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Embedding calculus and automorphisms of high dimensional manifolds

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Abstract

This thesis examines manifolds of dimension at least five, their automorphism groups, and moduli spaces thereof, by means of embedding calculus. The latter is a homotopy theoretic approximation of manifolds, which, informally, remembers the homotopy type of all spaces of framed configurations in a manifold M, together with the natural point-splitting and forgetful maps between them.

In Paper A, we study the groups $\pi_0\mathrm{Homeo}(M)$ and $\pi_0\mathrm{Diff}(M)$, for a given smooth manifold M of sufficiently high dimension, and under some additional technical conditions. In particular, we show that $\pi_0\mathrm{Homeo}(M)$ is a residually finite group, which, combined with a theorem of Sullivan, implies it is an arithmetic group. In contrast, we isolate the obstruction for residual finiteness of $\pi_0\mathrm{Diff}(M)$, which came out in the form of exotic spheres via the extension by identity morphism $\pi_0\mathrm{Diff}_\partial(D^d) \to \pi_0\mathrm{Diff}(M)$, for some embedded disc $D^d \subset M$.

In joint work with Manuel Krannich and Alexander Kupers, we investigate in Paper B whether the equivalence class of the truncated $\mathcal{D}$isc-presheaf associated to a manifold is an invariant that can distinguish exotic spheres. We give a complete answer in the cases $d \not\equiv 1 \pmod 4$ and $k < \infty$, in terms of the Kervaire-Milnor exact sequence.

Sammenfatning

Denne afhandling undersøger mangfoldigheder af dimension mindst fem, deres automorfigrupper og tilhørende modulirum ved hjælp af indlejringskalkulus. Det sidstnævnte er en homotopiteoretisk approksimation af mangfoldigheder, som løst formuleret husker homotopitypen af alle rum af indrammede konfigurationer i en mangfoldighed M, sammen med de naturlige punktsplitnings- og glemsomme afbildninger imellem dem.

I artikel A undersøger vi grupperne $\pi_0\mathrm{Homeo}(M)$ og $\pi_0\mathrm{Diff}(M)$ for en given glat mangfoldighed M af tilstrækkelig høj dimension. Vi viser blandt andet, at $\pi_0\mathrm{Homeo}(M)$ er en residuelt endelig gruppe, hvilket, kombineret med en sætning af Sullivan, indebærer, at den er en aritmetisk gruppe. I diffeomorfitilfældet isolerer vi obstruktionen for residual endelighed af $\pi_0\mathrm{Diff}(M)$, som viser sig i form af eksotiske sfærer via homomorfien defineret ved at udvide med identiteten $\pi_0\mathrm{Diff}_\partial(D^d) \to \pi_0\mathrm{Diff}(M)$ for en indlejret disk $D^d \subset M$.

I fælles arbejde med Manuel Krannich og Alexander Kupers undersøger vi i artikel B om ækvivalensklassen af det trunkerede $\mathcal{D}$isc-præknippe associeret til en mangfoldighed er en invariant, der kan skelne mellem eksotiske sfærer. Vi giver et fuldstændigt svar i tilfældene $d \not\equiv 1 \pmod 4$ og $k < \infty$, udtrykt ved Kervaire-Milnor's eksakte følge.

Thesis statement

This thesis consists of an introduction and two articles, both of which have been submitted to journals. The introduction is solely written by the author of the thesis. The articles consist of:

A. Residual finiteness of some automorphism groups of high dimensional manifolds. Publicly available on arXiV: 2410.08902.
B. Framed configuration spaces and exotic spheres (joint with Manuel Krannich and Alexander Kupers). Publicly available on arXiV: 2509.19074.

Contents

Introduction

"Misery is manifold."

—Edgar Allan Poe, *Berenice*

Background and motivation

The results obtained in this thesis have been largely the consequence of an attempt of combining two approaches that have proved to be individually fruitful in the study of high dimensional manifold theory over the last few decades: *moduli spaces of manifolds* and *embedding calculus*. In this present introduction, we give an overview of the developments of each of the above topics.

Moduli spaces of manifolds. Given a manifold M, the topological groups $\mathrm{Diff}(M)$ of diffeomorphisms of M and its topological analogue, $\mathrm{Homeo}(M)$ of homeomorphisms, occupy a central position in geometric topology. Their classifying spaces, $\mathrm{BDiff}(M)$ and $\mathrm{BHomeo}(M)$ are equally as important, as they serve as universal objects for smooth and topological manifold bundles with fibre M, respectively.

Classically, these spaces were studied through *surgery theory*. Tracing back to Kervaire and Milnor's breakthrough work on exotic spheres [KM63], the strategy of surgery theory breaks into two steps, starting with the introduction of larger spaces $\mathrm{B}\widetilde{\mathrm{Diff}}(M)$ and $\mathrm{B}\widetilde{\mathrm{Homeo}}(M)$ of *block diffeomorphisms and block homeomorphisms*. The difference between these moduli spaces and the moduli space $\mathrm{BAut}(M)$, the classifying space of the space of homotopy automorphisms of M, can be understood by means of algebraic L-theory, via the surgery fibration [Qui70]. The second step of this approach consists of studying the difference between $\mathrm{BDiff}(M)$, $\mathrm{BHomeo}(M)$ and their block analogues: in a range, this difference is captured by Waldhausen's K-theory of spaces [WW88]. While none of the results in this thesis particularly pertains to surgery theory, many ideas and strategies have been influenced by it.

The second strategy in the study of moduli spaces of manifolds has its origins in *cobordism theory* at large. Originating from the landmark Madsen-Weiss proof of the Mumford conjecture [MW07], the determination of the homotopy type of the cobordism category in [GMTW09], and the works of Galatius and Randal-Williams in the high dimensional setting [GRW14, GRW18, GRW17, GRW20], this approach culminated in a detailed description of the cohomology group $H^*(\mathrm{BDiff}^+(M))$ of characteristic classes of oriented smooth manifold bundles, where M is an even dimensional oriented manifold (of dimension not equal to 4) in a certain range of degrees. The range in this setup is given by the *genus*: for $g \in \mathbb{N}$, we define W_g^n as the g-fold connected sum of copies of $S^n \times S^n$. The genus of M is then defined as

$$g(M) \coloneqq \max\{g \in \mathbb{N} \mid \text{there exists } M' \text{ such that } M \cong M' \# W_g^n\}$$

A parametrised version of classical Pontrjagin-Thom theory yields, for an arbitrary connected compact $2n$-dimensional manifold M with non-empty boundary, a map

$$\mathrm{colim}_g \left(\mathrm{BDiff}^+_\partial(M \# W^n_{g,1})\right) \to (\Omega^\infty \mathbf{MT}\Theta)\,/\mathrm{hAut}(u_M)$$

which was shown to be a homology equivalence on the components it hits; here, Θ is the tangential structure obtained as the n-th stage Moore-Postnikov truncation of the map $M \to \mathrm{BO}(2n)$ classifying the tangent bundle of M, and u_M is the map $u_M : B_\Theta \to \mathrm{BO}(2n)$ in that Moore-Postnikov system, see [**GRW17**, Corollary 1.9]. It is noteworthy to point out that the target of the above map, the infinite loop space of a Madsen-Tillmann spectrum, is amenable to homology computations; thus, the above homology equivalence yields a complete description of the homology of the stable moduli space of manifolds. The determination of the unstable homology from the stable one follows from the various homological stability results for groups of diffeomorphisms listed above in the program of Galatius and Randal-Williams; the stability range is expressed in terms of the genus $g(M)$.

Embedding calculus. For $d \in \mathbb{N}$, we define $\mathcal{M}\mathrm{an}_d$ to be the ∞-category consisting of smooth manifolds, not necessarily compact, of dimension d with $\partial M = \emptyset$ as objects and smooth embeddings as morphisms. The groupoid core of the above category splits off as

$$\mathcal{M}\mathrm{an}_d^{\simeq} \simeq \coprod_{[M]} \mathrm{BDiff}(M)$$

where the disjoint union runs over the set of diffeomorphism classes of manifolds. As a consequence, the category $\mathcal{M}\mathrm{an}_d$ elevates itself to an interesting object of study. *Embedding calculus* can be perceived as a homotopy theoretic gadget that approximates the category of manifolds, which remembers more information than the classical surgery theoretic approximation discussed above.

More precisely, we let $\mathcal{D}\mathrm{isc}_d \subset \mathcal{M}\mathrm{an}_d$ be the full subcategory of the category of manifolds on those objects that are of the form $\sqcup_\ell \mathbb{R}^d$ for some $\ell \in \mathbb{N}$. The category of discs admits a filtration by arity: that is, for a finite $k \in \mathbb{N}$, we can consider the full subcategory of $\mathcal{D}\mathrm{isc}_{d,\leq k}$ on objects of the form $\sqcup_\ell \mathbb{R}^d$, for $\ell \leq k$. We thus obtain a tower of fully faithful inclusions

$$\mathcal{D}\mathrm{isc}_{d,\leq 1} \subset \mathcal{D}\mathrm{isc}_{d,\leq 2} \subset \cdots \subset \mathrm{colim}_k \mathcal{D}\mathrm{isc}_{d,\leq k} \simeq \mathcal{D}\mathrm{isc}_d \subset \mathcal{M}\mathrm{an}_d$$

For $k \in \mathbb{N} \cup \{\infty\}$, we define ι_k as the inclusion $\mathcal{D}\mathrm{isc}_{d,\leq k} \hookrightarrow \mathcal{M}\mathrm{an}_d$, and we define a functor

$$\iota_k^* E : \mathcal{M}\mathrm{an}_d \to \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k}) \coloneqq \mathrm{Fun}\left(\mathcal{D}\mathrm{isc}^{\mathrm{op}}_{d,\leq k}, \mathcal{S}\right)$$

as the composite

$$\mathcal{M}\mathrm{an}_d \hookrightarrow \mathrm{PSh}(\mathcal{M}\mathrm{an}_d) \xrightarrow{\iota_k^*} \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k})$$

where the left morphism is the Yoneda embedding. This construction yields a tower of functors

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{D}\mathrm{isc}_d) \arrow[r] & \cdots \arrow[r] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq 2}) \arrow[r] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq 1}) \\ \mathcal{M}\mathrm{an}_d \arrow[u] \arrow[urr] \arrow[urrr, bend right] & & & \end{tikzcd} \tag{1}$$

We list two key characteristics of the above tower:

(i) The functor $\iota_1^* E : \mathcal{M}\mathrm{an}_d \to \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq 1}) \simeq \mathrm{PSh}(\mathrm{BO}(d))$, sends a manifold to its frame bundle $\mathrm{Fr}(M)$;

(ii) The differences of the stages of the tower admit a very concrete description, in the sense that the category of lifts of $X \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k-1})$ along the restriction $\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k}) \to \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k-1})$ has a concrete description in terms of X.

On mapping spaces, the diagram (1) induces the *embedding calculus tower*. For M and N two smooth, d-dimensional manifolds, and $k \in \mathbb{N} \cup \{\infty\}$, we define

$$T_k\mathrm{Emb}(M,N) \coloneqq \mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k})}(E_M, E_N)$$

and

$$T_k\mathrm{Diff}(M) \coloneqq \mathrm{Aut}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k})}(E_M)$$

The above two spaces play a central role throughout the current thesis.

Variants of embedding calculus that account for manifolds with boundary, tangential structures, topological locally flat embeddings of smoothable topological manifolds and embeddings in positive codimension are well-established by now; see for instance [**KK24b**, §4 and 5]. Historically, embedding calculus was developed by Goodwillie, Klein and Weiss [**Goo90, Wei99, GW99, GKW01**], and was specifically aimed at studying embeddings in codimension at least 3. The main result in that direction is that given a smooth d-manifold M, and a smooth manifold W such that $\dim(W) - \dim(M) \geq 3$, the map

$$\mathrm{Emb}(M,W) \to T_\infty\mathrm{Emb}(M,W)$$

is an equivalence, in which case we say that the embedding calculus tower *converges*. The analogous statement for manifolds with boundary and embeddings rel. boundary also holds by their work.

In recent years, interest in codimension 0 embedding calculus has grown considerably within the high dimensional manifold theory community. Codimension 0 embedding calculus has been a crucial input in the following list of remarkable works: finiteness results of moduli spaces of manifolds [**Kup19, Wei21**], disc-structure spaces [**KK22**], Pontrjagin-Weiss classes for topological manifold bundles [**KK25**], detection of exotic smooth structures [**KK24a**], the Alexander trick for contractible manifolds and homology spheres [**GRW24**], diffeomorphisms of discs [**KRW25**], to name but a few. The two articles making up the present thesis consist of a modest addition to this literature.

Summary of results

We present a brief description of the results of both papers. We refer to the introductions of each of these papers for a more detailed exposition.

Mapping class groups. For M a smooth manifold, the *smooth mapping class group* $\pi_0\mathrm{Diff}(M)$ of smooth isotopy classes of diffeomorphisms of M, and its topological analogue, the *topological mapping class group* $\pi_0\mathrm{Homeo}(M)$, occupy a central part of the two articles in this thesis. By an application of parametrized smoothing theory [**KS77**, Essay 5, §3], one observes that the natural map $\pi_0\mathrm{Diff}(M) \to \pi_0\mathrm{Homeo}(M)$ has finite kernel and finite index in a large variety of cases, for instance when M is compact and of dimension at least 5. In these cases, experience has taught us that these two discrete groups behave fairly similarly. The first article of this thesis is concerned with finding a concrete group theoretic difference between them. This difference came out in the form of residual finiteness: for a discrete group G, we define its *finite residual* $\mathrm{fr}(G)$ as

$$\mathrm{fr}(G) \coloneqq \bigcap_{H \trianglelefteq G,\, [G:H]<\infty} H$$

and we say a group G is *residually finite* precisely when $\mathrm{fr}(G)$ is the trivial subgroup.

In [**KRW20**], an interesting relation was obtained between finite residuals of smooth mapping class groups and the Kervaire-Milnor groups of exotic spheres [**KM63**], which we present in the following. For $d \in \mathbb{N}$, we define Θ_d as the group of exotic spheres up to diffeomorphisms. We may consider the following group homomorphisms

$$\pi_0\mathrm{Diff}_\partial(D^{d-1}) \xrightarrow{\mathrm{ext}_{S^{d-1}}} \pi_0\mathrm{Diff}^+(S^{d-1}) \xrightarrow{\mathrm{tw}} \Theta_d$$

where $\mathrm{ext}_{S^{d-1}}$ is defined as extension by identity after a choice of one (and hence any) orientation preserving embedding of D^{d-1} in S^{d-1}, and where tw sends an orientation preserving diffeomorphism f of the $(d-1)$-sphere to the diffeomorphism class of the twisted sphere $D^d \cup_f D^d$. The morphism $\mathrm{ext}_{S^{d-1}}$ is an isomorphism by an application of the parametrized isotopy extension theorem, while the morphism tw is an isomorphism for $d \geq 6$, by an application of the h-cobordism theorem and Cerf's pseudo-iosotopy theorem.

With the above isomorphism $\mathrm{tw} \circ \mathrm{ext}_{S^d}$ at hand, we define the following morphism: for any d-dimensional connected oriented smooth manifold M, let ext_M be the morphism

$$\mathrm{ext}_M \colon \Theta_{d+1} \to \pi_0\mathrm{Diff}^+(M)$$

obtained as the extension by identity after choice of an orientation preserving embedding $D^d \hookrightarrow M$. In [**KRW20**], it was shown that the morphism ext_M obstructs residual finiteness of $\pi_0\mathrm{Diff}^+(M)$ for certain choices of smooth manifolds M, namely $M = W_g^n$, for $g \geq 2$ and certain values of n; we later extend this result in the second paper to include all odd n. The following consists of the main result of Paper A

Theorem A. *Let M be a closed, smooth 2-connected manifold of dimension $d \geq 6$.*

(i) The topological mapping class groups $\pi_0\mathrm{Homeo}(M)$ and $\pi_0\mathrm{Homeo}^+(M)$ are residually finite groups;

(ii) The finite residuals $\mathrm{fr}(\pi_0\mathrm{Diff}(M))$ and $\mathrm{fr}(\pi_0\mathrm{Diff}^+(M))$ are contained in the image of the morphism ext_M. In other words, any diffeomorphism of M that is in the finite residual of the mapping class group is isotopic to a diffeomorphism supported in a disc in M.

The above theorem is proved by means of codimension 0 embedding calculus. A sequence of intermediary groups are defined, namely the *T_k-mapping class groups*, defined as $\pi_0 T_k\mathrm{Diff}(M)$ (and various analogues for manifolds with boundary). The technical heart of the first article is showing that $\pi_0 T_k\mathrm{Diff}(M)$ is indeed residually finite, for M a 2-connected, smooth closed manifold of dimension at least 6.

Exotic spheres and configuration spaces. In joint work with Manuel Krannich and Alexander Kupers, and taking advantage of the above group theoretic difference between $\pi_0\mathrm{Diff}(W_g^n)$ and $\pi_0 T_k\mathrm{Diff}(W_g^n)$, together with the role that the group of exotic spheres plays in regards to the finite residual of $\pi_0\mathrm{Diff}(W_g^n)$, we turn our attention in the second article to the question of detectability of exotic smooth structures on spheres in the category of truncated $\mathcal{D}$isc-presheaves. We give a complete answer to this question when $d \not\equiv 1 \pmod 4$, in terms of the Kervaire-Milnor exact sequence.

More precisely, the isomorphism class of the $\mathcal{D}$isc-presheaf E_M associated to a smooth manifold M can be viewed as a diffeomorphism invariant, and as such, it is natural to inquire about the robustness of this invariant. Currently, there are no known examples of two non-diffeomorphic closed manifolds M_0 and M_1 of dimension at least 5 such that $E_{M_0} \simeq E_{M_1}$[1]. The situation is different in the cases of k-truncated $\mathcal{D}$isc-presheaves, for finite k; this is a corollary of our result, which we state in the following. The input of our work is again the set of exotic spheres up to diffeomorphisms; by Kervaire and Milnor's work [**KM63**], the set Θ_d of d-dimensional exotic spheres admits the structure of a finite abelian

[1]In dimension 4, such examples exist, see [**KK24a**]

group under connect sum, and fits into an exact sequence[2]

$$0 \to \mathrm{bP}_{d+1} \to \Theta_d \to \operatorname{coker} J_d \to 0$$

where bP_{d+1} is the subgroup consisting of those exotic spheres that bound parallelisable compact manifolds. The following is the main result of Paper B

Theorem B. *For exotic spheres Σ_0 and Σ_1 of dimension $d \not\equiv 1 \pmod 4$, and for $1 < k < \infty$, we have $\iota_k^* E_{\Sigma_0} \simeq \iota_k^* E_{\Sigma_1} \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\leq k})$ if and only if $\Sigma_0 \# \overline{\Sigma_1}$ bounds a parallelisable compact $(d+1)$-dimensional manifold.*

Original motivation and future directions

The first part of this section introduces a project undertaken during the writing of this thesis, and vaguely related to its content. The second consists of a strategy, obtained after discussions with Manuel Krannich, regarding generalizations of some of the results of Paper A.

From tangential structures to embeddings. $\mathcal{D}$isc-presheaves can be viewed as a tool for relating various moduli spaces of manifolds of interest; in this section, we introduce these moduli spaces, and point out how $\mathcal{D}$isc-presheaves could serve as a tool for interpolating between them.

For a closed smooth manifold M, the space of embeddings

$$\mathrm{Emb}(M, \mathbb{R}^\infty) \coloneqq \operatorname{colim}_N \left(\mathrm{Emb}(M, \mathbb{R}^N)\right)$$

is non-empty and contractible by the Whitney embedding theorem, and furthermore admits a free action of $\mathrm{Diff}(M)$ by pre-composition. As a consequence, the moduli space $\mathrm{BDiff}(M)$ may be modeled as the space

$$\mathrm{BDiff}(M) \simeq \mathrm{Emb}(M, \mathbb{R}^\infty)/\mathrm{Diff}(M)$$

By changing the target in the above embedding space to some fixed background manifold X, we may consider the *moduli space of submanifolds of type M* in a fixed background manifolds X, namely

$$\mathcal{M}(M; X) \coloneqq \mathrm{Emb}(M, X)/\mathrm{Diff}(M)$$

An important variant of the above moduli spaces accounts for *tangential structures*. For Θ a space admitting an action by $\mathrm{GL}_d(\mathbb{R}^d)$, a Θ-tangential structure on a manifold M is the datum of a $\mathrm{GL}_d(\mathbb{R})$-equivariant map $\mathrm{Fr}(M) \to \Theta$, where $\mathrm{Fr}(M)$ is the frame bundle of M, endowed with the obvious $\mathrm{GL}_d(\mathbb{R})$ action. We define the moduli space of Θ-framed manifolds of type M as

$$\mathrm{BDiff}^\Theta(M) \coloneqq \left(\mathrm{Map}^{\mathrm{GL}_d(\mathbb{R})}(\mathrm{Fr}(M), \Theta)\right)/\mathrm{Diff}(M)$$

Taking derivatives at the origin yields a map

$$\mathrm{Emb}(\mathbb{R}^d, \mathbb{R}^d) \to \mathrm{GL}_d(\mathbb{R})$$

which can easily be seen to be an equivalence. From this, one can derive an equivalence

$$\mathrm{PSh}(\mathrm{BGL}_d(\mathbb{R})) \simeq \mathrm{PSh}(\mathcal{D}\mathrm{isc}_d^{\leq 1})$$

[2]There are finitely many exceptions to the Kervaire-Milnor exact sequence precisely when the Kervaire invariant 1 admits a solution; in these cases, the exact sequence extends once to the right to a $\mathbb{Z}/2\mathbb{Z}$, but this will be irrelevant to our discussion

where we refer to [**KK24b**, Thm A] for more details. As a consequence of the above equivalence of categories, one observes:

$$\begin{aligned}\mathrm{BDiff}^{\Theta}(M) &\simeq \left(\mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq 1})}(\iota_1^* E_M, \Theta)\right)/\mathrm{Diff}(M)\\ &\simeq \left(\mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d})}(E_M, (\iota_1)_*\Theta)\right)/\mathrm{Diff}(M)\end{aligned}$$

This motivates the following definition: given $\Lambda \in \mathrm{PSh}(\mathcal{D}\mathrm{isc})$, not necessarily of the form $(\iota_1)_*\Theta$ as above, we define $\mathrm{BDiff}^{\Lambda}(M)$ as

$$\mathrm{BDiff}^{\Lambda}(M) \coloneqq \left(\mathrm{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d})}(E_M, \Lambda)\right)/\mathrm{Diff}(M)$$

The above object interpolates between the moduli spaces presented above, in the sense that

- For $\Lambda_\Theta = (\iota_1)_*\Theta$ where Θ is a space with a $\mathrm{GL}_d(\mathbb{R})$-action, $\mathrm{BDiff}^{\Lambda_\Theta}(M) \simeq \mathrm{BDiff}^{\Theta}(M)$;
- For $\Lambda_X = E_X$, the d-dimensional $\mathcal{D}$isc-presheaf associated to a manifold X of dimension at least $d+3$, $\mathrm{BDiff}^{\Lambda_X}(M) \simeq \mathcal{M}(M;X)$, which follows from the convergence of the embedding calculus tower in codimension 3.

It is possible to modify the above definitions to account for manifolds with boundary, following [**KK22**, **KK24b**]. The upshot of the above discussion is that we may now tackle questions about moduli spaces of submanifolds of a background manifold in the same manner and with the same techniques we use in the study of moduli spaces of Θ-framed manifolds. In particular, two questions stand out.

Question 0.1. *Can one show homological stability for the system* $\{\mathcal{M}(W_{g,1};X)\}_{g\in\mathbb{N}}$ *of type* $W_{g,1} = \#_g(S^n \times S^n)\setminus D$ *in a sufficiently high codimension manifold* X*, by switching to the moduli spaces* $\mathrm{BDiff}^{\Lambda_X}_{\partial}(W_{g,1})$ *and using a modification of the usual proof for* $\mathrm{BDiff}^{\Theta}_{\partial}(W_{g,1})$[3]*?*

Question 0.2. *Can one define a cobordism category* $\mathcal{B}\mathrm{ord}^{\Lambda}_d$ *of* Λ*-framed manifolds that interpolates between* Θ*-framed cobordism categories and embedded cobordism categories*[4]*?*

A strategy for some generalizations. We recall the main result of Paper A. Given a smoothable, 2-connected topological manifold M of dimension at least 6, the topological mapping class group $\pi_0\mathrm{Homeo}(M)$ of isotopy classes of homeomorphisms, is a residually finite group, hence arithmetic. The conditions of dimensionality and connectivity of M, while necessary for the proof as presented in the paper, seem somewhat restrictive. During discussions with Manuel Krannich, a strategy was born with the specific aim of generalizing to the situation where M is assumed to be smoothable, simply connected, and of dimension at least 5.

Motivated by §4 of Paper B, we claim that one may construct a commutative square

$$\begin{tikzcd} \mathrm{BHomeo}(M) \arrow[r, "1"] \arrow[d, "4"'] & \mathrm{BT_2Homeo}(M) \arrow[d, "2"] \\ \mathrm{B}\widetilde{\mathrm{Homeo}}(M) \arrow[r, "3"'] & \mathcal{N}(M)/\mathrm{Aut}(M) \end{tikzcd}$$

in $\mathcal{S}/\mathrm{BAut}(M)$, where

- $\mathrm{T_2Homeo}(M)$ is the space of automorphisms of E^t_M in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^t_{d,\leq 2})$, the topological 2-truncated $\mathcal{D}$isc-presheaf category;

[3] The only positive result of homological stability for moduli of submanifolds is obtained in the case of surfaces in a 5-manifold, in [**CMRW17**].

[4] Our working definition of the embedded cobordism category is as in [**RW11**].

- $\mathcal{N}(M)/\mathrm{Aut}(M)$ is the homotopy quotient of the space of normal invariants of M by the group of homotopy automorphisms of M;
- The map ① is induced by the functor $\iota_2^* E^t : \mathcal{M}\mathrm{an}_d^t \to \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq 2}^t)$ from the category of d-dimensional topological manifolds and locally flat topological embeddings to the category of 2-truncated topologica $\mathcal{D}$isc-presheaves;
- The map ② could be constructed following §4 of Paper B, noting that the stable collapse map of a manifold only depends on its 2-truncated $\mathcal{D}$isc-presheaf, including the case of topological manifolds (see Remark 4.8 *loc.cit.*);
- ③ is the map obtained from the map $\mathcal{S}(M) \to \mathcal{N}(M)$ in the surgery fibration upon passing to the homotopy quotient by the group $\mathrm{Aut}(M)$;
- ④ is the natural map from the moduli space of topological manifolds of type M to its block analogue.

Once such a commutative square is set-up, it follows rather readily that $\pi_0\mathrm{Homeo}(M)$ is a residually finite whenever M is closed, smoothable, simply connected and of dimension at least 5.

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In the last two years, and in the midst of the war of aggression waged on my home country of Lebanon and on the neighboring land of Palestine[5], mathematics, while offering at times a temporary escape, was not the first thing on my mind for a significant period of time. I wish to thank all the people who offered their support during that period, particularly Filip, Florian, Oscar, Bella and Carlotta.

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أتقدم بخالص الشكر والامتنان إلى والدتي آمال، إلى والدي إلياس، إلى أخي عماد، إلى أختي نادين ونسرين، إلى يعقوب، وسيم ، وليد ، منى، جورج ورفائيل، على دعمهم الذي كان له الأثر الأكبر في إنجاز هذا العمل. إن كلمات الشكر لتقف عاجزة أمام ما قدّموه لي من دعم معنوي وعاطفي لا يقدّر بثمن.

أختم بكلمة تقدير لأهل فلسطين الأبية، لقضيتهم النبيلة، ولمقاومتهم الأبدية. لعلّنا نعي يوماً على أرض البرتقال الحزينة، حرةً وأبية.

Fadi Mezher
Copenhagen, September 2025

[5]Lest this paragraph be misconstrued as a political statement, we strive to devoid it of such content, following some generous advice.

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Paper A

RESIDUAL FINITENESS OF SOME AUTOMORPHISM GROUPS OF HIGH DIMENSIONAL MANIFOLDS

FADI MEZHER

Abstract. We show that for a smooth, closed 2-connected manifold M of dimension $d \geq 6$, the topological mapping class group $\pi_0 \operatorname{Homeo}(M)$ is residually finite, in contrast to the situation for the smooth mapping class group $\pi_0 \operatorname{Diff}(M)$. Combined with a result of Sullivan, this implies that $\pi_0 \operatorname{Homeo}(M)$ is an arithmetic group. The proof uses embedding calculus, and is of independent interest: we show that the T_k-mapping class group, $\pi_0 T_k \operatorname{Diff}(M)$, is residually finite, for all $k \in \mathbb{N}$. The statement on the topological mapping class group is then deduced from the Weiss fibre sequence, convergence of the embedding calculus tower and smoothing theory.

1. Introduction

In his seminal work [Sul77], Sullivan shows, among other things, a structural result on mapping class groups of smooth manifolds. Combining surgery theory and rational homotopy theory, Sullivan shows that, for M a closed, simply connected manifold of dimension $d \geq 5$, $\pi_0 \operatorname{Diff}(M)$, the group of isotopy classes of diffeomorphisms of M, is *commensurable up to finite kernel* to an arithmetic group (see §4.5 for more details on this equivalence relation).

The equivalence relation of commensurability up to finite kernel differs from the classical notion of commensurability, and studying this difference is initiated in [KRW20]. Among groups commensurable up to finite kernel to an arithmetic group, being arithmetic is equivalent to another classical group theoretic notion, that of *residual finiteness.* A group G is said to be residually finite if its finite residual $\operatorname{fr}(G)$, the intersection of all finite index normal subgroups of G, is the trivial group; see Definition 2.5 for further details. Building on an example of Deligne of a $\mathbb{Z}$-extension of the symplectic group $\operatorname{Sp}_{2g}(\mathbb{Z})$ which is not residually finite, Krannich and Randal-Williams show that not all smooth mapping class groups of high dimensional manifolds are residually finite in [KRW20]. They consider the following example: let $W_g^n \coloneqq \#_g(S^n \times S^n)$ where we assume $g \geq 5$, and fix an embedding $D^{2n} \hookrightarrow W_g^n$. Extension by the identity yields a group homomorphism

$$\pi_0 \operatorname{Diff}_\partial(D^{2n}) \longrightarrow \pi_0 \operatorname{Diff}(W_g^n)$$

where the domain may be identified with the group of exotic spheres Θ_{2n+1}. Whenever $n = 5 \pmod 8$, Krannich and Randal-Williams show that the subgroup bP_{2n+2} of exotic spheres bounding a parallelisable compact manifold embeds into $\operatorname{fr}(\pi_0 \operatorname{Diff}(W_g^n))$. For these values of n, bP_{2n+2} is known to be non-trivial [KM63]; this thus yields a family of examples of closed smooth manifolds with non-residually finite smooth mapping class groups.

Main results. Given the smooth nature of the above counterexamples, one may wonder what happens in the category of topological manifolds. The following three theorems constitute three of the main results of this work

Theorem A. *Let M be a smoothable, closed 2-connected topological manifold of dimension $d \geq 6$. Then, $\pi_0 \operatorname{Homeo}(M)$ and $\pi_0 \operatorname{Homeo}^+(M)$ are residually finite groups. For W a smoothable, 2-connected compact manifold of dimension $d \geq 5$, such that $\partial W \neq \emptyset$, then $\pi_0 \operatorname{Homeo}_\partial(W)$ is residually finite.*

This is shown in the text as Theorem 4.9 for the closed case, and Theorem 4.16 for the boundary case. As a corollary, we obtain the following (in the text Theorem 4.14)

Theorem B. *Let M be a smoothable, closed 2-connected topological manifold of dimension $d \geq 6$. Then, $\pi_0 \operatorname{Homeo}(M)$ and $\pi_0 \operatorname{Homeo}^+(M)$ are arithmetic groups.*

We furthermore show that the failure of residual finiteness of smooth mapping class groups is entirely due to exotic spheres, as occurring in Krannich and Randal-Williams' example. Indeed, we show the following (as Theorem 4.18 in the text)

Theorem C. *Let M be a smooth, closed 2-connected manifold of dimension $d \geq 6$, and fix an embedded disc $D^d \subset M$. Then,*

(i) $\operatorname{fr}(\pi_0 \operatorname{Diff}^+(M)) \subseteq \operatorname{Im}(\pi_0 \operatorname{Diff}_\partial(D^d) \to \pi_0 \operatorname{Diff}^+(M))$*, where the map* $\pi_0 \operatorname{Diff}_\partial(D^d) \to \pi_0 \operatorname{Diff}^+(M)$ *is given by extension by the identity;*

(ii) $\operatorname{fr}(\pi_0 \operatorname{Diff}(M)) \subseteq \operatorname{Im}(\pi_0 \operatorname{Diff}_\partial(D^d) \to \pi_0 \operatorname{Diff}(M))$*, where the map* $\pi_0 \operatorname{Diff}_\partial(D^d) \to \pi_0 \operatorname{Diff}(M)$ *is given by extension by the identity*

Outline of proof. The proof of the above statements is broken into various key steps, which we outline here. We begin by pointing out that Theorem A implies both Theorem B and Theorem C: see §4.5 for the first implication, and the proof of Theorem 4.18 for the second. We therefore focus on presenting an outline of the proof of Theorem A.

Weiss fibre sequences. The first key tool is the Weiss fibre sequence, as developed in [Kup19, Wei21]. For a smooth manifold M of dimension d, let M° denote the manifold with boundary $M \setminus \operatorname{int}(D^d)$, where $D^d \hookrightarrow M$ is a fixed embedding of a d-dimensional disc. We then have a commutative diagram

$$\begin{tikzcd}
B\operatorname{Diff}_\partial(D^d) \arrow[r] \arrow[d] & B\operatorname{Diff}_\partial(M^\circ) \arrow[r] \arrow[d] & B\operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ) \arrow[d, "\Psi"] \\
* \simeq B\operatorname{Homeo}_\partial(D^d) \arrow[r] & B\operatorname{Homeo}_\partial(M^\circ) \arrow[r, "\simeq"'] & B\operatorname{Emb}^{\mathrm{Top},\cong}_{\partial/2}(M^\circ, M^\circ)
\end{tikzcd}$$

where both rows are fibre sequences, the smooth and topological *Weiss fibre sequences* (§4.1), and where $\operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$ denotes the space of smooth embeddings that are isotopic to a diffeomorphism, and that restrict to the identity on a neighbourhood around the lower hemisphere $D^{d-1}_- \subset S^{d-1} \cong \partial M^\circ$; $\operatorname{Emb}^{\mathrm{Top},\cong}_{\partial/2}$ is its topological analogue. By the Alexander trick, $B\operatorname{Homeo}_\partial(D^d)$ is contractible, so that $B\operatorname{Homeo}_\partial(M^\circ) \simeq B\operatorname{Emb}^{\mathrm{Top},\cong}_{\partial/2}(M^\circ, M^\circ)$. Following [Kup19, Wei21], we study the space of self-embeddings relative half the boundary using embedding calculus.

Embedding calculus. We consider the ∞-category Manf_d consisting of d-dimensional smooth manifolds with empty boundary as objects, and spaces of embeddings as 1-morphisms. We can consider the full subcategory of Manf_d, Disc_d, on those objects of the form $\sqcup_\ell \mathbb{R}^d$, which further decomposes into full subcategories $\{\mathrm{Disc}_d^{\leq k}\}_{k \in \mathbb{N}}$, where for a given $k \in \mathbb{N}$, the objects of $\mathrm{Disc}_d^{\leq k}$ are precisely $\sqcup_\ell \mathbb{R}^d$, for $\ell \leq k$. We denote the inclusions $\mathrm{Disc}_d^{\leq k} \hookrightarrow \mathrm{Manf}_d$ by ι_k, for $k \in \mathbb{N} \cup \{\infty\}$. Thus, we define a tower of functors via

$$\iota_k^* E \colon \mathrm{Manf}_d \xrightarrow{h} \mathrm{Psh}(\mathrm{Manf}_d) \xrightarrow{\iota_k^*} \mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$$

with the convention that $\iota_\infty^* E_M$ is simply denoted by E_M. Concretely, given a smooth d-manifold M, E_M is the disc-presheaf $\sqcup_\ell \mathbb{R}^d \mapsto \operatorname{Emb}(\sqcup_\ell \mathbb{R}^d, M)$. For each $k \in \mathbb{N} \cup \{\infty\}$, and smooth manifolds $M, N \in \mathrm{Manf}_d$, we define

$$T_k \operatorname{Emb}(M, N) \coloneqq \operatorname{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\leq k})}(E_M, E_N)$$

and

$$T_k \operatorname{Diff}(M) \coloneqq \operatorname{Map}^{\simeq}_{\operatorname{Psh}(\operatorname{Disc}^{\leq k}_d)}(E_M, E_M)$$

On mapping spaces, the above tower of functors yields the *embedding tower.* This consists of a tower $\{T_k \operatorname{Emb}(M,N)\}_{k\in\mathbb{N}}$, receiving a map from $\operatorname{Emb}(M,N)$. We say the embedding tower *converges* if the map

$$\operatorname{Emb}(M,N) \longrightarrow T_\infty \operatorname{Emb}(M,N) \simeq \lim_k T_k \operatorname{Emb}(M,N)$$

is an equivalence. By the celebrated result of Goodwillie-Weiss [GW99, Corollary 2.5], this holds in the case where $h\dim(M) \leq d-3$, where $h\dim$ denotes the handle dimension of M.

A setup for embedding calculus relative boundary can be formulated, and the analogous convergence result holds; see, for instance, [GW99, Chapter 5] or [Kup19, §3.3.2]. We shall make use of this setup as follows. Given M a closed, smooth 2-connected manifold of dimension $d \geq 6$, a standard Morse theoretic argument shows that $M^\circ \coloneqq M \setminus \operatorname{int}(D^d)$ admits a handle decomposition of handle dimension at most $d-3$, relative half the boundary; thus, the embedding calculus tower for $\operatorname{Emb}_{\partial/2}(M^\circ, M^\circ) \simeq \operatorname{Emb}_\partial(M^*, M^*)$, where $M^* \coloneqq M^\circ \setminus \partial/2$, converges. Consequently, one can infer information on $\operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$ from embedding calculus. As convergence of embedding calculus is not known in the topological category, we restrict ourselves to the case of smoothable manifolds and use smoothing theory to infer information in the topological setting. As the class of residually finite groups is closed under taking limits, the convergence of embedding calculus allows us to infer that $\pi_0 \operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$ is residually finite, by showing that $\pi_0 T_k \operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$ is residually finite. The following is the technical heart of this paper (in the text as Theorems 3.4 and 3.16 and Proposition 4.5).

Theorem D. *For M a smooth, closed 2-connected manifold of dimension $d \geq 5$, and for any $k \in \mathbb{N}$, the groups $\pi_0 T_k \operatorname{Diff}(M)$ and $\pi_0 T_k \operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$ are residually finite.*

Profinite homotopy theory. Theorem D above builds on a known result, observed by Serre in [Ser79, p. 108], as a corollary of Sullivan's work on profinite completion of spaces, namely [Sul74, thm 3.2]; we give a careful and modern exposition of its proof, that is adaptable to embedding calculus.

Theorem E (Serre-Sullivan)**.** *Let X be a finite, simply connected CW-complex. Then, $\pi_0 \operatorname{hoAut}(X)$ is a residually finite group.*

As alluded to in the above, this follows from *profinite homotopy theory.* Roughly speaking, profinite homotopy theory attempts to lift the group profinite completion functor to spaces. We begin by briefly recalling group profinite completion. We consider the functor $\widehat{(-)}\colon \operatorname{Grp} \to \operatorname{Pro}(\operatorname{Grp}^{\mathrm{fin}})$, where Grp denotes the 1-category of groups and group morphisms, and $\operatorname{Grp}^{\mathrm{fin}}$ the full subcategory thereof, consisting of finite groups; the functor is defined by sending a group G to the functor $F \mapsto \operatorname{Hom}(G,F) \in \operatorname{Fun}(\operatorname{Grp}^{\mathrm{fin}}, \operatorname{Grp})^{\mathrm{op}}$. The latter is classically denoted by $\widehat{G}$, and coincides with the cofiltered system of finite quotients of G. The above functor admits a right adjoint Mat^g, the *materialisation functor*, sending a pro-object of finite groups F to its limit in groups. The composite $\operatorname{Mat}^g \circ \widehat{(-)}\colon \operatorname{Grp} \to \operatorname{Grp}$ we denote by Φ^g. The unit of the adjunction $\widehat{(-)} \dashv \operatorname{Mat}^g$ yields, for every group G, a map $\eta_G\colon G \to \Phi^g G$, called the *profinite completion map.* More concretely, $\Phi^g G$ is the limit over the diagram $\{G/N_i\}$ over $\{N_i\}$, the cofiltered system of finite index subgroups of G. Thus, $\ker(\eta_G\colon G \to \Phi^g)$ coincides with the finite residual $\operatorname{fr}(G)$; consequently, a group G is residually finite if and only if the map $\eta_G\colon G \to \Phi^g G$ is injective, which is the key connection between residually finite groups and group profinite completion.

As is the case for groups, one can also approximate spaces by certain spaces with a finiteness condition. We say a space X is *π-finite* if

- $\pi_0 X$ is a finite set;
- There exists some $N \in \mathbb{N}$ such that for all $x \in X$, and all $k \geq N$, $\pi_k(X, x) = 0$;
- $\pi_n(X, x)$ is finite for all $n \in \mathbb{N}$ and $x \in X$.

We denote $\mathcal{S}_\pi \subset \mathcal{S}$ for the full subcategory of the ∞-category of spaces consisiting of π-finite spaces. We define a functor $\widehat{(-)}\colon \mathcal{S} \to \mathrm{Pro}(\mathcal{S}_\pi), X \mapsto (F \mapsto \mathrm{Map}_{\mathcal{S}}(X, F))$. This functor similarly admits a right adjoint $\mathrm{Mat}\colon \mathrm{Pro}(\mathcal{S}_\pi) \to \mathcal{S}$, obtained by applying the limit in $\mathcal{S}$ of an object $X \in \mathrm{Pro}(\mathcal{S}_\pi)$. We denote the composite $\mathrm{Mat} \circ \widehat{(-)}$ by Φ^s, which we call the *finite completion functor* following Sullivan. The unit of the adjunction $\widehat{(-)} \dashv \mathrm{Mat}$ yields, for each X, a natural map $\eta_X\colon X \to \Phi^s X$. Residual finiteness of $\pi_0\,\mathrm{hoAut}(X)$ then follows from [Sul74, thm 3.2], stating that the map

$$\mathrm{Map}(Y, Z) \to \mathrm{Map}(Y, \Phi^s Z)$$

is π_0-injective, for nice enough spaces Y and Z. The key input of Sullivan's proof of the above statement, which we study in some detail, is that the functor Φ^s comes close to preserving finite limits when restricted to nice enough spaces, namely componentwise nilpotent spaces of finite type (Theorem 2.18 and Corollary 2.20).

Theorem F. *Let $\mathcal{F}\colon \mathcal{D} \to \mathcal{S}^{\mathrm{nil,ft}}$ be a functor from a finite ∞-category to the category of componentwise nilpotent spaces of finite type. Then, the canonical coassembly morphism*

$$\mathrm{coass}\colon \Phi^s \lim_{\mathcal{D}} \mathcal{F} \to \lim_{\mathcal{D}} \Phi^s \mathcal{F}$$

induces an isomorphism on π_n, for all $n \geq 1$, based at points coming from $\lim_{\mathcal{D}} \mathcal{F}$ via the maps of the following commutative diagram

$$\begin{tikzcd} & \lim_{\mathcal{D}} \mathcal{F} \arrow[dl, "\eta"'] \arrow[dr, "\lim \eta"] & \\ \Phi^s \lim_{\mathcal{D}} \mathcal{F} \arrow[rr, "\mathrm{coass}"'] & & \lim_{\mathcal{D}} \Phi^s \mathcal{F} \end{tikzcd}$$

where all limits are taken in $\mathcal{S}$.

We now fix M a closed, 2-connected smooth manifold, and study residual finiteness of $\pi_0 T_k\,\mathrm{Diff}(M)$. Given $\iota_k^* E_M \in \mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$, we may consider the finite and profinite completions pointwise. As in the setting of homotopy automorphisms, residual finiteness follows from the π_0-injectivity of the map

$$\mathrm{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\leq k})}(\iota_k^* E_M, \iota_k^* E_M) \to \mathrm{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\leq k})}(\iota_k^* E_M, \iota_k^* \Phi^s E_M)$$

given by composition with the map $\iota_k^* E_M \to \iota_k^* \Phi^s E_M$. This is shown in the text as Theorem 3.7, and occupies the technical heart of the paper.

Smoothing theory. Convergence of the embedding tower thus implies that $\pi_0\,\mathrm{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$ is residually finite. In order to deduce the same for the topological analogue, we use smoothing theory as follows: the fibre of the map

$$B\,\mathrm{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ) \to B\,\mathrm{Emb}^{\cong,\mathrm{Top}}_{\partial/2}(M^\circ, M^\circ) \simeq B\,\mathrm{Homeo}_\partial(M^\circ)$$

can be described as a collection of components of the space of sections $\Gamma_{\partial/2}(\xi_M)$ of a bundle over M° with fibres $\mathrm{Top}(d)/O(d)$, relative some fixed section over $\partial/2$. One can show that $\Gamma_{\partial/2}(\xi_M)$ has finitely many components, and that on each of these components, the fundamental group is a finite group. The long exact sequence on homotopy groups associated to the above fibre sequence thus yields an exact sequence

$$\pi_1 \Gamma_{\partial/2}(\xi_M) \to \pi_0\,\mathrm{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ) \to \pi_0\,\mathrm{Homeo}_\partial(M^\circ) \to \pi_0 \Gamma_{\partial/2}(\xi_M)$$

where exactness on the last term is in the category of pointed sets. It now follows from elementary group theoretic arguments (Lemmas 2.9 and 2.10) that $\pi_0 \operatorname{Homeo}_\partial(M^\circ)$ is residually finite. Finally, to glue back in the deleted disc, we use parametrized isotopy extension again to obtain a fibre sequence

$$\operatorname{Homeo}_\partial(M^\circ) \longrightarrow \operatorname{Homeo}(M) \longrightarrow \operatorname{Emb}^{\operatorname{Top}}(D, M)$$

where the base space of the fibration is equivalent to the topological frame bundle of M, which in particular has $\pi_0 \operatorname{Fr}^{\operatorname{Top}}(M) \cong \mathbb{Z}/2\mathbb{Z}$, and $\pi_1 \operatorname{Fr}^{\operatorname{Top}}(M) \cong \mathbb{Z}/2\mathbb{Z}$; again, applications of Lemmas 2.9 and 2.10 implies $\pi_0 \operatorname{Homeo}(M)$ is indeed residually finite.

