{B_u}(y), \tag{E.11}$$

$$v_2^B(x) := \frac{g(x)(1+\omega(x))}{\chi * \chi(x/\ell_{\rm GP})}, \qquad w_{2,B_u}(x,y) := \chi_{B_u}(x) v_2^B(x-y) \chi_{B_u}(y), \tag{E.12}$$

where we see that $w_B, w_{1,B}, w_{2,B}$ are localized versions of $v, g, (1+\omega)g$, respectively.

For the kinetic energy, the localization to the small boxes is contained in the lemma below.

Lemma Appendix E.1. *There exists a constant $b > 0$ such that, for $s > 0$ small enough, the periodic Laplacian on Λ satisfies*

$$-\Delta \geq |B|^{-1} \int_{\Lambda} \mathcal{T}_u \, \mathrm{d}u + \frac{b}{\ell^2} Q_{\Lambda}, \tag{E.13}$$

where Q_Λ is the projector outside the condensate of the box Λ, and where the new kinetic energy has the form

$$\mathcal{T}_u := Q_{B_u} \chi_{B_u} \Big(-\Delta_{\mathbb{R}^d} - s^{-2} \ell_{\mathrm{GP}}^{-2} \Big)_+ \chi_{B_u} Q_{B_u} + b \ell_{\mathrm{GP}}^{-2} Q_{B_u}. \tag{E.14}$$

Proof. The proof can be found in [43, Lemma 3.3]. □

Since we do not know how the particles distribute in the boxes, we introduce a chemical potential ρ_μ. We will impose $\rho_\mu = \rho$ to be coherent with the original density. In this way we can define the grand canonical large box Hamiltonian, on the sector with n bosons, as

$$\mathcal{H}_\Lambda(\rho_\mu)_n := \sum_{j=1}^{n} \Big(-\Delta_j - \rho_\mu \int_{\mathbb{R}^d} g(x_j - y) \mathrm{d}y \Big) + \sum_{i<j}^{n} v(x_i - x_j). \tag{E.15}$$

The small-box Hamiltonian $\mathcal{H}_B$ which acts on $\mathscr{F}_s(L^2(B_u))$ is

$$\mathcal{H}_{B_u}(\rho_\mu)_n := \sum_{j=1}^{n} \Big(\mathcal{T}_{j,u} - \rho_\mu \int_{\mathbb{R}^d} w_{1,B_u}(x_j, y) \mathrm{d}y \Big) + \sum_{i<j}^{n} w_{B_u}(x_i, x_j). \tag{E.16}$$

Joining Lemma Appendix E.1 and a direct calculation for the potential, we obtain the relation between the last two Hamiltonians in the theorem below.

Theorem Appendix E.2.

$$\mathcal{H}_\Lambda(\rho_\mu)_n \geq \sum_{j=1}^{n} \frac{b}{\ell^2} Q_{\Lambda,j} + \frac{1}{|B|} \int_\Lambda \mathcal{H}_{B_u}(\rho_\mu)_n \mathrm{d}u. \tag{E.17}$$

A lower bound for $\mathcal{H}_{B_u}$ gives a lower bound for $\mathcal{H}_\Lambda(\rho_\mu)_n$ still conserving the contribution from the spectral gap. In the next proposition we give a lower bound for $\mathcal{H}_{B_u}$. The proof, that we omit, is identical to the one given in [43] for the $3D$ case (see also [10, Appendix B] and [18, Appendix D]).

Proposition Appendix E.1. *Assume the conditions in Appendix F are true, then there exists a constant $C_B > 0$ such that, for sufficiently small values of $\rho_\mu a^d$,*

$$\mathcal{H}_B(\rho_\mu)_n \geq -\frac{1}{2} \rho_\mu^2 |B| \widehat{g}(0) - C_B \rho_\mu^2 |B| \widehat{g}(0) \lambda_d^{\rho_\mu}, \tag{E.18}$$

where $\lambda_d^{\rho_\mu}$ has the same expression as λ_d^{LHY}, with ρ_μ in place of ρ.

Plugging the result of this last proposition into (E.17), and since all the $\mathcal{H}_{B_u}$ are unitarily equivalent, we get a lower bound for the large box Hamiltonian, contained in the next theorem.

Theorem Appendix E.3. *We have the following lower bound for the large box Hamiltonian*

$$\mathcal{H}_\Lambda(\rho_\mu)_n \geq \frac{b}{2\ell^2} n_+ - \rho_\mu^2 |\Lambda| \widehat{g}(0) \Big(\frac{1}{2} + C_B \lambda_d^{\rho_\mu} \Big). \tag{E.19}$$

To lower bound the large box Hamiltonian by the spectral gap plus the energy contribution up to the Lee-Huang-Yang level, allows us to finally prove the bound on the number of excitations for states of low energy.

Proof. [Proof of Theorem Appendix E.1] We only sketch the proof, details can be found in [18, Appendix D] and [10, Appendix B]. Choosing $\rho_\mu = \rho$ we have that the original large box Hamiltonian can be expressed, in relation to the grand canonical one, as

$$\mathcal{H}_N = \mathcal{H}_\Lambda(\rho)_N + \rho \widehat{g}(0) N. \tag{E.20}$$

Therefore, comparing the upper bound from the assumption (E.1) on Ψ and the lower bound from Theorem Appendix E.3, we get

$$\begin{aligned} \frac{b}{2\ell^2}\langle n_+ \rangle_\Psi + \frac{1}{2}\rho^2 |\Lambda| \widehat{g}(0) - C_B \rho^2 |\Lambda| \widehat{g}(0) \lambda_d^{\mathrm{LHY}} & \\ \leq \langle \mathcal{H} \rangle_\Psi \leq \frac{1}{2}\rho^2 |\Lambda| \widehat{g}(0) + C_B \rho^2 |\Lambda| \widehat{g}(0) \lambda_d^{\mathrm{LHY}}&, \end{aligned} \tag{E.21}$$

which yields, for n_+,

$$\frac{b}{2\ell^2}\langle n_+ \rangle_\Psi \leq 2 C_B \rho^2 |\Lambda| \widehat{g}(0) \lambda_d^{\mathrm{LHY}}, \tag{E.22}$$

giving the desired bound.

The bound of n_+^H follows from the one of n_+ and a lower bound on the Hamiltonian in the large box Λ, and we give a sketch of the proof below.

We write the Laplacian in second quantization and on the N boson space as

$$-\Delta = \sum_{k \in \Lambda^*} \tau_k a_k^\dagger a_k + b \frac{K_L^2}{\ell^2} n_+^H, \tag{E.23}$$

where, for a $b < \frac{1}{100}$,

$$\tau_k := |k|^2 - b \mathbb{1}_{[K_L \ell^{-1}, +\infty)}(k) \frac{K_L^2}{\ell^2}, \tag{E.24}$$

isolating, in this way, the spectral gap for high momenta. Thanks to this observation and Proposition 2.1, the Hamiltonian acting on the N Fock space sector can be bounded as

$$\begin{aligned} \mathcal{H}_n \geq \mathcal{K}_{\mathrm{quad}} + b \frac{K_L^2}{\ell^2} n_+^H + \frac{n_0(n_0-1)}{2|\Lambda|} \big(\widehat{g}(0) + \widehat{g\omega}(0) \big) \\ + \mathcal{Q}_3^{\mathrm{ren}} + \mathcal{Q}_4^{\mathrm{ren}} - C n \widehat{g}(0) \frac{n_+}{|\Lambda|}, \end{aligned}$$

where by $\mathcal{K}_{\text{quad}}$ we denoted the quadratic part of the Hamiltonian in $a_k^{\#}$:

$$\mathcal{K}_{\text{quad}} := \sum_{k\in\Lambda^*} \tau_k a_k^\dagger a_k + \frac{1}{2|\Lambda|}\sum_{k\in\Lambda^*} \widehat{g}_k (a_0^\dagger a_0^\dagger a_k a_{-k} + h.c.). \tag{E.25}$$

Here we do not need to reach the Lee-Huang-Yang precision, therefore we do not have to work with soft pairs and the bound on $\mathcal{Q}_3^{\text{ren}}$ and $\mathcal{Q}_4^{\text{ren}}$ is easier. It is obtained by an application of a Cauchy-Schwarz inequality on $\mathcal{Q}_3^{\text{ren}}$ and estimating the missing terms to reconstruct $\mathcal{Q}_4^{\text{ren}}$ in a similar way as in (2.12):

$$\mathcal{Q}_3^{\text{ren}} + \frac{1}{2}\mathcal{Q}_4^{\text{ren}} \geq -C\frac{n_0}{|\Lambda|} n_+ \widehat{g}(0). \tag{E.26}$$

We introduce a new pair of creation and annihilation operators

$$b_k := a_0^\dagger a_k, \qquad b_k^\dagger := a_0 a_k^\dagger, \tag{E.27}$$

and adding and subtracting

$$A_0 := \frac{\widehat{g}(0)}{2|\Lambda|}\sum_{k\in\Lambda^*}(b_k^\dagger b_k + b_{-k}^\dagger b_{-k}), \tag{E.28}$$

where $|A_0| \leq CN\widehat{g}(0)\frac{n_+}{|\Lambda|}$, we get

$$\mathcal{K}_{\text{quad}} + A_0 \geq \frac{1}{2|\Lambda|}\sum_{k\in\Lambda^*}\Big(\mathcal{A}_k (b_k^\dagger b_k + b_{-k}^\dagger b_{-k}) + \widehat{g}_k (b_k^\dagger b_{-k}^\dagger + b_{-k} b_k)\Big)$$

where $\mathcal{A}_k := \frac{|\Lambda|}{(N+1)}\tau_k + \widehat{g}(0)$. By the standard Bogoliubov theory of diagonalization and recalling the definition of G_d in (1.18), we bound the previous expression by the Bogoliubov integral

$$\mathcal{K}_{\text{quad}} + A_0 \geq I(d) - \frac{N(N+1)}{2|\Lambda|}\widehat{g\omega}(0), \tag{E.29}$$

with

$$I(d) := -\frac{N}{2(2\pi)^d}\int_{\mathbb{R}^d}\Big(\mathcal{A}_k - \sqrt{\mathcal{A}_k^2 - \widehat{g}_k^2} - \frac{N+1}{|\Lambda|}G_d(k)\Big)\mathrm{d}k. \tag{E.30}$$

We calculate the integral in a similar way as in Lemma Appendix C.1, splitting into two regions for momenta higher or lower than $K_L\ell^{-1}$, obtaining, since $K_\ell \ll K_L$, that there exists a $C > 0$, such that

$$I(3) \geq -C\frac{N(N+1)}{|\Lambda|}\widehat{g}(0)\sqrt{\rho\widehat{g}(0)^3}\frac{K_L}{K_\ell}, \qquad I(2) \geq -C\frac{N(N+1)}{|\Lambda|}\widehat{g}(0)^2, \tag{E.31}$$

Collecting the inequalities (E.31), the bound on A_0 and (E.26), using the bound we obtained for n_+ and considering the quadratic form of the $N-$particle state Ψ from the assumptions, we get the following lower bound for the Hamiltonian:

$$\langle\mathcal{H}\rangle_\Psi \geq b\frac{K_L^2}{\ell^2}\langle n_+^H\rangle_\Psi + \frac{1}{2}\langle\mathcal{Q}_4^{\text{ren}}\rangle_\Psi + \frac{1}{2}\rho N\widehat{g}(0)\times\begin{cases}\Big(1 - C\sqrt{\rho a^3}\frac{K_L}{K_\ell}\Big), & \text{for } d=3,\\ \Big(1 - C\widehat{g}(0)\Big), & \text{for } d=2,\end{cases} \tag{E.32}$$

which, together with the assumption (E.1) on Ψ, gives the bounds

$$\langle \mathcal{Q}_4^{\mathrm{ren}} \rangle_\Psi \leq C\rho N \widehat{g}(0)\sqrt{\rho a^3}, \tag{E.33}$$

$$b\frac{K_L^2}{2\ell^2}\langle n_+^H \rangle_\Psi \leq C\rho N \widehat{g}(0) \times \begin{cases} C\sqrt{\rho a^3}\dfrac{K_L}{K_\ell}, & \text{for } d=3, \\ C\widehat{g}(0), & \text{for } d=2, \end{cases} \tag{E.34}$$

from which the bounds on n_+^H and $\mathcal{Q}_4^{\mathrm{ren}}$ follow. □

Appendix F. Parameters

In this appendix we list the parameters needed in the proof and the relations they have to satisfy. Finally, in (F.15) below we give a concrete choice satisfying those conditions. Throughout all the paper, the following parameters are used

$$\varepsilon_K, \varepsilon_{\mathrm{gap}} \ll 1 \ll \mathcal{M}, K_\ell, K_L, K_H, \tag{F.1}$$

We use the notation $A \ll B$ to mean

$$A \ll B \;\Leftrightarrow\; \begin{cases} A \leq C(\rho a^3)^\zeta B, & \text{if } d=3, \\ A \leq C\delta^\zeta B, & \text{if } d=2. \end{cases} \tag{F.2}$$

for a constant $C > 0$ and a $\zeta > 0$.

Recall that K_L and K_H define the sets of low and high momenta respectively. They must satisfy

$$K_\ell \ll K_\ell^4 \ll K_L \ll K_H. \tag{F.3}$$

The chain of conditions is important in many inequalities throughout all the paper. $\mathcal{M}$ is the bound on n_+^L that we allow our states to satisfy. Our localization result on n_+^L, Theorem Appendix B.1, respectively requires in equation (B.20) and in equation (B.18)

$$\mathcal{M} \gg \ell\rho^{1/2}K_H^{1/2}\|v\|_1^{1/2}, \tag{F.4}$$

and

$$\mathcal{M} \gg \ell^2\rho^2|\Lambda|\widehat{g}(0)\lambda_d^{\mathrm{LHY}}. \tag{F.5}$$

The parameter $\mathcal{M}$ has to be smaller than the total number of particles according to the following condition

$$\frac{\mathcal{M}}{N} \ll \left(\frac{K_\ell}{K_L}\right)^4 \ll 1, \tag{F.6}$$

where the last inequality follows from (F.3) using (F.1). The errors when localizing the 3Q terms in Proposition 5.1 require the following condition

$$\frac{\mathcal{M}}{N}K_H^d \ll 1. \tag{F.7}$$

When dealing with the 3Q terms, we need a small fraction $\varepsilon_K \ll 1$ of $\mathcal{K}_H^{\text{diag}}$ to control some errors. This coefficient needs to be large enough,

$$\varepsilon_K^2 \gg \ell^{8-d}\rho^3\widehat{g}(0)^4 K_H^{-8}K_L^d\mathcal{M}. \tag{F.8}$$

Other errors from 3Q are controlled by n_+^H using

$$K_\ell^2 \gg \ell^{4-d}\rho\widehat{g}(0)^2 K_H^{-2}K_L^d\mathcal{M}, \tag{F.9}$$

or by a fraction $\varepsilon_{\text{gap}} \ll 1$ of the spectral gap, which needs to satisfy

$$\rho^3\widehat{g}(0)^4\ell^{8-d}K_H^{d-6} \ll \varepsilon_{\text{gap}}, \tag{F.10}$$

$$\varepsilon_K\ell^2\rho\widehat{g}(0) + \mathcal{E}_d\ell^2\rho + K_\ell^2K_H^{-1} \ll \varepsilon_{\text{gap}}, \tag{F.11}$$

$$\frac{R}{\ell}K_L\ell^2\rho\widehat{g}(0) \ll \varepsilon_{\text{gap}}, \tag{F.12}$$

where $\mathcal{E}_d$ is the error from Lemma Appendix A.1.

We explain here how to get explicit choices of parameters, starting from any box Λ satisfying

$$K_\ell \ll \begin{cases} \delta^{-\frac{1}{26}}, & \text{if } d=2, \\ (\rho a^3)^{-\frac{1}{28}}, & \text{if } d=3. \end{cases} \tag{F.13}$$

Given such a K_ℓ, there exists an $\varepsilon \in (0,1)$ small enough such that

$$\begin{cases} K_\ell^{-26-19\varepsilon} \gg \delta, & \text{if } d=2, \\ K_\ell^{-28-16\varepsilon} \gg \rho a^3, & \text{if } d=3. \end{cases} \tag{F.14}$$

Then, with the choice

$$\begin{aligned} &K_L = K_\ell^{4+2\varepsilon}, \qquad K_H = K_\ell^{4+3\varepsilon}, \qquad \tfrac{\mathcal{M}}{N} = K_\ell^{-12-10\varepsilon}, \\ &\varepsilon_{\text{gap}} = K_\ell^{-2},\ \varepsilon_K = K_\ell^{-18+2d+(d-16)\varepsilon}, \end{aligned} \tag{F.15}$$

all the conditions (F.3), (F.4), (F.6), (F.7), (F.9), (F.8), (F.10), (F.11), (F.12) are satisfied, for potentials satisfying $\|v\|_1 \le C$ and $\rho\widehat{g}(0)R^2 \le K_\ell^{-9}$.

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Chapter 5

Paper: The free energy of dilute Bose gases at low temperatures interacting via strong potentials

This chapter contains the paper [Fou+24a] by Fournais, Girardot, Morin, Olivieri, Triay and the author. It provides a lower bound for eq. (1.3). The paper is included in its pre-print version, which can be found at https://doi.org/10.48550/arXiv.2408.14222. It can be located within the thesis by the colour ■) at the top of the page.

The free energy of dilute Bose gases at low temperatures interacting via strong potentials

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August 27, 2024

Abstract

We consider a dilute Bose gas in the thermodynamic limit and prove a lower bound on the free energy for low temperatures which is in agreement with the conjecture of Lee-Huang-Yang on the excitation spectrum of the system. Combining techniques of [15] and [17], we give a simpler and shorter proof resolving the case of strong interactions, including the hard-core potential.

1 Introduction

The mathematical analysis of the interacting Bose gas has been subject of intense scrutiny and important breakthroughs in the last decades. In particular, much effort has been dedicated to rigorously justifying the seminal work of Lee, Huang and Yang [20] where they compute the spectral properties of the dilute Bose gas. A corner stone of this program has been the proof of the leading term of the ground state energy density in [24], wherein they complement Dyson's upper bound [10] from 40 years before with a matching lower bound. The successes of the approaches initiated around that time are well described in the lecture notes [11].

According to [20] a dilute gas of Bosons behave similarly as a system of independent (quantum) harmonic oscillators following Bogoliubov's dispersion relation. Interestingly, the authors of [20] gave their names to the correction to the leading order of the ground state energy density - the famous Lee-Huang-Yang term - which corresponds to the zero point energy of such a system, and was rigorously derived in [29, upper bound] and [14, 15, lower bound].

To see the effect of the rest of the spectrum, it is necessary to consider the system at positive temperature and, for example, compute its free energy per unit volume. This was recently investigated in the work [17] (see also [18] for the matching upper bound) where a two-term asymptotic expansion for the free energy per unit volume was proved for temperatures that are low but for which the thermal correction is of the order of the Lee-Huang-Yang (LHY) term.

Since the foundational work [20], a special role has been played by the hard core (or hard sphere) potential. This potential is often chosen for its simplicity as it depends on one parameter only (its scattering length, which equals its radius) and because the system in the dilute regime is not expected to depend on the microscopic details of the potential, at least to some order. However, its rigorous mathematical analysis is very challenging since it strongly penalises approximation schemes by having infinite integral. It is therefore an important testing problem for techniques in the area and for the idea of *universality*, i.e. that the scattering length should be the main parameter governing the dilute regime.

The purpose of the current paper is two-fold. First of all we extend the result of [17] to the case of interaction potentials with large (possibly infinite) integral, thereby in particular for the first time rigorously obtaining the lower bound on the free energy of the Bose gas in the case of hard core interactions. The second purpose is expository. By combining the Neumann localization approach of [17] (see also [6]) with the techniques of [14, 15], we provide a fairly simple and short proof of lower bounds in the thermodynamic limit in a generality allowing for very singular potentials. Our proof is clearly shorter than [15] which is restricted to $T = 0$ (although we here have slightly more restrictive assumptions on the potential) and is also arguably shorter and simpler than [17].

We will now define the model considered and state the result of the paper. We consider a system of N interacting, non-relativistic bosons in a large box of sidelength L in 3 dimensions governed by the

Hamiltonian

$$H_{N,L} = \sum_{i=1}^{N} -\Delta_i + \sum_{i<j} V(x_i - x_j). \tag{1.1}$$

The operator $H_{N,L}$ acts on the bosonic Hilbert space $\otimes_s^N L^2(\Lambda_L) = L^2_{\rm sym}(\Lambda^N)$, with $\Lambda_L := [0,L]^3$. For concreteness, we will take Neumann boundary conditions in order to realize $H_{N,L}$ as a self-adjoint operator. We define the ground state energy $E(L,N)$ and the free energy $F(L,N)$ at temperature $T > 0$ by

$$E(L,N) = \inf \operatorname{Spec} H_{N,L}, \qquad F(L,N) = -T \log \operatorname{Tr}\big(e^{-\frac{H_{N,L}}{T}}\big). \tag{1.2}$$

We recall that the free energy is the minimum of the variational problem

$$F(L,N) = \inf_{\Gamma} \Big\{ \operatorname{Tr}\big(H_N \Gamma\big) - T\, S(\Gamma) \Big\}, \tag{1.3}$$

where the infimum is taken over trace-class operators Γ on $L^2_{\rm sym}(\Lambda^N)$ such that $0 \le \Gamma \le 1$ and $\operatorname{Tr}\Gamma = 1$ and $S(\Gamma) = -\operatorname{Tr}(\Gamma \ln \Gamma)$ is the entropy of Γ. The particle density for finite systems is $\rho = N/L^3$, and we define the ground state energy density $e(\rho)$ and the free energy density $f(\rho,T)$ by

$$e(\rho) = \lim_{\substack{N\to\infty \\ N/L^3 \to \rho}} \frac{E(L,N)}{L^3}, \qquad f(\rho,T) = \lim_{\substack{N\to\infty \\ N/L^3 \to \rho}} \frac{F(L,N)}{L^3}. \tag{1.4}$$

It is standard [27] that the limits exist and are independent of both the boundary conditions and the sequence of (N,L) as long as $N/L^3 \to \rho$.

The main result of the paper is the following.

Theorem 1.1. *(Free energy in the thermodynamic limit) For all $C_0 > 0$ there exists $C > 0$ such that, for $\eta > 0$ small enough the following holds. Let V be a non-negative, radially symmetric and non-increasing potential with scattering length a and compact support of radius $R \le C_0 a$. Then for $0 \le \rho a^3 \le C^{-1}$, $\nu \in (0, \frac{\eta}{3})$ and $0 \le T \le \rho a (\rho a^3)^{-\nu}$,*

$$f(\rho,T) \ge 4\pi a \rho^2 \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3}\right) + \frac{T^{5/2}}{(2\pi)^3} \int_{\mathbb{R}^3} \log\left(1 - e^{-\sqrt{p^4 + \frac{16\pi\rho a}{T} p^2}}\right) \mathrm{d}p - C(\rho a)^{5/2} (\rho a^3)^{\eta/4}. \tag{1.5}$$

Remark 1.2.

- *The result on the free energy in Theorem 1.1 confirms the picture of [20]. According to this, the energy of a large dilute system should approximately be given by*

$$E(L,N) \approx 4\pi N \rho a \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3}\right). \tag{1.6}$$

  Furthermore, the excitation spectrum for low-lying eigenvalues should be given by the Bogoliubov dispersion relation

$$\sum n_p \sqrt{p^4 + 16\pi \rho a p^2},$$

  where $p \in \frac{2\pi}{L}\mathbb{Z}^3 \setminus \{0\}$ denotes the momentum of an excitation, and $n_p \in \mathbb{N}$ denotes the number of bosonic excitations with that momentum. Combining these two pieces of information leads to the expression given in Theorem 1.1. Note that the expansion (1.5) *is not expected to hold when the temperature reaches the transition temperature for Bose-Einstein condensation, which should be of order $\rho a (\rho a^3)^{-1/3}$. A leading order analysis in this temperature regime has been carried out in [28, 30].*

- *As mentioned above, the leading term in* (1.6) *was proved by [10, 24] in the thermodynamic limit. Further progress was [12] (upper bound to second order for soft potentials) and [16, 7] (two term asymptotics for very soft potentials). The upper bound in the thermodynamic limit consistent with the two-term asymptotics* (1.6) *was given by [29] (see also [4, 3]). The lower bound in the thermodynamic limit was proved in [14, 15].*

  There is extensive work on confined systems, in particular in the Gross-Pitaevskii scaling regime. We will not review this here but only refer to [2] for an overview and further references (see also [5, 8] for recent developments).

The energy of the dilute Bose gas is strongly dependent on the dimension of the ambient space, with the discussion above concerning only the physical $d = 3$. Also the 1 and 2-dimensional situations are interesting and physically realizable. The 1-dimensional gas with δ interactions was treated by [22] and more recently [1] treated the case of general interactions. In 2 dimensions, the leading order energy was proved by [25], and the correction term analogous to the Lee-Huang-Yang term was proved recently in [13]. The free energy in 2-dimensions was analyzed to leading order in [26, 9].

The main part of our analysis relies on estimates on the free energy on boxes Λ_ℓ with

$$\ell = K_\ell \frac{1}{\sqrt{\rho a}}, \qquad K_\ell = (\rho a^3)^{-\eta},$$

where $\eta > 0$ is the parameter in Theorem 1.1. It is reasonably straightforward to prove Theorem 1.1 based on a result for the localized free energy given as Theorem 1.3 below and the subadditivity of the free energy. The details of the proof of Theorem 1.1 based on Theorem 1.3 are given in Appendix B.

Before stating the result on the localised energy, let us recall that we defined our operator $H_{N,L}$ to have Neumann boundary conditions. At the scale $L = \ell$ the effect of boundary conditions is important, and the boundary conditions are also relevant for the proof of Theorem 1.1 based on Theorem 1.3.

Theorem 1.3. *(Free energy on Λ_ℓ) For all $C_0 > 0$, there exists $C > 0$ such that, such that for $0 < \eta < \frac{1}{1026}$ the following holds. Let V be a positive, radially symmetric and non-increasing potential with scattering length a, and compact support of radius $R \leq C_0 a$. Then for all ρ such that $0 < \rho a^3 \leq C^{-1}$, $0 \leq T \leq \rho a (\rho a^3)^{-\nu}$ with $\nu < \eta/3$ and $n \leq 20\rho\ell^3$ we have*

$$F(\ell, n) \geq F_{\text{Bog}}(\ell, n) - C\ell^3 (\rho a)^{5/2} (\rho a^3)^{\eta/4}, \tag{1.7}$$

where

$$F_{\text{Bog}}(\ell, n) = 4\pi \rho_{n,\ell}^2 a \ell^3 \Big(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho_{n,\ell} a^3}\Big) + T \sum_{p \in \frac{\pi}{\ell} \mathbf{N}_0^3} \log\Big(1 - e^{-\frac{1}{T}\sqrt{p^4 + 16\pi a \rho_{n,\ell} p^2}}\Big), \tag{1.8}$$

with $\rho_{n,\ell} = n\ell^{-3}$.

In the following Section 2 we streamline the proof of Theorem 1.3 by shifting the proofs of the main technical lemmas to the next sections. Since most of the paper will be focused on the box of size ℓ we will often simplify notation and write $\Lambda = \Lambda_\ell$.

2 Free energy on Λ_ℓ

This section is devoted to the proof of Theorem 1.3 (given at the end of this section) which is the main ingredient in the proof of our main result Theorem 1.1.

2.1 Replacement by an integrable potential

We begin by replacing the initial interaction potential V by a suitable integrable potential $v \leq V$, which obviously lowers the free energy. We need however to control the difference between the scattering lengths to ensure that the error coming from the substitution in the leading term does not affect the correction term.

Let us first recall some facts about the scattering problem. We denote by φ_V the radial solution in $\mathbb{R}^3$ to the zero-energy scattering equation

$$-\Delta \varphi_V + \frac{1}{2} V \varphi_V = 0, \tag{2.1}$$

normalized such that $\lim_{|x|\to\infty} \varphi_V(x) = 1$. Of particular interest throughout the paper are also the functions

$$\omega_V := 1 - \varphi_V, \qquad g_V := V\varphi_V = V(1 - \omega_V). \tag{2.2}$$

By these functions, it is possible to rewrite equation (2.1) as

$$-\Delta \omega_V = \frac{1}{2} g_V, \tag{2.3}$$

and from this, by the divergence theorem, recover the scattering length by

$$\widehat{g}_V(0) = \int g_V(x) \mathrm{d}x = 8\pi a. \tag{2.4}$$

We refer to Section 3 for more details.

Proposition 2.1 (Replacement by an integrable potential)**.** *There exists $C > 0$ such that the following holds. Let V be a positive, radially symmetric and decreasing potential with scattering length a, and compact support of radius $0 \leq R \leq C_0 a$. Then we can construct another positive and radially symmetric potential $v \leq V$ with scattering length $a(v)$ satisfying*

$$0 \leq a - a(v) \leq a\sqrt{\rho a^3} K_\ell^{-1}, \qquad \int v(x)\mathrm{d}x \leq C(\rho a)^{-\frac{1}{2}} K_\ell, \qquad v \leq \ell^2 a^{-4}, \tag{2.5}$$

and

$$g_v(y) \leq C v(x) \quad \text{for} \quad |x| \leq |y|, \tag{2.6}$$

where $g_v = v\varphi_v$ and φ_v is the scattering solution for v.

The proof is given in Section 3. We denote here by $F(\ell,n)[V]$ and $F_{\mathrm{Bog}}(\ell,n)[a]$ the free energy (1.3) associated to the potential V in the box Λ and the expression (1.8) with scattering length a, respectively. Assuming that the desired estimate (1.7) holds for the potential v and using that the free energy $F(\ell,n)[V]$ is a non-decreasing function of V, we obtain

$$\begin{aligned} F(\ell,n)[V] \geq F(\ell,n)[v] &\geq F_{\mathrm{Bog}}(\ell,n)[a(v)] - C\ell^3(\rho a(v))^{5/2}(\rho a(v)^3)^{\eta/4} \\ &\geq F_{\mathrm{Bog}}(\ell,n)[a] - C\ell^3(\rho a)^{5/2}(\rho a^3)^{\eta/4} - C\left(\ell^3\rho^2 + T^{3/2}\rho\ell^3\right)|a(v) - a| \\ &\geq F_{\mathrm{Bog}}(\ell,n)[a] - C\ell^3(\rho a)^{5/2}(\rho a^3)^{\eta/4}, \end{aligned}$$

where we used (2.5) to obtain the second line and [17, Lemma 9.1] for the replacement of $a(v)$ by a in the sum appearing in $F_{\mathrm{Bog}}(\ell,n)$.

Therefore, it is enough to prove Theorem 1.3 for potentials v satisfying (2.5) and (2.6). In the sequel we will abuse notation and write $F(\ell,n)$ for $F(\ell,n)[v]$, and also write g,ω and φ for g_v,ω_v and φ_v.

2.2 Splitting of the potential energy

Let us introduce the projections on and outside the condensate,

$$P = \frac{1}{\ell^3}|1\rangle\langle 1|, \qquad Q = \mathbf{1} - P, \tag{2.7}$$

acting on $L^2(\Lambda)$. If $\Phi \in L^2(\Lambda^N)$ and $1 \leq i \leq N$, we denote by $P_i\Phi$ and $Q_i\Phi$ the action of P and Q on the variable x_i. We also denote the number of particles in the condensate and the number of excited particles respectively by

$$n_0 = \sum_{i=1}^N P_i, \qquad n_+ = \sum_{i=1}^N Q_i = N - n_0. \tag{2.8}$$

The strategy carried out in [21] and used in many other works, is to expand $\mathbf{1}_{(L^2)^{\otimes_s N}} = \prod_{i=1}^N (P_i + Q_i)$ which leads to identifying $(L^2)^{\otimes_s N}$ with a truncated Fock space over the excitation space $\mathrm{Ran}\, Q$ where the vacuum vector is the pure condensate $|\frac{1}{\ell^{3/2}}\rangle^{\otimes_s N}$. Having factored out the condensate, it remains to extract the correlation energy. This strategy does not work for hard core potentials because the vacuum (the pure condensate) does not even belong to the domain of the Hamiltonian: one needs to factor out parts of the correlations first. In this spirit, the strategy initiated in [14] and then [15, 13] is to *renormalize* the potential energy from the beginning by using the identity

$$\mathbf{1} = (Q_iQ_j + \omega(x_i - x_j)(\mathbf{1} - Q_iQ_j)) + \varphi(x_i - x_j)(\mathbf{1} - Q_iQ_j). \tag{2.9}$$

Lemma 2.2. *For all positive and radially symmetric potentials v, we have the identity*

$$\sum_{i<j} v(x_i - x_j) = Q_0^{\mathrm{ren}} + Q_1^{\mathrm{ren}} + Q_2^{\mathrm{ren}} + Q_3^{\mathrm{ren}} + Q_4^{\mathrm{ren}}$$

where

$$Q_0^{\text{ren}} := \frac{1}{2}\sum_{i\neq j} P_iP_j(g+g\omega)(x_i-x_j)P_jP_i. \tag{2.10}$$

$$Q_1^{\text{ren}} := \sum_{i\neq j} \big(Q_iP_j(g+g\omega)(x_i-x_j)P_jP_i + \text{h.c.}\big), \tag{2.11}$$

$$Q_2^{\text{ren}} := \sum_{i\neq j} P_iQ_j(g+g\omega)(x_i-x_j)P_jQ_i + \sum_{i\neq j} P_iQ_j(g+g\omega)(x_i-x_j)Q_jP_i + \frac{1}{2}\sum_{i\neq j} P_iP_jg(x_i-x_j)Q_jQ_i + \text{h.c.}, \tag{2.12}$$

$$Q_3^{\text{ren}} := \sum_{i\neq j} P_iQ_jg(x_i-x_j)Q_jQ_i + \text{h.c.}, \tag{2.13}$$

$$Q_4^{\text{ren}} := \frac{1}{2}\sum_{i\neq j} \Pi_{ij}^* v(x_i-x_j)\Pi_{ij}\,, \tag{2.14}$$

with

$$\Pi_{ij} := Q_jQ_i + \omega(x_i-x_j)\,(P_jP_i + P_jQ_i + Q_jP_i)\,.$$

Proof. This is proved by inserting (2.9) in each term of the potential $\sum_{i<j} v(x_i-x_j)$ and expanding using that $1-Q_iQ_j = P_iP_j + P_iQ_j + Q_iP_j$. We recall that $g = v(1-\omega)$. □

2.3 Spectral gaps

Many of our estimates will rely on analysis in momentum space. In this article, the momenta are elements of $\frac{\pi}{\ell}\mathbb{N}_0^3$, and we define low and high momenta by

$$\mathcal{P}_L := \Big\{p\in\frac{\pi}{\ell}\mathbb{N}_0^3,\ 0<|p|\le K_H\ell^{-1}\Big\}, \qquad \mathcal{P}_H := \Big\{p\in\frac{\pi}{\ell}\mathbb{N}_0^3,\ |p|>K_H\ell^{-1}\Big\}, \tag{2.15}$$

where K_H is a large ρ-dependent parameter. In the theorems below we always write our assumptions on K_H. The final choice of K_H, and a few other parameters, will be specified in (2.42) below. We define the corresponding localized projectors by

$$Q^L := \mathbf{1}_{\mathcal{P}_L}(\sqrt{-\Delta}), \qquad Q^H := \mathbf{1}_{\mathcal{P}_H}(\sqrt{-\Delta}), \tag{2.16}$$

where Δ is the Neumann Laplacian on Λ. The number of high and low excitations are respectively given by

$$n_+^H := \sum_{j=1}^N Q_j^H, \qquad n_+^L := \sum_{j=1}^N Q_j^L, \tag{2.17}$$

both acting on $L^2_{\text{sym}}(\Lambda^N)$. These definitions imply for instance that $P + Q_L + Q_H = 1$ and that $n_+^L + n_+^H = n_+$.

Many of our error terms will be bounded by n_+, n_+^H or n_+^L. To control these errors, we will extract some positive quantities from the kinetic energy, refered to as *spectral gaps*. They are gathered into one operator

$$G := \frac{\pi n_+}{4\ell^2} + \frac{K_H n_+^H}{2\ell^2} + \frac{\pi n_+^L n_+}{4\mathcal{M}\ell^2} + \frac{K_H n_+^L n_+^H}{2\mathcal{M}\ell^2}, \tag{2.18}$$

where $\mathcal{M}$ is some large ρ-dependent parameter, which will need to satisfy specific conditions in later theorems. These spectral gaps are extracted from the kinetic energy, from which will only remain

$$\sum_{j=1}^N \mathcal{T}_j = \sum_{j=1}^N -\Delta_j - \frac{\pi n_+}{2\ell^2} - \frac{K_H n_+^H}{\ell^2},$$

with

$$\mathcal{T} = -\Delta - \frac{\pi}{2\ell^2}Q - \frac{K_H}{\ell^2}Q^H \ge 0. \tag{2.19}$$

More precisely, we obtain the following result.

Theorem 2.3. *There exist $C > 0$ such that the following holds. Let v be as in Proposition 2.1, $\rho\ell^3(\rho a^3)^\alpha \leq N \leq 20\rho\ell^3$, $T \leq \rho a(\rho a^3)^{-\nu}$, with $\alpha + 5\nu/2 < 6/17$ and $\mathcal{M} \geq \rho\ell^3(\rho a^3)^\gamma$, for some $\gamma > 0$, then for $\rho a^3 \leq C^{-1}$ we have*

$$F(\ell, N) \geq \inf_\Gamma \left\{ \mathrm{Tr}\big(H_N^{\mathrm{mod}}\Gamma + T\,\Gamma \ln \Gamma + G\Gamma\big) \right\} - C\ell^3(\rho a)^{5/2}(\rho a^3)^{1/18-2\gamma-\nu} K_H^3,$$

where the infimum is taken over trace-one operators $\Gamma \geq 0$. Here the modified Hamiltonian is

$$H_N^{\mathrm{mod}} = \sum_{i=1}^N \mathcal{T}_i + \sum_{i<j} v(x_i - x_j), \tag{2.20}$$

and G was defined in (2.18).

The proof of Theorem 2.3 is given in Section 4 and relies on rough a priori bounds on the numbers of excitations in Λ_ℓ.

2.4 Localization of the 3Q term

Here we show that the main contribution of Q_3^{ren} defined in (2.13) comes from the creation and annihilation of the so-called "soft pairs" where one particle is in the condensate and the other has low but non-zero momentum.

Lemma 2.4. *Let v be as in Proposition 2.1. For all $\varepsilon > 0$, there exists a $C > 0$ such that the following holds. Assume that $\rho a^3 \leq C^{-1}$, then for all $K_H \geq CK_\ell^4$,*

$$Q_3^{\mathrm{ren}} \geq Q_{3,L} - \varepsilon\left(Q_4^{\mathrm{ren}} + \frac{1}{\ell^2}n_+ + \frac{K_H}{\ell^2}n_+^H\right),$$

with

$$Q_{3,L} := \sum_{i \neq j} P_i Q_j^L g(x_i - x_j) Q_i Q_j + \mathrm{h.c.} \tag{2.21}$$

The last two error terms will be absorbed by a fraction of the spectral gaps G by taking ε sufficiently small but fixed. The proof of Lemma 2.4 is given in Section 7.

2.5 Symmetrization

To deal with Neumann boundary conditions, an important idea of [17] is to symmetrize the potential using a mirroring technique to make it commute with the Neumann momentum. The mirror transformations p_z, for $z \in \mathbb{Z}^3$, are defined by

$$(p_z(x))_i = (-1)^{z_i}\left(x_i - \frac{\ell}{2}\right) + \frac{\ell}{2} + \ell z_i, \qquad i = 1, 2, 3. \tag{2.22}$$

Note that in particular $p_z(\Lambda) = \{x + \ell z \ : \ x \in \Lambda\}$. For $p \in \frac{\pi}{\ell}\mathbb{N}_0^3$, we denote by

$$u_p(x) = \frac{1}{\sqrt{|\Lambda|}} \prod_{i=1}^3 c_{p_i} \cos(p_i x_i), \qquad \text{where} \quad c_{p_i} = \begin{cases} 1 & \text{if} \quad p_i = 0, \\ \sqrt{2} & \text{if} \quad p_i \neq 0, \end{cases} \tag{2.23}$$

the normalized eigenbasis of the Neumann Laplacian on Λ. For any $f \in L^1(\Lambda)$, the symmetrization of f is defined by

$$f^{\mathrm{s}}(x, y) := \sum_{z \in \mathbb{Z}^3} f(p_z(x) - y) \tag{2.24}$$

for a.e. $x, y \in \Lambda$. If f is radial, as shown in [17, Lemma 3.2], the operator with kernel $f^{\mathrm{s}}(x, y)$ commutes with the Neumann basis.

Lemma 2.5. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be radial and integrable with* $\mathrm{supp}(f) \subset B(0, R)$ *for some $R \leq \ell/2$. Then for $p, q \in \frac{\pi}{\ell}\mathbb{N}^3$ we have*

$$\int_{\Lambda^2} f^{\mathrm{s}}(x, y) u_p(x) u_q(y) \mathrm{d}x\mathrm{d}y = \delta_{p,q} \int_{\mathbb{R}^3} f(x) \prod_{i=1}^3 \cos(p_i x_i)\mathrm{d}x = \delta_{p,q}\widehat{f}(p). \tag{2.25}$$

Here and through all the paper $\widehat{f}$ is the Fourier transform of f defined by

$$\widehat{f}(p) = \int_{\mathbb{R}^3} f(x)e^{-ipx}\mathrm{d}x.$$

Note that since $\operatorname{supp} g \subset \operatorname{supp} v \subset B(0,R)$, the sum defining g^s is finite, and g^s agrees with g except when x or y is at distance R from the boundary of Λ. We then define the symmetrized operators

$$Q_0^{\mathrm{sym}} := \frac{1}{2}\sum_{i\neq j} P_iP_j(g+g\omega)^{\mathrm{s}}(x_i,x_j)P_iP_j, \tag{2.26}$$

$$Q_1^{\mathrm{sym}} := \sum_{i\neq j} Q_iP_j(g+g\omega)^{\mathrm{s}}(x_i-x_j)P_jP_i + \mathrm{h.c.}, \tag{2.27}$$

$$Q_2^{\mathrm{sym}} := \sum_{i\neq j} P_iQ_j(g+g\omega)^{\mathrm{s}}(x_i-x_j)P_jQ_i + \sum_{i\neq j} P_iQ_j(g+g\omega)^{\mathrm{s}}(x_i-x_j)Q_jP_i \tag{2.28}$$

$$+ \frac{1}{2}\sum_{i\neq j} P_iP_jg^s(x_i-x_j)Q_jQ_i + \mathrm{h.c.},$$

$$Q_{3,L}^{\mathrm{sym}} := \sum_{i\neq j} P_iQ_j^Lg^{\mathrm{s}}(x_i,x_j)Q_iQ_j + \mathrm{h.c.} \tag{2.29}$$

Theorem 2.6. *For all $\varepsilon > 0$, there exists a $C > 0$ such that the following holds. Let v be as in Proposition 2.1 and assume that $\rho a^3 \leq C^{-1}$, $K_H \geq CK_\ell^4$, and $K_\ell K_H^3 \leq (\rho a^3)^{-\frac{1}{2}}$, we have*

$$H_N^{\mathrm{mod}} \geq H_N^{\mathrm{sym}} - CN\rho a\frac{R}{\ell} - \varepsilon G,$$

where

$$H_N^{\mathrm{sym}} := \sum_{i=1}^N \mathcal{T}_i + Q_0^{\mathrm{sym}} + Q_2^{\mathrm{sym}} + Q_{3,L}^{\mathrm{sym}}. \tag{2.30}$$

The proof of Theorem 2.6 is given in Section 5.

Remark 2.7. *The assumption that V is non-decreasing is only used here. It ensures that v satisfies the condition* (2.6) *which is needed to estimate some error terms arising from the symmetrization of the term Q_2^{ren}.*

2.6 C-number substitution

From now on, we work with the symmetrized Hamiltonian H_N^{sym} in second quantization. Recall the basis $\{u_p\}_p$ defined in (2.23). We introduce the bosonic Fock space

$$\mathscr{F}(L^2(\Lambda)) = \bigoplus_{n=0}^{\infty} L^2(\Lambda)^{\otimes_s n}, \tag{2.31}$$

and the associated creation and annihilation operators that satisfy the canonical commutation relations

$$a_p = a(u_p), \qquad a_p^* = a^*(u_p), \qquad [a_p, a_q^*] = \delta_{p,q} \tag{2.32}$$

for all $p,q \in \Lambda^* = \frac{\pi}{\ell}\mathbb{N}_0^3$. It is convenient to extend this definition to $p = (p_1,p_2,p_3) \in \frac{\pi}{\ell}\mathbb{Z}^3$, by denoting $u_p = u_{(|p_1|,|p_2|,|p_3|)}$ and $a_p = a(u_p) = a_{(|p_1|,|p_2|,|p_3|)}$. We also define $\Lambda_+^* = \frac{\pi}{\ell}\mathbb{N}_0^3 \setminus \{0\}$ and the set of generalized low momenta as

$$\mathcal{P}_L^{\mathbb{Z}} = \{p \in \frac{\pi}{\ell}\mathbb{Z}^3, \quad 0 < |p| \leq K_H\ell^{-1}\}.$$

With this notation we obtain

Lemma 2.8 (Second quantization in the Neumann basis)**.** *We have the following identity on the sector of $\mathscr{F}(L^2(\Lambda))$ with N bosons:*

$$\begin{aligned} H_N^{\mathrm{sym}} &= \frac{\widehat{g}(0)(N(N-1) - n_+(n_+ - 1))}{2|\Lambda|} + \frac{\widehat{g\omega}(0)}{2|\Lambda|} a_0^* a_0^* a_0 a_0 \\ &+ \sum_{p \in \Lambda_+^*} \Big(\tau(p) a_p^* a_p + \frac{\widehat{g}(p)}{|\Lambda|} a_0^* a_p^* a_p a_0 + \frac{\widehat{g}(p)}{2|\Lambda|} \big(a_0^* a_0^* a_p a_p + \mathrm{h.c.} \big) \Big) \\ &+ \frac{1}{|\Lambda|} \sum_{\substack{k \in \Lambda_+^*,\, p \in \mathcal{P}_L^{\mathbb{Z}} \\ p \neq k}} c(p,k) \widehat{g}(k) \big(a_0^* a_p^* a_{p-k} a_k + \mathrm{h.c.} \big) + \frac{1}{|\Lambda|} \sum_{p \in \Lambda_+^*} \Big((\widehat{g\omega}(0) + \widehat{g\omega}(p)) a_0^* a_p^* a_p a_0 \Big). \end{aligned} \tag{2.33}$$

where

$$\tau(p) = |p|^2 - \frac{\pi}{2\ell^2} \mathbf{1}_{\{p \neq 0\}} - \frac{K_H}{\ell^2} \mathbf{1}_{\{p \in \mathcal{P}_H\}} \tag{2.34}$$

is the symbol of the kinetic energy $\mathcal{T}$ and, recalling (2.23)*, $c(p,k)$ are the normalizing factors given by*

$$c(p,k) := \prod_{i=1}^{3} \frac{c_{k_i - p_i}}{c_{p_i} c_{k_i}}. \tag{2.35}$$

Note that if none of the indices i in the product is 0*, then the above constant is $\frac{1}{\sqrt{8}}$.*

The proof of Lemma 2.8 is given in Section 6. Following Bogoliubov's approximation and using [23], we perform a c-number substitution, effectively replacing a_0 by a complex number z and a_0^* by $\overline{z}$. The transformed Hamiltonian acts on the excitation Fock space,

$$\mathscr{F}^\perp = \mathscr{F}\big(\{1\}^\perp\big).$$

We obtain the following lower bound on the free energy of H_N^{sym}.

Theorem 2.9. *For any $m > 3$ there exist $C > 0$ and $\varepsilon > 0$ such that the following holds. Let v be as in Proposition 2.1 and assume $\rho a^3 \leq C^{-1}$. Then for all $0 \leq 10\mu \leq \ell^{-2}$, $C \leq N \leq 20\rho\ell^3$, $0 \leq T \leq C\rho a(\rho a^3)^{-\nu}$, $C \leq \mathcal{M} \leq C^{-1}\ell/a$, and $K_H \geq CK_\ell^4$, $K_\ell K_H^3 \leq (\rho a^3)^{-\frac{1}{2}}$, we have*

$$-T \log \mathrm{Tr}\Big(e^{-\frac{1}{T}(H_N^{\mathrm{sym}} + \frac{1}{2}G)}\Big) \geq 4\pi a \frac{N^2}{|\Lambda|} - T \log \int_{\mathbb{C}} \mathrm{Tr}_{\mathscr{F}^\perp} \Big(e^{-\frac{1}{T}(\mathcal{H}_\mu(z) + \varepsilon \mathcal{G}(z))}\Big) e^{-\frac{\mu}{T}N} \mathrm{d}z - C\rho a,$$

where

$$\begin{aligned} \mathcal{H}_\mu(z) &= \sum_{p \in \Lambda_+^*} \Big(\tau(p) a_p^* a_p + \frac{|z|^2}{|\Lambda|} \widehat{g}(p) a_p^* a_p + \frac{1}{2|\Lambda|} \widehat{g}(p) (\overline{z}^2 a_p a_p + z^2 a_p^* a_p^*) \Big) \\ &- \mu |z|^2 + 8\pi a (\rho a^3)^{\frac{1}{4}} \Big(\frac{|z|^4}{|\Lambda|} - \rho N \Big) \\ &+ \frac{1}{2|\Lambda|} \widehat{g\omega}(0) |z|^4 + \frac{|z|^2}{|\Lambda|} \sum_{p \in \Lambda_+^*} (\widehat{g\omega}(0) + \widehat{g\omega}(p)) a_p^* a_p + Q_{3,L}^{\mathrm{sym}}(z), \end{aligned} \tag{2.36}$$

$$Q_{3,L}^{\mathrm{sym}}(z) = \frac{1}{|\Lambda|} \sum_{\substack{k \in \Lambda_+^*, p \in \mathcal{P}_L^{\mathbb{Z}} \\ p \neq k}} c(p,k) \widehat{g}(k) \big(\overline{z} a_p^* a_{p-k} a_k + \mathrm{h.c.} \big), \tag{2.37}$$

and

$$\mathcal{G}(z) = \frac{\pi^2 n_+}{2\ell^2} + \frac{K_H n_+^H}{\ell^2} + \frac{n_+^L n_+}{\mathcal{M}\ell^2} + \frac{K_H n_+^L n_+^H}{\mathcal{M}\ell^2} + \frac{\rho a}{N^m} (|z|^{2m} + |z|^{2m-2} n_+ + |z|^{2m-4} n_+^2). \tag{2.38}$$

The proof of Theorem 2.9, including details of the c-number substitution, is given in Section 6.

Remark 2.10.

1. *In the physically relevant region $|z|^2 \simeq N$ we control the number of particles using the chemical potential μ. To estimate the contribution of the physically non-relevant region $|z|^2 \gg N$, we will use the last terms in* (2.38)*.*

2. *The term proportional to* $(\rho a^3)^{\frac14}$ *in* (2.36) *is artificially added to the energy to provide convexity in the variable* $|z|^2$ *(see* (2.46)*).*

For $|z|^2 \geq K_\ell^{1/4}N = (\rho a^3)^{-\eta/4}N$ we have a simple lower bound on the Hamiltonian.

Lemma 2.11. *Under the assumptions of Theorem 2.9, if* $|z|^2 \geq K_\ell^{1/4}N$ *and* $m > 2\eta^{-1} + 14$ *such that* $K_H \leq K_\ell^{\frac{m+1}{12}}$, *then there exists a constant* $c > 0$ *such that*

$$\mathcal{H}_\mu(z) + \varepsilon\mathcal{G}(z) \geq \frac12 \sum_{p\in\Lambda_+^*} \tau(p) a_p^* a_p + c\rho a K_\ell^{\frac{m-1}{4}} \frac{|z|^2}{N},$$

where we recall that $\tau(p)$ *is given by (2.34).*

The proof is given at the end of Section 6. It remains to deal with $|z|^2 \leq K_\ell^{1/4}N$.

2.7 Bogoliubov diagonalization

We now diagonalize the main quadratic part of the Hamiltonian $\mathcal{H}_\mu(z)$ appearing in the first line of (2.36). The second line will be estimated later. For $|z|^2 \leq K_\ell^{1/4}N$, using the CCR (2.32), we obtain the identity

$$\begin{aligned}&\sum_{p\in\Lambda_+^*} (\tau(p) + \rho_z \widehat{g}(p)) a_p^* a_p + \frac{1}{2|\Lambda|}\widehat{g}(p)(\overline{z}^2 a_p a_p + z^2 a_p^* a_p^*)\\ &= \sum_{p\in\Lambda_+^*} D_p(z) b_p^* b_p + \frac12 \sum_{p\in\Lambda_+^*}\Big(\sqrt{\tau(p)^2 + 2\rho_z\widehat{g}(p)\tau(p)} - \tau(p) - \rho_z\widehat{g}(p)\Big),\end{aligned} \tag{2.39}$$

where we denoted $\rho_z = \frac{|z|^2}{|\Lambda|}$ and

$$D_p(z) := \sqrt{\tau(p)^2 + 2\tau(p)\rho_z\widehat{g}(p)}, \qquad b_p := \frac{a_p + \alpha_p a_p^*}{\sqrt{1-\alpha_p^2}}, \qquad \alpha_p := \frac{\tau(p) + \rho_z\widehat{g}(p) - D_p(z)}{\rho_z \widehat{g}(p)}. \tag{2.40}$$

One easily checks that the argument of the square root is non-negative using that $|\widehat{g}(p) - \widehat{g}(0)| \leq R^2\widehat{g}(0)|p|^2 \leq Ca^3p^2$.

2.8 Contribution of the 3Q terms.

It remains to bound the terms in the second line of (2.36), this is done in the following lemma.

Theorem 2.12. *Under the assumptions of Theorem 2.9 and if* $K_\ell^{5/4}K_H^2 \leq C^{-1}(\rho a^3)^{-\frac12}$ *and* $K_H \geq K_\ell^4$, *then for all* $|z|^2 \leq K_\ell^{1/4}N$ *and all* $\mathcal{M} \leq C^{-1}\rho\ell^3 K_H^{-3}K_\ell^{-17/4}$ *we have*

$$(1-K_H^{-1})\sum_{k\in\mathcal{P}_H} D_k(z) b_k^* b_k + \frac{|z|^2}{|\Lambda|}\sum_{p\in\Lambda_+^*}\Big((\widehat{g\omega}(0) + \widehat{g\omega}(p))a_p^*a_p\Big) + Q_{3,L}^{\rm sym}(z) \geq -\varepsilon\mathcal{G}(z) - CN\rho a\sqrt{\rho a^3}K_\ell^{-1}.$$

Theorem 2.12 is proven in Section 7. As a corollary, and combining with the Bogoliubov diagonalization, we obtain the following lower bound on $\mathcal{H}_\mu(z)$.

Corollary 2.13. *Under the assumptions of Theorem 2.9 and Theorem 2.12, for all* $|z|^2 \leq K_\ell^{1/4}N$ *we have*

$$\begin{aligned}\mathcal{H}_\mu(z) + \varepsilon\mathcal{G}(z) \geq{}& 4\pi|z|^2\rho_z a\cdot\frac{128}{15\sqrt{\pi}}\sqrt{\rho_z a^3} + \sum_{p\in\Lambda_+^*}\tilde{D}_p(z)b_p^*b_p - \mu|z|^2\\ &+ 8\pi a\Big(\frac{|z|^4}{|\Lambda|} - \rho N\Big)(\rho a^3)^{\frac14} - C\ell^3(\rho a)^{5/2}K_\ell^{-1/4}\end{aligned}$$

where

$$\tilde{D}_p(z) = \begin{cases} D_p(z) & \text{if } \; p \notin \mathcal{P}_H,\\ K_H^{-1}D_p(z) & \text{if } \; p\in\mathcal{P}_H.\end{cases} \tag{2.41}$$

Proof. We start from (2.36) and diagonalize the main quadratic part using (2.39) and then bound the last two terms using Theorem 2.12. This results in the bound

$$\mathcal{H}_\mu(z) + \varepsilon\mathcal{G}(z) \geq \frac{1}{2}\sum_{p\in\Lambda_+^*}\Big(\sqrt{\tau(p)^2 + 2\rho_z\widehat{g}(p)\tau(p)} - \tau(p) - \rho_z\widehat{g}(p)\Big) + \frac{1}{2|\Lambda|}\widehat{g\omega}(0)|z|^4$$
$$+ \sum_{p\in\Lambda_+^*}\tilde{D}_p(z)b_p^*b_p - \mu|z|^2 + \widehat{g}(0)(\rho a^3)^{\frac{1}{4}}\Big(\frac{|z|^4}{|\Lambda|} - \rho N\Big) - CN\rho a\sqrt{\rho a^3}K_\ell^{-1}.$$

We then approximate the sum by an integral according to Lemma A.1, and the largest error is of order $\ell^3(\rho a)^{5/2}K_\ell^{-1/4}$. □

2.9 Proof of Theorem 1.3

We make the following choice for the parameters, recalling that $K_\ell = (\rho a^3)^{-\eta}$,

$$K_H = K_\ell^5, \qquad \gamma = 20\eta, \qquad \alpha = \frac{1}{4} + \frac{\eta}{2}, \qquad \mathcal{M} = \rho\ell^3 K_\ell^{-21}, \qquad m = 10\eta^{-1}. \tag{2.42}$$

In particular we have $\alpha + \frac{5\nu}{2} < \frac{6}{17}$ for all $\nu < \frac{\eta}{3}$ and $\eta < \frac{1}{1026}$.

Case $N \leq (\rho a^3)^\alpha \rho\ell^3$. Discarding the interaction for a lower bound, we find similarly as in (4.7),

$$F(\ell,N) \geq T\sum_{p\in\Lambda_+^*}\log(1 - e^{-\frac{p^2}{T}}) \geq T\sum_{p\in\Lambda_+^*}\log\Big(1 - e^{\frac{-1}{T}\sqrt{p^4 + 16\pi aN\ell^{-3}p^2}}\Big) - C\frac{aN}{T\ell^3}T^{5/2}\ell^3$$
$$\geq F_{\mathrm{Bog}}(\ell,N) - C\ell^3(\rho a)^{5/2}\Big((\rho a^3)^{2\alpha - 1/2} + (\rho a^3)^{\alpha - 3\nu/2}\Big), \tag{2.43}$$

where we used [17, Eq. (8.16)] to estimate the difference between the two sums. Note that we need $\alpha > 1/4$ for the error term to be subleading compared to LHY order. We choose $\alpha = 1/4 + \eta/2$ where we recall that $K_\ell = (\rho a^3)^{-\eta}$, so that $(\rho a^3)^{2\alpha-1/2} + (\rho a^3)^{\alpha - 3\nu/2} = K_\ell^{-1} + (\rho a^3)^{1/4+\eta/2-3\nu/2} \leq CK_\ell^{-1}$.

Case $N > (\rho a^3)^\alpha\rho\ell^3$. We combine Theorems 2.3, 2.6 and 2.9, to obtain the lower bound

$$F(\ell,N) \geq 4\pi a\frac{N^2}{|\Lambda|} - T\log\Big[\int_{\mathbb{C}}\mathrm{Tr}_{\mathscr{F}^\perp}\Big(e^{-\frac{1}{T}(\mathcal{H}_\mu(z)+\varepsilon\mathcal{G}(z))}\Big)e^{-\frac{\mu}{T}N}\mathrm{d}z\Big] - \mathcal{E}. \tag{2.44}$$

with

$$\mathcal{E} \leq C\ell^3(\rho a)^{5/2}(\rho a^3)^{1/18-2\gamma-\nu}K_H^3 + CK_\ell^2\rho a,$$

which holds for $K_H \geq CK_\ell^4$, $K_\ell K_H^3 \leq (\rho a^3)^{-\frac{1}{2}}$, $K_H \leq \sqrt{\ell/a} = (\rho a^3)^{-1/4-\eta/2}$ and since $\mathcal{M} \leq C^{-1}\rho\ell^3K_H^{-3}K_\ell^{-17/4}$. These conditions are satisfied with our choice of parameters, and the error is $\mathcal{E} \leq C\ell^3(\rho a)^{5/2}K_\ell^{-1}$.

Let us decompose the integral in (2.44) as $\int_{\mathbb{C}} = \int_{|z|^2\leq K_\ell^{1/4}N} + \int_{|z|^2 > K_\ell^{1/4}N} = X + Y$. We will use that if $-T\log X \geq Z$ and $-T\log Y \geq Z$, then

$$-T\log(X+Y) \geq -T\log(2e^{-Z/T}) = Z - T\log 2. \tag{2.45}$$

For the relevant region of z, where $|z|^2 \leq K_\ell^{1/4}N$, we can use Corollary 2.13, which gives

$$\mathcal{H}_\mu(z) + \varepsilon\mathcal{G}(z) + \mu N \geq 4\pi|z|^2\rho_z a\cdot\frac{128}{15\sqrt{\pi}}\sqrt{\rho_z a^3} + \sum_{p\in\Lambda_+^*}\tilde{D}_p(z)b_p^*b_p - \mu|z|^2 + 8\pi a(\rho a^3)^{\frac{1}{4}}\rho_z|z|^2$$
$$+ N(\mu - 8\pi\rho a(\rho a^3)^{\frac{1}{4}}) - \mathcal{E}'$$

with

$$\mathcal{E}' \leq C\ell^3(\rho a)^{5/2}K_\ell^{-1/4}.$$

Now using Lemma A.2, we obtain

$$-T\log\mathrm{Tr}_{\mathscr{F}^\perp}\Big(e^{-\frac{1}{T}\sum_{p\in\Lambda_+^*}\tilde{D}_p(z)b_p^*b_p}\Big) = T\sum_{p\in\Lambda_+^*}\log\big(1 - e^{-\frac{1}{T}\tilde{D}_p(z)}\big) \geq T\sum_{p\in\Lambda_+^*}\log(1 - e^{-\frac{1}{T}\omega_p(z)}) - C\ell^3(\rho a)^3$$

and denoting $\omega_p(z) = \sqrt{p^4 + 16\pi a|z|^2\ell^{-3}p^2}$, we have

$$-T\log \mathrm{Tr}_{\mathscr{F}^\perp}\left(e^{-\frac{1}{T}(\mathcal{H}_\mu(z)+\varepsilon\mathcal{G}(z)+\mu N)}\right) \geq F(|z|^2) + N(\mu - 8\pi\rho a(\rho a^3)^{\frac14}) - \mathcal{E}'$$

with

$$F(|z|^2) := 8\pi a(\rho a^3)^{\frac14}\rho_z|z|^2 + 4\pi|z|^2\rho_z a\cdot\frac{128}{15\sqrt{\pi}}\sqrt{\rho_z a^3} + T\sum_{p\in\Lambda_+^*}\log(1-e^{-\frac1T\omega_p(z)}) - \mu|z|^2, \tag{2.46}$$

By Lemma B.1, we deduce that F is convex for ρa^3 and η small enough. Choosing μ so that $F'(N)=0$, F achieves its minimum at $|z|^2 = N$. We obtain

$$\begin{aligned}&-T\log\int_{|z|^2<K_\ell^{1/4}N}\mathrm{Tr}_{\mathscr{F}^\perp}\left(e^{-\frac1T(\mathcal{H}_\mu(z)+\varepsilon\mathcal{G}(z))}\right)e^{-\frac{\mu}{T}N}\mathrm{d}z\\ &\geq 4\pi a\frac{N^2}{|\Lambda|}\cdot\frac{128}{15\sqrt{\pi}}\sqrt{\frac{Na^3}{|\Lambda|}} + T\sum_{p\in\Lambda_+^*}\log(1-e^{-\frac1T\omega_p(\sqrt{N})}) - \mathcal{E}' - T\log(CK_\ell^{1/4}N).\end{aligned} \tag{2.47}$$

On the other hand when $|z|^2\geq K_\ell^{1/4}N$, and since our choice of m satisfies both $m > 2\eta^{-1}+14$ and $K_H \leq K_\ell^{\frac{m+1}{12}}$, we use Lemma 2.11 to obtain

$$\begin{aligned}&-T\log\int_{|z|^2\geq K_\ell^{1/4}N}\mathrm{Tr}_{\mathscr{F}^\perp}\left(e^{-\frac1T(\mathcal{H}_\mu(z)+\varepsilon\mathcal{G}(z))}\right)e^{-\frac{\mu}{T}N}\mathrm{d}z\\ &\geq T\sum_{p\in\Lambda_+^*}\log(1-e^{-\frac{\tau(p)}{2T}}) + \mu N - T\log\int_{|z|^2\geq K_\ell^{1/4}N}\exp\Big(-\frac{c\rho a|z|^2}{TNK_\ell}\Big)\mathrm{d}z\\ &\geq T\sum_{p\in\Lambda_+^*}\log(1-e^{-\frac{p^2}{4T}}) + \mu N - T\log\bigg(\frac{CTN}{\rho aK_\ell^{\frac{m-1}{4}}}\exp\Big(-\frac{c\rho aK_\ell^{\frac m4}}{T}\Big)\bigg)\\ &\geq -CT^{5/2}\ell^3 - T\log\left(\frac{CTN}{\rho aK_\ell^{\frac{m-1}{4}}}\right) + c\rho aK_\ell^{\frac m4},\end{aligned}$$

for ρa^3 small enough. We used that $\tau(p)\geq p^2/2$ and that $\mu\geq 0$. The above is clearly bigger than the right-hand side of (2.47) for ρa^3 small enough due to the constraint on N and the assumptions on m. Thus for ρa^3 small enough, using (2.45), we obtain

$$\begin{aligned}-T\log\Big[\int_{\mathbb{C}}\mathrm{Tr}_{\mathscr{F}^\perp}\left(e^{-\frac1T(\mathcal{H}_\mu(z)+\varepsilon\mathcal{G}(z))}\right)e^{-\frac{\mu}{T}N}\mathrm{d}z\Big] &\geq 4\pi a\frac{N^2}{|\Lambda|}\cdot\frac{128}{15\sqrt{\pi}}\sqrt{\frac{Na^3}{|\Lambda|}} + T\sum_{p\in\Lambda_+^*}\log(1-e^{-\frac1T\omega_p(\sqrt{N})})\\ &\quad - \mathcal{E}' - T\log(CK_\ell^{1/4}N).\end{aligned}$$

We combine the above with (2.44) to obtain for $N > (\rho a^3)^\alpha\rho\ell^3$

$$F(\ell,N)\geq 4\pi a\frac{N^2}{|\Lambda|}\Big(1+\frac{128}{15\sqrt{\pi}}\sqrt{\frac{Na^3}{|\Lambda|}}\Big) + T\sum_{p\in\Lambda^*}\log(1-e^{-\frac1T\omega_p(\sqrt{N})}) - CT\log(N) - \mathcal{E} - \mathcal{E}'. \tag{2.48}$$

We have

$$\begin{aligned}CT\log(N)+\mathcal{E}+\mathcal{E}' &\leq C\ell^3\Big((\rho a)^{5/2}(\rho a^3)^{1/18-2\gamma-\nu}K_H^3 + \frac{T}{\ell^3}\log N + (\rho a)^{5/2}K_\ell^{-1/4}\Big)\\ &\leq C\ell^3(\rho a)^{5/2}\left(K_\ell^{-1/4} + (\rho a^3)^{-\nu}K_\ell^{-3}|\log(\rho a^3)|\right)\\ &\leq C\ell^3(\rho a)^{5/2}K_\ell^{-1/4}.\end{aligned}$$

Combining the cases $N\leq(\rho a^3)^\alpha\rho\ell^3$ and $N>(\rho a^3)^\alpha\rho\ell^3$ finishes the proof of Theorem 1.3.

3 Approximation by integrable potentials

This section is devoted to the proof of Proposition 2.1. We begin by recalling the definition of the scattering length and related quantities. For more details see [11].

Definition 3.1. *Let $V : \mathbb{R}^3 \to \mathbb{R}^+ \cup \{\infty\}$ be measurable and radial with support in $B(0,R)$. The scattering length $a = a(V)$ is defined as*

$$4\pi a = \inf\Big\{ \int |\nabla\varphi|^2 + \frac{1}{2}V|\varphi|^2 \mathrm{d}x \,\Big|\, \varphi \in \dot{H}^1(\mathbb{R}^3), \quad \lim_{|x|\to\infty} \varphi(x) = 1 \Big\}, \tag{3.1}$$

An important special case is the hard core potential of radius $R > 0$

$$V_{\mathrm{hc}}(x) := \begin{cases} +\infty, & |x| \leq R, \\ 0, & |x| > R. \end{cases} \tag{3.2}$$

For this special potential it is not difficult to see that $a(V_{\mathrm{hc}}) = R$. For general potentials, inserting the test function $\max\{0, 1 - \frac{R}{|x|}\}$, we find $a \leq R$. It is also easy to verify that a is an increasing function of V. The minimizer φ_V solves the corresponding Euler-Lagrange equation

$$-\Delta\varphi_V + \frac{1}{2}V\varphi_V = 0, \tag{3.3}$$

in a weak sense. It is easy to check with Newton's theorem that

$$\varphi_V(x) = 1 - \frac{a}{|x|}, \quad \text{for } |x| \geq R, \tag{3.4}$$

and furthermore φ_V is non-decreasing, non-negative and radial. We will also use the following standard monotonicity result from [11, Lemma C.2]: if $V_1 \geq V_2 \geq 0$, then $\varphi_{V_1}(x) \leq \varphi_{V_2}(x)$ for all x. We will omit the v from the notation of the scattering length and write $\varphi := \varphi_V$, if the potential is clear from the context. We then recall the notation

$$\omega = 1 - \varphi, \qquad g = V\varphi = V(1-\omega). \tag{3.5}$$

Clearly,

$$-\Delta\omega = \frac{1}{2}g, \qquad \text{and} \qquad \widehat{g}(0) = \int g \,\mathrm{d}x = 8\pi a. \tag{3.6}$$

Having introduced the necessary theory and notation we may provide the proof of Proposition 2.1. The proof will rely on the following two lemmas.

Lemma 3.2. *For all $v_1 \geq v_2 \geq 0$ and $v' \geq 0$ we have*

$$a(v_1) - a(v_2) \geq a(v_1 + v') - a(v_2 + v').$$

Proof. For all $t \in [0,1]$, we introduce φ_1^t and φ_2^t as the scattering solutions for $v_1 + tv'$ and $v_2 + tv'$ respectively. By definition we have for $j = 1, 2$,

$$4\pi a(v_j + tv') = \int \Big(|\nabla\varphi_j^t|^2 + \frac{1}{2}(v_j + tv')|\varphi_j^t|^2\Big)\mathrm{d}x.$$

In particular, rearranging the terms we find

$$\begin{aligned} 4\pi a(v_1 + tv') - 4\pi a(v_2 + tv') = &\int \Big(\nabla\varphi_1^t \cdot \nabla(\varphi_1^t - \varphi_2^t) + \frac{1}{2}(v_1 + tv')\varphi_1^t(\varphi_1^t - \varphi_2^t)\Big)\mathrm{d}x \\ &- \int \Big(\nabla\varphi_2^t \cdot \nabla(\varphi_2^t - \varphi_1^t) + \frac{1}{2}(v_2 + tv')\varphi_2^t(\varphi_2^t - \varphi_1^t)\Big)\mathrm{d}x \\ &+ \frac{1}{2}\int (v_1 - v_2)\varphi_1^t\varphi_2^t \,\mathrm{d}x. \end{aligned}$$

When we integrate by parts, we find that the two first lines vanish by (3.3). For instance,

$$\int \Big(\nabla\varphi_1^t \cdot \nabla(\varphi_1^t - \varphi_2^t) + \frac{1}{2}(v_1 + tv')\varphi_1^t(\varphi_1^t - \varphi_2^t)\Big)\mathrm{d}x = \int (-\Delta\varphi_1^t + \frac{1}{2}(v_1 + tv')\varphi_1^t)(\varphi_1^t - \varphi_2^t)\mathrm{d}x = 0.$$

Therefore,

$$a(v_1+tv')-a(v_2+tv')=\frac{1}{8\pi}\int(v_1-v_2)\varphi_1^t\varphi_2^t\mathrm{d}x,$$

for all $t\in[0,1]$. Comparing with $t=0$ we deduce

$$a(v_1+tv')-a(v_2+tv')=a(v_1)-a(v_2)+\frac{1}{8\pi}\int(v_1-v_2)(\varphi_1^t\varphi_2^t-\varphi_1^0\varphi_2^0)\mathrm{d}x.$$

Since $v_j+tv'\geq v_j$, we have $0\leq\varphi_j^t\leq\varphi_j^0$ pointwise, and thus the last integral is negative. The result follows with $t=1$. □

Lemma 3.3. *Given $K>0$ and a non-increasing potential v, the potential* $\min(v,K)$ *satisfies*

$$a(V)\geq a(\min(V,K))\geq a(V)-\frac{2\sqrt{2}}{\sqrt{K}}$$

Proof. Using that V is decreasing, we get $\{V>K\}=B(0,R_K)$ for some $R_K\geq 0$. Note that the hard core potential with radius R_K has scattering length R_K. Then comparing to the hard core potential, we have

$$a(V)-a(\min(V,K))\leq a(V\mathbf{1}_{B(0,R_K)})-a(K\mathbf{1}_{B(0,R_K)})\leq R_K-a(K\mathbf{1}_{B(0,R_K)}),\tag{3.7}$$

where in the first inequality we used Lemma 3.2 with $V'=V\mathbf{1}_{B(0,R_K)^c}$, and in the second inequality that $V\mathbf{1}_{B(0,R_K)}$ is smaller than the hardcore potential V_{hc} of radius R_K. Now one can compute $a_K:=a(K\mathbf{1}_{B(0,R_K)})$ by solving

$$-\Delta\varphi+\frac{1}{2}K\mathbf{1}_{B(0,R_K)}\varphi=0,$$

to find that there is a $c>0$ such that

$$\varphi(x)=\begin{cases}c\frac{\sinh(\sqrt{\frac{K}{2}}|x|)}{|x|}, & |x|\leq R_K,\\ 1-\frac{a_K}{|x|}, & |x|\geq R_K.\end{cases}$$

Knowing that φ is continuous and differentiable yields we find that

$$R_K-a_K=\frac{1-e^{-2\gamma}}{\gamma-1+e^{-2\gamma}(\gamma+1)}a_K\qquad\text{where}\quad\gamma=\sqrt{\frac{K}{2}}R_K.\tag{3.8}$$

Lastly we observe that for $\gamma\geq 2$ the right-hand side of (3.8) is bounded by $\frac{2}{\gamma}a_K$, and if $\gamma\leq 2$ we have $R_K\leq\frac{2\sqrt{2}}{\sqrt{K}}$. Thus combining (3.7) with (3.8), using $\frac{a_K}{R_K}\leq 1$, we get

$$a(V)-a(\min(V,K))\leq R_K-a_K\leq\frac{2\sqrt{2}}{\sqrt{K}}.$$

□

Before giving the proof of Proposition 2.1, we explain the construction of v in the case where $V=V_{\mathrm{hc}}$ is the hard core potential of radius a. In [15] it was explained that to get (2.5) the (almost optimal) approximation of V_{hc} is by a 'thin shell' potential supported on the annulus $A:=\{a-a^2\ell^{-1}\leq|x|\leq a\}$ and of height ℓ^2a^{-4}. However, this potential clearly doesn't satisfy (2.6), since it vanishes inside the shell. The remedy is to fill the inside of the shell, without changing too much the L^1 norm. Therefore, the final choice of v is

$$v=\ell a^{-3}\mathbf{1}_{B(0,a-a^2\ell^{-1})}+\ell^2a^{-4}\mathbf{1}_A,\qquad\text{when } V=V_{hc}\text{ is the hard core.}$$

Let us now give the details starting from an arbitrary V. For convenience we include the following lemma.

Lemma 3.4. *There exists a universal constant $C_0>0$ such that the following is true. Suppose that $V:\mathbb{R}^3\to[0,\infty]$ is radial and of class L^1 with compact support of radius R and that $S\geq 0$. If $S\geq\frac{\int V}{8\pi a(V)}$ define the potential $V_S:=V$. If $S<\frac{\int V}{8\pi a(V)}$ define*

$$V_S:=V\mathbf{1}_{[R_S,\infty)},$$

with R_S chosen so that $\int V_S=8\pi Sa(V)$. Then V_S satisfies

$$a(V)\geq a(V_S)\geq a(V)(1-C_0S^{-1}).$$

This is a reformulation of [15, Lemma 3.3] using the fact that the proof gives the explicit construction of v_S. Using this Lemma we can prove Proposition 2.1.

Proof of Proposition 2.1. By Lemma 3.3, we may assume that $V \leq K$, with $K = \ell^2 a^{-4}$. This choice guarantees that the change to the scattering length is of order $a^2\ell^{-1}$. By Lemma 3.4, we can find $0 < R_S < R$ such that the potential

$$V_S(x) = \begin{cases} V(x) & |x| \geq R_S \\ 0 & |x| < R_S \end{cases}$$

has integral $\int V_S \leq 8\pi S a(V)$ and satisfies

$$a(V) \geq a(V_S) \geq a(V)(1 - C_0 S^{-1}). \tag{3.9}$$

Note that this error is small enough with the choice $S = \ell/a(V)$. However note that V_S does not satisfy (2.6), so it requires further modifications. First, we extend slightly V_S and define

$$w_S = V_S(R_S)\mathbf{1}_{[R_S-\varepsilon, R_S]} + V_S, \tag{3.10}$$

where we used the convention $\mathbf{1}_I(x) = \mathbf{1}_{\{|x|\in I\}}$ for a subset $I \subset \mathbb{R}$, and where $\varepsilon > 0$ will be chosen later. Let $g_S = w_S \varphi_{w_S}$, and let x_0 be the maximal point of g_S,

$$g_S(x_0) = \sup g_S.$$

Note that $x_0 \geq R_S$ by construction. Then we define our potential v by

$$v := \min\big(g_S(x_0), M\big)\mathbf{1}_{[0,R_S-\varepsilon]} + w_S, \tag{3.11}$$

where the parameter M will be chosen to bound the L^1 norm of v. Indeed, choosing $\varepsilon = a(V)^2\ell^{-1}$ and $M = \ell R_S^{-3}$,

$$\int v \leq C R_S^3 M + C a(V)^2 \ell^{-1} R_S^2 V_S(R_S) + \int V_S \leq C\ell,$$

where we used that $V_S(R_S) \leq K = \ell^2 a(V)^{-4}$, $R_S \leq R \leq C a(V)$ and $\int V_S \leq 8\pi\ell$. We can compare the scattering lengths of v and v using that $V \geq v \geq V_S$ and therefore, we get from (3.9) that

$$a(V) \geq a(v) \geq a(V)(1 - C a(V)/\ell).$$

Let us now show that the potential v satisfies (2.6). We consider $|x| < |y|$.

- If $|x| \geq R_S - \varepsilon$ then v is non-increasing on this region, and therefore since $\varphi_v \leq 1$ we have

$$v(x) \geq v(y) \geq v(y)\varphi_v(y) = g_v(y).$$

- When $|x| \leq R_S - \varepsilon$. Notice that $v\varphi_v$ is increasing on $[0, R_S]$. Therefore,

$$g_v(y) \leq \sup_{|x|\geq R_S} v\varphi_v \leq \sup_{|x|\geq R_S} w_S \varphi_{w_S} = g_S(x_0), \tag{3.12}$$

where the last inequality follows since $\varphi_{w_S} \geq \varphi_v$. If $g_S(x_0) < M$ then $g_S(x_0) = v(x)$ and we are done. In the other case, when $M \leq g_S(x_0)$ we introduce an auxiliary potential,

$$v_0 = v(x_0)\mathbf{1}_{[R_S-\varepsilon, |x_0|]}. \tag{3.13}$$

We observe that $v_0 \leq w_S$ and this implies that $\varphi_{w_S} \leq \varphi_{v_0}$. Therefore

$$g_S(x_0) \leq V(x_0)\varphi_{v_0}(x_0) = V(x_0)\Big(1 - \frac{a(v_0)}{|x_0|}\Big). \tag{3.14}$$

We estimate the difference between $a(v_0)$ and $|x_0|$ identified with the scattering length of the hard-core of radius $|x_0|$. We use Lemma 3.3 with $K = v(x_0)$ to cut the hard-core potential, and then compare the resulting potential to v_0 using Lemma 3.4 (with $8\pi S|x_0| = \int v_0$) to get

$$|x_0| - a(v_0) \leq \frac{C}{\sqrt{v(x_0)}} + C\frac{|x_0|^2}{\int v_0}.$$

Inserting this bound in (3.14) we get

$$g_S(x_0) \le C\frac{\sqrt{v(x_0)}}{|x_0|} + Ca(V)\frac{V(x_0)}{\int v_0} \le C\frac{\ell}{a(V)^2 R_S} + C\frac{a(V)}{\min(\varepsilon, R_S)R_S^2} \le CM,$$

where we used $V(x_0) \le \ell^2 a(V)^{-4}$, $\varepsilon = a(V)^2/\ell$, and $R_S \le |x_0| \le R \le Ca(V)$. Therefore,

$$g_S(x_0) \le CM = Cv(x)$$

which combined with (3.12) gives the desired bound.

□

4 Localization of large matrices: Proof of Theorem 2.3

One of the main ingredients to prove Theorem 2.3 is a rough condensation estimate for low energy states. This is by now a well known result and its proof can be found in [11], and the technique was also crucial in [17]. At positive temperature, we need to extend this property to mixed states.

Lemma 4.1 (Condensation of low energy states)**.** *There exists a $C > 0$ such that the following holds. Assume that $\rho\ell^3(\rho a^3)^\alpha \le N \le 20\rho\ell^3$, $T \le \rho a(\rho a^3)^{-\nu}$, with $\alpha + 5\nu/2 < 6/17$ and $(\rho a^3) \le C^{-1}$. Let Γ be a trace-class operator on $L^2(\Lambda^N)$ such that $\Gamma \ge 0$, $\mathrm{Tr}\,\Gamma = 1$, and*

$$\mathrm{Tr}(H_N\Gamma) \le 4\pi N^2 \ell^{-3} a(1 + (\rho a^3)^{\frac{1}{17}}). \tag{4.1}$$

Then we have

$$\mathrm{Tr}(n_+\Gamma) \le CNK_\ell^2(\rho a^3)^{\frac{1}{17}}. \tag{4.2}$$

In particular, this holds for the Gibbs state

$$\Gamma_0 = \frac{e^{-\frac{H_N}{T}}}{\mathrm{Tr}(e^{-\frac{H_N}{T}})}. \tag{4.3}$$

Proof. The result follows from the following lower bound, which holds as long as $N(\rho a^3)^{\frac{1}{17}} \ge 1$,

$$H_N \ge 4\pi a\frac{N^2}{\ell^3}(1 - C(\rho a^3)^{\frac{1}{17}}) + C'\frac{n_+}{\ell^2}, \tag{4.4}$$

and which can be found in [11, Lemma 4.1 and Lemma 5.2]. The constraint on α ensures in particular the condition on N. Together with the upper bound (4.1) we deduce

$$\mathrm{Tr}\Big(\frac{n_+}{\ell^2}\Gamma\Big) \le CN\rho a(\rho a^3)^{\frac{1}{17}}, \tag{4.5}$$

which is the expected condensation estimate. It remains to show that Γ_0 satisfies the upper bound (4.1), which is not obvious due to the entropy term. To prove this, we use the upper bound on the free energy from [11],

$$\mathrm{Tr}(H_N\Gamma_0 + T\,\Gamma_0 \ln \Gamma_0) \le \inf \sigma(H_N) \le 4\pi\rho Na(1 + C(\rho a^3)^{1/3}). \tag{4.6}$$

The upper bound on the ground state energy was proven by Dyson [10] (see also [3] for an improvement to the order of the LHY correction, but here we do not need such precise estimates). Therefore, using (4.6) and the Gibbs variational principle we can bound the energy for any $0 < \varepsilon < 1$,

$$\begin{aligned}\mathrm{Tr}(H_N\Gamma_0) &\le (1+\varepsilon)\mathrm{Tr}(H_N\Gamma_0 + T\,\Gamma_0\ln\Gamma_0) - \big(\varepsilon\mathrm{Tr}(-\Delta_{\mathbb{R}^{3N}}\Gamma_0) + (1+\varepsilon)T\,\mathrm{Tr}(\,\Gamma_0\ln\Gamma_0)\big)\\ &\le (1+\varepsilon)4\pi\rho Na(1 + C(\rho a^3)^{1/3}) + (1+\varepsilon)T\log\mathrm{Tr}\big(e^{-\frac{\varepsilon}{(1+\varepsilon)T}(-\Delta_{\mathbb{R}^{3N}})}\big),\end{aligned} \tag{4.7}$$

In the second term, the free energy of the ideal gas is bounded by

$$T\log\mathrm{Tr}_{\mathscr{F}_+^{\le N}}\big(e^{-\frac{\varepsilon}{(1+\varepsilon)T}\sum_{p\neq 0}p^2 a_p^* a_p}\big) \le T\sum_{p\in\ell^{-1}\mathbb{N}^3\setminus\{0\}}\log\big(1 - e^{-\frac{\varepsilon}{(1+\varepsilon)T}p^2}\big) \le C\varepsilon^{-3/2}T^{5/2}\ell^3,$$

where $\mathscr{F}_+^{\leq N} = \bigoplus_{n=0}^{N} (\operatorname{Ran} Q)^{\otimes n} \simeq L^2(\Lambda^N)$ is the truncated bosonic Fock space of excitations. Using that $T \leq C\rho a(\rho a^3)^{-\nu}$ and $N \geq \rho \ell^3 (\rho a^3)^\alpha$, we obtain

$$\begin{aligned}
\operatorname{Tr}(H_N \Gamma_0) &\leq 4\pi\rho N a(1 + C(\rho a^3)^{1/3} + C(\varepsilon + \varepsilon^{-3/2} \frac{T^{5/2}\ell^3}{N\rho a})) \\
&\leq 4\pi\rho N a(1 + C(\rho a^3)^{1/3} + C(\varepsilon + \varepsilon^{-3/2}(\rho a^3)^{1/2-\alpha-5\nu/2})) \\
&\leq 4\pi\rho N a(1 + C(\rho a^3)^{1/3} + C(\rho a^3)^{(1-2\alpha-5\nu)/5}),
\end{aligned}$$

where we optimized in ε to obtain the last inequality. The condition that $\alpha + 5\nu/2 < 6/17$ leads to the estimate (4.1). □

Theorem 2.3 follows from Proposition 4.2 below, together with the a priori bounds on n_+ from Lemma 4.1. The proof of Proposition 4.2 is inspired by the localization of large matrices as in [15], and simplified in [13]. It is also similar to the bounds in [19, Proposition 21]. It can be interpreted as an analogue of the standard IMS localization formula. It roughly says that, for a lower bound, we can restrict the estimates states $\Gamma_{\mathcal{M}}$ which have bounded number of low excitations.

Proposition 4.2 (Localization to $\{n_+^L \leq \mathcal{M}\}$)**.** *Let Γ_0 be the Gibbs state associated to H_N. Let $\mathcal{M} \geq N(\rho a^3)^\gamma$, for some $\gamma > 0$, then there exists a trace class operator $\Gamma_{\mathcal{M}}$ on $L^2(\Lambda^N)$ with $\Gamma_{\mathcal{M}} \geq 0$ and trace 1 such that*

$$\mathbf{1}_{\{n_+^L \leq \mathcal{M}\}} \Gamma_{\mathcal{M}} \mathbf{1}_{\{n_+^L \leq \mathcal{M}\}} = \Gamma_{\mathcal{M}}, \tag{4.8}$$

and

$$\begin{aligned}
F(\ell, N) \geq \operatorname{Tr}\big(H_N \Gamma_{\mathcal{M}} + T\,\Gamma_{\mathcal{M}} \ln \Gamma_{\mathcal{M}}\big) &- C\frac{1}{\mathcal{M}^2}\big(\operatorname{Tr}(H_N\Gamma_0) + \|v\|_1 K_H^3 \ell^{-3} N \operatorname{Tr}(n_+\Gamma_0)\big) \\
&- CT\frac{\operatorname{Tr}(n_+\Gamma_0)}{\mathcal{M}}\left(1 + \left|\log \frac{\operatorname{Tr}(n_+\Gamma_0)}{\mathcal{M}}\right|\right).
\end{aligned}$$

Theorem 2.3 follows from this proposition.