Remark on notation. It is common to use the notation $\widehat{X}$ to denote profinite completion of an object X, be it a set, group or space. In this work, the notation $\widehat{X}$ is only reserved for Pro-objects. To make clear the distinction between the different types of objects in use, the endofunctor on our favorite category obtained after materialisation of the profinite completion is denoted Φ, with a decoration to emphasize which object we are looking at. The choice of Φ is to remind us that this functor is called *finite completion*, following Sullivan. We will mainly be interested in three functors:

- $\Phi^g \colon \mathrm{Grp} \longrightarrow \mathrm{Grp}$;
- $\Phi^{tg} \colon \mathrm{Grp} \longrightarrow \mathrm{TopGrp}$;
- $\Phi^s \colon \mathcal{S} \longrightarrow \mathcal{S}$

the first of which is the group finite completion, whereas the second also remembers the canonical topological group structure obtained on the finite completion $\Phi^g G$ of a group G, and the last is the finite completion functor on spaces.

Conventions. Throughout this paper, we use the language of ∞-categories. However, for the more geometrically minded reader, ∞-categories, while being sometimes more convenient, can be substituted by more classical homotopy theoretic language without much trouble. The reader is encouraged to think of classical homotopy theory for the idea and intuition, and ∞-categories for the convenience. We will throughout attempt to point out classical analogues of some stated ∞-categorical facts or constructions.

Furthermore, throughout the work, the sets of smooth embeddings and diffeomorphisms of smooth manifolds are endowed with the C^∞-topology, while sets of topological embeddings and homeomorphisms are endowed with the compact-open topology.

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Contents

2. Homotopy automorphisms

In this section, we lay out a careful exposition for a proof of the following theorem, with a view towards later generalizations. This theorem was observed by Serre in [Ser79, p. 108], building on work of Sullivan, namely [Sul74, thm 3.2].

Theorem 2.1 (Serre-Sullivan)**.** *For X a simply connected, finite CW complex, the group $\pi_0 \operatorname{hoAut}(X)$ is residually finite.*

As a build-up for the proof, we review some constructions on groups and spaces that are crucial for the argument: the classical theory of profinite groups and group profinite completion, Sullivan's theory of profinite completions of spaces, along with a functorial setup for some classical constructions in homotopy theory.

2.1. Profinite completion of groups.

One approach to studying groups is by approximating them by their finite quotients. For a group G, consider the cofiltered system $\{G/N\}$, where N ranges over all finite index, normal subgroups of G. Equivalently, one may look at all group morphisms $G \to F$, where F is a finite group. The categorical notion that captures this type of object is called a *Pro-object*; namely, if $\mathrm{Grp}^{\mathrm{fin}}$ denotes the 1-category of finite groups and group morphisms, a Pro-object in $\mathrm{Grp}^{\mathrm{fin}}$ is then an object in $\mathrm{Fun}(\mathrm{Grp}^{\mathrm{fin}}, \mathrm{Set})^{\mathrm{op}}$, that preserves finite limits. The *(group) profinite completion* functor is the functor $\mathrm{Grp} \to \mathrm{Pro}(\mathrm{Grp}^{\mathrm{fin}})$, $G \mapsto (F \mapsto \mathrm{Hom}(G, F))$. The *group materialisation functor* is defined to be the functor $\mathrm{Mat}^g \colon \mathrm{Pro}(\mathrm{Grp}^{\mathrm{fin}}) \to \mathrm{Grp}$, sending a pro-object of finite groups to its limit in groups, and we have an adjunction $\widehat{(-)} \dashv \mathrm{Mat}^g$. The composite

$$\mathrm{Grp} \to \mathrm{Pro}(\mathrm{Grp}^{\mathrm{fin}}) \to \mathrm{Grp}$$

is denoted by Φ^g. The unit of the above adjunction gives a natural transformation $\mathrm{id} \Rightarrow \Phi^g$, and for a given group G, we denote by η_G the associated map $\eta_G \colon G \to \Phi^g G$. Note that for a group G, the group $\Phi^g G$ carries a canonical structure of a topological group, given by the subgroup topology it inherits from $\Pi(G/N)$, where the product is taken over the same indexing set as above, namely over all finite quotients of G, and where each factor is endowed with the discrete topology. When endowed with this topology, we obtain a functor $\mathrm{Grp} \to \mathrm{TopGrp}$, the category of topological groups and continuous group homomorphisms; this functor is the *finite completion* functor of groups, which we shall denote Φ^{tg} (classically, Φ^{tg} is referred to as profinite completion, and denoted by $\widehat{(-)}$; we call it finite completion to stay consistent with Sullivan's terminology later). Such a topological group has a fairly explicit description in terms of its topology: it is compact, Hausdorff and totally disconnected. We thus define

Definition 2.2. A topological group G is said to be *profinite* if it is a compact, Hausdorff totally disconnected group.

Let ProfGrp be the full subcategory of TopGrp consisting of profinite topological groups. The functor Φ^{tg} does indeed land in the above full subcategory, as can be seen through the following classical result

Theorem 2.3. *A topological group is profinite if and only if it can be written as a cofiltered limit of finite groups (with its canonical topology).*

We also obtain an adjunction $\Phi^{tg} \dashv \mathrm{fgt}$ between the group finite completion functor and the forgetful functor. This adjunction thus allows us to write the following universal property

Theorem 2.4 (Universal property of group finite completion)**.** *Let G be a group, and let H be a profinite topological group. Given a group homomorphism $f \colon G \to H$, there exists a unique continuous group homomorphism $\Phi^{tg} G \to H$ making the following diagram*

$$\begin{tikzcd} G \arrow[r] \arrow[dr, "f"'] & \Phi^{tg}G \arrow[d, dashed] \\ & H \end{tikzcd}$$

commute.

2.1.1. *Residual finiteness.* The functor Φ^g can furthermore be used to define *residual finiteness*, the main group theoretic notion under investigation in this study.

Definition 2.5. A group G is said to be *residually finite* if the canonical group homomorphism

$$G \to \Phi^g G$$

is injective; equivalently, if the intersection of all finite index, normal subgroups of G is the trivial subgroup. In other words, given any $g \in G$ such that $g \neq 1$, there exists some finite group F_g and a group homomorphism $\varphi_g \colon G \longrightarrow F_g$ such that $\varphi_g(g) \neq 1 \in F_g$; we will be informally referring to this property by saying that $g \in G$ can be *detected by finite groups.*

For G a group, we denote $\mathrm{fr}(G)$ for the *finite residual* of G, namely, the intersection of all finite index, normal subgroups of G. From the definitions, it follows that $\mathrm{fr}(G) = \ker(G \longrightarrow \Phi^g G)$, and that G is residually finite if and only if $\mathrm{fr}(G)$ is the trivial subgroup.

Example 2.6. The following classes of groups are residually finite

- Finite groups;
- Free groups;
- Finitely generated nilpotent groups;
- Finitely generated subgroups of general linear groups;

The class of residually finite groups is closed under taking subgroups and inverse limits. However, it is not closed under extensions:

Example 2.7. The following example is due to Deligne [Del78]. We consider the Lie group $\mathrm{Sp}_{2g}(\mathbb{R})$. We recall that $\pi_1 \mathrm{Sp}_{2g}(\mathbb{R}) \cong \mathbb{Z}$; let $\widetilde{\mathrm{Sp}}_{2g}(\mathbb{Z})$ denote the pullback of the universal cover of $\mathrm{Sp}_{2g}(\mathbb{R})$ along the inclusion map $\mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{R})$. We thus obtain a central extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathrm{Sp}}_{2g}(\mathbb{Z}) \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z}) \longrightarrow 1$$

For $g \geq 2$, Deligne shows that the finite residual of $\widetilde{\mathrm{Sp}}_{2g}(\mathbb{Z})$ is $2\mathbb{Z}$.

The above example was used in [KRW20], to show that $\pi_0 \operatorname{Diff}(\#_g(S^n \times S^n))$ is not residually finite for certain $n \in \mathbb{N}$. Combining their result with one of the main results of this paper (Theorem 4.9), we obtain a similar example:

Example 2.8. Let $n = 5 \pmod 8$ and $g \geq 5$, so that $\pi_0 \operatorname{Diff}(W_g^n)$ is not residually finite, where $W_g^n := \#_g(S^n \times S^n)$. One of the main results of this work, which is shown much later, namely Theorem 4.9, implies that $\pi_0 \operatorname{Homeo}(W_g^n)$ is residually finite. By smoothing theory, we can describe the fibre F of the forgetful map

$$B \operatorname{Diff}(W_g^n) \longrightarrow B \operatorname{Homeo}(W_g^n)$$

as a collection of components of a space of sections $\Gamma(\xi)$ of a bundle ξ over W_g^n with fibres $\mathrm{Top}(d)/O(d)$ as in [KS77, Essay V. §3,4], which in particular has finite π_1 on each of its components. Thus, we obtain an exact sequence

$$\pi_1 F \longrightarrow \pi_0 \operatorname{Diff}(W_g^n) \longrightarrow \pi_0 \operatorname{Homeo}(W_g^n)$$

and restricting to the image and kernel, we obtain an extension of a residually finite group by a finite group, which is not residually finite.

The following two elementary lemmas concern certain properties of that class of groups, that prove to be necessary in some arguments later.

Lemma 2.9. *Let G be a residually finite group, and let N be a finite normal subgroup. Then, G/N is again residually finite.*

Proof. Let $x \in G/N$ be a non-zero element; we work to find a normal finite index subgroup of G/N that does not contain x. Consider the set $\{\widetilde{x_i}\}_{i\in\{1,\cdots,n\}}$ of lifts of x via the canonical morphism $p : G \to G/N$, where n is the order of the finite group N. Since G is residually finite, it follows that for each i, there exists H_i normal, finite index subgroup of G that does not contain $\widetilde{x_i}$. Consequently, $H := \cap_i H_i$ is again a finite index normal subgroup with the further property that $\widetilde{x_i} \notin H$, for all i. Then, $p(H)$ is a normal subgroup of G/N which does not contain x. To see that $p(H)$ is a finite index subgroup, we observe that

$$\left({}^{G}\!\!\diagup_{N}\right)\!\diagup_{p(H)} = \left({}^{G}\!\!\diagup_{N}\right)\!\diagup_{\left({}^{NH}\!\!\diagup_{N}\right)} \cong {}^{G}\!\!\diagup_{NH}$$

where the latter is finite as it receives a surjection from the finite group G/H. □

We next see that residual finiteness can be already detected at the level of finite index subgroups.

Lemma 2.10. *Let G be a group, and let H be a finite index subgroup of G that is residually finite. Then, G is also residually finite.*

Proof. As normality is not transitive for subgroups, we use the equivalent formulation of residual finiteness: we have to check that the intersection of all finite index subgroups of G is the trivial group. Let $g \neq 0 \in G$. If $g \notin H$, then H is a finite index subgroup of G not containing g. Otherwise, $g \in H$, and $g \neq 0$. Since H was itself residually finite, there exists some finite index subgroup K of H such that $g \notin K$, and as H is finite index in G, then so too is K. Consequently, K is a finite index subgroup of G not containing g, and thus G is residually finite. □

2.1.2. *Exactness properties of profinite completions of groups.* In this section, we study the exactness of the profinite completion functor of groups, on a certain subclass of groups. The result is stated as [HR79, thm 2.2], which they credit to Bousfield and Kan [BK72], a proof of which can be also found in [Sch78, §5.3]. Let NilGrp^{fg} be the full subcategory of Grp consisting of finitely generated, nilpotent groups.

Theorem 2.11. *The functors Φ^g and Φ^{tg} are exact when restricted to* NilGrp^{fg}.

Remark 2.12. Exactness in the above theorem means that the functor sends short exact sequences to short exact sequences. For the category of compact Hausdorff topological groups, a short exact sequence is a triple (G_1, G_2, G_3) of topological groups together with consecutive continuous morphisms

$$1 \to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to 1$$

that is an exact sequence on the underlying groups. It was shown in the above references that Φ^{tg} sends a short exact sequence of finitely generated nilpotent groups to a short exact sequence of compact Hausdorff topological groups. Among the topological groups that we are interested in, namely topologically finitely generated profinite groups, the two notions of exactness agree: given a triple (K, G, H) of topologically finitely generated profinite groups, and maps $K \to G \to H$, exactness as underlying discrete groups is equivalent to exactness in the category of compact Hausdorff groups. This can be seen as a corollary of [NS07, thm 1.1]. However, we strive to make our presentation independent of *loc. cit.*, and focus on the functor Φ^g.

For the purposes of later application, we show that Theorem 2.11 implies that arbitrary long exact sequences of finitely generated nilpotent groups are indeed sent to long exact sequences by the group profinite completion functor. Such considerations would be automatic should the category NilGrp^{fg} be an abelian category, which it is not, and consequently further arguments are in order.

Lemma 2.13. *Consider a long exact sequence*

$$\cdots \xrightarrow{\varphi_2} G_2 \xrightarrow{\varphi_1} G_1 \xrightarrow{\varphi_0} G_0$$

of finitely generated nilpotent groups. Then, the sequence

$$\cdots \xrightarrow{\Phi^g \varphi_2} \Phi^g G_2 \xrightarrow{\Phi^g \varphi_1} \Phi^g G_1 \xrightarrow{\Phi^g \varphi_0} \Phi^g G_0$$

is again exact.

Proof. We begin by showing that Φ^g preserves injections on the category of finitely generated, nilpotent groups. This is clear when restricted to finitely generated abelian groups. Fix now finitely generated nilpotent groups H and G, and an injective group morphism $1 \to H \xrightarrow{\varphi} G$. As G is a finitely generated nilpotent group, it can be obtained by a finite iteration of central extensions, starting from finitely generated abelian groups; in other words, there exists a finite collection $\{A_0, A_0'\} \cup \{A_i\}_{1\le i\le n}$ of finitely generated abelian groups and finitely generated groups $\{G_i\}_{0\le i\le n}$ that fit into central extensions

$$1 \to A_0' \to G_0 \to A_0 \to 1$$

and for all $1 \le i \le n-1$, central extensions

$$1 \to A_i \to G_{i+1} \to G_i \to 1$$

and finally, such that $G_n \cong G$. We proceed by induction. We say that each G_i is $(i+1)$-centrally extended from abelian groups: i.e. G_0 is once centrally extended from abelian groups, G_1 is twice centrally extended from abelian groups, and so on. We show that if Φ^g preserves injections when mapping into any group that is i-centrally extended from abelian groups, then it also does for mapping into $(i+1)$-centrally extended from abelian groups. To see this, we consider the following diagram

$$\begin{tikzcd}
1 \arrow[r] & \ker(f\circ\varphi) \arrow[r] \arrow[d, hook] & H \arrow[r] \arrow[d, "\varphi"] & \mathrm{Im}(f\circ\varphi) \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & A_i \arrow[r, "g"'] & G_{i+1} \arrow[r, "f"'] & G_i \arrow[r] & 1
\end{tikzcd}$$

where A_i is finitely generated abelian, G_i is i-centrally extended from abelian groups and G_{i+1} is consequently $(i+1)$-centrally extended from abelian groups. The two outermost vertical maps are injective by definition; thus, the group $\ker(f\circ\varphi)$ is finitely generated abelian. By Theorem 2.11, it follows that we obtain a morphism of exact sequences after applying Φ^g. The outermost maps remain injective, by the induction hypothesis for the rihtmost vertical morphism, and since Φ^g preserves injections on finitely generated abelian groups for the leftmost morphism. As a consequence, the middle vertical map $\Phi^g\varphi\colon \Phi^g H \to \Phi^g G_{i+1}$ is injective; consequently, the claim follows by induction.

We now turn our attention to the following question. Given finitely generated nilpotent groups G, H and a morphism $\varphi\colon G \to H$, how does $\Phi^g(\mathrm{Im}(\varphi))$ compare to $\mathrm{Im}(\Phi^g\varphi)$, and $\Phi^g \ker(\varphi)$ to $\ker(\Phi^g\varphi)$? We show that the natural comparison maps are indeed isomorphisms in the setting of finitely generated nilpotent groups. We begin by noting that the image of the continuous group morphism $\Phi^{tg}\varphi\colon \Phi^{tg}G \to \Phi^{tg}H$ is a closed subgroup of the profinite topological group $\Phi^{tg}H$, and its kernel a closed subgroup of $\Phi^{tg}G_1$; thus, both the kernel and image carry a profinite topological group structure, and by the universal property of profinite completions, we obtain the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \Phi^g \ker(\varphi) \arrow[r] \arrow[d] & \Phi^g G \arrow[r] \arrow[d, equal] & \Phi^g \mathrm{Im}(\varphi) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \ker \Phi^g \varphi \arrow[r] & \Phi^g G \arrow[r] & \mathrm{Im}(\Phi^g \varphi) \arrow[r] & 1
\end{tikzcd}$$

We observe that the lower row in the diagram is an exact sequence by definition. As for the upper row, we note that subgroups of finitely generated nilpotent groups are again finitely generated nilpotent, and exactness follows from Theorem 2.11. As a consequence, it follows that $\Phi^g\mathrm{Im}(\varphi) \to \mathrm{Im}(\Phi^g\varphi)$ is surjective. We argue that it is injective; once done, it also follows that $\Phi^g \ker(\varphi) \cong \ker(\Phi^g\varphi)$. We have a commutative triangle

$$\begin{tikzcd}
\Phi^g\mathrm{Im}(\varphi) \arrow[r] \arrow[d] & \mathrm{Im}(\Phi^g\varphi) \arrow[dl, hook] \\
\Phi^g H &
\end{tikzcd}$$

Since Φ^g preserves injections on finitely generated nilpotent groups, it follows that $\Phi^g\mathrm{Im}(\varphi) \to \Phi^g H$ is again injective, and thus it follows that $\Phi^g\mathrm{Im}(\varphi) \to \mathrm{Im}(\Phi^g\varphi)$ is injective, hence an isomorphism.

Consider now a long exact sequence

$$\cdots \to G_2 \xrightarrow{\varphi_1} G_1 \xrightarrow{\varphi_0} G_0$$

of finitely generated, nilpotent groups. To show that the sequence obtained by applying Φ^g to the above is exact, we use the above observation that $\Phi^g\mathrm{Im}(\varphi_n) \cong \mathrm{Im}(\Phi^g\varphi_n)$, and $\Phi^g \ker(\varphi) \cong \ker(\Phi^g\varphi)$, for all $n \geq 0$. Thus, we may extend the above long exact sequence to the right, as follows

$$\cdots G_2 \xrightarrow{\varphi_1} G_1 \xrightarrow{\varphi_0} \mathrm{Im}(\varphi_0) \to 1 \to 1 \cdots$$

and we would like to show that applying Φ^g yields a long exact sequence. The claim follows from the following, more general statement. Let

$$\cdots \to H_{-2} \xrightarrow{\varphi_{-1}} H_{-1} \xrightarrow{\varphi_0} H_0 \xrightarrow{\varphi_1} H_1 \xrightarrow{\varphi_2} H_2 \to \cdots$$

be a long exact sequence of finitely generated, nilpotent groups, that is infinite in both directions. The claim is that Theorem 2.11 implies that such two-sided long exact sequences are sent to long exact sequences. To see this, consider the node $H_{k-1} \xrightarrow{\varphi_k} H_k \xrightarrow{\varphi_{k+1}} H_{k+1}$. We consider the diagram

$$\begin{tikzcd}
\ker\varphi_k \arrow[dr] & & & & \mathrm{Im}(\varphi_{k+1}) \arrow[dr] & & \\
& H_{k-1} \arrow[rr, "\varphi_k"] \arrow[dr] & & H_k \arrow[rr, "\varphi_{k+1}"] \arrow[ur] & & H_{k+1} \arrow[dr] & \\
& & \mathrm{Im}\varphi_k \arrow[ur] & & & & H_{k+1}/\mathrm{Im}(\varphi_{k+1})
\end{tikzcd}$$

where all the diagonals are short exact sequences. We observe that the quotient $H_{k+1}/\mathrm{Im}(\varphi_{k+1})$ is again a group, since $\mathrm{Im}(\varphi_{k+1})$ is a normal subgroup of H_{k+1}, as being equal to the kernel of the successive map; this was the key reason to extend the exact sequence to the right. Furthermore, all groups involved in the above diagram are finitely generated, nilpotent. Applying Φ^g to the above diagram, and using Theorem 2.11, it follows that Φ^g preserves two-sided long exact sequences, and the claim follows. $\square$

2.2. **Profinite completion of spaces.**

2.2.1. *Definitions and main properties.* In [Sul74], Sullivan considers an analogous situation of profinite completion as above, but this time in the setting of spaces. The idea is to approximate a space X by certain spaces with a finiteness assumption, namely

Definition 2.14 (π-finite spaces)**.** A space X is said to be π-finite if

- it has a finite set of path components;
- there exists $k \in \mathbb{N}$, such that for all $n \geq k$ and all $x \in X$, $\pi_n(X, x)$ is trivial;
- all homotopy groups are finite groups.

Roughly speaking, Sullivan considers the profinite completion of X as the limit of π-finite spaces F equipped with all possible maps from X; we briefly recall this construction, following [Lur18, Appendix E]. The following is by no means a complete exposition, and we refer to Lurie's Appendix E for more details. Let $\mathcal{S}_\pi \subset \mathcal{S}$ be the full subcategory of spaces consisting of π-finite spaces, and consider the category $\mathrm{Pro}(\mathcal{S}_\pi)$ of pro-objects in $\mathcal{S}_\pi$, i.e. the full subcategory of $\mathrm{Fun}(\mathcal{S}_\pi, \mathcal{S})^{\mathrm{op}}$ of functors preserving finite limits; objects in that category are called *profinite spaces*, and we emphasize that such an object is *not* a space, but merely a formal limit of spaces. Given a space $X \in \mathcal{S}$, we may associate to it the functor $\mathcal{S}_\pi \to \mathcal{S}$, $F \mapsto \mathrm{Map}(X, F)$. This association defines a functor, which we call the *pro-π-finite completion*, and denote by $\widehat{(-)}\colon \mathcal{S} \to \mathrm{Pro}(\mathcal{S}_\pi)$. For a space X, the pro-object $\widehat{X}$ can be identified with the diagram $\{X_\alpha\}$ indexed by maps $X \to X_\alpha$, where X_α is a π-finite space. The *materialisation* functor is then the functor $\mathrm{Mat}\colon \mathrm{Pro}(\mathcal{S}_\pi) \to \mathcal{S}$, sending a pro-object $\{X_\alpha\}$ to its limit in spaces. The above defines an adjoint pair $\widehat{(-)} \dashv \mathrm{Mat}$. The composite

$$\mathcal{S} \xrightarrow{\widehat{(-)}} \mathrm{Pro}(\mathcal{S}_\pi) \xrightarrow{\mathrm{Mat}} \mathcal{S}$$

is denoted by Φ^s, and called *finite completion*, following Sullivan. The unit of the adjunction gives rise a to natural transformation $\mathrm{id} \Rightarrow \Phi^s$; for a space X, we denote by η_X the associated map $\eta_X\colon X \to \Phi^s X$.

The first relation between profinite completion of spaces and that of groups is given as follows. Following Lurie, given a profinite space X and a point $x \in X$, we define $\pi_n(X, x) \coloneqq \pi_n(\mathrm{Mat}(X), x)$. By [Lur18, Corollary E.5.2.4, Remark E.5.2.5], the observation is then that $\pi_n(X, x)$ carries a canonical structure of a profinite group. Thus, if we start with a space X and choose a basepoint, let $\eta_X\colon X \to \Phi^s X$ be the canonical map given by the counit of the adjunction, $\mathrm{id} \Rightarrow \Phi^s$. Then, $\pi_n(\Phi^s X, \eta_X(x))$ carries a canonical structure of profinite group; in particular, via the universal property of group profinite completion, we obtain a commutative diagram

$$\begin{tikzcd} \pi_n(X,x) \arrow[r] \arrow[dr] & \pi_n(\Phi^s X, \eta_X(x)) \\ & \Phi^g \pi_n(X,x) \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

It is natural to then ask: for which spaces X is the comparison map $\Phi^g \pi_n(X, x) \to \pi_n(\Phi^s X, \eta_X(x))$ an isomorphism? This is the content of [Sul74, thm 3.1].

Definition 2.15. Let $\mathcal{C}$ be a subclass of the class of groups. A connected space X is said to be π_1-$\mathcal{C}$ of finite type, if, for all choices of basepoints $x \in X$, $\pi_1(X, x)$ is a $\mathcal{C}$-group, and $\pi_n(X, x)$ is a finitely generated group for all $n \in \mathbb{N}$. An arbitrary space X is said to be componentwise π_1-$\mathcal{C}$ of finite type, if each of its connected components is π_1-$\mathcal{C}$ of finite type.

Let $\mathcal{S}^{\pi_1\text{-gd,ft}}$ be the full subcategory of spaces on those objects which are componentwise π_1-$\mathcal{C}_{gd}$ of finite type, where $\mathcal{C}_{gd}$ is the class of "good" groups in the sense of Sullivan, namely all groups commensurable to a solvable group, with all subgroups finitely generated. We emphasize that we require no further condition on the action of π_1 on the higher homotopy groups.

Theorem 2.16 (Sullivan)**.** *Let $X \in \mathcal{S}^{\pi_1\text{-gd,ft}}$ and fix $x \in X$. Then, the comparison map $\Phi^g\pi_n(X,x) \to \pi_n(\Phi^s X, \eta_X(x))$ is an isomorphism, for all $n \geq 1$.*

In other words, the map $\eta_X : X \to \Phi^s X$ exhibits the underlying group of the profinite completion of the homotopy groups of X as the homotopy groups of the space $\Phi^s X$, whenever X is a reasonable enough space. This class of spaces includes, among others, componentwise nilpotent spaces of finite type; these will play a crucial role in what follows.

Remark 2.17. Given spaces $X, Y \in \mathcal{S}^{\pi_1\text{gd,ft}}$, and a map $f : X \to Y$, the natural transformation id $\Rightarrow \Phi^s$ yields a commutative square

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \Phi^s X \arrow[d, "\Phi^s f"] \\ Y \arrow[r, "\eta_Y"'] & \Phi^s Y \end{tikzcd}$$

For all basepoints $x \in X$ and $y := f(x) \in Y$, Theorem 2.16 implies that for all $n \geq 1$, the square of groups obtained by applying π_n is isomorphic, in the category of squares of groups, to the square

$$\begin{tikzcd} \pi_n(X,x) \arrow[r] \arrow[d, "\pi_n(f)"'] & \Phi^g\pi_n(X,x) \cong \pi_n(\Phi^s X, \eta_X(x)) \arrow[d, "\pi_n(\Phi^s f)"] \\ \pi_n(Y,y) \arrow[r] & \Phi^g\pi_n(Y,y) \cong \pi_n(\Phi^s Y, \eta_Y(y)) \end{tikzcd}$$

A careful reading of Sullivan's proof shows that the group morphism $\pi_n(\Phi^s f) : \pi_n(\Phi^s X, \eta_X(x)) \to \pi_n(\Phi^s Y, \eta_Y(y))$ agrees with the morphism $\Phi^g(\pi_n f) : \Phi^g\pi_n(X,x) \to \Phi^g\pi_n(Y,y)$. This could also be seen as a corollary of [NS07, thm 1.1].

2.2.2. *Finite completion and finite limits.* In the following, we study how close the functor Φ^s comes to preserving finite limits. Observe that Φ^s depends on the functor $\mathrm{Map}(-, X)$, so that preservation of any kinds of limits will not follow from a formal argument. Recall that preserving finite limits is equivalent to preserving the terminal object and pullbacks; we study how Φ^s behaves with pullbacks. Let $X \to Z \leftarrow Y$ be a cospan in $\mathcal{S}^{\mathrm{nil,ft}}$, the category of finite type nilpotent spaces, and let P be its pullback. We can apply Φ^s to the cospan and obtain $\Phi^s X \to \Phi^s Z \leftarrow \Phi^s Y$, and let P' denote its pullback. We have a commutative diagram

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ \Phi^s P \arrow[rr] & & P' \end{tikzcd}$$

The functor preserving finite limits is then equivalent to the map $\Phi^s P \to P'$ being an equivalence, for all such P and P'. We show that the functor comes close to preserving finite limits, when restricted to $\mathcal{S}^{\mathrm{nil,ft}}$. The aim of this section is to show the following

Theorem 2.18. *Consider a cospan $X \to Z \leftarrow Y$ of componentwise nilpotent, finite type spaces, and let P denote the pullback. Let P' denote the pullback of the cospan $\Phi^s X \to \Phi^s Z \leftarrow \Phi^s Y$, so that we have a commutative triangle*

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ \Phi^s P \arrow[rr] & & P' \end{tikzcd}$$

Then, for all basepoints coming from P via the above diagram, the map $\Phi^s P \to P'$ induces an isomorphism on π_n, $n \geq 1$.

We begin by showing a special case of the above, for fibre sequences.

Lemma 2.19. *Let $F \to E \to B$ be a fibre sequence, where E and B are connected, nilpotent spaces of finite type. Denote by F' the fibre of the map $\Phi^s E \to \Phi^s B$. Then, the natural map $\Phi^s F \to F'$ induces an isomorphism on π_n for all $n \geq 1$, with basepoints coming from F.*

Proof. We begin by observing that F is itself a componentwise nilpotent space of finite type. We have a commutative diagram

$$\begin{tikzcd} \Phi^s F \arrow[r] \arrow[d] & \Phi^s E \arrow[r] \arrow[d, equal] & \Phi^s B \arrow[d, equal] \\ F' \arrow[r] & \Phi^s E \arrow[r] & \Phi^s B \end{tikzcd}$$

Since F is a nilpotent space of finite type, it follows that $\pi_n(\Phi^s F, \eta_F(x)) \cong \Phi^g \pi_n(F, x)$, where $x \in F$, and this exhibits the induced map on π_n of the map $\eta_F \colon F \to \Phi^s F$ as the (group) profinite completion of $\pi_n(F, x)$; the same holds for the other two spaces, E and B. By Lemma 2.13, it follows that applying Φ^g to the long exact sequence of homotopy groups of the fibre sequence $F \to E \to B$ yields again a long exact sequence of homotopy groups; thus, even though the top row of the above diagram is not a fibre sequence, we do have a long exact sequence of homotopy groups based at points coming from F, E and B respectively. We thus obtain a map of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}(\Phi^s B, \eta_B(b)) \arrow[r] \arrow[d, equal] & \pi_n(\Phi^s F, \eta_F(f)) \arrow[r] \arrow[d] & \pi_n(\Phi^s E, \eta_E(e)) \arrow[r] \arrow[d, equal] & \pi_n(\Phi^s B, \eta_B(b)) \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r] & \pi_{n+1}(\Phi^s B, \eta_B(b)) \arrow[r] & \pi_n(F', \eta'(f)) \arrow[r] & \pi_n(\Phi^s E, \eta_E(e)) \arrow[r] & \pi_n(\Phi^s B, \eta_B(b)) \arrow[r] & \cdots \end{tikzcd}$$

where $\eta' \colon F \to F'$ is the natural map between the pullbacks. The claim now follows from the five lemma. $\square$

With the above lemma at hand, we can prove Theorem 2.18.

Proof of Theorem 2.18. The strategy is to now deduce the general claim about pullbacks from Lemma 2.19. Consider

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

where X, Y and Z are connected, nilpotent spaces of finite type. Let P' be the pullback of the associated cospan obtained after applying Φ^s. We aim to show that $\Phi^s P \to P'$ induces isomorphisms on homotopy groups, based at points from P, using Lemma 2.19. Denote by $\{Z_n\}_{n \in \mathbb{N}}$ the principal refinement of the Postnikov tower of Z. In particular, we note that $Z_1 \simeq K(A, 1)$, for some finitely generated abelian group. Furthermore, there is a sequence K_n of Eilenberg-Maclane spaces associated to a finitely generated abelian group A_n (we omit declaring the degree of the Eilenberg-Maclane space, for clarity), fitting into pullback squares

$$\begin{tikzcd} Z_n \arrow[r] \arrow[d] & * \arrow[d] \\ Z_{n-1} \arrow[r] & \Omega K_n \end{tikzcd}$$

For each $n \in \mathbb{N}$, we consider the pullback $P_n \coloneqq X \times_{Z_n} Y$ of the cospan

$$X \to Z_n \leftarrow Y$$

obtained by composition with the canonical map $Z \to Z_n$, and observe that since the pullback functor $\mathrm{cospan}(\mathcal{S}) \to \mathcal{S}$ preserves limits, it follows that $P \simeq \lim_n P_n$. Furthermore, we observe that we have a pullback square

$$\begin{tikzcd} P_{n+1} \arrow[r] \arrow[d] & * \arrow[d] \\ P_n \arrow[r] & K_n \end{tikzcd}$$

for each $n \in \mathbb{N}$. In particular, for each $k \in \mathbb{N}$, $\pi_k P$ can be read at a finite stage of the inverse system $\{P_n\}$, namely there is a $N_k \in \mathbb{N}$ such that $P \to P_{N_k}$ induces an isomorphism on all homotopy groups below degree k. As P is a componetnwise nilpotent space of finite type, Lemma 2.27 implies that $\{\Phi^s P_n\}_{n\in\mathbb{N}}$ is again the principal refinement of the Postnikov system of $\Phi^s P$, on the components coming from the image of the map $\eta_P \colon P \to \Phi^s P$, Lemma 2.27. Consequently, the same analysis as above applies, and we obtain an inverse system $\{P'_n\}_{n\in\mathbb{N}}$ via

$$P'_n \coloneqq \Phi^s X \times_{\Phi^s Z_n} \Phi^s Y$$

and observe that $P' \to \lim_n P'_n$ is again an equivalence. We similarly have pullback squares

$$\begin{tikzcd} P'_{n+1} \arrow[r] \arrow[d] & * \arrow[d] \\ P'_n \arrow[r] & \Phi^s K_n \end{tikzcd}$$

for all $n \in \mathbb{N}$, where we recall that $\Phi^s K_n$ is equivalent to the Eilnberg-Maclane space obtained by applying Φ^g to the non-trivial homotopy group of K_n.

Fix $k \in \mathbb{N}$, and let $N_k \in \mathbb{N}$ be the integer after which both the inverse systems $\{P_n\}$ and $\{P'_n\}$ induce isomorphisms on π_k. We now consider the following commutative diagram

$$\begin{tikzcd} \Phi^s P \arrow[r] \arrow[dddd] & \Phi^s(X \times_{Z_{N_k}} Y) \simeq \Phi^s(* \times_{K_{N_k}} (X \times_{Z_{N_{k}-1}} Y)) \arrow[d] \\ & * \times_{\Phi^s K_{N_k}} (\Phi^s(* \times_{K_{N_k-1}} (X \times_{Z_{N_k-2}} Y)) \arrow[d] \\ & \vdots \arrow[d] \\ & \\ P' \arrow[r] & * \times_{\Phi^s K_{N_k}} (* \times_{\Phi^s K_{N_k-1}} (* \cdots * \times_{\Phi^s K_1} (\Phi^s X \times \Phi^s Y))) \end{tikzcd}$$

Without further reference, all homotopy groups under consideration are based at points coming from the image of P via the canonical maps. By assumption, the lower horizontal map induces isomorphisms on all π_n, for $n \leq k$. Using Lemma 2.19, the same holds for the top horizontal map, and that furthermore, all the rightmost vertical maps are isomorphisms on all homotopy groups. Thus, the left vertical map induces an isomorphism on π_n, for all $n \leq k$. This being true for all k, the claim follows. □

From Theorem 2.18, the claim about finite limit follows. Recall that a finite limit is a limit over a category which is the nerve of a finite simplicial set, i.e. consisting of finitely many degeneracies. Given a pushout for the indexing category

$$\begin{tikzcd} S \arrow[r] \arrow[d] & D \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

the limit of the functor over this pushout is precisely the pullback of the evaluations. By finite induction, we obtain the following

Corollary 2.20. *Let $F\colon \mathcal{D} \to \mathcal{S}^{nil,ft}$ be a functor from a finite diagram of componentwise nilpotent spaces of finite type. Then, the canonical map*

$$\Phi^s(\lim_{\mathcal{D}} F) \to \lim_{\mathcal{D}} \Phi^s F$$

induces an isomorphism on π_n for $n \geq 1$, based at points coming from $\lim_{\mathcal{D}} F$ (via the canonical maps), where all the limits are taken in $\mathcal{S}$.

Rather informally speaking, the above is a categorical reformulation of the following classical argument, that is used by Sullivan in [Sul74, p. 28]. Let X be a finite CW-complex, and let Y be a nilpotent, finite type space. We would like to show that $\pi_n \operatorname{Map}(X, \Phi^s Y)$, based at the a map $\eta_Y \circ f\colon X \to \Phi^s Y$, obtained as the composition

$$\eta_Y \circ f\colon X \xrightarrow{f} Y \xrightarrow{\eta_Y} \Phi^s Y$$

for some map $f\colon X \to Y$, can be canonically identified with $\pi_n \Phi^s \operatorname{Map}(X, Y)$, based at $\eta_{\operatorname{Map}(X,Y)}(f)$. This is shown by induction on the finite cells of X, as follows. We consider the situation where $X = X' \cup e_n$, for some n-cell. Then, we have a fibration

$$F \to \operatorname{Map}(X, Y) \xrightarrow{\text{res}} \operatorname{Map}(X', Y)$$

where the fibre $F \coloneqq \operatorname{fib}_{f|X'}(\text{res})$ can be identified with $\Omega^n Y$, after choice of a nullhomotopy of $f|\partial e_n$. Thus, we obtain an exact sequence

$$\cdots \to \pi_2 Y^{X'} \to \pi_{n+1} Y \to \pi_1 Y^X \to \pi_1 Y^{X'} \to \pi_n Y$$

where the basepoints of the mapping spaces are taken to be f and its restriction to X', correspondingly. The induction hypothesis is then that $\pi_1 Y^{X'}$ is a finitely generated, nilpotent group, and that its higher homotopy groups are finitely generated, abelian groups; that is, $Y^{X'}$ is a nilpotent space of finite type. It can further be shown that the extension above on π_1 is central; thus, it follows that $\pi_1 Y^X$ is again a finitely generated nilpotent group (as being centrally extended from a finitely generated nilpotent group by a finitely generated abelian group), and that all the higher homotopy groups are finitely generated. Exactness of the group profinite completion functor on that class of groups yields the identification $\pi_n \operatorname{Map}(X, \Phi^s Y) \cong \Phi^g \pi_n \operatorname{Map}(X, Y)$ on the basepoints specified above.

2.2.3. *Diagrams of finite completions.* The following theorem is crucial for the proof of residual finiteness of the automorphism groups under consideration in this work.

Theorem 2.21. *Let $\mathcal{C}$ be an arbitrary ∞-category, and consider a space valued functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{S}^{\text{fin}}_{>1}$ landing in the full subcategory of 1-connected finite spaces (i.e. simply connected finite CW complexes), and denote by $\widehat{\mathcal{F}}$ the composition of $\mathcal{F}$ with the profinite completion functor. Then, the group*

$$\pi_0 \operatorname{Map}^{\simeq}_{\operatorname{Fun}(\mathcal{C}, \operatorname{Pro}(\mathcal{S}_\pi))}(\widehat{\mathcal{F}}, \widehat{\mathcal{F}})$$

can be promoted to a profinite topological group.

The proof of the above theorem is of categorical nature, and consequently we delay it to Appendix A. Let us first highlight how the above theorem is crucial for the later sections. The following corrollary is an immediate application of the universal property of profinite completion of groups (Theorem 2.4).

Corollary 2.22. *Let $\mathcal{F}$ be as in Theorem 2.21. Then, we have the following commutative diagram*

$$\begin{tikzcd}
\pi_0 \operatorname{Map}^{\simeq}_{\operatorname{Fun}(\mathcal{C},\mathcal{S})}(\mathcal{F},\mathcal{F}) \arrow[r] \arrow[d] \arrow[dr] & \Phi^{g}\pi_0 \operatorname{Map}^{\simeq}_{\operatorname{Fun}(\mathcal{C},\mathcal{S})}(\mathcal{F},\mathcal{F}) \arrow[d, "\exists!"] \\
\pi_0 \operatorname{Map}^{\simeq}_{\operatorname{Fun}(\mathcal{C},\mathcal{S})}(\Phi^{s}\circ\mathcal{F},\Phi^{s}\circ\mathcal{F}) & \pi_0 \operatorname{Map}^{\simeq}_{\operatorname{Fun}(\mathcal{C},\operatorname{Pro}(\mathcal{S}_\pi))}(\widehat{\mathcal{F}},\widehat{\mathcal{F}}) \arrow[l]
\end{tikzcd}$$

Corollary 2.22 serves as a blueprint to showing residual finiteness: in order to show that the top horizontal map is injective (which is equivalent to the top left group being residually finite), it suffices to show that the left vertical map is injective. The latter is amenable to more homotopy theoretic methods. In fact, injectivity of that map in the setting of embedding calculus follows from obstruction theory, and this is the topic for the later sections.

2.3. **Postnikov decomposition.** The main idea of *Postnikov theory* is to study a space X via a particular sequence of truncations: for a given $n \in \mathbb{N}$, one can define a space $\tau_{\leq n}X$ receiving a map from X, satisfying the following two properties

(i) The map $X \longrightarrow \tau_{\leq n}X$ induces an isomorphism on π_i, for $0 \leq i \leq n$
(ii) $\pi_i(\tau_{\leq n}X, x) = 0$ for all $i > n$ and $x \in X$.

The collection $\{\tau_{\leq n}X\}_{n\in\mathbb{N}}$ forms an inverse system approximating X, in the sense that the induced map $X \longrightarrow \lim_n \tau_{\leq n}X$ is an equivalence. A key property of the Postnikov decomposition of a space X is that $\tau_{\leq n+1}X$ is built out of $\tau_{\leq n}X$ and some cohomological datum, the *Postnikov k-invariant*. We now study how to do this construction functorially in X. Classically, this cohomology class is constructed by transgression on the Leray-Serre spectral sequence for the fibration $K(\pi_{n+1}X, n+1) \longrightarrow \tau_{\leq n+1}X \longrightarrow \tau_{\leq n}X$. An ∞-categorical setup can be found in [Pst23, §2], which we follow.

2.3.1. *Non-principal case.* A crucial ingredient is the *Grothendieck construction*: for a space X, there is an equivalence of ∞-categories

$$\mathcal{S}_{/X} \xrightarrow{\simeq} \operatorname{Fun}(X,\mathcal{S})$$

which informally sends a map $f : Y \longrightarrow X$ to the functor $x \mapsto \operatorname{fib}_x(f)$. Furthermore, for $n \geq 1$, let $\mathcal{EM}^{\simeq}_n \subset \mathcal{S}^{\simeq}$ be the subcategory consisting of spaces that have the homotopy type of a $K(G,n)$ where G is a group (abelian when $n \geq 2$). Let $\pi_n(X,x)\colon \tau_{\leq 1}X \longrightarrow \operatorname{Grp}^{\cong}$ be the classifying map for the π_1-action on π_n, where $\operatorname{Grp}^{\cong}$ is the 1-category of groups and group isomorphisms. We then define $K_n(X)$ as the pullback in $\mathcal{S}$ (where we identify the ∞-category of spaces with the ∞-category of ∞-groupoids)

$$\begin{tikzcd}
K_n(X) \arrow[r] \arrow[d] & \mathcal{EM}^{\simeq}_{n+1} \arrow[d, "\pi_{n+1}"] \\
\tau_{\leq 1}X \arrow[r, "\pi_{n+1}(X,-)"'] \arrow[u, bend left, dashed, "s_n"] & \mathcal{Ab}^{\cong} \arrow[u, bend left, dashed, "B^{n+1}"]
\end{tikzcd}$$

where $\mathcal{Ab}^{\cong}$ is the full subcategory of $\operatorname{Grp}^{\cong}$ of abelian groups, and where we note that the right vertical map is well-defined, since an Eilenberg-Maclane space of type $K(G,n+1)$ is simply connected. The map admits a section given by the delooping functor $B^{n+1}\colon \mathcal{Ab}^{\cong} \longrightarrow \mathcal{EM}^{\simeq}_{n+1}$, which thus gives a section $s_n\colon \tau_{\leq 1}X \longrightarrow K_n(X)$. The map $\tau_{\leq n}X \longrightarrow \mathcal{EM}^{\simeq}_{n+1}$, obtained as the Grothendieck construction on the fibration $\tau_{\leq n+1}X \longrightarrow \tau_{\leq n}X$, and the truncation map $\tau_{\leq n}X \longrightarrow \tau_{\leq 1}X$ commute after passing to $\mathcal{Ab}^{\cong}$, and thus give rise to a map $\kappa_n : \tau_{\leq n}X \longrightarrow K_n(X)$. The key observation is then the following

Lemma 2.23. *For $X \in \mathcal{S}$ and $n \geq 1$, we have a pullback square*

$$\begin{tikzcd} \tau_{\leq n+1}X \arrow[r] \arrow[d, "p_{n+1}"'] & \tau_{\leq 1}X \arrow[d, "\mathcal{s}_n"] \\ \tau_{\leq n}X \arrow[r, "\kappa_n"'] & K_n(X) \end{tikzcd}$$

Proof. We first observe that the fibre of the functor $\pi_{n+1}\colon \mathcal{EM}^{\simeq}_{n+1} \to \mathcal{Ab}^{\cong}$ at $\pi_{n+1}(X,x) \in \mathcal{Ab}^{\cong}$ is $BK(\pi_{n+1}(X,x), n+1) \simeq K(\pi_{n+1}X, n+2)$. Now, as $\mathcal{s}_n\colon \tau_{\leq 1}X \to K_n(X)$ is a section to the map $K_n(X) \to \tau_{\leq 1}X$, the homotopy type of the fibre at any basepoint (since $K_n(X)$ is connected) is therefore $\Omega K(\pi_{n+1}(X,x), n+2) \simeq K(\pi_{n+1}(X,x), n+1)$. We thus obtain the commutative diagram

$$\begin{tikzcd} K(\pi_{n+1}(X,x), n+1) \arrow[d] & K(\pi_{n+1}(X,x), n+1) \arrow[d] \\ \tau_{\leq n+1}X \arrow[r] \arrow[d] & \tau_{\leq 1}X \arrow[d, "\mathcal{s}_n"] \\ \tau_{\leq n}X \arrow[r, "\kappa_n"'] & K_n(X) \end{tikzcd}$$

which induces an equivalence on the fibres, as the induced map is an isomorphism on π_{n+1} by construction. □

It follows readily that the above construction is functorial in equivalences, and we obtain a functor $\mathcal{S}^{\simeq} \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{S})$, i.e. a functor into the category of commutative squares.