Proof of Theorem 2.3. Using the a priori bounds on n_+ from Lemma 4.1, we have for $\mathcal{M} \geq N(\rho a^3)^\gamma$,

$$\frac{\operatorname{Tr}(n_+\Gamma_0)}{\mathcal{M}} \leq (\rho a^3)^{1/17-\gamma} K_\ell^2. \tag{4.9}$$

We use Proposition 4.2, and bound the error terms using (4.9) and the upper bound (4.1) on the energy of Γ_0,

$$\begin{aligned}
&F(\ell, N) - \operatorname{Tr}\big(H_N\Gamma_{\mathcal{M}} + T\,\Gamma_{\mathcal{M}} \ln \Gamma_{\mathcal{M}}\big) \\
&\geq -C\ell^{-3}\Big(a(\rho a^3)^{-2\gamma} + \|v\|_1 K_H^3 (\rho a^3)^{1/17-2\gamma} K_\ell^2\Big) - CT(\rho a^3)^{1/17-\gamma} K_\ell^2 |\log(\rho a^3)| \\
&\geq -C\ell^3 (\rho a)^{5/2}\Big((\rho a^3)^{1/2-2\gamma} K_\ell^{-6} + K_H^3 K_\ell^{-3} (\rho a^3)^{1/17-2\gamma} + (\rho a^3)^{1/17-\gamma-\nu} K_\ell^{-1} |\log(\rho a^3)|\Big)
\end{aligned}$$

where we used that $\ell = K_\ell (\rho a)^{-1/2}$, $T \leq C(\rho a)(\rho a^3)^{-\nu}$ and that $\|v\|_1 \leq \ell$. For ρa^3 small enough and using that $K_\ell \geq 1$, this is smaller than the claimed error term of Theorem 2.3. It remains to extract the spectral gaps. By definition of H_N^{mod} in (2.20) we have

$$\begin{aligned}
\operatorname{Tr}\big(H_N \Gamma_{\mathcal{M}}\big) &= \operatorname{Tr}\Big(\Big(H_N^{\mathrm{mod}} + \frac{\pi}{2\ell^2} n_+ + \frac{K_H}{\ell^2} n_+^H\Big)\Gamma_{\mathcal{M}}\Big) \\
&\geq \operatorname{Tr}\Big(\Big(H_N^{\mathrm{mod}} + \frac{\pi}{4\ell^2} n_+ + \frac{K_H}{2\ell^2} n_+^H + \frac{\pi n_+^L n_+}{4\mathcal{M}\ell^2} + \frac{K_H n_+^L n_+^H}{2\mathcal{M}\ell^2}\Big)\Gamma_{\mathcal{M}}\Big) \\
&= \operatorname{Tr}\Big(\big(H_N^{\mathrm{mod}} + G\big)\Gamma_{\mathcal{M}}\Big).
\end{aligned}$$

This concludes the proof of Theorem 2.3. □

4.1 Proof of Proposition 4.2

The rest of this section is dedicated to the proof of Proposition 4.2. It will follow from the Lemmas 4.3 and 4.4 below, both of which are adapted from [13].

Lemma 4.3. *Let $\theta : \mathbb{R} \to [0,1]$ be any compactly supported smooth function such that $\theta(s) = 1$ for $|s| < \frac{1}{8}$ and $\theta(s) = 0$ for $|s| > \frac{1}{4}$. For any $\mathcal{M} \geq 1$, define $c_{\mathcal{M}} > 0$ and $\theta_{\mathcal{M}}$ such that*

$$\theta_{\mathcal{M}}(s) = c_{\mathcal{M}}\theta\Big(\frac{s}{\mathcal{M}}\Big), \qquad \sum_{s\in\mathbb{Z}} \theta_{\mathcal{M}}(s)^2 = 1.$$

Then there exists a $C > 0$ depending only on θ such that, for any normalized state Γ,

$$\mathrm{Tr}(H_N\Gamma) \geq \sum_{m\in\mathbb{Z}} \mathrm{Tr}(H_N\Gamma_m) - \frac{C}{\mathcal{M}^2}(|\mathrm{Tr}(d_1\Gamma)| + |\mathrm{Tr}(d_2\Gamma)|) \tag{4.10}$$

where $\Gamma_m = \theta_{\mathcal{M}}(n_+^L - m)\Gamma\theta_{\mathcal{M}}(n_+^L - m)$ and

$$d_1 = \sum_{i\neq j} Q_i^L(1 - Q_j^L)v(x_i - x_j)[Q_i^L Q_j^L + (1 - Q_j^L)(1 - Q_j^L)] + \mathrm{h.c.}, \tag{4.11}$$

$$d_2 = \frac{1}{2}\sum_{i\neq j} Q_i^L Q_j^L v(x_i - x_j)(1 - Q_i^L)(1 - Q_j^L) + \mathrm{h.c.} \tag{4.12}$$

Proof. Notice that H_N only contains terms that change n_+^L by $0, \pm 1$ or ± 2. Therefore, we write our operator as $H_N = \sum_{|k|\leq 2} \mathcal{H}^{(k)}$, with $\mathcal{H}^{(k)} n_+^L = (n_+^L + k)\mathcal{H}^{(k)}$. Moreover, one easily checks that for $k = 1, 2$,

$$\mathcal{H}^{(k)} + \mathcal{H}^{(-k)} = d_k.$$

For $|k| \leq 2$, we have

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \mathrm{Tr}(\mathcal{H}^{(k)}\Gamma_m) &= \sum_{m\in\mathbb{Z}} \mathrm{Tr}(\theta_{\mathcal{M}}(n_+^L - m)\mathcal{H}^{(k)}\theta_{\mathcal{M}}(n_+^L - m)\Gamma) \\ &= \sum_{m\in\mathbb{Z}} \mathrm{Tr}(\theta_{\mathcal{M}}(n_+^L - m)\theta_{\mathcal{M}}(n_+^L - m + k)\mathcal{H}^{(k)}\Gamma) \\ &= \sum_{m\in\mathbb{Z}} \theta_{\mathcal{M}}(m)\theta_{\mathcal{M}}(m + k)\mathrm{Tr}(\mathcal{H}^{(k)}\Gamma),\end{aligned}$$

where we used the commutation property of $\mathcal{H}^{(k)}$ and that the function $\sum_{m\in\mathbb{Z}} \theta_{\mathcal{M}}(X - m)\theta_{\mathcal{M}}(X - m + k)$ is constant on $\mathbb{Z}$. Using that $\sum_{s\in\mathbb{Z}} \theta_{\mathcal{M}}(s)^2 = 1$, we obtain that

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \mathrm{Tr}(H_N\Gamma_m) &= \mathrm{Tr}(H_N\Gamma) + \sum_{|k|\leq 2} \delta_k \mathrm{Tr}(\mathcal{H}^k\Gamma) \\ &= \mathrm{Tr}(H_N\Gamma) + \delta_1\mathrm{Tr}(d_1\Gamma) + \delta_2\mathrm{Tr}(d_2\Gamma),\end{aligned}$$

with

$$\delta_k = \sum_{m\in\mathbb{Z}} \big(\theta_{\mathcal{M}}(m)\theta_{\mathcal{M}}(m + k) - \theta_{\mathcal{M}}(m)^2\big) = -\frac{1}{2}\sum_{m\in\mathbb{Z}} \big(\theta_{\mathcal{M}}(m) - \theta_{\mathcal{M}}(m + k)\big)^2.$$

It remains to prove that $|\delta_k| \leq C\mathcal{M}^{-2}$. This follows from the fact that θ is smooth and has support in $[-1/4, 1/4]$, so that we can restrict the sum to $m \in \big[-\frac{\mathcal{M}}{2}, \frac{\mathcal{M}}{2}\big]$, and $c_{\mathcal{M}} > C^{-1}\mathcal{M}$. □

To estimate the error in (4.10), we need the following bounds on d_1 and d_2.

Lemma 4.4. *There exists a universal constant $C > 0$ such that, for any trace-class operator Γ with $\Gamma \geq 0$ and $\mathrm{Tr}\,\Gamma = 1$ we have*

$$|\mathrm{Tr}(d_1\Gamma)| + |\mathrm{Tr}(d_2\Gamma)| \leq C\mathrm{Tr}\Big(\sum_{i\neq j} v(x_i - x_j)\Gamma\Big) + C\|v\|_1 K_H^3 \ell^{-3} N\mathrm{Tr}\big(n_+\Gamma\big). \tag{4.13}$$

Proof. We start by noting the following bound. For $x, y \in \Lambda$, denoting $v_y(x) = v(x-y)$, we have

$$Q^L v_y Q^L = \sum_{0<|p|,|q|<K_H} \Big(\int_\Lambda v_y u_p u_q \Big) |u_p\rangle \langle u_q| \le C \frac{K_H^3}{\ell^3} \|v\|_1 n_+, \tag{4.14}$$

where $\{u_p\}$ is the Neumann basis defined in (2.23). Therefore, expanding all the prefactors in (4.11), we can bound d_k, $k \in \{1,2\}$, using the Cauchy-Schwarz inequality by

$$\begin{aligned} \pm d_k &\le C \sum_{i \neq j} v(x_i - x_j) + Q_i^L v(x_i - x_j) Q_i^L + Q_i^L Q_j^L v(x_i - x_j) Q_i^L Q_j^L \\ &\le C \sum_{i \neq j} v(x_i - x_j) + C K_H^3 \frac{\|v\|_{L^1}}{\ell^3} n_+ N. \end{aligned}$$

From which the claim follows. □

Now we can combine Lemmas 4.3 and 4.4 to prove Proposition 4.2.

Proof of Proposition 4.2. Let $\Gamma_0 = e^{-H_N/T}/\mathrm{Tr}\big(e^{-H_N/T}\big)$ be the Gibbs state. Lemma 4.3 gives

$$\mathrm{Tr}(H_N \Gamma_0) \ge \mathrm{Tr}(H_N \Gamma_\le) + \mathrm{Tr}(H_N \Gamma_>) - \frac{C}{\mathcal{M}^2} (|\mathrm{Tr}(d_1 \Gamma_0)| + |\mathrm{Tr}(d_2 \Gamma_0)|), \tag{4.15}$$

where

$$\Gamma_\le := \sum_{|m| \le \mathcal{M}/8} \Gamma_m, \qquad \Gamma_> := \sum_{|m| > \mathcal{M}/8} \Gamma_m. \tag{4.16}$$

Denoting by $\alpha_\Gamma := \mathrm{Tr}(\Gamma_>)$, we have $1 - \alpha_\Gamma = \mathrm{Tr}(\Gamma_\le)$, and from the sub-additivity of the entropy $S(\Gamma) = -\mathrm{Tr}\,(\Gamma \ln \Gamma)$ we have

$$S(\Gamma_0) \le S(\Gamma_\le) + S(\Gamma_>) = \alpha_\Gamma S\Big(\frac{\Gamma_>}{\alpha_\Gamma}\Big) + (1-\alpha_\Gamma) S\Big(\frac{\Gamma_\le}{1-\alpha_\Gamma}\Big) - \alpha_\Gamma \log(\alpha_\Gamma) - (1-\alpha_\Gamma)\log(1-\alpha_\Gamma). \tag{4.17}$$

Combining (4.15) and (4.17) we obtain

$$\begin{aligned} F(\ell, N) &= \mathrm{Tr}(H_N \Gamma_0) - T S(\Gamma_0) \\ &\ge (1-\alpha_\Gamma) \Big[\mathrm{Tr}\Big(H_N \frac{\Gamma_\le}{1-\alpha_\Gamma}\Big) - T S\Big(\frac{\Gamma_\le}{1-\alpha_\Gamma}\Big) \Big] + \alpha_\Gamma \Big[\mathrm{Tr}\Big(H_N \frac{\Gamma_>}{\alpha_\Gamma}\Big) - T S\Big(\frac{\Gamma_>}{\alpha_\Gamma}\Big) \Big] \\ &\quad - \frac{C}{\mathcal{M}^2} (|\mathrm{Tr}(d_1 \Gamma_0)| + |\mathrm{Tr}(d_2 \Gamma_0)|) + T \alpha_\Gamma \log(\alpha_\Gamma) + T(1-\alpha_\Gamma)\log(1-\alpha_\Gamma). \end{aligned}$$

By the variational principle, the second term above is bigger than $\alpha_\Gamma F(\ell, N)$, and subtracting this quantity on both sides and dividing by $1 - \alpha_\Gamma$, we obtain

$$\begin{aligned} F(\ell, N) &\ge \mathrm{Tr}\Big(H_N \frac{\Gamma_\le}{1-\alpha_\Gamma}\Big) - T S\Big(\frac{\Gamma_\le}{1-\alpha_\Gamma}\Big) - \frac{C}{(1-\alpha_\Gamma)\mathcal{M}^2} (|\mathrm{Tr}(d_1 \Gamma_0)| + |\mathrm{Tr}(d_2 \Gamma_0)|) \\ &\quad + T \frac{\alpha_\Gamma}{1-\alpha_\Gamma} \log(\alpha_\Gamma) + T \log(1-\alpha_\Gamma). \end{aligned}$$

Lastly we see that

$$\alpha_\Gamma = \mathrm{Tr}(\Gamma_>) \le \mathrm{Tr}(\mathbf{1}_{\{n_+ \ge \frac{\mathcal{M}}{8}\}} \Gamma) \le \mathrm{Tr}\Big(8 \frac{n_+}{\mathcal{M}} \Gamma\Big).$$

Defining $\Gamma_{\mathcal{M}} = \frac{\Gamma_\le}{1-\alpha_\Gamma}$ and using the bounds in Lemma 4.4, we obtain the claim. □

5 Symmetrization: Proof of Theorem 2.6

In this section we prove Theorem 2.6, which estimates the error made replacing the Hamiltonian H_N^{mod} in (2.20) by its symmetrized version H_N^{sym}. We begin by gathering some important properties of the symmetrization. Recall the definition (2.22) of the symmetry p_z, for $z \in \mathbb{Z}^3$, and the symmetrized version f^{s} of a function f in (2.24).

Lemma 5.1. *For all $z \in \mathbb{Z}^3$, x, $y \in \Lambda$ and $f : \mathbb{R}^3 \to \mathbb{R}$ radial such that with $\mathrm{supp}(f) \subset B(0, R)$ for some $R \le \ell/2$, we have*

1. *$f(p_z(x) - y) = 0$ if one of the $|z_i| \ge 2$,*

2. $f^{\mathrm{s}}(x,y)=f^{\mathrm{s}}(y,x)$,
3. $|p_z(x)-y|\geq|x-y|$,
4. $u_p(p_z(x))=u_p(x)$, *for* $p\in\frac{\pi}{\ell}\mathbb{N}_0^3$.

Proof. Property 1 is a consequence of the assumption on the support of f. Property 2 follows from that f is radial and $|p_z(x)-y|=|p_z(y)-x|$. Property 3 is obvious from the definition of p_z, observing that x, $y\in\Lambda$. The last property follows from that the definition (2.23) of the Neumann eigenbasis (u_p), noting that $((-1)^{z+1}+1)/2+z\in2\mathbb{Z}$ for all $z\in\mathbb{Z}$. □

We now proceed to the proof of Theorem 2.6. From (2.20) and (2.30), we need to find a lower bound on

$$H_N^{\mathrm{mod}}-H_N^{\mathrm{sym}}=\sum_{j=0}^{3}(Q_j^{\mathrm{ren}}-Q_j^{\mathrm{sym}})+Q_4^{\mathrm{ren}},$$

where the Q_j^{ren} were defined in (2.10)-(2.14) and the Q_j^{sym} were defined in (2.26)-(2.29).

Symmetrization of Q_0^{ren}. From $P=\frac{1}{\ell^3}|1\rangle\langle1|\leq\mathbf{1}$, we obtain

$$\begin{aligned}0\leq Q_0^{\mathrm{sym}}-Q_0^{\mathrm{ren}}&=\frac12\sum_{i\neq j}P_iP_j(g^{\mathrm{s}}-g+(g\omega)^{\mathrm{s}}-g\omega)P_jP_i\\&=\frac12\sum_{z\neq0}\sum_{i\neq j}P_iP_j\big(g(p_z(x_i)-x_j)+(g\omega)(p_z(x_i)-x_j)\big)P_jP_i\\&\leq\frac{CN^2}{|\Lambda|^2}\sum_{z\neq0}\int_{\Lambda^2}g(p_z(x)-y)\mathrm{d}x\mathrm{d}y,\end{aligned}$$

where we first used the definition of g^{s} in (2.24) and that $g\omega\leq g$. Since $\operatorname{supp}g\subset\{|x|\leq R\}$, we can restrict the sum to $|z|\leq1$ and the integration over x to $\Lambda\setminus\Lambda_R$, with $\Lambda_R=[R,\ell-R]^3$. We obtain that

$$Q_0^{\mathrm{sym}}-Q_0^{\mathrm{ren}}\leq\frac{CN^2}{|\Lambda|^2}\sum_{|z|\leq1}\int_{\Lambda^2}\mathbf{1}_{\{\Lambda\setminus\Lambda_R\}}(x)g(p_z(x)-y)\mathrm{d}x\mathrm{d}y\leq CN\rho\widehat{g}(0)\frac{R}{\ell}.\tag{5.1}$$

Symmetrization of Q_2^{ren}. Recall the definition of Q_2^{ren} in (2.12). We have three different kinds of terms from $Q_2^{\mathrm{ren}}-Q_2^{\mathrm{ren}}$ to treat, which we denote in a self-explanatory way by $PQPQ$, $PQQP$ and $PPQQ$. We start with

$$\begin{aligned}PQQP:=\sum_{i\neq j}P_iQ_j(g^{\mathrm{s}}-g+(g\omega)^{\mathrm{s}}-g\omega)Q_jP_i&\leq\frac{2}{|\Lambda|}\sum_{z\neq0}\sum_{i\neq j}Q_j\int_\Lambda g(p_z(x_j)-y)\mathrm{d}yQ_jP_i\\&\leq C\frac{N}{|\Lambda|}\widehat{g}(0)\sum_{j=1}^{N}Q_j\mathbf{1}_{\{\Lambda\setminus\Lambda_R\}}(x_j)Q_j.\end{aligned}$$

where we used the same properties as above plus the symmetry $|p_z(x)-y|=|p_z(y)-x|$. To estimate this last sum, we first isolate the high momenta using $Q=Q^H+Q^L$ defined in (2.16) to get

$$PQQP\leq C\rho\widehat{g}(0)\sum_{j=1}^{N}Q_j^L\mathbf{1}_{\{\Lambda\setminus\Lambda_R\}}(x_j)Q_j^L+C\rho\widehat{g}(0)n_+^H.\tag{5.2}$$

For the low momenta part, we decompose with respect to the basis (u_p) introduced in (2.23). Using the Cauchy-Schwarz inequality, we obtain

$$Q^L\mathbf{1}_{\{\Lambda\setminus\Lambda_R\}}Q^L=\sum_{p,q\in\mathcal{P}_L}\Big(\int_{\Lambda\setminus\Lambda_R}\overline{u_p}u_q\Big)|u_p\rangle\langle u_q|\leq C\frac{R}{\ell}|\mathcal{P}_L|Q^L=C\frac{R}{\ell}K_H^3Q^L$$

where we used $|u_p(x)u_{p'}(x)|\leq C|\Lambda|^{-1}$ and $|\Lambda\setminus\Lambda_R|\leq CR\ell^2$. Inserting the above in (5.2), we deduce that

$$PQQP\leq C\rho\widehat{g}(0)\frac{R}{\ell}K_H^3n_+^L+C\rho\widehat{g}(0)n_+^H.\tag{5.3}$$

The next term to bound in Q_2^{ren} is the $PQPQ$. By the Cauchy-Schwarz inequality $P_iQ_jP_jQ_i \leq P_iQ_jQ_jP_i + Q_iP_jP_jQ_i$ and the symmetry of g^{s} and $(g\omega)^{\mathrm{s}}$ we obtain the same bound as (5.3).

In order to bound $QQPP$, we need to use Q_4^{ren}. To this end we reconstruct the projector $\Pi_{ij} = Q_jQ_i + \omega(x_i - x_j)\,(P_jP_i + P_jQ_i + Q_jP_i)$ appearing in the definition of Q_4^{ren}. We have

$$\begin{aligned} QQPP &:= \sum_{i\neq j} Q_iQ_j(g^{\mathrm{s}} - g)P_jP_i + \mathrm{h.c.} \\ &= \Big(\sum_{i\neq j} \Pi_{ij}(g^{\mathrm{s}} - g)P_jP_i + \mathrm{h.c.}\Big) - \Big(\sum_{i\neq j}(Q_iP_j + P_iQ_j + P_iP_j)\omega(g^{\mathrm{s}} - g)P_jP_i + \mathrm{h.c.}\Big) \end{aligned}$$

We can bound the second term by $Q_0^{\mathrm{sym}} - Q_0^{\mathrm{ren}}$ and $PQQP$, which have already been estimated, using the Cauchy-Schwarz inequalities and $\omega \leq 1$. The first term is also bounded using the Cauchy-Schwarz inequality, giving

$$QQPP \leq C\varepsilon^{-1}N\rho\widehat{g}(0)\frac{R}{\ell} + C\rho\widehat{g}(0)\frac{R}{\ell}K_H^3 n_+^L + C\rho\widehat{g}(0)n_+^H + \varepsilon\sum_{i\neq j}\Pi_{ij}^*\omega(g^{\mathrm{s}} - g)\Pi_{ij},$$

for all $\varepsilon > 0$. The last term is estimated using that Q_4^{ren} and $\omega \leq 1$,

$$g(p_z(x) - y) \leq C_0 v(x - y) \tag{5.4}$$

for some $C_0 > 0$, which follows from that g and v satisfy (2.6), and $|p_z(x) - y| \geq |x - y|$ from Lemma 5.1. Choosing $\varepsilon = C_0/100$, we conclude that

$$Q_2^{\mathrm{sym}} - Q_2^{\mathrm{ren}} \leq C\rho\widehat{g}(0)\frac{R}{\ell}K_H^3 n_+^L + C\rho\widehat{g}(0)n_+^H + CN\rho\widehat{g}(0)\frac{R}{\ell} + \frac{1}{100}Q_4^{\mathrm{ren}} \tag{5.5}$$

Note that the first two errors can be estimated by a fraction of the gap operator G defined in (2.18), as soon as $K_H \geq CK_\ell^2$ and $K_\ell K_H^3 \leq (\rho a^3)^{-\frac{1}{2}}$. The last fraction of Q_4^{ren} is absorbed by the positive $\frac{1}{2}Q_4^{\mathrm{ren}}$ in H_N^{mod}.

Symmetrization of Q_1^{ren}. From the Cauchy-Schwarz inequality, we have

$$Q_1^{\mathrm{sym}} - Q_1^{\mathrm{ren}} \leq C(QPPQ + Q_0^{\mathrm{sym}} - Q_0^{\mathrm{ren}}) \leq C\rho a(\frac{R}{\ell}N + \frac{R}{\ell}K_H^3 n_+^L + n_+^H). \tag{5.6}$$

which we already estimated in (5.1) and (5.3), and that are absorbed by a fraction of the gap operator G for ρa^3 small enough. It remains to note that $Q_1^{\mathrm{sym}} = 0$, indeed

$$\begin{aligned} Q_1^{\mathrm{sym}} &= \sum_z\sum_{i\neq j} P_iP_jg(p_z(x_i) - x_j)P_iQ_j + \mathrm{h.c.} = \frac{1}{|\Lambda|}\sum_z\sum_{i\neq j} P_j\int_\Lambda g(p_z(x) - x_j)\mathrm{d}x P_iQ_j + \mathrm{h.c.} \\ &= \frac{1}{|\Lambda|}\sum_{i\neq j} P_j\int_{\mathbb{R}^3} g(x - x_j)\mathrm{d}x P_iQ_j + \mathrm{h.c.} = \frac{\widehat{g}(0)}{|\Lambda|}\sum_{i\neq j} P_iP_jQ_j = 0. \end{aligned}$$

Symmetrization of Q_3^{ren}. In view of Lemma 2.4, it is enough to estimate

$$Q_{3,L}^{\mathrm{sym}} - Q_{3,L} = \sum_{z\neq 0}\sum_{i\neq j}(P_iQ_j^L g(p_z(x_i) - x_j)Q_iQ_j + \mathrm{h.c.}).$$

For the rest of the proof we use the notation $g_{\neq 0} := \sum_{z\neq 0} g(p_z(x_i) - x_j)$. We want to reconstruct Q_4^{ren} as before, we write

$$Q_{3,L}^{\mathrm{sym}} - Q_{3,L} = \Big(\sum_{i\neq j} P_iQ_j^L g_{\neq 0}\Pi_{ij} + \mathrm{h.c.}\Big) - \Big(\sum_{i\neq j} P_iQ_j^L g_{\neq 0}\omega(P_iP_j + P_iQ_j + Q_iP_j) + \mathrm{h.c.}\Big) \tag{5.7}$$

From the Cauchy-Schwarz inequality we obtain, as before using (5.4), that

$$Q_{3,L}^{\mathrm{sym}} - Q_{3,L} \leq C(QPPQ + Q_0^{\mathrm{sym}} - Q_0^{\mathrm{ren}}) + \frac{1}{100}Q_4^{\mathrm{ren}},$$

where the first term is estimated in (5.6). This concludes the proof of Theorem 2.6. □

6 C-number substitution

In this section we prove Lemma 2.8, Theorem 2.9 and Lemma 2.11.

6.1 Proof of Lemma 2.8

The second quantization of the kinetic energy is immediate. Then by definition of n_+ and from Lemma 2.5 we have

$$Q_0^{\rm sym} = \frac{\widehat{g}(0) + \widehat{g\omega}(0)}{2|\Lambda|}(N - n_+)(N - n_+ - 1).$$

Again using Lemma 2.5, we have

$$Q_2^{\rm sym} = \frac{1}{|\Lambda|}\sum_{p\in\Lambda_+^*}\Big(g\widehat{(1+\omega)}(p) + g\widehat{(1+\omega)}(0)\Big)\, a_0^* a_p^* a_p a_0 + \frac{1}{2}\widehat{g}(p)\big(a_0^* a_0^* a_p a_p + \text{h.c.}\big)$$

From that $(N-n_+)(N-n_+-1) = N(N-1) - 2n_+(N-n_+) - n_+(n_+-1)$ and that $\sum_{p\in\Lambda_+^*} a_0^* a_p^* a_p a_0 = n_+(N-n_+)$ we recover the first two lines of (2.33). The second quantization of the $Q_{3,L}^{\rm sym}$ is not expressed as simply. We start from

$$Q_{3,L}^{\rm sym} = \sum_{q,p,k,s\in\Lambda^*}\langle u_q\otimes u_p|P_x Q_y^L g^s(x,y) Q_x Q_y|u_k\otimes u_s\rangle\, a_q^* a_p^* a_k a_s + \text{h.c.} \tag{6.1}$$

where we observe that we can restrict the sum to $q=0$, $p\in\mathcal{P}_L$ and $k,s\in\Lambda_+^*$. We are left to compute the integral

$$\langle u_0\otimes u_p|P_x Q_y^L g^s(x,y) Q_x Q_y|u_k\otimes u_s\rangle = \frac{1}{|\Lambda|^{1/2}}\int_{\Lambda^2} u_p(y) g^s(x,y) u_k(x) u_s(y)dxdy.$$

Using the trigonometric formulas we obtain

$$\begin{aligned} u_p(y)u_s(y) &= \frac{1}{2^3|\Lambda|}\prod_{i=1}^3 c_{p_i}c_{s_i}(\cos((p_i - s_i)y_i) + \cos((p_i+s_i)y_i)) \\ &= \frac{1}{2^3|\Lambda|^{1/2}}\prod_{i=1}^3 c_{p_i}c_{s_i}\sum_{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}}\frac{u_{p_1+\sigma_1 s_1, p_2+\sigma_2 s_2, p_3+\sigma_3 s_3}}{c_{p_1+\sigma_1 s_1}c_{p_2+\sigma_2 s_2}c_{p_3+\sigma_3 s_3}}, \end{aligned}$$

where c_k was defined in (2.23). Using Lemma 2.5 we obtain

$$\begin{aligned} &\int_{\Lambda^2} u_p(y) g^s(x,y) u_k(x) u_s(y) dxdy \\ &\quad = \frac{1}{2^3|\Lambda|^{1/2}}\prod_{i=1}^3 c_{p_i}c_{s_i}\sum_{\sigma_1,\sigma_2,\sigma_3\in\{\pm\}}\frac{\delta_{k_1,|p_1+\sigma_1 s_1|}\delta_{k_2,|p_2+\sigma_2 s_2|}\delta_{k_3,|p_3+\sigma_3 s_3|}}{c_{|p_1+\sigma_1 s_1|}c_{|p_2\sigma_2 s_2|}c_{|p_3+\sigma_3 s_3|}}\widehat{g}(k). \end{aligned}$$

Noting that we can replace $\sum_{p\in\mathcal{P}_L}\sum_{\{\pm\}}$ by $\sum_{p\in\mathcal{P}_L^{\mathbb{Z}}}$, using the convention $a_p = a(u_p) = a_{(|p_1|,|p_2|,|p_3|)}$ and changing variable $s = k - p$ we obtain

$$Q_{3,L}^{\rm sym} = \frac{1}{|\Lambda|}\sum_{p\in P_L^{\mathbb{Z}}, k\in\Lambda_+^*\setminus\{p\}} c(p,k)\widehat{g}(k) a_0^* a_p^* a_k a_{p-k} + \text{h.c.} \tag{6.2}$$

where $c(p,k)$ is given by

$$c(p,k) = \prod_{i=1}^3 \frac{c_{p_i}c_{k_i-p_i}}{2c_{k_i}}2^{\delta_{p_i,0}} = \prod_{i=1}^3\frac{c_{k_i-p_i}}{c_{p_i}c_{k_i}}. \tag{6.3}$$

□

6.2 Proof of Theorem 2.9

The c-number substitution is an important step of Bogoliubov's approach to the dilute Bose gas and it is well known that it can be performed in a rigorous way. We start from Lemma 2.8, where the Hamiltonian $H_N^{\rm sym}$ has been written in second quantization (2.33). It is initially defined on $L^2(\Lambda)^N$ but naturally extends to the Fock space $\mathscr{F}(L^2(\Lambda))$.

We subtract the leading order of the energy and introduce a chemical potential μ, which we will choose later, in order to finely tune the number of particles. Thus, we define

$$\begin{aligned}
\mathcal{H}_\mu^{\rm sym} :=& \frac{\widehat{g\omega}(0)}{2|\Lambda|}a_0^*a_0^*a_0a_0 + \sum_{p\in\Lambda_+^*}\Big(\tau(p)a_p^*a_p + \frac{\widehat{g}(p)}{|\Lambda|}a_0^*a_p^*a_pa_0 + \frac{\widehat{g}(p)}{2|\Lambda|}\big(a_0^*a_0^*a_pa_p + \text{h.c.}\big)\Big) \\
&+ \frac{1}{|\Lambda|}\sum_{k\in\Lambda_+^*,\, p\in\mathcal{P}_L^{\mathbb{Z}}} c(p,k)\widehat{g}(k)\big(a_0^*a_p^*a_{p-k}a_k + \text{h.c.}\big) + \frac{1}{|\Lambda|}\sum_{p\in\Lambda_+^*}(\widehat{g\omega}(0) + \widehat{g\omega}(p))a_0^*a_p^*a_pa_0 \\
&- \mu\mathcal{N} + \rho a\frac{\mathcal{N}^m}{N^m} + \frac{8\pi a(\rho a^3)^{\frac14}}{|\Lambda|}(n_0^2 - N^2) \\
=& H_N^{\rm sym} - \frac{\widehat{g}(0)(N(N-1) - n_+(n_+-1))}{2|\Lambda|} - \mu\mathcal{N} + \rho a\frac{\mathcal{N}^m}{N^m} + \frac{8\pi a(\rho a^3)^{\frac14}}{|\Lambda|}(n_0^2 - N^2)
\end{aligned} \tag{6.4}$$

where the last term, which is negative on $L^2(\Lambda)^N$, is added to ensure convexity in "n_0" (see Remark 2.10) and we introduced the number operator which acts on the Fock space on each sector as

$$(\mathcal{N}\Psi)^{(N)} = N\Psi^{(N)}, \qquad \text{for any } \Psi\in\mathscr{F}(L^2(\Lambda)). \tag{6.5}$$

Note that

$$\widehat{g}(0)n_+^2|\Lambda|^{-1} \le C\rho a n_+^H + Ca\ell^{-3}n_+^Ln_+ \le C(\rho a\ell^2K_H^{-1} + \mathcal{M}a\ell^{-1})G$$

where G is the gap operator defined in (2.18). Using that, from assumptions, $CK_\ell^4 \le K_H$ and $\mathcal{M} \le C^{-1}\ell/a$, we obtain that for ρa^3 small enough,

$$-T\log\mathrm{Tr}_{L^2(\Lambda^N)}e^{-\frac1T(H_N^{\rm sym}+\frac12G)} \ge 4\pi a\frac{N^2}{|\Lambda|} - T\log\mathrm{Tr}_{\mathscr{F}(L^2(\Lambda))}e^{-\frac1T(\mathcal{H}_\mu^{\rm sym}+\frac14G)} + \mu N - C\rho a. \tag{6.6}$$

In the above we artificially inserted $\frac{\mathcal{N}^m}{N^m}$ (which is equal to 1 on the N particle sector) at the expanse of an error of order ρa. This will ensure that we never have too many particles after the c-number substitution via (6.12) and will be used to prove Lemma 2.11.

We follow [23] and perform the c-number substitution. The decomposition $L^2(\Lambda) = \mathrm{Ran}P\oplus\mathrm{Ran}Q$ leads to the splitting of the bosonic Fock space $\mathscr{F}(L^2(\Lambda)) = \mathscr{F}(\mathrm{Ran}P)\otimes\mathscr{F}(\mathrm{Ran}Q)$. Denoting by Ω the vacuum vector, we introduce the class of coherent states in $\mathscr{F}(\mathrm{Ran}P)$, labeled by $z\in\mathbb{C}$,

$$|z\rangle = e^{-\left(\frac{|z|^2}{2}+za_0^\dagger\right)}\,\Omega, \tag{6.7}$$

which are eigenvectors for the annihilation operator of the condensate. One easily checks that

$$a_0|z\rangle = z\,|z\rangle \quad \text{and} \quad 1 = \int_{\mathbb{C}}|z\rangle\langle z|\,\mathrm{d}z \tag{6.8}$$

where $\mathrm{d}z = \pi^{-1}\mathrm{d}x\mathrm{d}y$, $z = x+iy$. For any $\Psi\in\mathscr{F}(L^2(\Lambda))$, we denote by $\Phi(z) = \langle z|\Psi\rangle$ the partial inner product, which is in $\mathscr{F}(\mathrm{Ran}Q)$. We write any monomial in a_0 and a_0^* as a polynomial sum of terms in anti-normal order thanks to the commutation rules. Using theses definitions and the inequality [23, Eq.(7)] we are allowed then to replace a_0 with a complex number (called upper symbol) according to the following substitution rules

$$\begin{aligned}
&a_0 \mapsto z, \qquad && a_0a_0 \mapsto z^2, \qquad && a_0a_0^* \mapsto |z|^2, \\
&a_0^* \mapsto \overline{z}, \qquad && a_0^*a_0^* \mapsto \overline{z}^2, \qquad && a_0a_0a_0^*a_0^* \mapsto |z|^4
\end{aligned} \tag{6.9}$$

which also give the substitutions

$$a_0^*a_0 \mapsto |z|^2 - 1, \qquad a_0^*a_0^*a_0a_0 \mapsto |z|^4 - 4|z|^2 + 2 \tag{6.10}$$

before integrating over z in front of the trace in (6.6). For the specific case of a power of the n_0, we use again the commutation rules to write it as polynomial of anti-normal ordered monomials in a_0, a_0^* and apply the rules (6.9) to define the polynomial in $|z|^2$

$$n_0^m = (a_0^* a_0)^m \mapsto p_m(z) = |z|^{2m} + \text{smaller order terms}, \tag{6.11}$$

where the smaller order terms have explicit, constant coefficients.

We first bound below the new term using that

$$\rho a \frac{\mathcal{N}^m}{N^m} \geq \rho a \frac{n_0^m + n_0^{m-1} n_+ + n_0^{m-2} n_+^2}{N^m}. \tag{6.12}$$

The upper symbol of the right hand side is

$$\frac{\rho a}{N^m}\big(p_m(z) + p_{m-1}(z) n_+ + p_{m-2}(z) n_+^2\big). \tag{6.13}$$

We observe that, for $|z|^2 > N$, (and N sufficiently large) the polynomials satisfy the bound $p_m(z) \geq \frac{1}{2}|z|^{2m}$ and therefore, in this region, (6.13) can be bounded from below by

$$\frac{\rho a}{2N^m}\big(|z|^{2m} + |z|^{2m-2} n_+ + |z|^{2m-4} n_+^2\big). \tag{6.14}$$

When $|z|^2 \leq N$, the error made replacing (6.13) by (6.14) is of order $N^{-1}\rho a$. The last term in (6.4) becomes

$$\frac{8\pi a(\rho a^3)^{\frac14}}{|\Lambda|}(n_0^2 - N^2) \mapsto \frac{8\pi a(\rho a^3)^{\frac14}}{|\Lambda|}(|z|^4 - 3|z|^2 + 1 - N^2). \tag{6.15}$$

Using that, for $m > 3$,

$$3\frac{a}{|\Lambda|}|z|^2 - 2 \leq C\frac{a}{|\Lambda|}\left(K_\ell N \mathbf{1}_{|z|^2 \leq K_\ell N} + \frac{|z|^{2m}}{N^m} N K_\ell^{1-m} \mathbf{1}_{|z|^2 > K_\ell N}\right) \leq C\rho a\left(K_\ell + \frac{|z|^{2m}}{N^m}\right) \tag{6.16}$$

we recover the last term of (2.36) up to an error $C\rho a$ (because from the assumptions $K_\ell \leq (\rho a^3)^{-1/26}$) and a fraction of the gap $\mathcal{G}(z)$.

When proceeding to the c-number substitution in the other terms of the Hamiltonian $\mathcal{H}_\mu^{\rm sym}$ we obtain errors from the terms proportional to $a_0^* a_0$ and $a_0^* a_0^* a_0 a_0$, which are

$$\mathcal{E} = \widehat{g\omega}(0)\frac{2 - 4|z|^2}{2|\Lambda|} - \sum_{p \in \Lambda_+^*} \frac{\widehat{g}(p) + \widehat{g\omega}(0) + \widehat{g\omega}(p)}{|\Lambda|} a_p^* a_p - \mu(n_+ - 1). \tag{6.17}$$

Note that $|\widehat{g\omega}(p)| + |\widehat{g}(p)| \leq Ca$. Therefore, we can estimate the first term above as in (6.16). The second and third terms are estimated by by $C(a|\Lambda|^{-1} + \mu) n_+ \leq \ell^{-2} n_+$ which is absorbed by a fraction of the gap (recall that $0 \leq 10\mu \leq \ell^{-2}$ by assumption). In the end, we recover $\mathcal{H}_\mu(z)$ as well as $\mathcal{G}(z)$ and the inequality stated in Theorem 2.9. □

6.3 Proof of Lemma 2.11

We conclude this section with the proof of Lemma 2.11, which gives a lower bound for the Hamiltonian in the physically irrelevant region $|z|^2 \geq K_\ell^{1/4} N$. Starting from the Hamiltonian (2.36), we use that

$$\frac{|z|^2}{|\Lambda|}\sum_{p \in \Lambda_+^*}\big(\widehat{g}(p) + \widehat{g\omega}(p)\big) a_p^* a_p \leq C\rho a |z|^2 \frac{n_+}{N}$$

and we drop all the non-negative terms except the kinetic energy and the spectral gaps. Using that $|z|^2 \geq K_\ell^{1/4} N$, we obtain

$$\begin{aligned}\mathcal{H}_\mu(z) + \mathcal{G}(z) \geq \sum_{p \in \Lambda^*}\Big(\tau(p) a_p^* a_p + \frac{\widehat{g}(p)}{2|\Lambda|}(z^2 a_p^* a_p^* + h.c)\Big) + Q_{3,L}^{\rm sym}(z) - \mu|z|^2 \\ + \frac{\rho a}{2N^m}(|z|^{2m} + |z|^{2m-2} n_+ + |z|^{2m-4} n_+^2).\end{aligned} \tag{6.18}$$

We now show that a fraction of this last term absorbs all the negative terms of (6.18). First note that $\mu \le C\ell^{-2} \le K_\ell^{-1}\rho a$. To bound $Q_{3,L}^{\rm sym}$ we use the Cauchy-Schwarz inequality, we obtain for all $\delta > 0$

$$\begin{aligned}\pm Q_{3,L}^{\rm sym} &\le \delta \sum_{p\in P_L^{\mathbb{Z}}, k\in\Lambda^*\cup\{0\}} k^2 a_k^* a_k + \delta^{-1}|z|^2 \frac{1}{|\Lambda|^2} \sum_{p\in P_L^{\mathbb{Z}}, k\in\Lambda_+^*} \frac{\widehat{g}(k)^2}{k^2} a_p^* a_{p-k} a_{p-k}^* a_p \\ &\le C\delta|\mathcal{P}_L| \sum_{p\in\Lambda^*} \tau_p a_p^* a_p + \delta^{-1}\frac{C|z|^2}{\ell^3}(an_+ + a^2\ell^{-1}n_+^2) \\ &\le \frac{1}{2}\sum_{p\in\Lambda^*} \tau_p a_p^* a_p + C\rho a K_H^3 K_\ell^{-2} a\ell^{-1}|z|^2(n_+ + a\ell^{-1}n_+^2)\end{aligned}$$

where we used (7.12) to estimate the sum and chose $\delta = \frac{1}{2}C^{-1}|\mathcal{P}_L|^{-1}$. The second term above is bounded by a fraction of the last term in (6.18) using the assumptions on m. Recalling that τ_p is given by (2.34), we can replace it by p^2 by absorbing the negative $n_+\ell^{-2}$ and $K_H n_+^H \ell^{-2}$ terms, and we are left with

$$\sum_{p\in\Lambda^*}\Big(\frac{1}{2}p^2 a_p^* a_p + \frac{\widehat{g}(p)}{2|\Lambda|}(z^2 a_p^* a_p^* + h.c)\Big) = \sum_{p\in\Lambda^*}\frac{1}{2}p^2 d_p^* d_p - \frac{|z|^4}{|\Lambda|^2}\sum_{p\in\Lambda^*}\frac{\widehat{g}(p)^2}{2p^2} a_p a_p^* \tag{6.19}$$

where

$$d_p = \frac{z^2\widehat{g}(p)}{|\Lambda|p^2} a_p^* + a_p.$$

Then the first term of (6.19) is positive and dropped for a lower bound and the second is bounded by a fraction of the last term of (6.18) if $m > 2\eta^{-1} + 14$, similarly as before. □

7 Bounds on the 3Q terms

In this section we prove the bounds on the 3Q terms stated in Lemma 2.4 and Theorem 2.12. We start with Lemma 2.4, which estimates the error made by replacing $Q_3^{\rm ren}$ in (2.13) by $Q_{3,L}$ in (2.21).

Proof of Lemma 2.4. From the definitions we have

$$Q_3^{\rm ren} - Q_{3,L} = \sum_{i\ne j}(P_i Q_j^H g(x_i - x_j) Q_j Q_i + \text{h.c.}). \tag{7.1}$$

In the right-hand side we aim at reconstructing $Q_4^{\rm ren}$ as

$$\sum_{i\ne j}(P_i Q_j^H g Q_j Q_i + \text{h.c.}) = \sum_{i\ne j}\big(P_i Q_j^H g\, \Pi_{ij} + \text{h.c.}\big) - \sum_{i\ne j} P_i Q_j^H g\omega(P_j P_i + P_j Q_i + Q_j P_i) + \text{h.c.} \tag{7.2}$$

We use a weighted Cauchy-Schwarz inequality on both terms. Using that $g \le v$, the first term in (7.2) is controlled by

$$C\delta^{-1}\sum_{i\ne j} P_i Q_j^H g Q_j^H P_i + \delta Q_4^{\rm ren} = C\delta^{-1}\widehat{g}(0)\frac{n_0 n_+^H}{|\Lambda|} + \delta Q_4^{\rm ren}$$

for all $\delta > 0$. The other terms can be estimated similarly to above. For instance,

$$\begin{aligned}\sum_{i\ne j}(P_i Q_j^H g\omega Q_j P_i + \text{h.c.}) &\le C\delta^{-1}\sum_{i\ne j} P_i Q_j^H g\omega Q_j^H P_i + \delta\sum_{i\ne j} P_i Q_j g\omega Q_j P_i \\ &\le C\widehat{g}(0)\frac{n_0}{|\Lambda|}\big(\delta^{-1}n_+^H + \delta n_+\big),\end{aligned} \tag{7.3}$$

where we used $\widehat{g\omega}(0) \le \widehat{g}(0)$. We collect the previous inequalities to obtain

$$|\langle Q_3^{\rm ren}\rangle_\Psi - \langle Q_{3,L}\rangle_\Psi| \le \delta\langle Q_4^{\rm ren}\rangle_\Psi + C\rho\widehat{g}(0)\big(\delta\langle n_+\rangle_\Psi + \delta^{-1}\langle n_+^H\rangle_\Psi\big), \tag{7.4}$$

where we bounded $n_0 \le N$. Choosing $\delta = \varepsilon K_\ell^{-2}$, the two last terms are bounded by spectral gaps if $K_H \ge CK_\ell^4$. □

Then $Q_{3,L}$ was symmetrized in Theorem 2.6, leading after c-number substitution to $Q_{3,L}^{\rm sym}(z)$, defined in (2.37). The following Lemma shows that in $Q_{3,L}^{\rm sym}(z)$, only the soft pairs contribute, up to errors controlled by spectral gaps.

Lemma 7.1. *For all $\varepsilon > 0$ there exists a $C > 0$ such that the following holds. Let v be a positive, radially symmetric potential with scattering length a and assume $\rho a^3 \le C^{-1}$. Then for all $|z|^2 \le K_\ell^{1/4} N$ and $\mathcal{M} \le C^{-1}\rho\ell^3 K_H^{-3} K_\ell^{-17/4}$ we have*

$$Q_{3,L}^{\rm sym}(z) - Q_3^{\rm soft}(z) \ge -\varepsilon \mathcal{G}(z). \tag{7.5}$$

where

$$Q_3^{\rm soft}(z) = \frac{1}{|\Lambda|} \sum_{p\in\mathcal{P}_L^{\mathbb{Z}}, k\in\mathcal{P}_H} c(p,k)\widehat{g}(k)\overline{z}a_p^* a_{p-k}a_k + \text{h.c.}, \tag{7.6}$$

and the spectral gaps $\mathcal{G}(z)$ are defined in (2.38).

Proof. Recalling the definition of $Q_{3,L}^{\rm sym}(z)$ in (2.37) with $\mathcal{P}_L^{\mathbb{Z}} = \{p \in \frac{\pi}{\ell}\mathbb{Z}^3,\ \ 0 < |p| \le \frac{K_H}{\ell}\}$ we take the difference

$$Q_{3,L}^{\rm sym}(z) - Q_3^{\rm soft}(z) = \frac{1}{|\Lambda|} \sum_{\substack{p\in\mathcal{P}_L^{\mathbb{Z}}, k\in\mathcal{P}_L\\ p\neq k}} c(p,k)\widehat{g}(k)\big(\overline{z}a_p^* a_{p-k}a_k + \text{h.c.}\big). \tag{7.7}$$

We use the Cauchy-Schwarz inequality with weight $\delta > 0$ and deduce

$$Q_{3,L}^{\rm sym}(z) - Q_3^{\rm soft}(z) \ge -C\frac{\widehat{g}(0)}{|\Lambda|} \sum_{\substack{p\in\mathcal{P}_L^{\mathbb{Z}}, k\in\mathcal{P}_L\\ p\neq k}} \big(\delta|z|^2 a_p^* a_p + \delta^{-1} a_k^* a_{p-k}^* a_{p-k} a_k\big), \tag{7.8}$$

where we bounded the $c(p,q)$ by a constant.

Recall that the commutation relation $[a_p, a_q^*] = \delta_{p,q}$ only holds when p and q are in $\pi\ell^{-1}\mathbb{N}_0^3$. To deal with the negative components that can appear due to the set where p belongs, we use that $a_p = a_{(|p_1|,|p_2|,|p_3|)}$. In particular, note that

$$\sum_{p\in\mathcal{P}_L^{\mathbb{Z}}} a_p^* a_p \le 8 \sum_{p\in\mathcal{P}_L} a_p^* a_p, \tag{7.9}$$

and therefore the first term of (7.8) is bounded by n_+ and a cardinal of $\mathcal{P}_L$. Similarly in the second term, we bound the p-sum by Cn_+ and the k-sum by n_+^L. Choosing $\delta = \varepsilon C^{-1} K_\ell^{-9/4} K_H^{-3}$ and using that $\rho_z \le \rho K_\ell^{1/4}$ we get

$$Q_{3,L}^{\rm sym}(z) - Q_3^{\rm soft}(z) \ge -C\delta\rho\widehat{g}(0)K_\ell^{\frac{1}{4}} K_H^3 n_+ - C\delta^{-1}\rho\widehat{g}(0)\frac{n_+ n_+^L}{\rho\ell^3} \tag{7.10}$$

$$\ge -\varepsilon\frac{n_+}{\ell^2} - C\frac{K_H^3 K_\ell^5}{\varepsilon\rho\ell^3}\frac{n_+ n_+^L}{\ell^2}. \tag{7.11}$$

These errors can be absorbed in spectral gaps if $\mathcal{M} \le C^{-1}\rho\ell^3 K_H^{-3} K_\ell^{-17/4}$. □

In order to prove Theorem 2.12, we need the following approximations of $\widehat{g\omega}(0)$.

Lemma 7.2. *The following estimates hold.*

1. *For all $\Psi \in \mathscr{F}^\perp$,*

$$\Big|\sum_{p\in\Lambda_+^*} \widehat{g\omega}(p)\langle a_p^* a_p\rangle_\Psi - \widehat{g\omega}(0)\langle n_+\rangle_\Psi\Big| \le C\widehat{g}(0)\langle n_+^H\rangle_\Psi + C\widehat{g}(0)K_H^2 R^2 \ell^{-2}\langle n_+\rangle_\Psi.$$

2. *Moreover,*

$$\Big|\widehat{g\omega}(0) - \frac{1}{8|\Lambda|}\sum_{k\in\frac{\pi}{\ell}\mathbb{Z}^3\setminus\{0\}} \frac{\widehat{g}(k)^2}{2k^2}\Big| \le Ca^2\ell^{-1} \tag{7.12}$$

$$\frac{1}{|\Lambda|}\sum_{k\in\mathcal{P}_L^{\mathbb{Z}}} \frac{\widehat{g}(k)^2}{2k^2} \le CK_H a^2 \ell^{-1}, \tag{7.13}$$

Proof. 1. For the first estimate we split the sum into high and low momenta,

$$\sum_{p\in\Lambda_+^*} \widehat{g\omega}(p) a_p^* a_p =: \mathcal{S}(\mathcal{P}_L) + \mathcal{S}(\mathcal{P}_H). \tag{7.14}$$

The high momenta part is controlled by n_+^H,

$$\mathcal{S}(\mathcal{P}_H) = \sum_{p\in\mathcal{P}_H} \widehat{g\omega}(p) a_p^* a_p \leq \widehat{g}(0) n_+^H, \tag{7.15}$$

where we used $\widehat{g\omega}(0) \leq \widehat{g}(0)$. For low momenta, we use a Taylor approximation $\widehat{g\omega}(p) \simeq \widehat{g\omega}(0) + \mathcal{O}(p^2 \widehat{g}(0) R^2)$, which follows using $\widehat{g\omega}'(0) = 0$ (by radiality) and $|\widehat{g\omega}''(p)| \leq \widehat{g}(0) R^2$ (since $\mathrm{supp}(g) \subset B(0,R)$). This way

$$\big|\mathcal{S}(\mathcal{P}_L) - \widehat{g\omega}(0) n_+^L\big| \leq C\widehat{g}(0) R^2 \sum_{p\in\mathcal{P}_L} |p|^2 a_p^* a_p \leq C\widehat{g}(0) K_H^2 R^2 \ell^{-2} n_+. \tag{7.16}$$

Finally, the remaining $\widehat{g\omega}(0) n_+^H$ is controlled by $\widehat{g}(0) n_+^H$.

2. To prove the second estimate, we introduce a cutoff version of the scattering function. Let χ be a smooth and radial function so that $0 \leq \chi \leq 1$ and $\chi(x) = 1$ for $|x| \leq 1/3$ and $\chi(x) = 0$ for $|x| > 1/2$. Let us define $\omega_c(x) = \omega(x)\chi(x/\ell)$, it has support inside $\Lambda_{\ell/2}$ and satisfies

$$-\Delta\omega_c = \frac{1}{2} g - \frac{a}{\ell^3} \left(\frac{\chi''}{|\cdot|}\right)(x\ell^{-1}).$$

Denoting $U = \frac{\chi''}{|\cdot|}$, which is smooth and compactly supported, we have

$$p^2 \widehat{\omega}_c(p) = \frac{1}{2}\widehat{g}(p) - a\widehat{U}(\ell p).$$

Using that $\chi(x/\ell) = 1$ for $x \in \mathrm{supp}\, g$ we have that $\widehat{g\omega}(0) = \widehat{g\omega_c}(0)$. The Plancherel formula then gives

$$\left| \int g\omega_c - \frac{1}{8\ell^3} \sum_{k\in\frac{\pi}{\ell}\mathbb{Z}^3\setminus\{0\}} \frac{\widehat{g}(k)^2}{2|k|^2} \right| \leq \frac{1}{8\ell^3} \sum_{k\in\frac{\pi}{\ell}\mathbb{Z}^3\setminus\{0\}} a \left| \frac{\widehat{g}(k)\widehat{U}(k\ell)}{|k|^2} \right| \leq Ca^2 \ell^{-1},$$

where we used that $\|\widehat{g}\|_\infty \leq Ca$. This proves (7.12). The bound (7.13) follows by using again that $\|\widehat{g}\|_\infty \leq Ca$ and the definition of $\mathcal{P}_L^{\mathbb{Z}}$. □

Now we can prove Theorem 2.12, combining Q_3^{soft} with the remaining quadratic part of the Hamiltonian after the Bogoliubov diagonalization. This process also uses a fraction of the diagonalized Hamiltonian.

Proof of Theorem 2.12. First of all, we can use Lemma 7.1 to replace $Q_{3,L}^{\mathrm{sym}}(z)$ by $Q_3^{\mathrm{soft}}(z)$ defined in (7.6). In this expression we replace a_k by b_k using (2.40) and find

$$\begin{aligned} Q_3^{\mathrm{soft}}(z) &= \frac{1}{|\Lambda|} \sum_{\substack{k\in\mathcal{P}_H \\ p\in\mathcal{P}_L^{\mathbb{Z}}}} \frac{c(p,k)\widehat{g}(k)}{\sqrt{1-\alpha_k^2}} \big(\overline{z} a_p^* a_{p-k} b_k + \mathrm{h.c.}\big) - \frac{1}{|\Lambda|} \sum_{\substack{k\in\mathcal{P}_H \\ p\in\mathcal{P}_L^{\mathbb{Z}}}} \frac{c(p,k)\widehat{g}(k)\alpha_k}{\sqrt{1-\alpha_k^2}} \big(\overline{z} a_p^* a_{p-k} b_k^* + \mathrm{h.c.}\big). \\ &=: \mathcal{T}_1 - \mathcal{T}_\alpha. \end{aligned} \tag{7.17}$$

The second term $\mathcal{T}_\alpha$ is an error, which we can bound as follows, using a Cauchy-Schwarz inequality,

$$\begin{aligned} \mathcal{T}_\alpha &= \frac{1}{|\Lambda|} \sum_{\substack{k\in\mathcal{P}_H \\ p\in\mathcal{P}_L^{\mathbb{Z}}}} \frac{c(p,k)\widehat{g}(k)\alpha_k}{\sqrt{1-\alpha_k^2}} \big(\overline{z} a_p^* a_{p-k} b_k^* + \mathrm{h.c.}\big) \\ &\leq \frac{C}{|\Lambda|} \sum_{\substack{k\in\mathcal{P}_H \\ p\in\mathcal{P}_L^{\mathbb{Z}}}} \frac{\widehat{g}(k)\alpha_k}{\sqrt{1-\alpha_k^2}} \big(\rho a k^{-2} K_H^{-2} K_\ell^{-4} |z|^2 b_k b_k^* + (\rho a)^{-1} k^2 K_H^2 K_\ell^4 a_p^* a_{p-k} a_{p-k}^* a_p\big). \end{aligned}$$

We then use that $|\alpha_k| \leq C\rho_z a k^{-2} \leq CK_\ell^{1/4}\rho a k^{-2}$ and $|\mathcal{P}_L^{\mathbb{Z}}| \leq CK_H^3$ to obtain

$$\begin{aligned}\mathcal{T}_\alpha &\leq C\frac{N\rho^2 a^3 K_H K_\ell^{-4+1/2}}{|\Lambda|}\sum_{k\in\mathcal{P}_H}\frac{1}{k^4}(b_k^* b_k + 1) + C\frac{aK_H^2 K_\ell^5}{|\Lambda|}\sum_{\substack{k\in\mathcal{P}_H\\ p\in\mathcal{P}_L^{\mathbb{Z}}}} a_p^* a_p(a_{p-k}^* a_{p-k}+1)\\ &\leq CK_H^{-5}K_\ell^{5/2}\sum_{k\in\mathcal{P}_H} k^2 b_k^* b_k + CN\rho a\sqrt{\rho a^3}K_\ell^{-1} + C\mathcal{M}\frac{aK_H^2K_\ell^5}{\ell}\frac{n_+^L n_+}{\mathcal{M}\ell^2},\end{aligned}$$

where in the last inequality we used $k^6 \geq K_H^6\ell^{-6}$. Since $CK_H^{-5}K_\ell^{5/2} \leq K_H^{-1}$ and $k^2 \leq CD_k$, the first term is absorbed in a fraction of the Bogoliubov Hamiltonian, and the last term is absorbed in a fraction of the spectral gaps if $\mathcal{M} \leq C^{-1}\rho\ell^3 K_\ell^{-7}K_H^{-2}$, which is guaranteed by the assumptions on $\mathcal{M}$ and $K_H \geq K_\ell^4$. We can now focus in $\mathcal{T}_1$. Using the remaining fraction of the diagonalized Hamiltonian, we complete a square to find

$$\mathcal{T}_1 + (1-2K_H^{-1})\sum_{k\in\mathcal{P}_H} D_k b_k^* b_k = \sum_{k\in\mathcal{P}_H}(1-2K_H^{-1})D_k(b_k+A_k)^*(b_k+A_k) - \sum_{k\in\mathcal{P}_H}(1-2K_H^{-1})D_k A_k^* A_k, \tag{7.18}$$

where

$$A_k = \frac{z\widehat{g}(k)}{|\Lambda|(1-2K_H^{-1})D_k\sqrt{1-\alpha_k^2}}\sum_{p\in\mathcal{P}_L^{\mathbb{Z}}} c(p,k)a_{p-k}^* a_p.$$

The first term in (7.18) is positive and can be dropped for a lower bound. We are left with a term in $A_k^* A_k$, which we can rewrite in normal order as

$$\begin{aligned}(1-2K_H^{-1})\sum_{k\in\mathcal{P}_H} D_k A_k^* A_k = \sum_{k\in\mathcal{P}_H}\frac{\rho_z\widehat{g}(k)^2}{|\Lambda|(1-2K_H^{-1})D_k(1-\alpha_k^2)}\Big(&\sum_{p,s\in\mathcal{P}_L^{\mathbb{Z}}} c(p,k)c(s,k)a_p^*[a_{p-k},a_{s-k}^*]a_s\\ &+\sum_{p,s\in\mathcal{P}_L^{\mathbb{Z}}} c(p,k)c(s,k)a_p^* a_{s-k}^* a_{p-k}a_s\Big).\end{aligned} \tag{7.19}$$

We call the two terms of above $\mathcal{T}_c$ for the commutator term and $\mathcal{T}_0$ for the other one, so that

$$\mathcal{T}_1 + (1-2K_H^{-1})\sum_{k\in\mathcal{P}_H} D_k b_k^* b_k \geq -\mathcal{T}_c - \mathcal{T}_0. \tag{7.20}$$

We start by estimating the main term $\mathcal{T}_c$, and then we bound the error term $\mathcal{T}_0$.

Commutator term $\mathcal{T}_c$. Recall that the commutation $[a_p, a_q^*] = \delta_{p,q}$ only applies when $p,q \in \frac{\pi}{\ell}\mathbb{N}_0^3$. In the above commutators, due to the sets on which we sum, this may not be the case. We can however use that $a_p = a_{p^+}$ where $p^+ = (|p_1|,|p_2|,|p_3|)$, and deduce that

$$[a_{p-k}, a_{s-k}^*] \neq 0 \Leftrightarrow \Big(p_j = s_j \quad\text{or}\quad 2k_j = p_j + s_j, \quad \forall j = 1,2,3.\Big) \tag{7.21}$$

For p, $s \in \mathcal{P}_L$, the second case in (7.21) implies $|k_j| \leq K_H\ell^{-1}$. Therefore, if $p \neq s$,

$$[a_{p-k}, a_{s-k}^*] \neq 0 \Rightarrow \Big(|k_j| \leq K_H\ell^{-1} \quad \text{for some } j\Big)$$

and we deduce

$$\begin{aligned}\mathcal{T}_c &= \sum_{k\in\mathcal{P}_H}\frac{\rho_z\widehat{g}(k)^2}{|\Lambda|(1-2K_H^{-1})D_k(1-\alpha_k^2)}\sum_{p,s\in\mathcal{P}_L^{\mathbb{Z}}} c(p,k)c(s,k)a_p^*[a_{p-k},a_{s-k}^*]a_s\\ &\leq \sum_{k\in\mathcal{P}_H}\frac{\rho_z\widehat{g}(k)^2}{|\Lambda|(1-2K_H^{-1})D_k(1-\alpha_k^2)}\sum_{p\in\mathcal{P}_L^{\mathbb{Z}}} c(p,k)^2 a_p^* a_p + \sum_{k\in\mathcal{P}_H}\frac{C\rho_z\widehat{g}(k)^2\mathbf{1}_{\{|k_1|\leq K_H\ell^{-1}\}}}{|\Lambda|(1-2K_H^{-1})D_k(1-\alpha_k^2)}\sum_{p\in\mathcal{P}_L^{\mathbb{Z}}} a_p^* a_{s_{p,k}},\end{aligned} \tag{7.22}$$

where the components of $s_{p,k}$ are either equal to p_j or $2k_j - p_j$. In any case, we can always bound the last p-sum by n_+ using a Cauchy-Schwarz inequality. We also use $CD_k(1-\alpha_k^2) \geq k^2$ to get

$$\mathcal{T}_c \leq \big(1+CK_H^{-1}\big)\frac{1}{|\Lambda|}\sum_{k\in\mathcal{P}_H}\frac{\rho_z\widehat{g}(k)^2}{k^2}\sum_{p\in\mathcal{P}_L^{\mathbb{Z}}} c(p,k)^2 a_p^* a_p + \frac{C}{|\Lambda|}\sum_{k\in\mathcal{P}_H}\frac{\rho_z\widehat{g}(k)^2}{k^2}\mathbf{1}_{\{|k_1|\leq K_H\ell^{-1}\}}n_+. \tag{7.23}$$

We recall that the normalization coefficients $c(p,k) \in [\frac{1}{\sqrt{8}}, \sqrt{8}]$ are defined in (6.3). We write them as $c(p,k) = \frac{c_{p-k}}{c_k c_p}$ with the notation $c_k = c_{k_1} c_{k_2} c_{k_3}$. Note that the c_p's (defined in (2.23)) are such that

$$\sum_{p \in \mathcal{P}_L^{\mathbb{Z}}} c_p^{-2} a_p^* a_p = \sum_{p \in \mathcal{P}_L} a_p^* a_p = n_+^L. \tag{7.24}$$

We can also bound $c_{p-k}^2 \leq 8$ and we deduce

$$\mathcal{T}_c \leq \left(1 + CK_H^{-1}\right) \frac{1}{|\Lambda|} \sum_{k \in \mathcal{P}_H} \frac{8\rho_z \widehat{g}(k)^2}{c_k^2 k^2} n_+^L + \frac{C}{|\Lambda|} \sum_{k \in \mathcal{P}_H} \frac{\rho_z \widehat{g}(k)^2}{k^2} \mathbf{1}_{\{|k_1| \leq K_H \ell^{-1}\}} n_+. \tag{7.25}$$

Similarly as in (7.24), we can complete the k-sum to $\mathcal{P}_H^{\mathbb{Z}}$ up to an extra factor c_k^{-2}. Moreover, $c_k^4 = 8^2$ unless at least one of the components k_j vanishes. Therefore, the terms for which $c_k^4 \neq 8^2$ can be controlled by the last term in (7.25), and we obtain

$$\mathcal{T}_c \leq \left(1 + CK_H^{-1}\right) \frac{1}{|\Lambda|} \sum_{k \in \mathcal{P}_H^{\mathbb{Z}}} \frac{\rho_z \widehat{g}(k)^2}{8k^2} n_+^L + \frac{C}{|\Lambda|} \sum_{k \in \mathcal{P}_H} \frac{\rho_z \widehat{g}(k)^2}{k^2} \mathbf{1}_{\{|k_1| \leq K_H \ell^{-1}\}} n_+. \tag{7.26}$$

In the first term we use point 2. of Lemma 7.2 to replace the k-sum by $2\widehat{g\omega}(0) \leq C\rho a \leq C\ell^{-2} K_\ell^2$. The second term is bounded in Lemma 7.3 below, and we get

$$\mathcal{T}_c \leq 2\rho_z \widehat{g\omega}(0) n_+ + C\Big(K_\ell^2 K_H^{-1} + (\rho a^3)^{1/2} K_H^2\Big) K_\ell^{1/4} \ell^{-2} n_+. \tag{7.27}$$

By point 1. of Lemma 7.2, the first term of above is precisely the remaining quadratic term we want to cancel in Theorem 2.12. The second one is absorbed in spectral gaps when $K_H \geq K_\ell^4$ and $K_H^2 K_\ell^{1/4} \leq C^{-1} (\rho a^3)^{-\frac{1}{2}}$.

The error term $\mathcal{T}_0$. It only remains to control $\mathcal{T}_0$. We use similar bounds as for $\mathcal{T}_c$ and a Cauchy Schwarz inequality and find

$$\begin{aligned} \mathcal{T}_0 &= \sum_{k \in \mathcal{P}_H} \frac{\rho_z \widehat{g}(k)^2}{|\Lambda|(1 - 2K_H^{-1}) D_k (1 - \alpha_k^2)} \sum_{p,s \in \mathcal{P}_L^{\mathbb{Z}}} c(p,k) c(s,k) a_p^* a_{s-k}^* a_{p-k} a_s \\ &\leq \frac{C}{|\Lambda|} \sum_{k \in \mathcal{P}_H} \frac{\rho_z \widehat{g}(0)^2}{k^2} \sum_{p,s \in \mathcal{P}_L^{\mathbb{Z}}} a_p^* a_{s-k}^* a_{s-k} a_p \leq C \frac{\widehat{g}(0) K_\ell^3}{|\Lambda| K_H^2} \sum_{k \in \mathcal{P}_H} \sum_{p,s \in \mathcal{P}_L^{\mathbb{Z}}} a_p^* a_{s-k}^* a_{s-k} a_p, \end{aligned} \tag{7.28}$$

where in the second inequality we used $|k| \geq K_H \ell^{-1}$ and $\rho_z \leq 2\rho$. The k-sum can be bounded by n_+, the p-sum by n_+^L, and remains the cardinal of $\mathcal{P}_L^{\mathbb{Z}}$, i.e.

$$\mathcal{T}_0 \leq \frac{C}{|\Lambda|} \widehat{g}(0) K_\ell^3 K_H n_+ n_+^L. \tag{7.29}$$

This is absorbed in spectral gaps under the condition $\mathcal{M} \leq C^{-1} \rho \ell^3 K_\ell^{-5} K_H^{-1}$, for C large enough. □

Lemma 7.3. *Under the assumptions of Theorem 2.12 we have*

$$\frac{1}{|\Lambda|} \sum_{k \in \mathcal{P}_H} \frac{\rho \widehat{g}(k)^2}{k^2} \mathbf{1}_{\{|k_1| \leq K_H \ell^{-1}\}} \leq C\ell^{-2} (\rho a^3)^{1/2} K_H^2.$$

Proof. We first remove the very high momenta from the sum, for $|k| > K_0 \ell^{-1}$, with $K_0 > 0$ to be chosen later,

$$\frac{1}{|\Lambda|} \sum_{|k| > \frac{K_0}{\ell}} \frac{\rho \widehat{g}(k)^2}{k^2} \mathbf{1}_{\{|k_1| \leq K_H \ell^{-1}\}} \leq \frac{C\rho \ell^2 \|g\|_2^2}{K_0^2} \leq C \frac{\rho \ell^4}{K_0^2 a^3} = C\ell^{-2} K_\ell^6 (\rho a^3)^{-2} K_0^{-2} \tag{7.30}$$

where in the last inequality we used that $\|g\|_2^2 \leq \|g\|_\infty \|g\|_1 \leq C\ell^2 a^{-3}$. We now deal with the rest

$$\frac{1}{|\Lambda|} \sum_{|k| \leq \frac{K_0}{\ell}} \frac{\rho \widehat{g}(k)^2}{k^2} \mathbf{1}_{\{|k_1| \leq K_H \ell^{-1}\}} \leq C\rho a^2 \ell^{-1} K_H \log K_0 = C\ell^{-2} (\rho a^3)^{1/2} K_H K_\ell^{-1} \log K_0.$$

Choosing $K_0 = K_\ell^3 (\rho a^3)^{-5/4}$ and using that $\log K_0 \leq K_H$ concludes the proof. □

A Estimates on the LHY terms

Lemma A.1. *There exists a $C > 0$ such that, if $|z|^2 \leq K_\ell^{1/4} N \leq C\rho\ell^3 K_\ell^{1/4}$, we have*

$$\rho_z^2 \widehat{g\omega}(0) + \frac{1}{|\Lambda|} \sum_{p \in \Lambda_+^*} \Big(\sqrt{\tau(p)^2 + 2\rho_z \widehat{g}(p)\tau(p)} - \tau(p) - \rho_z \widehat{g}(p) \Big) = 8\pi(\rho_z a)^{5/2} \frac{128}{15\sqrt{\pi}} + \mathcal{O}((\rho a)^{5/2} K_\ell^{-1/4}),$$

We recall that $\tau(p)$ is defined in (2.34).

Proof. Using (7.12) and that $|p^{-2} - \tau_p^{-1}| \leq Cp^{-4}\ell^{-2}(1 + K_H \mathbf{1}_{p \in \mathcal{P}_H})$, we can rewrite the left-hand side of the above equation as

$$\sum_{p \in \Lambda_+^*} \Big(\sqrt{\tau(p)^2 + 2\rho_z \widehat{g}(p)\tau(p)} - \tau(p) - \rho_z \widehat{g}(p) + \frac{(\rho_z \widehat{g}(p))^2}{2\tau(p)} \Big) + \mathcal{O}(|\Lambda|(\rho a)^{5/2} K_\ell^{-1/2}),$$

where the error can be absorbed in $\mathcal{E}$. To estimate the sum, we define $G(t) = \sqrt{1+2t} - 1 - t + t^2/2 \geq 0$, which is such that $xG(y/x) = \sqrt{x^2 + 2xy} - x - y - x/(2y)$. Let us introduce a cut-off $1 < K < K_H$ which we will choose at the end. Using that $G(t) \leq Ct^3$, we have for $\rho_z \leq K_\ell^{1/4}\rho$,

$$\sum_{|p| > K\ell^{-1}} \tau(p) G\Big(\frac{\rho_z \widehat{g}(p)}{\tau(p)}\Big) + \sum_{|p| > K\ell^{-1}} p^2 G\Big(\frac{8\pi\rho_z a}{p^2}\Big) \leq CK_\ell^{3/4}(\rho a)^3 \sum_{|p| > K\ell^{-1}} \frac{1}{p^4} \leq C\ell^3 (\rho a)^{5/2} K_\ell^{7/4} K^{-1},$$

where we recall that the sums are over $p \in 2\pi\ell^{-1}\mathbb{Z}^3$. Let us now deal with $|p| \leq K\ell^{-1}$. Note that $G(t) \leq Ct^2$ and $|G'(t)| \leq C(1+t)$ for some $C > 0$ and all $t \geq 0$, so that

$$\begin{aligned} \Big|(\tau(p) - p^2) G\Big(\frac{\rho_z \widehat{g}(p)}{\tau(p)}\Big)\Big| &\leq CK_\ell^{1/2}(\rho a)^2 \ell^{-2} p^{-4} \\ p^2 \Big| G\Big(\frac{\rho_z \widehat{g}(p)}{\tau(p)}\Big) - G\Big(\frac{8\pi a \rho_z}{p^2}\Big)\Big| &\leq CK_\ell^{1/4}(\rho a) p^2 (1 + K_\ell^{1/4}\rho a p^{-2})(p^{-4}\ell^{-2} + R^2) \end{aligned}$$

where we used that $|\tau(p) - p^2| \leq C\ell^{-2}$ for $|p| \leq K\ell^{-1}$ and that $|\widehat{g}(p) - \widehat{g}(0)| \leq R^2 \widehat{g}(0)|p|^2$. We obtain that

$$\begin{aligned} \Big| \sum_{|p| \leq K\ell^{-1}} \tau(p) G\Big(\frac{\rho_z \widehat{g}(p)}{\tau(p)}\Big) - p^2 G\Big(\frac{8\pi\rho_z a}{p^2}\Big) \Big| &\leq CK_\ell^{1/4}(\rho a) \Big(\ell^{-2} R^2 K^5 + \rho a R^2 K_\ell^{1/4} K^3 + K + K_\ell^{9/4} \Big) \\ &\leq C\ell^3 (\rho a)^{5/2} K_\ell^{-\frac{11}{4}} \Big(\rho a^3 K^5 K_\ell^{-2} + \rho a^3 K^3 K_\ell^{1/2} + K + K_\ell^{9/4} \Big) \end{aligned}$$

where we used that $R \leq Ca$. Choosing $K = K_\ell^2$ and using that $K_\ell \leq C(\rho a^3)^{1/10}$ we can absorb the error terms in $\mathcal{E}$. Next, we approximate the sum by the integral, recalling that $\Lambda^* = \pi\ell^{-1}\mathbb{N}_0^3$, one easily checks that

$$\Big| \frac{\pi^3}{\ell^3} \sum_{p \in \Lambda_+^*} p^2 G\Big(\frac{8\pi\rho_z a}{p^2}\Big) - 8 \int_{\mathbb{R}_+^3} p^2 G\Big(\frac{8\pi\rho_z a}{p^2}\Big) dp \Big| \leq C(\rho_z a)^3 \ell^{-1} \leq CK_\ell^{-1/4}(\rho a)^{5/2}.$$

Finally, it is a standard result that $\int_{\mathbb{R}^3} p^2 G\Big(\frac{8\pi\rho_z a}{p^2}\Big) = -64\pi^4 \frac{128}{15\sqrt{\pi}} (\rho_z a)^{5/2}$ (see for instance [14]). ☐

Recall that

$$\tilde{D}_p(z) = \begin{cases} D_p(z) & \text{if } \ p \notin \mathcal{P}_H, \\ K_H^{-1} D_p(z) & \text{if } \ p \in \mathcal{P}_H. \end{cases}, \qquad D_p(z) := \sqrt{\tau(p)^2 + 2\tau(p)|z|^2 \ell^{-3} \widehat{g}(p)}. \tag{A.1}$$

Lemma A.2. *If $\nu < 2\eta < 1/5$ and $|z|^2 \leq CK_\ell^{1/4}\rho\ell^3$, then we have*

$$T \sum_{p \in \Lambda_+^*} \log(1 - e^{-\frac{1}{T}\tilde{D}_p(z)}) \geq T \sum_{p \in \Lambda_+^*} \log(1 - e^{-\frac{1}{T}\omega_p(z)}) - C|\Lambda|(\rho a)^3,$$

where $\omega_p(z) = \sqrt{p^4 + 16\pi a|z|^2\ell^{-3}p^2}$.

Proof. First note that $D_p(z) \geq p^2/4$ which gives

$$T \sum_{p\in\mathcal{P}_H} \log(1 - e^{-\frac{1}{T}\tilde{D}_p(z)}) \geq -T \sum_{p\in\mathcal{P}_H} e^{-p^2/(4TK_H)} \geq -CT^{5/2}\ell^3 K_H^{3/2} e^{-CK_H/(T\ell^2)},$$

which is smaller than the expected error since $K_H/(T\ell^2)$ is big. Similarly, we can neglect the term $T\sum_{p\in\mathcal{P}_H} \log(1-e^{-\frac{1}{T}\omega_p(z)})$. Now note that there is a constant $C>0$ such that, for all $p \in \mathcal{P}_L$,

$$\left|D_p - \omega_p\right| \leq C\Big(1 + \frac{\sqrt{|z|^2\ell^{-3}a}}{|p|}\Big)|p^2 - \tau(p)| \leq C\frac{\sqrt{\rho a}K_\ell^{\frac{1}{8}}}{|p|}\frac{1}{\ell^2} \tag{A.2}$$

Moreover, denoting $G(x) = \log(1-e^{-x})$, we have $G'(x) = \frac{1}{e^x - 1}$ and a Taylor expansion gives

$$\begin{aligned} T\sum_{p\in\mathcal{P}_L} \log(1-e^{-\frac{1}{T}D_p(z)}) - T\sum_{p\in\mathcal{P}_L}\log(1-e^{-\frac{1}{T}\omega_p(z)}) &\geq -C\frac{\sqrt{\rho a}K_\ell^{1/8}}{\ell^2}\sum_{p\in\mathcal{P}_L}\frac{1}{|p|(e^{\frac{p^2}{4T}}-1)} \\ &\geq -C\frac{\sqrt{\rho a}K_\ell^{1/8}}{\ell^2}T^2\ell^3\log(T^{1/2}\ell) \\ &\geq -C|\Lambda|\log(\rho a^3)(\rho a^3)^{7/2+15\eta/8-2\nu} \end{aligned}$$

and the result follows from the assumptions on η and ν. □

B Reduction to small boxes: proof of Theorem 1.1

In this section we prove Theorem 1.1 in the thermodynamic limit using the lower bound on small boxes from Theorem 1.3. We follow the proof of [17, Theorem 1.1]. It applies to our case, but we want to keep track of our parameters. The limit (1.4) is independent of the choice of N and L, we can therefore choose L such that L/ℓ is an integer and define $M = (L/\ell)^3$. We can now divide Λ_L into M translates of Λ_ℓ. From [17, Eq. (9.6)], we obtain immediately that, for all $\mu \in \mathbb{R}$,

$$F(L,N) \geq -TM\log\sum_{n=0}^{N} e^{-\frac{1}{T}(F(\ell,n)-\mu n)} + \mu\rho L^3. \tag{B.1}$$

For $n \leq 20\rho\ell^3$, we use Theorem 1.3 to obtain

$$F(\ell,n) - \mu n \geq F_{\mathrm{Bog}}(\ell,n) - \mu n - C\ell^3(\rho a)^{5/2}(\rho a^3)^{\eta/4},$$

where we recall that

$$F_{\mathrm{Bog}}(\ell,n) = 4\pi\rho_{n,\ell}^2 a\ell^3\Big(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho_{n,\ell}a^3}\Big) + T\sum_{p\in\Lambda_+^*}\log\Big(1 - e^{-\frac{1}{T}\sqrt{p^4+16\pi a\rho_{n,\ell}p^2}}\Big).$$

It is easy to check that it is convex in n for ρa^3 small enough using to the following lemma.

Lemma B.1. *There exists $C>0$ such that for all $\rho \geq 0$, $\ell, T > 0$,*

$$0 \leq \frac{\partial}{\partial\rho}T\sum_{p\in\Lambda_+^*}\log(1-e^{-T^{-1}\sqrt{p^4+16\pi a\rho p^2}}) \leq CT^{5/2}\ell^3, \tag{B.2}$$

$$0 \leq -\frac{\partial^2}{\partial\rho^2}T\sum_{p\in\Lambda_+^*}\log(1-e^{-T^{-1}\sqrt{p^4+16\pi a\rho p^2}}) \leq CT^{5/2}\ell^3. \tag{B.3}$$

Choosing μ so that $\frac{\partial}{\partial n}F_{\mathrm{Bog}}(\ell,\rho\ell^3) = \mu$, it follows that

$$F_{\mathrm{Bog}}(\ell,n) - \mu n \geq F_{\mathrm{Bog}}(\ell,\rho\ell^3) - \mu\rho\ell^3 \tag{B.4}$$

for all $n \geq 1$ and one can easily check from (B.2) that

$$|\mu(\rho,a,T,\ell) - 8\pi a\rho| \leq C\rho a(\rho a^3)^{1/2-5\nu/2} \tag{B.5}$$

which gives

$$F(\ell,n)-\mu n\geq F_{\mathrm{Bog}}(\ell,\rho\ell^3)-\mu\rho\ell^3-C\ell^3(\rho a)^{5/2}(\rho a^3)^{\eta/4}. \tag{B.6}$$

For $n>20\rho\ell^3$, denoting $n_0=\lfloor 5\rho\ell^3\rfloor$ and $r_0=n-\left\lfloor\frac{n}{n_0}\right\rfloor n_0$, we use the subadditivity of the free energy $F(\ell,n)$ to obtain

$$\begin{aligned}F(\ell,n)-\mu n&\geq\left\lfloor\frac{n}{n_0}\right\rfloor(F(\ell,n_0)-\mu n_0)+(F(\ell,r_0)-\mu r_0)\\&\geq\left\lfloor\frac{n}{n_0}\right\rfloor(F_{\mathrm{Bog}}(\ell,n_0)-\mu n_0)+(F_{\mathrm{Bog}}(\ell,r_0)-\mu r_0)-C\left\lfloor\frac{n}{n_0}\right\rfloor\ell^3(\rho a)^{5/2}(\rho a^3)^{\eta/4},\end{aligned}$$

where we used Theorem 1.3 to obtain the second inequality, since $n_0,r\leq 20\rho\ell^3$. To estimate the first term, we use the bound (B.5) on μ and that, for $T\leq C\rho a(\rho a^3)^{-\nu}$,

$$F_{\mathrm{Bog}}(\ell,n_0)-\mu n_0\geq 4\pi a\big(\frac{n_0}{\ell^3}\big)^2\ell^3-8\pi a\rho n_0-Ca\rho^2\ell^3(\rho a^3)^{1/2-5\nu/2}\geq 10\pi a\rho^2\ell^3,$$

for ρa^3 small enough. Therefore, estimating $F_{\mathrm{Bog}}(\ell,r_0)$ with (B.4) we obtain for $n>n_0$,

$$\begin{aligned}F(\ell,n)-\mu n&\geq F_{\mathrm{Bog}}(\ell,\rho\ell^3)-\mu\rho\ell^3+10\left\lfloor\frac{n}{n_0}\right\rfloor\pi a\rho^2\ell^3\\&\geq F_{\mathrm{Bog}}(\ell,\rho\ell^3)-\mu\rho\ell^3+na\rho.\end{aligned}\tag{B.7}$$

Hence, from (B.1) and the bounds (B.6) and (B.7), we obtain

$$\begin{aligned}\frac{1}{L^3}F(L,N)-\frac{1}{\ell^3}F_{\mathrm{Bog}}(\ell,\rho\ell^3)&\geq-\frac{T}{\ell^3}\log\left(\sum_{n\leq 20\rho\ell^3}e^{C\frac{1}{T}\ell^3(\rho a)^{5/2}(\rho a^3)^{\eta/4}}+\sum_{n>20\rho\ell^3}e^{-n\frac{\rho a}{T}}\right)\\&\geq-(\rho a)^{5/2}(\rho a^3)^{-\nu}K_\ell^{-3}\log(n_0e^{C\ell^3\frac{(\rho a)^{5/2}}{T}(\rho a^3)^{\eta/4}}+e^{-n_0\frac{\rho a}{T}}/(1-e^{-\frac{\rho a}{T}}))\\&\geq-C(\rho a)^{5/2}(\rho a^3)^{\eta/4}+C\frac{T}{\ell^3}\log(\rho a^3).\end{aligned}$$

where we used that $e^{-n_0\frac{\rho a}{T}}/(1-e^{-\frac{\rho a}{T}})\leq 1$ and that $n_0\simeq(\rho a^3)^{-1/2}K_\ell^3$. Now from [17, Lemma 9.1], we obtain

$$\begin{aligned}&\left|\frac{T}{\ell^3}\sum_{p\in\frac{\pi}{\ell}\mathbf{N}_0^3}\log\left(1-e^{-\frac{1}{T}\sqrt{p^4+16\pi a\rho p^2}}\right)-\frac{T^{5/2}}{(2\pi)^3}\int_{\mathbb{R}^3}\log\left(1-e^{-\sqrt{p^4+\frac{16\pi\rho a}{T}p^2}}\right)\mathrm{d}p\right|\\&\leq C(T\ell^2)^{-1/2}T^{5/2}\leq C(\rho a)^{5/2}(\rho a^3)^{-2\nu}K_\ell^{-1}.\end{aligned}$$

From which Theorem 1.1 follows since $\nu<\eta/3$. □

Proof of Lemma B.1. Let us define $w_p(\rho)=\sqrt{p^4+16\pi\rho aT^{-1}p^2}$ and $G(x)=\log(1-e^{-x})$. For all p we have

$$w_p'(\rho)=\frac{8\pi aT^{-1}p^2}{w_p(\rho)},\qquad w_p''(\rho)=-\frac{\left(8\pi aT^{-1}p^2\right)^2}{w_p(\rho)^3}$$

and $G'(x)=1/(e^x-1)$, $G''(x)=-e^x/(e^x-1)^2$. We deduce that

$$0\leq\frac{\partial}{\partial\rho}G(w_p(\rho))=\frac{8\pi aT^{-1}p^2}{w_p(\rho)^2}\frac{w_p(\rho)}{e^{w_p(\rho)}-1}\leq Ce^{-p^2}$$

and that

$$\begin{aligned}0\geq\frac{\partial^2}{\partial\rho^2}G(w_p(\rho))&=-\frac{\left(8\pi aT^{-1}p^2\right)^2}{w_p(\rho)^4}\frac{w_p^2(\rho)}{(e^{w_p(\rho)}-1)^2}\left(e^{w_p(\rho)}+\frac{e^{w_p(\rho)}-1}{w_p(\rho)}\right)\\&\geq-Ce^{-w_p(\rho)}(w_p(\rho)+1)^2\geq-Ce^{-p^2/2}(p^2+1)^2,\end{aligned}$$

where we used that $w_p(\rho)^2\geq 16\pi aT^{-1}p^2$, that $(e^x-1)/x\leq e^x$ and that $x/(e^x-1))\leq Ce^{-x}(x+1)$. The result follows by estimating the associated Riemann sum. □

Acknowledgements This work was partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 470903074 – TRR 352. S.F., L.J., L.M., and M.O. were partially supported by the grant 0135-00166B from the Independent Research Fund Denmark and by the Villum Centre of Excellence for the Mathematics of Quantum Theory (QMATH) with Grant No.10059. M.O. was partially supported by the European Union's HORIZON-MSCA-2022-PF-01 grant agreement, project number: 101103304 (UniBoGas). Furthermore, S.F., L.M. and M.O. were partially funded by the European Union. The author T.G. was partially funded by the grant HE ERC MATCH-G.A. n. 101117299-CUP: D13C23002820006, Macroscopic properties of interacting bosons: a unified approach to the Thermodynamic Challenge, as well as the mathematics area of GSSI. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

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Chapter 6

A second order upper bound to the free energy of the two dimensional Bose Gas.