2.3.2. *Principal case.* The passage from $\tau_{\leq n}X$ to $\tau_{\leq n+1}X$ involved a cohomology class

$$[\kappa_n] \in H^{n+2}(\tau_{\leq n}X; \underline{\pi_{n+1}X})$$

where the local coefficient system is induced from the action of $\pi_1 X$ on $\pi_n X$, and such a group is usually unwieldy for computations. However, when the action of π_1 is trivial, it becomes significantly easier to grasp these cohomology groups. We first begin in the case of a 1-connected space X; in this setting, one has $\tau_{\leq 1}X \simeq *$, and thus the map $\tau_{\leq n+1}X \to \tau_{\leq n}X$ is a principal fibration fitting into the following pullback diagram

$$\begin{tikzcd} \tau_{\leq n+1}X \arrow[r] \arrow[d] & * \arrow[d] \\ \tau_{\leq n}X \arrow[r, "\kappa_n"'] & K(\pi_{n+1}X, n+2) \end{tikzcd}$$

Similarly, if $\pi_1(X,x)$ is non-trivial but acts trivially on $\pi_n(X,x)$ for all $x \in X$, one obtains a decomposition $K_n(X) \simeq K(\pi_{n+1}(X,x), n+2) \times \tau_{\leq 1}X$ and a trivialization of the fibration $\tau_{\leq 1}X \to K_n(X)$, and we recover the same pullback diagram for the simply connected case by gluing the two pullback diagrams

$$\begin{tikzcd} \tau_{\leq n+1}X \arrow[r] \arrow[d] & \tau_{\leq 1}X \arrow[r] \arrow[d, "*\times \mathrm{id}"] & * \arrow[d] \\ \tau_{\leq n}X \arrow[r, "\kappa_n"'] & K(\pi_{n+1}X, n+2) \times \tau_{\leq 1}X \arrow[r, "\mathrm{proj}_1"'] & K(\pi_{n+1}X, n+2) \end{tikzcd}$$

This defines a functor from the category of simple spaces to the category of towers consisting of principal fibrations (we note that the k-invariants form additional data).

2.3.3. *Nilpotent case.* Among spaces whose Postnikov tower is not principal, one particularly stands out for being close enough: *nilpotent spaces*. We give a brief reminder of this notion. Fix a group π and a $\mathbb{Z}\pi$-module A, i.e. an abelian group endowed with an action of π.

Definition 2.24 (Lower central series of action)**.** The lower central series of a $\mathbb{Z}\pi$-module A is a sequence of submodules

$$\cdots \subset \Gamma_\pi^r(A) \subset \Gamma_\pi^{r-1}(A) \subset \cdots \subset \Gamma_\pi^2(A) \subset A$$

where $\Gamma_\pi^2(A)$ is generated by elements of the form $g \cdot x - x$, with $g \in \pi$ and $x \in A$, and where $\Gamma_\pi^r(A)$ is inductively defined by

$$\Gamma_\pi^r(A) \coloneqq \Gamma_\pi^2(\Gamma_\pi^{r-1}(A))$$

The action of π on A is *nilpotent* if its lower central series is eventually trivial, i.e. if there exists some $r \in \mathbb{N}$ so that $\Gamma_\pi^r(A)$ is the trivial module.

Definition 2.25 (Nilpotent space)**.** A space X is nilpotent if for all $x \in X$, the group $\pi_1(X, x)$ is a nilpotent group, and if the action of $\pi_1(X, x)$ on $\pi_n(X, x)$ is nilpotent for all $n \geq 2$.

Let X be a nilpotent space. Then, for all $n \in \mathbb{N}$, the maps

$$\tau_{\leq n} X \longrightarrow \tau_{\leq n-1} X$$

occuring in the Postnikov tower can be rewritten as a composition of finitely many maps, each of which is a principal fibration. In fact, the converse is true: a space X is nilpotent if and only if each map $\tau_{\leq n} X \longrightarrow \tau_{\leq n-1} X$ can be decomposed as above. We call the tower thus obtained a *principal refinement* of the Postnikov tower.

Remark 2.26. The above construction can be made functorial in the ∞-category of spaces; however, this will not be needed for our purposes, and we will be treating the principal refinement following classical homotopy theory.

We now record a lemma highlighting the behaviour of (refined) Postnikov towers under finite completions, for nilpotent spaces of finite type; such a technical result will prove useful in the proofs of the following sections.

Lemma 2.27. *Let X be a connected, nilpotent space of finite type, and let $\{X_n\}_{n\in\mathbb{N}}$ be a refinement of its Postnikov tower. Then, $\{\Phi^s X_n\}$ is a refined Postnikov tower for $\Phi^s X$, where the fibre of the map $X_{n+1} \longrightarrow X_n$ is an Eilenberg-Maclane space, obtained from the fibre of $X_{n+1} \longrightarrow X_n$ by group profinite completion.*

Proof. This follows immediately from Lemma 2.19. □

2.3.4. *Moore-Postnikov decomposition.* Given connected spaces X, Y and a map $f\colon X \longrightarrow Y$, the *Moore-Postnikov decomposition of f* is a sequence of spaces $\{Z_n\}_{n\in\mathbb{N}}$ along with maps $g_n : X \longrightarrow Z_n$, $h_n : Z_n \longrightarrow Y$ and $p_n : Z_{n+1} \longrightarrow Z_n$ fitting into the following commutative diagram

$$\begin{tikzcd}
X \arrow[d, "g_n"] \arrow[dr, bend left, "g_{n+1}"] \arrow[dd, bend right, "f"'] & \\
Z_n \arrow[d, "h_n"] & Z_{n+1} \arrow[l, "p_n"] \arrow[dl, bend left, "h_{n+1}"] \\
Y &
\end{tikzcd}$$

uniquely characterized by the following properties

(i) g_n is n-connected: it induces an isomorphism on homotopy groups π_k for $k < n$, and a surjection for $k = n$;
(ii) h_n is n-truncated (sometimes also referred to as n-co-connected in geometric contexts): it induces an isomorphism on π_k for $k > n$, and an injection on $k = n$;

Given a morphism between two maps $f : X \longrightarrow Y$, $f' : X' \longrightarrow Y'$ in $\mathrm{Ar}(\mathcal{S})$, namely a commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

we obtain maps $Z_n \to Z'_n$ between their Moore-Postnikov truncations, fitting into the above square. This suggests that the Moore-Postnikov decomposition defines a system of functors

$$\mathrm{MP}_n : \mathrm{Ar}(\mathcal{S}) \to \mathcal{S}$$

which to an object in the arrow category of spaces, namely a map $X \to Y$, associates its Moore-Postnikov truncation in degree n. To see that this describes a functor, we observe that by [Lur09, Example 5.2.8.16] that (i), (ii) in the above describe a factorisation system.

2.4. **Homotopy mapping class groups.** We are now ready to show Theorem 2.1 of Serre and Sullivan. Let X be a simply connected finite CW-complex. Consider the following diagram

$$\begin{tikzcd} \pi_0 \operatorname{hoAut}(X) \arrow[r] \arrow[d] \arrow[dr] & \Phi^g \pi_0 \operatorname{hoAut}(X) \arrow[d, dashed, "\exists!"] \\ \pi_0 \operatorname{hoAut}(\Phi^s X) & \pi_0 \operatorname{Map}^{\simeq}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{X}, \widehat{X}) \arrow[l] \end{tikzcd}$$

The existence (and uniqueness) of the rightmost vertical morphism follows from the universal property of profinite completion of groups, and Theorem 2.21. The purpose of the above diagram is that injectivity of $\pi_0 \operatorname{hoAut}(X) \to \Phi^g \pi_0 \operatorname{hoAut}(X)$ follows from injectivity of $\pi_0 \operatorname{hoAut}(X) \to \pi_0 \operatorname{hoAut}(\Phi^s X)$; the latter follows from a more general theorem of Sullivan, namely [Sul74, thm 3.2] (Theorem 2.28), stating that for reasonable spaces X, Y, the map $\operatorname{Map}(X, Y) \to \operatorname{Map}(X, \Phi^s Y)$, obtained by composition with the map $\eta_Y : Y \to \Phi^s Y$, is π_0-injective; we rewrite the proof in what follows, as we will be using a similar strategy in later proofs.

Theorem 2.28 (Sullivan)**.** *Let Y be a finite CW complex, and let B be a nilpotent space of finite type. Then, the map $\operatorname{Map}(Y, B) \to \operatorname{Map}(Y, \Phi^s B)$, given by composition with the finite completion $B \to \Phi^s B$, induces an injective map on π_0.*

Proof. Let $\{B_n\}_{n\in\mathbb{N}}$ be a principal refinement of the Postnikov decomposition of B, and observe that $\{\Phi^s B_n\}_{n\in\mathbb{N}}$ yields a principal Postnikov decomposition of $\Phi^s B$, by Lemma 2.27. As Y is a finite CW complex (hence its cohomology is eventually trivial), any map $f : Y \to B$ is uniquely determined by the composition $f_N : Y \to B \to \tau_{\leq N} B$ for some $N \in \mathbb{N}$, in the sense that for all $k \geq N$, one has unique lifts up to homotopy:

$$\begin{tikzcd} & \tau_{\leq k+1} B \arrow[d] \\ Y \arrow[ur, "\exists!"] \arrow[r, "f_k"'] & \tau_{\leq k} B \end{tikzcd}$$

Consequently, given $f \not\simeq g : Y \to B$, we may consider f and g as maps to $\tau_{\leq N} B$ for some large N; the proof of injectivity then follows by working inductively on the Postnikov tower. Consider the following situation:

$$\begin{tikzcd} & \Omega K \arrow[d] & & \\ & B_{n+1} \arrow[rr] \arrow[d] & & * \arrow[d] \\ Y \arrow[uur, dashed, bend left, "\Delta(f,g)"] \arrow[ur, bend left, "g"] \arrow[ur, bend right, "f"'] \arrow[r, "\varphi"'] & B_n \arrow[rr, "\kappa_n"'] & & K_n := K(\pi_{n+1}, n+2) \end{tikzcd}$$

We first describe the above diagram. The right hand square is a pullback square. We fix a map $\varphi : Y \to B_n$, and take two lifts f, g of φ over B_{n+1}. Each of the previous two lifts yields a nullhomotopy of the composition $\kappa_n \circ \varphi$, which is equivalent to the data of two maps $N(f), N(g) \colon CY \to K$ from the cone of Y. Gluing the above two maps from the cone, we obtain a map from the suspension $\Sigma Y \to K$, which by adjunction yields the map $\Delta(f,g) \colon Y \to \Omega K$ (which is rightly thought of as the difference between f and g). The cohomology class represented by this map detects when f is homotopic to g as lifts: $f \simeq g$ as lifts if and only if $[\Delta(f,g)] = 0$. However, one can also obtain information on when f and g are homotopic, but not necessarily through lifts, from the cohomology class $[\Delta(f,g)]$. Indeed, we obtain the following criterion:

$$f \simeq g \iff [\Delta(f,g)] \in \operatorname{Im}\left(\pi_0\Omega_\varphi \operatorname{Map}(Y, B_n) \xrightarrow{\Omega\kappa_n} \pi_0\Omega_{\kappa_n\circ\varphi} \operatorname{Map}(Y,K)\right)$$

Additionally, $\pi_0\Omega_\varphi \operatorname{Map}(Y, B_n)$ and $\pi_0\Omega_{\kappa_n\circ\varphi} \operatorname{Map}(Y,K)$ carry group structures by concatenation of loops, and the map $\Omega\kappa_n$ induces a group homomorphism between them. By the Brown representability theorem, we also see that $\pi_0\Omega_{\kappa_n\circ\varphi} \operatorname{Map}(Y,K) \cong H^{n+1}(Y, \pi_{n+1})$.

We can repeat the same diagram after finite completion $\Phi^s B$ on the target, and composition with the natural map $\eta_B \colon B \to \Phi^s B$. We obtain the same diagram above, where B_i is replaced by $\Phi^s B_i$, and K by $\Phi^s K$, and where Lemma 2.27 provides the required pullback square. Our aim is now to show that if $f \not\simeq g$, then $\eta_B \circ f \not\simeq \eta_B \circ g$.

Consider the following diagram

$$(1) \qquad \begin{tikzcd} G \coloneqq \pi_0\Omega_\varphi \operatorname{Map}(Y, B_n) \arrow[r] \arrow[d, "\Omega\kappa_n"'] & G' \coloneqq \pi_0\Omega_{\eta_B\circ\varphi} \operatorname{Map}(Y, \Phi^s B_n) \arrow[d, "\Omega\eta_B\circ\kappa_n"] \\ H^{n+1}(Y, \pi_{n+1}) \arrow[r] \arrow[d] & H^{n+1}(Y, \Phi^g \pi_{n+1}) \arrow[d] \\ \operatorname{coker} \Omega\kappa_n \arrow[r] & \operatorname{coker} \Omega\eta_B \circ \kappa_n \end{tikzcd}$$

Note that $f \not\simeq g$ is equivalent to $\overline{[\Delta(f,g)]} \neq 0$ in $\operatorname{coker} \Omega\kappa_n$ (and similarly for $\eta_B \circ f, \eta_B \circ g$), and that $\Delta(f,g)$ maps to $\Delta(\eta_B \circ f, \eta_B \circ g)$ via the middle horizontal morphism of the diagram. Consequently, our aim reduces to showing that the group homomorphism $\operatorname{coker} \Omega\kappa_n \to \operatorname{coker} \Omega\eta_B\kappa_n$ is injective. To this end, we consider the following

- *Claim*: the upper square in diagram 1 is isomorphic, in the category $\operatorname{Sq}(\mathrm{Grp})$ of squares in groups, to the square

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[d, "\Omega\kappa_n"'] & \Phi^g G \arrow[d, "\Phi^g(\Omega\kappa_n)"] \\ H^{n+1}(Y, \pi_{n+1}) \arrow[r, "\eta_{H^{n+1}}"'] & \Phi^g H^{n+1}(Y, \pi_{n+1}) \end{tikzcd}$$

We delay the proof of the claim to the next paragraph, and show how this already implies that the map $\operatorname{coker} \Omega\kappa_n \to \operatorname{coker} \Omega\Phi^s\kappa_n$ is injective, as desired. We begin by observing that we have an isomorphism in the arrow category in groups $\operatorname{Ar}(\mathrm{Grp})$ between the arrow $(\operatorname{coker} \Omega\kappa_n \to \operatorname{coker} \Omega\eta_B \circ \kappa_n)$, and the arrow $(\operatorname{coker} \Omega\kappa_n \to \operatorname{coker} \Phi^g(\Omega\kappa_n))$. The profinite completion functor (remembering the topology) $\Phi^{tg} \colon \mathrm{Grp} \to \mathrm{ProfGrp}$ is left adjoint to the forgetful functor $\mathrm{ProfGrp} \to \mathrm{Grp}$, and is therefore right exact. Furthermore, as profinite groups are compact, Hausdorff totally disconnected topological groups, the forgetful functor $\mathrm{ProfGrp} \to \mathrm{Grp}$ is exact, so that the functor $\Phi^g \colon \mathrm{Grp} \to \mathrm{Grp}$ remains right exact. Consequently, the latter arrow is itself isomorphic to the arrow $\operatorname{coker} \Omega\kappa_n \xrightarrow{\eta} \Phi^g \operatorname{coker} \Omega\kappa_n$, where η is the finite completion morphism. Now, as $\operatorname{coker} \Omega\kappa_n$ is a finitely generated abelian group (as being the quotient of the cohomology group, itself a finitely generated abelian group), it is in particular residually finite, whence injectivity of all the three previous arrows. It therefore follows that if $f \neq g$, then $\eta \circ f \neq \eta \circ g$, and the proof follows by finite induction.

We now prove the claim. We begin by observing that the upper square diagram 1 is obtained by applying π_1 to the following square

$$\begin{tikzcd}
\mathrm{Map}(Y, B_n) \arrow[r, "\eta_B \circ -"] \arrow[d, "\kappa_n \circ -"'] & \mathrm{Map}(Y, \Phi^s B_n) \arrow[d, "\Phi^s \kappa_n \circ -"] \\
\mathrm{Map}(Y, K_n) \arrow[r, "\eta_K \circ -"'] & \mathrm{Map}(Y, \Phi^s K_n)
\end{tikzcd}$$

where the basepoint at the top left corner is chosen to be $\varphi \in \mathrm{Map}(Y, B_n)$ (and all other basepoints are taken to be the images of φ by the corresponding maps). Given spaces X, Z, we write the mapping space $\mathrm{Map}(X, Z)$ as the following limit

$$\mathrm{Map}(X, Z) \simeq \lim_X Z$$

and note that in the case X is a finite CW complex, the above is a finite limit. Rephrased in these terms, the above comutative squares spells out to the following

$$\begin{tikzcd}
\lim_Y B_n \arrow[r, "\lim \eta_B"] \arrow[d, "\lim \kappa_n"'] & \lim_Y \Phi^s B_n \arrow[d, "\lim \Phi^s \kappa_n"] \\
\lim_Y K_n \arrow[r, "\lim \eta_K"'] & \lim_Y \Phi^s K_n
\end{tikzcd}$$

Coassembly yields a natural transformation $\mathrm{coass}\colon \Phi^s \lim_Y \Rightarrow \lim_Y \Phi^s$, which consequently yields the following commutative diagram

$$\begin{tikzcd}
& \Phi^s \lim_Y B_n \arrow[dr, "\mathrm{coass}"] \arrow[dd, dotted, "\Phi^s \lim \kappa_n"] & \\
\lim_Y B_n \arrow[ur, "\eta"] \arrow[rr, "\lim \eta_B"] \arrow[dd, "\lim \kappa_n"'] & & \lim_Y \Phi^s B_n \arrow[dd, "\lim \Phi^s \kappa_n"] \\
& \Phi^s \lim_Y K_n \arrow[dr, "\mathrm{coass}"] & \\
\lim_Y K_n \arrow[ur, "\eta"'] \arrow[rr, "\lim \eta_K"'] & & \lim_Y \Phi^s K_n
\end{tikzcd}$$

By Theorem 2.18 and Corollary 2.20, both the coassembly maps in the above diagram induce isomorphisms on π_n, for all $n \geq 1$, and in particular on π_1, at the basepoint corresponding to the image of $\varphi \in \mathrm{Map}(Y, B_n)$; this follows since B_n and K_n are nilpotent spaces of finite types. Theorem 2.16 identifies the arrow in groups $(\pi_1(\lim_Y B_n, \varphi) \to \pi_1(\Phi^s \lim_Y B_n, \eta\varphi))$ as $\eta\colon \pi_1(\lim_Y B_n, \varphi) \to \Phi^g \pi_1(\lim_Y B_n, \varphi)$, the group finite completion map (and the analogous statement for K_n), while Remark 2.17 identifies the induced morphism $\pi_1(\Phi^s \lim \kappa_n)$ as $\Phi^g \pi_1(\lim \kappa_n)$. This consequently shows the claim, and the proof follows by finite induction. □

Remark 2.29. Although not relevant for our purposes, the above theorem can, without much trouble, be seen as a special case of a more general statement about section spaces of bundles. Indeed, given a bundle $\xi\colon E \to B$ over a finite CW complex B, with fibre F a componentwise nilpotent space of finite type, the map on section spaces

$$\Gamma(\xi) \to \Gamma(\Phi^s_{/B}\xi)$$

can be shown to be π_0-injective, where $\Phi^s_{/B}$ is the fibrewise finite completion: via the Grothendieck construction, ξ can be identified with a functor in $\mathrm{Fun}(B, \mathcal{S})$, where B is viewed as an ∞-groupoid, and $\Phi^s_{/B}$ is given by composing the above functor with the functor $\Phi^s\colon \mathcal{S} \to \mathcal{S}$. The above theorem is then the special case when ξ is trivialisable.

3. T_k-mapping class groups

In this chapter, we generalize Sullivan's result to the setting of mapping spaces in categories of presheaves on discs. Given a smooth d-dimensional manifold M, and $k \in \mathbb{N} \cup \{\infty\}$, we define

$$T_k \operatorname{Diff}(M) \coloneqq \operatorname{Map}^{\simeq}_{\mathrm{Psh}(\mathrm{Disc}_d^{\leq k})}(E_M, E_M)$$

where E_M is the disc-presheaf $\sqcup_\ell \mathbb{R}^d \mapsto \operatorname{Emb}(\sqcup_\ell \mathbb{R}^d, M)$. The goal of this section is to use techniques similar to those of the previous section in order to show that the group $\pi_0 T_k \operatorname{Diff}(M)$ is residually finite, when $k < \infty$, and M is closed and 2-connected.

3.1. **Embedding calculus with tangential structures.** Let Manf_d be the topologically enriched category of smooth d-dimensional manifolds (not necessarily compact) with empty boundary, and smooth embeddings, endowed with the C^∞-topology; via the coherent nerve, we view this category as an ∞-category. As a full subcategory, we can consider $\mathrm{Disc}_d \subset \mathrm{Manf}_d$, where objects are $\sqcup_\ell \mathbb{R}^d$, for all $\ell \in \mathbb{N}$; we denote by ι_∞ the inclusion $\mathrm{Disc}_d \hookrightarrow \mathrm{Manf}_d$. This category has an obvious filtration: for each $k \in \mathbb{N}$ we can further consider the full subcategory $\mathrm{Disc}_d^{\leq k}$ consisting of objects of the form $\sqcup_k \mathbb{R}^d$, where $\ell \leq k$; we denote the inclusions $\mathrm{Disc}_d^{\leq k} \hookrightarrow \mathrm{Disc}_d$ by ι_k. We define a functor $E\colon \mathrm{Manf}_d \to \mathrm{Psh}(\mathrm{Disc}_d)$ as

$$E\colon \mathrm{Manf}_d \xrightarrow{h} \mathrm{Psh}(\mathrm{Manf}_d) \xrightarrow{\iota_\infty^*} \mathrm{Psh}(\mathrm{Disc}_d)$$

where h is the Yoneda embedding. Concretely, given $M \in \mathrm{Manf}_d$, E_M is the disc-presheaf $\sqcup_\ell \mathbb{R}^d \mapsto \operatorname{Emb}(\sqcup_\ell \mathbb{R}^d, M)$. We can further compose E by the restriction functor ι_k^*, and obtain a functor $\mathrm{Manf}_d \to \mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$. We thus obtain a tower of functors

$$\begin{tikzcd}
\mathrm{Psh}(\mathrm{Disc}_d) \arrow[r] & \cdots \arrow[r] & \mathrm{Psh}(\mathrm{Disc}_d^{\leq k}) \arrow[r] & \cdots \arrow[r] & \mathrm{Psh}(\mathrm{Disc}_d^{\leq 1}) \\
 & & & & \mathrm{Manf}_d \arrow[u, "\iota_1^* E"'] \arrow[ull, "\iota_k^* E"] \arrow[ullll, bend left, "E"]
\end{tikzcd}$$

Given a smooth d-manifold M, and a smooth manifold N of dimension $\geq d$, we define

$$T_k \operatorname{Emb}(M, N) \coloneqq \operatorname{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\leq k})}(\iota_k^* E_M, \iota_k^* E_N^d)$$

for all $k \in \mathbb{N} \cup \{\infty\}$, where E_N^d is the disc-preseheaf sending $\sqcup_\ell \mathbb{R}^d \mapsto \operatorname{Emb}(\sqcup_\ell \mathbb{R}^d, N)$. On mapping spaces, the above tower of functors gives a tower of spaces, the *embedding calculus tower*, $\{T_k \operatorname{Emb}(M, N)\}_{k \in \mathbb{N}}$, receiving a map from $\operatorname{Emb}(M, N)$. We say the tower *converges* if the map

$$\operatorname{Emb}(M, N) \to T_\infty \operatorname{Emb}(M, N)$$

is an equivalence. The following theorem stands as a fundamental theorem of embedding calculus

Theorem 3.1 (Goodwillie-Weiss). *Let M, N be two smooth manifolds of dimension d. If $h\dim M \le d-3$, then $\mathrm{Emb}(M,N) \to T_\infty \mathrm{Emb}(M,N)$ is an equivalence, where $h\dim M$ is the handle dimension of M.*

The advantage of the above theorem is to give more homotopy theoretic methods to study spaces of embeddings. In particular, given a fixed embedding $f\colon M \hookrightarrow N$, we can describe the fibre of the map

$$T_{k+1}\mathrm{Emb}(M,N) \to T_k \mathrm{Emb}(M,N)$$

as a certain space of sections of a bundle over the unordered configuration space $U\mathrm{Conf}_k(M)$ of M relative a fixed section on the fat diagonal, where the fibres are built out of cubes of configuration spaces of N (see, for instance, the final paragraph of [Wei99], or [KK24, thm 4.9]). This fibre is denoted $L_k \mathrm{Emb}(M,N)_f$, and called the *layer of the tower.*
Given a smooth manifold M of dimension d, and $k \in \mathbb{N} \cup \{\infty\}$, the main object under investigation in this work is

$$T_k \mathrm{Diff}(M) \coloneqq \mathrm{Map}^{\simeq}_{\mathrm{Psh}(\mathrm{Disc}_d^{\le k})}(\iota_k^* E_M, \iota_k^* E_M)$$

We refer to the group of components of the above space the *T_k-mapping class group* of M.

It is possible to set up embedding calculus to include manifolds with boundary, and boundary conditions. We shall return to this once needed later.

We now study a method of imposing tangential structures on manifolds, and applying embedding calculus, where now all embeddings are required to preserve those tangential structures. Fix a dimension $d \in \mathbb{N}$, and let $\Theta \in \mathrm{Psh}(BO(d))$ be a space with a Borel action of $O(d)$. Denote by $\xi_\Theta\colon B_\Theta \to BO(d)$ the associated Borel construction through the Grothendieck equivalence

$$\mathrm{Psh}(BO(d)) \simeq \mathcal{S}/_{BO(d)}$$

We define Manf_d^Θ as the following pullback in ∞-categories

$$\begin{tikzcd} \mathrm{Manf}_d^\Theta \arrow[r] \arrow[d] & \mathcal{S}/_{B_\Theta} \arrow[d, "\xi_\Theta \circ -"] \\ \mathrm{Manf}_d \arrow[r] & \mathcal{S}/_{BO(d)} \end{tikzcd}$$

where the lower horizontal functor sends a manifold M to the map $M \to BO(d)$ classifying the tangent bundle of M.

We may further restrict to discs. Given any tangential structure Θ and $k \in \mathbb{N}$, we define $\mathrm{Disc}_d^{\Theta,\le k}$ as the following pullback in ∞-categories

$$\begin{tikzcd} \mathrm{Disc}_d^{\Theta,\le k} \arrow[r] \arrow[d] & \mathcal{S}/_{B_\Theta} \arrow[d, "\xi_\Theta \circ -"] \\ \mathrm{Disc}_d^{\le k} \arrow[r] & \mathcal{S}/_{BO(d)} \end{tikzcd}$$

By the universal property of presheaf categories [Lur09, 5.1.5.6], we obtain a colimit preserving functor $\mathrm{Psh}(\mathrm{Disc}_d^{\le k}) \to \mathcal{S}/BO(d)$, which sits in the following commuting diagram

$$\begin{tikzcd} \mathrm{Disc}_d^{\Theta,\leq k} \arrow[r] \arrow[d] & \mathcal{S}_{/B\Theta} \arrow[d] \\ \mathrm{Psh}(\mathrm{Disc}_d^{\leq k}) \arrow[r] & \mathcal{S}_{/BO(d)} \end{tikzcd}$$

and which in turn yields a functor $\mathrm{Disc}_d^{\Theta,\leq k} \to \mathcal{C}$, where $\mathcal{C}$ is the pullback of the cospan

$$\mathrm{Psh}(\mathrm{Disc}_d^{\leq k}) \to \mathcal{S}_{/BO(d)} \leftarrow \mathcal{S}_{/B\Theta}$$

Note that $\mathcal{C}$ again admits all colimits. Using once more the universal property of presheaf categories [Lur09, 5.1.5.6], we obtain a colimit preserving functor $\mathrm{Psh}(\mathrm{Disc}_d^{\Theta,\leq k}) \to \mathcal{C}$, which thus yields a commutative diagram

$$\begin{tikzcd} \mathrm{Psh}(\mathrm{Disc}_d^{\Theta,\leq k}) \arrow[r] \arrow[d] & \mathcal{S}_{/B\Theta} \arrow[d] \\ \mathrm{Psh}(\mathrm{Disc}_d^{\leq k}) \arrow[r] & \mathcal{S}_{/BO(d)} \end{tikzcd}$$

The left vertical functor is obtained by left Kan extension

$$\begin{tikzcd} \mathrm{Disc}_d^{\Theta,\leq k} \arrow[r] \arrow[d, hook] & \mathrm{Disc}_d^{\leq k} \arrow[r, hook] & \mathrm{Psh}(\mathrm{Disc}_d^{\leq k}) \\ \mathrm{Psh}(\mathrm{Disc}_d^{\Theta,\leq k}) \arrow[urr, dashed] & & \end{tikzcd}$$

We may further consider the full subcategory of $\mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$ given by elements E_M for M a d-dimensional manifold with empty boundary, which we denote by $T_k\mathrm{Manf}_d$. We may also consider the full subcategory $T_k\mathrm{Manf}_d^{\Theta}$ given by pullback

$$\begin{tikzcd} T_k\mathrm{Manf}_d^{\Theta} \arrow[r] \arrow[d] & \mathrm{Psh}(\mathrm{Disc}_d^{\Theta,\leq k}) \arrow[d] \\ T_k\mathrm{Manf}_d \arrow[r, hook] & \mathrm{Psh}(\mathrm{Disc}_d^{\leq k}) \end{tikzcd}$$

For a d-dimensional smooth manifold M together with a fixed tangential Θ-structure ℓ on it, we define

$$T_k \,\mathrm{Diff}^{\Theta}(M, \ell) := \mathrm{Map}^{\simeq}_{\mathrm{Psh}(\mathrm{Disc}_d^{\Theta,\leq k})}(E_M^{\Theta}, E_M^{\Theta})$$

The following result, due to Krannich and Kupers [KK24, thm 4.15], gives a smoothing theory description on the level of tangential structures.

Theorem 3.2 (Krannich-Kupers). *Let $\Theta \in \mathrm{Psh}(BO(d))$. For each $k \in \mathbb{N} \cup \{\infty\}$, every square in the following commutative diagram is a pullback square of ∞-categories*

$$\begin{tikzcd} \mathrm{Disc}_d^{\Theta,\leq k} \arrow[r] \arrow[d] & \mathrm{Manf}_d^{\Theta} \arrow[r] \arrow[d] & \mathrm{Psh}(\mathrm{Disc}_d^{\Theta,\leq k}) \arrow[r] \arrow[d] & \mathcal{S}_{/B\Theta} \arrow[d] \\ \mathrm{Disc}_d^{\leq k} \arrow[r, hook] & \mathrm{Manf}_d \arrow[r, "\iota_k^* E"'] & \mathrm{Psh}(\mathrm{Disc}_d^{\leq k}) \arrow[r] & \mathcal{S}_{/BO(d)} \end{tikzcd}$$

As a corollary, we obtain the following

Corollary 3.3. *Let $\Theta \in \mathrm{Psh}(BO(d))$ be a tangential structure, and fix a smooth, d-dimensional manifold with empty boundary, together with a fixed Θ-stucture ℓ on it. The fibre of the map*

$$BT_k \,\mathrm{Diff}^{\Theta}(M, \ell) \to BT_k \,\mathrm{Diff}(M)$$

is a collection of components of the space of tangential Θ*-structures on* M*, i.e. the space*

$$\mathrm{Map}^{O(d)}(\mathrm{Fr}(M), \Theta)$$

3.2. **Spin T_k-mapping class groups.** Consider a closed, smooth 2-connected d-manifold, together with a choice $\mathfrak{s}$ of a spin structure on it, and define $E_M^{\mathrm{Spin}} \in \mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin}})$ by $\sqcup_k \mathbb{R}^d \mapsto \mathrm{Emb}^{\mathrm{Spin}}(\sqcup_k \mathbb{R}^d, M)$. The aim of this section is to show the following.

Theorem 3.4. *Let* $(M, \mathfrak{s})$ *be a closed, smooth* 2*-connected* d*-manifold, together with a choice of a spin structure on it. Then, for all* $k \in \mathbb{N}$*,*

$$\pi_0 T_k \operatorname{Diff}^{\mathrm{Spin}}(M) := \pi_0 \operatorname{Map}^{\simeq}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin}, \leq k})}(\iota_k^* E_M^{\mathrm{Spin}}, \iota_k^* E_M^{\mathrm{Spin}})$$

is a residually finite group.

We begin with some notation. Given a space valued diagram $X \in \mathrm{Fun}(\mathcal{C}, \mathcal{S})$, we define its *profinite completion* $\widehat{X}$ to be performed pointwise, namely as the following composition

$$\widehat{X} \colon \mathcal{C} \xrightarrow{X} \mathcal{S} \to \mathrm{Pro}(\mathcal{S}_\pi)$$

We may further compose with the materialisation functor $\mathrm{Mat} : \mathrm{Pro}(\mathcal{S}_\pi) \to \mathcal{S}$, and keeping with our conventions, we denote the composition as $\Phi^s X \in \mathrm{Fun}(\mathcal{C}, \mathcal{S})$. Similarly, we denote by $\tau_{\leq n} X$ the composition

$$\tau_{\leq n} X \colon \mathcal{C} \xrightarrow{X} \mathcal{S} \xrightarrow{\tau_{\leq n}} \mathcal{S}$$

In analogy to the similar situation in spaces, we call $\{\tau_{\leq n} X\}_{n \in \mathbb{N}}$ the *Postnikov decomposition* of X. A natural transformation $\alpha : X \to Y$ is said to be *principal* if it is a principal fibration pointwise. A Postnikov decomposition of X is then said to be *principal* if all transformations $\tau_{\leq n+1} X \to \tau_{\leq n} X$ are principal.

Lemma 3.5. *For all* $k \in \mathbb{N}$*, and for* $(M, \mathfrak{s})$ *as above,* $\mathrm{Emb}^{\mathrm{Spin}}(\sqcup_k \mathbb{R}^d, M)$ *has the homotopy type of a simply connected, finite CW complex.*

Proof. Recall that $\mathrm{Emb}^{\mathrm{Spin}}(\sqcup_k \mathbb{R}^d, M)$ sits in the following homotopy pullback square

$$\begin{tikzcd}
\mathrm{Emb}^{\mathrm{Spin}}(\sqcup_k \mathbb{R}^d, M) \arrow[r] \arrow[d] & \mathrm{Fr}^{\mathrm{Spin}}(M)^k \arrow[d] \\
\mathrm{Conf}_k(M) \arrow[r, hook] & M^k
\end{tikzcd}$$

The rightmost vertical map is the cartesian product of the spin frame bundle of M, which exists as M is assumed to be 2-connected and thus admits a spin structure. Furthermore, we observe that the vertical fibre $\prod_k \mathrm{Spin}(d)$ in the above is homotopy equivalent to a finite CW complex; indeed, $\mathrm{Spin}(d)$ is the double cover of $SO(d)$ (universal for $d \geq 3$), a compact Lie group. Additionally, note that $M \setminus \sqcup_k *$ is again a 2-connected manifold, for all k. Induction on the Fadell-Neuwirth fibre bundle $M \setminus \sqcup_{k-1} * \to \mathrm{Conf}_k(M) \to \mathrm{Conf}_{k-1}(M)$ thus yields the result: one first observes that $\pi_1 \mathrm{Conf}_k(M)$ and $\pi_2 \mathrm{Conf}_k(M)$ are trivial for all $k \in \mathbb{N}$, so that the long exact sequence on homotopy groups applied to the fibration $\prod_k \mathrm{Spin}(d) \to \mathrm{Emb}^{\mathrm{Spin}}(\sqcup_k \mathbb{R}^d, M) \to \mathrm{Conf}_k(M)$ implies that the space in question is simply connected. That it has the homotopy type of a finite CW complex also follows from the fact that it sits as the total space of a fiber bundle whose fibre and base space have the homotopy type of finite CW complexes. Indeed, $\mathrm{Conf}_k(M)$ is homotopy equivalent to a finite CW complex for all k, since it is homeomorphic to the interior of a compact topological manifold with boundary via the Fulton-MacPherson compactification. □

The above analysis thus yields the following.

Corollary 3.6. *The presheaf $E_M^{\mathrm{Spin}} \in \mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin}})$ admits a principal Postnikov decomposition. In particular, so do all restrictions $\iota_k^* E_M \in \mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})$, for $k \in \mathbb{N}$.*

Using the above corollary, we show the following proposition, which is a generalization of Sullivan's theorem (Theorem 2.28) to the case of certain diagrams of spaces.

Theorem 3.7. *Let M be a closed, smooth 2-connected d-manifold, where $d \geq 4$. Then, for every $k \in \mathbb{N}$, the composition map*

$$\mathrm{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\iota_k^* E_M^{\mathrm{Spin}}, \iota_k^* E_M^{\mathrm{Spin}}) \to \mathrm{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\iota_k^* E_M^{\mathrm{Spin}}, \iota_k^* \Phi^s E_M^{\mathrm{Spin}})$$

induces an injection on π_0.

We present a proof for $k = 1$, and delay the case $k \geq 2$ for the following section, as some more technology will have to go into a proof in that setting. We point out that the following proof is precisely an equivariant version of the proof of Theorem 2.28, and therefore we will be following the aforementioned proof quite closely.

Proof for $k = 1$. Observe that $\iota_1^* E_M^{\mathrm{Spin}} \simeq \mathrm{Fr}^{\mathrm{Spin}}(M)$, the spin frame bundle of M. We furthermore observe that $\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq 1}) \simeq \mathrm{Psh}(B\mathrm{Spin}(d))$, the category of spaces carrying an action by the Lie group $\mathrm{Spin}(d)$. The strategy to show injectivity is to work up the Postnikov ladder: given a $\mathrm{Spin}(d)$-equivariant map $\varphi \colon \mathrm{Fr}^{\mathrm{Spin}}(M) \to \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M)$, we consider two non-homotopic lifts to $\tau_{\leq n+1} \mathrm{Fr}^{\mathrm{Spin}}(M)$. The key claim is that if we compose with the finite completion on the target, the two lifts remain non-homotopic. As $\mathrm{Fr}^{\mathrm{Spin}}(M)$ is a finite $\mathrm{Spin}(d)$-CW complex, cohomological boundedness implies that for some $N \in \mathbb{N}$, there are unique lifts to $\tau_{\leq \ell} \mathrm{Fr}^{\mathrm{Spin}}(M)$, for $\ell \geq N$; thus, the claim follows by finite induction from the above key claim. Consider the situation of the following diagram in $\mathrm{Psh}(B\mathrm{Spin}(d))$

$$\begin{tikzcd}
& \tau_{\leq n+1} \mathrm{Fr}^{\mathrm{Spin}}(M) \arrow[r] \arrow[d] & * \arrow[d] \\
\mathrm{Fr}^{\mathrm{Spin}}(M) \arrow[ur, dashed, bend left, "f"] \arrow[ur, dashed, "g"'] \arrow[r, "\varphi"'] & \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M) \arrow[r, "\kappa_n"'] & K := K(\pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M), n+2)
\end{tikzcd}$$

where the square is a pullback square in $\mathrm{Psh}(B\mathrm{Spin}(d))$, and where the action of $\mathrm{Spin}(d)$ on the Eilenberg-Maclane space in the lower right corner is trivial. We fix a map $\varphi : \mathrm{Fr}^{\mathrm{Spin}}(M) \to \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M)$, and consider two lifts f and g. The goal is to study how differen f and g can be. As the right hand side square is a pullback square, each of these two lifts is equivalent to a nullhomotopy of $\kappa_n \circ \varphi$. Using the Grothendieck construction $\mathrm{Psh}(B\mathrm{Spin}(d)) \simeq \mathcal{S}/B\mathrm{Spin}(d)$, and since the action of $\mathrm{Spin}(d)$ on the Eilenberg-Maclane space is trivial, we obtain equivalences

$$\begin{aligned}
\mathrm{Map}^{\mathrm{Spin}(d)}(\mathrm{Fr}^{\mathrm{Spin}}(M), K) &\simeq \mathrm{Map}_{\mathcal{S}/B\mathrm{Spin}(d)}\left(\mathrm{Fr}^{\mathrm{Spin}}(M)//\mathrm{Spin}(d), K \times B\mathrm{Spin}(d)\right) \\
&\simeq \mathrm{Map}(M, K)
\end{aligned}$$

Consequently, it follows that a nulhomotopy of $\kappa_n \circ \varphi$ through $\mathrm{Spin}(d)$-equivariant maps, is equivalent to a nullhomotopy of the corresponding map between M and K in $\mathcal{S}$. For the lifts f, g, we get maps $N(f), N(g) : CM \to K$ from the cone of M, which can be glued to a map $\Sigma M \to K$, resulting in a map $\Delta(f,g) \colon M \to \Omega K$, whose corresponding cohomology class $[\Delta(f,g)] \in H^{n+1}(M; \pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M))$ characterizes when f and g are homotopic through lifts:

$$f \simeq g \text{ as lifts } \iff [\Delta(f,g)] = 0 \in H^{n+1}(M; \pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M))$$

The cohomology class $[\Delta(f,g)]$ can similarly characterize when f and g are homotopic:

$$f \simeq g \iff \Delta(f,g) \in \mathrm{Im}\left(G \xrightarrow{\Omega\kappa_n} H^{n+1}(M; \pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M))\right)$$

where $G \coloneqq \pi_0 \Omega_\varphi \operatorname{Map}^{\mathrm{Spin}}(\mathrm{Fr}^{\mathrm{Spin}}(M), \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M))$. We now apply finite completion on the target, and observe that we obtain the same pullback square above, along with the lifting problem for the map

$$\eta \circ \varphi : \mathrm{Fr}^{\mathrm{Spin}}(M) \xrightarrow{\varphi} \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M) \xrightarrow{\eta} \Phi^s \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M)$$

Since $\pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M)$ is a finitely generated group, we know that

$$\Phi^s K(\pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M), n+2) \simeq K\left(\Phi^g \pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M), n+2\right)$$

Given two lifts f, g prior to finite completion, we obtain two corresponding lifts of $\eta \circ \varphi$ upon profinite completion, namely $\eta \circ f, \eta \circ g$ respectively. Composition with finite completion

$$H^{n+1}(M; \pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M))$$

maps the obstruction $\Delta(f, g)$ to the corresponding obstruction $\Delta(\eta \circ f, \eta \circ g)$. We further denote G' to be the group

$$G' \coloneqq \pi_0 \Omega_{\eta \circ \varphi} \operatorname{Map}^{\mathrm{Spin}}(\mathrm{Fr}^{\mathrm{Spin}}(M), \Phi^s \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M))$$

We now consider the diagram

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "\Omega\kappa_n"'] & G' \arrow[d, "\Omega\eta\circ\kappa_n"] \\
H^{n+1}(M; \pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M)) \arrow[r] \arrow[d, two heads] & H^{n+1}(M; \Phi^g \pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M)) \arrow[d, two heads] \\
\operatorname{coker} \Omega\kappa_n \arrow[r] & \operatorname{coker}(\Omega\eta \circ \kappa_n)
\end{tikzcd}$$

Similarly to the proof of Theorem 2.28, the argument boils down to showing the lower horizontal map is injective. As $\operatorname{coker} \Omega\kappa_n$ is a finitely generated abelian group - which we recall are residually finite - it would suffice to show that $\operatorname{coker} \Omega\Phi^s\kappa_n$ is isomorphic to the underlying group of the profinite completion of $\operatorname{coker} \Omega\kappa_n$, and that the map is given by profinite completion. This follows if we can show that the top square is isomorphic, in the category $\mathrm{Sq}(\mathrm{Grp})$ of squares in groups, to the square

$$\begin{tikzcd}
G \arrow[r, "\eta"] \arrow[d, "\Omega\kappa_n"'] & \Phi^g G \arrow[d, "\Phi^g \Omega(\eta\circ\kappa_n)"] \\
H^{n+1}(M; \pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M)) \arrow[r, "\eta"'] & \Phi^g H^{n+1}(M; \pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M))
\end{tikzcd}$$

obtained applying the natural transformation $\mathrm{id} \Rightarrow \Phi^g$ to the arrow

$$G \xrightarrow{\Omega\kappa_n} H^{n+1}(M; \pi_{n+1} \mathrm{Fr}^{\mathrm{Spin}}(M))$$

we refer to the proof of Theorem 2.28 for further details.

We show how this works for the arrow $G \to G'$ first; the statement for the square works in the same way, as showcased in the proof of Theorem 2.28. By the Grothendieck equivalence $\mathrm{Psh}(B\mathrm{Spin}(d)) \simeq \mathcal{S}_{/B\mathrm{Spin}(d)}$, we note that $\mathrm{Fr}^{\mathrm{Spin}}(M)$ is mapped to $M \to B\mathrm{Spin}(d)$, classifying the spin tangent bundle. As $M \simeq \varinjlim_M *$ in $\mathcal{S}$, this gives a description of $M \to B\mathrm{Spin}(d)$ as $\varinjlim_M (* \to B\mathrm{Spin}(d))$, since colimits in overcategories are computed in the underlying category. Going back via the Grothendieck equivalence, we obtain

$$\mathrm{Fr}^{\mathrm{Spin}}(M) \simeq \varinjlim_M \mathrm{Spin}(d)$$

where the diagram is given by sending $p \in M$ to $\mathrm{Spin}(T_p M)$. The above equivalence is the categorical analogue of the fact that $\mathrm{Fr}^{\mathrm{Spin}}(M)$ admits a free, finite $\mathrm{Spin}(d)$-CW-structure, obtained by pulling back a finite CW-structure on M against the principal $\mathrm{Spin}(d)$-bundle $\mathrm{Fr}^{\mathrm{Spin}}(M) \to M$. We thus obtain the equivalences

$$\begin{aligned} \mathrm{Map}^{\mathrm{Spin}}(\mathrm{Fr}^{\mathrm{Spin}}(M), \Phi^s \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M)) &\simeq \mathrm{Map}^{\mathrm{Spin}}(\underset{M}{\underrightarrow{\mathrm{colim}}}\, \mathrm{Spin}(d), \Phi^s \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M)) \\ &\simeq \lim_M \mathrm{Map}^{\mathrm{Spin}}(\mathrm{Spin}(d), \Phi^s \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M)) \\ &\simeq \lim_M \mathrm{Map}_{\mathcal{S}}(*, \Phi^s \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M)) \\ &\simeq \lim_M \Phi^s \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M) \end{aligned}$$

We observe that $\tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M)$ is a simply connected space of finite type, and that the above limit is a finite limit, as M is a closed manifold; thus, Corollary 2.20 applies, and supplies an isomorphism on π_1

$$\Phi^s \lim_M \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M) \to \lim_M \Phi^s \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M)$$

on choices of basepoints coming from the images of $\varphi \in \lim_M \tau_{\leq n} \mathrm{Fr}^{\mathrm{Spin}}(M)$ via the two canonical maps. Theorem 2.16 now implies that $G' \cong \Phi^g G$, and that the map $G \to G' \xrightarrow{\cong} \Phi^g G$ agrees with the finite completion map. The remainder of the proof carries through as for the proof of Theorem 2.28. □

3.3. **Restricted Postnikov truncations.** We now aim for a proof of Theorem 3.7 for $k \geq 2$. In order to do this, we wish to construct a *restricted Postnikov decomposition* for $X \in \mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$, namely an inverse system $\{\widetilde{\tau_{\leq n}} X\}_{n \in \mathbb{N}}$, along with morphisms $X \to \widetilde{\tau_{\leq n}} X$ satisfying the following

(i) The induced map
$$X \to \lim_n \widetilde{\tau_{\leq n}} X$$
is an equivalence

(ii) For $\ell < k$, and for all $n \in \mathbb{N}$, the map
$$X(\sqcup_\ell \mathbb{R}^d) \xrightarrow{\simeq} \widetilde{\tau_{\leq n}} X(\sqcup_\ell \mathbb{R}^d)$$
is an equivalence

(iii) The tower evaluated on $\sqcup_k \mathbb{R}^d$ is given by a sequence of fibrations with fibre Eilenberg-MacLane spaces.

Construction 3.8. Any $X \in \mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$ comes equipped with a canonical morphism $X \to (\iota_{k-1})_* \iota_{k-1}^* X$. From X, we define a functor

$$X/T_{k-1}X \colon (\mathrm{Disc}_d^{\leq k})^{\mathrm{op}} \to \mathrm{Ar}(\mathcal{S})$$

by $D \mapsto \Big(X(D) \to (\iota_{k-1})_* \iota_{k-1}^* X(D)\Big)$. For each $n \in \{-1, 0\} \cup \mathbb{N}$, we define $\widetilde{\tau_{\leq n}} X$ to be the composition

$$\widetilde{\tau_{\leq n}} X \colon (\mathrm{Disc}_d^{\leq k})^{\mathrm{op}} \xrightarrow{X/T_{k-1}X} \mathrm{Ar}(\mathcal{S}) \xrightarrow{\mathrm{MP}_n} \mathcal{S}$$

where MP_n is the n-th stage Moore-Postnikov functor of §2.3.4.

The following is immediate from the above construction.

Lemma 3.9. *The inverse system $\widetilde{\tau_{\leq n}} X$ is a restricted Postnikov decomposition of the presheaf $X \in \mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$, i.e. satisfies (i), (ii) and (iii) from above.*

Remark 3.10. The above construction can be seen as a certain localization onto a full-subcategory of $\mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$: for each $n \in \{-1, 0\} \cup \mathbb{N}$, we say X is *n-truncated over* $\mathrm{Disc}_d^{\leq k-1}$ if the map $X(\sqcup_k \mathbb{R}^d) \to (\iota_{k-1})_* \iota_{k-1}^* X(\sqcup_k \mathbb{R}^d)$ is n-truncated. $\mathcal{C}_n$ is then defined to be the full subcategory of $\mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$ of those objects $X \in \mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$ which are n-truncated over $\mathrm{Disc}_d^{\leq k}$. Observe that for any $X \in \mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$, $\widetilde{\tau_{\leq n}} X \in \mathcal{C}_n$. We thus define a direct system of full subcategories $\mathcal{C}_n$ of $\mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$ such that $\mathcal{C}_{-1} \simeq \mathrm{Psh}(\mathrm{Disc}_d^{\leq k-1})$, $\varinjlim_n \mathcal{C}_n \simeq \mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$, and the map $\mathcal{C}_{-1} \to \varinjlim_n \mathcal{C}_n$ agrees with the inclusion functor $\mathrm{Psh}(\mathrm{Disc}_d^{\leq 1}) \hookrightarrow \mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$. For each n, the inclusion $c_n \colon \mathcal{C}_n \hookrightarrow \mathrm{Psh}(\mathrm{Disc}_d^{\leq k})$ admits a left adjoint $(c_n)_* \colon \mathrm{Psh}(\mathrm{Disc}_d^{\leq k}) \to \mathcal{C}_n$, and it can be shown that $(c_n)_* X \simeq \widetilde{\tau}_{\leq n} X$.

Recollement/Reedy extensions, following [KK24, Theorem 4.9 (i)] and [HRS], allow us to show the following key step. Let $\mathrm{Disc}_d^{=k,\mathrm{Spin}}$ be the category consisting of a single object $\sqcup_k \mathbb{R}^d$, where the morphism space is $\mathrm{Emb}^{\mathrm{Spin}}_{\pi_0\text{-inj}}(\sqcup_k \mathbb{R}^d, \sqcup_k \mathbb{R}^d)$ of those embeddings that induce a bijection on π_0. We observe that $\mathrm{Disc}_d^{=k,\mathrm{Spin}}$ is equivalent to the category BG_k, where $G_k \coloneqq \mathfrak{S}_k \ltimes \mathrm{Spin}(d)^k$, the action given by permutation of factors. There is an inclusion functor $j : \mathrm{Disc}_d^{=k,\mathrm{Spin}} \to \mathrm{Disc}_d^{\leq k,\mathrm{Spin}}$, which induces a restriction functor $j^* \colon \mathrm{Psh}(\mathrm{Disc}_d^{\leq k,\mathrm{Spin}}) \to \mathrm{Psh}(\mathrm{Disc}_d^{=k,\mathrm{Spin}})$.

Theorem 3.11. *We have the following pullback diagram of* ∞*-categories*

$$\begin{tikzcd}
\mathrm{Psh}(\mathrm{Disc}_d^{\leq k,\mathrm{Spin}}) \arrow[r, "1"] \arrow[d, "\iota_{k-1}^*"'] & \mathrm{Fun}(\Delta^2, \mathrm{Psh}(\mathrm{Disc}_d^{=k,\mathrm{Spin}})) \arrow[d, "2"] \\
\mathrm{Psh}(\mathrm{Disc}_d^{\leq k-1,\mathrm{Spin}}) \arrow[r, "3"'] & \mathrm{Fun}(\Delta^1, \mathrm{Psh}(\mathrm{Disc}_d^{=k,\mathrm{Spin}}))
\end{tikzcd}$$

where the three other functors are the following

- 1: $\mathrm{Psh}(\mathrm{Disc}_d^{\leq k,\mathrm{Spin}}) \to \mathrm{Fun}(\Delta^2, \mathrm{Psh}(\mathrm{Disc}_d^{=k,\mathrm{Spin}}))$ *maps* $X \in \mathrm{Psh}(\mathrm{Disc}_d^{\leq k,\mathrm{Spin}})$ *to the triangle in* $\mathrm{Psh}(\mathrm{Disc}_d^{=k,\mathrm{Spin}})$

$$\begin{tikzcd}
j^*(\iota_{k-1})_! \iota_{k-1}^* X \arrow[r] \arrow[d] & j^* X \arrow[dl] \\
j^*(\iota_{k-1})_* \iota_{k-1}^* X &
\end{tikzcd}$$

- *The functor* 2: $\mathrm{Fun}(\Delta^2, \mathrm{Psh}(\mathrm{Disc}_d^{=k,\mathrm{Spin}})) \to \mathrm{Fun}(\Delta^1, \mathrm{Psh}(\mathrm{Disc}_d^{=k,\mathrm{Spin}}))$ *is given by precomposition with the map* $\Delta^1 \to \Delta^2$, $0 \mapsto 0$ *and* $1 \mapsto 2$.
- *The functor* 3: $\mathrm{Psh}(\mathrm{Disc}_d^{\leq k-1,\mathrm{Spin}}) \to \mathrm{Fun}(\Delta^1, \mathrm{Psh}(\mathrm{Disc}_d^{=k,\mathrm{Spin}}))$ *maps* X *to the arrow*

$$j^*(\iota_{k-1})_! X \to j^*(\iota_{k-1})_* X$$

Remark 3.12. We emphasize that the above theorem works in general for all tangential structures $\Theta \in \mathrm{Psh}(BO(d))$.

We now combine the previous pullback square with Construction 3.8. For $X \in \mathrm{Psh}(\mathrm{Disc}_d^{\leq k,\mathrm{Spin}})$, let $\{\widetilde{\tau_{\leq n}} X\}$ be its restricted Postnikov decomposition. In our aim of studying self-maps of X in $\mathrm{Psh}(\mathrm{Disc}_d^{\leq k,\mathrm{Spin}})$, we observe that

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\leq k,\mathrm{Spin}})}(X, X) &\simeq \mathrm{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\leq k,\mathrm{Spin}})}(X, \lim_n \widetilde{\tau_{\leq n}} X) \\
&\simeq \lim_n \mathrm{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\leq k,\mathrm{Spin}})}(X, \widetilde{\tau_{\leq n}} X)
\end{aligned}$$

It would be ideal for our purposes if the consecutive stages of the above limit are governed, as for usual mapping spaces, by some cohomology group; the following result is a step in this direction, and follows readily from Theorem 3.11.

Proposition 3.13. *For any $X \in \mathrm{Psh}(\mathrm{Disc}_d^{\leq k,\mathrm{Spin}})$, evaluation at $D_k := \sqcup_k \mathbb{R}^d$ yields an equivalence*

$$\mathrm{Lift}_{\mathrm{Psh}(\mathrm{Disc}_d^{\leq k,\mathrm{Spin}})}\left(\begin{tikzcd} & \widetilde{\tau_{\leq n+1}}X \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & \widetilde{\tau_{\leq n}}X \end{tikzcd}\right) \xrightarrow{\simeq} \mathrm{Lift}_{G_k}\left(\begin{tikzcd} j^*(\iota_1)_!\iota_1^*X(D_k) \arrow[r] \arrow[d] & j^*\widetilde{\tau_{\leq n+1}}X(D_k) \arrow[d] \\ j^*X(D_k) \arrow[r] \arrow[ur, dashed] & j^*\widetilde{\tau_{\leq n}}X(D_k) \end{tikzcd}\right)$$

where $G_k := \mathfrak{S}_k \ltimes \mathrm{Spin}(d)^k \simeq \mathrm{Emb}^{\mathrm{Spin}}_{\pi_0\text{-}inj}(\sqcup_k\mathbb{R}^d, \sqcup_k\mathbb{R}^d)$, and where the target space denotes the space of G_k-equivariant lifts.

We now specialize to the case of $X = E_M^{\mathrm{Spin}}$ for a 2-connected closed manifold. We first observe the following. For a closed manifold, we let $\overline{\mathrm{Conf}}_k(M)$ denote the Fulton-MacPherson compactification of $\mathrm{Conf}_k(M)$; this is a compact manifold with corners containing $\mathrm{Conf}_k(M)$ as its interior. Let $\partial\overline{\mathrm{Conf}}_k(M)$ denote its boundary.

Lemma 3.14. *For M a 2-connected closed manifold, and for $D_k := \sqcup_k\mathbb{R}^d$, $j^*(\iota_{k-1})_!\iota_{k-1}^*E_M^{\mathrm{Spin}}(D_k)$ sits in the following pullback diagram*

$$\begin{tikzcd} j^*(\iota_{k-1})_!\iota_{k-1}^*E_M^{\mathrm{Spin}}(D_k) \arrow[r] \arrow[d] & \mathrm{Emb}^{\mathrm{Spin}}(D_k, M) \arrow[d, "\mathrm{ev}_0"] \\ \partial\overline{\mathrm{Conf}}_k(M) \arrow[r, "\partial"'] & \mathrm{Conf}_k(M) \end{tikzcd}$$

where the bottom arrow is defined after choice of a boundary collar on $\overline{\mathrm{Conf}}_k(M)$.

Proof. This follows from [KK24, Proposition 5.12]. □

It follows in particular that the pair $(j^*E_M^{\mathrm{Spin}}(D_k), j^*(\iota_{k-1})_!\iota_{k-1}^*E_M^{\mathrm{Spin}}(D_k))$, for $D_k = \sqcup_k\mathbb{R}^d$, is equivalent to a finite, free G_k-CW pair, which, for simplicity, we henceforth denote by

$$(E_M^{\mathrm{Spin}}(D_k), \partial E_M^{\mathrm{Spin}}(D_k))$$

We now start with a warm-up to the proof of Theorem 3.7. The main advantage of Proposition 3.13 is that it enables us to study a space of lifts of maps in a complicated category in terms of obstruction theory of equivariant spaces. We now carry out this analysis for $X = E_M^{\mathrm{Spin}}$. For ease of notation, we let D_k denote the disjoint union of k-many discs, and we again let $G_k := \mathfrak{S}_k \ltimes \mathrm{Spin}(d)^k$. We first observe that $\pi_1 E_M^{\mathrm{Spin}}(D_k) \cong 1$, so that the fibration $\widetilde{\tau_{\leq n+1}}E_M^{\mathrm{Spin}}(D_k) \to \widetilde{\tau_{\leq n}}E_M^{\mathrm{Spin}}(D_k)$ may be delooped to a pullback diagram in $\mathrm{Psh}(BG_k)$

$$\begin{tikzcd} \widetilde{\tau_{\leq n+1}}E_M^{\mathrm{Spin}}(D_k) \arrow[r] \arrow[d] & * \arrow[d] \\ \widetilde{\tau_{\leq n}}E_M^{\mathrm{Spin}}(D_k) \arrow[r, "\kappa_n"'] & K(\pi_{n+1}F, n+2) \end{tikzcd}$$

where F is the fibre of the map $E_M^{\mathrm{Spin}}(D_k) \to (\iota_{k-1})_*\iota_{k-1}^*E_M^{\mathrm{Spin}}(D_k)$, on which G_k acts. Observe now that $\pi_{n+1}F$ is a finitely generated group, which admits a potentially non-trivial action of G_k, as G_k is not connected. We now fix a pair of maps $(\varphi, \partial\varphi) : (E_M^{\mathrm{Spin}}(D_k), \partial E_M^{\mathrm{Spin}}(D_k)) \to (\widetilde{\tau_{\leq n}}E_M^{\mathrm{Spin}}(D_k), \widetilde{\tau_{\leq n+1}}E_M^{\mathrm{Spin}}(D_k))$, where $\partial E_M^{\mathrm{Spin}}(D_k)$ is to be understood as in Lemma 3.14. We are thus looking at the following lifting problem in $\mathrm{Psh}(BG_k)$

$$\begin{tikzcd}
\partial E_M^{\mathrm{Spin}}(D_k) \arrow[r, "\partial\varphi"] \arrow[d] & \widetilde{\tau_{\leq n+1}} E_M^{\mathrm{Spin}}(D_k) \arrow[r] \arrow[d] & * \arrow[d] \\
E_M^{\mathrm{Spin}}(D_k) \arrow[r, "\varphi"'] \arrow[ur, dashed, "f"] & \widetilde{\tau_{\leq n}} E_M^{\mathrm{Spin}}(D_k) \arrow[r, "\kappa_n"'] & K(\pi_{n+1}F, n+2)
\end{tikzcd}$$

As the right hand square is a pullback square, a lift f is therefore equivalent to a null-homotopy of $\kappa_n \circ \varphi$ relative to $\partial E_M^{\mathrm{Spin}}(D_k)$, or equivalently, a map from the relative cone of $(E_M^{\mathrm{Spin}}(D_k), \partial E_M^{\mathrm{Spin}}(D_k))$ to $K(\pi_{n+1}F, n+2)$. Given two lifts f, g, we may glue the two maps from the cone to obtain a map from the suspension, which by adjunction yields a relative map $\Delta(f,g)\colon (E_M^{\mathrm{Spin}}(D_k), \partial E_M^{\mathrm{Spin}}(D_k)) \to (K(\pi_{n+1}F, n+1), *)$. Considered up to homotopy, this gives a class $[\Delta(f,g)] \in H^{n+1}_{G_k}(E_M^{\mathrm{Spin}}(D_k), \partial E_M^{\mathrm{Spin}}(D_k); \pi_{n+1}F)$ in the relative G_k-equivariant cohomology group with coefficients in the $\mathbb{Z}G_k$-module $\pi_{n+1}F$. We note that the above cohomology group is a finitely generated abelian group, as the pair is equivalent to a finite, free G_k-CW pair, and $\pi_{n+1}F$ is finitely generated. The class $\Delta(f,g)$ satisfies, as usual, the following property

$$[\Delta(f,g)] = 0 \iff f \simeq g \text{ through equivariant lifts}$$

As in the previous setting, we are after a subgroup K of $H^{n+1}_{G_k}(E_M^{\mathrm{Spin}}(D_k), \partial E_M^{\mathrm{Spin}}(D_k); \pi_{n+1}F)$, with the further property that

$$f \simeq g \iff [\Delta(f,g)] \in K$$

It can be readily seen that such a subgroup is given by the image of

$$\pi_0 \Omega_{(\varphi,\partial\varphi)} \operatorname{Map}_{\mathrm{Ar}(\mathrm{Psh}(BG_k))}\Big((E_M^{\mathrm{Spin}}(D_k), \partial E_M^{\mathrm{Spin}}(D_k)), (\widetilde{\tau_{\leq n}} E_M^{\mathrm{Spin}}(D_k), \widetilde{\tau_{\leq n+1}} E_M^{\mathrm{Spin}}(D_k))\Big)$$

in the equivariant cohomology group by the obvious composition morphism. We can repeat the above after finite completion on the target. We first record some technicalities that prove useful.

We now treat the case of relative mapping spaces.

Lemma 3.15. *The maps*

$$\Phi^s \operatorname{Map}^{G_k}\left(\begin{array}{c} \partial E_M^{\mathrm{Spin}}(D_k) \\ \big\downarrow \\ E_M^{\mathrm{Spin}}(D_k) \end{array}, \begin{array}{c} \widetilde{\tau_{\leq n+1}} E_M^{\mathrm{Spin}}(D_k) \\ \big\downarrow \\ \widetilde{\tau_{\leq n+1}} E_M^{\mathrm{Spin}} \end{array}\right) \to \operatorname{Map}^{G_k}\left(\begin{array}{c} \partial E_M^{\mathrm{Spin}}(D_k) \\ \big\downarrow \\ E_M^{\mathrm{Spin}}(D_k) \end{array}, \begin{array}{c} \Phi^s \widetilde{\tau_{\leq n+1}} E_M^{\mathrm{Spin}}(D_k) \\ \big\downarrow \\ \Phi^s \widetilde{\tau_{\leq n+1}} E_M^{\mathrm{Spin}} \end{array}\right)$$

and

$$\Phi^s \operatorname{Map}^{G_k}\left(\begin{array}{c} \partial E_M^{\mathrm{Spin}}(D_k) \\ \big\downarrow \\ E_M^{\mathrm{Spin}}(D_k) \end{array}, \begin{array}{c} * \\ \big\downarrow \\ K(\pi_{n+1}F, n+2) \end{array}\right) \to \operatorname{Map}^{G_k}\left(\begin{array}{c} \partial E_M^{\mathrm{Spin}}(D_k) \\ \big\downarrow \\ E_M^{\mathrm{Spin}}(D_k) \end{array}, \begin{array}{c} * \\ \big\downarrow \\ \Phi^s K(\pi_{n+1}F, n+2) \end{array}\right)$$

induce isomorphisms on all π_n, $n \geq 1$, on basepoints coming from the image of the corresponding spaces before finite completions. In the above, the mapping spaces are taken in the category $\mathrm{Fun}(\Delta^1, \mathrm{Psh}(BG_k))$, the category of arrows in spaces with an action of G_k.

Proof. We prove it for one of the above spaces, as the next one works in the exact same way. By the description of mapping spaces in arrow categories, the mapping space in consideration can be seen as the pullback of the following cospan

$$\begin{tikzcd}
 & \operatorname{Map}^{G_k}(\partial E_M^{\mathrm{Spin}}(D_k), \Phi^s \widetilde{\tau_{\leq n+1}} E_M^{\mathrm{Spin}}(D_k)) \arrow[d] \\
\operatorname{Map}^{G_k}(E_M^{\mathrm{Spin}}(D_k), \Phi^s \widetilde{\tau_{\leq n}} E_M^{\mathrm{Spin}}(D_k)) \arrow[r] & \operatorname{Map}^{G_k}(\partial E_M^{\mathrm{Spin}}(D_k), \Phi^s \widetilde{\tau_{\leq n}} E_M^{\mathrm{Spin}}(D_k))
\end{tikzcd}$$

where the maps are the obvious post/pre-composition, and where we observe that all the spaces in the above diagram can be written as finite limits of the finite completion of finite type, simply connected spaces. The result follows from iterated application of Theorem 2.18. □

We now proceed to show Theorem 3.7.

Proof of Theorem 3.7. The proof proceeds by induction, and follows a strategy similar to that of Sullivan's proof. Let $n \in \mathbb{N}$, and consider the following lifting problem

$$\begin{tikzcd} & \widetilde{\tau_{\leq n+1} E_M^{\mathrm{Spin}}} \arrow[d] \\ E_M^{\mathrm{Spin}} \arrow[r, "\varphi"'] \arrow[ur, dashed] & \widetilde{\tau_{\leq n} E_M^{\mathrm{Spin}}} \end{tikzcd}$$

By Proposition 3.13 and Lemma 3.14, this translates to a lifting problem in $\mathrm{Psh}(BG_k)$, where $G_k := \mathfrak{S}_k \ltimes \mathrm{Spin}(d)^k$:

$$\begin{tikzcd} \partial E_M^{\mathrm{Spin}}(D_k) \arrow[r, "\partial\varphi"] \arrow[d] & \widetilde{\tau_{\leq n+1} E_M^{\mathrm{Spin}}}(D_k) \arrow[r] \arrow[d] & * \arrow[d] \\ E_M^{\mathrm{Spin}}(D_k) \arrow[r, "\varphi"'] \arrow[ur, dashed] & \widetilde{\tau_{\leq n} E_M^{\mathrm{Spin}}}(D_k) \arrow[r] & K(\pi_{n+1}F, n+2) \end{tikzcd}$$

Given two lifts f, g, the above analysis yields a cohomology class in the G_k-equivariant cohomology group

$$[\Delta(f,g)] \in H^{n+1}_{G_k}(E_M^{\mathrm{Spin}}(D_k), \partial E_M^{\mathrm{Spin}}(D_k); \pi_{n+1}F)$$

whose vanishing implies the equivalence of f and g as lifts (we refer to the discussion after Lemma 3.14 for more details). For the sake of ease of notation, we denote the above equivariant cohomology group by H throughout the proof. We also consider the group

$$G := \pi_0 \Omega_{(\varphi,\partial\varphi)} \mathrm{Map}_{\mathrm{Ar}(\mathrm{Psh}(BG_k))}\left((E_M^{\mathrm{Spin}}(D_k), \partial E_M^{\mathrm{Spin}}(D_k)), (\widetilde{\tau_{\leq n} E_M^{\mathrm{Spin}}}(D_k), \widetilde{\tau_{\leq n+1} E_M^{\mathrm{Spin}}}(D_k))\right)$$

which maps to H by post-composition with the right hand square of the above diagram; we call the group morphism $\alpha \colon G \to H$. We observe that

$$f \simeq g \iff [\Delta(f,g)] \in \mathrm{Im}(G \to H)$$

We may repeat the same analysis after composing with finite completion on the target. Namely, given the above lifts f, g, we get lifts $\eta \circ f, \eta \circ g$ fitting in the following diagram

$$\begin{tikzcd} \partial E_M^{\mathrm{Spin}}(D_k) \arrow[r, "\partial\eta\circ\varphi"] \arrow[d] & \Phi^s \widetilde{\tau_{\leq n+1} E_M^{\mathrm{Spin}}}(D_k) \arrow[r] \arrow[d] & * \arrow[d] \\ E_M^{\mathrm{Spin}}(D_k) \arrow[r, "\Phi\varphi"'] \arrow[ur, dashed] & \Phi^s \widetilde{\tau_{\leq n} E_M^{\mathrm{Spin}}}(D_k) \arrow[r] & \Phi^s K(\pi_{n+1}F, n+2) \end{tikzcd}$$

The rightmost square is again a pullback square, and since $\pi_{n+1}F$ is a finitely generated abelian group, we furthermore obtain an equivalence

$$\Phi^s K(\pi_{n+1}F, n+2) \simeq K((\Phi^g \pi_{n+1}F), n+2)$$

We thus obtain a cohomology class

$$[\Delta(\eta \circ f, \eta \circ g)] \in H^{n+1}_{G_k}\left(E_M^{\mathrm{Spin}}(D_k), \partial E_M^{\mathrm{Spin}}(D_k); \Phi^g \pi_{n+1}F\right)$$

whose vanishing implies that $\eta \circ f, \eta \circ g$ are equivalent as equivariant lifts. Again for ease of notation, we denote the above cohomology group by H'. We define G' analogously to G, by finite completion on the target:

$$G' := \pi_0 \Omega_{(\eta\varphi, \partial\eta\varphi)} \operatorname{Map}_{\mathrm{Ar}(\mathrm{Psh}(BG_k))} \Big((E_M^{\mathrm{Spin}}(D_k), \partial E_M^{\mathrm{Spin}}(D_k)), (\Phi^s \widetilde{\tau_{\leq n} E_M^{\mathrm{Spin}}}(D_k), \Phi^s \widetilde{\tau_{\leq n+1} E_M^{\mathrm{Spin}}}(D_k)) \Big)$$

The above group maps to H' by composition with the pullback square on the target, and we call the group homomorphism α'. Again, obstruction theory tells us that

$$\Phi^s f \simeq \Phi^s g \iff [\Delta(\eta \circ f, \eta \circ g)] \in \operatorname{Im}(G' \to H')$$

The theorem follows by induction once we show that if $f \not\simeq g$, then $\eta \circ f \not\simeq \eta \circ g$. To that end, we observe that we have a commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\alpha"'] & G' \arrow[d, "\alpha'"] \\ H \arrow[r] \arrow[d, "p"'] & H' \arrow[d, "p'"] \\ \operatorname{coker}(\alpha) \arrow[r] & \operatorname{coker}(\alpha') \end{tikzcd} \tag{2}$$

such that $[\Delta(f,g)]$ is mapped to $[\Delta(\eta \circ f, \eta \circ g)]$ via the middle horizontal map. The claim follows if we can show that the lower horizontal map is injective. Indeed, $f \not\simeq g$ is equivalent to $p([\Delta(f,g)]) \neq 0$ in $\operatorname{coker}(\alpha)$, and hence if the map $\operatorname{coker}(\alpha) \to \operatorname{coker}(\alpha')$ is injective, we obtain that $p'([\Delta(\eta \circ f, \eta \circ g)]) \neq 0$ in $\operatorname{coker}(\alpha')$, which in turn is equivalent to $\eta \circ f \not\simeq \eta \circ g$. As H is a finitely generated abelian group, so is $\operatorname{coker}(\alpha)$. In particular, as finitely generated abelian groups are residually finite, the map $\operatorname{coker}(\alpha) \to \operatorname{coker}(\alpha')$ would be injective if we can show that the arrow $\operatorname{coker}(\alpha) \to \operatorname{coker}(\alpha')$ is isomorphic in the category $\operatorname{Fun}(\Delta^1, \mathrm{Grp})$ of arrows in groups, to the arrow $\operatorname{coker}(\alpha) \xrightarrow{\eta} \Phi^g \operatorname{coker}(\alpha)$, which is injective by residual finiteness of finitely generated abelian groups. To that end, we show that the top square of diagram 2 is isomorphic, in the category $\mathrm{Sq}(\mathrm{Grp})$ of squares in groups, to the square

$$\begin{tikzcd} G \arrow[r, "\eta_G"] \arrow[d, "\alpha"'] & \Phi^g G \arrow[d, "\Phi^g \alpha"] \\ H \arrow[r, "\eta_H"'] & \Phi^g H \end{tikzcd}$$

By Lemma 3.15, we observe that G' and H' are the fundamental groups (based at a point coming from G and H, through the canonical map) of a finite limit of the finite completion of some (componentwise) finite type, nilpotent space. Then, Corollary 2.20 implies that the map $G \to G'$ and $H \to H'$ are identified with the group profinite completion maps. Consequently, the remainder of the proof of Theorem 2.28 carries through, and we obtain the desired result.

□

Having finished the technical heart of the argument, we can now assemble the pieces and show that $\pi_0 T_k \operatorname{Diff}^{\mathrm{Spin}}(M)$ is a residually finite group, for every k.

Proof of Theorem 3.4. Consider the following diagram

$$\begin{tikzcd}
\pi_0 \operatorname{Map}^{\simeq}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\iota_k^* E_M^{\mathrm{Spin}}, \iota_k^* E_M^{\mathrm{Spin}}) \arrow[r] \arrow[d] \arrow[dr] & \Phi^g \pi_0 \operatorname{Map}^{\simeq}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\iota_k^* E_M^{\mathrm{Spin}}, \iota_k^* E_M^{\mathrm{Spin}}) \arrow[d, dashed, "\exists!"] \\
\pi_0 \operatorname{Map}^{\simeq}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\Phi^s \iota_k^* E_M^{\mathrm{Spin}}, \Phi^s \iota_k^* E_M^{\mathrm{Spin}}) & \pi_0 \operatorname{Map}^{\simeq}_{\mathrm{Fun}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k}, \mathrm{Pro}(\mathcal{S}_\pi))}(\iota_k^* \widehat{E}_M^{\mathrm{Spin}}, \iota_k^* \widehat{E}_M^{\mathrm{Spin}}) \arrow[l]
\end{tikzcd}$$

The existence (and uniqueness) of the right vertical map follows from Theorem 2.21, and the universal property of profinite completion of groups. Consequently, the injectivity of the top horizontal morphism follows from the injectivity of the leftmost vertical morphism. Furthermore, we observe that from the canonical morphism $E_M^{\mathrm{Spin}} \to \Phi^s E_M^{\mathrm{Spin}}$, we obtain the following commuting diagram

$$\begin{tikzcd}
\operatorname{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\iota_k^* E_M^{\mathrm{Spin}}, \iota_k^* E_M^{\mathrm{Spin}}) \arrow[r] \arrow[dr] & \operatorname{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\Phi \iota_k^* E_M^{\mathrm{Spin}}, \Phi \iota_k^* E_M^{\mathrm{Spin}}) \arrow[d] \\
& \operatorname{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\iota_k^* E_M^{\mathrm{Spin}}, \Phi \iota_k^* E_M^{\mathrm{Spin}})
\end{tikzcd}$$

We deduce the injectivity of the horizontal map on π_0 from the injectivity of the diagonal one, as given by Theorem 3.7. Finally, the injectivty on the invertible parts of $T_k \operatorname{Emb}^{\mathrm{Spin}}(M, M)$ follows from the following commutative diagram

$$\begin{tikzcd}
\pi_0 \operatorname{Map}^{\simeq}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\iota_k^* E_M^{\mathrm{Spin}}, \iota_k^* E_M^{\mathrm{Spin}}) \arrow[r] \arrow[d, hook] & \pi_0 \operatorname{Map}^{\simeq}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\Phi^s \iota_k^* E_M^{\mathrm{Spin}}, \Phi^s \iota_k^* E_M^{\mathrm{Spin}}) \arrow[d, hook] \\
\pi_0 \operatorname{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\iota_k^* E_M^{\mathrm{Spin}}, \iota_k^* E_M^{\mathrm{Spin}}) \arrow[r, hook] & \pi_0 \operatorname{Map}_{\mathrm{Psh}(\mathrm{Disc}_d^{\mathrm{Spin},\leq k})}(\Phi^s \iota_k^* E_M^{\mathrm{Spin}}, \Phi^s \iota_k^* E_M^{\mathrm{Spin}})
\end{tikzcd}$$

□

3.4. **Main result: residual finiteness.** We are now ready to prove the main theorem of the section.

Theorem 3.16. *Let M be a smooth, 2-connected closed manifold of dimension $d \geq 5$. Then, $\pi_0 T_k \operatorname{Diff}^\theta(M)$ is a residually finite group, for θ the spin, oriented or non-oriented tangential structure.*

Proof. We fix $\mathfrak{s}$ a Spin-structure on M, and let ℓ be the associated Θ-structure. By Corollary 3.3, the fibre F of the map

$$BT_k \operatorname{Diff}^{\mathrm{Spin}}(M, \mathfrak{s}) \to BT_k \operatorname{Diff}^\theta(M, \ell)$$

over the identity is a collection of components of the space of lifts

$$\operatorname{Lift}\left(\begin{tikzcd}
& B\mathrm{Spin}(d) \arrow[d] \\
M \arrow[ur, dashed] \arrow[r, "\ell"'] & B\Theta(d)
\end{tikzcd}\right)$$

and it is fairly straigthforward to see that both π_0 and π_1 (at each basepoint) of the above space are finite. We thus have an exact sequence

$$\pi_1 F \to \pi_1 BT_k \operatorname{Diff}^{\mathrm{Spin}}(M) \xrightarrow{f} \pi_1 BT_k \operatorname{Diff}^{\Theta(d)}(M) \to \pi_0 F$$

By Theorem 3.4 and Lemma 2.9, we know that $\pi_1 BT_k \operatorname{Diff}^{\Theta(d)}/\operatorname{Im}(f)$ is a residually finite group whenever M is 2-connected, and can furthermore be identified with a finite index subgroup of $\pi_1 BT_k \operatorname{Diff}^{\Theta(d)}(M)$. Thus, by Lemma 2.10, it also follows that $\pi_1 BT_k \operatorname{Diff}^{\Theta(d)}(M)$ is residually finite. The latter is isomorphic to $\pi_0 T_k \operatorname{Diff}^{\Theta(d)}(M)$, and the result follows.

Remark 3.17. Via smoothing theory for embedding calculus, as in [KK24, Theorem 4.15], one concludes that $\pi_0 T_k \mathrm{Homeo}(M)$ is residually finite, under the same assumptions on M, where $T_k \operatorname{Homeo}(M)$ is the space of automorphisms of $\iota_k^* E_M$ in the category of $\operatorname{Psh}(\operatorname{Disc}_d^{\mathrm{Top},\leq k})$.

□

3.5. **Remarks on mapping class groups of W_g^n and disc-structure spaces.** In [KK22], Krannich and Kupers study the fibre of the embedding calculus map

$$\mathcal{S}^{\mathrm{Disc}}(M) := \operatorname{fib}_{E_M}\left(\operatorname{Manf}_d^{\cong} \to \operatorname{Psh}(\operatorname{Disc}_d)^{\cong}\right)$$

where M is a smooth, closed manifold, and where $\operatorname{Manf}_d^{\cong}$ denotes the groupoid core of the ∞-category of smooth, d-dimensional manifold with empty boundary and embeddings as spaces of 1-morphisms. The fibre over E_M is called the *disc-structure space of* M. Two important properties of the above space are

- $\mathcal{S}^{\mathrm{Disc}}(M)$ is not contractible, when M admits a spin structure, and $\dim M \geq 8$.
- $\mathcal{S}^{\mathrm{Disc}}(M)$ only depends on the tangential 2-type of M.

In particular, in the case $M = W_g^n := \#_g(S^n \times S^n)$ where $n > 3$, we know that $\operatorname{Emb}(W_g^n, W_g^n) \to T_\infty \operatorname{Emb}(W_g^n, W_g^n)$ is not an equivalence. Using Theorem 3.16, we observe that the non-equivalence can be already detected on π_0.

Proposition 3.18. *For $n = 5 \pmod 8$ and $g \geq 5$, the group homomorphism $\pi_0 \operatorname{Diff}^\theta(W_g^n) \to \pi_0 T_\infty \operatorname{Diff}^\theta(W_g^n)$ induced by the embedding calculus tower is not an isomorphism, for θ either the unoriented or oriented tangential structure.*

Proof. We begin by showing that $\pi_1 T_k \operatorname{Diff}^\theta(M)$ is finitely generated for all $k \in \mathbb{N}$, where the basepoint here is taken to be the identity. Indeed, we know that $T_1 \operatorname{Emb}^\theta(M, M) \simeq \operatorname{Bun}^\theta(TM, TM)$, which sits in a fibration

$$\operatorname{Map}(M, G_\theta(d)) \to \operatorname{Bun}^\theta(TM, TM) \to \operatorname{Map}(M, M)$$

where the fibre is taken over the identity, and where $G_\theta(d)$ is $O(d)$ in the unoriented case, $SO(d)$ in the oriented case. By [Kup19, Lemma 2.21], the base and the fibre of the above fibration are have finitely generated homotopy groups on each component (we regard the mapping space as a space of sections of the trivial bundle). Consequently, the fundamental group of the total space is also finitely generated. Furthermore, we have a fibration

$$L_k \operatorname{Emb}^\theta(M, M) \to T_k \operatorname{Emb}^\theta(M, M) \to T_{k-1} \operatorname{Emb}^\theta(M, M)$$

where L_k is the layer of the embedding tower, obtained as a space of relative sections of a fibration over $\operatorname{Conf}_k(M)$ relative the fat diagonal; its homotopy groups can be seen to be finitely generated, as follows again from [Kup19, Lemma 2.21]. By finite induction, we conclude that $\pi_1 T_k \operatorname{Emb}^\theta(M, M)$ is indeed finitely generated, and hence so is $\pi_1 T_k \operatorname{Diff}^\theta(M)$. The same holds for the other path components, as $T_k \operatorname{Diff}^\theta(M)$ is a grouplike E_1-space. Now, we observe that the Milnor exact sequence

$$1 \to \lim_k^1 \pi_1 T_k \operatorname{Diff}^\Theta(M) \to \pi_0 T_\infty \operatorname{Diff}^\theta(M) \to \lim_k \pi_0 T_k \operatorname{Diff}^\theta(M) \to 1$$

which is a priori an exact sequence of pointed sets, is indeed an exact sequence of groups. Indeed, as $T_\infty \operatorname{Diff}^\theta(M)$ and $\lim_k T_k \operatorname{Diff}^\theta(M)$ are grouplike E_1-spaces, and the map $T_\infty \operatorname{Diff}^\theta(M) \to \lim_k T_k \operatorname{Diff}^\theta(M)$ is an E_1-map, the above exact sequence occurs as the Milnor exact sequence computing the fundamental groups of their deloopings $BT_\infty \operatorname{Diff}^\theta(M)$

and $BT_k \operatorname{Diff}^\theta(M)$, based at the identity; furthermore, the $\lim^1$ term now consists of abelian groups. By [MP11, thm 2.3.3], $\lim^1_k \pi_1 T_k \operatorname{Diff}^\theta(M)$ is either trivial or uncountable. In the case it is trivial, then $\pi_0 T_\infty \operatorname{Diff}^\theta(M) \cong \lim_k \pi_0 T_k \operatorname{Diff}^\theta(M)$. By Theorem 3.16, the latter is again residually finite. Observe that $\operatorname{Emb}^\theta(W_g^n) = \operatorname{Diff}^\theta(M)$, as any self-embedding of a closed manifold is necessarily surjective. If the map $\operatorname{Emb}^\theta(W_g^n) \to T_\infty \operatorname{Emb}^\theta(W_g^n)$ was indeed an equivalence, it induces an equivalence $\operatorname{Diff}^\theta(M) \to T_\infty \operatorname{Diff}^\theta(M)$ which in turn induces an isomorphism of groups on π_0. In particular, this would imply that $\pi_0 \operatorname{Diff}^\theta(W_g^n)$ is residually finite, which is not the case as follows from [KRW20].

In the case $\lim^1_k \pi_1 T_k \operatorname{Diff}^\theta(M)$ is non-trivial, it follows from the Milnor exact sequence that $\pi_0 T_\infty \operatorname{Diff}^\theta(M)$ is uncountable, and hence cannot be isomorphic to $\pi_0 \operatorname{Diff}^\theta(M)$, a finitely generated group, as can be seen from [Sul77, thm 13.3]. □

We observe that in the above proof, $\pi_1 T_k \operatorname{Diff}^\theta(M)$ is shown to be a finitely generated group, and by [MP11, thm 2.3.3], it follows that $\lim^1 \pi_1 T_k \operatorname{Diff}^\theta(M)$ is a divisible group. In particular, we obtain the following

Corollary 3.19. *In the above setting,* $\lim^1 \pi_1 T_k \operatorname{Diff}^\theta(M)$ *embeds, via the Milnor exact sequence, into the finite residual of* $\pi_0 T_\infty \operatorname{Diff}^\theta(M)$.

In the case the above $\lim^1$ vanishes, one obtains an interesting application to the disc-structure space $\mathcal{S}_\partial^{\mathrm{Disc}}(D^{2n})$. We restrict ourselves to the case $n = 5 \pmod 8$, and consider the following diagram

$$\begin{tikzcd}
 & & \mathrm{bP}_{2n+2} \arrow[dll, dashed, "\exists"'] \arrow[d, hook] \arrow[drr, "0"] & & \\
 & & \pi_0 \operatorname{Diff}(W_g^n) \arrow[d] & & \\
1 \arrow[r] & \lim^1 \pi_1 T_k \operatorname{Diff}(W_g^n) \arrow[r] & \pi_0 T_\infty \operatorname{Diff}(W_g^n) \arrow[r] & \lim_k \pi_0 T_k \operatorname{Diff}(W_g^n) \arrow[r] & 1
\end{tikzcd}$$

We observe that the map $\mathrm{bP}_{2n+2} \to \lim_k \pi_0 T_k \operatorname{Diff}(M)$, obtained by restricting the embedding calculus map, is trivial. This follows from the fact that bP_{2n+2} is embedded as a subgroup of the finite residual of $\pi_0 \operatorname{Diff}(W_g^n)$. In case $\mathrm{bP}_{2n+2} \to \lim_k \pi_0 T_k \operatorname{Diff}(W_g^n)$ is non-trivial, and since $\lim_k \pi_0 T_k \operatorname{Diff}(W_g^n)$ is itself residually finite, then some non-trivial bP-sphere will be detected by some map $\pi_0 \operatorname{Diff}(W_g^n) \to F$ where F is a finite group, contradicting the fact that bP_{2n+2} is indeed in the finite residual of $\pi_0 \operatorname{Diff}(W_g^n)$. Thus, we obtain a lift $\mathrm{bP}_{2n+2} \to \lim^1 \pi_1 T_k \operatorname{Diff}(W_g^n)$; if the latter is trivial, then $\mathrm{bP}_{2n+2} \leq \ker(\pi_0 \operatorname{Diff}(W_g^n) \to \pi_0 T_\infty \operatorname{Diff}(W_g^n)) = \operatorname{Im}(\pi_1 \mathcal{S}^{\mathrm{Disc}}(W_g^n) \to \pi_0 \operatorname{Diff}(W_g^n))$, i.e. every bP-sphere can be hit by an element in $\pi_1 \mathcal{S}^{\mathrm{Disc}}(W_g^n)$, and in particular, $\mathcal{S}^{\mathrm{Disc}}(W_g^n) \simeq \mathcal{S}_\partial^{\mathrm{Disc}}(D^{2n})$ is not simply connected.

4. Topological mapping class groups

Combining the Weiss fibre sequence and convergence results for embedding calculus relative half the boundary, Theorem 3.16 implies that $\pi_0 \operatorname{Homeo}(M)$ is residually finite, whenever M is a smoothable closed manifold of dimension $d \geq 6$. This is the content of the current section. We begin by setting up the Wess fibre sequence, both in the smooth and the topological categories; embedding calculus allows us to infer some consequences on embeddings relative boundary conditions. We then use smoothing theory in order to describe the difference between the smooth and topological Weiss fibre sequences.

4.1. **The Weiss fibre sequence.** Let M be a smooth, closed 2-connected manifold of dimension $d \geq 6$, and fix a smooth embedding $D^d \hookrightarrow M$. In what follows, we denote by M° as the manifold $M \setminus \mathrm{Int}(D)$, with boundary diffeomorphic to S^{d-1}. We decompose the boundary sphere into upper and lower hemispheres as

$$S^{d-1} = D_+^{d-1} \cup D_-^{d-1}$$

We fix a collar $D_+^{d-1} \times I \hookrightarrow M^\circ$, and denote $P \coloneqq M^\circ \setminus (\mathrm{int}(D_+^{d-1} \times I))$. We furtermore denote D_-^{d-1} by $\partial/2$. For $\mathrm{Cat} \in \{O, \mathrm{Top}\}$, the parametrized isotopy extension theorem yields a fibre sequence

$$\mathrm{Aut}_\partial^{\mathrm{Cat}}(M^\circ \setminus \mathrm{int}(P)) \to \mathrm{Aut}_\partial^{\mathrm{Cat}}(M^\circ) \to \mathrm{Emb}_{\partial/2}^{\mathrm{Cat},\cong}(P, M^\circ)$$

where the base is the space of Cat-embeddings of P into M, relative $\partial/2$, which extend to a diffeomorphism of M°, and where $\mathrm{Aut}^{\mathrm{Cat}}$ denotes the space of automorphisms in the corresponding category. We observe that $M^\circ \setminus \mathrm{int}(P) \cong D_-^{d-1} \times I \cong D^d$. Finally, another application of parametrized isotopy extension and contractibility of spaces of neighborhood collars implies that the map

$$\mathrm{Emb}_{\partial/2}^{\cong,\mathrm{Cat}}(M^\circ, M^\circ) \to \mathrm{Emb}_{\partial/2}^{\cong,\mathrm{Cat}}(P, M^\circ)$$

is an equivalence, where the target is the space of Cat-embeddings, relative half the boundary, that extend to an automorphism of M°. We thus obtain a fibre sequence of E_1-spaces

$$\mathrm{Aut}_\partial^{\mathrm{Cat}}(D^d) \to \mathrm{Aut}_\partial^{\mathrm{Cat}}(M^\circ) \to \mathrm{Emb}_{\partial/2}^{\cong,\mathrm{Cat}}(M^\circ, M^\circ)$$

which upon delooping, yields the *Weiss fibre sequence*. It is perhaps noteworthy to mention that the Weiss fibre sequence can be delooped once, as in [Kup19]; this will however not be necessary for our purposes. Forgetting smooth structures yields a map of fibre sequences

$$\begin{tikzcd}
B\,\mathrm{Diff}_\partial(D^d) \arrow[r] \arrow[d] & B\,\mathrm{Diff}_\partial(M^\circ) \arrow[r] \arrow[d] & B\,\mathrm{Emb}_{\partial/2}^{\cong}(M^\circ, M^\circ) \arrow[d, "\Psi"] \\
* \simeq B\,\mathrm{Homeo}_\partial(D^d) \arrow[r] & B\,\mathrm{Homeo}_\partial(M^\circ) \arrow[r, "\simeq"] & B\,\mathrm{Emb}_{\partial/2}^{\cong,\mathrm{Top}}(M^\circ, M^\circ)
\end{tikzcd}$$

By the Alexander trick, we observe that the map $B\,\mathrm{Homeo}_\partial(M^\circ) \xrightarrow{\simeq} B\,\mathrm{Emb}_{\partial/2}^{\cong,\mathrm{Top}}(M^\circ, M^\circ)$ is a homotopy equivalence, which is a crucial observation to deduce results on homeomorphism groups via embedding calculus. We next study the fibre of the map Ψ; this is the subject of parametrized smoothing theory, relative half the boundary.