This chapter contains an in progress article. Its written by Haberberger and the author. The paper provides an upper bound on eq. (1.4). It can be located within the thesis by the colour ■ at the top of the page.

A second order upper bound to the free energy of the two dimensional Bose gas

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August 30, 2025

1 Introduction

1.1 Setting and Main Result

We consider the Hamiltonian

$$H_n = \sum_{i=1}^{n} -\Delta_i + \sum_{i<j}^{n} v(x_i - x_j) \tag{1.1}$$

acting on the symmetric $n-$particle Hilbert space $L_s^2(\Lambda^n)$ where $\Lambda = [-\frac{L}{2}, \frac{L}{2}]^2$ is a torus. Here $v \geqslant 0$ and radial. We will then compute the free energy at temperature $T \geqslant 0$ given by

$$E(L,T) = \inf_{\Gamma} \mathrm{Tr}(H\Gamma) - TS(\Gamma) \tag{1.2}$$

where the infimum is taken over density matrices and

$$S(\Gamma) = -\,\mathrm{Tr}(\Gamma \log(\Gamma)).$$

The Berezinskii–Kosterlitz–Thouless critical temperature is given by

$$T_c = \frac{4\pi\rho}{\log|\log(\rho\mathfrak{a}^2)|}.$$

By variational principle the minimizer is given by

$$\Gamma_T = \frac{e^{-\beta H}}{Tr(e^{-\beta H})}$$

and its energy

$$E(L,T) = \mathrm{Tr}(H\Gamma_T) - TS(\Gamma_T) = -T\log(\mathrm{Tr}(e^{-\beta H})) \tag{1.3}$$

In these notes we come up with a trial state with the correct energy on a smaller box. The Trial state we will use is the following

$$\Gamma_0 = \frac{F\Gamma_B F}{\mathrm{Tr}\,(F\Gamma_B F)}, \quad \Gamma_B = \frac{W_{N_0} e^{\mathcal{B}} e^{-\beta\sum_{p\in\Lambda_+^*} D_p a_p^* a_p} \mathbb{1}_{\{\mathcal{N}_0=0\}} e^{-\mathcal{B}} W_{N_0}^*}{\mathrm{Tr}\left(e^{-\beta\sum_{p\in\Lambda_+^*} D_p a_p^* a_p} \mathbb{1}_{\{\mathcal{N}_0=0\}}\right)}, \tag{1.4}$$

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with

$$D_p = \sqrt{p^4 + 2p^2\rho_0\widehat{g}(p)}, \tag{1.5}$$

W_{N_0} a Weyl transformation creating the condensate and $\mathcal{B}$ a Bogoliubov transformation. The Weyl transform will ensure that our final state has exactly the number of particles needed, in order to use the Legendre transform to go back to the canonical setting. We denote by $a_p = a\big(|\Lambda_L|^{-1/2}e^{ip\cdot}\big)$ for $p \in \Lambda^* := \frac{2\pi}{L}\mathbb{Z}^2$ the usual annihilation operator and analogously the creation operator a_p^*. They satisfy the canonical commutation relation

$$[a_p, a_q^*] = \delta_{p,q}, \quad [a_p, a_q] = 0 = [a_p^*, a_q^*].$$

Moreover, we denote the set of excited momenta by $\Lambda_+^* := \Lambda^* \setminus \{0\}$ and the excited Fock space by $\mathcal{F}^\perp$. We will construct a trial state Γ_0 on $\mathcal{F}(L^2(\Lambda))$ for the periodic, grand-canonical Hamiltonian on the box Λ

$$\mathcal{H} = \mathcal{H}_v = \bigoplus_{n=0}^{\infty} H_n = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{1}{2|\Lambda|}\sum_{p,q,k\in\Lambda^*} \widehat{v}(k) a_p^* a_{q+k}^* a_q a_{p+k}. \tag{1.6}$$

All the log we use are natural log. We use the following definition of the Fourier transform

$$\widehat{f}(p) = \int_{\mathbb{R}^2} f(x)e^{-ipx}\mathrm{d}x, \quad f(x) = \int_{\mathbb{R}^2} \widehat{f}(p)e^{ipx}\frac{\mathrm{d}p}{(2\pi)^2}.$$

Theorem 1. *For a radial positive potential v with compact support R and scattering length a. There exist a constant c (only depending on the support and scattering length of v) such that for $T \leqslant cT_c = c\beta_c^{-1}$ and $\rho a^2 \leqslant c$ we have*

$$f^2(\rho, T) \leqslant 2\pi\rho^2\delta\left(1 + \big(\Gamma + \frac{1}{4} + \frac{\log(\pi)}{2}\big)\delta\right) + \frac{T^2}{(2\pi)^2}\int_{\mathbb{R}^2}\log\left(1 - e^{-\sqrt{p^4 + \frac{8\pi\rho\delta}{T}p^2}}\right)$$
$$+ c^{-1}\rho^2\delta\left(\delta^2|\log(\delta)| + \frac{T^2}{T_c^2}\right).$$

Here $\Gamma = 0.577\ldots$ is the Euler-Mascheroni constant and

$$\delta = \frac{2}{|\log(\rho a^2)| + |\log\big(|\log(\rho a^2)|\big)|}.$$

Remark. • The above result yields a better precision than [11] for $T \leqslant \rho\sqrt{Y}$, For larger temperatures the term $\frac{T^2}{T_c^2}$ becomes too big. In fact the above expansion is equal to the expansion of [11] if the constant in front of $\frac{T^2}{T_c^2}$ was 1, for our state, however, the constant is 2, implying that for larger temperatures the Bogoliubov transform is incorrect.

1.2 Proof of Theorem 1

In this section we collect the main lemmata and prove Theorem 1.

Lemma 2. *For a compactly supported radial v with scattering length $\mathfrak{a} > 0$ and $L = \rho^{-\frac{1}{2}}Y^{-\alpha}$ with $\alpha \geqslant \frac{5}{2}$ and $T \leqslant cT_c$ for some c small enough. For any $\widetilde{\rho} \in [\rho, \rho(1+Y^2)]$ we can choose W_{N_0} in Γ_0 from (1.4) and it satisfies the energy bound*

$$\mathrm{Tr}\,(\mathcal{H}\Gamma_0) \leqslant 2\pi L^2\rho^2\delta\left(1 + \delta\Big(\gamma + \frac{1}{4} + \frac{\log(\pi)}{2}\Big)\right) + \mathrm{Tr}_{\,\mathcal{F}^\perp}\left(\sum_{p\in\Lambda_+^*} D_p a_p^* a_p e^{-\beta\sum_{p\in\Lambda_+{}^*} D_p a_p^* a_p}\right)$$
$$+ CL^2 Y|\log(Y)|^2(\rho Y + T)^2,$$

Moreover, we have

$$\mathrm{Tr}\,(\mathcal{N}\Gamma_0) = L^2\widetilde{\rho}. \tag{1.7}$$

Lemma 3. *Let* $h(x) = -x\log(x), x \in [0,1]$ *and* $T \ll T_c$. *Then there is a constant* $C > 0$ *such that*

$$-TS(\Gamma_0) \leqslant -TS(\Gamma_B) + CL^2T^2|\log(Y)|^2\|\Gamma_0 - \Gamma_B\|_1 + Th(\|\Gamma_0 - \Gamma_B\|_1) + L^3T^3Y^{1/2}b.$$

Moreover,

$$\|\Gamma_0 - \Gamma_B\|_1 \leqslant C\sqrt{b^2\rho N}.$$

We can extend the trial state Γ_0 to the thermodynamic box Ω by first obtaining a Dirichlet version of Γ_0 on a slightly smaller box and then gluing copies of them together. This is a standard technique and we have the following result.

Proposition 4. *Let* $T \leqslant cT_c$ *and let* $0 < R < \ell < L$ *such that* $\mathrm{supp}(v) \subset B(0,R)$. *Let* Γ_L *be a density matrix on the Fock space* $\mathcal{F}(\Lambda_L)$, *i.e. a non-negative operator on* $\mathcal{F}(\Lambda_L)$ *with* $\mathrm{Tr}\,(\Gamma_L) = 1$, *satisfying periodic boundary conditions and* $\mathrm{Tr}\,(\mathcal{N}\Gamma_L) < \infty$. *If*

$$\rho = \frac{1}{(L+2\ell+R)^2}\mathrm{Tr}\,(\mathcal{N}\Gamma_L), \tag{1.8}$$

then there exists a constant $C > 0$ *such that*

$$f(\rho,T) \leqslant \frac{1}{(L+2\ell+R)^2}\left[\mathrm{Tr}\,(\mathcal{H}\Gamma_L) - TS(\Gamma_L)\right] + \frac{C}{L^3\ell}\mathrm{Tr}\,(\mathcal{N}\Gamma_L).$$

We are now ready to prove Theorem 1.

Proof of Theorem 1. We take $\ell = \frac{1}{4}Y^{\alpha-1/2}L$. Then

$$\frac{L^2}{(L+\ell+R)^2} \leqslant 1 + Y^2. \tag{1.9}$$

Thus according to Lemma 2 we may take the Γ_0 satisfying (1.8). Thus Proposition 4 implies

$$f(\rho,T) \leqslant \frac{\mathrm{Tr}\,(\mathcal{H}\Gamma_0) - TS(\Gamma_0)}{L^2} + C\frac{\rho}{L\ell}. \tag{1.10}$$

and Theorem 1 follows by combining Lemma 2 and Lemma 3 and noting $\rho/(L\ell) = \rho^2Y^{\alpha+1/2}$. □

2 Scattering Length

Definition 5. Let $v : \mathbb{R}^2 \to \mathbb{R}$ be positive, radial and compactly supported. We define its scattering length $\mathfrak{a}(v)$ by the variational problem

$$\frac{2\pi}{\log(\frac{R}{\mathfrak{a}(v)})} = \inf\left\{\int_{B(0,R)} |\nabla\phi|^2 + \frac{1}{2}v|\phi|^2\mathrm{d}x \;\middle|\; \phi \in H^1(B(0,R)), \phi_{|\partial B(0,R)} = 1\right\} \tag{2.1}$$

for R such that $\mathrm{supp(v)} \subset \mathrm{B(0,R)}$.

One readily checks that the variational problem (2.1) has a unique minimizer ϕ_R. We have $0 \leqslant \phi_R \leqslant 1$, ϕ_R is radial and satisfies the scattering equation in $B(0,R)$

$$(-\Delta + \frac{1}{2}v)\phi = 0 \tag{2.2}$$

in a distributional sense. Let $\text{supp}(v) \subset B(0, R_0) \subset B(0, R)$. Then from the fundamental solution of the Laplace equation in two dimensions we obtain

$$\phi_R(x) = \frac{\log(|x|/\mathfrak{a})}{\log(R/\mathfrak{a})}, \; R_0 < |x| < R. \tag{2.3}$$

By comparing with the solution in $B(0, R_0)$, we find that $\mathfrak{a}$ is independent of R.

Our proof works with a wide range of R, but choosing a specific $\widetilde{R}$ depending on $\rho\mathfrak{a}^2$ simplifies the computations. Moreover, we introduce the variable $Y := |\log \rho\mathfrak{a}^2|^{-1}$ for convenience, though also for historical reasons as it is the small expansion parameter appearing in [10].

$$\widetilde{R}/\mathfrak{a} = \sqrt{\frac{|\log(\rho\mathfrak{a}^2)|}{\rho\mathfrak{a}^2}} = \frac{1}{\sqrt{\rho\mathfrak{a}^2 Y}} \gg 1, \quad \delta = \frac{1}{\log(\widetilde{R}/\mathfrak{a})} = \frac{2}{|\log(\rho\mathfrak{a}^2 Y)|}.$$

Let $\widetilde{\phi}$ be an extension of $\phi_{\widetilde{R}}$ to the full space, that is $\widetilde{\phi}(x) = \begin{cases} \phi_{\widetilde{R}}(x), & |x| < \widetilde{R} \\ \delta \log(|x|/\mathfrak{a}), & |x| \geqslant \widetilde{R}. \end{cases}$ In particular $\widetilde{\phi}$ satisfies the scattering equation (2.2) on the full space. We further define $\omega = 1 - \widetilde{\phi}$ and

$$g := \widetilde{\phi} v. \tag{2.4}$$

In the following lemma we collect some facts about g, see also [6, Chapter 3.4].

Lemma 6. *It holds that*

$$\widehat{g}(0) = 4\pi\delta \tag{2.5}$$

and

$$\widehat{g\omega}(0) = \int_{\mathbb{R}^2} \frac{\widehat{g}(p)^2 - \widehat{g}(0)^2 \mathbb{1}_{\{|p| \leqslant 2e^{-\gamma - 1/\delta}\mathfrak{a}^{-1}\}}}{2p^2} \frac{\mathrm{d}p}{(2\pi)^2} \tag{2.6}$$

Proof. We compute with partial integration

$$\widehat{g}(0) = 2 \int_{B(0,\widetilde{R})} \Delta\phi_{\widetilde{R}} = 2 \int_{\partial B(0,\widetilde{R})} \nabla\phi_{\widetilde{R}} \cdot \vec{n} \mathrm{d}S = 4\pi\delta.$$

The second identity (2.6) requires some work, mainly since $\widehat{\omega}$ is a distribution and has a singular part at 0, different than in the three-dimensional case, where p^{-2} is locally integrable. From (2.2) we find $-\Delta\omega = \frac{g}{2}$. Thus $\widehat{\omega}(u) = \int_{\mathbb{R}^2} \frac{\widehat{g}(p)}{2p^2} u(p)\mathrm{d}p$ for all test functions $u \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$. We write $\omega = -\delta \log(e^{-1/\delta}|x|/\mathfrak{a}) + \widetilde{\omega}$ so that $\widetilde{\omega} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Let $u \in C_c^\infty(\mathbb{R}^2)$ be a test function. Then for all $\varepsilon > 0$ we may write $u = u_\varepsilon + (u - u_\varepsilon)$ for some $u_\varepsilon \in C_c^\infty(B(0, \varepsilon))$ such that $u - u_\varepsilon \in C_c^\infty(\mathbb{R}^2 \setminus \{0\})$. We obtain

$$\langle \widehat{\omega}, u \rangle = \widehat{\widetilde{\omega}}(u_\varepsilon) - \delta \, \langle \widehat{\log(e^{-1/\delta}|x|/\mathfrak{a})}, u_\varepsilon \rangle + \int_{\mathbb{R}^2} \frac{\widehat{g}(p)}{2p^2} (u(p) - u_\varepsilon(p))\mathrm{d}p.$$

We immediately get $\widehat{\widetilde{\omega}}(u_\varepsilon) \xrightarrow{\varepsilon \to 0} 0$. Moreover, one can calculate the distributional Fourier transformation of the logarithm, see e.g. [15, 9.8d)], and for $u \in C_c^\infty(\mathbb{R}^2)$ one obtains

$$-\delta \, \langle \widehat{\log(e^{-1/\delta}|x|/\mathfrak{a})}, u \rangle = 4\pi\delta \int_{\mathbb{R}^2} \frac{u(p) - u(0)\mathbb{1}_{\{|p| \leqslant 2e^{-\gamma} e^{-1/\delta}\mathfrak{a}^{-1}\}}}{2p^2} \mathrm{d}p.$$

Since $\widehat{g}(0) = 4\pi\delta$ this implies

$$\langle \widehat{\omega}, u \rangle = \int_{\mathbb{R}^2} \frac{\widehat{g}(p)u(p) - \widehat{g}(0)u(0)\mathbb{1}_{\{|p| \leqslant 2e^{-\gamma-1/\delta}\mathfrak{a}^{-1}\}}}{2p^2} \mathrm{d}p + \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \frac{u_\varepsilon(p)}{2p^2} (\widehat{g}(0) - \widehat{g}(p)) \mathrm{d}p$$
$$= \int_{\mathbb{R}^2} \frac{\widehat{g}(p)u(p) - \widehat{g}(0)u(0)\mathbb{1}_{\{|p| \leqslant 2e^{-\gamma-1/\delta}\mathfrak{a}^{-1}\}}}{2p^2} \mathrm{d}p,$$

where we used that $u_\varepsilon \to 0$ almost everyhwere and that $\frac{\widehat{g}(0)-\widehat{g}(p)}{p^2}$ is locally integrable. This implies (2.6) by noting that $\widehat{g\omega}(0) = \langle \omega, g \rangle = \frac{1}{(2\pi)^2} \langle \widehat{\omega}, \widehat{g} \rangle$.

□

2.1 Jastrow Factor

In our trial state we will use a "Jastrow factor" $F : \mathcal{F} \to \mathcal{F}$. It was introduced in [9] and for example used by Dyson [3] in his proof of an upper bound for the ground state energy of a Dilute Bose gas in 3D. It was also used with great success in the study of the 3D ground state [1] and [1] as well as in 2D [7]. The idea is to implement correlations, thus "softening" the potential, while controlling its action by the inequality $F_n \leqslant 1$. F is given by

$$F_n(x) = \prod_{i<j}^{n} f(x_i - x_j), \qquad x \in \mathbb{R}^{2n}, \tag{2.7}$$

where f is a truncated, and therefore well-behaved, scattering solution

$$f(x) = \begin{cases} \phi_b(x), & |x| < b \\ 1, & |x| \geqslant b, \end{cases} \tag{2.8}$$

where ϕ_b is the scattering solution (2.3) that is normalized to 1 at $|x| = b$. We will choose b as

$$b/\mathfrak{a} = \frac{Y^m}{\sqrt{\rho \mathfrak{a}^2}}, \qquad m \in \mathbb{N}, \tag{2.9}$$

with m to be fixed later. Here we think of m large. Thus, b is much bigger than the scattering length $\mathfrak{a}$ however much smaller than the average particle distance $\rho^{-1/2}$. When we conjugate the Hamiltonian H with F we obtain an effective potential whose properties we summarize in the following lemma.

Lemma 7. *Let $v \colon \mathbb{R}^2 \to \mathbb{R}$ be positive, radial and compactly supported with scattering length $\mathfrak{a}$, and let f be its associated scattering solution truncated at b such that* $\mathrm{supp}(v) \subset B(0, b)$*. Then*

$$(-2\Delta + v)f = \delta_{|x|=b} \frac{2}{b \log(\frac{b}{a})} =: \widetilde{v},$$

in a distributional sense. Furthermore, the scattering lengths of v and $\widetilde{v}$ agree. Lastly, if $\widetilde{g}$ is given as in (2.4) *with respect to $\widetilde{v}$ and b as in* (2.9) *then*

$$\int \widetilde{v} \leqslant 2Y, \qquad \int \widetilde{v} - \widetilde{g} \leqslant 2mY^2 |\log(Y)|. \tag{2.10}$$

Both $\widetilde{v}$ and $\widetilde{g}$ are uniform measures on a sphere and thus their Fourier transforms are Bessel functions and we get the following decay

$$|\widehat{\widetilde{v}}(p)| \leqslant C \frac{\widehat{\widetilde{v}}(0)}{\sqrt{b|p|}} \tag{2.11}$$

Proof. The proof is a straightforward computation, see also [7, Lemma 3.10]. □

3 Hamiltonian part

In this section, we provide the proof of Lemma 2. This involves finding a state Γ_0 with the correct energy and enough particles. The method follows that of [7], however having to include excitations we need to keep track of terms of the Hamiltonian, which in the setting [7] would simply vanish when acting on the ground state. We consider a box of fixed length $\Lambda = [-\frac{L}{2}, -\frac{L}{2}]$ with

$$L = \rho^{-\frac{1}{2}} Y^{-\alpha}, \qquad \alpha > 3/2. \tag{3.1}$$

Moreover, we introduce $N = \rho L^2$ for convenience.

3.1 Bogoliubov trial state

We first use a Weyl transformation with a parameter N_0, to be specified in (3.5), to factor out the condensate

$$W_{N_0} = \exp[\sqrt{N_0}(a_0^* - a_0)].$$

Here one should think of N_0 being approximately the number of particles in the condensate. In particular $N_0 \approx N$. We readily obtain

$$W_{N_0}^* a_p W_{N_0} = \delta_{p,0}\sqrt{N_0} + a_p. \tag{3.2}$$

Next, we aim to approximately diagonalize the resulting excitation Hamiltonian. Since the Jastrow factor will soften the potential, a quadratic Bogoliubov transformation is sufficient. Thus we consider

$$\mathcal{B} = \frac{1}{2}\sum_{p\in\Lambda_+^*} \varphi_p a_p^* a_{-p}^* - \text{h.c.}$$

We choose the kernel φ as the minimizer of the main term of the energy, as was done in [6] following ideas from [4]: $\tanh(2\varphi_p) = -\frac{\rho_0 \widehat{g}(p)}{p^2+\rho_0\widehat{g}(p)}$ so that for $p \neq 0$

$$\begin{aligned} e^{-\mathcal{B}} a_p e^{\mathcal{B}} &= a_p \sqrt{\frac{1}{2}\left(\frac{p^2+\rho_0\widehat{g}(p)}{D_p}+1\right)} - a_{-p}^* \sqrt{\frac{1}{2}\left(\frac{p^2+\rho_0\widehat{g}(p)}{D_p}-1\right)} \\ &=: c_p a_p + s_p a_{-p}^*, \end{aligned} \tag{3.3}$$

with $\rho_0 := \frac{N_0}{|\Lambda|}$. Here $D_p = \sqrt{p^4 + 2p^2\rho_0\widehat{g}(p)}$ and D_p is well defined due to $\widehat{g}(0) \geqslant 0$ and the continuity of $\widehat{g}$. In particular there is a $C > 0$ such that $\widehat{g}(p) > 0$ for $|p| \leqslant C$. Moreover, this choice will turn out to naturally satisfy $c_p s_p \approx -\rho_0\widehat{\omega}(p)$, see [14, Section 4] for a nice heuristic argument.

For convenience we introduce $H_D := \sum_{p\in\Lambda_+^*} D_p a_p^* a_p$ and its Gibbs state

$$\Gamma_D = \frac{e^{-\beta H_D}\mathbb{1}_{\{\mathcal{N}_0=0\}}}{\mathrm{Tr}_{\mathcal{F}^\perp}(e^{-\beta H_D})}. \tag{3.4}$$

Furthermore, recall (1.4)

$$\Gamma_B = W_{N_0} e^{\mathcal{B}} \Gamma_D e^{-\mathcal{B}} W_{N_0}^*.$$

From (3.2), the fact that Γ_D satisfies Wicks theorem and does not contain any particles in the zero mode we find $\mathrm{Tr}\,(\mathcal{N}\Gamma_B) = N_0 + \mathrm{Tr}\,(e^{-\mathcal{B}}\mathcal{N}_+ e^{\mathcal{B}}\Gamma_D)$ with $\mathcal{N}_+ = \sum_{p\in\Lambda_+^*} a_p^* a_p$. This motivates the choice

$$N_0 := N - \mathrm{Tr}\,(e^{-\mathcal{B}}\mathcal{N}_+ e^{\mathcal{B}}\Gamma_D) \tag{3.5}$$

so that $\mathrm{Tr}\,(\mathcal{N}\Gamma_B) = N$. The following proposition ensures that $N_0 \geqslant 0$ and quantifies $N_0 \approx N$.

Proposition 8. *Let c_p and s_p be given according to* (3.3) *then*

$$\operatorname{Tr}(\mathcal{N}\Gamma_B) = N_0 + \sum_{p\in\Lambda_+^*} s_p^2 + \sum_{p\in\Lambda_+^*} (s_p^2 + c_p^2) \operatorname{Tr}(a_p^* a_p \Gamma_D). \tag{3.6}$$

Moreover,

$$\sum_{p\in\Lambda_+^*} s_p^2 \leqslant CNY. \tag{3.7}$$

Lastly for $T \leqslant \rho$ we have

$$\sum_{p\in\Lambda_+^*} (s_p^2 + c_p^2) \operatorname{Tr}(a_p^* a_p \Gamma_D) \leqslant CNT|\log(Y)|\rho^{-1}. \tag{3.8}$$

In particular, if $T \ll T_c$ we find $0 \leqslant N_0 \leqslant N$ and $|N - N_0| \leqslant CN(Y + T|\log(Y)|\rho^{-1})$.

Proof. From (3.5) and (3.3) we readliy obtain (3.6), where we used that $\operatorname{Tr}(a_p^* a_{-p}^* \Gamma_D) = 0$ since Γ_D is the Gibbs state of a diagonal Operator.

To show (3.7) we split the sum as follows

$$\begin{aligned}
\sum_{p\in\Lambda_+^*} s_p^2 &= \left(\sum_{p^2 \leqslant \rho_0 \widehat{g}(0)} + \sum_{p^2 > \rho_0 \widehat{g}(0)} \right) \frac{p^2 + \rho_0 \widehat{g}(p) - D_p}{2D_p} \\
&\leqslant \sum_{p^2 \leqslant \rho_0 \widehat{g}(0)} \left[\frac{p^2 + \rho_0 \widehat{g}(p)}{2|p|\sqrt{2\rho_0 \widehat{g}(p)}} \right] + C \sum_{p^2 > \rho_0 \widehat{g}(0)} \left[1 + \rho_0 \widehat{g}(p) p^{-2} - \sqrt{1 + 2\rho_0 \widehat{g}(p) p^{-2}} \right] \\
&\leqslant CL^2 \rho_0 \widehat{g}(0) \leqslant CNY,
\end{aligned}$$

where we used the elementary inequality $1 + x - \sqrt{1 + 2x} \leqslant x^2,\ x \geqslant -(\sqrt{2} - 1)$.

Let us consider the bound (3.8). We first note that $s_p^2 + c_p^2 = (p^2 + \rho_0 \widehat{g}(p))/D_p$ and that a standard ideal gas computation shows $\operatorname{Tr}(a_p^* a_p \Gamma_D) = \frac{1}{e^{\beta D_p} - 1}$. This yields

$$\sum_{p\in\Lambda_+^*} (s_p^2 + c_p^2) \operatorname{Tr}(a_p^* a_p \Gamma_D) = \sum_{p\in\Lambda_+^*} \frac{p^2 + \rho_0 \widehat{g}(p)}{D_p} \frac{1}{e^{\beta D_p} - 1}.$$

We separate the sum as before. For $p^2 \leqslant \rho_0 \widehat{g}(0)$ we find

$$\begin{aligned}
\sum_{p^2 \leqslant \rho_0 \widehat{g}(0)} \frac{p^2 + \rho_0 \widehat{g}(p)}{D_p} \frac{1}{e^{\beta D_p} - 1} &\leqslant \sum_{p^2 \leqslant \rho_0 \widehat{g}(0)} \left[\frac{1}{e^{\beta p^2} - 1} + \frac{\sqrt{\rho_0 \widehat{g}(p)}}{2|p|} \frac{1}{e^{\beta |p| \sqrt{\rho_0 \widehat{g}(p)}} - 1} \right] \\
\leqslant 2T \sum_{p^2 \leqslant \rho_0 \widehat{g}(0)} p^{-2} &\leqslant CL^2 T \int_{2\pi L^{-1}}^{\sqrt{\rho_0 \widehat{g}(0)}} |p|^{-1} \mathrm{d}p \leqslant CL^2 T \log(L^2 \rho_0 \widehat{g}(0)/(2\pi)).
\end{aligned}$$

For $|p| > \sqrt{\rho_0 \widehat{g}(0)}$ we use $p^2/2 \leqslant D_p \leqslant \sqrt{3} p^2$ and obtain

$$\begin{aligned}
\sum_{|p| > \sqrt{\rho_0 \widehat{g}(0)}} \frac{p^2 + \rho_0 \widehat{g}(p)}{D_p} \frac{1}{e^{\beta D_p} - 1} &\leqslant 4 \sum_{|p| > \sqrt{\rho_0 \widehat{g}(0)}} \frac{1}{e^{\beta p^2/2} - 1} \leqslant CL^2 T \int_{\sqrt{\beta \rho_0 \widehat{g}(0)}}^{\infty} \frac{|p|}{e^{p^2/2} - 1} \mathrm{d}p \\
&= CL^2 T |\log(1 - e^{-\beta \rho_0 \widehat{g}(0)/2})| \leqslant CL^2 T \max\{|\log(\beta \rho_0 \widehat{g}(0)/2)|, 1\}.
\end{aligned}$$

Combining these two inequalities and using our choice of L, (3.1) and the condition $T \leqslant \rho$ concludes the proof. □

3.2 Bogoliubov Diagonalization

Proposition 9. *Recall the periodic Hamiltonian $\mathcal{H}$ (1.6). Let $v \geqslant 0$ be a radial potential supported in $B(0,R)$ for some $R > 0$ and assume that v has positive scattering length $\mathfrak{a}$. We further assume $\rho R^2 \leqslant Y$ and*

$$|\widehat{g}(p)| \leqslant c \frac{\widehat{g}(0)}{\sqrt{R|p|}} \quad \textit{for} \quad |p| \geqslant \mathfrak{a}^{-1}. \tag{3.9}$$

Let Γ_B be the state given by (1.4), then there exist constants $C, c > 0$ such that for $T \leqslant cT_c$ and $Y \leqslant c$ we have

$$\mathrm{Tr}(\mathcal{H}\Gamma_B) \leqslant 2\pi N\rho\delta \left(1 + \delta\Big(\gamma + \frac{1}{4} + \frac{\log(\pi)}{2}\Big)\right) + \mathrm{Tr}_{\mathcal{F}^\perp}(H_D\Gamma_D) + C\,\mathrm{Tr}_{\mathcal{F}^\perp}\big((Q_0 + Q_2 + Q_4)\Gamma_D\big)$$

with

$$Q_0 = N\rho Y^3 + \widehat{v}(0)\rho N Y^2 (Y^{\alpha - 3/2}|\log(Y)| + 1) + \rho(N - N_0 + NY)(\widehat{v}(0) - \widehat{g}(0))$$
$$Q_2 = \rho\left((\widehat{v}(0) - \widehat{g}(0)) + \widehat{v}(0)Y + \widehat{v}(0)Y^{\alpha+1/2}|\log(Y)|\right) \sum_{p\in\Lambda_+^*} (s_p^2 + c_p^2) a_p^* a_p$$
$$Q_4 = \frac{\widehat{v}(0)}{|\Lambda|}\Big(\sum_{p\in\Lambda_+^*} c_p^2 a_p^* a_p\Big)^2.$$

Proof. From (1.6) and (3.2) we obtain

$$\begin{aligned} W_{N_0}^* \mathcal{H} W_{N_0} = &\sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{N_0^2}{2|\Lambda|}\widehat{v}(0) + \frac{N_0^{3/2}}{2|\Lambda|}\widehat{v}(0)(a_0^* + a_0) + \frac{N_0}{|\Lambda|}\sum_{p\in\Lambda^*}\big(\widehat{v}(p) + \widehat{v}(0)\big)a_p^* a_p \\ &+ \frac{N_0}{2|\Lambda|}\sum_{p\in\Lambda^*}\widehat{v}(p) a_p^* a_{-p}^* + \mathrm{h.c.} + \frac{N_0^{1/2}}{|\Lambda|}\sum_{p,q\in\Lambda^*}\widehat{v}(p) a_{p+q}^* a_p a_q + \mathrm{h.c.} \\ &+ \frac{1}{2|\Lambda|}\sum_{p,q,k\in\Lambda^*}\widehat{v}(k) a_p^* a_{q+k}^* a_q a_{p+k}. \end{aligned} \tag{3.10}$$

Therefore, the odd terms, i.e., the third term in the first line and the last term in the second line vanish in the trace. Moreover, Γ_D does not have any particles in the zero mode so that all sums restrict to Λ_+^*.

Recall (3.5) $N_0 = N - \mathrm{Tr}\,(e^{-\mathcal{B}}\mathcal{N}_+ e^{\mathcal{B}}\Gamma_D)$, which allows us to combine the second term with a part of the last term in the first line.

$$\mathrm{Tr}\left(\Big(\frac{N_0^2}{2|\Lambda|} + \frac{N_0}{|\Lambda|}\sum_{p\in\Lambda_+^*} a_p^* a_p\Big) e^{-\mathcal{B}}\Gamma_D e^{\mathcal{B}}\right) = \frac{N^2}{2|\Lambda|} - \frac{1}{2|\Lambda|}\mathrm{Tr}(e^{-\mathcal{B}}\mathcal{N}_+ e^{\mathcal{B}}\Gamma_D)^2 \leqslant \frac{N^2}{2|\Lambda|}. \tag{3.11}$$

We now apply the Bogoliubov transformation on (3.10) using (3.3) and commuting to normal order. We also use that Γ_D is the Gibbs state of a diagonal operator so $\mathrm{Tr}\,(a_p^* a_q^* \Gamma_D) = 0$ for all

p, q and $\mathrm{Tr}\,(a_p^* a_q \Gamma_D) = 0$ for $p \neq q$. A straightforward calculation then shows the upper bound

$$\mathrm{Tr}(\mathcal{H}\Gamma_B) \leqslant \widehat{v}(0)\frac{N\rho}{2} + \frac{1}{2|\Lambda|}\sum_{p,k\in\Lambda_+^*} \widehat{v}(k)c_p s_p c_{p+k}s_{p+k} + \sum_{p\in\Lambda_+^*} p^2 s_p^2 + \rho_0\widehat{v}(p)s_p^2 + \rho_0\widehat{v}(p)s_p c_p$$
$$+ \sum_{p\in\Lambda_+^*}\left((p^2+\rho_0\widehat{v}(p))(s_p^2+c_p^2) + 2\rho_0\widehat{v}(p)c_p s_p + \frac{2}{|\Lambda|}\sum_{\substack{k\in\Lambda^*:\\ p+k\neq 0}} \widehat{v}(k)c_p s_p c_{p+k}s_{p+k}\right)\mathrm{Tr}(a_p^* a_p\Gamma_D)$$
$$+ \frac{C\widehat{v}(0)}{|\Lambda|}\left(\sum_{p\in\Lambda_+^*} s_p^2 + \sum_{p\in\Lambda_+^*} c_p^2\mathrm{Tr}\,(a_p^* a_p \Gamma_D)\right)^2, \tag{3.12}$$

where we also used the simple inequalities $|\widehat{v}(k)| \leqslant \widehat{v}(0)$ and $s_q^2 \leqslant c_q^2$.

Let us simplify the constant term in (3.12). First, we note that we can replace all sums in the first line by integrals up to an error of order $\widehat{v}(0)L^{-1}N\rho^{1/2}Y^{1/2}|\log(Y)| = \widehat{v}(0)\rho N Y^{\alpha+1/2}|\log(Y)|$. This approximation follows from the standard technique of passing from sums to integrals in the limit; We use the mean value theorem and can bound the error in terms of bounds on the gradient of the considered function. For further details, see [6, Appendix G]. In particular, this estimate makes use of

$$\sum_{p\in\Lambda_+^*} |s_p c_p| \leqslant CN, \tag{3.13}$$

which can be shown as in the proof of Proposition 8 while using the decay assumption on g (3.9).

Secondly, as mentioned earlier, we know $c_p s_p \approx -\rho_0\widehat{\omega}(p)$ and therefore we write

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2} c_p s_p \widehat{v}(k) c_{p+k}s_{p+k}\mathrm{d}p\mathrm{d}k &= \frac{1}{2}\langle cs, \widehat{v} * (cs)\rangle \\
&= \frac{1}{2}\langle cs + \rho_0\widehat{\omega}, \widehat{v} * (cs + \rho_0\widehat{\omega})\rangle - \langle cs, \widehat{v} * \rho_0\widehat{\omega}\rangle - \frac{1}{2}\langle \rho_0\widehat{\omega}, \widehat{v} * \rho_0\widehat{\omega}\rangle \\
&= \frac{1}{2}\langle cs + \rho_0\widehat{\omega}, \widehat{v} * (cs + \rho_0\widehat{\omega})\rangle - \rho_0(2\pi)^2\langle cs, \widehat{v\omega}\rangle - \frac{\rho_0^2(2\pi)^4}{2}\widehat{v\omega^2}(0).
\end{aligned}$$

According to [6, Lemma 4.7] the first term in the last line is bounded by $C\widehat{v}(0)\rho^2 Y^2$ so that

$$\begin{aligned}
&\widehat{v}(0)\frac{N\rho}{2} + \frac{1}{2|\Lambda|}\sum_{p,k\in\Lambda_+^*}\widehat{v}(k)c_p s_p c_{p+k}s_{p+k} + \sum_{p\in\Lambda_+^*} p^2 s_p^2 + \rho_0\widehat{v}(p)s_p^2 + \rho_0\widehat{v}(p)s_p c_p \\
&\leqslant \widehat{v}(0)\frac{N\rho}{2} - \frac{N_0\rho_0}{2}\widehat{v\omega^2}(0) + |\Lambda|\int_{\mathbb{R}^2} p^2 s_p^2 + \rho_0\widehat{v}(p)s_p^2 + \rho_0 c_p s_p \widehat{g}(p)\frac{\mathrm{d}p}{(2\pi)^2} \\
&\quad + C\widehat{v}(0)\rho N Y^2(Y^{\alpha-3/2}|\log(Y)| + 1) \\
&= \widehat{g}(0)\frac{N\rho}{2} + \frac{N_0\rho_0}{2}\widehat{g\omega}(0) + |\Lambda|\int_{\mathbb{R}^2} p^2 s_p^2 + \rho_0\widehat{g}(p)s_p^2 + \rho_0 c_p s_p\widehat{g}(p)\frac{\mathrm{d}p}{(2\pi)^2} \\
&\quad + C\widehat{v}(0)\rho N Y^2(Y^{\alpha-3/2}|\log(Y)| + 1) + \frac{N\rho - N_0\rho_0}{2}\widehat{v\omega}(0) + |\Lambda|\rho_0\int_{\mathbb{R}^2}(\widehat{v}(p) - \widehat{g}(p))s_p^2\frac{\mathrm{d}p}{(2\pi)^2} \\
&\leqslant 2\pi\delta N\rho + \frac{|\Lambda|}{2}\left(\rho_0^2\widehat{g\omega}(0) + \int_{\mathbb{R}^2} D_p - p^2 - \rho_0\widehat{g}(p)\frac{\mathrm{d}p}{(2\pi)^2}\right) \\
&\quad + C\widehat{v}(0)\rho N Y^2(Y^{\alpha-3/2}|\log(Y)| + 1) + C\rho(N - N_0 + NY)(\widehat{v}(0) - \widehat{g}(0)),
\end{aligned}$$

where we used
$$p^2 s_p^2 + \rho_0 \widehat{g}(p) s_p^2 + \rho_0 \widehat{g}(p) s_p c_p = \frac{1}{2}(D_p - p^2 - \rho_0 \widehat{g}(p))$$
and (3.7). We use the integral representation of $\widehat{g\omega}(0)$, see (2.6), and we may replace $\widehat{g}(p)$ by $\widehat{g}(0)$ everywhere up to an error of $\rho^2 Y^3$, see [6, Proposition C.3]. Then we can explicitly compute
$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^2} D_p - p^2 - \rho_0 \widehat{g}(0) + \rho_0^2 \frac{\widehat{g}(0)^2}{2p^2} \mathbb{1}_{\{|p|>2e^{-\gamma-\delta^{-1}}\mathfrak{a}^{-1}\}} \frac{\mathrm{d}p}{(2\pi)^2} \\
&= \frac{(\rho_0 \widehat{g}(0))^2}{16\pi}\left(2\gamma + \frac{1}{2} + \log(\pi) + \log(\frac{\rho_0 \delta}{2\rho Y})\right) \\
&\leqslant 2\pi\rho^2\delta^2\left(\gamma + \frac{1}{4} + \frac{\log(\pi)}{2}\right),
\end{aligned}$$
where we used $\rho_0 \leqslant \rho, \delta \leqslant 2Y$. We obtain
$$\begin{aligned}
&\widehat{v}(0)\frac{N\rho}{2} + \frac{1}{2|\Lambda|}\sum_{p,k\in\Lambda_+^*} \widehat{v}(k) c_p s_p c_{p+k} s_{p+k} + \sum_{p\in\Lambda_+^*} p^2 s_p^2 + \rho_0 \widehat{v}(p) s_p^2 + \rho_0 \widehat{v}(p) s_p c_p \\
&\leqslant 2\pi N\rho\delta\left(1 + \delta\left(2\gamma + \frac{1}{2} + \log(\pi)\right)\right) \qquad (3.14)\\
&\quad + CN\rho Y^3 + C\widehat{v}(0)\rho N Y^2 (Y^{\alpha-3/2}|\log(Y)| + 1) + C\rho(N - N_0 + NY)(\widehat{v}(0) - \widehat{g}(0)).
\end{aligned}$$

Returning to the second line of (3.12) we find that
$$\left|\frac{1}{|\Lambda|}\sum_{\substack{k\in\Lambda^*:\\ k+p\neq 0}} \widehat{v}(k) c_{p+k} s_{p+k} + \rho_0 \widehat{v\omega}(p)\right| \leqslant C\widehat{v}(0)\rho Y + C\widehat{v}(0)\rho Y^{\alpha+1/2}|\log(Y)|, \qquad (3.15)$$
which follows as before by replacing sums with integrals and using that
$$|\langle cs + \rho_0 \widehat{\omega}, \widehat{v}(p - \cdot)\rangle| \leqslant C\widehat{v}(0)\rho Y$$
uniformly in p, which is shown in the first step of the proof of [6, Lemma 4.7]. Inserting (3.15) in the second line of (3.12) using
$$(p^2 + \rho_0 \widehat{g}(p))(s_p^2 + c_p^2) + 2\rho_0 \widehat{g}(p) c_p s_p = D_p \qquad (3.16)$$
yields
$$\begin{aligned}
&\sum_{p\in\Lambda_+^*}\left((p^2 + \rho_0 \widehat{v}(p))(s_p^2 + c_p^2) + 2\rho_0 \widehat{v}(p) c_p s_p + \frac{2}{|\Lambda|}\sum_{\substack{k\in\Lambda^*:\\ p+k\neq 0}} \widehat{v}(k) c_p s_p c_{p+k} s_{p+k}\right) \mathrm{Tr}(a_p^* a_p \Gamma_D) \\
&\leqslant \sum_{p\in\Lambda_+^*} D_p \,\mathrm{Tr}(a_p^* a_p \Gamma_D) \\
&\quad + C\rho\left((\widehat{v}(0) - \widehat{g}(0)) + \widehat{v}(0) Y + \widehat{v}(0) Y^{\alpha+1/2}|\log(Y)|\right)\sum_{p\in\Lambda_+^*}(s_p^2 + c_p^2)\,\mathrm{Tr}(a_p^* a_p \Gamma_D).
\end{aligned}$$
Inserting this inequality and (3.14) in (3.12) implies the result. □

In the next subsection we will soften the potential. The following Corollary, which applies to soft potentials, is immediate from Proposition 8 and Proposition 9.

Corollary 10. *Under the assumptions of Proposition 9, if we further assume that*

$$\widehat{v}(0) \leqslant CY, \qquad \widehat{v}(0) - \widehat{g}(0) \leqslant CY^2 \log(Y) \tag{3.17}$$

and that $\alpha > 3/2$. *Then*

$$\begin{aligned} \mathrm{Tr}(\mathcal{H}\Gamma_B) \leqslant 2\pi N\rho\delta \left(1 + \delta\Big(\gamma + \frac{1}{4} + \frac{\log(\pi)}{2}\Big)\right) + \mathrm{Tr}_{\mathcal{F}^\perp}(H_D\Gamma_D) \\ + CN\rho Y\big(Y|\log(Y)| + T|\log(Y)|\rho^{-1}\big)^2. \end{aligned} \tag{3.18}$$

3.3 Soften the Hamiltonian

Observe that Corollary 10 implies Lemma 2 if we assume that the potential v is soft, i.e., that it satisfies (3.17). Therefore, we employ the Jastrow factor (2.7), which "softens the potential". This method was in particular used in the 3D Gross-Pitaevskii scaling [1] and for 2D in [7]. Due to the positive temperature we need to keep track on the higher order excitations the Jastrow factor creates.
In this subsection we explicitly display the potential in the Hamiltonian as $\mathcal{H}_v$, recall (1.6).