4.2. **Parameterized smoothing theory rel. half boundary.** Smoothing theory is the study of the space of smooth structures on a topological manifold. Given a smoothable manifold M with empty boundary, smoothing theory describes the fibre (over the identity) of the map $B\,\mathrm{Diff}(M) \to B\,\mathrm{Homeo}(M)$ as a space of sections of a bundle over M with fibres $\mathrm{Top}(d)/\mathrm{O}(d)$, and this space is amenable to computations using homotopical methods. The proof of the following can be extracted from the proof of [Kup19, Corollary 5.15].

Lemma 4.1. *Let W be a smooth compact manifold with boundary $\partial W \cong S^{d-1}$, and let $D^{d-1} \subset S^{d-1}$ denote the lower hemisphere. Let $\varphi : B\,\mathrm{Emb}_{\partial/2}^{\cong}(W) \to B\,\mathrm{Emb}_{\partial/2}^{\cong,\mathrm{Top}}(W)$ be the forgetful map. Then, the fibre over the identity is given by*

$$\mathrm{fib}_{\mathrm{id}}(\varphi) \simeq \Gamma_{\partial/2}(E \to W)$$

where the right hand side is a space of sections of a bundle over W with fibres equivalent to $\mathrm{Top}(d)/\mathrm{O}(d)$, relative a fixed section on the lower hemisphere.

Proof. By [Las76, Corollary 2], the natural inclusion maps of Cat-embeddings spaces into Cat-immersion spaces, for $\mathrm{Cat} \in \{\mathrm{Top}, O\}$, induce an equivalence on fibres

$$\mathrm{fib}_{\mathrm{id}}\left(\mathrm{Emb}_{\partial/2}^{\cong}(M^\circ, M^\circ) \to \mathrm{Emb}_{\partial/2}^{\cong,\mathrm{Top}}(M^\circ, M^\circ)\right) \simeq \mathrm{fib}_{\mathrm{id}}\left(\mathrm{Imm}_{\partial/2}^{\cong}(M^\circ, M^\circ) \to \mathrm{Imm}_{\partial/2}^{\cong,\mathrm{Top}}(M^\circ, M^\circ)\right)$$

where the fibre is taken over the identity. By immersion theory, the fibre on the right hand side can be identified with the fibre of a map

$$\Gamma_{\partial/2}(E^{O} \to M^{\circ}) \to \Gamma_{\partial/2}(E^{\mathrm{Top}} \to M^{\circ})$$

where $E^{\mathrm{Cat}} \to M^{\circ}$ is a bundle with fibres $\mathrm{Cat}(d)$, over some fixed section in the base. The result thus readily follows. ◻

As observed previously, one can show that the homotopy groups of these section spaces are finitely generated, at all choices of basepoints. However, one further thing can be said about its set of connected components and the fundamental group over each of them: they are indeed finite. First, we recall that by [KS77, p.246], we have the following isomorphisms

$$\pi_k\,(\mathrm{Top/O}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & k=3 \\ \Theta_k & \text{else} \end{cases}$$

where Θ_k are the groups of exotic spheres. We also note that the map

$$\mathrm{Top}(d)/\mathrm{O}(d) \to \mathrm{Top/O}$$

is $(d+2)$-connected, for $d \geq 5$, again by [KS77, p.246]. In particular, $\pi_k\mathrm{Top}(d)/\mathrm{O}(d)$ is finite for $k \leq d+1$. By obstruction theory, we have the following.

Proposition 4.2. *Let W be a compact d-manifold with boundary $\partial W \cong S^{d-1}$, and let $\partial/2 \coloneqq D^{d-1}_{-} \subset S^{d-1}$ be the lower hemisphere. Then, the space $\Gamma_{\partial/2}(E \to W)$ has finitely many path components, and for any choice of basepoint, its fundamental group is finite.*

Proof. The claim regarding path components follows readily from obstruction theory. For π_1, we fix a section $s \in \Gamma_{\partial/2}(E \to W)$ as a basepoint. We begin by observing that $(W, \partial/2)$ is equivalent, as a pair, to a finite, d-dimensional CW-pair, i.e. a finite relative CW-complex where all cells have dimension $\leq d$. Given $(X, \partial/2)$ a skeleton of $(W, \partial/2)$, attaching an n-cell yields the following fibre sequence by restriction

$$\Gamma_{\partial D^n}(E|D^n \to D^n) \to \Gamma_{\partial/2}(E|X \cup D^n \to X \cup D^n) \xrightarrow{res} \Gamma_{\partial/2}(E|X \to X)$$

where the fibre is taken over the restriction of the section s to $\Gamma_{\partial/2}(E|X \to X)$. The fibre of the above fibre sequence can be identified with $\Omega^n(\mathrm{Top}(d)/O(d))$, where $n \leq d$; thus, its fundamental group is again finite, on all possible choices of basepoints. The statement thus follows from finite induction, as on fundamental groups, the above fibre sequence exhibits $\pi_1\Gamma_{\partial/2}(E|X \cup D^n \to X \cup D^n)$ as an extension of a finite group by a finite group. ◻

4.3. **Embedding calculus rel. half boundary.** We set up embedding calculus to study embedding spaces of manifolds, fixing boundary conditions. This section follows closely the account of [Kup19, §3.3.2].

Fix P, a closed $(d-1)$-dimensional smooth manifold. We consider the topological category $\mathrm{Manf}_{d,P}$, consisting of smooth d-dimensional manifolds (not necessarily compact) with boundary diffeomorphic to P, and where morphisms are taken to be embeddings preserving the boundary identifications; we also view $\mathrm{Manf}_{d,P}$ as an ∞-category via coherent nerve. Inside $\mathrm{Manf}_{d,P}$, we can consider the collection of full subcategories $\mathrm{Disc}^{\leq k}_{d,P}$, for $k \in \mathbb{N} \cup \{\infty\}$, on objects of the form $(P \times [0,\infty)) \sqcup (\sqcup_{\ell}\mathbb{R}^d)$, for $\ell \leq k$. Yoneda followed by restriction defines a tower of functors

$$\mathrm{Manf}_{d,P} \to \mathrm{Psh}(\mathrm{Disc}^{\leq k}_{d,P})$$

which, on morphism spaces, yields the *rel ∂ embedding tower*, $\mathrm{Emb}_{\partial}(-,-) \to T_k\,\mathrm{Emb}_{\partial}(-,-)$. The same convergence theorem relative boundary also holds with the same codimension estimates, [GW99, Chapter 5] and [Kup19, §3.3.2].

The above setting is in fact well-suited for our purposes. Let M be a closed, smooth d-dimensional manifold, and as above, denote M° to be the manifold obtained from M by removing the interior of some embedded disc. We note that $\partial M^\circ \cong S^{d-1}$, which decomposes into $S^{d-1} = D^{d-1}_- \cup D^{d-1}_+$, where we let $\partial/2$ stand for the lower hemisphere. The space we are interested in is precisely $\mathrm{Emb}_{\partial/2}(M^\circ, M^\circ)$ of those self embeddings that are the identity on $\partial/2$. We let $M^* \coloneqq M^\circ \setminus D^{d-1}_+$.

Lemma 4.3. *In the above setting, M° is isotopy equivalent to M^*, relative $\partial/2$. We consequently obtain an equivalence*

$$\mathrm{Emb}_{\partial/2}(M^\circ, M^\circ) \simeq \mathrm{Emb}_{\partial}(M^*, M^*)$$

We observe that with the above equivalence, we have a method of attacking the embedding space by means of embedding calculus relative to the full boundary of M^*. We consequently obtain a tower $\{T_k \,\mathrm{Emb}_{\partial/2}(M^\circ, M^\circ)\}_{k\in\mathbb{N}}$ receiving maps from $\mathrm{Emb}_{\partial/2}(M^\circ, M^\circ)$.

Proposition 4.4. *If M is a 2-connected closed manifold, then the embedding calculus tower for $\mathrm{Emb}_{\partial/2}(M^\circ, M^\circ)$ converges.*

This is proved as follows: using a Morse theoretic argument, one can show that M admits a handle decomposition of dimension $\leq d-3$, relative $\partial/2$ [Kup19, Lemma 3.14]. Convergence of embedding calculus for embeddings relative half the boundary can be deduced from [KK24, thm 6.3], and the increasing connectivity of the layers of the tower can be deduced from Remark 6.11 (ii), *loc. cit.*

With the above observation in mind, we work towards showing Proposition 4.5 below.

Proposition 4.5. *For M a closed, 2-connected smooth manifold, and $k \in \mathbb{N}$, the group*

$$\pi_0 T_k \,\mathrm{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$$

is residually finite.

Proof. The following proof follows from the proofs of Theorem 3.7 and Theorem 3.16 above, with the modifications that we present here. We recall that $T_k \,\mathrm{Emb}_{\partial/2}(M^\circ, M^\circ)$ was defined as the space $T_k \,\mathrm{Emb}_\partial(M^*, M^*)$, where M^* is the manifold $M \setminus D^{d-1}_+$ with boundary $S^{d-1} \setminus D^{d-1}_+$, i.e. the endomorphism space in the category $\mathrm{Psh}(\mathrm{Disc}^{\leq k}_{d,\partial})$ of the object $\iota_k^* E_{M^*}$. Following the proof of Theorem 3.16, it suffices to consider residual finiteness in the case where we remember spin structures. In that setting, we fix once and for all a spin structure on a collar of $S^{d-1} \setminus D^{d-1}_+$, and endow M^* a spin structure extending the fixed spin structure. We now claim that pointwise, $\iota_k^* E^{\mathrm{Spin}}_{M^*}$ is a 1-connected finite space. To see this, we recall that objects of $\mathrm{Disc}^{\leq k}_{d,\partial}$ are of the form $(\partial \times [0,\infty)) \sqcup (\sqcup_\ell \mathbb{R}^d)$, where $\ell \leq k$. The restriction map

$$\mathrm{Emb}^{\mathrm{Spin}}_\partial(\partial \times [0,\infty) \sqcup (\sqcup_\ell \mathbb{R}^d), M^*) \to \mathrm{Emb}^{\mathrm{Spin}}(\sqcup_\ell \mathbb{R}^d, \mathrm{Int}(M^*))$$

is an equivalence, by the contractibility of the space of boundary collars, and the space on the right hand side is a simply connected, finite space as follows from Lemma 3.5. Furthermore, we consider the following diagram

$$\begin{tikzcd}
\mathrm{Psh}(\mathrm{Disc}^{\mathrm{Spin},\leq k}_{d,\partial}) \arrow[r, "①"] \arrow[d, "\iota^*_{k-1}"'] & \mathrm{Fun}([2], \mathrm{Psh}(\mathrm{Disc}^{\mathrm{Spin},=k}_{d,\partial})) \arrow[d, "③"] \\
\mathrm{Psh}(\mathrm{Disc}^{\mathrm{Spin},\leq k-1}_{d,\partial}) \arrow[r, "②"'] & \mathrm{Fun}([1], \mathrm{Psh}(\mathrm{Disc}^{\mathrm{Spin},=k}_{d,\partial}))
\end{tikzcd} \tag{3}$$

where

- ι^*_{k-1} is induced by the inclusion $\iota_{k-1} \colon \mathrm{Disc}^{\mathrm{Spin},\leq k-1}_{d,\partial} \hookrightarrow \mathrm{Disc}^{\leq k}_{d,\partial}$;

- The category $\mathrm{Disc}_{d,\partial}^{\mathrm{Spin},=k}$ consits of a single object $D_k^\partial := (\partial \times [0,\infty)) \sqcup (\sqcup_k \mathbb{R}^d)$, and where the space of morphisms is $\mathrm{Emb}_\partial^{\mathrm{Spin},\pi_0\text{-inj}}(D_k^\partial, D_k^\partial)$, which is equivalent to the group $\mathfrak{S}_k \ltimes \mathrm{Spin}(d)^k$.
- The functor ① sends $X \in \mathrm{Psh}(\mathrm{Disc}_{d,\partial}^{\mathrm{Spin},\leq k})$ to the commutative triangle

$$\begin{tikzcd} j^*(\iota_{k-1})_!\iota_{k-1}^*X(D_k^\partial) \arrow[r] \arrow[dr] & j^*X(D_k^\partial) \arrow[d] \\ & j^*(\iota_{k-1})_*\iota_{k-1}^*X(D_k^\partial) \end{tikzcd}$$

  and where $j\colon \mathrm{Disc}_{d,\partial}^{\mathrm{Spin}=k} \to \mathrm{Disc}_{d,\partial}^{\mathrm{Spin},\leq k}$ is the obvious inclusion functor.
- The functor ② sends $X \in \mathrm{Psh}(\mathrm{Disc}_{d,\partial}^{\mathrm{Spin},\leq k-1})$ to the arrow

$$j^*(\iota_{k-1})_!X(D_k^\partial) \to j^*(\iota_{k-1})_*X(D_k^\partial)$$

- The funtor ③ is induced by the map $[1] \to [2]$, $0 \mapsto 0$ and $1 \mapsto 2$.

By [KK24, thm 4.15], we conclude that diagram (3) is a pullback square. Furthermore, for $n \in \mathbb{N}$, we consider the restricted Postnikov truncation of $X \in \mathrm{Psh}(\mathrm{Disc}_{d,\partial}^{\mathrm{Spin},\leq k})$, as constructed in Construction 3.8, namely as the composite

$$\widetilde{\tau_{\leq n}}X\colon \left(\mathrm{Disc}_{d,\partial}^{\mathrm{Spin},\leq k}\right)^{\mathrm{op}} \xrightarrow{X/T_{k-1}X} \mathrm{Ar}(\mathcal{S}) \xrightarrow{\mathrm{MP}_n} \mathcal{S}$$

where the first functor sends D to the arrow $X(D) \to (\iota_{k-1})_*\iota_{k-1}^*X(D)$, and where the second functor is the n-th stage Moore-Postnikov truncation. As per the proof of Theorem 3.16, residual finiteness of the above group follows from the π_0-injectivity of the map

$$\mathrm{Map}_{\mathrm{Psh}(\mathrm{Disc}_{d,\partial}^{\mathrm{Spin},\leq k})}(E_{M^*}, E_{M^*}) \to \mathrm{Map}_{\mathrm{Psh}(\mathrm{Disc}_{d,\partial}^{\mathrm{Spin},\leq k})}(E_{M^*}, \Phi^s \circ E_{M^*})$$

induced by composition with the map $\eta\colon E_{M^*} \to \Phi^s \circ E_{M^*}$. By diagram (3), we have an equivalence

$$\mathrm{Lift}_{\mathrm{Psh}(\mathrm{Disc}_{d,\partial}^{\mathrm{Spin},\leq k})}\left(\begin{tikzcd} & \widetilde{\tau_{n+1}}E_{M^*} \arrow[d] \\ E_{M^*} \arrow[r] \arrow[ur, dashed] & \widetilde{\tau_{\leq n}}E_{M^*} \end{tikzcd}\right) \xrightarrow{\simeq} \mathrm{Map}^{G_k}\left(\begin{tikzcd} j^*(\iota_{k-1})_!\iota_{k-1}^*E_{M^*} \arrow[r, dashed] \arrow[d] & j^*\widetilde{\tau_{\leq n+1}}E_{M^*} \arrow[d] \\ j^*E_{M^*} \arrow[r, dashed] & j^*\widetilde{\tau_{\leq n}}E_{M^*} \end{tikzcd}\right)$$

where $G_k := \mathfrak{S}_k \ltimes \mathrm{Spin}(d)^k$, and where the mapping space on the right hand side is the mapping space in the arrow category $\mathrm{Fun}(\Delta^1, \mathrm{Psh}(BG_k))$ of spaces with an action of G_k. We observe that the pair $(j^*E_{M^*}, j^*(\iota_{k-1})_!\iota_{k-1}^*E_{M^*})$ is a finite, free G_k-CW pair, as can be seen from [KK24, Proposition 5.15]. The remainder of the obstruction theoretic proof of Theorem 3.7 carries through. □

As a consequence of Proposition 4.5, we obtain the following residual finiteness result

Proposition 4.6. *The group $\pi_0 \mathrm{Emb}_{\partial/2}^{\cong}(M^\circ, M^\circ)$ is residually finite.*

Proof. By Proposition 4.4, the embedding calculus map is an equivalence, and in particular induces a group isomorphism

$$\pi_0 \mathrm{Emb}_{\partial/2}^{\cong}(M^\circ, M^\circ) \xrightarrow{\cong} \pi_0 T_\infty \mathrm{Emb}_{\partial/2}^{\cong}(M^\circ, M^\circ)$$

We identify $T_\infty \mathrm{Emb}_{\partial/2}^{\cong}(M^\circ, M^\circ)$ with $\lim_{k\in\mathbb{N}} T_k \mathrm{Emb}_{\partial/2}^{\cong}(M^\circ, M^\circ)$. By the Milnor exact sequence, we obtain a short exact sequence of groups

$$1 \to \lim_k^1 \pi_1 T_k \mathrm{Emb}_{\partial/2}^{\cong}(M^\circ, M^\circ) \to \pi_0 T_\infty \mathrm{Emb}_{\partial/2}^{\cong}(M^\circ, M^\circ) \to \lim_{k\in\mathbb{N}} \pi_0 T_k \mathrm{Emb}_{\partial/2}^{\cong}(M^\circ, M^\circ) \to 1$$

By the proof of the convergence of the embedding tower, we know that the connectivity of the fibres L_k over the identity of the maps strictly increases with k (see for instance, [KK24, Remark 6.11 (ii)]). Consequently, the system $\{\pi_1 T_k \operatorname{Emb}^{\cong}(M^\circ, M^\circ)\}_{k\in\mathbb{N}}$ eventually consists of isomorphisms, so that in particular it satisfies the Mittag-Leffler condition. As a consequence, $\lim^1$ vanishes, and the above short exact sequence yields an isomorphism

$$\pi_0 T_\infty \operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ) \xrightarrow{\cong} \lim_{k\in\mathbb{N}} T_k \operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$$

As the class of residually finite groups is closed under inverse limits, the claim follows. □

Remark 4.7. The above proof generalizes to the situation where we consider a smooth, compact 2-connected manifold W of dimension $d \geq 5$, where $\partial W \neq \emptyset$, with the following technical condition. Let $\partial_0 W \subset \partial W$ be a compact, codimension 0 submanifold of the boundary, such that $\partial_1 W \hookrightarrow W$ is 2-connected, where $\partial_1 W := \partial W \setminus \partial_0 W$. In that case, the same proof of Proposition 4.6 implies that $\pi_0 T_k \operatorname{Emb}^{\cong}_{\partial_0}(W, W)$ is residually finite, and [KK24, Remark 6.11 (ii)] yields convergence of embedding calculus rel ∂_1. In particular, we conclude that $\pi_0 \operatorname{Emb}^{\cong}_{\partial_0}(W, W)$ is residually finite. This is developped in more details in the proof of Theorem 4.16 below.

4.4. **Residual finiteness of topological mapping class groups.** We now combine the three previous sections to show that topological mapping class groups are indeed residually finite.

Theorem 4.8. *For any 2-connected smoothable closed topological manifold M of dimension $d \geq 6$, the group $\pi_0 \operatorname{Homeo}_\partial(M^\circ) \cong \pi_1 B \operatorname{Emb}^{\cong,\mathrm{Top}}_{\partial/2}(M^\circ)$ is residually finite.*

Proof. We consider the smooth and topological Weiss fibre sequences, and the forgetful map between them

$$\begin{tikzcd}
 & & \operatorname{fib}(\Psi) \simeq \Gamma_{\partial/2}(\xi_M \to M) \arrow[d, "\iota"] \\
B\operatorname{Diff}_\partial(D^d) \arrow[r] \arrow[d] & B\operatorname{Diff}_\partial(M^\circ) \arrow[r] \arrow[d] & B\operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ) \arrow[d, "\Psi"] \\
* \simeq B\operatorname{Homeo}_\partial(D^d) \arrow[r] & B\operatorname{Homeo}_\partial(M^\circ) \arrow[r, "\simeq"'] & B\operatorname{Emb}^{\cong,\mathrm{Top}}_{\partial/2}(M^\circ, M^\circ)
\end{tikzcd}$$

By Lemma 4.1 and Proposition 4.2, the fibre of the map Ψ has a finite set of components, and on each of them, π_1 is a finite group. The long exact sequence on homotopy groups thus yields an exact sequence

$$\pi_1\Gamma_\partial(\xi_M) \xrightarrow{\iota_*} \pi_1 B\operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ) \xrightarrow{\Psi_*} \pi_1 B\operatorname{Emb}^{\cong,\mathrm{Top}}_{\partial/2}(M^\circ, M^\circ) \xrightarrow{\partial} \pi_0\Gamma_{\partial/2}(\xi_M)$$

where exactness on ∂ is exactness as pointed sets. We note that $\pi_1 B\operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$ is residually finite by Proposition 4.6. Furthermore, $\operatorname{Im}(\iota_*) = \ker \Psi_*$ is a finite, normal subgroup of $\pi_1 B\operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$. Thus, by Lemma 2.9, it follows that

$$\pi_1 B\operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)\big/\operatorname{Im}(\iota_*)$$

is residually finite. Ψ_* embedds this group into a finite index subgroup of $\pi_1 B\operatorname{Emb}^{\cong,\mathrm{Top}}_{\partial/2}(M^\circ, M^\circ)$. Thus, by Lemma 2.10, it follows that $\pi_1 B\operatorname{Emb}^{\cong,\mathrm{Top}}_{\partial/2}(M^\circ, M^\circ)$ is residually finite. The Alexander trick yields an equivalence

$$B\operatorname{Homeo}_\partial(M^\circ) \xrightarrow{\simeq} B\operatorname{Emb}^{\cong,\mathrm{Top}}_{\partial/2}(M^\circ, M^\circ)$$

and thus identifies their fundamental groups, implying the desired result. □

In order to now study $\pi_0 \operatorname{Homeo}(M)$ from $\pi_0 \operatorname{Homeo}_\partial(M^\circ)$, we use isotopy extension to obtain the fibre sequence

$$\operatorname{Homeo}_\partial(M^\circ) \to \operatorname{Homeo}(M) \to \operatorname{Emb}^{\mathrm{Top}}(D^d, M)$$

We note that $\operatorname{Emb}^{\mathrm{Top}}(D^d, M) \simeq \operatorname{Fr}^{\mathrm{Top}}(M)$; consequently, $\pi_0 \operatorname{Emb}^{\mathrm{Top}}(D^d, M) \cong \mathbb{Z}/2\mathbb{Z}$, and on each basepoint, $\pi_1 \operatorname{Emb}(D^d, M) \cong \mathbb{Z}/2\mathbb{Z}$. Running the long exact sequence on homotopy groups, we obtain an exact sequence

$$\mathbb{Z}/2\mathbb{Z} \to \pi_0 \operatorname{Homeo}_\partial(M^\circ) \to \pi_0 \operatorname{Homeo}(M) \to \mathbb{Z}/2\mathbb{Z}$$

As a corollary of Lemmas 2.9 and 2.10, we obtain the following

Theorem 4.9. *For M a 2-connected closed, smooth manifold, the group $\pi_0 \operatorname{Homeo}(M)$ is residually finite.*

Remark 4.10. It readily follows that the same holds for $\pi_0 \operatorname{Homeo}^+(M)$.

4.5. **Arithmeticity of topological mapping class groups.** We follow [Kup24b, §2] for all conventions regarding arithmetic groups. We consider two equivalence relations on the class of groups. The first is generated by

(i) isomorphisms;
(ii) passing to finite index subgroups;
(iii) taking quotients by finite, normal subgroups

Two groups in the same equivalence class of the above relation are said to be *commensurable up to finite kernel*. Dropping condition (iii), the equivalence relation thus obtained is the classical notion of *commensurability*. In [Sul77, thm 13.3], Sullivan shows that smooth mapping class groups of closed, simply connected smooth manifolds of dimension $d \geq 5$ are commensurable up to finite kernel to an arithmetic group.

Theorem 4.11 (Sullivan)**.** *For a closed, smooth simply connected manifold M of dimension $d \geq 5$, the group $\pi_0 \operatorname{Diff}(M)$ is commensurable up to finite kernel to an arithmetic group.*

To go from the smooth category to the topological, we use smoothing theory as follows

Lemma 4.12. *For a smoothable, closed topological manifold M of dimension $d \geq 5$, $\pi_0 \operatorname{Homeo}(M)$ and $\pi_0 \operatorname{Diff}(M)$ are commensurable up to finite kernel.*

Proof. Using Kirby-Siebenmann's bundle theorem [KS77, Essay V, §3], we have a fibre sequence

$$F \to B\operatorname{Diff}(M) \to B\operatorname{Homeo}(M)$$

where the fibre F is given as a collection of components of the space $\Gamma(\xi_M)$ of sections of a bundle over M with fibre $\operatorname{Top}(d)/O(d)$. By obstruction theory, and using the fact that $\pi_k(\operatorname{Top}(d)/O(d))$ is a finite group for $k \leq d+1$, it follows that $\pi_0\Gamma(\xi_M)$ is finite, and for all basepoints, $\pi_1\Gamma(\xi_M)$ is a finite group. Let $\operatorname{Homeo}^{\mathrm{sm}}(M)$ be the subgroup of $\operatorname{Homeo}(M)$ of those homeomorphisms isotopic to a diffeomorphism. Using the long exact sequence on homotopy groups associated to the above fibre sequence, we have a short exact sequence

$$1 \to \ker p \to \pi_0 \operatorname{Diff}(M) \xrightarrow{p} \pi_0 \operatorname{Homeo}^{\mathrm{sm}}(M) \to 1$$

where $\ker p \leq \pi_1 F$ is a finite normal subgroup of $\pi_0 \operatorname{Diff}(M)$. Thus,

$$\pi_0 \operatorname{Homeo}^{\mathrm{sm}}(M) \cong \pi_0 \operatorname{Diff}(M) / \ker p$$

and consequently, $\pi_0 \operatorname{Homeo}^{\mathrm{sm}}(M)$ is commensurable up to finite kernel to $\pi_0 \operatorname{Diff}(M)$. Again using the same long exact sequence as above, we notice that the quotient

$$\pi_0 \operatorname{Homeo}(M) / \pi_0 \operatorname{Homeo}^{\mathrm{sm}}(M) \subset \pi_0\Gamma(\xi_M)$$

is a finite set; thus, $\pi_0 \operatorname{Homeo}^{\mathrm{sm}}(M)$ is a finite index subgroup of $\pi_0 \operatorname{Homeo}(M)$. We thus conclude that $\pi_0 \operatorname{Diff}(M)$ and $\pi_0 \operatorname{Homeo}(M)$ are commensurable up to finite kernel. $\square$

As a corollary, we obtain

Corollary 4.13. *Let M be a smoothable closed manifold of dimension $d \geq 5$. Then, $\pi_0 \operatorname{Homeo}(M)$ is commensurable up to finite kernel to an arithmetic group.*

However, as observed in [KRW20, Remark, p.470], commensurability and commensurability up to finite kernel agree among residually finite groups. Indeed, if G is a residually finite group, and K is a finite, normal subgroup, then G and G/K are commensurable. To see this, we observe that for any $g \neq 1$ in G, there exists a finite group F_g and a group morphism $\varphi_g \colon G \to F_g$ such that $\varphi_g(g) \neq 1 \in F_g$. In particular, for a finite normal subgroup $K \leq G$, we can find a finite group F and a group morphism $\varphi \colon G \to F$ such that the product map $p \times \varphi \colon G \to (G/K) \times F$ is injective. The image of the above group morphism is isomorphic to G, and is a finite index subgroup of $(G/K) \times F$. Finally, G/K is evidently a finite index subgroup of $(G/K) \times F$, and as a consequence, G is commensurable to G/K. Combining Theorem 4.9 and Corollary 4.13, we obtain the following

Theorem 4.14. *For M a smoothable, 2-connected closed topological manifold of dimension $d \geq 6$, $\pi_0 \operatorname{Homeo}(M)$ is an arithmetic group.*

Remark 4.15. Similar considerations also show that $\pi_0 \operatorname{Homeo}^+(M)$ is arithmetic.

4.6. **Compact manifolds with boundary.** All of our results have so far focused on the case of closed manifolds: the strategy was to delete the interior of a disc inside a closed manifold M, and study embedding calculus relative boundary for this manifold. We study here generalizations to the case of compact manifolds with non-empty boundaries. Let W be a compact, d-dimensional manifold of dimension $d \geq 5$, such that $\partial W \neq \emptyset$. We furthermore assume that W is 2-connected. The aim of this remark is to show the following

Theorem 4.16. *Let W be a smoothable, 2-connected compact manifold of dimension $d \geq 5$, with $\partial W \neq \emptyset$. Then, $\pi_0 \operatorname{Homeo}_\partial(W)$ is residually finite.*

Proof. We fix a smooth embedding $D^{d-1} \hookrightarrow \partial W$, and denote $\partial_1 W \coloneqq \partial W \setminus D^{d-1}$. The aim is to show that $\pi_0 \operatorname{Emb}^{\cong}_{\partial_1}(W, W)$, the group of path components of the space of embeddings fixing ∂_1 that are isotopic to diffeormophisms, is residually finite. Indeed, the smooth and topological Weiss fibre sequences yields a commutative diagram

$$\begin{tikzcd}
B\operatorname{Diff}_\partial(D^d) \arrow[r] \arrow[d] & B\operatorname{Diff}_\partial(W) \arrow[r] \arrow[d] & B\operatorname{Emb}^{\cong}_{\partial_1}(W,W) \arrow[d] \\
B\operatorname{Homeo}_\partial(D^d) \simeq * \arrow[r] & B\operatorname{Homeo}_\partial(W) \arrow[r, "\simeq"'] & B\operatorname{Emb}^{\cong,\mathrm{Top}}_{\partial_1}(W,W)
\end{tikzcd}$$

The fibre of the rightmost vertical map is understood by smoothing theory, and has finite π_0, and finite fundamental groups over each component (Lemma 4.1 and Proposition 4.2). Thus, residual finiteness of $\pi_0 \operatorname{Emb}^{\cong}_{\partial_1}(W, W)$ implies the residual finiteness of $\pi_0 \operatorname{Emb}^{\cong,\mathrm{Top}}_{\partial_1}(W, W) \cong \pi_0 \operatorname{Homeo}_\partial(W)$, by Lemmas 2.9 and 2.10.

We now focus on showing that $\pi_0 \operatorname{Emb}^{\cong}_{\partial_1}(W, W)$ is residually finite. This follows the same embedding calculus strategy, and we first consider the case where we remember spin structures. Fix a $\operatorname{Spin}(d)$ structure on a collar of ∂_1, and consider embedding calculus rel. boundary $\operatorname{Psh}(\operatorname{Disc}^{\mathrm{Spin},\leq k}_{d,\partial_1})$ where $\operatorname{Disc}^{\mathrm{Spin},\leq k}_{d,\partial_1}$ is the category consisiting of objects of the form $\partial_1 \times [0, \infty) \times \sqcup_\ell \mathbb{R}^d$, for $\ell \leq k$, along with choices of spin structures extending the fixed spin structure on $\partial_1 \times [0, \infty)$, and embeddings rel. ∂_1 respecting the spin structures.

By [KK24, thm 4.20], we obtain a pullback square of ∞-categories

$$\begin{tikzcd}
\mathrm{Psh}(\mathrm{Disc}^{\mathrm{Spin},\leq k}_{d,\partial_1}) \arrow[r, "①"] \arrow[d, "\iota^*_{k-1}"'] & \mathrm{Fun}([2],\mathrm{Psh}(\mathrm{Disc}^{\mathrm{Spin},=k}_{d,\partial_1})) \arrow[d, "③"] \\
\mathrm{Psh}(\mathrm{Disc}^{\mathrm{Spin},\leq k-1}_{d,\partial_1}) \arrow[r, "②"'] & \mathrm{Fun}([1],\mathrm{Psh}(\mathrm{Disc}^{\mathrm{Spin},=k}_{d,\partial_1}))
\end{tikzcd} \tag{4}$$

where ①, ②, ③ are as in the proof of Proposition 4.5. We denote $T_k\,\mathrm{Emb}^{\mathrm{Spin},\cong}_{\partial_1}(W,W)$ as the space of automorphisms of $\iota^*_k E_{W^*}$ in the category $\mathrm{Psh}(\mathrm{Disc}^{\mathrm{Spin},\leq k}_{d,\partial_1})$, where $W^* \coloneqq W \setminus D^{d-1}$. Pointwise, this presheaf has the homotopy type of a simply connected, finite CW-complex (Lemma 3.5). In order to run the same obstruction theoretic proof as in Theorem 3.7, we observe that diagram (4) yields an equivalence

$$\mathrm{Lift}_{\mathrm{Psh}(\mathrm{Disc}^{\mathrm{Spin},\leq k}_{d,\partial})}\left(\begin{tikzcd} & \widetilde{\tau_{n+1}}E_{W^*} \arrow[d] \\ E_{W^*} \arrow[r] \arrow[ur, dashed] & \widetilde{\tau_{\leq n}}E_{W^*} \end{tikzcd}\right) \xrightarrow{\simeq} \mathrm{Map}^{G_k}\left(\begin{tikzcd} j^*(\iota_{k-1})_!\iota^*_{k-1}E_{W^*} \arrow[r, dashed] \arrow[d] & j^*\widetilde{\tau_{\leq n+1}}E_{W^*} \arrow[d] \\ j^*E_{W^*} \arrow[r, dashed] & j^*\widetilde{\tau_{\leq n}}E_{W^*} \end{tikzcd}\right)$$

G_k in the above denotes the group $\mathfrak{S}_k \ltimes \mathrm{Spin}(d)$, and the space on the right hand side is the space of morphism in the arrow category $\mathrm{Fun}(\Delta^1, \mathrm{Psh}(BG_k))$ of spaces with an action of G_k, where j^* is the restriction functor along the inclusion $BG_k \to \mathrm{Disc}^{\mathrm{Spin},\leq k}_{d,\partial_1}$. The pair $(j^*E_{W^*}, j^*(\iota_{k-1})E_{W^*})$ is equivalent to a finite, G_k-CW pair, as follows from [KK24, Proposition 5.15]. The remainder of the obstruction theoretic proof of Theorem 3.7 follows, and implies that $\pi_0 T_k\,\mathrm{Emb}^{\mathrm{Spin},\cong}_{\partial_1}(W^*,W^*)$ is residually finite, for all $k \in \mathbb{N}$. Convergence of embedding calculus, along with increasing connectivity of the layer ([KK24, Remark 6.11(ii)]) implies the same for $\pi_0\,\mathrm{Emb}^{\mathrm{Spin}}_{\partial_1}(W^*,W^*)$. The proof of Theorem 3.16 allows us to forget the spin structures, and implies that $\pi_0\,\mathrm{Emb}^{\cong}_{\partial_1}(W^*,W^*)$ is residually finite, as required. □

4.7. **Finite residuals of smooth mapping class groups.** As mentioned previously, the smooth mapping class group $\pi_0\,\mathrm{Diff}(M)$ of a smooth manifold M may fail to be residually finite, as was shown in [KRW20]. In the situation of the counterexample in *loc. cit.*, we fix an embedding $D^{2n} \hookrightarrow W^n_g \coloneqq \#_g(S^n \times S^n)$, where $n = 5 \pmod 8$, for $g \geq 5$. Extension by the identity yields a morphism

$$\pi_0\,\mathrm{Diff}_\partial(D^{2n}) \longrightarrow \pi_0\,\mathrm{Diff}(W^n_g)$$

which embeds the subgroup $\mathrm{bP}_{2n+2} \leq \Theta_{2n+1} \cong \pi_0\,\mathrm{Diff}_\partial(D^{2n})$ (non-trivial for these values of n by [KM63]) to the finite residual of $\pi_0\,\mathrm{Diff}(W^n_g)$. The following two results imply that this is the only phenomenon preventing residual finiteness. We split it into two statements, one regarding smooth mapping class groups relative boundary, and one for smooth mapping class groups of closed manifolds.

Lemma 4.17. *Let M be a closed, 2-connected smooth manifold of dimension $d \geq 6$, and denote $M^\circ \coloneqq M \setminus \mathrm{Int}(D^d)$, for some disc $D^d \subset M$. Fix another embedding $D^d \hookrightarrow \mathrm{Int}(M^\circ)$. Then, $\mathrm{fr}(\pi_0\,\mathrm{Diff}_\partial(M^\circ)) \subseteq \mathrm{Im}(\pi_0\,\mathrm{Diff}_\partial(D^d) \to \pi_0\,\mathrm{Diff}_\partial(M^\circ))$, where $\mathrm{fr}(\pi_0\,\mathrm{Diff}_\partial(M^\circ))$ is the finite residual, and where the map $\pi_0\,\mathrm{Diff}_\partial(D^d) \to \pi_0\,\mathrm{Diff}_\partial(M^\circ)$ is extension by identity.*

Proof. Let M° denote $M \setminus \mathrm{Int}(D^d)$, where $D^d \hookrightarrow M$ is a fixed embedded disc. We also fix an embedding $D^d \hookrightarrow \mathrm{Int}(M^\circ)$. The proof boils down to showing that the quotient group

$$G \coloneqq \pi_0\,\mathrm{Diff}_\partial(M^\circ) \Big/ \Big(\mathrm{Im}(\pi_0\,\mathrm{Diff}_\partial(D^d) \to \pi_0\,\mathrm{Diff}_\partial(M^\circ)\Big)$$

is a residually finite group: indeed, if that is the case, we take $g \in \pi_0\,\mathrm{Diff}_\partial(M^\circ)$ which does not lie in the image of $\pi_0\,\mathrm{Diff}_\partial(D^d)$. This g maps to a non-trivial element in the quotient group, and if the quotient group G is residually finite, this element can be detected by a finite group, hence is not in $\mathrm{fr}(\pi_0 \operatorname{Diff}_\partial(M^\circ))$. To show that G is residually finite, we consider the smooth Weiss fibre sequence

$$B\operatorname{Diff}_\partial(D^d) \to B\operatorname{Diff}_\partial(M^\circ) \to B\operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$$

which, on homotopy groups, yields an exact sequence

$$\pi_0 \operatorname{Diff}_\partial(D^d) \to \pi_0 \operatorname{Diff}_\partial(M^\circ) \to \pi_0 \operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$$

As a consequence, we observe that the image of the morphism $\pi_0 \operatorname{Diff}_\partial(D^d) \to \pi_0 \operatorname{Diff}_\partial(M^\circ)$ is a normal subgroup, and hence the quotient is indeed a group. By Proposition 4.6, we know that $\pi_0 \operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$ is residually finite. Thus, the quotient group G, which is identified as a subgroup of $\pi_0 \operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ)$, is consequently itself a residually finite group, as subgroups of residually finite groups are residually finite. □

We now study the case of the mapping class groups of M, $\pi_0 \operatorname{Diff}(M)$.

Theorem 4.18. *Let M be a smooth, closed 2-connected manifold of dimension $d \geq 6$, and fix an embedded disc $D^d \subset M$. Then,*

(i) $\mathrm{fr}(\pi_0 \operatorname{Diff}^+(M)) \subseteq \mathrm{Im}(\pi_0 \operatorname{Diff}_\partial(D^d) \to \pi_0 \operatorname{Diff}^+(M))$, *where the map* $\pi_0 \operatorname{Diff}_\partial(D^d) \to \pi_0 \operatorname{Diff}^+(M)$ *is given by extension by the identity;*

(ii) $\mathrm{fr}(\pi_0 \operatorname{Diff}(M)) \subseteq \mathrm{Im}(\pi_0 \operatorname{Diff}_\partial(D^d) \to \pi_0 \operatorname{Diff}(M))$, *where the map* $\pi_0 \operatorname{Diff}_\partial(D^d) \to \pi_0 \operatorname{Diff}(M)$ *is given by extension by the identity*

Proof. We begin by showing (i). For the proof, we make use of the following three fibre sequences below, where M is a smooth, 2-connected closed manifold, and where M° denotes, as usual, M with the interior of an embedded disc deleted; this new disc is taken to be disjoint from the fixed disc $D^d \subset M$. These fibre sequences are the following:

$$B\operatorname{Diff}_\partial(D^d) \to B\operatorname{Diff}_\partial(M^\circ) \to B\operatorname{Emb}^{\cong}_{\partial/2}(M^\circ, M^\circ) \tag{5}$$

$$B\operatorname{Diff}_\partial(M^\circ) \to B\operatorname{Diff}^+(M, *) \to BSO(d) \tag{6}$$

$$M \to B\operatorname{Diff}^+(M, *) \to B\operatorname{Diff}^+(M) \tag{7}$$

The fibre sequence (5) is the Weiss fibre sequence, and fibre sequences (6) and (7) are as in [Kra21, equation (6) and (10)]; $\operatorname{Diff}^+(M, *)$ denotes the group of orientation preserving diffeomorphisms that fix a basepoint of M, and the morphism $\operatorname{Diff}_\partial(M^\circ) \to \operatorname{Diff}^+(M, *)$ is given by extension by the identity. As $BSO(d)$ is 1-connected, and M is 2-connected, it follows that the morphism $B\operatorname{Diff}_\partial(M) \to B\operatorname{Diff}^+(M)$ is surjective on π_1; therefore, the morphism $\pi_0 \operatorname{Diff}_\partial(M^\circ) \to \pi_0 \operatorname{Diff}^+(M)$ is surjective. Additionally, we observe that the image of $\pi_0 \operatorname{Diff}_\partial(D^d) \cong \Theta_{d+1}$ in $\pi_0 \operatorname{Diff}_\partial(M^\circ)$ is normal, as can be seen from the exact sequence on π_1 for the fibre sequence (5). Thus, the image of $\pi_0 \operatorname{Diff}_\partial(D^d)$ in $\pi_0 \operatorname{Diff}^+(M)$ is also normal, as surjective group homormophisms send normal subgroups to normal subgroups. We thus obtain a group morphism

$$\frac{\pi_0 \operatorname{Diff}_\partial(M^\circ)}{\Theta_{d+1}} \to \frac{\pi_0 \operatorname{Diff}^+(M)}{\Theta_{d+1}}$$

where by the quotient by Θ_{d+1}, we mean the quotient by the image of Θ_{d+1} in each of the respective groups. By Lemma 4.17, we know that the group on the left hand side is residually finite. We claim that the above morphism is an isomorphism. Surjectivity follows from the fact that the group morphism $\pi_0 \operatorname{Diff}_\partial(M^\circ) \to \pi_0 \operatorname{Diff}^+(M)$ is surjective prior to taking quotients. For injectivity, we consider an element $[g] \in \frac{\pi_0 \operatorname{Diff}_\partial(M^\circ)}{\Theta_{d+1}}$, represented by an isotopy class of diffeomorphisms in $\pi_0 \operatorname{Diff}_\partial(M^\circ)$, that is mapped to the trivial element in $\frac{\pi_0 \operatorname{Diff}^+(M)}{\Theta_{d+1}}$. Thus, the diffeomorphism $g \cup_\partial \mathrm{id}$, given by extension by the identity applied to g, is in the image of Θ_{d+1}. However, we have a commutative diagram

$$\begin{tikzcd} \Theta_{d+1} \cong \pi_0 \operatorname{Diff}_\partial(D^d) \arrow[r] \arrow[dr] & \pi_0 \operatorname{Diff}_\partial(M^\circ) \arrow[d] \\ & \pi_0 \operatorname{Diff}^+(M) \end{tikzcd}$$

where all morphisms are given by extensions by the identity. Thus, g is itself in the image of Θ_{d+1} in $\pi_0 \operatorname{Diff}_\partial(M^\circ)$, so that $[g] = 0$ in the quotient. Consequently, $\frac{\pi_0 \operatorname{Diff}^+(M)}{\Theta_{d+1}}$ is residually finite, and (i) follows.

We now show (ii). The non-oriented analogue of fibre sequence (7) holds, namely we have a fibre sequence

$$M \to B \operatorname{Diff}(M, *) \to B \operatorname{Diff}(M) \tag{8}$$

where $\operatorname{Diff}(M, *)$ is the group of diffeomorphisms preserving a basepoint $* \in M$. As M is assumed 2-connected, the group morphism $\pi_0 \operatorname{Diff}(M, *) \to \pi_0 \operatorname{Diff}(M)$ is an isomorphism. As a consequence, we observe that $\operatorname{Im}(\Theta_{d+1} \to \pi_0 \operatorname{Diff}(M))$ is a normal subgroup of $\pi_0 \operatorname{Diff}(M)$: indeed, any diffeomorphism of M is isotopic to a diffeomorphism that fixes a point, and hence fixes a neighborhood disc around that point setwise. As a consequence, we have a group morphism

$$\frac{\pi_0 \operatorname{Diff}_\partial(M^\circ)}{\Theta_{d+1}} \to \frac{\pi_0 \operatorname{Diff}(M)}{\Theta_{d+1}}$$

where we use the same convention regarding the quotient by Θ_{d+1} as in the proof of (i). The above group morphism is injective, for the same argument above; however, it may fail to be surjective. Nonetheless, it embedds $\frac{\pi_0 \operatorname{Diff}_\partial(M^\circ)}{\Theta_{d+1}}$ as a finite index subgroup of $\frac{\pi_0 \operatorname{Diff}(M)}{\Theta_{d+1}}$. As the domain of the map is residually finite, Lemma 2.10 implies the target is also residually finite, and (ii) follows. □

As a corollary, it follows that $\pi_0 \operatorname{Diff}(M)$ and $\pi_0 \operatorname{Diff}^+(M)$ are residually finite, for M a closed smooth 2-connected manifold of dimension $d \geq 5$, whenever Θ_{d+1} is the trivial group. By [BHHM20, Corollary 1.3], this holds for $d \in \{5, 11, 55, 60\}$, for instance.