Lemma 11. *Let v be as in Theorem 1 and Γ a periodic density matrix on $\mathcal{F}$. Let F be the Jastrow factor from* (2.7). *Then*

$$\mathrm{Tr}\,(\mathcal{H}_v F\Gamma F) \leqslant \mathrm{Tr}\,(\Gamma\mathcal{H}_{\widetilde{v}}) - \mathrm{Tr}\,(\Gamma\mathcal{R}),$$

where

$$\widetilde{v} = (-2\Delta + v)f = \frac{2}{b\log(\frac{b}{\mathfrak{a}})}\delta_{|x|=b} \quad \textit{and} \quad \mathcal{R} = \bigoplus_{n=0}^{\infty} R_n,\ R_n = F_n^2 \sum_{i\neq j\neq k}^{n} \frac{(\nabla f)_{ij}}{f_{ij}} \cdot \frac{(\nabla f)_{ik}}{f_{ik}}$$

with the notation $\{i \neq j \neq k\} = \{$*pairwise distinct indices* i, j, k *running from* 1 *to* $n\}$.

Proof. Let P_n be the projection onto the $n-$particle sector $L^2_s(\Lambda^n)$. Then P_n commutes with $\mathcal{H}$ and F so that

$$\mathrm{Tr}\,(\mathcal{H}_v F\Gamma F) = \sum_{n=0}^{\infty} \mathrm{Tr}\,(F_n\mathcal{H}_v F_n P_n\Gamma P_n). \tag{3.19}$$

$P_n\Gamma P_n$ is a density matrix on $L^2_s(\Lambda^n)$ thus we may write $P_n\Gamma P_n = \sum_{m\in\mathbb{N}} \lambda_{n,m}|\phi^n_m\rangle\langle\phi^n_m|$ for some $\phi^n_m \in L^2_s(\Lambda^n), \lambda_{n,m} \geqslant 0$. Therefore, by linearity, it is enough to considernthe following expression for $\phi \in L^2_s(\Lambda^n)$

$$\begin{aligned} &\mathrm{Tr}\,(F_n\mathcal{H}_v F_n|\phi\rangle\langle\phi|) = \langle\mathcal{H}_v F_n\phi, F_n\phi\rangle \\ &= \sum_{i=1}^{n}\int_{\Lambda^n} F_n^2|\nabla_i\phi|^2 + |\phi|^2|\nabla_i F_n|^2 + (\overline{\phi}\nabla_i\phi + \phi\nabla_i\overline{\phi})\cdot F_n\nabla_i F_n + \sum_{i<j}^{n}\int_{\Lambda^n} v(x_i - x_j)F_n^2|\phi|^2. \end{aligned} \tag{3.20}$$

Partial integration yields

$$\int_{\Lambda^n} \overline{\phi}\nabla_i\phi\cdot F_n\nabla_i F_n = -\int_{\Lambda^n} \phi\nabla_i\overline{\phi}\cdot F_n\nabla_i F_n - |\phi|^2|\nabla_i F_n|^2 + |\phi|^2 F_n(-\Delta_i F_n). \tag{3.21}$$

From the definition $F_n = \prod_{i<j}^n f_{ij}$ we find

$$-\sum_{i=1}^{n}\Delta_i F_n = F_n\sum_{i\neq j}^{n}\frac{-\Delta_i f_{ij}}{f_{ij}} - F_n\sum_{i\neq j\neq k}^{n}\frac{\nabla_i f_{ij}}{f_{ij}}\cdot\frac{\nabla_i f_{ik}}{f_{ik}}. \tag{3.22}$$

We insert (3.21) and (3.22) into (3.20), use $F_n \leqslant f_{ij} \leqslant 1$ and obtain

$$\begin{aligned}\operatorname{Tr}\left(F_n \mathcal{H}_v F_n |\phi\rangle\langle\phi|\right) &\leqslant \sum_{i=1}^{n} \langle \phi, -\Delta_i \phi\rangle + \sum_{i<j}^{n} \int_{\Lambda^n} |\phi|^2 \big((v-2\Delta)f\big)(x_i - x_j) - \langle \phi, R_n \phi\rangle \\ &= \operatorname{Tr}\left(\mathcal{H}_{\tilde{v}} |\phi\rangle\langle\phi|\right) - \operatorname{Tr}\left(R_n |\phi\rangle\langle\phi|\right).\end{aligned}$$

Inserting this into (3.19) concludes the proof. □

The next lemma is the final ingredient for Lemma 2 and shows in particular that we can bound the error $\mathcal{R}$ and the difference between $\operatorname{Tr}(F\Gamma_B F)$ and $\operatorname{Tr}(\Gamma_B)$.

Lemma 12. *Let v be as in Theorem 1, Γ_B the state (1.4), F the Jastrow factor from (2.7) and $\mathcal{R}$ the operator defined in Lemma 11. Then for $T \ll T_c$*

$$\operatorname{Tr}(F\Gamma_B F) \geqslant 1 - Cb^2\rho N, \tag{3.23}$$

$$\operatorname{Tr}(\mathcal{N} F\Gamma_B F) \geqslant N(1 - C\rho b^2 N), \tag{3.24}$$

$$Tr(\mathcal{R}\Gamma_B) \leqslant Cb^2\rho^2 N. \tag{3.25}$$

Remark. From $N = \rho L^2 = Y^{-\alpha}$ we find $b^2\rho N = Y^{m-\alpha}$ so that the corrections are small for $m > \alpha$.

Proof. From the inequality

$$F_n^2 = \prod_{i<j}^{n} (1 - (1 - (f_{ij})^2)) \geqslant 1 - \sum_{i<j}^{n} 1 - (f_{ij})^2 =: 1 - \sum_{i<j}^{n} u_{ij}, \tag{3.26}$$

we find

$$\operatorname{Tr}(F\Gamma_B F) \geqslant 1 - \operatorname{Tr}(\Gamma_B U_u),$$

where the operator U_u in second quantized form is given by

$$U_u = \frac{1}{2|\Lambda|} \sum_{p,q,k \in \Lambda^*} \widehat{u}(k) a_p^* a_{q+k}^* a_q a_{p+k}. \tag{3.27}$$

We can bound $\operatorname{Tr}(\Gamma_B U_u)$ by computing the action of the Weyl and the Bogoliubov transformation on U_u, which was done in the proof of Proposition 9, and estimating terms with (3.13)

$$\operatorname{Tr}(\Gamma_B U_u) \leqslant C \frac{\widehat{u}(0)}{|\Lambda|} \left(N + \sum_p (c_p^2 + s_p^2) \operatorname{Tr}(a_p^* a_p \Gamma_D) \right)^2. \tag{3.28}$$

We can bound $\widehat{u}(0)$ by using the explicit formula for f (2.8) when $|x| \geqslant R$, and $|f| \leqslant 1$ for $|x| \leqslant R$

$$\widehat{u}(0) \leqslant \pi R^2 + \int_{R \leqslant |x| \leqslant b} 1 - \frac{\log(|x|/\mathfrak{a})^2}{\log(b/\mathfrak{a})^2} dx \leqslant CR^2 + \frac{\pi b^2}{\log(b/\mathfrak{a})} \leqslant Cb^2, \tag{3.29}$$

where we used that $b \gg a$. From (3.8) and $T \ll T_c$ we find (3.23).

Let us turn our attention to (3.24). From the fact that F commutes with the number operator and inequality (3.26) we obtain

$$\operatorname{Tr}(\mathcal{N} F\Gamma_B F) \geqslant \operatorname{Tr}(\mathcal{N}\Gamma_B) - \operatorname{Tr}(\mathcal{N} U \Gamma_B) = N - \operatorname{Tr}(\mathcal{N} U \Gamma_B).$$

By a similar computation as before we find

$$\operatorname{Tr}(\mathcal{N}U\Gamma_B) \leqslant C\frac{\widehat{u}(0)}{|\Lambda|}\left(N+\sum_p (c_p^2+s_p^2)\operatorname{Tr}(a_p^*a_p\Gamma_D)\right)^3 \leqslant Cb^2\rho N^2.$$

To show (3.25) we use $R_n \leqslant \sum_{i\neq j\neq k}^n |(\nabla f)_{ij}||(\nabla f)_{ik}|$, which implies the following inequality in the second quantization formalism

$$\mathcal{R} \leqslant \frac{1}{|\Lambda|^2}\sum_{p,q,r,s,k\in\Lambda^*} \widehat{|\nabla f|}(k)\widehat{|\nabla f|}(s)a_p^*a_q^*a_r^*a_{r-s}a_{q-k}a_{p+k+s}.$$

By the same argument as above we find

$$\operatorname{Tr}(\Gamma_B\mathcal{R}) \leqslant C\frac{\widehat{|\nabla f|}(0)^2}{|\Lambda|^2}\left(N+\sum_p (c_p^2+s_p^2)\operatorname{Tr}(a_p^*a_p\Gamma_D)\right)^3.$$

From a Cauchy–Schwarz inequality and the definition of f we find

$$\widehat{|\nabla f|}(0)^2 \leqslant Cb^2\int_{|x|\leqslant b} |\nabla f|^2 \leqslant Cb^2,$$

where the last inequality used $f=\phi_b$ on $\{|x|\leqslant b\}$ and that ϕ_b is the minimizer of (2.1). (3.8) then implies (3.25). □

Proof of Lemma 2. Recall the definition of Γ_0 from (1.4). We pick b that appears in the definition of Jastrow factor as

$$b=\rho^{-\frac{1}{2}}Y^{\alpha+2}. \tag{3.30}$$

Then by Lemmata 11 and 12 we find that

$$\operatorname{Tr}(\mathcal{H}_v\Gamma_0) \leqslant \big(\operatorname{Tr}(\mathcal{H}_{\widetilde{v}}\Gamma_B)-\operatorname{Tr}(\Gamma_B\mathcal{R})\big)\frac{1}{\operatorname{Tr}(F\Gamma_BF)} \leqslant \operatorname{Tr}(\mathcal{H}_{\widetilde{v}}\Gamma_B)(1+CY^4)+CN\rho Y^4,$$

where $\widetilde{v}$ was defined in Lemma 7. In particular it has the same scattering length as v and satisfies the conditions of Corollary 10 from which we find

$$\begin{aligned}\operatorname{Tr}(\mathcal{H}_v\Gamma_0) \leqslant{}& 2\pi N\rho\delta\left(1+\delta\big(2\gamma+\frac{1}{2}+\log(\pi)\big)\right)+\operatorname{Tr}_{\mathcal{F}^\perp}(H_D\Gamma_D)\\ &+CN\rho Y\big(Y|\log(Y)|+T|\log(Y)|\rho^{-1}\big)^2.\end{aligned} \tag{3.31}$$

In order to ensure (1.7) we change W_{N_0} and keep the Bogoliubov tranform fixed. By Stones theorem

$$N_0 \mapsto \operatorname{Tr}(\mathcal{N}\Gamma_0) \tag{3.32}$$

is continuous in N_0 and by (3.24) we can make it satisfy

$$\operatorname{Tr}(\mathcal{N}\Gamma_0)=L^2\widetilde{\rho} \tag{3.33}$$

and for $\widetilde{\rho}\leqslant\rho(1+Y^2)$ our state still satisfies (3.31).

□

4 Proof of Lemma 3

In the previous section we saw that we could soften the potential by using the Jastrow factor F, see Lemma 11. In this section we show that the Jastrow factor does not change the entropy much. This essentially follows from a Fannes type inequality [5] (see also Thm 11.6 in [12]), which allows us to control the entropy difference by the trace difference. However, the Fannes inequality only holds on finite spaces. Thus we truncate the density matrices by restricting to eigenfunction with at most M particles in each state and no particles with momenta higher than $\sqrt{K}$. We obtain the following lemma, which implies Lemma 3 by choosing appropriate K and M.

Lemma 13. *Let $K \geqslant 0$, $M \in \mathbb{N}$, $b^2\rho N \ll 1$ and $T \ll T_c$. Then for Γ_0 and Γ_B given as in (1.4) there is a constant $C > 0$ such that*

$$\begin{aligned} -TS(\Gamma_0) \leqslant & -TS(\Gamma_B) + CL^2T^2(L^2T+1)\left(M^{-1} + e^{-\beta K/2}\right) + CL^2T^2\beta K\log(M)\|\Gamma_0 - \Gamma_B\|_1 \\ & + Th(\|\Gamma_0 - \Gamma_B\|_1), \end{aligned} \tag{4.1}$$

where $h(x) = -x\log(x), x \in [0,1]$. Moreover,

$$\|\Gamma_0 - \Gamma_B\|_1 \leqslant C\sqrt{b^2\rho N}. \tag{4.2}$$

Proof of Lemma 3. Let $M = Y^{-1/2}L/b, \beta K = \log(M)$. Our previous choices $L = \rho^{-1/2}Y^{-\alpha}, b = \rho^{-1/2}Y^m$ then imply the Lemma. □

Proof of Lemma 13. Let us start by showing (4.2), i.e., that Γ_0 and Γ_B are close in trace norm. Recall (1.4)

$$\Gamma_0 = \frac{F\Gamma_B F}{\mathrm{Tr}\,(F\Gamma_B F)}, \quad \Gamma_B = \frac{W_{N_0}e^{\mathcal{B}}e^{-\beta\sum_{p\in\Lambda_+^*}D_p a_p^* a_p}\mathbb{1}_{\{\mathcal{N}_0=0\}}e^{-\mathcal{B}}W_{N_0}^*}{\mathrm{Tr}\,\left(e^{-\beta\sum_{p\in\Lambda_+^*}D_p a_p^* a_p}\mathbb{1}_{\{\mathcal{N}_0=0\}}\right)}.$$

Due to the spectral theorem we may write

$$\Gamma_B = \sum_{i\geqslant 1}\lambda_i|\Psi_i\rangle\langle\Psi_i|, \quad \Gamma_0 = \frac{1}{\mathrm{Tr}\,(F\Gamma_B F)}\sum_{i\geqslant 1}\lambda_i|F\Psi_i\rangle\langle F\Psi_i| = \sum_{i\geqslant 1}\lambda_i^F|\Psi_i^F\rangle\langle\Psi_i^F|$$

for decreasing sequences of eigenvalues $\lambda_i, \lambda_i^F \geqslant 0$, $i \in \mathbb{N}$ and normalized eigenfunctions $\Psi_i, \Psi_i^F \in \mathcal{F}(L^2(\Lambda))$ of Γ_B respectively Γ_0. The triangle inequality yields

$$\|\Gamma_0 - \Gamma_B\|_1 \leqslant |1 - \mathrm{Tr}\,(F\Gamma_B F)|\|\Gamma_0\|_1 + \sum_{i\geqslant 1}\lambda_i\||\Psi_i\rangle\langle\Psi_i| - |F\Psi_i\rangle\langle F\Psi_i|\|_1. \tag{4.3}$$

We further estimate the second term on the right-hand side as follows

$$\||\Psi_i\rangle\langle\Psi_i| - |F\Psi_i\rangle\langle F\Psi_i|\|_1 \leqslant 2\|(1-F)\Psi_i\| \leqslant 2\sqrt{\langle\Psi_i, U_{1-f}\Psi_i\rangle},$$

where we used $1 - F_n \leqslant \sum_{i<j}^n 1 - f_{ij}$ for $n \in \mathbb{N}$, recall (3.26), and where U_{1-f} was defined in (3.27). Then, applying the Jensen inequality in (4.3) gives

$$\|\Gamma_0 - \Gamma_B\|_1 \leqslant |1 - \mathrm{Tr}\,(F\Gamma_B F)| + 2\sqrt{\mathrm{Tr}\,(U_{1-f}\Gamma_B)}.$$

We bound the first term with (3.23). The second term can be bounded with (3.28), where as in (3.29) we may show $\widehat{1-f}(0) \leqslant Cb^2$. We find

$$\|\Gamma_0 - \Gamma_B\|_1 \leqslant Cb^2\rho N + C\sqrt{b^2\rho N}.$$

By the choice of our parameters we get (4.2).

Next, we want to show the entropy inequality (4.1). As explained before we need to truncate the problem to a finite dimensional one. Thus let $I \subset \mathbb{N}$ a finite subset and note that the entropy function $h(x) = -x\log(x)$ is concave and positive on the unit interval $[0,1]$. We find

$$S(\Gamma_0) = \sum_{i\geqslant 1} h(\lambda_i^F) \geqslant \sum_{i\in I} h(\lambda_i^F) = S(\Gamma_B) - \sum_{i\notin I} h(\lambda_i) + \sum_{i\in I}\big[h(\lambda_i^F) - h(\lambda_i)\big]. \tag{4.4}$$

Our choice of I will be such that the second term on the right-hand side is small, that is, I includes all entropy-relevant momenta. The third term is bounded with a Fannes type inequality and will only depend logarithmically on the cardinality of I.

Let us start with the third term. We want to show the following Fannes type inequality.

$$\left|\sum_{i\in I}\big[h(\lambda_i^F) - h(\lambda_i)\big]\right| \leqslant \log(|I|)\|\Gamma_0 - \Gamma_B\|_1 + h\big(\|\Gamma_0 - \Gamma_B\|_1\big). \tag{4.5}$$

Since we are not in the setting of the reference [5] we provide a complete proof. As a first step we show

$$\sum_{i\geqslant 1} |\lambda_i^F - \lambda_i| \leqslant \|\Gamma_0 - \Gamma_B\|_1. \tag{4.6}$$

Indeed, this is a very general statement. We may write $\Gamma_0 - \Gamma_B = P - Q$, where P, Q are the projections onto the positive respectively negative eigenspace of $\Gamma_0 - \Gamma_B$. Let $\mu_i(\cdot)$ denote the $i-$th largest eigenvalue of an operator. Then from the definition of λ_i, λ_i^F we find $\lambda_i^F = \mu_i(\Gamma_0) \leqslant \mu_i(Q + \Gamma_0) = \mu_i(P + \Gamma_B) \geqslant \mu_i(\Gamma_B) = \lambda_i$. We obtain

$$\mu_i(Q + \Gamma_0) + \mu_i(P + \Gamma_B) \geqslant 2\max\{\lambda_i^F, \lambda_i\} = \lambda_i^F + \lambda_i + |\lambda_i^F - \lambda_i|.$$

We conclude the proof of (4.6) as follows

$$\|\Gamma_0 - \Gamma_B\|_1 = \mathrm{Tr}\,(P+Q) \geqslant \sum_{i\geqslant 1}\big[\lambda_i^F + \lambda_i + |\lambda_i^F - \lambda_i|\big] - \mathrm{Tr}\,(\Gamma_0 + \Gamma_B) = \sum_{i\geqslant 1} |\lambda_i^F - \lambda_i|.$$

We can now show (4.5) by using $|\lambda_j^F - \lambda_j| \leqslant \sum_{i\in I}|\lambda_i^F - \lambda_i| \leqslant \|\Gamma_0 - \Gamma_B\|_1 \ll 1$ for all $j \in I$, which follows from (4.6) and (4.2), as well as the elementary observations that h is concave, monotone on $[0, 1/e]$ and that for $x, y \in [0,1]$ with $|x-y| \leqslant \frac{1}{2}$ we have $|h(x) - h(y)| \leqslant h(|x-y|)$.

$$\begin{aligned}\left|\sum_{i\in I}\big[h(\lambda_i^F) - h(\lambda_i)\big]\right| &\leqslant |I|\, h\left(\frac{\sum_{i\in I}|\lambda_i^F - \lambda_i|}{|I|}\right) = \log(|I|)\sum_{i\in I}|\lambda_i^F - \lambda_i| + h\left(\sum_{i\in I}|\lambda_i^F - \lambda_i|\right) \\ &\leqslant \log(|I|)\|\Gamma_0 - \Gamma_B\|_1 + h\big(\|\Gamma_0 - \Gamma_B\|_1\big).\end{aligned}$$

As a last step, we show that the second term on the right-hand side of (4.4) is small, i.e. that we did not make too much of an error when truncating the density matrices. We use the explicit knowledge of the eigenvalues of Γ_B; since Γ_B is unitarily equivalent to

$$\Gamma_D = \frac{e^{-\beta\sum_{p\in\Lambda_+^*} D_p a_p^* a_p}\mathbb{1}_{\{\mathcal{N}_0 = 0\}}}{\mathrm{Tr}_{\mathcal{F}^\perp}\left(e^{-\beta\sum_{p\in\Lambda_+^*} D_p a_p^* a_p}\right)},$$

each eigenvalue λ_i of Γ_B uniquely corresponds to a finite sequence $\{n_p\}_{p\in\Lambda_+^*}, n_p \in \mathbb{N}_0$ with

$$\lambda_i = \frac{e^{-\beta\sum_{p\in\Lambda_+^*} D_p n_p}}{\mathrm{Tr}_{\mathcal{F}^\perp}\left(e^{-\beta\sum_{p\in\Lambda_+^*} D_p a_p^* a_p}\right)}.$$

For $K \geqslant 0, M \in \mathbb{N}$ we define $\{n_p\}_{p\in\Lambda_+^*} \in \mathcal{I}$ if and only if $n_p \leqslant M$ for all p and $n_p = 0$ for all $|p| > \sqrt{K}$. We then define $i \in I$ if and only if its corresponding sequence $\{n_p\}_{p\in\Lambda_+^*} \in \mathcal{I}$. In other words, $i \in I$ if and only if its corresponding eigenfunction has at most M particles in each state and no particles with momenta higher than $\sqrt{K}$. We compute

$$\sum_{i\notin I} h(\lambda_i) = \beta \frac{\sum_{\{n_p\}\notin\mathcal{I}} \sum_{p\in\Lambda_+^*} D_p n_p e^{-\beta\sum_{q\in\Lambda_+^*} D_q n_q}}{\mathrm{Tr}_{\mathcal{F}^\perp}\left(e^{-\beta\sum_{p\in\Lambda_+^*} D_p a_p^* a_p}\right)} + \log\left(\mathrm{Tr}\left(e^{-\beta\sum_{p\in\Lambda_+^*} D_p a_p^* a_p}\mathbb{1}_{\{\mathcal{N}_0=0\}}\right)\right) \frac{\sum_{\{n_p\}\notin\mathcal{I}} e^{-\beta\sum_{p\in\Lambda_+^*} D_p n_p}}{\mathrm{Tr}_{\mathcal{F}^\perp}\left(e^{-\beta\sum_{p\in\Lambda_+^*} D_p a_p^* a_p}\right)}. \tag{4.7}$$

From $\mathcal{I}^c \subset \bigcup_{|q|\leqslant\sqrt{K}}\{n_q > M\} \cup \bigcup_{|q|>\sqrt{K}}\{n_q > 0\}$ it follows that

$$\begin{aligned}\sum_{\{n_p\}\notin\mathcal{I}} e^{-\beta\sum_{p\in\Lambda_+^*} D_p n_p} &\leqslant \sum_{|q|\leqslant\sqrt{K}} \sum_{\substack{\{n_p\}:\\ n_q>M}} e^{-\beta\sum_{p\in\Lambda_+^*} D_p n_p} + \sum_{|q|>\sqrt{K}} \sum_{\substack{\{n_p\}:\\ n_q>0}} e^{-\beta\sum_{p\in\Lambda_+^*} D_p n_p} \\ &= \prod_{p\in\Lambda_+^*} \frac{1}{1-e^{-\beta D_p}} \left[\sum_{|q|\leqslant\sqrt{K}} e^{-\beta D_q(M+1)} + \sum_{|q|>\sqrt{K}} e^{-\beta D_q}\right],\end{aligned} \tag{4.8}$$

where the last step is a standard ideal gas computation. This ideal gas computation also shows that $\prod_{p\in\Lambda_+^*} \frac{1}{1-e^{-\beta D_p}} = \mathrm{Tr}_{\mathcal{F}^\perp}\left(e^{-\beta\sum_{p\in\Lambda_+^*} D_p a_p^* a_p}\right)$ and

$$\log\left(\mathrm{Tr}_{\mathcal{F}^\perp}\left(e^{-\beta\sum_{p\in\Lambda_+^*} D_p a_p^* a_p}\right)\right) = -\sum_{p\in\Lambda_+^*} \log\left(1-e^{-\beta D_p}\right) \leqslant C\frac{L^2}{\beta}, \tag{4.9}$$

where in the last step we bounded the sum by an integral. We may also bound the sums in (4.8) with integrals and find

$$\sum_{|q|\leqslant\sqrt{K}} e^{-\beta D_q(M+1)} \leqslant C\frac{L^2}{\beta M} \quad \text{and} \quad \sum_{|q|>\sqrt{K}} e^{-\beta D_q} \leqslant C\frac{L^2}{\beta} e^{-\beta K}. \tag{4.10}$$

Combining (4.8)–(4.10) shows that we may bound the second term on the right-hand side of (4.7) by $CL^2TL^2T\left(M^{-1} + e^{-\beta K}\right)$. We proceed similar for the fist term on the right-hand side of (4.7) and find

$$\beta\frac{\sum_{\{n_p\}\notin\mathcal{I}}\sum_{p\in\Lambda_+^*}D_pn_pe^{-\beta\sum_{q\in\Lambda_+^*}D_qn_q}}{\mathrm{Tr}_{\mathcal{F}^\perp}\left(e^{-\beta\sum_{p\in\Lambda_+^*}D_pa_p^*a_p}\right)}$$

$$\leqslant \beta\sum_{|q|\leqslant\sqrt{K}}\left(\sum_{p\in\Lambda_+^*\setminus\{q\}}\left[\frac{D_p}{e^{\beta D_p}-1}\right]e^{-\beta D_q(M+1)}+D_q\frac{\sum_{n=M+1}^\infty ne^{-\beta D_q n}}{1/(1-e^{-\beta D_q})}\right)$$

$$+\beta\sum_{|q|>\sqrt{K}}\left(\sum_{p\in\Lambda_+^*\setminus\{q\}}\left[\frac{D_p}{e^{\beta D_p}-1}\right]e^{-\beta D_q}+D_q\frac{\sum_{n=1}^\infty ne^{-\beta D_q n}}{1/(1-e^{-\beta D_q})}\right)$$

$$\leqslant CL^2T\sum_{|q|\leqslant\sqrt{K}}e^{-\beta D_q(M+1)}+\sum_{|q|\leqslant\sqrt{K}}D_qe^{-\beta D_q(M+1)}\left((M+1)+\frac{1}{e^{\beta D_q}-1}\right)$$

$$+CL^2T\sum_{|q|>\sqrt{K}}e^{-\beta D_q}+\sum_{|q|>\sqrt{K}}\frac{D_q}{e^{\beta D_q}-1}$$

$$\leqslant CL^2T\left(\frac{L^2T+1}{M}+(L^2T+1)e^{-\beta K/2}\right).$$

Inserting this into (4.7) concludes the bound of the second term in (4.4)

$$\sum_{i\notin I}h(\lambda_i)\leqslant CL^2T(L^2T+1)(M^{-1}+e^{-\beta K/2}).$$

Moreover, we can estimate $|I|\leqslant M^{CL^2K}$ so that from (4.5) we obtain

$$\left|\sum_{i\in I}\left[h(\lambda_i^F)-h(\lambda_i)\right]\right|\leqslant CL^2K\log(M)\|\Gamma_0-\Gamma_B\|_1+h(\|\Gamma_0-\Gamma_B\|_1).$$

Inserting this into (4.4) yields Lemma 13. □

5 Proof of Proposition 4

The proof of Proposition 4 is standard and can be found in [8], respectively [2] for the three dimensional case and its adaption to two dimensions is straightforward. We outline the steps of the proof that need to be modified. The next lemma says that we can extend a periodic state on a fixed box to a state with Dirichlet boundary conditions on the thermodynamic box. The second lemma states, that the information we have on grand canonical states is enough to make a statement about the canonical free energy.

Lemma 14. *Let Γ_L be a normalized density matrix on $\mathcal{F}(\Lambda_L)$ with periodic boundary conditions. Then there is a density matrix $\Gamma_{\widetilde{L}}^D$ that satisfies Dirichlet boundary conditions on the thermodynamic box $\Lambda_{\widetilde{L}}$ such that*

$$\mathrm{Tr}\,\Gamma_{\widetilde{L}}^D=1,\quad \mathrm{Tr}\,(\mathcal{N}\Gamma_{\widetilde{L}}^D)=t^2\mathrm{Tr}\,(\mathcal{N}\Gamma_{L+2\ell}^D),\quad S(\Gamma_{\widetilde{L}}^D)=t^2S(\Gamma_{L+2\ell}^D).$$

and

$$\mathrm{Tr}\,(\mathcal{H}\Gamma_{\widetilde{L}}^D)\leqslant t^2\left(\mathrm{Tr}\,(\mathcal{H}\Gamma_L)+\frac{C}{L\ell}\mathrm{Tr}\,(\mathcal{N}\Gamma_L)\right).$$

Here $\widetilde{L}=t(L+2\ell+R)$ with $R>0$ such that $\mathrm{supp}(v)\subset B(0,R)$.

Proof sketch. From periodic to Dirichlet on fixed boxes. See [2, Lemma A.1] and [8, Lemma 27]. We write Γ_L in its eigenbasis $\Gamma_L = \sum_{j\in\mathbb{N}} \lambda_j |\Psi_j\rangle\langle\Psi_j|$. Then we extend each periodic eigenfunction Ψ_j to a function with Dirichlet boundary conditions on the bigger box $\Lambda_{L+2\ell}$ by multiplying it with an appropriate cosine in all directions. We combine the Dirichlet functions to obtain a state $\Gamma^D_{L+2\ell}$ on $\mathcal{F}(\Lambda_{L+2\ell})$ with Dirichlet boundary conditions and one can verify that

$$S(\Gamma^D_{L+2\ell,u}) = S(\Gamma_L), \quad \mathrm{Tr}\,(\Gamma^D_{L+2\ell,u}\mathcal{N}^j) = \mathrm{Tr}\,(\Gamma_L\mathcal{N}^j)\ \forall j \in \mathbb{N}.$$

and

$$\mathrm{Tr}\,(\mathcal{H}\Gamma^D_{L+2\ell,\overline{u}}) \leqslant \mathrm{Tr}\,(\mathcal{H}\Gamma_L) + \frac{C}{L\ell}\mathrm{Tr}\,(\mathcal{N}\Gamma_L).$$

Extension to the thermodynamic box. See [2, Lemma A.2] and [8, Chapter B.2]. We take the previously constructed state and copy it t^2 times. Additionally, we insert small corridors of size R between the boxes so that the different boxes do not interact with each other. We obtain a thermodynamic trial state $\Gamma^D_{\widetilde{L}}$ and its computation of the number of particles, the entropy and the energy is almost trivial since the different boxes do not interact with each other. □

Lemma 15. *(From grand canonical to canonical) Let $\Gamma^D_{\widetilde{L}}$ be a sequence of normalized states with Dirichlet boundary conditions on $\Lambda_{\widetilde{L}}$ for $\widetilde{L}\infty$ and*

$$\rho = \frac{\mathrm{Tr}\,(\mathcal{N}\Gamma^D_{\widetilde{L}})}{|\Lambda_{\widetilde{L}}|}, \qquad \forall \widetilde{L}.$$

Then

$$f(\widetilde{\rho}, T) \leqslant \liminf_{\widetilde{L}\to\infty} \frac{\mathrm{Tr}\,(\mathcal{H}\Gamma^D_{\widetilde{L}}) - TS(\Gamma^D_{\widetilde{L}})}{|\Lambda_{\widetilde{L}}|}.$$

Proof. We use the equivalence of the canonical and grand-canonical ensemble, see e.g.[13]. The (grand-canonical) pressure is defined as

$$p(\mu, T) := -\lim_{\widetilde{L}\to\infty} |\Lambda_{\widetilde{L}}|^{-1} \inf\{\mathrm{Tr}\,(\mathcal{H} - \mu\mathcal{N})\Gamma) - TS(\Gamma) \mid \Gamma : \mathcal{F} \to \mathcal{F}, \Gamma \geqslant 0, \mathrm{Tr}\,\Gamma = 1\}$$

so that

$$-p(\mu, T) \leqslant \liminf_{\widetilde{L}\to\infty} \frac{\mathrm{Tr}\,(\mathcal{H}\Gamma^D_{\widetilde{L}}) - TS(\Gamma^D_{\widetilde{L}})}{|\Lambda_{\widetilde{L}}|} - \rho\mu.$$

From the equivalence of ensembles:

$$f(\rho, T) = \sup_{\mu}\{\rho\mu - p(\mu, T)\}$$

we obtain

$$f(\widetilde{\rho}, T) \leqslant \liminf_{\widetilde{L}\to\infty} \frac{\mathrm{Tr}\,(\mathcal{H}\Gamma^D_{\widetilde{L}}) - TS(\Gamma^D_{\widetilde{L}})}{|\Lambda_{\widetilde{L}}|}.$$

□

Proof of Proposition 4. This is a direct application of Lemma 14 and Lemma 15. □

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Chapter 7

Paper: Ground state energy of a dilute Bose gas with three-body hard-core interactions

This chapter contains the paper [JV24], of Visconti and the author. The paper provides a short argument using a generalisation of techniques implemented by Dyson [Dys57] to derive an upper bound. The paper is included in its pre-print version, which can be found at https://doi.org/10.48550/arXiv.2406.09019. It can be located within the thesis by the colour ■ at the top of the page.

GROUND STATE ENERGY OF A DILUTE BOSE GAS WITH THREE-BODY HARD-CORE INTERACTIONS

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Abstract

We consider a gas of bosons interacting through a three-body hard-core potential in the thermodynamic limit. We derive an upper bound on the ground state energy of the system at the leading order using a Jastrow factor. Our result matches the lower bound proven by Nam–Ricaud–Triay [15] and therefore resolves the leading order. Moreover, a straightforward adaptation of our proof can be used for systems interacting via combined two-body and three-body interactions to generalise [22, Theorem 1.2] to hard-core potentials.

Contents

1. Introduction

A system of N bosons trapped in a box $\Lambda_L := [0,L]^3$ interacting via three-body interactions can be described by the Hamiltonian operator

$$H_{N,L} = \sum_{i=1}^{N} -\Delta_{x_i} + \sum_{1\leq i<j<k\leq N} w(x_i - x_j, x_i - x_k) \tag{1}$$

acting on the Hilbert space $L^2_s(\Lambda_L^N)$ - the subspace of $L^2(\Lambda_L^N)$ consisting of functions that are symmetric with respect to permutations of the N particles. Such systems have received a lot of attention in recent years and have been the subject of many mathematical works [1, 5–7, 12, 13, 17–19, 21, 23, 24].

In [15], Nam–Ricaud–Triay proved that for a nonnegative, compactly supported potential $w \in L^\infty(\mathbb{R}^6)$, the Hamiltonian (1) satisfies

$$\lim_{\substack{N,L\to\infty \\ N/L^3\to\rho}} \frac{\inf \sigma(H_{N,L})}{N} = \frac{1}{6}\rho^2 b_{\mathcal{M}}(w)(1 + O(Y^\nu)) \tag{2}$$

when $Y := \rho b_{\mathcal{M}}(w)^{3/4} \to 0$, for some constant $\nu > 0$. Here, $b_{\mathcal{M}}(w)$ is the scattering energy associated to w (see [16]). This was then improved in [22], where it was shown that (2) holds for $w \geq 0$ compactly supported and satisfying

$$\|w\|_{L^2L^1} := \left(\int_{\mathbb{R}^3} \|w(x,\cdot)\|^2_{L^1(\mathbb{R}^3)} \, \mathrm{d}x \right)^{1/2} < \infty.$$

It was also shown in [22] that (2) holds with an error in $o(1)$ for w of class L^1. The goal of this paper is to prove that (2) remains valid for particles interacting with a hard-core potential.

We consider a gas of N bosons with three-body hard-core interactions in $\Lambda_L = [0,L]^3$. We are looking for an upper bound on the ground state energy

$$E_{N,L} = \inf \frac{\left\langle \Psi, \sum_{i=1}^N -\Delta_{x_i} \Psi \right\rangle}{\|\Psi\|^2}, \tag{3}$$

with the infimum taken over all $\Psi \in L^2_s(\Lambda_L^N)$ satisfying the three-body hard-core condition $\Psi(x_1,\ldots,x_N) = 0$ if there exist $i,j,k \in \{1,\ldots,N\}$, $i \neq j \neq k \neq i$ with $|(x_i - x_j, x_i - x_k, x_j - x_k)|/\sqrt{3} \leq \mathfrak{a}$. [1] Here, $|\cdot|$ is the euclidean norm in $\mathbb{R}^9$ and $\sum_{i=1}^N -\Delta_{x_i}$ is to be understood in the quadratic form sense in $H^1_s(\Lambda_L^N)$. Note that the scattering energy associated to the hard-core potential

$$w_{\mathrm{hc}}(x-y, x-z) = \begin{cases} +\infty & \text{if } |(x-y, x-z, y-z)|/\sqrt{3} \leq \mathfrak{a}, \\ 0 & \text{otherwise} \end{cases}$$

is given by

$$b_{\mathcal{M}}(w_{\mathrm{hc}}) = \frac{64}{3\sqrt{3}} \pi^2 \mathfrak{a}^4.$$

Theorem 1. *There exists $C > 0$ (independent of $\mathfrak{a}$ and ρ) such that*

$$\lim_{\substack{N,L\to\infty \\ N/L^3 \to \rho}} \frac{E_{N,L}}{N} = \frac{32}{9\sqrt{3}} \pi^2 \rho^2 \mathfrak{a}^4 \left(1 + C\left(\rho \mathfrak{a}^3\right)^\nu\right) \tag{4}$$

for all $\rho\mathfrak{a}^3$ small enough and for some $\nu > 0$.

The matching lower bound was proven in [15]. Here are some remarks on the result:

(1) The strategy of the proof of Theorem 1 can be used to generalise (2) to w of class L^1 with an error uniform in w assuming that $R_0/b_{\mathcal{M}}(w)^{1/4}$ remains bounded, where R_0 is the range of w.

(2) In [14, Conjecture 8], a heuristic approach was used to predict that the ground state energy of a system described by the Hamiltonian (1) should satisfy

$$\lim_{\substack{N,L\to\infty \\ N/L^3 \to \rho}} \frac{\inf \sigma(H_{N,L})}{N} = \frac{1}{6} \rho^2 b_{\mathcal{M}}(w)(1 + C(w)\rho + o(\rho)) \tag{5}$$

in the low density regime $\rho \to 0$, for some constant $C(w)$ depending only on w. The error yielded by the proof of Theorem 1 is of order $(\rho\mathfrak{a}^3)^{4/7} \gg \rho\mathfrak{a}^3$, meaning that it does not capture the error at the correct order. The same problem arises in the two-body case when only using cancellations between the numerator and the denominator similar to the ones used in (18)–(20) (see for example [3]). To extract an error at the correct order in the two-body

[1] Though there is no canonical choice for the three-body hard-core potential, the present choice is motivated by the Physics literature (see e.g. [8, 20]).

case one needs to push the analysis much further and identify additional cancellations, as was done in [2].

(3) A straightforward adaptation of the proof of Theorem 1 can also be used to derive a correct upper bound at the first order of the ground state energy of a system interacting via two-body and three-body interactions. More specifically, the ground state energy $E'_{N,L}$ of a system of N bosons interacting via a two-body hard-core potential of radius $\mathfrak{a}_2$ and a three-body hard-core potential of radius $\mathfrak{a}_3 > \mathfrak{a}_2$ in Λ_L is such that

$$\lim_{\substack{N,L\to\infty \\ N/L^3\to\rho}} \frac{E'_{N,L}}{N} \leq \left(4\pi\rho\mathfrak{a}_2 + \frac{32}{9\sqrt{3}}\pi^2\rho^2\mathfrak{a}_3^4\right)\left(1 + C\left(\rho\mathfrak{a}_2^3\right)^{1/3} + C\left(\rho\mathfrak{a}_3^3\right)^{4/7}\right)$$

for $\rho\mathfrak{a}_2^3$ and $\rho\mathfrak{a}_3^3$ small enough. To prove this one considers a trial state of the form

$$\Psi(x_1,\dots,x_N) = \prod_{1\leq i<j\leq N} \widetilde{f}_{\ell_1}(x_i - x_j) \prod_{1\leq i<j<k\leq N} f_{\ell_2}(x_i - x_j, x_i - x_k),$$

where $\widetilde{f}_{\ell_1}$ describes the two-body correlations up to a distance ℓ_1 and f_{ℓ_2} describes the three-body correlations up to a distance ℓ_2. This generalises [22, Theorem 1.2] to hard-core potentials.

2. Scattering properties of the three-body hard-core potential

Since we are considering a dilute Gas the correlation structure is encoded in the zero-scattering problem

$$(-\Delta_x - \Delta_y - \Delta_z) f(x-y, x-z) = 0 \tag{6}$$

on $\left(\mathbb{R}^3\right)^3$, where f satisfies the conditions $f(x-y,x-z) = 0$ if $|(x-y, x-z, y-z)|/\sqrt{3} \leq \mathfrak{a}$ and $f(\mathbf{x}) \to 1$ as $|\mathbf{x}| \to \infty$. Note that f satisfies the three-body symmetry properties

$$f(x,y) = f(y,x) \quad \text{and} \quad f(x-y,x-z) = f(y-x,y-z) = f(z-x,z-y), \tag{7}$$

for all $x,y,z \in \mathbb{R}^3$.

By removing the centre of mass using the change of variables

$$r_1 = \frac{1}{3}(x+y+z), \quad r_2 = x-y, \quad \text{and} \quad r_3 = x-z, \tag{8}$$

we find that the scattering problem (6) is equivalent to the modified zero-scattering problem

$$-2\Delta_{\mathcal{M}} f(r_2, r_3) = 0 \tag{9}$$

on $\left(\mathbb{R}^3\right)^2$, with f satisfying the conditions $f(r_2,r_3) = 0$ if $\left|\mathcal{M}^{-1}(r_2,r_3)\right| \leq \sqrt{2}\mathfrak{a}$ and $f(\mathbf{x}) \to 1$ as $|\mathbf{x}| \to \infty$. Here we introduced the modified Laplacian

$$-\Delta_{\mathcal{M}} = -|\mathcal{M}\nabla_{\mathbb{R}^6}|^2 = -\operatorname{div}_{\mathbb{R}^6}(\mathcal{M}^2\nabla_{\mathbb{R}^6}),$$

where the matrix $\mathcal{M} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3$ is given by

$$\mathcal{M} := \left(\frac{1}{2}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right)^{1/2} = \frac{1}{2\sqrt{2}}\begin{pmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ \sqrt{3}-1 & \sqrt{3}+1 \end{pmatrix},$$

with inverse

$$\mathcal{M}^{-1} = \left(\frac{2}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\right)^{1/2} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} \end{pmatrix}$$

(see [16] for a more in depth discussion on the matter). Note that $\det\mathcal{M} = \sqrt{3}/2$.

Let f denote a solution to (9) and define $\widetilde{f} := f(\mathcal{M}\cdot)$. Then, $\widetilde{f}$ solves

$$-\Delta\widetilde{f} = 0$$

on $\mathbb{R}^6$, with the conditions $\widetilde{f}(\mathbf{x}) = 0$ for $|\mathbf{x}| \leq \sqrt{2}\mathfrak{a}$ and $\widetilde{f}(\mathbf{x}) \to 1$ as $|\mathbf{x}| \to \infty$. By rewriting the previous problem in hyperspherical coordinates we find that (9) has for unique solution

$$f(\mathbf{x}) = \begin{cases} 1 - \dfrac{4\mathfrak{a}^4}{|\mathcal{M}^{-1}\mathbf{x}|^4} & \text{if } |\mathcal{M}^{-1}\mathbf{x}| > \sqrt{2}\mathfrak{a}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us also define

$$\omega(\mathbf{x}) := 1 - f(\mathbf{x}),$$

for all $\mathbf{x} \in \mathbb{R}^6$.