Appendix A. Diagrams of finite completion

The content of this appendix builds up to a proof of Theorem 2.21 and Corollary 2.22. We wish to thank Thomas Blom for the main ideas presented here.

Definition A.1 (Tensoring/cotensoring)**.** We consider the following two functors

$$\begin{aligned} \mathcal{S} \times \mathrm{Pro}(\mathcal{S}_\pi) &\to \mathrm{Pro}(\mathcal{S}_\pi) \\ (K, X) &\mapsto K \otimes X \coloneqq \varinjlim_K X \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}^{\mathrm{op}} \times \mathrm{Pro}(\mathcal{S}_\pi) &\to \mathrm{Pro}(\mathcal{S}_\pi) \\ (K, X) &\mapsto X^K \coloneqq \lim_K X \end{aligned}$$

both of which are taken over the constant diagrams.

The following couple of lemmas consist of showing that the above tensoring-cotensoring define a $\mathrm{Pro}(\mathcal{S}_\pi)$-enriched mapping space between $\widehat{X}$ and $\widehat{Y}$, whenever X and Y are finite, simply connected spaces, which recovers $\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{X}, \widehat{Y})$ upon materialisation.

Lemma A.2. *Let K be a space, and X a profinite space. Then,*

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{K}, X) \simeq \mathrm{Mat}(X^K)$$

Proof. By adjunction, we have an equivalence

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{K}, X) \simeq \mathrm{Map}_{\mathcal{S}}(K, \mathrm{Mat}(X))$$

In spaces, we may write K as a colimit over the constant diagram

$$K \simeq \varinjlim_K *$$

which in turns gives an equivalence

$$\mathrm{Map}_{\mathcal{S}}(K, \mathrm{Mat}(X)) \simeq \lim_K \mathrm{Map}_{\mathcal{S}}(*, \mathrm{Mat}(X)) \simeq \lim_K \mathrm{Mat}(X)$$

As materialisation is a right adjoint, it preserves limits, and we consequently obtain the desired equivalence

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{K}, X) \simeq \mathrm{Mat}(X^K)$$

□

Lemma A.3. *Let $X \in \mathrm{Pro}(\mathcal{S}_\pi)$, and fix K a simply connected, finite space. Then, $K \otimes X \simeq \widehat{K} \times X$, naturally in K and X.*

Proof. By definition, $X \coloneqq \lim_i X_i$, where X_i is a cofiltered system of π-finite spaces, and where the limit is meant as a formal limit. Then, since cofiltered limits commute with finite colimits, we get the equivalence

$$\begin{aligned} K \otimes X &\simeq \varinjlim_K \lim_i X_i \\ &\simeq \lim_i \varinjlim_K X_i \simeq \lim_i K \otimes X_i \end{aligned}$$

We now check that for a fixed π-finite space X_i, $K \otimes X_i \simeq X_i \times \widehat{K}$ where we view X_i as a constant Pro-object in $\mathcal{S}_\pi$. By the Yoneda lemma, it suffices to show that for all $Y \in \mathrm{Pro}(\mathcal{S}_\pi)$, we have a natural equivalence

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(K \otimes X_i, Y) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{K} \times X_i, Y)$$

which we may further restrict to the case where Y is a π-finite space, viewed as a constant Pro-object in $\mathcal{S}_\pi$. We have the following chain of natural equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(K \otimes X_i, Y) &\overset{①}{\simeq} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(X_i, Y^K) \\
&\overset{②}{\simeq} \mathrm{Map}_{\mathcal{S}}(X_i, \mathrm{Mat}(Y^K)) \\
&\overset{③}{\simeq} \mathrm{Map}_{\mathcal{S}}(X_i, \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{K}, Y)) \\
&\overset{④}{\simeq} \mathrm{Map}_{\mathcal{S}}(X_i, \mathrm{Map}_{\mathcal{S}}(K, Y)) \\
&\overset{⑤}{\simeq} \mathrm{Map}_{\mathcal{S}}(X_i \times K, Y) \\
&\overset{⑥}{\simeq} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{K \times X_i}, Y)
\end{aligned}$$

where

- ① is given by the tensoring-cotensoring adjunction;
- ② follows from the fact that $X_i = \widehat{X_i}$, as X_i is a π-finite space;
- ③ follows from Lemma A.2;
- ④ is obtained from the adjunction $\widehat{(-)} \dashv \mathrm{Mat}$ adjunction, along with the fact that $\mathrm{Mat}(\widehat{Y}) \simeq Y$, since Y is π-finite;
- ⑤ follows from the usual tensor-hom adjunction in $\mathcal{S}$;
- ⑥ is again obtained from the adjunction $\widehat{(-)} \dashv \mathrm{Mat}$

As a consequence, we obtain an equivalence $K \otimes X_i \simeq \widehat{K \times X_i}$. We conclude by showing that the natural map

$$\widehat{K \times X_i} \longrightarrow \widehat{K} \times \widehat{X_i} = \widehat{K} \times X_i$$

is an equivalence. First, observe that $\pi_0(K \times X_i) \cong \pi_0 X_i$ is a finite set, as K is a connected space. As (space) finite completion induces the set profinite completion on π_0, it follows that $\pi_0(\widehat{K \times X_i}) \cong \pi_0(\widehat{K} \times X_i) \cong \pi_0 X_i$. As the functor $\mathrm{Mat}\colon \mathrm{Pro}(\mathcal{S}_\pi) \longrightarrow \mathcal{S}$ is conservative [Lur18, thm E.3.1.6], it follows that the above map is an equivalence of profinite spaces if it induces an isomorphism on all homotopy groups, and all choices of basepoints. Now, we note that $K \times X_i$ satisfies Sullivan's conditions for [Sul74, thm 3.1] (??), and we see that (omitting basepoint notation) the map induces the morphism

$$\Phi^g(\pi_n(K \times X_i)) \longrightarrow \Phi^g \pi_n K \times \Phi^g \pi_n X_i$$

Since group profinite completion preserves finite products, it follows that the above is an isomorphism, for all $n \in \mathbb{N}$, and thus that $\widehat{K \times X_i} \simeq \widehat{K} \times X_i$. Combining with the above, we conclude that $K \otimes X_i \simeq \widehat{K} \times X_i$, and thus that $K \otimes X \simeq \widehat{K} \times X$, as desired. □

While showing the above lemma, we also show a separate result of interest; we record it as a separate lemma and reprove it, and point out the similarity between the proofs.

Lemma A.4. *Let $X, Y \in \mathcal{S}$ be arbitrary spaces. Then, there are equivalences*

$$X \otimes \widehat{Y} \simeq \widehat{X \times Y} \simeq Y \otimes \widehat{X}$$

that are natural in both variables.

Proof. It suffices to show that we have a natural equivalence $X \otimes \widehat{Y} \simeq \widehat{X \times Y}$. Using the Yoneda lemma, we restrict ourselves to showing that for all $Z \in \mathrm{Pro}(\mathcal{S}_\pi)$, there are natural equivalences

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(X \otimes \widehat{Y}, Z) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{X \times Y}, Z)$$

As Z can be written as a limit of π-finite spaces $\{Z_\alpha\}$, and as we are mapping into a limit, we may further restrict to the case $Z \in \mathcal{S}_\pi$. We consider the following chain of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(X \otimes \widehat{Y}, Z) &\overset{①}{\simeq} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{Y}, Z^X) \\
&\overset{②}{\simeq} \mathrm{Map}_{\mathcal{S}}(Y, \mathrm{Mat}(Z^X)) \\
&\overset{③}{\simeq} \mathrm{Map}_{\mathcal{S}}(Y, \mathrm{Map}_{\mathcal{S}}(X, Z)) \\
&\overset{④}{\simeq} \mathrm{Map}_{\mathcal{S}}(X \times Y, Z) \\
&\overset{⑤}{\simeq} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{X \times Y}, Z)
\end{aligned}$$

where

- ① follows from the tensoring-cotensoring adjunction;
- ② follows from the adjunction $\widehat{(-)} \dashv \mathrm{Mat}$;
- ③ follows from Lemma A.2;
- ④ follows from the tensor-hom adjunction in $\mathcal{S}$;
- ⑤ follows from the adjunction $\widehat{(-)} \dashv \mathrm{Mat}$, along with the fact that Z is π-finite

□

We combine the previous two lemmas to obtain an interesting result; it will not play an important role in what follows, but is worth mentioning.

Lemma A.5. *Let $X \in \mathcal{S}$ be an arbitrary space, and let $Y \in \mathcal{S}^{\mathrm{fin}}_{>1}$ be a finite, 1-connected space. The comparison map*

$$\widehat{X \times Y} \to \widehat{X} \times \widehat{Y}$$

is an equivalence of profinite spaces.

Proof. By Lemma A.4, we have an equivalence

$$Y \otimes \widehat{X} \simeq \widehat{X \times Y}$$

By Lemma A.3, we have an equivalence

$$Y \otimes \widehat{X} \simeq \widehat{Y} \times \widehat{X}$$

and by chasing through the proofs of the above two lemmas, we see that the equivalence is indeed given by the natural map

$$\widehat{X \times Y} \to \widehat{X} \times \widehat{Y}$$

□

Using Lemma A.3, one can construct a composition morphism on the cotensoring.

Construction A.6. Let X, Y and Z be simply connected, finite spaces. We recall that we have a composition morphism

$$-\circ-\colon\ \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{X}, \widehat{Y}) \times \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{Y}, \widehat{Z}) \longrightarrow \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{X}, \widehat{Z})$$

Similarly, we know that $\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{X}, \widehat{Y}) \simeq \mathrm{Mat}(\widehat{Y}^X)$, where $\widehat{Y}^X$ is the cotensoring. We construct a map

$$-\widehat{\circ}-\colon \widehat{Y}^X \times \widehat{Z}^Y \to \widehat{Z}^X$$

as follows. We first observe that we have an equivalence

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{Y}^X \times \widehat{Z}^Y, \widehat{Z}^X) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(X \otimes (\widehat{Y}^X \times \widehat{Z}^Y), \widehat{Z})$$

and using this equivalence, we construct the desired composition as an element of the mapping space on the right hand side. This is done as follows

$$\begin{aligned} X \otimes (\widehat{Y}^X \times \widehat{Z}^Y) &\overset{①}{\simeq} \widehat{X} \times \widehat{Y}^X \times \widehat{Z}^Y \\ &\xrightarrow{②} \widehat{Y} \times \widehat{Z}^Y \\ &\xrightarrow{③} \widehat{Z} \end{aligned}$$

where ① follows from Lemma A.3, and where ② and ③ are given by the corresponding image of the identity morphism via the following equivalence

$$\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{Y}^X, \widehat{Y}^X) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(X \otimes \widehat{Y}^X, \widehat{Y}) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{X} \times \widehat{Y}^X, \widehat{Y})$$

We have thus constructed, for simply connected finite spaces X, Y and Z, a map

$$-\widehat{\circ}-: \widehat{Y}^X \times \widehat{Z}^Y \to \widehat{Z}^X$$

We now aim at generalizing the above composition morphism to the case of profinite completion of space-valued functors. This will be done using the universal properties of endomorphism objects of Lurie [Lur17, §4.7.1]. The setup is as follows: for $\mathcal{C}$ an arbitrary ∞-category, and a functor $\mathcal{F} : \mathcal{C} \to \mathcal{S}^{\mathrm{fin}}_{>1}$, we denote $\widehat{\mathcal{F}}$ to be the composite

$$\widehat{\mathcal{F}} : \mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{S}^{\mathrm{fin}}_{>1} \xrightarrow{\widehat{(-)}} \mathrm{Pro}(\mathcal{S}_\pi)$$

We then show that there is an E_1-algebra $\widehat{\mathcal{F}}^{\mathcal{F}}$ in $\mathrm{Pro}(\mathcal{S}_\pi)$ which, upon materialisation, is equivalent to the space of endomorphisms

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(\widehat{\mathcal{F}}, \widehat{\mathcal{F}})$$

as spaces, and such that both compositions agree.

We begin by constructing the above object, first as an object of $\mathrm{Pro}(\mathcal{S}_\pi)$, and show that its materialisation is equivalent to the mapping space. We later use Lurie's theory of endomorphism objects to check that the object defined promotes to an E_1-algebra in $\mathrm{Pro}(\mathcal{S}_\pi)$, and compare the multiplications of $\mathrm{Mat}(\widehat{\mathcal{F}}^{\mathcal{F}})$ to that of $\mathrm{End}(\widehat{\mathcal{F}})$.

Corollary A.7. *Let $\mathcal{C}$ be an arbitrary ∞-category, and let $\mathcal{F} : \mathcal{C} \to \mathcal{S}$, $\mathcal{G} : \mathcal{C} \to \mathrm{Pro}(\mathcal{S}_\pi)$ be two functors. Then, there exists an object in $\mathrm{Pro}(\mathcal{S}_\pi)$, denoted $\mathcal{G}^{\mathcal{F}}$, such that*

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(\widehat{\mathcal{F}}, \mathcal{G}) \simeq \mathrm{Mat}(\mathcal{G}^{\mathcal{F}})$$

Proof. We rewrite the mapping space in the category $\mathrm{Fun}(\mathcal{C}, \mathrm{Pro}(\mathcal{S}_\pi))$ as a limit over the twisted arrow category of $\mathcal{C}$ ([GHN20, Prop. 5.1])

$$\begin{aligned} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(\widehat{\mathcal{F}}, \mathcal{G}) &\simeq \lim_{(a\to b)\in \mathrm{Tw}(\mathcal{C})} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{\mathcal{F}(a)}, \mathcal{G}(b)) \\ &\simeq \lim_{(a\to b)\in \mathrm{Tw}(\mathcal{C})} \mathrm{Mat}(\mathcal{G}(b)^{\mathcal{F}(a)}) \end{aligned}$$

Defining $\mathcal{G}^{\mathcal{F}} \in \mathrm{Pro}(\mathcal{S}_\pi)$ as

$$\mathcal{G}^{\mathcal{F}} \coloneqq \lim_{(a\to b)\in \mathrm{Tw}(\mathcal{C})} \mathcal{G}(b)^{\mathcal{F}(a)}$$

the claim follows from the fact that Mat is a right adjoint, hence preserves all limits. □

We now proceed as follows: we fix $\mathcal{F} : \mathcal{C} \to \mathcal{S}^{\mathrm{fin}}_{>1}$, a functor from an arbitrary category $\mathcal{C}$ to the category of finite, simply connected spaces; we denote by $\widehat{\mathcal{F}}$ the composite

$$\mathcal{C} \xrightarrow{\mathcal{F}} \mathcal{S}^{\mathrm{fin}}_{>1} \xrightarrow{\widehat{(-)}} \mathrm{Pro}(\mathcal{S}_\pi)$$

We consider $\widehat{\mathcal{F}}^{\mathcal{F}}$ as the object constructed in the above proof, namely

$$\widehat{\mathcal{F}}^{\mathcal{F}} \coloneqq \lim_{(a\to b)\in \mathrm{Tw}(\mathcal{C})} \widehat{\mathcal{F}(b)}^{\mathcal{F}(a)}$$

By Corollary A.7, we already know that

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(\widehat{\mathcal{F}},\widehat{\mathcal{F}}) \simeq \mathrm{Mat}(\widehat{\mathcal{F}}^{\mathcal{F}})$$

We furthermore observe that the space on the left hand side admits a structure of E_1-algebra in $\mathcal{S}$ (with respect to the cartesian monoidal structure), where the multiplication is given by composition of natural transformations $\widehat{\mathcal{F}} \Rightarrow \widehat{\mathcal{F}}$. The aim in what follows is to first show that $\widehat{\mathcal{F}}^{\mathcal{F}}$ promotes to an E_1-algebra in $\mathrm{Pro}(\mathcal{S}_\pi)$ (with respect to the cartesian monoidal structure), and then show that the multiplication map $\widehat{\mu}\colon \widehat{\mathcal{F}}^{\mathcal{F}} \times \widehat{\mathcal{F}}^{\mathcal{F}} \to \widehat{\mathcal{F}}^{\mathcal{F}}$, obtained as part of the datum of the E_1-algebra structure on $\widehat{\mathcal{F}}^{\mathcal{F}}$, agrees with the usual composition multiplication on $\mathrm{End}(\mathcal{F})$ upon materialisation; note that Mat, being a right adjoint, in particular preserves finite products. This requires the technology of [Lur17, §4.7.1]. In what follows, we give a brief summary of the required results.
Let $\mathcal{C}$ be a monoidal ∞-category, and let $\mathcal{M}$ be an ∞-category left-tensored over $\mathcal{C}$. Given an object $M \in \mathcal{M}$, an action of an object $C \in \mathcal{C}$ on M is precisely the datum of a map

$$C \otimes M \to M$$

in $\mathcal{M}$. The endomorphism object of $\mathrm{End}(M)$ is then defined to be universal among objects acting on M in the following sense: if for any $C \in \mathcal{C}$, we have an equivalence

$$\mathrm{Map}_{\mathcal{C}}(C, \mathrm{End}(M)) \simeq \mathrm{Map}_{\mathcal{M}}(C \otimes M, M)$$

which is natural in C. Should such an object exits, Lurie shows it admits the structure of an E_1-algebra in $\mathcal{C}$ ([Lur17, Corollary 4.7.1.40]).

In our setting, we consider the monoidal category $(\mathrm{Pro}(\mathcal{S}_\pi), \times)$, and view $\mathrm{Fun}(\mathcal{C}, \mathrm{Pro}(\mathcal{S}_\pi))$ as left-tensored over $\mathrm{Pro}(\mathcal{S}_\pi)$, via

$$(X, F) \mapsto X \times F,\ X \in \mathrm{Pro}(\mathcal{S}_\pi),\ F \in \mathrm{Fun}(\mathcal{C}, \mathrm{Pro}(\mathcal{S}_\pi))$$

Lemma A.8. *Let $\mathcal{C}$ be an arbitrary ∞-category, and fix a functor $\mathcal{F} : \mathcal{C} \to \mathcal{S}^{\mathrm{fin}}_{>1}$ landing in the full subcategory of finite, simply connected spaces. Then, $\widehat{\mathcal{F}}^{\mathcal{F}}$ is an endomorphism object for $\widehat{\mathcal{F}}$ in* $\mathrm{Pro}(\mathcal{S}_\pi)$.

Proof. We have to check the universal property for endomorphism objects. Namely, we need to check that for all $X \in \mathrm{Pro}(\mathcal{S}_\pi)$, we have equivalences

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(X \times \widehat{\mathcal{F}},\widehat{\mathcal{F}}) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(X, \widehat{\mathcal{F}}^{\mathcal{F}})$$

that are natural in X. For this, we consider the following list of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(X \times \widehat{\mathcal{F}},\widehat{\mathcal{F}}) &\overset{①}{\simeq} \lim_{(a\to b)\mathrm{Tw}(\mathcal{C})} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(X \times \widehat{\mathcal{F}(a)}, \widehat{\mathcal{F}(b)}) \\
&\overset{②}{\simeq} \lim_{(a\to b)\mathrm{Tw}(\mathcal{C})} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(X \otimes \mathcal{F}(a), \widehat{\mathcal{F}(b)}) \\
&\overset{③}{\simeq} \lim_{(a\to b)\mathrm{Tw}(\mathcal{C})} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(X, \widehat{\mathcal{F}(b)}^{\mathcal{F}(a)}) \\
&\overset{④}{\simeq} \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(X, \lim_{(a\to b)\mathrm{Tw}(\mathcal{C})} \widehat{\mathcal{F}(b)}^{\mathcal{F}(a)}) \\
&= \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(X, \widehat{\mathcal{F}}^{\mathcal{F}})
\end{aligned}$$

where

- ① follows from the limit description of mapping spaces in functor categories;

- ② follows from Lemma A.3;
- ③ follows from the tensoring-cotensoring adjunction;
- ④ follows after pulling the limit into the mapping space

As all the above equivalences are natural in X, it follows that $\widehat{\mathcal{F}}^{\mathcal{F}}$ is an endomorphism object for $\widehat{\mathcal{F}}$, and in particular promotes to an E_1-algebra in $(\mathrm{Pro}(\mathcal{S}_\pi), \times)$. □

We now stick to the situation of Lemma A.8. We fix a functor $\mathcal{F} : \mathcal{C} \to \mathcal{S}^{\mathrm{fin}}_{>1}$ from an ∞-category $\mathcal{C}$ to the ∞-category of finite, 1-connected spaces. Using Lemma A.8, we obtain an evaluation morphism

$$\mathrm{ev} \colon \widehat{\mathcal{F}}^{\mathcal{F}} \times \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$$

obtained explicitly as the adjoint of the identity map via the equivalence described in the proof, namely

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(\widehat{\mathcal{F}}^{\mathcal{F}} \times \widehat{\mathcal{F}}, \widehat{\mathcal{F}}) \simeq \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{\mathcal{F}}^{\mathcal{F}}, \widehat{\mathcal{F}}^{\mathcal{F}})$$

The multiplication morphism of the E_1-algebra $\widehat{\mathcal{F}}^{\mathcal{F}}$ is explicitly described as the adjoint of the composition

$$\widehat{\mathcal{F}}^{\mathcal{F}} \times \widehat{\mathcal{F}}^{\mathcal{F}} \times \widehat{\mathcal{F}} \xrightarrow{\mathrm{id}\times\mathrm{ev}} \widehat{\mathcal{F}}^{\mathcal{F}} \times \widehat{\mathcal{F}} \xrightarrow{\mathrm{ev}} \widehat{\mathcal{F}}$$

which we henceforth denote

$$\widehat{\mu} \colon \widehat{\mathcal{F}}^{\mathcal{F}} \times \widehat{\mathcal{F}}^{\mathcal{F}} \to \widehat{\mathcal{F}}^{\mathcal{F}}$$

We denote $\mathrm{End}(\widehat{\mathcal{F}})$ as the space of endomorphisms

$$\mathrm{End}(\widehat{\mathcal{F}}) \coloneqq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(\widehat{\mathcal{F}}, \widehat{\mathcal{F}})$$

We note that the above is indeed an endomorphism object of $\widehat{\mathcal{F}}$, with respect to the following tensoring: we consider the monoidal ∞-category $(\mathcal{S}, \times)$, and view $\mathrm{Fun}(\mathcal{C}, \mathrm{Pro}(\mathcal{S}_\pi))$ as left-tensored over $\mathcal{S}$ via:

$$(X, \mathcal{G}) \mapsto X \otimes \mathcal{G}, X \in \mathcal{S}, \mathcal{G} \in \mathrm{Fun}(\mathcal{C}, \mathrm{Pro}(\mathcal{S}_\pi))$$

Similarly to the proof of Lemma A.8, we have natural equivalences

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(X \otimes \widehat{\mathcal{F}}, \widehat{\mathcal{F}}) \simeq \mathrm{Map}_{\mathcal{S}}(X, \mathrm{End}(\widehat{\mathcal{F}})) \tag{9}$$

exhibiting $\mathrm{End}(\widehat{\mathcal{F}})$ as the endomorphism object of $\widehat{\mathcal{F}}$ in $\mathcal{S}$. The composition

$$- \circ - \colon \mathrm{End}(\widehat{\mathcal{F}}) \times \mathrm{End}(\widehat{\mathcal{F}}) \to \mathrm{End}(\widehat{\mathcal{F}})$$

thus occurs as the adjoint of applying twice the action map

$$\mathrm{End}(\widehat{\mathcal{F}}) \otimes \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$$

given by the adjoint of the identity $\mathrm{id} \colon \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$ in equation 9. Consequently, to compare the multiplication maps, it suffices to compare the action maps. The counit of the adjunction $\widehat{(-)} \dashv \mathrm{Mat}$ yields a map

$$\mathrm{counit} \colon \widehat{\mathrm{Mat}(\widehat{\mathcal{F}}^{\mathcal{F}})} \to \widehat{\mathcal{F}}^{\mathcal{F}}$$

We consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(\widehat{\mathcal{F}}^{\mathcal{F}} \times \widehat{\mathcal{F}}, \widehat{\mathcal{F}}) \arrow[r, dash, "\simeq", "①"'] \arrow[d, "-\circ(\mathrm{counit}\times\mathrm{id})"'] & \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{\mathcal{F}}^{\mathcal{F}}, \widehat{\mathcal{F}}^{\mathcal{F}}) \arrow[d, "-\circ(\mathrm{counit})"'] \arrow[dd, bend left=60, "\mathrm{Mat}"] \\
\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(\widehat{\mathrm{Mat}(\widehat{\mathcal{F}}^{\mathcal{F}})} \times \widehat{\mathcal{F}}, \widehat{\mathcal{F}}) \arrow[r, dash, "②", "\simeq"'] \arrow[d, dash, "\simeq"', "④"] & \mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{\mathrm{Mat}(\widehat{\mathcal{F}}^{\mathcal{F}})}, \widehat{\mathcal{F}}^{\mathcal{F}}) \arrow[d, dash, "\simeq"', "③"] \\
\mathrm{Map}_{\mathrm{Fun}(\mathcal{C},\mathrm{Pro}(\mathcal{S}_\pi))}(\mathrm{End}(\widehat{\mathcal{F}}) \otimes \widehat{\mathcal{F}}, \widehat{\mathcal{F}}) \arrow[r, dash, "\simeq", "⑤"'] & \mathrm{Map}_{\mathcal{S}}(\mathrm{End}(\widehat{\mathcal{F}}), \mathrm{End}(\widehat{\mathcal{F}}))
\end{tikzcd}$$

where

- ① and ② are the natural equivalences constructed in the proof of Lemma A.8;
- ③ follows from the adjunction $\widehat{(-)} \dashv \mathrm{Mat}$, and Corollary A.7;
- ④ is obtained by combining the equivalences

$$\mathrm{End}(\widehat{\mathcal{F}}) \otimes \widehat{\mathcal{F}} \simeq \mathcal{F} \otimes \widehat{\mathrm{End}(\widehat{\mathcal{F}})} \simeq \widehat{\mathrm{Mat}(\widehat{\mathcal{F}}^{\mathcal{F}})} \times \widehat{\mathcal{F}}$$

which follow from Lemmas A.3 and A.4

- ⑤ follows from equation 9

With the above diagram at hand, we see that the action map $\mathrm{ev}\colon \widehat{\mathcal{F}}^{\mathcal{F}} \times \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$ exhibiting $\widehat{\mathcal{F}}^{\mathcal{F}}$ as an endomorphism object of $\widehat{\mathcal{F}}$ in $\mathrm{Pro}(\mathcal{S}_\pi)$, and which is adjoint to the identity in $\mathrm{Map}_{\mathrm{Pro}(\mathcal{S}_\pi)}(\widehat{\mathcal{F}}^{\mathcal{F}}, \widehat{\mathcal{F}}^{\mathcal{F}})$, is mapped by materialisation to the action map $\mathrm{End}(\widehat{\mathcal{F}}) \otimes \widehat{\mathcal{F}} \to \widehat{\mathcal{F}}$, exhibiting $\mathrm{End}(\widehat{\mathcal{F}})$ as an endomorphism object of $\widehat{\mathcal{F}}$ in $\mathcal{S}$, and which is adjoint to the identity in $\mathrm{Map}_{\mathcal{S}}(\mathrm{End}(\widehat{\mathcal{F}}), \widehat{\mathcal{F}})$. As a consequence, the two multiplication morphisms on $\mathrm{End}(\widehat{\mathcal{F}}) \simeq \mathrm{Mat}(\widehat{\mathcal{F}}^{\mathcal{F}})$, namely the usual composition $- \circ -$ and $\mathrm{Mat}(\widehat{\mu})$ agree.

Combining the results of the section, we have shown the following

Lemma A.9. *Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{S}^{\mathrm{fin}}_{>1}$. Then, we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{Mat}(\widehat{\mathcal{F}}^{\mathcal{F}}) \times \mathrm{Mat}(\widehat{\mathcal{F}}^{\mathcal{F}}) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Mat}(\widehat{\mu})"'] & \mathrm{End}(\widehat{\mathcal{F}}) \times \mathrm{End}(\widehat{\mathcal{F}}) \arrow[d, "- \circ -"] \\
\mathrm{Mat}(\widehat{\mathcal{F}}^{\mathcal{F}}) \arrow[r, "\simeq"'] & \mathrm{End}(\widehat{\mathcal{F}})
\end{tikzcd}$$

We observe that the techinques described above are better aimed at showing that $\pi_0\mathrm{End}(\widehat{\mathcal{F}})$ admits the structure of a monoid in profinite spaces. The following lemma serves as a bridge to conclude that $\pi_0\mathrm{Aut}(\widehat{\mathcal{F}})$, i.e. the group of units of the above monoid, admits the structure of a profinite group compatible with that of its monoid.

Lemma A.10. *Let M be a monoid in $\mathrm{Pro}(\mathcal{S}_\pi)$, i.e. a topological monoid whose underlying topological space is a Stone space. Then, $u(M)$, its group of units, admits the structure of a profinite group via the subspace topology.*

Proof. We begin by observing that the category of profinite monoids, i.e. pro-objects in the category $\mathrm{Mon}^{\mathrm{fin}}$, is equivalent to the category $\mathrm{Prof}(\mathrm{Mon})$, the category of topological monoids with underlying topological space a compact, Hausdorff, totally disconnected space; this follows from [Joh82, Chapter VI, §2.9]. More precisely, the functor $\mathrm{Pro}(\mathrm{Mon}^{\mathrm{fin}}) \to \mathrm{Prof}(\mathrm{Mon})$, sending a cofiltered system of finite monoids to its limit, where we endow the limit with the profinite topology, is an equivalence of categories. The anologous statement in the case of profinite groups is classical, but a proof can also be found in *loc.cit.* Consider the adjunction $\iota \dashv u$:

$$\begin{tikzcd}
\mathrm{Grp} \arrow[r, hook, "\iota"] & \mathrm{Mon} \arrow[l, bend left, "u"]
\end{tikzcd}$$

where Grp is the category of discrete groups and group homomorphisms, Mon is the category of discrete monoids and monoid homomorphisms, ι is the inclusion functor and u is the group of units functor. The above adjunction restricts to the analogous one on finite groups and monoids. Applying $\mathrm{Pro}\colon \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$, we obtain a diagram

$$\begin{tikzcd}
\mathrm{Pro}(\mathrm{Grp}) \arrow[r, "\mathrm{Pro}(\iota)"] & \mathrm{Pro}(\mathrm{Mon}) \arrow[l, bend left, "\mathrm{Pro}(u)"]
\end{tikzcd}$$

which can also be seen to be an adjunction, as ι itself preserves all limits. As a consequence, we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Prof}(\mathrm{Mon}) \simeq \mathrm{Pro}(\mathrm{Mon}^{\mathrm{fin}}) \arrow[r, "\mathrm{Pro}(u)"] \arrow[d] & \mathrm{Pro}(\mathrm{Grp}^{\mathrm{fin}}) \simeq \mathrm{Prof}(\mathrm{Grp}) \arrow[d] \\ \mathrm{Mon} \arrow[r, "u"'] & \mathrm{Grp} \end{tikzcd}$$

which in particular implies that given a profinite monoid M, the group of units of its underlying discrete monoid lifts to profinite groups via the above diagram. □

We are now ready to show Theorem 2.21.

Proof of Theorem 2.21. The goal is to show that

$$G \coloneqq \pi_0 \operatorname{Map}^{\simeq}_{\operatorname{Fun}(\mathcal{C},\operatorname{Pro}(\mathcal{S}_\pi))}(\widehat{\mathcal{F}}, \widehat{\mathcal{F}})$$

admits the structure of a profinite group. Let $M \coloneqq \pi_0 \operatorname{Map}_{\operatorname{Fun}(\mathcal{C},\operatorname{Pro}(\mathcal{S}_\pi))}(\widehat{\mathcal{F}}, \widehat{\mathcal{F}})$, and observe that G is the group of units of M. Lemma A.10 gives therefore a reduction to showing that M is a profinite monoid. The latter fact follows readily from Lemma A.9. □

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Paper B

FRAMED CONFIGURATION SPACES AND EXOTIC SPHERES

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Abstract. We determine when an exotic sphere Σ of dimension $d \not\equiv 1 \pmod 4$ can be detected through the homotopy type of its truncated $\mathcal{D}$isc-presheaf. The latter records the diagram of framed configuration spaces of bounded cardinality in Σ with natural point-forgetting and -splitting maps between them. Our proof involves three ingredients that could be of independent interest: a gluing result for $\mathcal{D}$isc-presheaves of manifolds divided into two codimension zero submanifolds, a version of Atiyah duality in the context of $\mathcal{D}$isc-presheaves, and a computation of the finite residual of the mapping class group of the connected sums $\sharp^g(S^{2k+1} \times S^{2k+1})$.

Contents

1. Introduction

How "much" of a closed d-dimensional manifold is seen by the homotopy types of its configuration spaces? There are several ways to make this question precise, and to some of them partial answers have been given [Lev95, AK04, LS05, AS22, KK24a, KK24b]. In this work we answer an instance of this question in the case of exotic spheres.

To explain the result, we write $\mathcal{M}\mathrm{an}_d$ for the ∞-category with smooth manifolds M as objects and spaces of smooth embeddings $\mathrm{Emb}(M,N)$ as morphisms. This has a full subcategory $\mathcal{D}\mathrm{isc}_d \subseteq \mathcal{M}\mathrm{an}_d$ spanned by the manifolds $S \times \mathbf{R}^d$ for finite sets S. To a smooth d-manifold M, we can associate a presheaf E_M on $\mathcal{D}\mathrm{isc}_d$ with values in the ∞-category of spaces $\mathcal{S}$, given by

$$\mathcal{D}\mathrm{isc}_d^{\mathrm{op}} \ni S \times \mathbf{R}^d \overset{E_M}{\longmapsto} \mathrm{Emb}(S \times \mathbf{R}^d, M) \in \mathcal{S}, \tag{1}$$

and this construction extends—via postcomposition—to a functor out of $\mathcal{M}\mathrm{an}_d$

$$\mathcal{M}\mathrm{an}_d \ni M \overset{E}{\longmapsto} E_M \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}_d)$$

with values in the ∞-category $\mathrm{PSh}(\mathcal{D}\mathrm{isc}_d) \coloneqq \mathrm{Fun}(\mathcal{D}\mathrm{isc}_d^{\mathrm{op}}, \mathcal{S})$ of presheaves. The individual values of the presheaf E_M at objects $S \times \mathbf{R}^d$ are equivalent to the *framed configuration spaces*,

$$\mathrm{Emb}(S \times \mathbf{R}^d, M) \simeq F_S^{\mathrm{fr}}(M) \coloneqq \left\{(m_s, T_{m_s}M \overset{\phi_s}{\cong} \mathbf{R}^d)_{s \in S} \in \mathrm{Fr}(TM)^S \mid m_i \neq m_j \text{ for } i \neq j\right\}, \tag{2}$$

so $E_M \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}_d)$ encodes, roughly speaking, the homotopy types of all framed configuration spaces in M together with natural maps between them (e.g. the value of E_M at $(\iota \times \mathrm{id}_{\mathbf{R}^d})\colon S \times \mathbf{R} \hookrightarrow S' \times \mathbf{R}^d$ for an injection $\iota\colon S \hookrightarrow S'$ corresponds under (2) to forgetting some of the points). The equivalence class of the presheaf $E_M \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}_d)$ is a diffeomorphism-invariant of M from which many other invariants can be extracted, including the *factorisation*

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homology of M with coefficients in any framed E_d-algebra [Sal01, AF15, Lur17] or the *embedding calculus tower* for spaces of embeddings into or out of M [Wei99, BdBW13]. In fact, for $d \geq 5$, there is still no known example of two closed non-diffeomorphic d-manifolds M and N such that E_M and E_N are equivalent in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}_d)$[1].

There are several variants of the presheaf E_M one can consider, e.g. one for topological manifolds M based on a variant of the ∞-category $\mathcal{D}\mathrm{isc}_d$ involving topological embeddings, or one for manifolds M equipped with a tangential structure: for instance, if M comes equipped with an orientation one can consider the analogue $\mathcal{D}\mathrm{isc}_d^+$ of $\mathcal{D}\mathrm{isc}_d$ where one restricts to orientation-preserving embeddings, and the presheaf $E_M \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}_d^+)$ defined as in (1) but using spaces of orientation-preserving embeddings. As another variant, instead of considering presheaves on the full ∞-category $\mathcal{D}\mathrm{isc}_d$, one may for fixed $k \geq 1$ restrict to the full subcategory $\mathcal{D}\mathrm{isc}_{d,\leq k} \subseteq \mathcal{D}\mathrm{isc}_d$ spanned by $S \times \mathbf{R}^d$ for $|S| \leq k$ and consider the presheaf $E_M \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k})$ obtained from E_M by restriction; this is called the *k-truncated $\mathcal{D}$isc-presheaf* of M. In terms of diagrams of framed configuration spaces, passing to the truncated setting amounts to imposing an upper bound on the number of points.

The main result. In this work, we study the question which homotopy d-spheres can be distinguished by their truncated $\mathcal{D}\mathrm{isc}^+$-presheaves. Recall that a *homotopy d-sphere* is a closed smooth d-manifold Σ that is homotopy equivalent to a standard d-sphere, and thus—by the solution to the topological Poincaré conjecture—also homeomorphic to it. To state our main result, we write $M \sharp N$ for the connected sum of two oriented d-manifolds M and N, and $\overline{M}$ for M equipped with the opposite orientation.

Theorem A. *For oriented homotopy d-spheres Σ_0 and Σ_1 with $d \not\equiv 1 \pmod 4$ and $1 < k < \infty$, we have $E_{\Sigma_0} \simeq E_{\Sigma_1}$ in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k}^+)$ if and only if $\Sigma_0 \sharp \overline{\Sigma}_1$ bounds a compact parallelisable manifold.*

Remark.

(i) Theorem A answers [KK24a, Question 5.5] in many cases and its proof also yields a partial answer to Question 5.6 loc.cit..
(ii) The question which homotopy d-spheres bound compact parallelisable manifold has been studied extensively in the past and features in Kervaire–Milnor's classification of homotopy spheres [KM63]. For example, from the latter one can deduce that there are 16256 oriented homotopy 15-spheres up to orientation-preserving diffeomorphism, of which precisely half bound compact parallelisable manifolds. By Theorem A, this implies that also precisely half of them have the property that they can be distinguished from the standard 15-sphere by their k-truncated $\mathcal{D}\mathrm{isc}^+$-presheaf for any $1 < k < \infty$.

On the assumptions. We comment on the assumptions on d and k in Theorem A:

The case $k = 1$. Two oriented d-manifolds M and N have equivalent k-truncated $\mathcal{D}\mathrm{isc}^+$-presheaves for $k = 1$ if and only if there is a *tangential homotopy equivalence* between them, i.e. a homotopy equivalence $\varphi \colon M \to N$ covered by an orientation-preserving fibrewise isomorphism of the tangent bundles [KK24c, Proposition 5.10]. Since it is known that any two oriented homotopy d-spheres Σ_0 and Σ_1 are tangentially homotopy equivalent (see e.g. [RP80, Lemma 1.1]), one gets that $E_{\Sigma_0} \simeq E_{\Sigma_1}$ in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k}^+)$ for $k = 1$ and *all* Σ_0 and Σ_1.

Remark. As any homotopy equivalence between homotopy spheres is homotopic to a homeomorphism, one can even choose a tangential homotopy equivalence $\varphi \colon \Sigma_0 \to \Sigma_1$ that is a homeomorphism. The latter induces homotopy equivalences $F_S^{\mathrm{fr}}(\Sigma_0) \simeq F_S^{\mathrm{fr}}(\Sigma_1)$ between all framed configuration spaces, so in view of (2), this shows that the truncated $\mathcal{D}$isc-presheaves E_{Σ_0} and E_{Σ_1} can never be distinguished by considering any of their individual values.

[1] For $d = 4$ such examples are known by [KK24a, Theorem B].

The case $d \equiv 1 \pmod 4$. Our proof of the "only if" direction in Theorem A also goes through in the excluded dimensions $d \equiv 1 \pmod 4$ (see Section 2.2). The "if" direction however (i.e. whether $E_{\Sigma_0} \simeq E_{\Sigma_1}$ if $\Sigma_0 \sharp \overline{\Sigma_1}$ bounds a compact parallelisable manifold) does not. By the solution of the Kervaire invariant one problem, it is known that for $d = 2^\ell - 3$ with $\ell \leq 7$, only the standard sphere bounds a compact parallelisable manifold, so there is nothing to show, but in all other dimensions $d \equiv 1 \pmod 4$, there is a single nontrivial oriented homotopy sphere that bounds a compact parallelisable manifold, called the *Kervaire sphere*. This leads us to ask:

Question 1.1. *For which* $m \geq 2$ *and* $2 \leq k < \infty$ *is there an equivalence* $E_{\Sigma_K} \simeq E_{S^{4m+1}}$ *in* $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\leq k})$ *where* Σ_K *is the* $(4m+1)$*-dimensional Kervaire sphere?*

The case $k = \infty$. In the excluded case $k = \infty$, the "only if" direction of Theorem A follows from the case $1 < k < \infty$, since an equivalence of nontruncated presheaves induces by restriction one of all their truncations. However, the "if"-direction does not follow: similarly to how there are non-equivalent spaces whose finite Postnikov truncations are all equivalent [Ada57], there may be non-equivalent presheaves X and Y in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_d)$ that are equivalent in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\leq k})$ for all $k < \infty$. Theorem A thus leaves open the question for which homotopy d-spheres Σ_0 and Σ_1 such that the connected sum $\Sigma_0 \sharp \overline{\Sigma}_1$ bounds a compact parallelisable manifold there exists an equivalence $E_{\Sigma_0} \simeq E_{\Sigma_1} \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_d)$, *without truncation.* In particular we ask:

Question 1.2. *For which oriented homotopy spheres* Σ *that bound a compact parallelisable manifold does there exist an equivalence* $E_\Sigma \simeq E_{S^d}$ *in* $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_d)$*?*

Ingredients in the proof. The proof of Theorem A involves three ingredients that could be of independent interest:

(i) A description of $\mathcal{D}$isc-presheaves of manifolds that are decomposed into two codimension 0 submanifolds that intersect in their common boundary (see Theorem 3.2).
(ii) A proof that the Atiyah duality equivalence $D(\Sigma^\infty_+ M) \simeq \mathrm{Th}(-TM)$ for closed d-manifolds is natural in equivalences of $\mathcal{D}$isc-presheaves (see Theorem 2.6 and Section 4), based on a variant of a construction due to Naef–Safranov [NS24, Section 4.2] (see Remark 4.7). This also suggests a relation between algebraic L-theory and the $\mathcal{D}$isc-structure spaces as introduced in [KK24b] (see Section 2.3).
(iii) A computation of the finite residual (the intersection of all finite-index normal subgroups) of the group of isotopy classes of orientation-preserving diffeomorphisms of the iterated connected sums $\sharp^g(S^{2m+1} \times S^{2m+1})$ (see Theorem 2.3 and Section 5).

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2. $\mathcal{D}$isc-presheaves of exotic spheres

In this section, we prove Theorem A (or rather a strengthening of it) assuming three ingredients (Theorems 2.2, 2.3, and 2.6) which are established in the subsequent three sections.

Convention. We work in the setting of ∞-categories throughout, so a "category" is always an ∞-category. Unless mentioned otherwise, manifolds and embeddings are always assumed to be smooth. Given a homotopy d-sphere Σ, we write $\Sigma \in \mathrm{bP}_{d+1}$ if Σ bounds a compact parallelisable d-manifold. To treat the categories $\mathcal{D}\mathrm{isc}_{d,\leq k}$ and $\mathcal{D}\mathrm{isc}_d$ from the introduction on the same footing, we write $\mathcal{D}\mathrm{isc}_{d,\leq\infty} := \mathcal{D}\mathrm{isc}_d$ (similarly for the oriented variant).

2.1. $\mathcal{D}$isc-**presheaves of** bP-**spheres.** The "if" direction of Theorem A follows from the following more general result by specialising to $M = S^d$:

Theorem 2.1. *For a connected oriented d-manifold M with $d \not\equiv 1 \pmod 4$ and two oriented homotopy d-spheres Σ_0, Σ_1 with $\Sigma_0 \sharp \overline{\Sigma_1} \in \mathrm{bP}_{d+1}$, there is for all $1 \le k < \infty$ an equivalence*

$$E_{M\sharp\Sigma_0} \simeq E_{M\sharp\Sigma_1} \quad \textit{in} \quad \mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\le k}).$$

In particular, for all $\Sigma \in \mathrm{bP}_{d+1}$, we have $E_{M\sharp\Sigma} \simeq E_M$ in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\le k})$ whenever $1 \le k < \infty$.

The proof of Theorem 2.1 has two main ingredients—one homotopy-theoretic and one group-theoretic. The homotopy-theoretic one—which we prove in Section 3—is the following criterion to show that two manifolds obtained by gluing together the same pair of manifolds along different diffeomorphisms of their boundaries have equivalent $\mathcal{D}$isc-presheaves:

Theorem 2.2. *Fix d-manifolds M_0 and M_1 and two diffeomorphisms $\varphi_0, \varphi_1 \colon \partial M_0 \to \partial M_1$ between their boundaries. If for some $1 \le k \le \infty$ we have*

$$[E_{\varphi_0}] = [E_{\varphi_1}] \in \pi_0 \operatorname{Map}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d-1,\le k})}(E_{\partial M_0}, E_{\partial M_1}),$$

then there is an equivalence

$$E_{M_0 \cup_{\varphi_0} M_1} \simeq E_{M_0 \cup_{\varphi_1} M_1} \quad \textit{in} \quad \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le k}).$$

Moreover, the analogous statement holds when M_0 and M_1 are oriented, the φ_i are orientation-reversing, and $\mathcal{D}\mathrm{isc}_{d-1,\le k}$ and $\mathcal{D}\mathrm{isc}_{d,\le k}$ are replaced by $\mathcal{D}\mathrm{isc}^+_{d-1,\le k}$ and $\mathcal{D}\mathrm{isc}^+_{d,\le k}$.