We shall need a truncated version of f with a cut-off. Let $\widetilde{\chi} \in C^\infty(\mathbb{R}^6; [0,1])$ be a radial function satisfying $\widetilde{\chi}(\mathbf{x}) = 1$ if $|\mathbf{x}| \leq 1/2$ and $\widetilde{\chi}(\mathbf{x}) = 0$ if $|\mathbf{x}| \geq 1$, and define $\chi := \widetilde{\chi}(\mathcal{M}^{-1}\cdot)$. For all $\ell \in (\mathfrak{a}, L)$, we define

$$\chi_\ell := \chi(\ell^{-1}\cdot), \quad \omega_\ell := \chi_\ell\omega, \quad \text{and} \quad f_\ell := 1 - \omega_\ell. \tag{10}$$

Note that ω, f, ω_ℓ and f_ℓ satisfy the three-body symmetry (7). Moreover, they have the following properties:

Lemma 2. *Let $\ell \in (\mathfrak{a}, L)$. Then, we have*

$$|\nabla f_\ell(\mathbf{x})| \leq C\mathfrak{a}^4\ell^{-1}\frac{\mathbb{1}_{\{C_1\ell\leq|\mathbf{x}|\leq C_2\ell\}}}{|\mathbf{x}|^4} \tag{11}$$

and

$$0 \leq 1 - f_\ell^2(\mathbf{x}) \leq C\mathfrak{a}^4\frac{\mathbb{1}_{\{C_1\mathfrak{a}\leq|\mathbf{x}|\leq C_2\ell\}}}{|\mathbf{x}|^4}, \tag{12}$$

for all $\mathbf{x} \in \mathbb{R}^6$. Here, C, C_1, C_2 are universal positive constants such that $C_1 < C_2$. Moreover,

$$\int_{\mathbb{R}^6} \mathrm{d}\mathbf{x}\,\left(|(\mathcal{M}\nabla_{\mathbb{R}^6}f_\ell)(\mathbf{x})|^2\right) \leq \frac{32}{3\sqrt{3}}\mathfrak{a}^4\left(1 + C\left(\frac{\mathfrak{a}}{\ell}\right)^4\right) \tag{13}$$

Furthermore, by defining $\widetilde{\ell} := \sqrt{3/2}\ell$ and

$$g_\ell(x) := \mathbb{1}_{\{|x|\geq\widetilde{\ell}\}}, \tag{14}$$

we have

$$f_\ell(x_1, x_2) \geq \max(g_\ell(x_1), g_\ell(x_2)), \tag{15}$$

for all $x_1, x_2 \in \mathbb{R}^3$.

Proof. Both (11) and (12) follow directly from the definition of f_ℓ and $|\nabla\chi(\mathbf{x})| \leq C\ell^{-1}\mathbb{1}_{\{\ell/2\leq|\mathcal{M}^{-1}\mathbf{x}|\leq\ell\}}$ and $\sigma(\mathcal{M}^{-1}) = \left\{\sqrt{2/3}, \sqrt{2}\right\}$. To compute (13) we first write

$$\begin{aligned}\int_{\mathbb{R}^6}\mathrm{d}\mathbf{x}\Big(|(\mathcal{M}\nabla_{\mathbb{R}^6}f_\ell)(\mathbf{x})|^2\Big) &= \int_{\mathbb{R}^6}\mathrm{d}\mathbf{x}\Big(|(\mathcal{M}\nabla_{\mathbb{R}^6}\omega)(\mathbf{x})|^2\chi_\ell(\mathbf{x})^2\Big) \\ &\quad + 2\int_{\mathbb{R}^6}\mathrm{d}\mathbf{x}\Big((\mathcal{M}\nabla_{\mathbb{R}^6}\omega)(\mathbf{x})\cdot(\mathcal{M}\nabla_{\mathbb{R}^6}\chi_\ell)(\mathbf{x})\omega_\ell(\mathbf{x})\Big) \\ &\quad + \int_{\mathbb{R}^6}\mathrm{d}\mathbf{x}\Big(\omega(\mathbf{x})^2|(\mathcal{M}\nabla_{\mathbb{R}^6}\chi_\ell)(\mathbf{x})|^2\Big).\end{aligned}$$

The only contribution of order $\mathfrak{a}^4$ comes from the first term. Indeed, using again $|\nabla\chi(\mathbf{x})| \le C\ell^{-1}\mathbb{1}_{\{\ell/2\le|\mathcal{M}^{-1}\mathbf{x}|\le\ell\}}$ and (11) we have

$$\int_{\mathbb{R}^6} \mathrm{d}\mathbf{x}\Big(2(\mathcal{M}\nabla_{\mathbb{R}^6}\omega)(\mathbf{x})\cdot(\mathcal{M}\nabla_{\mathbb{R}^6}\chi_\ell)(\mathbf{x})\omega_\ell(\mathbf{x}) + |\omega(\mathbf{x})^2|(\mathcal{M}\nabla_{\mathbb{R}^6}\chi_\ell)(\mathbf{x})|^2\Big) \le C\mathfrak{a}^4\left(\frac{\mathfrak{a}}{\ell}\right)^4.$$

Moreover, by writing $\omega = \widetilde{\omega}(\mathcal{M}^{-1}\cdot)$ with $\widetilde{\omega}(\mathbf{x}) = 4\mathfrak{a}^4/|\mathbf{x}|^4$, we get

$$\begin{aligned}\int_{\mathbb{R}^6} \mathrm{d}\mathbf{x}\Big(|(\mathcal{M}\nabla_{\mathbb{R}^6}\omega)(\mathbf{x})|^2\chi_\ell(\mathbf{x})^2\Big) &= \int_{\mathbb{R}^6}\mathrm{d}\mathbf{x}\Big(\,\big|(\nabla_{\mathbb{R}^6}\widetilde{\omega})(\mathcal{M}^{-1}\mathbf{x})\big|^2\,\widetilde{\chi}_\ell(\mathcal{M}^{-1}\mathbf{x})^2\Big)\\ &= \det\mathcal{M}\int_{\mathbb{R}^6}\mathrm{d}\mathbf{y}\Big(\,|(\nabla_{\mathbb{R}^6}\widetilde{\omega})(\mathbf{y})|^2\,\widetilde{\chi}_\ell(\mathbf{y})^2\Big)\\ &\le \frac{32}{3\sqrt{3}}\pi^2\mathfrak{a}^4.\end{aligned}$$

In the second equality we used the change of variables $\mathbf{y} = \mathcal{M}^{-1}\mathbf{x}$. In the last inequality we used $\nabla_{\mathbb{R}^6}\widetilde{\omega} = 0$ on $B(0,\sqrt{2}\mathfrak{a})$ and $\widetilde{\chi}_\ell(\mathbf{x}) \le \mathbb{1}_{\{|\mathbf{x}|\le\ell\}}$ and $\nabla_{\mathbf{y}}(1/|\mathbf{y}|^4) = -4\mathbf{y}/|\mathbf{y}|^6$ and $\det\mathcal{M} = \sqrt{3}/2$ and that the surface of the 5-dimensional sphere in $\mathbb{R}^6$ is given by $|\mathbb{S}^5| = 8\pi^2/3$. This proves (13).

Finally, notice that $f_\ell(x_1,x_2) = 1$ when $|\mathcal{M}^{-1}(x_1,x_2)|^{-1} \ge \ell$, which is true whenever $|x_1| \ge \widetilde{\ell}$ or $|x_2| \ge \widetilde{\ell}$. This immediately implies (15) and concludes the proof of Lemma 2. □

3. Proof of the upper bound

To get an upper bound on (3), we need to evaluate the energy on an appropriate trial state. To do so, we add correlations among particles to the uncorrelated state $\Psi_{N,L} \equiv 1$. Since correlations are produced mainly by three-body scattering events, we consider the trial state

$$\Psi_{N,L}(x_1,\ldots,x_N) = \prod_{1\le i<j<k\le N} f_\ell(x_i - x_j, x_i - x_k), \tag{16}$$

where ℓ is a parameter satisfying $\mathfrak{a} \ll \ell \ll L$ that will be fixed later; $\Psi_{N,L}$ is clearly an admissible state. The function f_ℓ defined in (10) describes the three-body correlations up to a distance ℓ. Such trial states have been first used in [4, 9, 11] and are usually referred to as Jastrow factors (in [10] Dyson worked with a nonsymmetric trial state describing only nearest neighbour correlations). For readability's sake we from now on write

$$f_{ijk} = f_\ell(x_i - x_j, x_i - x_k)$$

and

$$\nabla_i f_{ijk} = \nabla_{x_i} f_\ell(x_i - x_j, x_i - x_k) \tag{17}$$

for all $i,j,k \in \{1,\ldots,N\}$.

To compute the energy of the trial state (16), we first notice that

$$\nabla_{x_1}\Psi_{N,L}(x_1,\ldots,x_N) = \sum_{2\le p<q\le N}\frac{\nabla_1 f_{1pq}}{f_{1pq}}\prod_{1\le i<j<k\le N} f_{ijk},$$

which when combined with the three-body symmetry (7) implies

$$\frac{\Big\langle \Psi_{N,L}, \sum_{i=1}^N -\Delta_{x_i}\Psi_{N,L}\Big\rangle}{\|\Psi_{N,L}\|^2} = N\frac{\langle\nabla_{x_1}\Psi_{N,L},\nabla_{x_1}\Psi_{N,L}\rangle}{\|\Psi_{N,L}\|^2}$$

$$
\begin{aligned}
&= \frac{N(N-1)(N-2)}{3} \frac{\int \mathrm{d}\mathbf{x}_N \left(\frac{|\mathcal{M}\nabla f_{123}|^2}{f_{123}^2} \prod_{1\le i<j<k\le N} f_{ijk}^2 \right)}{\int \mathrm{d}\mathbf{x}_N \left(\prod_{1\le i<j<k\le N} f_{ijk}^2 \right)} \\
&\quad + N(N-1)(N-2)(N-3) \frac{\int \mathrm{d}\mathbf{x}_N \left(\frac{\nabla_1 f_{123}}{f_{123}} \cdot \frac{\nabla_1 f_{124}}{f_{124}} \prod_{1\le i<j<k\le N} f_{ijk}^2 \right)}{\int \mathrm{d}\mathbf{x}_N \left(\prod_{1\le i<j<k\le N} f_{ijk}^2 \right)} \\
&\quad + \frac{N(N-1)(N-2)(N-3)(N-4)}{4} \frac{\int \mathrm{d}\mathbf{x}_N \left(\frac{\nabla_1 f_{123}}{f_{123}} \cdot \frac{\nabla_1 f_{145}}{f_{145}} \prod_{1\le i<j<k\le N} f_{ijk}^2 \right)}{\int \mathrm{d}\mathbf{x}_N \left(\prod_{1\le i<j<k\le N} f_{ijk}^2 \right)} \\
&=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3.
\end{aligned}
$$

In the second equality we used

$$
\begin{aligned}
&|\nabla_{x_1} f_\ell(x_1 - x_2, x_1 - x_3)|^2 + |\nabla_{x_2} f_\ell(x_1 - x_2, x_1 - x_3)|^2 \\
&\qquad + |\nabla_{x_3} f_\ell(x_1 - x_2, x_1 - x_3)|^2 = 2\, |(\mathcal{M}\nabla_{\mathbb{R}^6} f_\ell)(x_1 - x_2, x_1 - x_3)|^2 .
\end{aligned}
$$

Let us now bound each term one by one. Thanks to (14), we have

$$
\prod_{3\le j<k\le N} f_\ell(x_1 - x_j, x_1 - x_k)^2 \ge \prod_{j=3}^{N} g_\ell(x_1 - x_j)
$$

and

$$
\prod_{3\le j<k\le N} f_\ell(x_2 - x_j, x_2 - x_k)^2 \ge \prod_{j=3}^{N} g_\ell(x_2 - x_j).
$$

Hence, by defining $u_\ell := 1 - f_\ell^2$ and $v_\ell := 1 - g_\ell$ we have the estimate

$$
1 - \sum_{j=3}^{N} v_{1j} - \sum_{j=3}^{N} v_{2j} - \sum_{k=3}^{N} u_{12k} \le \prod_{3\le j<k\le N} f_{1jk}^2 f_{2jk}^2 \prod_{k=3}^{N} f_{12k}^2 \le 1,
$$

where we used the short-hand notations $v_{ij} = v_\ell(x_i - x_j)$ and $u_{ijk} = u_\ell(x_i - x_j, x_i - x_k)$. This allows us to decouple the variables x_1 and x_2 in the numerator and in the denominator of $\mathcal{I}_1$; with (12) and (13) we obtain

$$
\begin{aligned}
\mathcal{I}_1 &\le \frac{N^3}{3} \frac{\int_{\mathbb{R}^6} \mathrm{d}\mathbf{x} \left(|(\mathcal{M}\nabla_{\mathbb{R}^6} f_\ell)(\mathbf{x})|^2 \right)}{L^6 - CL^3 N \int \mathrm{d}x (v_\ell(x)) - CN \int \mathrm{d}\mathbf{x}(u_\ell(\mathbf{x}))} \\
&\le \frac{32}{9\sqrt{3}} \pi^2 \frac{N\rho^2 \mathfrak{a}^4 (1 + C(\mathfrak{a}/\ell)^4)}{1 - C\rho\ell^3 - C\rho\ell^2\mathfrak{a}^4/L^3} \\
&\le \frac{32}{9\sqrt{3}} \pi^2 N\rho^2 \mathfrak{a}^4 \left(1 + C\left(\frac{\mathfrak{a}}{\ell}\right)^4 + C\rho\ell^3 \right),
\end{aligned}
\tag{18}
$$

under the assumption that $\rho\ell^3 \ll 1$ and $\mathfrak{a} \ll \ell \ll L$. In the last inequality we used $\ell^2\mathfrak{a}^4/L^3 \le \ell^3$. To bound $\mathcal{I}_2$ we similarly decouple the variables x_2, x_3 and x_4. Using again (11) and (12) we can bound

$$
\begin{aligned}
\mathcal{I}_2 &\le CN^4 \frac{\int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z (|\nabla f_\ell(x,y)| \cdot |\nabla f_\ell(x,z)|)}{L^9 - CNL^6 \int \mathrm{d}x(v_\ell(x)) - CNL^3 \int \mathrm{d}\mathbf{x}(u_\ell(\mathbf{x}))} \\
&\le CN\rho^2\mathfrak{a}^4 \left[\rho\mathfrak{a}^4\ell^{-1}\right] \left(1 + C\rho\ell^3\right) \\
&\le CN\rho^2\mathfrak{a}^4 \left(\rho\ell^3\right),
\end{aligned}
\tag{19}
$$

when $\rho\ell^3 \ll 1$ and $\mathfrak{a} \ll \ell \ll L$. Analogously, we bound $\mathcal{I}_3$ by decoupling the variables x_1, x_2 and x_4. Namely, using once more (11) and (12) we get

$$\begin{aligned}\mathcal{I}_3 &\leq CN^5 \frac{\left(\int \mathrm{d}\mathbf{x} |\nabla f_\ell(\mathbf{x})|\right)^2}{L^{12} - CNL^9 \int \mathrm{d}x(v_\ell(x)) - CNL^6 \int \mathrm{d}\mathbf{x}(u_\ell(\mathbf{x}))} \\ &\leq CN\rho^2\mathfrak{a}^4 \left[\rho^2\mathfrak{a}^4\ell^2\right]\left(1 + C\rho\ell^3\right) \\ &\leq CN\rho^2\mathfrak{a}^4\left(\rho\ell^3\right) \end{aligned} \tag{20}$$

again under the condition that $\rho\ell^3 \ll 1$ and $\mathfrak{a} \ll \ell \ll L$. From (18)–(20) we conclude that

$$E_{N,L} \leq \frac{32}{9\sqrt{3}}\pi^2 N\rho^2\mathfrak{a}^4\left(1 + C\left(\frac{\mathfrak{a}}{\ell}\right)^4 + C\rho\ell^3\right).$$

Taking $\ell = \mathfrak{a}\left(\rho\mathfrak{a}^3\right)^{-1/7}$ finishes the proof of Theorem 1.

Acknowledgments.

We thank Arnaud Triay for his precious feedback. L. J. was partially supported by the European Union. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them. L. J. was partially supported by the Villum Centre of Excellence for the Mathematics of Quantum Theory (QMATH) with Grant No.10059. L. J. was supported by the grant 0135-00166B from Independent Research Fund Denmark. F. L. A. V. acknowledges partial support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the TRR 352 Project ID. 470903074 and by the European Research Council through the ERC CoG RAMBAS Project Nr. 101044249.

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Chapter 8

Derivation of Hartree theory for two-dimensional attractive Bose gases in very dilute regime

This chapter contains the paper , of Visconti and the author. Here we provide a proof of convergence towards hartree energy in much more singular regmies than previously considered. The paper is included in its pre-print version, which can be found at https://doi.org/10.48550/arXiv.2410.23150. It can be located within the thesis by the colour ■ at the top of the page.

DERIVATION OF HARTREE THEORY FOR TWO-DIMENSIONAL ATTRACTIVE BOSE GASES IN VERY DILUTE REGIME

LUKAS JUNGE AND FRANÇOIS L. A. VISCONTI

Abstract. We study the ground state energy of trapped two-dimensional Bose gases with mean-field type interactions that can be attractive. We prove the stability of second kind of the many-body system and the convergence of the ground state energy per particle to that of a non-linear Schrödinger (NLS) energy functional. Notably, we can take any polynomial scaling of the interaction, and even exponential scalings arbitrarily close to the Gross–Pitaevskii regime, which is a drastic improvement on the best-known result for systems with attractive interactions. As a consequence of the stability of second kind we also obtain Bose–Einstein condensation for the many-body ground states for a much improved range of the diluteness parameter.

Contents

1. Introduction

We consider N two-dimensional bosons interacting via a pair potential w_N and trapped by an external potential V. The system is described by the N-body Hamiltonian

$$H_N = \sum_{i=1}^{N} (-\Delta_{x_i} + V(x_i)) + \frac{1}{N-1} \sum_{1\leq i<j\leq N} w_N(x_i - x_j) \tag{1}$$

acting on

$$\mathfrak{H}^N = \bigotimes_{\text{sym}}^{N} \mathfrak{H},$$

the symmetric tensor product of N copies of the one-body Hilbert space $\mathfrak{H} := L^2(\mathbb{R}^2)$. The interaction potential w_N and the trapping potential V satisfy the following assumptions.

Assumption 1. The two-body interaction is either of the form

$$w_N = N^{2\beta} w(N^{\beta}\cdot), \tag{2}$$

for some fixed $\beta > 0$, or of the form

$$w_N = e^{2N^{\kappa}} w(e^{N^{\kappa}}\cdot), \tag{3}$$

for some $0 < \kappa < 1$. The function $w : \mathbb{R}^2 \to \mathbb{R}$ is fixed and satisfies

$$w \in L^1(\mathbb{R}^2) \cap L^{1+\eta}(\mathbb{R}^2) \quad \text{and} \quad w(x) = w(-x),$$

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for some $\eta > 0$. The external trapping potential $V : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$V(x) \geq C^{-1}|x|^s - c \tag{4}$$

for some constants $s > 0$ and $c, C > 0$.

Under these assumptions the Hamiltonian H_N is bounded from below with the core domain $\mathfrak{H}^N \cap C_c^\infty(\mathbb{R}^{2N})$ and can thus be extended to a self-adjoint operator by Friedrichs' method.

The system we are considering is a mean-field system and is expected to exhibit Bose–Einstein condensation, meaning that almost all particles would live in the same quantum state. In terms of wavefunctions this roughly translates to

$$\Psi(x_1, \ldots, x_N) \approx u^{\otimes N}(x_1, \ldots, x_N) \coloneqq u(x_1) \cdots u(x_N). \tag{5}$$

Taking the trial state wavefunction $u^{\otimes N}$, it is therefore natural to look at the Hartree energy functional

$$\mathcal{E}_N^{\mathrm{H}}[u] = \frac{\langle u^{\otimes N}, H_N u^{\otimes N} \rangle}{N} = \langle u, hu \rangle + \frac{1}{2} \int_{\mathbb{R}^2} (w_N * |u|^2)|u|^2, \tag{6}$$

where we defined

$$h \coloneqq -\Delta + V.$$

Since

$$w_N \rightharpoonup a\delta_0 \quad \text{with} \quad a \coloneqq \int_{\mathbb{R}^2} w$$

in the limit $N \to \infty$, the Hartree functional (6) *formally* converges to the non-linear Schrödinger (NLS) energy functional

$$\mathcal{E}^{\mathrm{nls}}[u] = \langle u, hu \rangle + \frac{a}{2} \int_{\mathbb{R}^2} |u|^4. \tag{7}$$

The speed at which the convergence of the Hartree functional (6) to the NLS functional (7) occurs is measured by the parameter β in the polynomial case and κ in the exponential case. In general, the faster the convergence, the harder the analysis.

For the repulsive case $w \geq 0$, it is known that the limiting energy functional (7) admits a subtle correction when the potential scales exponentially with N. Indeed, when taking $\kappa = 1$ in (3) and removing the mean-field factor $(N-1)^{-1}$ in (1), we obtain the so-called *Gross–Pitaevskii regime*, for which the convergence to point-wise like interactions still occurs, but where the limiting functional is now (7) with $a \approx 8\pi / \log(N/\mathfrak{a}^2)$, where $\mathfrak{a}$ is the scattering length of w [15–17]. This is because the ground state of (1) includes a non-trivial correction to the ansatz (5), in the form of a short-range correlation structure.

When w is not purely repulsive (e.g. $w \leq 0$), the problem is more difficult since we have to deal with the issue of the system's stability. More precisely, the energy $\mathcal{E}^{\mathrm{nls}}$ is bounded from below under the constraint $\|u\|_2 = 1$ if and only if

$$a \geq -a^*, \tag{8}$$

where a^* is the optimal constant of the Gagliardo–Nirenberg inequality

$$\left(\int_{\mathbb{R}^2} |\nabla u|^2 \right) \left(\int_{\mathbb{R}^2} |u|^2 \right) \geq \frac{a^*}{2} \int_{\mathbb{R}^2} |u|^4, \quad \forall u \in H^1\left(\mathbb{R}^2\right). \tag{9}$$

Thus, $a \geq -a^*$ is a necessary condition for stability [6, 18, 25, 26]. Furthermore, the Hartree energy functional (6) is stable in the limit $N \to \infty$ only if

$$\inf_{u \in H^1(\mathbb{R}^2)} \left(\frac{\displaystyle\int (w * |u|^2)|u|^2}{2\|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2} \right) \geq -1 \tag{10}$$

and we shall thus say that w is *Hartree stable* if (10) holds. Indeed, if (10) fails to hold, then the ground state energy of the Hartree functional (6) converges to $-\infty$ when $N \to \infty$, as was shown in [11, Proposition 2.3]. Hence, Hartree stability (10) is a *necessary* condition for stability of second kind [14] of the many-body system:

$$H_N \geq -CN. \tag{11}$$

The goal of the present paper is to show that the Hartree stability is also a *sufficient* condition to ensure the stability of second kind of the many-body system. For practical reasons, we need to assume the slightly stronger *strict Hartree stability*

$$\inf_{u \in H^1(\mathbb{R}^2)} \left(\frac{\displaystyle\int (w * |u|^2)|u|^2}{2\|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2} \right) > -1. \tag{12}$$

Note also that (12) in particular holds if

$$\int_{\mathbb{R}^2} w^- < a^*,$$

where $w^- := -\min(0, w)$ is the negative part of w. Such an assumption was for example considered in [20]. In the polynomial case (2) and under the condition (12), the many-body stability (11) has been proved in [10, 11] for $\beta > 0$ sufficiently small, and later in [20] for $0 < \beta < 1$. In the present work, we extend this stability result to all $\beta \in (0, \infty)$ and to exponential scalings (3) for all $\kappa \in (0, 1)$.

Alongside proving stability of second kind, we also prove the convergence of the many-body ground state energy to that of the NLS energy functional (7) for any $\beta \in (0, \infty)$ (or any $\kappa \in (0, 1)$). Consequently, we also obtain the convergence of the many-body ground states to those of the NLS functional (7). Though both the *defocusing* ($a \geq 0$) and *focusing* ($a \leq 0$) cases are covered, the main novelty of the paper lies in the latter case.

Acknowledgments. We thank Phan Thành Nam for his continuous support and his precious feedback. L. J. was partially supported by the European Union. Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them. L. J. was partially supported by the Villum Centre of Excellence for the Mathematics of Quantum Theory (QMATH) with Grant No.10059. L. J. was supported by the grant 0135-00166B from Independent Research Fund Denmark. F. L. A. V. acknowledges partial support by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the TRR 352 Project ID. 470903074 and by the European Research Council through the ERC CoG RAMBAS Project Nr. 101044249.

2. Main result

2.1. **Notations.**

For $\psi_1 \in \mathfrak{H}^{N_1}$ and $\psi_2 \in \mathfrak{H}^{N_2}$, we define the symmetric tensor product $\psi_1 \otimes_s \psi_2 \in \mathfrak{H}^{N_1+N_2}$ as follows:

$$\psi_1 \otimes_s \psi_2(x_1, \dots, x_{N_1+N_2}) := \frac{1}{\sqrt{N_1! N_2! (N_1+N_2)!}} \times \sum_{\sigma \in \mathcal{S}_{N_1+N_2}} \psi_1(x_{\sigma(1)}, \dots, x_{\sigma(N_1)}) \psi_2(x_{\sigma(N_1+1)}, \dots, x_{\sigma(N_1+N_2)}).$$

Here $\mathcal{S}_N$ is the group of permutations of $\{1, \dots, N\}$. We denote the i-fold tensor product of a vector $f \in \mathfrak{H}$ by $f^{\otimes i} \in \mathfrak{H}^i$, and the i-fold tensor product of an operator $A : \mathfrak{H} \to \mathfrak{H}$ by $A^{\otimes i}$. We denote by

$$\mathcal{S}(\mathfrak{X}) := \left\{ \Gamma \in \mathfrak{S}^1(\mathfrak{X}) : \Gamma = \Gamma^* \geq 0, \operatorname{Tr}_{\mathfrak{X}} \Gamma = 1 \right\}$$

the set of all states on a given Hilbert space $\mathfrak{X}$. Here $\mathfrak{S}^1(\mathfrak{X})$ is the space of all trace-class operators on $\mathfrak{X}$ [23]. The k-particle reduced density matrix of a given state $\Gamma \in \mathcal{S}(\mathfrak{H}^N)$ is obtained by taking the partial trace over all but the first k variables:

$$\Gamma^{(k)} := \mathrm{Tr}_{k+1\to N}(\Gamma).$$

Moreover, we denote the k-particle reduced density matrix of a normalised wavefunction $\Psi \in \mathfrak{H}^N$ by

$$\gamma_\Psi^{(k)} := \mathrm{Tr}_{k+1\to N} |\Psi\rangle\langle\Psi|.$$

2.2. **Statement of the main result.** We prove the convergence of the many-body ground state energy per particle

$$e_N := \frac{1}{N} \inf \sigma(H_N) = \frac{1}{N} \inf \left\{ \langle \Psi, H_N \Psi \rangle : \Psi \in \mathfrak{H}^N, \|\Psi\| = 1 \right\}$$

to that of the *NLS* functional (7)

$$e_{\mathrm{nls}} := \inf \left\{ \mathcal{E}^{\mathrm{nls}}[u] : u \in \mathfrak{H}, \|u\| = 1 \right\}.$$

The convergence of the ground states of H_N to those of $\mathcal{E}^{\mathrm{nls}}$ follows directly from the convergence of the energies, thanks to arguments from [11].

Theorem 2 (Convergence to NLS theory)**.** *Let w_N and V satisfy Assumption 1. Then,*

$$\boxed{\lim_{N\to\infty} e_N = e_{\mathrm{nls}} > -\infty}$$

and

$$\boxed{H_N \geq -CN,}$$

for some constant $C > 0$ that depends only on w and V. Moreover, for a sequence $\{\Psi_N\}_N$ of ground states of H_N, there exists a Borel probability measure μ supported on the minimisers of $\mathcal{E}^{nls}$ such that, along a subsequence,

$$\lim_{N\to\infty} \mathrm{Tr} \left| \gamma_{\Psi_N}^{(k)} - \int |u^{\otimes k}\rangle\langle u^{\otimes k}| \, \mathrm{d}\mu(u) \right| = 0, \quad \forall k \in \mathbb{N}.$$

If $\mathcal{E}^{nls}$ has a unique minimiser u_0 (up to a phase), then for the whole sequence

$$\lim_{N\to\infty} \mathrm{Tr} \left| \gamma_{\Psi_N}^{(k)} - |u_0^{\otimes k}\rangle\langle u_0^{\otimes k}| \right| = 0, \quad \forall k \in \mathbb{N}.$$

Remarks.

(1) Thanks to the diamagnetic inequality [13, Theorem 7.21], we can easily add an external magnetic field $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$ with $\mathbf{A} \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ to the system, meaning that $-\Delta_{x_j}$ can be replaced by $(\mathbf{i}\nabla_{x_j} + \mathbf{A}(x_j))^2$ in (1). Moreover, the same proof can also be used when working on the unit torus or in a finite box with Dirichlet boundary conditions instead of a trapping potential V. Heuristically, one can think of this as $s = \infty$ in (4).

(2) When considering the dynamics of a two-dimensional Bose gas with attractive interactions, that is when trying to prove that Bose–Einstein condensation is preserved by the Hamiltonian flow $e^{\mathbf{i}tH_N}$, one is also confronted with issues of stability of second kind. The best-known result in this direction is the range $0 < \beta < 1$ and follows from the method of [8] and the stability of second kind proven in [20]. It would be interesting to know whether Theorem 2 allows for an improvement of the previous result. More restrictive ranges had previously been obtained in [19] (without the use of (11)) and in [4] (for $s = 2$). See [5, 21] for related results in 3D. In the defocusing case $w \geq 0$, we refer to [7, 9] for results in 2D and [1, 4] for the effectively 2D dynamics of strongly confined 3D systems. We mention as well that the instability regime, that is $a < -a^*$ (see (9)), poses natural and interesting questions for the dynamics, namely whether Bose–Einstein

condensation persists until the blow-up of the Hartree solution. See [2] for new results in this direction.

(3) Stability of second kind is an issue as well for three-dimensional Bose gases with interaction potentials

$$N^{3\beta}w(N^{\beta}\cdot)$$

having an attractive part in the dilute regime $\beta > 1/3$. For stability of second kind to hold one must make the additional assumption that the potential is classically stable [11, 24]. Regrettably, a straightforward adaptation of the proof of Theorem 2 only works up to $\beta < 1/3$, thus failing to capture the dilute regime. The best-known results in the dilute regime are the ranges $1/3 < \beta < 1/3 + s/(45 + 42s)$ [24], where s is the exponent in (4), and $1/3 < \beta < 9/26$ [20].

2.3. **Strategy of the proof.** We wish to compare the many-body ground state energy per particle e_N to that of the Hartree functional (6)

$$e_N^{\mathrm{H}} = \inf\left\{\mathcal{E}_N^{\mathrm{H}}[u] : u \in \mathfrak{H}, \|u\| = 1\right\},$$

and then use the convergence

$$\lim_{N\to\infty} e_N^{\mathrm{H}} = e_{\mathrm{nls}}$$

(see [10, Lemma 7]). The upper bound $e_N \leq e_N^{\mathrm{H}}$ can immediately be obtained using the trial state $u^{\otimes N}$, and the difficult part is to prove a matching lower bound. Note that, for practical reasons, we shall not compare e_N to e_N^{H}, but to the ground state $\tilde{e}_N^{\mathrm{H}}$ of a slightly modified Hartree functional, which also converges to e_{nls}.

To prove the lower bound, we shall use a localisation technique in momentum space in order to reduce the infinite dimensional problem to multiple finite dimensional ones (similarly to [10, 11, 20]). Then, we shall apply the following quantitative version of the quantum de Finetti theorem [3, 12].

Theorem 3. *Given a Hilbert space $\mathfrak{X}$ of dimension d and a symmetric state $\Gamma_K \in S(\mathfrak{X}^K)$, there exists a probability measure μ on $S(\mathfrak{X})$ such that*

$$\left|\mathrm{Tr}\left[A \otimes B\left(\Gamma_K^{(2)} - \int_{S(\mathfrak{X})} \gamma^{\otimes 2} d\mu(\gamma)\right)\right]\right| \leq C\sqrt{\frac{\log d}{K}}\|A\|_{\mathrm{op}}\|B\|_{\mathrm{op}}$$

for all self-adjoint operators A and B on $\mathfrak{X}$, and for some universal constant $C > 0$.

Proof. In [3], the statement was proven with A, B replaced by quantum measurement. See [22, Proposition 3.3] and [20, Lemma 3.3] for an adaptation to self-adjoint operators. □

We shall apply Theorem 3 on energy subspaces of the one-body Schrödinger operator h acting on $\mathfrak{H}$ defined by the spectral projections

$$P_1 := \mathbb{1}_{\left\{\sqrt{h} < N^{\varepsilon}\right\}} \quad \text{and} \quad P_i := \mathbb{1}_{\left\{N^{(i-1)\varepsilon} \leq \sqrt{h} < N^{i\varepsilon}\right\}},\ 2 \leq i \leq M, \tag{13}$$

for some $\varepsilon, M > 0$. Note that thanks to Assumption (4) we have the Cwikel–Lieb–Rosenblum (CLR) type estimate

$$\dim\left(\mathbb{1}_{\left\{\sqrt{h} < N^{i\varepsilon}\right\}}\mathfrak{H}\right) \leq CN^{(2+4/s)i\varepsilon}, \tag{14}$$

for all $i \in \{1, \dots, M\}$ (see [11, Lemma 3.3] and references therein). When working on the unit torus or on a box with Dirichlet boundary conditions, the estimate (14) should be replaced by the usual Weyl asymptotic.

Before applying Theorem 3 we shall write the energy of a ground state Ψ of H_N as

$$\langle \Psi, H_N \Psi \rangle = \frac{N}{2}\,\mathrm{Tr}\left(H_{2,N}\Gamma^{(2)}\right)$$

using the two-particle reduced density matrix $\Gamma^{(2)}$ of Ψ and the two-body Hamiltonian

$$H_{2,N} \coloneqq h_1 + h_2 + w_N(x_1 - x_2) =: T + w_N.$$

Decomposing the identity according to the spectral projections defined in (13) we will then roughly get

$$\operatorname{Tr}\left(H_{2,N}\Gamma^{(2)}\right) \gtrsim \sum_{\substack{1\le i_1,i_2\le M\\ 1\le i_1',i_2'\le M}} \operatorname{Tr}\left(P_{i_1}\otimes P_{i_2} H_{2,N} P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}\right),$$

where the Sobolev inequality shall be used to neglect the terms containing the last spectral projection $P_{M+1} \coloneqq \mathbb{1}_{\left\{N^{M\varepsilon}\le\sqrt{h}\right\}}$.

After that, we will decompose the many-body state Ψ according to the occupancy of its energy levels (defined by the spectral projections 13), namely

$$\operatorname{Tr}\left(H_{2,N}\Gamma^{(2)}\right) \gtrsim \sum_{\underline{J},\underline{J}'} \sum_{\substack{1\le i_1,i_2\le M\\ 1\le i_1',i_2'\le M}} \operatorname{Tr}\left(P_{i_1}\otimes P_{i_2} H_{2,N} P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J},\underline{J}'}\right),$$

where the sum is taken over the multi-indices $\underline{J} = (j_1,\dots,j_{M+1})$ satisfying $|\underline{J}| = N$ and with

$$\Gamma_{\underline{J},\underline{J}'} = |\Psi_{\underline{J}}\rangle\langle\Psi_{\underline{J}'}| \quad \text{and} \quad \Psi_{\underline{J}} = P_1^{\otimes j_1}\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}} P_{M+1}^{\otimes j_{M+1}}\Psi.$$

Then, we shall define

$$i_{\max}(\underline{J}) \coloneqq \max\left\{i\in\{1,\dots,M\} : j_i \ge N^{1-\delta\varepsilon}\right\},$$

for some $\delta > 0$, and make an important distinction between the terms that satisfy $i_{\max}(\underline{J}) = i_{\max}(\underline{J}')$ and for which i_1, i_2, i_1' and i_2' are all less than $i_{\max}(\underline{J})$, and the other terms. In the latter case, we shall use Proposition 4 and Lemma 5 (see below) to show that their potential energy is much smaller than the overall kinetic energy of the system and that their contribution to the many-body energy is therefore negligible. Doing so, we shall be left with

$$\operatorname{Tr}\left(H_{2,N}\Gamma^{(2)}\right) \gtrsim \sum_{\substack{\underline{J},\underline{J}'\\ i_{\max}(\underline{J})=i_{\max}(\underline{J}')}} \sum_{\substack{i_1,i_2=1\\ i_1',i_2'=1}}^{i_{\max}(\underline{J})} \operatorname{Tr}\left(P_{i_1}\otimes P_{i_2} H_{2,N} P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J},\underline{J}'}\right), \tag{15}$$

where a small amount of the kinetic energy has been sacrificed to bound the error terms. Fixing $i = i_{\max}(\underline{J})$ and defining $\mathbb{P}_i = \sum_{k=1}^{i} P_k$, we can rewrite the sum over i_1, i_2, i_1', i_2' as

$$\sum_{\substack{i_1,i_2=1\\ i_1',i_2'=1}}^{i} \operatorname{Tr}\left(P_{i_1}\otimes P_{i_2} H_{2,N} P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J},\underline{J}'}\right) = \operatorname{Tr}\left(\mathbb{P}_i\otimes\mathbb{P}_i H_{2,N}\mathbb{P}_i\otimes\mathbb{P}_i\Gamma^{(2)}_{\underline{J},\underline{J}'}\right),$$

which is almost of the right form to apply Theorem 3 on $\mathfrak{X} = \mathbb{P}_i\mathfrak{H}$.

The only remaining obstacle is that, instead of having $\Gamma_{\underline{J},\underline{J}'}$, which is in general not a even a state, we would like to have a state belonging to $\mathcal{S}(\mathfrak{X}^K)$ for some K. In other words, we would like to know exactly how many particles have momenta in $\mathbb{P}_i$, and discard the information about the others. This can rigorously be done by not only fixing $i = i_{\max}(\underline{J})$, but also the number K of particles that have momenta in $\mathbb{P}_i$, and using Lemma 6 to construct a state $\gamma_{i,K} \in \mathcal{S}(\mathbb{P}_i^{\otimes K}\mathfrak{H}^K)$. Having done so, we shall get

$$\operatorname{Tr}\left(H_{2,N}\Gamma^{(2)}\right) \gtrsim \sum_{i=1}^{M}\sum_{K} \binom{N}{2}^{-1}\binom{K}{2}\|\Psi_{i,K}\|^2 \operatorname{Tr}\left(H_{2,N}\gamma^{(2)}_{i,K}\right), \tag{16}$$

where

$$\Psi_{i,K} := \sum_{\substack{\underline{J} \\ i_{\max}(\underline{J})=i \\ j_{\max}(\underline{J})=K}} \Psi_{\underline{J}} \quad \text{with} \quad j_{\max}(\underline{J}) := \sum_{k=1}^{i_{\max}(\underline{J})} j_k.$$

For each state $\gamma_{i,K}$, an application of Theorem 3 - which can be done since $\mathbb{P}_i\mathfrak{H}$ has finite dimension controlled by the CLR type estimate (14) - shall yield

$$\operatorname{Tr}\left(H_{2,N}\gamma_{i,K}^{(2)}\right) \gtrsim \int_{\mathcal{S}(\mathbb{P}_i\mathfrak{H})} \operatorname{Tr}\left(H_{2,N}\gamma^{\otimes 2}\right) \mathrm{d}\mu(\gamma) \gtrsim 2\tilde{e}_N^{\mathrm{H}}, \tag{17}$$

where $\tilde{e}_N^{\mathrm{H}}$ is the ground state energy of a slightly modified version of the Hartree functional 6 that satisfies

$$\tilde{e}_N^{\mathrm{H}} \geq -C \quad \text{and} \quad \lim_{N\to\infty} \tilde{e}_N^{\mathrm{H}} = e_{\mathrm{nls}}.$$

Finally, injecting (17) into (16) and controlling the error terms using Proposition 4 shall yield the desired result.

Proposition 4 (Plane wave estimate). *Let $\mathbf{e}_k$ be the multiplication operator on $\mathfrak{H}$ by $\cos(k\cdot x)$ or $\sin(k\cdot x)$. Let $P_i := \mathbb{1}_{\{\sqrt{h}<N^{i\varepsilon}\}}$. Then, for all $p \in \mathbb{N}_0$ and $k \in \mathbb{R}^2 \setminus \{0\}$,*

$$\pm P_i\mathbf{e}_k P_i \leq C_p \frac{N^{pi\varepsilon}}{|k|^p} P_i, \tag{18}$$

for N large enough and for $C_p > 0$ depending only on p. Consequently, for all $i_1, i_2 \in \{1,\ldots,M\}$, we have

$$P_{i_1} \otimes P_{i_2} |w_N(x-y)| P_{i_1} \otimes P_{i_2} \geq -CN^{2\min(i_1,i_2)\varepsilon} P_{i_1} \otimes P_{i_2}, \tag{19}$$

for N large enough and for some constant $C > 0$ (depending only on $\|w\|_{L^1}$).

Lemma 5. *Let P_1, P_2 and Q be orthogonal projections on $\mathfrak{H}$. Given a state*

$$\Gamma \in \mathcal{S}\left(P_1^{\otimes j_1} \otimes_s P_2^{\otimes j_2} \otimes_s Q^{\otimes(N-j_1-j_2)}\mathfrak{H}^N\right),$$

for some $0 \leq j_1, j_2 \leq N$, we have

$$\operatorname{Tr}\left(P_1\Gamma^{(1)}\right) = \frac{j_1}{N}, \quad \operatorname{Tr}\left(P_1^{\otimes 2}\Gamma^{(2)}\right) = \frac{j_1(j_1-1)}{N(N-1)} \tag{20}$$

and

$$\operatorname{Tr}\left(P_1 \otimes P_2\Gamma^{(2)}\right) = \frac{j_1 j_2}{N(N-1)}. \tag{21}$$

Lemma 6. *Let P and Q be orthogonal projections on $\mathfrak{H}$. Given a state*

$$\Gamma \in \mathcal{S}\left(P^{\otimes j} \otimes_s Q^{\otimes(N-j)}\mathfrak{H}^N\right),$$

for some $j \geq 1$, there exists another state

$$\Gamma_j \in \mathcal{S}\left(P^{\otimes j}\mathfrak{H}^j\right)$$

such that

$$P \otimes P\Gamma^{(2)} P \otimes P = \binom{N}{2}^{-1} \binom{j}{2} \Gamma_j^{(2)}. \tag{22}$$

Organisation of the paper. We prove Theorem 2 in Section 3. Then, we prove Proposition 4, as well as Lemmas 5 and 6 in Section 4. Lastly, an important technical lemma used in the proof of Theorem 2 is proven in Section 5.

3. Proof of Theorem 2

We begin the proof by splitting $L^2(\mathbb{R}^2)$ into $M+1$ annuli in momentum space according to

$$P_1 := \mathbb{1}_{\{\sqrt{h}<N^\varepsilon\}}, \quad P_i := \mathbb{1}_{\{N^{(i-1)\varepsilon}\le\sqrt{h}<N^{i\varepsilon}\}}, \quad 2\le i\le M \tag{23}$$

and

$$P_{M+1} := \mathbb{1}_{\{N^{M\varepsilon}\le\sqrt{h}\}}, \tag{24}$$

for some $\varepsilon, M>0$ that shall be chosen later. Moreover, we define

$$\mathbb{P}_j := \sum_{i=1}^{j} P_i \quad \text{and} \quad \mathbb{Q}_j := \sum_{i=j+1}^{M+1} P_i.$$

In the polynomial case, that is for w_N of the form (2), the parameter ε shall be universal, and M shall be a constant taken such that

$$2\beta + \frac{1+\eta}{\eta}\varepsilon < M\varepsilon < 4\beta + \frac{1+\eta}{\eta}\varepsilon, \tag{25}$$

where $\eta>0$ is such that $w\in L^{1+\eta}(\mathbb{R}^2)$. In the exponential case, that is w_N of the form (3), ε shall only depend on κ. The parameter M shall be taken such that

$$\frac{2N^\kappa}{\log N} + \frac{1+\eta}{\eta}\varepsilon < M\varepsilon < \frac{4N^\kappa}{\log N} + \frac{1+\eta}{\eta}\varepsilon, \tag{26}$$

where $\eta>0$ is again such that $w\in L^{1+\eta}(\mathbb{R}^2)$. In both (25) and (26), the lower bound arises when getting rid of the projection P_{M+1}, and the upper bound states that we do not want M too large.