The second ingredient in the proof of Theorem 2.1 is related to a certain group-theoretic difference between mapping class groups of manifolds and the analogue in the context of $\mathcal{D}$isc-presheaves, which was discovered in [Mez24]. To explain this, recall that a discrete group G is *residually finite* if its *finite residual*

$$\mathrm{fr}(G) := \left(\textstyle\bigcap_{G' \le G \text{ with } [G':G]<\infty} G'\right) \trianglelefteq G$$

vanishes. It was observed in [KRW20] that there are 2-connected closed high-dimensional manifolds M for which the group $\pi_0 \operatorname{Diff}^+(M)$ of isotopy classes of orientation preserving diffeomorphisms is *not* residually finite (more specifically this was shown for the iterated connected sums $W_g := \sharp^g(S^n \times S^n)$ for certain values of g and n). On the contrary, the analogous group $\pi_0 \operatorname{Aut}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le k})}(E_M)$ for the truncated $\mathcal{D}$isc-presheaf of M *is* residually finite for any $k < \infty$, as shown in [Mez24]. The proof of Theorem 2.2 will exploit this difference, based on an extension of the result from [KRW20] which we explain now. Recall that by Cerf's "pseudoisotopy-implies-isotopy" theorem and Smale's h-cobordism theorem, extending diffeomorphisms of D^{d-1} along the inclusion of a hemisphere $D^{d-1} \subset S^{d-1}$ via the identity, and gluing two copies of D^d together along a diffeomorphism of S^{d-1}, yields for $d \ge 6$ isomorphisms of abelian groups

$$\pi_0 \operatorname{Diff}_\partial(D^{d-1}) \overset{\text{ext}}{\cong} \pi_0 \operatorname{Diff}^+(S^{d-1}) \overset{\text{glue}}{\cong} \Theta_d. \tag{3}$$

where $\pi_0 \operatorname{Diff}_\partial(D^{d-1})$ is the group os isotopy classes of diffeomorphisms of a closed $(d-1)$-disc that pointwise fix the boundary and Θ_d is Kervaire–Milnor's group of oriented homotopy d-spheres [KM63]. For an oriented $(d-1)$-manifold M with an embedded codimension 0 disc $D^{d-1} \subset M$ compatible with the orientation, we write

$$\pi_0 \operatorname{Diff}_\partial(D^{d-1}) \xrightarrow{\text{ext}_M} \pi_0 \operatorname{Diff}^+_\partial(M)$$

for the morphism given by extending diffeomorphisms along $D^{d-1} \subset M$ by the identity. In these terms, the extension of the result of [KRW20] which we prove in Section 5 is:

Theorem 2.3. *For $n \ge 3$ odd and $g \ge 0$, setting $W_g := \sharp^g(S^n \times S^n)$, we have*

$$\mathrm{fr}\big(\pi_0 \operatorname{Diff}^+(W_g)\big) = \begin{cases} \operatorname{im}\Big(\mathrm{bP}_{2n+2} \le \Theta_d \overset{(3)}{\cong} \pi_0 \operatorname{Diff}_\partial(D^{2n}) \xrightarrow{\text{ext}_{W_g}} \pi_0 \operatorname{Diff}^+(W_g)\Big) & \textit{if } g \ge 2, \\ 0 & \textit{if } g < 2. \end{cases}$$

Remark 2.4. The morphism ext_{W_g} turns out to be injective (see Section 5.1 (b)), so in the first case of Theorem 2.3 we have $\mathrm{fr}(\pi_0\,\mathrm{Diff}^+(W_g)) \cong \mathrm{bP}_{2n+2}$.

Assuming the two ingredients Theorems 2.2 and 2.3, we finish the proof of Theorem 2.1:

Proof of Theorem 2.1. Throughout, we implicitly use the topological Poincaré conjecture to identify the topological manifold underlying a homotopy sphere with the standard sphere.

We first assume $d \leq 6$. For $d \leq 3$ or $d = 5, 6$, there is nothing to show since there are no nontrivial homotopy spheres. For $d = 4$ there may be, but we show that $E_{M\sharp\Sigma} \simeq E_M$ in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_d)$ for *all* oriented homotopy d-spheres Σ (and hence also in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\leq k})$ for any k, by restriction). This is closely related to [KK24a, Theorem B]. By smoothing theory for embedding calculus [KK24c, Theorem 5.21 (ii)], it suffices to show that the two lifts of the topological tangent bundle $M \to \mathrm{BSTop}(4)$ along $\mathrm{BSO}(4) \to \mathrm{BSTop}(4)$ induced by the smooth structure of M and of $M\sharp\Sigma$ are homotopic as lifts. For that it is enough to show that the two lifts of $D^4 \to \mathrm{BSTop}(4)$ induced by the smooth structure of D^4 and of $D^4\sharp\Sigma$ are homotopic as lifts, relative to the given lift on ∂D^4 induced by the standard smooth structure. But the obstruction for the existence of such a homotopy of lifts lies in $\pi_4(\mathrm{Top}(4)/\mathrm{O}(4))$ and this group vanishes, since $\pi_4(\mathrm{Top}(4)/\mathrm{O}(4)) \cong \pi_4(\mathrm{Top}/\mathrm{O})$ by [FQ90, Theorem 8.7A] and $\pi_4(\mathrm{Top}/\mathrm{O})$ vanishes by [KS77, V.5.0 (5)].

Turning to the case $d \geq 7$, we begin with a general observation: Given a decomposition $M = M_0 \cup M_1$ of an oriented d-manifold M into two codimension 0 submanifolds $M_0, M_1 \subset N$ that intersect in their common boundary $P \coloneqq \partial N_0 = \partial N_1$, then for any embedded disc $D^{d-1} \subset P$ compatible with the orientation, there is an orientation-preserving diffeomorphism $M\sharp\Sigma \cong M_0 \cup_{\mathrm{ext}_P(\mathrm{tw}(\Sigma))} M_1$ where $\mathrm{tw}\colon \Theta_d \to \pi_0\,\mathrm{Diff}_\partial(D^{d-1})$ is the inverse of the isomorphism from (3). Combining this with Theorem 2.2, when given oriented homotopy d-spheres Σ_0 and $\Sigma_1 \in \Theta_d$, to show $E_{M\sharp\Sigma_0} \simeq E_{M\sharp\Sigma_1}$ in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\leq k})$ for some fixed $1 \leq k \leq \infty$ it suffices to show that $\mathrm{tw}(\Sigma_0\sharp\overline{\Sigma_1}) = \mathrm{tw}(\Sigma_0) \circ \mathrm{tw}(\Sigma_1)^{-1}$ lies in the kernel of the composition of ext_P with the morphism $E\colon \pi_0\,\mathrm{Diff}^+(P) \to \pi_0\,\mathrm{Aut}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d-1,\leq k})}(E_P)$. Since $\mathrm{bP}_{d+1} = 0$ for d even by [KM63, Theorem 5.1, Lemma 2.3] and we assumed $d \not\equiv 1 \pmod 4$, this shows that in order to prove Theorem 2.1, it suffices to find for $d = 2n+1$ with $n \geq 3$ odd a decomposition $M = M_0 \cup M_1$ as above, such that the following composition is trivial for $1 \leq k < \infty$:

$$\mathrm{bP}_{2n+2} \leq \Theta_{2n+1} \overset{(3)}{\cong} \pi_0\,\mathrm{Diff}_\partial(D^{2n}) \xrightarrow{\mathrm{ext}_P} \pi_0\,\mathrm{Diff}^+(P) \xrightarrow{E} \pi_0\,\mathrm{Aut}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\leq k})}(E_P). \tag{4}$$

The decomposition we use is the following: for any fixed $g \geq 0$, choose an embedding of the iterated boundary connected sum $V_g \coloneqq \natural^g D^{n+1} \times S^n$ into M (such an embedding exists since V_g embeds into a closed disc D^{2n+1} which in turn embeds into any nonempty manifold), set M_0 to be the image of the embedding and M_1 to be the closure of its complement. The submanifolds M_0 and M_1 intersect in their common boundary which is diffeomorphic to $W_g = \partial(V_g) = \sharp^g S^n \times S^n$. It thus suffices to show that (4) is trivial for $P = W_g$ for *some* value of $g \geq 0$. This is true for any $g \geq 2$, since the composition $\mathrm{bP}_{2n+2} \to \pi_0\,\mathrm{Diff}^+(W_g)$ lands in the finite residual of $\pi_0\,\mathrm{Diff}^+(W_g)$ by Theorem 2.3 (it is in fact equal to it, but we will not need this), so as finite residuals are preserved by group homomorphisms, the image of the composition (4) is for $g \geq 2$ contained in the finite residual of $\pi_0\,\mathrm{Aut}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\leq k})}(E_{W_g})$. But the latter finite residual is trivial by [Mez24, Theorem 3.16], so the claim follows. □

2.2. $\mathcal{D}$isc-**presheaves of** $\mathrm{coker}(J)$-**spheres.** We now turn on the "only if" direction of Theorem A. In dimensions $d \leq 6$ any homotopy d-sphere bounds a compact parallelisable manifold (combine [KM63, p. 504] with [Ker69, Theorem 3]), so there is nothing to show. In dimensions $d \geq 7$, we use that by Kervaire–Milnor's exact sequence [KM63], the condition $\Sigma_0\sharp\overline{\Sigma}_1 \in \mathrm{bP}_{d+1}$ in Theorem A is equivalent to Σ_0 and Σ_1 having the same image under the morphism $\Theta_d \to \mathrm{coker}(J)_d$ from loc.cit., so the "only if" direction of Theorem A follows from:

Theorem 2.5. *For oriented homotopy d-spheres Σ_0 and Σ_1 whose associated oriented 2-truncated $\mathcal{D}$isc-presheaves $E_{\Sigma_0}, E_{\Sigma_1} \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\leq 2})$ are equivalent, we have*

$$[\Sigma_0] = [\Sigma_1] \in \mathrm{coker}(J)_d.$$

In particular, if $[\Sigma] \neq 0 \in \mathrm{coker}(J)_d$, then $E_\Sigma \not\simeq E_{S^d}$ in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\leq 2})$.

The main ingredient for proof of Theorem 2.5 says, informally speaking, that the Atiyah duality equivalence for manifolds can be made natural in equivalences of 2-truncated $\mathcal{D}$isc-presheaves. To explain the precise statement, recall that a k-truncated $\mathcal{D}$isc-presheaf $X \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k})$ has for $k \geq 1$ an underlying space, given by the orbits

$$|X| := X(\mathbf{R}^d)/\mathrm{Aut}(\mathbf{R}^d) \in \mathcal{S}$$

of the action of $\mathrm{Aut}(\mathbf{R}^d) := \mathrm{Aut}_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k})}(\mathbf{R}^d) \simeq \mathrm{O}(d)$ on $X(\mathbf{R}^d)$ by functoriality, and this space comes with a d-dimensional vector bundle ξ_X over it, classified by the map

$$|X| = X(\mathbf{R}^d)/\mathrm{Aut}(\mathbf{R}^d) \xrightarrow{\xi_X} */\mathrm{Aut}(\mathbf{R}^d) \simeq \mathrm{BO}(d). \tag{5}$$

If $X \simeq E_M$ for a smooth d-manifold M, then $|X| \simeq M$ and ξ_X models the tangent bundle TM (see [KK24c, Proposition 5.10]). If M is a *closed* manifold, then the Pontryagin–Thom collapse map induced by the choice of an embedding $M \subset \mathbf{R}^{d+k}$ for $k \gg 0$ gives a map $\mathbf{S} \to \mathrm{Th}(-TM)$ from the sphere spectrum to the Thom spectrum of the stable normal bundle of M, called the *stable collapse map*, which is natural in diffeomorphisms. In Section 4 we will show that it is even natural in equivalences of 2-truncated presheaves, in the sense of the following theorem. In the statement and henceforth, we write $\mathrm{map}(-,-)$ for mapping spectra and $D(-) := \mathrm{map}(-,\mathbf{S})$ for Spanier–Whitehead duals.

Theorem 2.6. *To any 2-truncated presheaf $X \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq 2})$, one can associate a zig-zag of maps of spectra which is natural in equivalences in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq 2})$,*

$$\mathbf{S} \to D(\Sigma^\infty_+|X|) \leftarrow Z_X \to \mathrm{Th}(-\xi_X) \quad \textit{for some spectrum } Z_X, \tag{6}$$

and has the property that if $X = E_M$ for a closed smooth d-manifold M, then the wrong-way map is an equivalence and the resulting map $\mathbf{S} \to \mathrm{Th}(-TM)$ agrees with the stable collapse map.

Remark 2.7. Some remarks on Theorem 2.6:

(i) The construction in the proof Theorem 2.6 is closely related to work of Naef–Safronov (c.f. [NS24, Section 4.2], and Remark 4.7 below). There is an alternative construction based on constructing collapse maps for $\mathcal{D}$isc-presheaves, inspired by [KK25, Section 4].

(ii) A related result was obtained by Prigge [Pri20, Theorem 7.1.8].

(iii) The first map in (6) is induced by the unique map $|X| \to *$, so considering only the right two maps in (6), Theorem 2.6 can be interpreted as saying that the Atiyah duality equivalence $D(\Sigma^\infty_+ M) \simeq \mathrm{Th}(-TM)$ can be recovered from the 2-truncated presheaf E_M associated to M, naturally in equivalence of presheaves.

(iv) A variant of Theorem 2.6 holds in the setting of topological manifolds (see Remark 4.8).

(v) Theorem 2.6 suggests the following notion: a k-truncated presheaf $X \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k})$ for $2 \leq k \leq \infty$ is *closed* if (a) $|X|$ lies in the full subcategory $\mathcal{S}^\omega \subset \mathcal{S}$ of compact objects in the ∞-category of spaces, (b) the wrong way map in (6) is an equivalence, (c) and the resulting map $c_X \colon \mathbf{S} \to \mathrm{Th}(-\xi_X)$ exhibits $-\xi_X$ as the dualising spectrum of $|X|$ (in the sense of e.g. [Lan22, p. 227]). In particular, in this case the underlying space $|X|$ is a d-dimensional Poincaré duality space (see p. 228-220 loc.cit.).

Assuming Theorem 2.6, we prove Theorem 2.5 on $\mathcal{D}$isc-presheaves of $\mathrm{coker}(J)$-spheres:

Proof of Theorem 2.5. We first recall the definition of the element $[\Sigma] \in \mathrm{coker}(J)_d$ associated to an oriented homotopy d-sphere (cf. [KM63, Section 4]): Given an oriented homotopy d-sphere Σ, consider its stable collapse map $\mathbf{S} \to \mathrm{Th}(-T\Sigma)$. Choosing a stable framing of Σ gives an equivalence $\mathrm{Th}(-T\Sigma) \simeq \mathbf{S}^{-d}$, so the stable collapse map gives an element in $\pi_0(\mathbf{S}^{-d}) \cong \pi_d(\mathbf{S})$.

Its image in $\mathrm{coker}(J)_d$ turns out to be independent of the choice of stable framing as long as the latter is chosen to be compatible with the given orientation of Σ; this defines the homomorphism $\Theta_d \to \mathrm{coker}(J)_d$. In particular, we see that the class $[\Sigma] \in \mathrm{coker}(J)_d$ only depends on, (a) a classifier of the stable oriented tangent bundle $T\Sigma\colon \Sigma \to \mathrm{BSO}$ and (b) the homotopy class of the stable collapse map $\mathrm{S} \to \mathrm{Th}(-T\Sigma)$. By the discussion above, in particular Theorem 2.6, both of these can be recovered from the equivalence class of the 2-truncated presheaf $E_\Sigma \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}^+_{d,\le 2})$, so the claim follows. □

2.3. **Digression: $\mathcal{D}$isc-structure spaces and L-theory.** Theorem 2.6 allows one to relate the $\mathcal{D}$isc-structure spaces from [KK24b] to algebraic L-theory. We intend to take up this direction in future work, but briefly sketch how this goes:

We write $\mathcal{P}\mathrm{oinc}_d^{\simeq}$ for the ∞-groupoid of d-dimensional Poincaré spaces (in the sense of e.g.[Lan22, p. 228-220]; this is a full subgroupoid of the core of the full subcategory $\mathcal{S}^\omega$ of compact objects in the ∞-category $\mathcal{S}$ of spaces) and $\mathcal{P}\mathrm{oinc}_d^{\nu,\simeq}$ for the ∞-groupoid of d-dimensional Poincaré space together with a stable vector bundle refinement of its Spivak fibration (this can be constructed as a full subgroupoid of the core of the pullback of the composition $\mathcal{S}^\omega_{/\mathbb{Z}\times\mathrm{BO}} \to \mathcal{S}^\omega_{/\mathrm{Sp}^{\simeq}} \to \mathrm{Sp}$ whose second functor is induced by taking colimits in the category Sp of spectra, along $\mathrm{Sp}_{\mathrm{S}/} \to \mathrm{Sp}$). In these terms Theorem 2.6 yields a lift $|{-}|^\nu\colon \mathrm{PSh}^{\mathrm{cl}}(\mathcal{D}\mathrm{isc}_d)^{\simeq} \to \mathcal{P}\mathrm{oinc}_d^{\nu,\simeq}$ of the functor $|{-}|\colon \mathrm{PSh}^{\mathrm{cl}}(\mathcal{D}\mathrm{isc}_d)^{\simeq} \to \mathcal{P}\mathrm{oinc}_d^{\simeq}$, where $\mathrm{PSh}^{\mathrm{cl}}(\mathcal{D}\mathrm{isc}_d) \subset \mathrm{PSh}(\mathcal{D}\mathrm{isc}_d)$ is the full subcategory of closed presheaves as in Remark 2.7 (v). Moreover, with some effort, one ought to be able to construct a commutative square

$$\begin{tikzcd} \mathcal{M}\mathrm{an}_d^{\mathrm{cl},\cong} \arrow[r, "E"] \arrow[d, "\mathrm{forget}"'] & \mathrm{PSh}^{\mathrm{cl}}(\mathcal{D}\mathrm{isc}_d)^{\simeq} \arrow[d, "|-|^\nu"] \\ \widetilde{\mathcal{M}\mathrm{an}}_d^{\mathrm{cl},\cong} \arrow[r, "\nu"] & \mathcal{P}\mathrm{oinc}_d^{\nu,\simeq} \end{tikzcd} \tag{7}$$

where $\mathcal{M}\mathrm{an}_d^{\mathrm{cl},\cong}$ (respectively $\widetilde{\mathcal{M}\mathrm{an}}_d^{\mathrm{cl},\cong}$) is the ∞-groupoid of closed d-dimensional smooth manifolds and spaces of diffeomorphisms (respectively block-diffeomorphisms) between them. The map ν is induced by taking stable normal bundles and constructed such that it agrees after taking fibres over $\mathcal{P}\mathrm{oinc}_d^{\simeq}$ with the normal invariant map in surgery theory. The fibre of the top map at E_M for $M \in \mathcal{M}\mathrm{an}_d^{\mathrm{cl},\cong}$ is by definition the $\mathcal{D}$isc-structure space $S^{\mathcal{D}\mathrm{isc}}(M)$ as introduced in [KK24b], so taking horizontal fibres yields a map

$$S^{\mathcal{D}\mathrm{isc}}(M) \longrightarrow \mathrm{fib}_{\nu(M)}(\widetilde{\mathcal{M}\mathrm{an}}_d^{\mathrm{cl},\cong} \to \mathcal{P}\mathrm{oinc}_d^{\nu,\simeq}).$$

Moreover, if $d \ge 5$, then by surgery theory, the collection of the components $(N, \nu(M) \simeq \nu(N))$ of the target whose underlying homotopy equivalence $M \simeq N$ is simple (with the finiteness structure on M and N induced from their manifold structures), is equivalent to loop space of the quadratic L-theory space $L^q(M)$ of M, so writing $S^{\mathcal{D}\mathrm{isc}}(M)^s \subset S^{\mathcal{D}\mathrm{isc}}(M)$ for the components of $(N, E_M \simeq E_N)$ whose underlying homotopy equivalence $M \simeq N$ is simple (note that we have $S^{\mathcal{D}\mathrm{isc}}(M) = S^{\mathcal{D}\mathrm{isc}}(M)^s$ if M is simply connected, or more generally—by [NS24, Corollary C]—if the Dennis trace $\mathrm{Wh}(\pi_1(M)) \to H_1(LM, M)$ is injective), one obtains a map of the form

$$S^{\mathcal{D}\mathrm{isc}}(M)^s \longrightarrow \Omega L^q(M). \tag{8}$$

As a result of [KK24b, Theorem A], the source of this map depends up to equivalence only on the tangential 2-type of M, and by the π-π-theorem the same holds for the target (in fact it only depends on the 1-truncation of M and the first Stiefel–Whitney class). It seems conceivable that the map (8) also only depends on the tangential 2-type of M.

3. Gluing $\mathcal{D}$isc-presheaves

This section serves to establish the first of the three ingredients assumed in Section 2: Theorem 2.2 on the dependence of $\mathcal{D}$isc-presheaves of glued manifolds on the gluing diffeomorphism. We will deduce this from a general gluing-result for presheaves.

3.1. Gluing $\mathcal{D}\mathrm{isc}$-presheaves.

3.1.1. *Manifolds with boundary.* Write $\mathcal{M}\mathrm{an}^\partial_d$ for the ∞-category whose objects are d-manifolds M (potentially non-compact and with boundary) and whose morphisms are spaces $\mathrm{Emb}^\partial(M,N)$ of embeddings $e\colon M \hookrightarrow N$ with $e^{-1}(\partial N) = \partial M$ (see Remark 3.4 for a precise construction of $\mathcal{M}\mathrm{an}^\partial_d$). It contains as a full subcategory the category $\mathcal{M}\mathrm{an}_d$ of d-manifolds without boundary and embeddings between them. Taking boundaries yields a functor $\partial\colon \mathcal{M}\mathrm{an}^\partial_d \to \mathcal{M}\mathrm{an}_{d-1}$ which has a fully faithful left adjoint $\kappa\colon \mathcal{M}\mathrm{an}_{d-1} \to \mathcal{M}\mathrm{an}^\partial_d$ given by taking half-open collars, i.e. it sends $P \in \mathcal{M}\mathrm{an}_{d-1}$ to $P \times [0,\infty)$. The reason that these functors are adjoint is that the restriction map $\mathrm{Emb}^\partial(P \times [0,\infty), M) \to \mathrm{Emb}(P, \partial M)$ is an equivalence for all $P \in \mathcal{M}\mathrm{an}_{d-1}$ and $M \in \mathcal{M}\mathrm{an}^\partial_d$ as its fibres are equivalent to spaces of collars, which are contractible. Note that the unit of this adjunction $\mathrm{id} \to \partial\kappa$ is an equivalence, as $P = \partial(P \times [0,\infty))$. Since ∂ is right adjoint to κ, the left Kan extension $\partial_!\colon \mathrm{PSh}(\mathcal{M}\mathrm{an}^\partial_d) \to \mathrm{PSh}(\mathcal{M}\mathrm{an}^\partial_{d-1})$ is right adjoint to $\kappa_!$, so since restriction is also left adjoint to left Kan extension, we get an equivalence $\partial_! \simeq \kappa^*$ of functors $\mathrm{PSh}(\mathcal{M}\mathrm{an}^\partial_d) \to \mathrm{PSh}(\mathcal{M}\mathrm{an}_{d-1})$.

3.1.2. $\mathcal{D}\mathrm{isc}$-*subcategories.* Writing $\mathbf{H}^d \coloneqq [0,\infty) \times \mathbf{R}^{d-1}$ for the d-dimensional halfspace, we consider the tower (in the sense of [KK24c, Section 1.2]) of full subcategories of $\mathcal{M}\mathrm{an}^\partial_d$

$$\mathcal{D}\mathrm{isc}^\partial_{d,\le 1} \subset \mathcal{D}\mathrm{isc}^\partial_{d,\le 2} \subset \cdots \subset \mathcal{D}\mathrm{isc}^\partial_{d,\le\infty} = \mathcal{D}\mathrm{isc}^\partial_d \tag{9}$$

where $\mathcal{D}\mathrm{isc}^\partial_{d,\le k} \subset \mathcal{M}\mathrm{an}^\partial_d$ is the full subcategory on those manifolds that are diffeomorphic to $S \times \mathbf{R}^d \sqcup T \times \mathbf{H}^{d-1}$ for finite sets S and T with $|S| + |T| \le k$. Note that intersecting (9) with $\mathcal{M}\mathrm{an}_d \subset \mathcal{M}\mathrm{an}^\partial_d$ yields the tower of full subcategories $\mathcal{D}\mathrm{isc}_{d,\le\bullet} \subset \mathcal{M}\mathrm{an}_d$ on manifolds diffeomorphic to $S \times \mathbf{R}^d$ with $|S| \le \bullet$, as in the introduction. The functor κ preserves these subcategories and gives a map of towers $\kappa\colon \mathcal{D}\mathrm{isc}_{d-1,\le\bullet} \to \mathcal{D}\mathrm{isc}^\partial_{d,\le\bullet}$, and thus a map of towers $\kappa^*\colon \mathrm{PSh}(\mathcal{D}\mathrm{isc}^\partial_{d,\le\bullet}) \to \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d-1,\le\bullet})$ by restriction. Note that the adjunction $\kappa \vdash \partial$ from Section 3.1.1 restricts to adjunctions $\kappa\colon \mathcal{D}\mathrm{isc}_{d-1,\le k} \rightleftarrows \mathcal{D}\mathrm{isc}^\partial_{d,\le k}\colon \partial$ for all k, so as above we obtain $\partial_! \simeq \kappa^*$ as functors $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^\partial_{d,\le k}) \to \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d-1,\le k})$. Writing $\iota^\partial\colon \mathcal{D}\mathrm{isc}^\partial_{d,\le\bullet} \hookrightarrow \mathcal{M}\mathrm{an}^\partial_d$ and $\iota\colon \mathcal{D}\mathrm{isc}_{d-1,\le\bullet} \hookrightarrow \mathcal{M}\mathrm{an}_{d-1}$ for the respective inclusions, this implies that the square

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{M}\mathrm{an}^\partial_d) \arrow[r, "(\iota^\partial)^*"] \arrow[d, "\partial_!"'] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}^\partial_{d,\le\bullet}) \arrow[d, "\partial_!"] \\ \mathrm{PSh}(\mathcal{M}\mathrm{an}_{d-1}) \arrow[r, "\iota^*"] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d-1,\le\bullet}) \end{tikzcd} \tag{10}$$

of towers of categories commutes in that the Beck–Chevalley transformation $\partial_!(\iota^\partial)^* \to \iota^*\partial_!$ is an equivalence. We denote by

$$E^\partial \coloneqq ((\iota^\partial)^* \circ y)\colon \mathcal{M}\mathrm{an}^\partial_d \longrightarrow \mathrm{PSh}(\mathcal{D}\mathrm{isc}^\partial_{d,\le\bullet})$$

the tower of restricted Yoneda embeddings.

3.1.3. *Gluing manifolds and* $\mathcal{D}\mathrm{isc}$-*presheaves.* Writing

$$\mathcal{M}\mathrm{an}^{\shortparallel}_d \coloneqq \mathcal{M}\mathrm{an}^\partial_d \times_{\mathcal{M}\mathrm{an}_{d-1}} \mathcal{M}\mathrm{an}^\partial_d$$

for the pullback of $\partial\colon \mathcal{M}\mathrm{an}^\partial_d \to \mathcal{M}\mathrm{an}_{d-1}$ along itself, naturality of the Yoneda embedding yields a functor $\mathcal{M}\mathrm{an}^{\shortparallel}_d \to \mathrm{PSh}(\mathcal{M}\mathrm{an}^\partial_d) \times_{\mathrm{PSh}(\mathcal{M}\mathrm{an}_{d-1})} \mathrm{PSh}(\mathcal{M}\mathrm{an}^\partial_d)$ to the pullback of the left vertical functor in (10) along itself.

Remark 3.1. An object in $\mathcal{M}\mathrm{an}^{\shortparallel}_d$ is given by a triple (M_0, M_1, φ) where the M_i are d-manifolds and $\varphi\colon \partial M_0 \to \partial M_1$ is a diffeomorphism between their boundaries. The spaces of morphisms in $\mathcal{M}\mathrm{an}^{\shortparallel}_d$ is given by the pullback of embedding spaces $\mathrm{Emb}^\partial(M_0,N_0) \times_{\mathrm{Emb}(\partial M_0, \partial N_1)} \mathrm{Emb}^\partial(M_1,N_1)$. Note that given a d-manifold W without boundary with a decomposition $W = W_0 \cup W_1$ into two codimension 0 submanifolds that intersect in their common boundary $P \coloneqq \partial_0 W = \partial_1 W$, we obtain an object $(W_0, W_1, \mathrm{id}_P)$, and it turns out that any object (M_0, M_1, φ) is equivalent to one of this form, by considering the glued manifold $W = M_0 \cup_\varphi M_1$.

We can postcompose the functor $\mathcal{M}\mathrm{an}^{\shortparallel}_d \to \mathrm{PSh}(\mathcal{M}\mathrm{an}^{\partial}_d) \times_{\mathrm{PSh}(\mathcal{M}\mathrm{an}_{d-1})} \mathrm{PSh}(\mathcal{M}\mathrm{an}^{\partial}_d)$ with the functor to the pullback of the right vertical functor in (10) along itself, induced by the commutativity of the square, to arrive at a functor of towers of the form

$$E^{\shortparallel}\colon \mathcal{M}\mathrm{an}^{\shortparallel}_d \longrightarrow \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\partial}_{d,\le\bullet}) \times_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d-1,\le\bullet})} \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\partial}_{d,\le\bullet}).$$

In arity $1 \le k \le \infty$, it sends a triple (M_0, M_1, φ) as in Remark 3.1 to the triple consisting of the presheaves $(E^{\partial}_M, E^{\partial}_N)$ in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\partial}_{d,\le k})$ and the equivalence $\partial_!(E^{\partial}_M) \simeq E_{\partial M} \simeq E_{\partial N} \simeq \partial_!(E^{\partial}_N)$ in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d-1,\le k})$ induced by the diffeomorphism $\partial M \cong \partial N$ and (10). The goal of the remainder of this section is to prove that the latter data $(E^{\partial}_M, E^{\partial}_N, E_{\partial M} \simeq E_{\partial N})$ is sufficient to reconstruct the presheaf $E_{M\cup_\partial N} \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le k})$ of the glued manifold. More concretely, gluing manifolds along their boundary yields a functor $\gamma\colon \mathcal{M}\mathrm{an}^{\shortparallel}_d \to \mathcal{M}\mathrm{an}$ (see Remark 3.4), and we will show:

Theorem 3.2. *There exists a commutative square of towers of categories*

$$\begin{tikzcd} \mathcal{M}\mathrm{an}^{\shortparallel}_d \arrow[r, "E^{\shortparallel}"] \arrow[d, "\gamma"'] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\partial}_{d,\le\bullet}) \times_{\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d-1,\le\bullet})} \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\partial}_{d,\le\bullet}) \arrow[d, "\gamma^{\mathcal{D}\mathrm{isc}}"] \\ \mathcal{M}\mathrm{an}_d \arrow[r, "E"] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le\bullet}) \end{tikzcd}$$

for some functor of towers $\gamma^{\mathcal{D}\mathrm{isc}}$ with the indicated source and target.

Remark 3.3.

(i) There is a variant of Theorem 3.2 for topological manifolds, by replacing all categories of (smooth) manifolds and (smooth) embeddings between them by the corresponding categories of topological manifolds and topological embeddings between them. The proof is the same as in the smooth setting.

(ii) From [KK24c, Theorem 5.3] one can extract a similar commutative square of towers

$$\begin{tikzcd} \mathcal{M}\mathrm{an}^{\shortparallel}_d \arrow[r] \arrow[d, "\gamma"'] & \mathrm{RMod}(\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le\bullet})) \times_{\mathrm{Ass}(\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le\bullet}))} \mathrm{LMod}(\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le\bullet})) \arrow[d, "\otimes"] \\ \mathcal{M}\mathrm{an}_d \arrow[r, "E"] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le\bullet}) \end{tikzcd}$$

where $\mathrm{LMod}(-)$, $\mathrm{RMod}(-)$, $\mathrm{Ass}(-)$ are the categories of left-modules, right-modules, and associative algebras in a symmetric monoidal category respectively, the symmetric monoidal structure on $\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le k})$ is a localisation of the Day convolution structure induced from the symmetric monoidal structure on $\mathcal{D}\mathrm{isc}_d$ by disjoint union, and $\otimes$ denotes the relative tensor product of modules. The square in Theorem 3.2 avoids algebras, modules, and tensor products of such, which turned out to be more convenient for the purpose of this work.

Remark 3.4 (Point-set models). Above we only gave informal definitions of the categories $\mathcal{M}\mathrm{an}^{\partial}_d$ and $\mathcal{M}\mathrm{an}_{d-1}$ in that we only described their objects and spaces of morphisms, and we also only informally specified the functors $\partial\colon \mathcal{M}\mathrm{an}^{\partial}_d \to \mathcal{M}\mathrm{an}_{d-1}$ and $\gamma\colon \mathcal{M}\mathrm{an}^{\partial}_d \times_{\mathcal{M}\mathrm{an}_{d-1}} \mathcal{M}\mathrm{an}^{\partial}_d \to \mathcal{M}\mathrm{an}_d$ (all other constructions, however, were formally obtained from these). Precise constructions of these categories and functors can be extracted from [KK24b, Section 3] as follows:

Recall (e.g. from [KK24c, Section 1.1]) that a *double category* $\mathcal{M}$ is a category-object in $\mathcal{C}\mathrm{at}_\infty$, i.e. a simplicial object $\mathcal{M} \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty)$ in categories that satisfies the Segal condition. It has a *category of objects* $\mathrm{ob}(\mathcal{M}) := \mathcal{M}_{[0]}$ and for $c, d \in \mathrm{ob}(\mathcal{M})$ a *category of morphisms* $\mathcal{M}_{c,d} := \{c\} \times_{\mathcal{M}_{[0]}} \mathcal{M}_{[1]} \times_{\mathcal{M}_{[0]}} \{d\}$ where the pullback is taken over the *source and target functors* induced by $0, 1\colon [0] \to [1]$. Letting source or target vary, we write $\mathcal{M}_{c,-} := \{c\} \times_{\mathcal{M}_{[0]}} \mathcal{M}_{[1]}$ and $\mathcal{M}_{-,d}\colon \mathcal{M}_{[1]} \times_{\mathcal{M}_{[0]}} \{d\}$, and these come with functors $t\colon \mathcal{M}_{c,-} \to \mathrm{ob}(\mathcal{M})$ and $s\colon \mathcal{M}_{-,d} \to \mathrm{ob}(\mathcal{M})$ by taking targets or sources, respectively. We have a *composition functor*

$$\mathcal{M}_{c,-} \times_{\mathrm{ob}(\mathcal{M})} \mathcal{M}_{-,d} \simeq \{c\} \times_{\mathcal{M}_{[0]}} \mathcal{M}_{[2]} \times_{\mathcal{M}_{[0]}} \{d\} \xrightarrow{(0\le 2)^*} \{c\} \times_{\mathcal{M}_{[0]}} \mathcal{M}_{[1]} \times_{\mathcal{M}_{[0]}} \{d\} = \mathcal{M}_{c,d}$$

where the first equivalence uses the Segal condition. The *opposite* $\mathcal{M}^{\mathrm{op}}$ of a double category $\mathcal{M}$ is the double category obtained by precomposing the simplicial object $\mathcal{M}$ with the functor $\mathrm{op}\colon \Delta \to \Delta$ with $\mathrm{op}([n]) = [n]$ and $\mathrm{op}(\alpha\colon [m] \to [n])(i) = n - \alpha(m-i))$. Note that one has $\mathrm{ob}(\mathcal{M}) = \mathrm{ob}(\mathcal{M}^{\mathrm{op}})$ and $\mathcal{M}^{\mathrm{op}}_{c,d} = \mathcal{M}_{d,c}$.

In [KK24b, Section 3, Steps ① and ⑤], we constructed a double category $\mathrm{nc}\mathcal{B}\mathrm{ord}_d$ of (possibly noncompact) $(d-1)$-manifolds and bordisms between them. It comes with an anti-involution $(-)^{\mathrm{rev}}\colon \mathrm{nc}\mathcal{B}\mathrm{ord}_d \to (\mathrm{nc}\mathcal{B}\mathrm{ord}_d)^{\mathrm{op}}$ given by “reversing bordisms”, which is—in the language of Section 3 Steps ① loc.cit.—induced by sending a $[p]$-walled d-manifold $(W \subset \mathbf{R}\times\mathbf{R}^\infty, \mu\colon [p] \to \mathbf{R})$ to $((-1\times_{\mathrm{id}_{\mathbf{R}^\infty}})(W), ((-1)\circ\mu\circ(i \mapsto p-i))\colon [p] \to \mathbf{R})$. Moreover, there is an equivalence $(\mathrm{nc}\mathcal{B}\mathrm{ord}_d)_{\varnothing,\varnothing} \simeq \mathrm{ob}(\mathrm{nc}\mathcal{B}\mathrm{ord}_{d+1})$ induced by sending a $[1]$-walled d-manifold $(W \subset \mathbf{R}\times\mathbf{R}^\infty, \mu\colon [1] \to \mathbf{R})$ to $(\mathbf{R}\times W|_{[\mu(0),\mu(1)]}, 0\colon [0] \to \mathbf{R})$. We set

$$\mathcal{M}\mathrm{an}_{d-1} \coloneqq \mathrm{ob}(\mathrm{nc}\mathcal{B}\mathrm{ord}_d) \quad \text{and} \quad \mathcal{M}\mathrm{an}^{\partial}_d \coloneqq (\mathrm{nc}\mathcal{B}\mathrm{ord}_d)_{\varnothing,-}.$$

The functor $\partial\colon \mathcal{M}\mathrm{an}^{\partial}_d \to \mathcal{M}\mathrm{an}_{d-1}$ is defined as $t\colon (\mathrm{nc}\mathcal{B}\mathrm{ord}_d)_{\varnothing,-} \to \mathrm{ob}(\mathrm{nc}\mathcal{B}\mathrm{ord}_d)$ and the functor $\gamma\colon \mathcal{M}\mathrm{an}^{\shortparallel}_d = \mathcal{M}\mathrm{an}^{\partial}_d \times_{\mathcal{M}\mathrm{an}_{d-1}} \mathcal{M}\mathrm{an}^{\partial}_d \to \mathcal{M}\mathrm{an}_d$ is given by the composition of the equivalence $(\mathrm{nc}\mathcal{B}\mathrm{ord}_d)_{\varnothing,-} \times_{\mathrm{ob}(\mathrm{nc}\mathcal{B}\mathrm{ord}_d)} (\mathrm{nc}\mathcal{B}\mathrm{ord}_d)_{\varnothing,-} \simeq (\mathrm{nc}\mathcal{B}\mathrm{ord}_d)_{\varnothing,-} \times_{\mathrm{ob}(\mathrm{nc}\mathcal{B}\mathrm{ord}_d)} (\mathrm{nc}\mathcal{B}\mathrm{ord}_d)_{-,\varnothing}$ which is induced by the anti-involution $(-)^{\mathrm{rev}}$ in the second argument, with the composition functor $(\mathrm{nc}\mathcal{B}\mathrm{ord}_d)_{\varnothing,-} \times_{\mathrm{ob}(\mathrm{nc}\mathcal{B}\mathrm{ord}_d)} (\mathrm{nc}\mathcal{B}\mathrm{ord}_d)_{-,\varnothing} \to (\mathrm{nc}\mathcal{B}\mathrm{ord}_d)_{\varnothing,\varnothing} \simeq \mathrm{ob}(\mathrm{nc}\mathcal{B}\mathrm{ord}_{d+1}) = \mathcal{M}\mathrm{an}_d$.

3.2. **Some category theory.** The proof of Theorem 3.2 relies on the following lemma:

Lemma 3.5. *Fix categories $\mathcal{D}_{01}, \mathcal{D}_0, \mathcal{D}_1$ and fully faithful functors $\kappa_i\colon \mathcal{D}_{01} \hookrightarrow \mathcal{D}_i$ for $i = 0,1$ that admit right adjoints $\partial_i\colon \mathcal{D}_i \to \mathcal{D}_{01}$.*

(i) The natural functor $\iota\colon \mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1 \to \mathcal{D}_0 \times_{\mathcal{D}_{01}} \mathcal{D}_1$ from the pushout of the κ_i to the pullback of the ∂_i, is fully faithful.

(ii) The value at an object $(d_0,d_1) \in \mathcal{D}_0 \times_{\mathcal{D}_{01}} \mathcal{D}_1$ of the unit $X \to \iota_*\iota^*X$ of the adjunction $\iota^*\colon \mathrm{PSh}(\mathcal{D}_0 \times_{\mathcal{D}_{01}} \mathcal{D}_1) \rightleftarrows \mathrm{PSh}(\mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1)\colon \iota_*$ between restriction and right Kan extension is naturally equivalent to the map from the top left-corner in the square

$$\begin{tikzcd} X(d_0,d_1) \arrow[r, "(\mathrm{id},\epsilon_1)^*"] \arrow[d, "(\epsilon_0,\mathrm{id})^*"'] & X(d_0, \kappa_1\partial_1(d_1)) \arrow[d, "(\epsilon_0,\mathrm{id})^*"] \\ X(\kappa_0\partial_0(d_0), d_1) \arrow[r, "(\mathrm{id},\epsilon_1)^*"] & X(\kappa_0\partial_0(d_0), \kappa_1\partial_1(d_1)) \end{tikzcd}$$

to the pullback of the remaining entries. Here the maps ϵ_i are the counits of $\kappa_i \vdash \partial_i$.

(iii) For full subcategories $\mathcal{D}'_{01} \subset \mathcal{D}_{01}$ and $\mathcal{D}'_i \subset \mathcal{D}_i$ for $i = 0,1$ to which the functors κ_i and ∂_i restrict, the diagram of categories of presheaves

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1) \arrow[r, "\subset^*"] \arrow[d, "\iota_*"'] & \mathrm{PSh}(\mathcal{D}'_0 \cup_{\mathcal{D}'_{01}} \mathcal{D}'_1) \arrow[d, "\iota'_*"] \\ \mathrm{PSh}(\mathcal{D}_0 \times_{\mathcal{D}_{01}} \mathcal{D}_1) \arrow[r, "\subset^*"] & \mathrm{PSh}(\mathcal{D}'_0 \times_{\mathcal{D}'_{01}} \mathcal{D}'_1) \end{tikzcd}$$

commutes, i.e. the Beck–Chevalley transformation $\subset^* \iota_* \to \iota'_* \subset^*$ is an equivalence.

(iv) In the situation of (iii), the unit $\mathrm{id} \to \iota'_*(\iota')^*$ is an equivalence on the essential image of the restricted Yoneda embedding $(\subset^* \circ y)\colon \mathcal{D}_0 \times_{\mathcal{D}_{01}} \mathcal{D}_1 \to \mathrm{PSh}(\mathcal{D}'_0 \times_{\mathcal{D}'_{01}} \mathcal{D}'_1)$.

Proof. We begin with a few preliminary observations:

(a) Using the adjunctions $\kappa_i \vdash \partial_i$ and that mapping spaces in pullbacks of categories are given by the pullbacks of mapping spaces, one sees that the two projections $\mathcal{D}_0 \times_{\mathcal{D}_{01}} \mathcal{D}_1 \to \mathcal{D}_i$ induce for $d_i, d'_i \in \mathcal{D}_i$ equivalences

$$\mathrm{Map}_{\mathcal{D}_0\times_{\mathcal{D}_{01}}\mathcal{D}_1}((d_0, \kappa_1\partial_0(d_0)), (d'_0, d'_1)) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{D}_0}(d_0, d'_0),$$
$$\mathrm{Map}_{\mathcal{D}_0\times_{\mathcal{D}_{01}}\mathcal{D}_1}((\kappa_0\partial_1(d_1), d_1), (d'_0, d'_1)) \xrightarrow{\simeq} \mathrm{Map}_{\mathcal{D}_1}(d_1, d'_1).$$

(b) The functor ι in (i) is induced by the two functors $(\mathrm{id}, \kappa_1\partial_0)\colon \mathcal{D}_0 \to \mathcal{D}_0 \times_{\mathcal{D}_{01}} \mathcal{D}_1$ and $(\kappa_0\partial_1, \mathrm{id})\colon \mathcal{D}_1 \to \mathcal{D}_0 \times_{\mathcal{D}_{01}} \mathcal{D}_1$ which are both fully faithful as a result of (a), and agree on $\mathcal{D}_{01}$ since $\partial_1\kappa_1 \simeq \mathrm{id} \simeq \partial_0\kappa_0$ as the κ_i were assumed to be fully faithful.
(c) Since fully faithful functors are preserved under pushouts [HRS25, Theorem 0.1], the inclusion functors $\mathcal{D}_i \to \mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1$ are fully faithful for $i = 0, 1$.