Let Ψ be the ground state of H_N defined in (1). Then, using the symmetry of Ψ and H_N, we have

$$\inf\sigma(H_N) = \langle\Psi, H_N\Psi\rangle = \frac{N}{2}\operatorname{Tr}\left(H_{2,N}\Gamma^{(2)}\right),$$

where

$$\Gamma := |\Psi\rangle\langle\Psi| \quad \text{and} \quad H_{2,N} := h_1 + h_2 + w_N(x_1-x_2) =: T + w_N.$$

Since we need to extract a small amount of the kinetic energy to control some error terms, we define

$$\tilde{H}_{2,N} := \left(1-\frac{1}{\log N}\right)T + w_N.$$

We claim that

$$\operatorname{Tr}\left(H_{2,N}\Gamma^{(2)}\right) \ge \operatorname{Tr}\left(\mathbb{P}_M\otimes\mathbb{P}_M\tilde{H}_{2,N}\mathbb{P}_M\otimes\mathbb{P}_M\Gamma^{(2)}\right) + \frac{C}{\log N}\operatorname{Tr}\left(h\Gamma^{(1)}\right) - CN^{-\varepsilon}. \tag{27}$$

Indeed, using the identity $\mathbb{1} = \mathbb{P}_M + P_{M+1}$, we see that, to prove (27), we need to bound terms of the form

$$\operatorname{Tr}\left(P_{M+1}\otimes\mathbb{P}_M w_N\mathbb{P}_M\otimes\mathbb{P}_M\Gamma^{(2)}\right),$$

and terms with more than one projection P_{M+1}, which are bounded in the same way. Thanks to the Hölder inequality and the Sobolev inequality, we have

$$\begin{aligned}
&\left|\operatorname{Tr}\left(P_{M+1}\otimes\mathbb{P}_M w_N\mathbb{P}_M\otimes\mathbb{P}_M\Gamma^{(2)}\right)\right| \\
&\qquad\le CN^{-\varepsilon}\operatorname{Tr}\left(P_{M+1}\otimes\mathbb{P}_M(N^{4\beta+2\varepsilon\frac{1+\eta}{\eta}}+h_1)P_{M+1}\otimes\mathbb{P}_M\Gamma^{(2)}\right) \\
&\qquad\quad + CN^{-\varepsilon}\operatorname{Tr}\left(\mathbb{P}_M\otimes\mathbb{P}_M(1+h_1)\mathbb{P}_M\otimes\mathbb{P}_M\Gamma^{(2)}\right) \\
&\qquad\le CN^{-\varepsilon}\operatorname{Tr}\left(P_{M+1}\otimes\mathbb{P}_M(1+h_1)P_{M+1}\otimes\mathbb{P}_M\Gamma^{(2)}\right) \\
&\qquad\quad + CN^{-\varepsilon}\operatorname{Tr}\left(\mathbb{P}_M\otimes\mathbb{P}_M(1+h_1)\mathbb{P}_M\otimes\mathbb{P}_M\Gamma^{(2)}\right).
\end{aligned}$$

In the first inequality we used the Sobolev inequality with weights

$$\|f\|_{L^p}^2 \leq C_p \tau^{-(p-2)/p}\left(\|f\|^2 + \tau\|\nabla f\|^2\right), \quad \forall f \in H^1(\mathbb{R}^2),$$

which holds for any $p \geq 2$ and for all $\tau > 0$, and that can be proven following the same proof as [13, Theorem 8.5]. In the second inequality we used (25) (or (26)) and the estimate $N^{2M\varepsilon} P_{M+1} \leq P_{M+1} h P_{M+1}$. This proves (27) for N large enough.

After that, we decompose Ψ according to the occupancy of the energy levels defined by the projections (23) and (24). For this, we define, for any multi-index $\underline{J} = (j_1, \ldots, j_{M+1})$ satisfying $|\underline{J}| = N$,

$$\Psi_{\underline{J}} := P_1^{\otimes j_1} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} P_{M+1}^{\otimes j_{M+1}} \Psi,$$

as well as

$$\Gamma_{\underline{J}} := \left|\Psi_{\underline{J}}\right\rangle \left\langle \Psi_{\underline{J}}\right| \quad \text{and} \quad \Gamma_{\underline{J},\underline{J}'} := \left|\Psi_{\underline{J}'}\right\rangle \left\langle \Psi_{\underline{J}}\right|.$$

Moreover, for any multi-index $\underline{J}$, we define

$$\underline{J}^{(i_1,i_2)} := \begin{cases} (j_1, \ldots, j_{i_1}+1, \ldots, j_{i_2}+1, \ldots, j_{M+1}) & \text{if } i_1 \neq i_2, \\ (j_1, \ldots, j_{i_1}+2, \ldots, j_{M+1}) & \text{otherwise.} \end{cases}$$

Then, using that

$$P_{i_1} \otimes P_{i_2} \Gamma^{(2)} P_{i'_1} \otimes P_{i'_2} = \sum_{\underline{J}} P_{i_1} \otimes P_{i_2} \Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i'_1,i'_2)}} P_{i'_1} \otimes P_{i'_2},$$

we can write

$$\begin{aligned} \operatorname{Tr}\left(H_{2,N}\Gamma^{(2)}\right) \geq \sum_{\underline{J}} \sum_{\substack{1\leq i_1,i_2\leq M \\ 1\leq i'_1,i'_2\leq M}} &\operatorname{Tr}\left(P_{i_1} \otimes P_{i_2} \tilde{H}_{2,N} P_{i'_1} \otimes P_{i'_2} \Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i'_1,i'_2)}}\right) \\ &+ \frac{C}{\log N} \operatorname{Tr}\left(h\Gamma^{(1)}\right) - CN^{-\varepsilon}, \end{aligned} \tag{28}$$

where we are summing over multi-indices $\underline{J}$ satisfying $|\underline{J}| = N - 2$.

Let us bound the first term in the right-hand side of (28). To do so, we define

$$i_{\max}(\underline{J}) := \max\left\{i \in \{1, \ldots, M\} : j_i \geq N^{1-\delta\varepsilon}\right\}, \tag{29}$$

for some $\delta > 0$ that shall be fixed later. We shall sometimes omit $\underline{J}$ and simply write $i_{\max}$ for readability's sake. We take the convention that $i_{\max}(\underline{J}) = 0$ if the set on the right-hand side is empty. Then, we make the following important distinction regarding the sum over i_1, i_2, i'_1, i'_2: either $i_{\max}(\underline{J}) = i_{\max}(\underline{J}')$ and $i_1, i_2, i'_1, i'_2 \leq i_{\max}(\underline{J})$ - in which case we shall use Theorem 3 - or not. In the latter case, we need to bound terms of the form

$$\operatorname{Tr}\left(P_{i_1} \otimes P_{i_2} w_N P_{i'_1} \otimes P_{i'_2} \Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i'_1,i'_2)}}\right),$$

where one of the four indices, say i_1, is such that $j_{i_1} < N^{1-\delta\varepsilon}$. This is done in the following lemma, whose proof is given in Section 5.

Lemma 7. *Take w_N as in Assumption 1, and M and ε satisfying either* (25) *or* (26)*. Let $\delta \geq 6$ and define $i_{\max}$ as in* (29) *for any given multi-index $\underline{J}$. Then,*

$$\begin{aligned} \sum_{\underline{J}} \sum_{i_1=i_{\max}+1}^{M} \sum_{i_2,i_3,i_4=1}^{M} &\left|\operatorname{Tr}\left(P_{i_1} \otimes P_{i_2} w_N P_{i'_1} \otimes P_{i'_2} \Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i'_1,i'_2)}}\right)\right| \\ &\leq C(N^{-\varepsilon} + N^{2\varepsilon-(1-\kappa)}) \operatorname{Tr}\left((1+h)\Gamma^{(1)}\right) \end{aligned} \tag{30}$$

for some constant C that depends only on $\|w\|_1$, and with $\kappa = 0$ in the polynomial case (25)*. Here, we are summing over all multi-indices $\underline{J}$ satisfying $|\underline{J}| = N - 2$.*

Injecting (30) into (28), we obtain

$$\operatorname{Tr}\left(H_{2,N}\Gamma^{(2)}\right) \geq \sum_{\underline{J}} \sum_{\substack{i_1,i_2=1\\ i_1',i_2'=1}}^{i_{\max}} \operatorname{Tr}\left(P_{i_1}\otimes P_{i_2}\tilde{H}_{2,N}P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right) + \frac{C}{\log N}\operatorname{Tr}\left(h\Gamma^{(1)}\right) - CN^{-\varepsilon} - CN^{2\varepsilon-(1-\kappa)}, \tag{31}$$

under the condition

$$\varepsilon < \frac{1-\kappa}{2}, \tag{32}$$

with $\kappa = 0$ in the polynomial case (2). We do not mention the condition (32) further since it will be fulfilled by our choice of ε.

Before we can apply Theorem 3, we need to do a bit of rewriting. We start by decomposing over the different values that $i_{\max}$ can take:

$$\sum_{\underline{J}} \sum_{\substack{i_1,i_2=1\\ i_1',i_2'=1}}^{i_{\max}} \operatorname{Tr}\left(P_{i_1}\otimes P_{i_2}\tilde{H}_{2,N}P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right) = \sum_{i=1}^{M} \sum_{\substack{i_1,i_2=1\\ i_1',i_2'=1}}^{i} \sum_{\substack{\underline{J}\\ i_{\max}(\underline{J})=i}} \operatorname{Tr}\left(P_{i_1}\otimes P_{i_2}\tilde{H}_{2,N}P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right).$$

Then, we use that, at fixed i_1, i_2, i_1', i_2', the sum over $\underline{J}$ can be rewritten as

$$\sum_{\substack{\underline{J}\\ i_{\max}(\underline{J})=i}} P_{i_1}\otimes P_{i_2}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}P_{i_1'}\otimes P_{i_2'} = \sum_{\substack{\underline{J}',\underline{J}''\\ i_{\max}(\underline{J}')=i\\ i_{\max}(\underline{J}'')=i}} P_{i_1}\otimes P_{i_2}\Gamma^{(2)}_{\underline{J}',\underline{J}''}P_{i_1'}\otimes P_{i_2'}, \tag{33}$$

where we are summing over $\underline{J}$'s satisfying $|\underline{J}| = N-2$ on the left-hand side, and over $\underline{J}', \underline{J}''$ satisfying $|\underline{J}'| = |\underline{J}''| = N$ on the right-hand side. Notice first that, due to the presence of the projections $P_{i_1}\otimes P_{i_2}$ and $P_{i_1'}\otimes P_{i_2'}$, the multi-index $\underline{J}''$ is entirely determined by the choice of $\underline{J}'$. Said differently, the double sum on the right-hand side is in essence only a single sum. Moreover, for every $\underline{J}'$, there is exactly one $\underline{J}$ such that $\underline{J}^{(i_1,i_2)} = \underline{J}'$. Combining the previous two arguments is enough to prove (33). We now define

$$j_{\max}(\underline{J}) \coloneqq \sum_{i=1}^{i_{\max}(\underline{J})} j_i,$$

as well as

$$\Psi_{i,K} \coloneqq \sum_{\substack{\underline{J}\\ i_{\max}(\underline{J})=i\\ j_{\max}(\underline{J})=K}} \Psi_{\underline{J}} \quad \text{and} \quad \Gamma_{i,K} \coloneqq |\Psi_{i,K}\rangle\langle\Psi_{i,K}|. \tag{34}$$

As it shall be useful later, we use the convention that, when $i = 0$, the condition $j_{\max} = K$ in the definition of $\Psi_{0,K}$ (34) should be replaced by $j_{M+1} = K$. With these new definitions, we may rewrite (33) as

$$\sum_{\substack{\underline{J}\\ i_{\max}(\underline{J})=i}} P_{i_1}\otimes P_{i_2}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}P_{i_1'}\otimes P_{i_2'} = \sum_{K} P_{i_1}\otimes P_{i_2}\Gamma^{(2)}_{i,K}P_{i_1'}\otimes P_{i_2'}.$$

Summing up, we have

$$\operatorname{Tr}\left(H_{2,N}\Gamma^{(2)}\right) \geq \sum_{i=1}^{M}\sum_{K}\operatorname{Tr}\left(\mathbb{P}_i\otimes\mathbb{P}_i\tilde{H}_{2,N}\mathbb{P}_i\otimes\mathbb{P}_i\Gamma^{(2)}_{i,K}\right) + \frac{C}{\log N}\operatorname{Tr}\left(h\Gamma^{(1)}\right) - CN^{-\varepsilon} - CN^{2\varepsilon-(1-\kappa)}, \tag{35}$$

where we are summing over $N - MN^{1-\delta\varepsilon} \leq K \leq N$.

Given that $\Psi_{i,K}$ is in $\mathbb{P}_i^{\otimes K} \otimes_{\mathrm{s}} \mathbb{Q}_i^{\otimes(N-K)}\mathfrak{H}^N$ and is symmetric, Lemma 6 allows us to find a symmetric state $\gamma_{i,K} \in \mathcal{S}\left(\mathbb{P}_i^{\otimes K}\mathfrak{H}^K\right)$ such that

$$\mathbb{P}_i\otimes\mathbb{P}_i\Gamma^{(2)}_{i,K}\mathbb{P}_i\otimes\mathbb{P}_i = \binom{N}{2}^{-1}\binom{K}{2}\|\Psi_{i,K}\|^2\gamma^{(2)}_{i,K}.$$

Then, writing $w_N(x-y)$ in Fourier, applying Theorem 3 and using Proposition 4 and the CLR type estimate (14) to bound the error terms, we obtain

$$\begin{aligned}\operatorname{Tr}\left(\tilde{H}_{2,N}\gamma^{(2)}_{i,K}\right) &= \int_{\mathcal{S}(\mathbb{P}_i\mathfrak{H})}\operatorname{Tr}\left(\tilde{H}_{2,N}\gamma^{\otimes 2}\right)\mathrm{d}\mu(\gamma) \\ &\quad + \left(1-\frac{1}{\log N}\right)\operatorname{Tr}\left(T\left[\gamma^{(2)}_{i,K} - \int_{\mathcal{S}(\mathbb{P}_i\mathfrak{H})}\gamma^{\otimes 2}\,\mathrm{d}\mu(\gamma)\right]\right) \\ &\quad + \int_{\mathbb{R}^2}\mathrm{d}k\,\hat{w}(N^{-\beta}k)\operatorname{Tr}\left(e^{\mathbf{i}k\cdot x}e^{-\mathbf{i}k\cdot y}\left[\gamma^{(2)}_{i,K} - \int_{\mathcal{S}(\mathbb{P}_i\mathfrak{H})}\gamma^{\otimes 2}\,\mathrm{d}\mu(\gamma)\right]\right) \\ &\geq \int_{\mathcal{S}(\mathbb{P}_i\mathfrak{H})}\operatorname{Tr}\left(\tilde{H}_{2,N}\gamma^{\otimes 2}\right)\mathrm{d}\mu(\gamma) \\ &\quad - C\sqrt{\frac{M\varepsilon\log N}{K}}\left(N^{2i\varepsilon} + \int_{\mathbb{R}^2}\mathrm{d}k\|\mathbb{P}_i e^{\mathbf{i}k\cdot x}\mathbb{P}_i\|^2_{\mathrm{op}}|\hat{w}(N^{-\beta}k)|\right) \\ &\geq 2\tilde{e}^{\mathrm{H}}_N - C\sqrt{\frac{M\varepsilon\log N}{K}}N^{2i\varepsilon},\end{aligned} \tag{36}$$

where $\tilde{e}^{\mathrm{H}}_N$ is the ground state energy of the Hartree functional (6) with h replaced by $(1-(\log N)^{-1})h$. To prove the last inequality, we decomposed the integral over k just as we did in the proof of Proposition 4. As a result of (35) and (36), we have

$$\begin{aligned}\operatorname{Tr}\left(H_{2,N}\Gamma^{(2)}\right) &\geq 2\tilde{e}^{\mathrm{H}}_N\sum_{i=1}^{M}\sum_{K}\binom{N}{2}^{-1}\binom{K}{2}\|\Psi_{i,K}\|^2 \\ &\quad - C\sum_{i=1}^{M}\sum_{K}\binom{N}{2}^{-1}\binom{K}{2}\|\Psi_{i,K}\|^2\sqrt{\frac{M\varepsilon\log N}{K}}N^{2i\varepsilon} \\ &\quad + \frac{C}{\log N}\operatorname{Tr}\left(h\Gamma^{(1)}\right) - CN^{-\varepsilon} - CN^{2\varepsilon-(1-\kappa)}.\end{aligned} \tag{37}$$

For the first term, we use the bound

$$\begin{aligned}1 \geq \sum_{i=1}^{M}\sum_{K}\binom{N}{2}^{-1}\binom{K}{2}\|\Psi_{i,K}\|^2 &= \sum_{i=1}^{M}\sum_{K}\operatorname{Tr}\left(\mathbb{P}_i\otimes\mathbb{P}_i\Gamma^{(2)}_{i,K}\right) \\ &= \sum_{i=0}^{M}\sum_{K}\|\Psi_{i,K}\|^2 - \sum_{K}\|\Psi_{0,K}\|^2\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{M} \sum_{K} \left(\operatorname{Tr}\left(\mathbb{Q}_i \Gamma^{(1)}_{i,K} \right) + \operatorname{Tr}\left(\mathbb{Q}_i \otimes \mathbb{P}_i \Gamma^{(2)}_{i,K} \right) \right) \\
&\geq 1 - \sum_{K} \|\Psi_{0,K}\|^2 - CN^{-\delta\varepsilon},
\end{aligned}
$$

where we used Lemma 5 to bound the error terms. Note that since the last sum is over $N - MN^{1-11\varepsilon} \leq K \leq N$ and $\tilde{e}^{\mathrm{H}}_N$ is bounded from below independently of N (see e.g. [20]), the term

$$
\tilde{e}^{\mathrm{H}}_N \sum_{K} \|\Psi_{0,K}\|^2 \leq CN^{-2M\varepsilon} \operatorname{Tr}\left(h\Gamma^{(1)} \right)
$$

is easily absorbed by the last term in (37). Lastly, using again $\binom{N}{2}^{-1}\binom{K}{2} \leq 1$, we can bound the second term in the right-hand side of (37) by

$$
\begin{aligned}
N^{\delta\varepsilon/2-1/2}\sqrt{M\varepsilon \log N} \sum_{i=1}^{M} \sum_{K} \binom{N}{2}^{-1} \binom{K}{2} \|\Psi_{i,K}\|^2 N^{2i\varepsilon} \\
&\leq N^{\delta\varepsilon/2-1/2}\sqrt{M\varepsilon \log N} \sum_{\underline{J}} \left\|\Psi_{\underline{J}}\right\|^2 N^{2i_{\max}(\underline{J})\varepsilon} \\
&\leq CN^{(3\delta/2+2)\varepsilon-1/2}\sqrt{M\varepsilon \log N} \sum_{\underline{J}} \left\|\Psi_{\underline{J}}\right\|^2 N^{2(i_{\max}(\underline{J})-1)\varepsilon} \frac{j_{i_{\max}(\underline{J})}}{N} \\
&\leq CN^{(3\delta/2+2)\varepsilon-1/2}\sqrt{M\varepsilon \log N} \operatorname{Tr}\left(h\Gamma^{(1)} \right).
\end{aligned}
$$

Choosing $\delta = 6$ and $\varepsilon < 1/22$ in the polynomial case (2), and $\delta = 6$ and $\varepsilon < (1-\kappa)/22$ in the exponential case (3), we see that the previous term can be absorbed by the last term in the the right-hand side of (37). Summing up, we have shown that

$$
\operatorname{Tr}\left(H_{2,N}\Gamma^{(2)} \right) \geq 2\tilde{e}^{\mathrm{H}}_N - CN^{-\nu},
$$

for N large enough and for some $\nu > 0$. We conclude the proof of Theorem 2 by using that $\tilde{e}^{\mathrm{H}}_N$ converges to e_{nls} as $N \to \infty$ (this follows directly from [10, Lemma 7]). Stability of second kind follows from the boundedness of $\tilde{e}^{\mathrm{H}}_N$ (see again [20]).

4. Plain wave estimate and other results

Proof of Proposition 4. Take $k \in \mathbb{R}^2 \setminus \{0\}$ and let us prove (18) for $\mathbf{e}_k = \cos(k \cdot x)$ ($\mathbf{e}_k = \sin(k \cdot x)$ being similar). The bound for $p = 0$ is just $|\mathbf{e}_k| \leq 1$. For $p = 1$, we take some smooth compactly supported function f and write

$$
\begin{aligned}
\langle P_i f, \cos(k \cdot x) P_i f \rangle &= \int_{\mathbb{R}^2} |P_i f|^2 \cos(k \cdot x)\, \mathrm{d}x = -\int \nabla |P_i f|^2 \frac{k \sin(k \cdot x)}{|k|^2}\, \mathrm{d}x \\
&\leq \frac{2}{|k|} \int_{\mathbb{R}^2} |\nabla |P_i f|| |P_i f|\, \mathrm{d}x \leq \frac{2}{|k|} \left(\int |\nabla P_i f|^2 \right)^{\frac{1}{2}} \|P_i f\| \\
&\leq C \frac{N^{i\varepsilon}}{|k|} \|P_i f\|^2,
\end{aligned}
$$

for N large enough. Here, we wrote $\cos(k \cdot x) = k \cdot \nabla \sin(k \cdot x)/|k|^2$ and used an integration by parts. Similarly, to prove (18) for $p \geq 2$ we do p integration by parts instead of a single one.

To prove (19), we first decompose w_N in Fourier space

$$
w_N(x-y) = \int_{\mathbb{R}^2} \mathrm{d}k \hat{w}(N^{-\beta}k) e^{\mathbf{i}k \cdot (x-y)}
$$

$$= \int_{\mathbb{R}^2} \mathrm{d}k \hat{w}(N^{-\beta}k) \left(\cos(k\cdot x)\cos(k\cdot y) + \sin(k\cdot x)\sin(k\cdot y)\right). \tag{38}$$

Conjugating the previous expression by $P_{i_1} \otimes P_{i_2}$, we see that we must bound terms of the form $P_i \mathbf{e}_k P_i$. Decomposing the integral (38) into integrals over $\{|k| \le N^{i_1\varepsilon}\}$, $\{N^{i_1\varepsilon} < |k|\}$ (taking $i_1 \le i_2$), and using the estimate (18) with $p = 0$ for the first integral and $p = 3$ for the second one, we obtain

$$\begin{aligned} P_{i_1} \otimes P_{i_2} w_N(x-y) P_{i_1} \otimes P_{i_2} \ge &- \int_{|k|\le N^{i_1\varepsilon}} \mathrm{d}k \left(|\hat{w}(N^{-\beta}k)| \right) P_{i_1} \otimes P_{i_2} \\ &- \int_{N^{i_1\varepsilon}<|k|} \mathrm{d}k \left(|\hat{w}(N^{-\beta}k)| \frac{N^{3i_1\varepsilon}}{|k|^3} \right) P_{i_1} \otimes P_{i_2}. \end{aligned}$$

Bounding $|\hat{w}|$ by $\|\hat{w}\|_{L^\infty} \le \|w\|_{L^1}$ and integrating over k directly implies (19), thereby finishing the proof of Proposition 4. □

We now prove Lemmas 5 and 6. Define the permutation operator $U_\sigma : \mathfrak{H}^N \to \mathfrak{H}^N$, for some $\sigma \in \mathcal{S}_N$, acting as

$$U_\sigma \Psi(x_1, \dots, x_N) = \Psi(x_{\sigma(1)}, \dots, _{\sigma(N)}),$$

for all $\Psi \in \mathfrak{H}^N$. Then, that a state $\Gamma \in \mathcal{S}(\mathfrak{H}^N)$ is symmetric can be restated as

$$U_\sigma \Gamma U_\sigma^* = \Gamma, \quad \forall \sigma \in \mathcal{S}_N.$$

Moreover, for any family $\{P_1, \dots, P_k\}$ of orthogonal projections on $\mathfrak{H}$, we can write

$$P_1^{\otimes n_1} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} P_k^{\otimes n_k} = \frac{1}{n_1! \cdots n_k!} \sum_{\sigma \in \mathcal{S}_N} U_\sigma P_1^{\otimes n_1} \otimes \cdots \otimes P_k^{\otimes n_k} U_\sigma^*, \tag{39}$$

for any nonnegative integers $n_1, \dots, n_k$ such that $\sum_{i=1}^k n_i = N$.

Proof of Lemma 5. We only prove (21) since (20) follows similarly. We begin by writing

$$\begin{aligned} \mathrm{Tr}\left(P_1 \otimes P_2 \Gamma^{(2)}\right) &= \mathrm{Tr}\left(P_1 \otimes P_2 \otimes \mathbb{1}^{\otimes(N-2)} \Gamma\right) \\ &= \mathrm{Tr}\left(P_1 \otimes P_2 \otimes (P_1 + P_2 + Q)^{\otimes(N-2)} \Gamma\right). \end{aligned}$$

Developing $(P_1 + P_2 + Q)^{\otimes(N-2)}$ and using that, since Γ acts on $P_1^{\otimes j_1} \otimes_{\mathrm{s}} P_2^{\otimes j_2} \otimes_{\mathrm{s}} Q^{\otimes(N-j_1-j_2)} \mathfrak{H}^N$, the only terms that do not vanish are the ones containing $j_1 - 1$ times P_1, $j_2 - 1$ times P_2 and $N - j_1 - j_2$ times Q, we obtain

$$\mathrm{Tr}\left(P_1 \otimes P_2 \Gamma^{(2)}\right) = \mathrm{Tr}\left(P_1 \otimes P_2 \otimes \left(P_1^{\otimes(j_1-1)} \otimes_{\mathrm{s}} P_2^{\otimes(j_2-1)} \otimes_{\mathrm{s}} Q^{\otimes(N-j_1-j_2)}\right) \Gamma\right).$$

Conjugating Γ by U_σ, which can be done for free because Γ is symmetric, this becomes

$$\begin{aligned} &\mathrm{Tr}\left(P_1 \otimes P_2 \Gamma^{(2)}\right) \\ &\qquad = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \mathrm{Tr}\left(U_\sigma^* P_1 \otimes P_2 \otimes \left(P_1^{\otimes(j_1-1)} \otimes_{\mathrm{s}} P_2^{\otimes(j_2-1)} \otimes_{\mathrm{s}} Q^{\otimes(N-j_1-j_2)}\right) U_\sigma \Gamma\right) \end{aligned}$$

Then, using the cyclicity of the trace as well as (39), we find

$$\begin{aligned} \mathrm{Tr}\left(P_1 \otimes P_2 \Gamma^{(2)}\right) = &\frac{1}{N!(j_1-1)!(j_2-1)!(N-j_1-j_2)!} \sum_{\sigma \in \mathcal{S}_N} \sum_{\pi \in \mathcal{S}_{N-2}} \\ &\mathrm{Tr}\left(U_\sigma^* P_1 \otimes P_2 \otimes U_\pi^* P_1^{\otimes(j_1-1)} \otimes P_2^{\otimes(j_2-1)} \otimes Q^{\otimes(N-j_1-j_2)} U_\pi U_\sigma \Gamma\right) \end{aligned}$$

$$= \frac{(N-2)!}{N!(j_1-1)!(j_2-1)!(N-j_1-j_2)!} \times \sum_{\sigma \in \mathcal{S}_N} \operatorname{Tr}\left(U_\sigma^* P_1^{\otimes j_1} \otimes P_2^{\otimes j_2} \otimes Q^{\otimes(N-j_1-j_2)} U_\sigma \Gamma\right)$$
$$= \frac{j_1 j_2}{N(N-1)} \operatorname{Tr}\left(P_1^{\otimes j_1} \otimes_{\mathrm{s}} P_2^{\otimes j_2} \otimes_{\mathrm{s}} Q^{\otimes(N-j_1-j_2)} \Gamma\right)$$
$$= \frac{j_1 j_2}{N(N-1)}.$$

□

Proof of Lemma 6. Define

$$\Gamma_j \coloneqq \binom{N}{j} \operatorname{Tr}_{j+1\to N}\left(P^{\otimes j} \otimes Q^{\otimes(N-j)} \Gamma\right).$$

Firstly, it is easy to verify that Γ_j is a symmetric state and that it lives in the desired space. Secondly, using again the symmetry of Γ and the cyclicity of the trace, we have

$$\Gamma_j^{(2)} = \binom{N}{j} \operatorname{Tr}_{3\to N}\left(P^{\otimes j} \otimes Q^{\otimes(N-j)} \Gamma\right)$$
$$= \binom{N}{j} \frac{1}{(N-2)!} \sum_{\sigma \in \mathcal{S}_{N-2}} \operatorname{Tr}_{3\to N}\left(P^{\otimes 2} \otimes U_\sigma^* P^{\otimes(j-2)} \otimes Q^{\otimes(N-j)} U_\sigma \Gamma\right)$$
$$= \binom{N}{j} \frac{(j-2)!(N-j)!}{(N-2)!} \operatorname{Tr}_{3\to N} P^2 \left(P^{\otimes 2} \otimes \left(P^{\otimes(j-2)} \otimes_{\mathrm{s}} Q^{\otimes(N-j)}\right) \Gamma\right) P^{\otimes 2}$$
$$= \frac{N(N-1)}{j(j-1)} P^{\otimes 2} \operatorname{Tr}_{3\to N}\left(P^{\otimes j} \otimes_{\mathrm{s}} Q^{\otimes(N-j)} \Gamma\right) P^{\otimes 2},$$

which is the desired claim. □

5. Proof of the main technical estimate: Lemma 7

This section is dedicated to the proof of Lemma 7. Although the proof used in the exponential case covers polynomial scalings as well, we provide first a simpler in that case for the reader's convenience.

Proof of Lemma 7 in the polynomial case. Thanks to the Cauchy–Schwarz inequality, Proposition 4 and Lemma 5, we have

$$\left|\operatorname{Tr}\left(P_{i_1} \otimes P_{i_2} w_N P_{i_1'} \otimes P_{i_2'} \Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right)\right|$$
$$\leq \frac{N^{3\varepsilon}}{2} \operatorname{Tr}\left(P_{i_1} \otimes P_{i_2} |w_N| P_{i_1} \otimes P_{i_2} \Gamma^{(2)}_{\underline{J}^{(i_1,i_2)}}\right) + \frac{N^{-3\varepsilon}}{2} \operatorname{Tr}\left(P_{i_1'} \otimes P_{i_2'} |w_N| P_{i_1'} \otimes P_{i_2'} \Gamma^{(2)}_{\underline{J}^{(i_1',i_2')}}\right)$$
$$\leq C N^{2\min(i_1,i_2)\varepsilon} N^{3\varepsilon} \frac{(j_{i_1}+1)(j_{i_2}+1)}{N(N-1)} \operatorname{Tr} \Gamma_{\underline{J}^{(i_1,i_2)}} + C N^{2\min(i_1',i_2')\varepsilon} N^{-3\varepsilon} \frac{(j_{i_1'}+1)(j_{i_2'}+1)}{N(N-1)} \operatorname{Tr} \Gamma_{\underline{J}^{(i_1',i_2')}},$$

On the one hand, using that $j_{i_1} < N^{1-\delta\varepsilon}$ and Lemma 5, we have

$$N^{2\min(i_1,i_2)\varepsilon} N^{3\varepsilon} \frac{(j_{i_1}+1)(j_{i_2}+1)}{N(N-1)} \operatorname{Tr} \Gamma_{\underline{J}^{(i_1,i_2)}} \leq N^{(5-\delta)\varepsilon} N^{2(i_2-1)\varepsilon} \frac{j_{i_2}+1}{N} \operatorname{Tr} \Gamma_{\underline{J}^{(i_1,i_2)}}$$

$$\leq CN^{(5-\delta)\varepsilon}\,\mathrm{Tr}\left(P_{i_2}(1+h)P_{i_2}\Gamma^{(1)}_{\underline{J}^{(i_1,i_2)}}\right).$$

On the other hand,

$$N^{2\min(i_1',i_2')\varepsilon}N^{-3\varepsilon}\frac{(j_{i_1'}+1)(j_{i_2'}+1)}{N(N-1)}\,\mathrm{Tr}\,\Gamma_{\underline{J}^{(i_1',i_2')}}$$
$$\leq CN^{-\varepsilon}\,\mathrm{Tr}\left(P_{i_1'}\otimes P_{i_2'}(1+h_1)P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1',i_2')}}\right).$$

Taking $\delta \geq 6$, and using

$$\sum_{\underline{J}} P_{i_2}\Gamma^{(1)}_{\underline{J}^{(i_1,i_2)}}P_{i_2} \leq P_{i_2}\Gamma^{(1)}P_{i_2}$$

and

$$\sum_{\underline{J}} P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1',i_2')}}P_{i_1'}\otimes P_{i_2'} = P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}P_{i_1'}\otimes P_{i_2'},$$

we therefore get

$$\sum_{\underline{J}}\left|\mathrm{Tr}\left(P_{i_1}\otimes P_{i_2}w_N P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{i_1',i_2'}}\right)\right| \leq CN^{-\varepsilon}\,\mathrm{Tr}\left(P_{i_2}(1+h)P_{i_2}\Gamma^{(1)}\right)$$
$$+ CN^{-\varepsilon}\,\mathrm{Tr}\left(P_{i_1'}\otimes P_{i_2'}(1+h_1)P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}\right).$$

Finally, we sum over i_1, i_2, i_1', i_2', and use that M is a constant to obtain (30). □

In the exponential case, M is no longer bounded, which prevents us from using the previous simple proof (except for κ small), and we consequently have to be much more precise.

Proof of Lemma 7 in the exponential case. We distinguish between $|i_1 - i_2| \leq 2$ and $|i_1 - i_2| > 2$, and similarly for i_1', i_2' and i_1, i_1'. Because we only to use $j_{i_1} \leq N^{1-\delta\varepsilon}$ when $|i_1 - i_2| \leq 2$, $|i_1' - i_2'| \leq 2$ and $|i_1 - i_1'| \leq 2$, we begin by treating the other cases. Namely, we first bound

$$\sum_{\underline{J}}\sum_{\substack{i_1\leq i_2\\ i_1+2<i_2}}\sum_{i_1'\leq i_2'}\left|\mathrm{Tr}\left(P_{i_1}\otimes P_{i_2}w_N P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right)\right| \tag{40}$$

and

$$\sum_{\underline{J}}\sum_{\substack{i_1\leq i_2\\ |i_1-i_2|\leq 2}}\sum_{\substack{i_1'\leq i_2'\\ |i_1'-i_2'|\leq 2\\ i_1+2<i_1'}}\left|\mathrm{Tr}\left(P_{i_1}\otimes P_{i_2}w_N P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right)\right|. \tag{41}$$

In both cases we can take the sum over i_1 to start from 1 rather than $i_{\max}+1$ for simplicity. Thanks to the Cauchy–Schwarz inequality, Lemma 5 and Proposition 4, we have

$$\left|\mathrm{Tr}\left(P_{i_1}\otimes P_{i_2}w_N P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right)\right|$$
$$\leq \tau CN^{(i_1+i_1')\varepsilon}\frac{(j_{i_1}+1)(j_{i_2}+1)}{N(N-1)}\,\mathrm{Tr}\,\Gamma_{\underline{J}^{(i_1,i_2)}} \tag{42}$$
$$+\tau^{-1}CN^{(i_1+i_1')\varepsilon}\frac{(j_{i_1'}+1)(j_{i_2'}+1)}{N(N-1)}\,\mathrm{Tr}\,\Gamma_{\underline{J}^{(i_1',i_2')}},$$

for all $\tau > 0$. Thus, an appropriate choice of τ and an application of Lemma 5 allow us to write

$$\left|\mathrm{Tr}\left(P_{i_1}\otimes P_{i_2}w_N P_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right)\right|$$

$$\leq CN^{(i_1-i_2+1)\varepsilon}N^{(i_1'-i_2'+1)\varepsilon}\operatorname{Tr}\left(P_{i_2}(1+h)P_{i_2}\otimes\left(P_{i_1'}+\frac{1}{N}\right)\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)}}\right)$$
$$+CN^{(i_1-i_2+1)\varepsilon}N^{(i_1'-i_2'+1)\varepsilon}\operatorname{Tr}\left(P_{i_2'}(1+h)P_{i_2'}\otimes\left(P_{i_1}+\frac{1}{N}\right)\Gamma^{(2)}_{\underline{J}^{(i_1',i_2')}}\right).$$

The reason we have the factor $\frac{1}{N}$ in the right-hand side is to deal with the possibility of j_{i_1} and $j_{i_1'}$ being null. Using that

$$\sum_{\underline{J}}P_{i_1}\otimes P_{i_2}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)}}P_{i_1}\otimes P_{i_2}=P_{i_1}\otimes P_{i_2}\Gamma^{(2)}P_{i_1}\otimes P_{i_2},\tag{43}$$

we may carry out the $\underline{J}$ sum in (40) first to obtain

$$\sum_{\underline{J}}\sum_{i_1+2<i_2}\sum_{i_1'\leq i_2'}\left|\operatorname{Tr}\left(P_{i_1}\otimes P_{i_2}w_NP_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right)\right|$$
$$\leq\sum_{i_1+2<i_2}\sum_{i_1'\leq i_2'}CN^{(i_1-i_2+1)\varepsilon}N^{(i_1'-i_2'+1)\varepsilon}\operatorname{Tr}\left(P_{i_2}hP_{i_2}\otimes\left(P_{i_1'}+\frac{1}{N}\right)\Gamma^{(2)}\right)$$
$$+\sum_{i_1+2<i_2}\sum_{i_1'\leq i_2'}CN^{(i_1-i_2+1)\varepsilon}N^{(i_1'-i_2'+1)\varepsilon}\operatorname{Tr}\left(P_{i_2'}hP_{i_2'}\otimes\left(P_{i_1}+\frac{1}{N}\right)\Gamma^{(2)}\right).$$

Then, carrying out the remaining four sums using the geometric series formula, the resolution of the identity

$$\sum_{i=1}^{M+1}P_i=1$$

and the fact that M satisfies (26),we obtain

$$\sum_{\underline{J}}\sum_{i_1+2<i_2}\sum_{i_1'\leq i_2'}\left|\operatorname{Tr}\left(P_{i_1}\otimes P_{i_2}w_NP_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right)\right|$$
$$\leq CN^{-\varepsilon}\operatorname{Tr}\left((1+h)\Gamma^{(1)}\right).$$

To bound (41), we again use (42) to write

$$\left|\operatorname{Tr}\left(P_{i_1}\otimes P_{i_2}w_NP_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right)\right|$$
$$\leq CN^{(i_1-i_1'+1)\varepsilon}N^{(i_1'-i_2'+1)\varepsilon}\operatorname{Tr}\left(P_{i_1'}(1+h)P_{i_1'}\otimes P_{i_1}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)}}\right)$$
$$+CN^{(i_1-i_1'+1)\varepsilon}N^{(i_1'-i_2'+1)\varepsilon}\operatorname{Tr}\left(P_{i_2'}(1+h)P_{i_2'}\otimes\left(P_{i_2}+\frac{1}{N}\right)\Gamma^{(2)}_{\underline{J}^{(i_1',i_2')}}\right),$$

when $j_{i_1'}\geq 1$. When $j_{i_1'}=0$, the previous expression cannot by made true by simply replacing $P_{i_1'}$ by $1/\sqrt{N}$, and we instead write

$$\left|\operatorname{Tr}\left(P_{i_1}\otimes P_{i_2}w_NP_{i_1'}\otimes P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1,i_2)},\underline{J}^{(i_1',i_2')}}\right)\right|$$
$$\leq CN^{2\varepsilon}N^{-(1+\kappa)/2}\operatorname{Tr}\left(P_{i_1}(1+h)P_{i_1}\Gamma^{(1)}_{\underline{J}^{(i_1,i_2)}}\right)$$
$$+CN^{2\varepsilon}N^{-(1-\kappa)/2}\operatorname{Tr}\left(\left(P_{i_2}+\frac{1}{N}\right)\otimes P_{i_2'}(1+h)P_{i_2'}\Gamma^{(2)}_{\underline{J}^{(i_1',i_2')}}\right).$$

Carrying out the sums similarly as before yields

$$\sum_{\underline{J}} \sum_{\substack{i_1 \le i_2 \\ |i_1 - i_2| \le 2}} \sum_{\substack{i_1' \le i_2' \\ |i_1' - i_2'| \le 2 \\ i_1 + 2 < i_1'}} \left| \operatorname{Tr} \left(P_{i_1} \otimes P_{i_2} w_N P_{i_1'} \otimes P_{i_2'} \Gamma^{(2)}_{\underline{J}^{(i_1,i_2)}, \underline{J}^{(i_1',i_2')}} \right) \right| \\ \le C(N^{-\varepsilon} + N^{2\varepsilon - (1-\kappa)/2}) \operatorname{Tr} \left((1+h)\Gamma^{(1)} \right).$$

What now remains is to deal with

$$\sum_{\underline{J}} \sum_{\substack{i_1 \le i_2 \\ |i_1 - i_2| \le 2}} \sum_{\substack{i_1' \le i_2' \\ |i_1' - i_2'| \le 2 \\ |i_1 - i_1'| \le 2}} \left| \operatorname{Tr} \left(P_{i_1} \otimes P_{i_2} w_N P_{i_1'} \otimes P_{i_2'} \Gamma^{(2)}_{\underline{J}^{(i_1,i_2)}, \underline{J}^{(i_1',i_2')}} \right) \right|,$$

where the sum over i_1 starts with $i_1 = i_{\max} + 1$, meaning that we always have $j_{i_1} \le N^{1-\delta\varepsilon}$. Using once more (42), as well as Lemma 5 and the fact that $j_{i_1} \le N^{1-\delta\varepsilon}$, we obtain

$$\left| \operatorname{Tr} \left(P_{i_1} \otimes P_{i_2} w_N P_{i_1'} \otimes P_{i_2'} \Gamma^{(2)}_{\underline{J}^{(i_1,i_2)}, \underline{J}^{(i_1',i_2')}} \right) \right| \\ \le C N^{(i_1 - i_2 + 1)\varepsilon} N^{(i_1' - i_2' + 1)\varepsilon} N^{-\delta\varepsilon/2} \operatorname{Tr} \left(P_{i_2} h P_{i_2} \Gamma^{(1)}_{\underline{J}^{(i_1,i_2)}} \right) \\ + C N^{(i_1 - i_2 + 1)\varepsilon} N^{(i_1' - i_2' + 1)\varepsilon} N^{-\delta\varepsilon/2} \operatorname{Tr} \left(P_{i_2'} h P_{i_2'} \Gamma^{(1)}_{\underline{J}^{(i_1',i_2')}} \right).$$

Carrying out the sums as above, we finally obtain

$$\sum_{\underline{J}} \sum_{\substack{i_1 \le i_2 \\ |i_1 - i_2| \le 2}} \sum_{\substack{i_1' \le i_2' \\ |i_1' - i_2'| \le 2 \\ |i_1 - i_1'| \le 2}} \left| \operatorname{Tr} \left(P_{i_1} \otimes P_{i_2} w_N P_{i_1'} \otimes P_{i_2'} \Gamma^{(2)}_{\underline{J}^{(i_1,i_2)}, \underline{J}^{(i_1',i_2')}} \right) \right| \\ \le C N^{2\varepsilon - \delta\varepsilon/2} \operatorname{Tr} \left((1+h)\Gamma^{(1)} \right).$$

Gathering the previous estimates and taking $\delta \ge 6$ yields (30). □

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