Combining (b) with (c) and observing that $\mathcal{D}_i \to \mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1$ for $i = 0, 1$ are jointly essentially surjective, to show the claim in (i) it suffices to show that for $d_i \in \mathcal{D}_i$, the functor induces an equivalence on the mapping space from d_0 to d_1 and on that from d_1 to d_0. By symmetry, it suffices to check the former. By Theorem 0.1 loc.cit., the map

$$\begin{aligned}|(\mathcal{D}_0)_{d_0/} \times_{\mathcal{D}_0} \mathcal{D}_{01} \times_{\mathcal{D}_1} (\mathcal{D}_1)_{/d_1}| \longrightarrow |(\mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1)_{d_0/} \times_{(\mathcal{D}_0\cup_{\mathcal{D}_{01}}\mathcal{D}_1)} (\mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1)_{/d_1}| \\ \simeq \mathrm{Map}_{\mathcal{D}_0\cup_{\mathcal{D}_{01}}\mathcal{D}_1}(d_0, d_1)\end{aligned} \tag{11}$$

induced by the inclusions of $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_{01}$ into $\mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1$ is an equivalence (for the equivalence, see Remark 3.4 loc.cit.). Here $|-|\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}$ denotes the left adjoint to the inclusion $\mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$. Now consider the commutative diagram of pullbacks of categories

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{D}_0}(d_0, \kappa_0\partial_1(d_1)) \arrow[r] \arrow[d] & (\mathcal{D}_0)_{d_0/} \times_{\mathcal{D}_0} (\mathcal{D}_{01} \times_{\mathcal{D}_1} (\mathcal{D}_1)_{/d_1}) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & (\mathcal{D}_0)_{d_0/} \arrow[d, "\mathrm{forget}"] \\
* \arrow[r, "(\partial_1(d_1), \kappa_1\partial_1(d_1) \xrightarrow{\epsilon_1} d_1)"] & \mathcal{D}_{01} \times_{\mathcal{D}_1} (\mathcal{D}_1)_{/d_1} \arrow[r, "\kappa_0\mathrm{pr}_1"] & \mathcal{D}_0.
\end{tikzcd}$$

The leftmost vertical map is a cocartesian fibration (it classifies the functor $\mathrm{Map}_{\mathcal{D}_0}(d_0, -)$), so as cocartesian fibrations are closed under pullback, pr_2 is one as well and is thus by [Lur09, 4.1.2.15] smooth in the sense of 4.1.2.9 loc.cit.. The bottom left horizontal map is the inclusion of a terminal object and therefore cofinal, so by 4.1.2.10 loc.cit. the upper left horizontal map is also cofinal and thus an equivalence after applying $|-|$ by 4.1.1.3 (3) loc.cit.. Combining this with the equivalence (11), we conclude that the composition

$$\mathrm{Map}_{\mathcal{D}_0}(d_0, \kappa_0\partial_1(d_1)) \to \mathrm{Map}_{\mathcal{D}_0\cup_{\mathcal{D}_{01}}\mathcal{D}_1}(d_0, \kappa_0\partial_1(d_1)) \simeq \mathrm{Map}_{\mathcal{D}_0\cup_{\mathcal{D}_{01}}\mathcal{D}_1}(d_0, \kappa_1\partial_1(d_1)) \to \mathrm{Map}_{\mathcal{D}_0\cup_{\mathcal{D}_{01}}\mathcal{D}_1}(d_0, d_1)$$

induced by the inclusion $\mathcal{D}_0 \to \mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1$ and the counit $\kappa_1\partial_1(d_1) \to d_1$, is an equivalence. This implies the claim in (i), since postcomposition of this composition with the map induced by ι is equivalent to the identity on $\mathrm{Map}_{\mathcal{D}_0}(d_0, \kappa_0\partial_1(d_1))$ as a result of (b).

We now turn to proving (ii). By the limit-formula for right Kan extensions, the unit map in the claim is naturally equivalent to the map $X(d_0, d_1) \to \lim_{(\iota(d)\to(d_0,d_1))} X(\iota(d))$ where the limit is taken over $(\mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1)_{/(d_0,d_1)} \coloneqq (\mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1) \times_{(\mathcal{D}_0\times_{\mathcal{D}_{01}}\mathcal{D}_1)} (\mathcal{D}_0 \times_{\mathcal{D}_{01}} \mathcal{D}_1)_{/(d_0,d_1)}$. The commutative square of fully faithful inclusions

$$\begin{tikzcd}
(\mathcal{D}_{01})_{/(d_0,d_1)} \arrow[r] \arrow[d] & (\mathcal{D}_1)_{/(d_0,d_1)} \arrow[d] \\
(\mathcal{D}_0)_{/(d_0,d_1)} \arrow[r] & (\mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1)_{/(d_0,d_1)}
\end{tikzcd} \tag{12}$$

is a pushout (e.g. by an application of (i)), so $\lim_{(\iota(d)\to(d_0,d_1))} X(\iota(d))$ is the pullback of the limits of the restriction of the diagram to the full subcategories in (12). The latter all have terminal objects induced by the units of the $(\kappa_i \dashv \partial_i)$-adjunction (for example $(\epsilon, \mathrm{id})\colon (\kappa_0\partial_0(d_0), d_1) \to (d_0, d_1)$ is a terminal object in $(\mathcal{D}_1)_{/(d_0,d_1)}$), so the claim follows.

We prove (iii) more generally: assume we have *maps of adjunctions* (maps of bicartesian fibrations over Δ^1 [Lur17, 4.7.4], e.g. obtained by restricting an adjunction to full subcategories)

$$\begin{tikzcd}
\mathcal{D}'_0 \arrow[r, shift right, "\partial'_0"'] \arrow[d, "\phi_0"'] & \mathcal{D}'_{01} \arrow[l, shift right, "\kappa'_0"'] \arrow[d, "\phi_{01}"] \\
\mathcal{D}_0 \arrow[r, shift right, "\partial_0"'] & \mathcal{D}_{01} \arrow[l, shift right, "\kappa_0"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{D}'_{01}\phi_0 \arrow[r, shift left, "\kappa'_1"] \arrow[d, "\phi_{01}"'] & \mathcal{D}'_1 \arrow[l, shift left, "\partial'_1"] \arrow[d, "\phi_1"] \\
s\mathcal{D}_{01} \arrow[r, shift left, "\kappa_1"] & \mathcal{D}_1 \arrow[l, shift left, "\partial_1"]
\end{tikzcd} \tag{13}$$

where the κ_i and κ'_i are fully faithful. In addition to commutativity of the four squares obtained from (13) by forgetting the κ_i and κ'_i or the ∂_i and ∂'_i, maps of adjunctions give compatibilities between the unit transformations, which yields a map of commutative diagrams

$$\text{from}\quad \begin{tikzcd} \mathcal{D}'_0 \arrow[d, "\mathrm{id}"] & \mathcal{D}'_{01} \arrow[l, "\kappa'_0"'] \arrow[r, "\kappa'_1"] \arrow[d, "\mathrm{id}"] & \mathcal{D}'_1 \arrow[d, "\mathrm{id}"] \\ \mathcal{D}'_0 \arrow[r, "\partial_0"'] & \mathcal{D}'_{01} & \mathcal{D}'_1 \arrow[l, "\partial_1"] \end{tikzcd} \quad\text{to}\quad \begin{tikzcd} \mathcal{D}_0 \arrow[d, "\mathrm{id}"] & \mathcal{D}_{01} \arrow[l, "\kappa_0"'] \arrow[r, "\kappa_1"] \arrow[d, "\mathrm{id}"] & \mathcal{D}_1 \arrow[d, "\mathrm{id}"] \\ \mathcal{D}_0 \arrow[r, "\partial_0"'] & \mathcal{D}'_{01} & \mathcal{D}_1. \arrow[l, "\partial_1"] \end{tikzcd}$$

This yields an equivalence $\iota\phi_\sqcup \simeq \phi_\times\iota'$ where $\phi_\sqcup$ and $\phi_\times$ are the induced functors between the pushouts of the top rows or the pullbacks of the bottom rows, respectively. The more general version of (iii) we show is that the diagram

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{D}_0 \cup_{\mathcal{D}_{01}} \mathcal{D}_1) \arrow[r, "\phi^*_\sqcup"] \arrow[d, "\iota_*"'] & \mathrm{PSh}(\mathcal{D}'_0 \cup_{\mathcal{D}'_{01}} \mathcal{D}'_1) \arrow[d, "\iota'_*"] \\ \mathrm{PSh}(\mathcal{D}_0 \times_{\mathcal{D}_{01}} \mathcal{D}_1) \arrow[r, "\phi^*_\times"] & \mathrm{PSh}(\mathcal{D}'_0 \times_{\mathcal{D}'_{01}} \mathcal{D}'_1), \end{tikzcd}$$

commutes, that is, the Beck–Chevalley transformation $\phi^*_\times\iota_* \to \iota'_*\phi^*_\sqcup$ is an equivalence. It suffices to prove this after precomposition with ι^* since the latter is essentially surjective as a result of (i). Then the transformation has the form $\phi^*_\times\iota_*\iota^* \to \iota'_*\phi^*_\sqcup\iota^* \simeq \iota'_*(\iota')^*\phi_\times$ and the claim that it is an equivalence follows from the description of $\iota'_*(\iota')^*$ and $\iota_*\iota^*$ from (ii).

Finally, (iv) follows by combining (ii), observation (a), and the fact that mapping spaces in pullbacks of categories are the pullbacks of the mapping spaces. □

3.3. **Proof of Theorem 3.2.** Equipped with Lemma 3.5, we now move towards the proof of Theorem 3.2. Setting $\mathcal{D}\mathrm{isc}^{\shortparallel}_d := \mathcal{D}\mathrm{isc}^\partial_d \times_{\mathcal{D}\mathrm{isc}_{d-1}} \mathcal{D}\mathrm{isc}^\partial_d$, the gluing functor $\gamma\colon \mathcal{M}\mathrm{an}^{\shortparallel}_d \to \mathcal{M}\mathrm{an}_d$ from Section 3.1.3 restricts to $\gamma\colon \mathcal{D}\mathrm{isc}^{\shortparallel}_d \to \mathcal{D}\mathrm{isc}_d$, so writing $\mathcal{D}\mathrm{isc}^{\shortparallel}_{d,\le\bullet} := \gamma^{-1}(\mathcal{D}\mathrm{isc}_{d,\le\bullet})$, we get a map $\gamma\colon \mathcal{D}\mathrm{isc}^{\shortparallel}_{d,\le\bullet} \to \mathcal{D}\mathrm{isc}_{d,\le\bullet}$ of towers. There is also a commutative square of towers

$$\begin{tikzcd} \mathcal{D}\mathrm{isc}_{d-1,\le\bullet} \arrow[r, "\kappa"] \arrow[d, "\kappa"'] & \mathcal{D}\mathrm{isc}^\partial_{d,\le\bullet} \arrow[d, "(\mathrm{id},\kappa\partial(-))"] \\ \mathcal{D}\mathrm{isc}^\partial_{d,\le\bullet} \arrow[r, "(\kappa\partial(-)\text{,}\mathrm{id})"'] & \mathcal{D}\mathrm{isc}^{\shortparallel}_{d,\le\bullet} \end{tikzcd}$$

which induces a map of towers $j\colon \mathcal{D}\mathrm{isc}^{\ell r}_{d,\le\bullet} \to \mathcal{D}\mathrm{isc}^{\shortparallel}_{d,\le\bullet}$ out of the tower of levelwise pushouts of categories $\mathcal{D}\mathrm{isc}^{\ell r}_{d,\le\bullet} := \mathcal{D}\mathrm{isc}^\partial_{d,\le\bullet} \cup_{\mathcal{D}\mathrm{isc}_{d-1,\le\bullet}} \mathcal{D}\mathrm{isc}^\partial_{d,\le\bullet}$. As further preparation for the proof of Theorem 3.2, we will show that left Kan extension along γ and right Kan extension along j interact well with restriction along the inclusions between the $\mathcal{D}\mathrm{isc}$-subcategories:

Lemma 3.6. *The following diagrams commute for* $1 \le k \le l \le \infty$

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\shortparallel}_{d,\le l}) \arrow[r, "\subset^*"] \arrow[d, "\gamma_!"'] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\shortparallel}_{d,\le k}) \arrow[d, "\gamma_!"] \\ \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le l}) \arrow[r, "\subset^*"] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le k}) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\ell r}_{d,\le l}) \arrow[r, "\subset^*"] \arrow[d, "j_*"'] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\ell r}_{d,\le k}) \arrow[d, "j_*"] \\ \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\shortparallel}_{d,\le l}) \arrow[r, "\subset^*"] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\shortparallel}_{d,\le k}) \end{tikzcd} \tag{14}$$

i.e. the Beck–Chevalley transformations $\gamma_!\subset^* \to \subset^*\gamma_!$ *and* $\mathrm{inc}^* j_* \to j_*\mathrm{inc}^*$ *are equivalences.*

Proof. The claim regarding the first square follows from the argument in the proof of [KK24c, Lemma 4.4] (in fact, it is a special case of it, using that source and target of $\gamma\colon \mathcal{D}\mathrm{isc}^{\shortparallel}_d \to \mathcal{D}\mathrm{isc}_d$ can be seen to be equivalent to the underlying categories of the symmetric monoidal envelope of unital operads, and the functor γ to be induced by a map of operads).

Regarding the second square, we first recall that $\mathcal{D}\mathrm{isc}^{\shortparallel}_{d,\le n}$ was defined as a certain full subcategory of the pullback $\mathcal{D}\mathrm{isc}^{\shortparallel}_d = \mathcal{D}\mathrm{isc}^\partial_d \times_{\mathcal{D}\mathrm{isc}_{d-1}} \mathcal{D}\mathrm{isc}^\partial_d$, namely the preimage of $\mathcal{D}\mathrm{isc}_{d,\le n}$ under γ. This is contained in the full subcategory $\mathcal{D}\mathrm{isc}^\partial_{d,\le n} \times_{\mathcal{D}\mathrm{isc}_{d-1,\le n}} \mathcal{D}\mathrm{isc}^\partial_{d,\le n} \subset \mathcal{D}\mathrm{isc}^{\shortparallel}_d$, but is strictly smaller. However, since right Kan extension along a full subcategory inclusion is fully faithful, it suffices to prove commutativity of the second square when replacing the subcategories $\mathcal{D}\mathrm{isc}^{||}_{d,\le n}$ with $\mathcal{D}\mathrm{isc}^{\partial}_{d,\le n} \times_{\mathcal{D}\mathrm{isc}_{d-1,\le n}} \mathcal{D}\mathrm{isc}^{\partial}_{d,\le n}$ for $n = k, l$, and then the claim becomes an instance of Lemma 3.5 (iii). □

As a result of Lemma 3.6, we have maps of towers

$$j_* \colon \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\ell r}_{d,\le\bullet}) \longrightarrow \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{||}_{d,\le\bullet}) \quad\text{and}\quad \gamma_! \colon \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{||}_{d,\le\bullet}) \to \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le\bullet}). \tag{15}$$

Now consider the following (potentially non-commutative) diagram of towers of categories

$$\begin{tikzcd} \mathcal{M}\mathrm{an}^{||}_d \arrow[r, "y"] \arrow[d, "\gamma"'] & \mathrm{PSh}(\mathcal{M}\mathrm{an}^{||}_d) \arrow[r, "(\iota^{||})^*"] \arrow[d, "\gamma_!"] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{||}_{d,\le\bullet}) \arrow[r, "j^*"] \arrow[d, "\gamma_!"] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}^{\ell r}_{d,\le\bullet}) \arrow[dl, bend left, "\gamma_! j_*"] \\ \mathcal{M}\mathrm{an}_d \arrow[r, "y"] & \mathrm{PSh}(\mathcal{M}\mathrm{an}_d) \arrow[r, "\iota^*"] & \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le\bullet}) & \end{tikzcd} \tag{16}$$

The bottom row agrees by definition with the bottom map in the claim of Theorem 3.2. Moreover, recalling the definition of the top map $E^{||}$ in Theorem 3.2 and using that $\mathrm{PSh}(-)\colon \mathcal{C}\mathrm{at}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$ sends pushouts to pullbacks, one sees that the top row in (16) agrees with $E^{||}$, so setting $\gamma^{\mathcal{D}\mathrm{isc}} \coloneqq \gamma_! j_*$ in order to prove Theorem 3.2 it suffices to show that the outer two compositions $\mathcal{M}\mathrm{an}^{||}_d \to \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\le\bullet})$ are equivalent as maps of towers. The two ingredients for this are:

Lemma 3.7. *The unit* $\mathrm{id} \to j_* j^*$ *is an equivalence on the essential image of* $(\iota^{||})^* \circ y$.

Proof. By an application of the triangle identities, one sees that it suffices to show the claim after replacing the tower $\mathcal{D}\mathrm{isc}^{||}_{d,\le\bullet}$ by the larger tower $\mathcal{D}\mathrm{isc}^{\partial}_{d,\le\bullet} \times_{\mathcal{D}\mathrm{isc}_{d-1,\le\bullet}} \mathcal{D}\mathrm{isc}^{\partial}_{d,\le\bullet}$ in which case it is an instance of Lemma 3.5 (iv). □

Lemma 3.8. *The second square in* (16) *commutes, i.e. the Beck–Chevalley transformation* $\gamma_!(\iota^{||})^* \to \iota^*\gamma_!$ *is an equivalence.*

Proof. Since we have already seen in (15) that $\gamma_!$ is a map of towers, it suffices to prove the statement for $\bullet = \infty$. Both sides of the Beck–Chevalley transformation $\delta\colon \gamma_!(\iota^{||})^* \to \iota^*\gamma_!$ commute with colimits, so since presheaf-categories are generated under colimits by representables, it suffices in view of Remark 3.1 to show that for each d-manifold M with no boundary and codimension 0 submanifolds $M_\ell, M_r \subset M$ that intersect in their common boundary $P \coloneqq \partial M_\ell = \partial M_r$, the transformation δ is an equivalence on $\mathrm{Map}_{\mathcal{M}\mathrm{an}^{||}_d}(-, M^{||}) \in \mathrm{PSh}(\mathcal{M}\mathrm{an}^{||}_d)$ for $M^{||} \coloneqq (M_\ell, M_r, \mathrm{id}) \in \mathcal{M}\mathrm{an}^{||}_d$. To do so, we consider the poset $\mathcal{O}^{||}_\infty(M)$ of open subsets $O \subset M$ such that the manifolds with boundary $O \cap M_\ell$ and $O \cap M_r$ are both contained in $\mathcal{D}\mathrm{isc}^{\partial}_d$. Inclusion gives a functor $\mathcal{O}^{||}_\infty(M) \to (\mathcal{M}\mathrm{an}^{||}_d)_{/M^{||}}$ that sends $O \in \mathcal{O}^{||}_\infty(M)$ to $O^{||} \coloneqq (O \cap M_\ell, O \cap M_r, \mathrm{id}_{O\cap P}) \subset M^{||}$. Now consider the commutative square in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}_d)$

$$\begin{tikzcd} \gamma_!(\iota^{||})^*\big(\mathrm{colim}_{O\in\mathcal{O}^{||}_\infty(M)} \mathrm{Map}_{\mathcal{M}\mathrm{an}^{||}_d}(-, O^{||})\big) \arrow[r, "①", "\delta"'] \arrow[d, "②"] & \iota^*\gamma_!\big(\mathrm{colim}_{O\in\mathcal{O}^{||}_\infty(M)} \mathrm{Map}_{\mathcal{M}\mathrm{an}^{||}_d}(-, O^{||})\big) \arrow[d, "③"] \\ \gamma_!(\iota^{||})^*\big(\mathrm{Map}_{\mathcal{M}\mathrm{an}^{||}_d}(-, M^{||})\big) \arrow[r, "④", "\delta"'] & \iota^*\gamma_!\big(\mathrm{Map}_{\mathcal{M}\mathrm{an}^{||}_d}(-, M^{||})\big). \end{tikzcd}$$

We will show that ①-③ are equivalences, which implies that ④ is one as well, so the claim will follow. To do this, we repeatedly use the facts that restriction and left Kan extension preserve colimits and that left Kan extension preserves representables. From these facts, together with the observation that $O^{||} \in \mathcal{D}\mathrm{isc}^{||}_d$, one sees that ① is equivalent to the identity on $\mathrm{colim}_{O\in\mathcal{O}^{||}_\infty(M)} \mathrm{Map}_{\mathcal{D}\mathrm{isc}_d}(-, O)$, so in particular an equivalence. Using the above facts again, the map ③ is equivalent to the map $\mathrm{colim}_{O\in\mathcal{O}^{||}_\infty(M)} \mathrm{Map}_{\mathcal{M}\mathrm{an}_d}(\iota(-), O) \to \mathrm{Map}_{\mathcal{M}\mathrm{an}_d}(\iota(-), M)$. Since $\{O\}_{O\in\mathcal{O}^{||}_\infty(M)}$ is a complete Weiss ∞-cover of M in the sense of [KK24a, Definition 6.3], the proof of Lemma 6.4 loc.cit. shows that this map is indeed an equivalence. Since it will be relevant later, recall that the key step in the proof of this result in loc.cit. is an application of [DI04, Proposition 4.6 (c)] to the open cover $\{F_n(O)\}_{O\in\mathcal{O}^{||}_\infty(M)}$ of the space

$F_n(M)$ of ordered configurations of n points in M, for $n \geq 0$, using that this is a complete cover in the sense of Definition 4.5 loc.cit.. We will now show by a similar argument that the map $\mathrm{colim}_{O \in \mathcal{O}^{\shortparallel}_\infty(M)} \mathrm{Map}_{\mathcal{M}\mathrm{an}^{\shortparallel}_d}(\iota^{\shortparallel}(-), O^{\shortparallel}) \to \mathrm{Map}_{\mathcal{M}\mathrm{an}^{\shortparallel}_d}(\iota^{\shortparallel}(-), M^{\shortparallel})$ is an equivalence, which will imply that $\gamma_!(-)$ of it—which is precisely ②—is an equivalence as well, so the claim will follow. Adapting the argument in the proof of [KK24a, Lemma 6.4] to this map reduces the claim to showing that for $r, s, t \geq 0$ the open cover $\{F_{r,s,t}(O)\}_{O \in \mathcal{O}^{\shortparallel}_\infty(M)}$ of the space $F_{r,s,t}(M)$ of ordered configurations of $r+s+t$ points in M where the first r points lie in $\mathrm{int}(M_\ell)$, the second s points in P, and the final t points in $\mathrm{int}(M_r)$, is a complete cover. But we have $F_{r,s,t}(O) = F_{r+s+t}(O) \cap F_{r,s,t}(M)$, so the claim follows from the corresponding fact for $\{F_{r+s+t}(O)\}_{O \in \mathcal{O}^{\shortparallel}_\infty(M)}$ we have already used above by observing that complete covers are preserved by taking intersections with a fixed subspace. □

We end the section by finishing the proof of Theorem 3.2 and deducing Theorem 2.2.

Proof of Theorem 3.2. By the discussion above the diagram (16), it suffices to show that the outermost two compositions in it agree. As a result of Lemma 3.7, it suffices to show that the two leftmost squares in the diagram commute. For the second square, this is Lemma 3.8 and for the first square it is an instance of the naturality of the Yoneda embedding. □

Proof of Theorem 2.2. For manifolds M_i and diffeomorphism φ_i as in the statement, we have that $E_{M_0 \cup_{\varphi_i} M_1}$ in $\mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k})$ for $i = 0, 1$ are the values of $(M_0, M_1, \varphi_i) \in \mathcal{M}\mathrm{an}^{\shortparallel}_d$ under the counterclockwise composition in the square of Theorem 3.2 for $\bullet = k$, so by commutativity they are also the values under the clockwise composition. But the assumption implies that their images $(E^{\partial}_{M_0}, E^{\partial}_{M_0}, E_{\varphi_i})$ under the top horizontal map $E^{\shortparallel}$ agree, so the claim follows. The addendum follows in the same way, using a variant of Theorem 3.2 for oriented manifolds which is proved in the same way as the non-oriented version. □

4. Stable collapse maps of $\mathcal{D}$isc-presheaves

In this section we establish another one of the three ingredients that we assumed in Section 2, namely Theorem 2.6 regarding the naturality of Atiyah duality in equivalences of 2-truncated presheaves. We adopt the notation from Section 2.2.

4.1. **Proof of Theorem 2.6.** We begin by fixing some notation:

(a) For a space B, we write $\mathcal{S}_{/B}$, $(\mathcal{S}_{/B})_*$, and Sp_B for the ∞-categories of spaces over B, retractive spaces over B (spaces over B equipped with a section), and parametrised spectra over B, respectively. By straightening, they are equivalent to the functor categories $\mathrm{Fun}(B, \mathcal{S})$, $\mathrm{Fun}(B, \mathcal{S}_*)$, and $\mathrm{Fun}(B, \mathrm{Sp})$, respectively.
(b) We denote various fibrewise constructions in $\mathcal{S}_{/B}$, $(\mathcal{S}_{/B})_*$, and Sp_B by adding a B-subscript. For instance, $\Sigma^\infty_B(-)\colon (\mathcal{S}_{/B})_* \to \mathrm{Sp}_B$ denotes taking fibrewise suspension spectrum, $(-)_{+,B}\colon \mathcal{S}_{/B} \to (\mathcal{S}_{/B})_*$ fibrewise adding a disjoint basepoint, $(-) \otimes_B (-)\colon \mathrm{Sp}_B \times \mathrm{Sp}_B \to \mathrm{Sp}_B$ fibrewise tensor product, $\mathrm{map}_B(-,-)\colon \mathrm{Sp}_B^{\mathrm{op}} \times \mathrm{Sp}_B \to \mathrm{Sp}_B$ fibrewise mapping spectra, and $D_B(-) = \mathrm{map}_B(-, \mathbf{S}_B)\colon \mathrm{Sp}_B^{\mathrm{op}} \to \mathrm{Sp}_B$ fibrewise Spanier–Whitehead dual given by taking fibrewise mapping spectra into the constant parametrized spectrum $\mathbf{S}_B$ with fibre $\mathbf{S}$.
(c) For a map of spaces $f\colon B \to C$, the restriction functor $f^*\colon \mathrm{Sp}_C \to \mathrm{Sp}_B$ has a left adjoint $f_!\colon \mathrm{Sp}_B \to \mathrm{Sp}_C$, given by left Kan extension under straightening. In particular, for the constant map $f = p\colon B \to *$ the adjoint $p_!\colon \mathrm{Sp}_B \to \mathrm{Sp}_* = \mathrm{Sp}$ is given by taking colimits.
(d) For a map $\varphi\colon V \to W$ in $\mathcal{S}_{/B}$, we denote the pushout in $\mathcal{S}$ of $V \to W$ along $V \to B$ as $C_B(\varphi)$ and think of it as a relative mapping cone. Note that $C_B(\varphi)$ comes with canonical maps to and from B, which turn it into a retractive space. This construction is natural in

that a commutative square in $\mathcal{S}_{/B}$

$$\begin{tikzcd} V' \arrow[r] \arrow[d, "\varphi"'] & V \arrow[d, "\varphi'"] \\ W' \arrow[r] & W \end{tikzcd} \tag{17}$$

induces a map $C_B(\varphi') \to C_B(\varphi)$ in $(\mathcal{S}_{/B})_*$. For $V' = \varnothing$ and $W' = W$ this in particular gives a map $W_{+,B} \to C_B(\varphi)$ n $(\mathcal{S}_{/B})_*$. An important class of examples for us is when φ is the projection $S(\xi) \to B$ of the $(d-1)$-dimensional spherical fibration of a d-dimensional vector bundle ξ over B given by removing the 0-section. In this case, $C_B(\varphi) \to B$ agrees with the retractive space given as the d-dimensional spherical fibration obtained from ξ by fibrewise one-point compactification, or equivalently obtained from $S(\xi)$ by fibrewise cone. Then $\Sigma^\infty_B C_B(\varphi)$ is the usual parametrised spectrum with fibres $\mathbf{S}^d$ associated to a d-dimensional vector bundle ξ, so $p_!(\Sigma^\infty_B C_B(\varphi)) \simeq \mathrm{Th}(\xi)$ is its Thom-spectrum.

Construction 4.1. Given a space B, we view $B \times B$ as a space over B via the projection to the first coordinate. Given $V \in \mathcal{S}_{/B}$ and a map $\varphi\colon V \to B \times B$ in $\mathcal{S}_{/B}$, we may apply $\Sigma^\infty_B(-)$ to the map $(B\times B)_{+,B} \to C_B(\varphi)$ from (d) to arrive at a map $p^*(\Sigma^\infty_+ B) \simeq \Sigma^\infty_B((B\times B)_{+,B}) \to \Sigma^\infty_B C_B(\varphi)$. Combining this with the fibrewise evaluation map $D_B(\Sigma^\infty_B C_B(\varphi)) \otimes_B \Sigma^\infty_B C_B(\varphi) \to p^*(\mathbf{S})$, we obtain a map $D_B(\Sigma^\infty_B C_B(\varphi)) \otimes_B p^*(\Sigma^\infty_+ B) \to p^*(\mathbf{S})$ which yields using the tensor-hom adjunction a map $D_B(\Sigma^\infty_B C_B(\varphi)) \to \mathrm{map}_B(p^*(\Sigma^\infty_+ B), p^*(\mathbf{S})) \simeq p^*\mathrm{map}(\Sigma^\infty_+ B, \mathbf{S}) = p^* D(\Sigma^\infty_+ B)$ which in turn yields via the $(p_! \dashv p^*)$-adjunction a map of spectra of the form

$$p_! D_B(\Sigma^\infty_B C_B(\varphi)) \longrightarrow D(\Sigma^\infty_+ B). \tag{18}$$

Construction 4.2. Suppose we are given a d-dimensional vector bundle $\xi\colon B \to \mathrm{BO}(d)$ over a space B and a commutative square in $\mathcal{S}$ of the form

$$\begin{tikzcd} S(\xi) \arrow[r] \arrow[d, "\pi"'] & C \arrow[d, "i"] \\ B \arrow[r, "\Delta"] & B \times B \end{tikzcd} \tag{19}$$

where $S(\xi)$ is the $(d-1)$-dimensional spherical fibration induced by ξ, the bottom horizontal map is the diagonal, and i is any map. Viewing this square as a square in $\mathcal{S}_{/B}$ via the projection of $B \times B$ onto the first coordinate, the naturality in (d) yields a map $C_B(S(\xi)) \to C_B(i)$ and thus by applying $p_! D_B(\Sigma^\infty_B(-))$ a map $p_! D_B(\Sigma^\infty_B C_B(i)) \to p_! D_B(\Sigma^\infty_B C_B(S(\xi))) \simeq \mathrm{Th}(-\xi)$ (see (d) for the final equivalence). This features in a natural zig-zag of spectra

$$\mathbf{S} \simeq D(\Sigma^\infty_+ *) \longrightarrow D(\Sigma^\infty_+ B) \longleftarrow p_! D_B(\Sigma^\infty_B C_B(i)) \longrightarrow \mathrm{Th}(-\xi) \tag{20}$$

where the first map results from applying $D(\Sigma^\infty_+(-))$ to the map $p\colon B \to *$, the second map is an instance of (18), and the final map is the one we just described.

Example 4.3. For us, the most important example of a square as in (19) is the square in $\mathcal{S}$

$$\begin{tikzcd} S(TM) \arrow[r] \arrow[d, "\pi"'] & M \times M \backslash \Delta \arrow[d, "\subset"] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd} \tag{21}$$

where $B = M$ is a d-manifold, $\xi = TM$ is its tangent bundle and $i\colon C \to B \times B$ is the inclusion $M \times M \backslash \Delta \subset M \times M$ of the complement of the diagonal. This square arises as follows: a choice of tubular neighbourhood of the diagonal $M \subset M \times M$ yields an embedding of manifold pairs $(TM, TM\backslash\{0\text{-section})) \hookrightarrow (M \times M, M \times M\backslash\Delta)$ from which one obtains (21) by using the equivalence of pairs of spaces $(TM, TM\backslash\{0\text{-section}\}) \simeq (M, S(TM))$ induced by the projection.

If M is a *closed* manifold, then the wrong-way map in the instance of the zig-zag (20) associated to (21) turns out to be an equivalence (see e.g. [ABG18, Proposition 4.12]) and the resulting map $\mathbf{S} \to \mathrm{Th}(-TM)$ agrees with the stable collapse map (see e.g. Section 4.3 loc.cit. or [NS24, Section 4.2] for explanations of this).

Example 4.4. Any k-truncated presheaf $X \in \mathrm{PSh}(\mathcal{D}\mathrm{isc}_{d,\leq k})$ for $k \geq 2$ also yields an example of a square as in (19): using the notation $|X|$ and ξ_X from Section 2.2 and writing $\mathrm{Map}(-,-)$ and $\mathrm{Aut}(-)$ for the mapping and automorphism spaces in $\mathcal{D}\mathrm{isc}_{d,\leq k}$, we have a square

$$\begin{tikzcd} S(\xi_X) \simeq \Big(\frac{\mathrm{Map}(\mathbf{R}^d,\mathbf{R}^d)^{\times 2}}{\mathrm{Aut}(\mathbf{R}^d)^{\times 2}} \times_{\mathrm{Aut}(\mathbf{R}^d)} X(\mathbf{R}^d)\Big) \arrow[r] \arrow[d, "\pi"'] & \frac{X(\mathbf{R}^d \sqcup \mathbf{R}^d)}{\mathrm{Aut}(\mathbf{R}^d)^{\times 2}} \arrow[d] \\ |X| \arrow[r, "\Delta"] & |X| \times |X|. \end{tikzcd} \tag{22}$$

obtained as follows: the map $\mathrm{Map}(\mathbf{R}^d \sqcup \mathbf{R}^d, \mathbf{R}^d) \to \mathrm{Map}(\mathbf{R}^d,\mathbf{R}^d)^{\times 2}$ given by precomposition with the inclusions, and the functoriality of X, yields a map of pairs

$$\begin{gathered} \Big(\big(\tfrac{\mathrm{Map}(\mathbf{R}^d,\mathbf{R}^d)}{\mathrm{Aut}(\mathbf{R}^d)}\big)^{\times 2} \times_{\mathrm{Aut}(\mathbf{R}^d)} X(\mathbf{R}^d),\ \tfrac{\mathrm{Map}(\mathbf{R}^d \sqcup \mathbf{R}^d,\mathbf{R}^d)}{\mathrm{Aut}(\mathbf{R}^d)^{\times 2}} \times_{\mathrm{Aut}(\mathbf{R}^d)} X(\mathbf{R}^d)\Big) \\ \downarrow \\ \Big(\big(\tfrac{X(\mathbf{R}^d)}{\mathrm{Aut}(\mathbf{R}^d)}\big)^{\times 2},\ \tfrac{X(\mathbf{R}^d \sqcup \mathbf{R}^d)}{\mathrm{Aut}(\mathbf{R}^d)^{\times 2}}\Big) \end{gathered} \tag{23}$$

from which one obtains (22) by using $\mathrm{Map}(\mathbf{R}^d,\mathbf{R}^d)/\mathrm{Aut}(\mathbf{R}^d) \simeq *$, $|X| = * \times_{\mathrm{Aut}(\mathbf{R}^d)} X(\mathbf{R}^d)$ together with the $\mathrm{GL}_d(\mathbf{R})$-equivariant equivalence $\mathrm{Map}(\mathbf{R}^d \sqcup \mathbf{R}^d, \mathbf{R}^d)/\mathrm{Aut}(\mathbf{R}^d)^{\times 2} \simeq \mathbf{R}^d \backslash \{0\}$ to identify the source in (23) with the pair $(S(\xi_X), |X|)$ given by the projection.

The previous two examples turn out to be closely related:

Lemma 4.5. *For a smooth d-manifold M, there is an equivalence of squares between* (21) *and* (22) *for $X = E_M$. More specifically, there is an equivalence of spherical fibrations $S(TM) \simeq S(\xi_{E_M})$ such that the resulting equivalences between the left vertical and bottom horizontal maps in* (21) *and* (22) *can be lifted to an equivalence of squares.*

Proof. To see this, one first uses that evaluation at the centre induces an equivalence between the map of pairs (23) for $X = E_M = \mathrm{Emb}(-, M)$ and the map of pairs

$$\begin{gathered} \Big(\mathrm{Emb}(*,\mathbf{R}^d)^{\times 2} \times_{\mathrm{Aut}(\mathbf{R}^d)} \mathrm{Emb}(\mathbf{R}^d, M), \mathrm{Emb}(* \sqcup *, \mathbf{R}^d) \times_{\mathrm{Aut}(\mathbf{R}^d)} \mathrm{Emb}(\mathbf{R}^d, M)\Big) \\ \downarrow \\ (\mathrm{Emb}(*, M)^{\times 2}, \mathrm{Emb}(* \sqcup *, M)) = (M \times M, M \times M \backslash \Delta). \end{gathered} \tag{24}$$

induced by composition of embeddings. It thus suffices to construct an equivalence of pairs

$$(TM, TM \backslash \{0\text{-section}\}) \simeq \Big(\mathrm{Emb}(*,\mathbf{R}^d)^{\times 2} \times_{\mathrm{Aut}(\mathbf{R}^d)} \mathrm{Emb}(\mathbf{R}^d, M), \mathrm{Emb}(* \sqcup *, \mathbf{R}^d) \times_{\mathrm{Aut}(\mathbf{R}^d)} \mathrm{Emb}(\mathbf{R}^d, M)\Big)$$

such that its postcomposition with (24) agrees with the map $(TM, TM \backslash \{0\text{-section}\}) \to (M \times M, M \times M \backslash \Delta)$ used in Example 4.3 (involving the choice of a tubular neighbourhood). By the uniqueness of tubular neighbourhoods, we may assume that the tubular neighbourhood $TM \hookrightarrow M \times M$ involved is on the first coordinate given by the projection and on the second coordinate given by a smooth retraction $e\colon TM \to M$ of the 0-section that is an embedding when restricted to each individual tangent space. Such a map e yields an $\mathrm{GL}_d(\mathbf{R})$-equivariant equivalence $\mathrm{Fr}(TM) \xrightarrow{\simeq} \mathrm{Emb}(\mathbf{R}^d, M)$ out of the frame bundle, which together with the $\mathrm{O}(d)$-equivariant equivalence of pairs $(\mathbf{R}^d, \mathbf{R}^d \backslash \{0\}) \xrightarrow{\simeq} (\mathrm{Emb}(*, \mathbf{R}^d)^{\times 2}, \mathrm{Emb}(* \sqcup *, \mathbf{R}^d))$ given by $\mathbf{R}^d \ni x \mapsto (0, x) \in \mathrm{Emb}(*, \mathbf{R}^d)^{\times 2}$ yields an equivalence of pairs of the form

$$\begin{gathered} (\mathbf{R}^d \times_{\mathrm{GL}_d(\mathbf{R})} \mathrm{Fr}(TM), \mathbf{R}^d \backslash \{0\} \times_{\mathrm{GL}_d(\mathbf{R})} \mathrm{Fr}(TM)) \\ \downarrow \simeq \\ \Big(\mathrm{Emb}(*,\mathbf{R}^d)^{\times 2} \times_{\mathrm{Aut}(\mathbf{R}^d)} \mathrm{Emb}(\mathbf{R}^d, M), \mathrm{Emb}(* \sqcup *, \mathbf{R}^d) \times_{\mathrm{Aut}(\mathbf{R}^d)} \mathrm{Emb}(\mathbf{R}^d, M)\Big) \end{gathered}$$

Going through the construction, the precomposition of this equivalence with the standard identification of its source with $(TM, TM \backslash 0\text{-section})$ has the desired properties. $\square$

Theorem 2.6 now follows by merely putting things together:

Proof of Theorem 2.6. Specialising (20) to Example 4.4 yields a natural zig-zag

$$\mathbf{S} \longrightarrow D(\Sigma^\infty_+|X|) \longleftarrow p_! D_B(\Sigma^\infty C_B(i)) \longrightarrow \mathrm{Th}(-\xi_X), \tag{25}$$

as in the claim. The asserted property of it in the case $X = E_M$ follows by combining Lemma 4.5 with the discussion at the end of Example 4.3. □

4.2. **Further remarks.** We end this section with some remarks on the proof of Theorem 2.6.

Remark 4.6 (Configuration space models). We constructed the square (21) from a strictly commutative square of topological spaces after replacing the lower left corner M up to equivalence with TM. There is also a way to model (21) as a strictly commutative square of topological spaces where one instead replaces the top map up to homotopy equivalence, but keeps the diagonal at the bottom (c.f. [KK24c, Remark 5.16(a)]), namely as

$$\begin{tikzcd} \partial\mathrm{FM}_2(M) \arrow[r, "\subset"] \arrow[d] & \mathrm{FM}_2(M) \arrow[d] \\ M \arrow[r, "\Delta"] & M \times M \end{tikzcd} \tag{26}$$

where $\mathrm{FM}_2(M)$ is the Fulton–MacPherson bordification of $F_2(M) = M \times M \backslash \Delta$ (see e.g. [Sin04]), $\partial\mathrm{FM}_2(M)$ is its boundary consisting of "infinitesimal configurations," and the left vertical map is the "macroscopic location" map.

Remark 4.7 (Work of Naef–Safronov). The proof of Theorem 2.6 is closely related to work of Naef and Safronov [NS24]. They explain in Section 4.2 loc.cit. how the square (26) for a closed smooth manifold M can be used to show that the constant parametrised spectrum $\mathbf{S}_M$ over M is relatively dualisable in the sense of Definition 1.16 loc.cit. with dual $\Sigma^\infty_M S(-TM)$. Our construction above uses the diagram square to construct the copairing—in their notation η—while they in Proposition 4.13 loc.cit. use it to construct the pairing—in their notation PT_Δ. These contain essentially the same information (see Remark 1.3 loc.cit.).

Remark 4.8 (Topological collapse maps). Theorem 2.6 also holds in the topological setting, that is, for $\mathcal{D}\mathrm{isc}_d$ replaced by the analogous category $\mathcal{D}\mathrm{isc}^t_d$ involving *topological* embeddings, M being a closed *topological* d-manifolds M, and ξ_X being an $\mathbf{R}^d$-bundle (a fibre bundle with fibre $\mathbf{R}^d$) instead of a vector bundle, from which one obtains a spherical fibration by removing a section. In fact, the version Theorem 2.6 for smooth manifolds can be deduced from that for topological ones by left Kan extending along the forgetful functor $\mathcal{D}\mathrm{isc}^t_{d,\leq 2} \to \mathcal{D}\mathrm{isc}_{d,\leq 2}$.

We briefly explain how to obtain the claimed extension of Theorem 2.6 to the topological setting: Construction 4.2 goes through verbatim for $\mathbf{R}^d$-bundles. In Example 4.3 one replaces TM by the *topological* tangent bundle T^tM, obtained by choosing using Kister's theorem [Kis64] a normal $\mathbf{R}^d$-bundle inside the normal microbundle of the diagonal $M \subset M \times M$; this yields a tubular neighbourhood with the property used in Lemma 4.5 by construction. With this choice, the proof of Lemma 4.5 goes through with minor changes. Finally, the reference [ABG18, Proposition 4.12] we cite in Example 4.3 smooth manifolds, but the proof can be extended to the topological setting:

One picks a locally flat embedding $M \subset \mathbf{R}^{d+k}$ with a normal $\mathbf{R}^k$-bundle ν_M for $k \gg 0$ [Hir66, Theorem (B)], which is unique up to isotopy after further stabilisations by Theorem (C) loc.cit.. The Pontryagin–Thom construction applied to $M \subseteq \mathbf{R}^{d+k}$ with normal bundle ν_M gives a pointed map $S^{d+k} \to \mathrm{Th}(\nu_M)$, which yields the stable collapse map $\mathbf{S} \to \mathrm{Th}(-TM)$. Performing the Pontryagin–Thom construction to both the inclusion $M \times M \subset \mathbf{R}^{d+k} \times M$ with normal bundle ν_M, as well as its precomposition with the diagonal inclusion gives a map $\mathrm{Th}(\nu_M) \wedge M_+ \to S^{d+k} \wedge M_+$. Postcomposing with the projection to S^{d+k}, yields a pairing $\mathrm{Th}(-T^tM) \otimes \Sigma^\infty M_+ \to \mathbf{S}$, which is the evaluation map that exhibits $\mathrm{Th}(-T^tM)$ as the Spanier–Whitehead dual of $\Sigma^\infty M_+$. Using this, the proof of [ABG18, Proposition 4.12] goes through and shows that the wrong-way map in (20) is in the case of Example 4.3 an equivalence and that the constructed map $\mathbb{S} \to \mathrm{Th}(-T^t M)$ agrees with the stable collapse map.

5. Finite residuals of mapping class groups

In this section we establish the remaining ingredient that we assumed in Section 2 and we have not proved yet, namely Theorem 2.3 regarding the finite residual of the group

$$\Gamma_g := \pi_0 \operatorname{Diff}^+(W_g)$$

of isotopy classes of orientation-preserving diffeomorphisms (we omit the dependence on n in the notation) of the g-fold connected sum $W_g := \sharp^g(S^n \times S^n)$ for $n \geq 3$ odd. The main input for this is an algebraic description of these mapping class groups which was established in [Kra20]. We will also use the following properties of finite residuals $\mathrm{fr}(G)$ of discrete groups G (see Section 2.1 for the definition):

Lemma 5.1.

(i) *For a group homomorphism $\varphi\colon E \to E'$, we have $\mathrm{fr}(E) \subseteq \varphi^{-1}(\mathrm{fr}(E'))$.*

(ii) *For a monomorphism of groups $\varphi\colon E \hookrightarrow E'$ such that its image $\varphi(E) \leq E'$ has finite index, we have $\mathrm{fr}(E) = \varphi^{-1}(\mathrm{fr}(E'))$.*

(iii) *Fix a group G with finite abelianisation and $\mathrm{fr}(G) = 0$, a torsion-free abelian group A, and two central extensions of G by a fixed abelian group A,*

$$0 \longrightarrow A \xrightarrow{\iota_i} E_i \longrightarrow G \longrightarrow 0 \quad \text{for } i = 0, 1,$$

such that their extension classes in $\mathrm{H}^2(G;A)$ agree in $\mathrm{Hom}(\mathrm{H}_2(G), A)$. Then the finite residuals of E_0 and E_1 agree in the sense that

$$\mathrm{fr}(E_i) \subset \iota_i(A) \quad \text{for } i = 0, 1 \quad \text{and} \quad \iota_0^{-1}(\mathrm{fr}(E_0)) = \iota_1^{-1}(\mathrm{fr}(E_1)).$$

(iv) *Fix a group G with finitely generated abelianisation and $\mathrm{fr}(G) = 0$, a finitely generated abelian group A, and a central extension $0 \to A \to E \to G \to 0$ whose extension class in $\mathrm{H}^2(G;A)$ is trivial in $\mathrm{Hom}(\mathrm{H}_2(G), A)$. Then $\mathrm{fr}(E) = 0$.*

Proof. Item (i) and (ii) follow directly from the definition of the finite residual. The claim $\mathrm{fr}(E_i) \subset \iota_i(A)$ in (iii) follows from $\mathrm{fr}(G) = 0$ and (i) applied to $E_i \to G$, so we are left to show $\iota_0^{-1}(\mathrm{fr}(E_0)) = \iota_1^{-1}(\mathrm{fr}(E_1))$. To do so, we write k for the (finite) order of $\mathrm{H}_1(G)$ and $\lambda_i \in \mathrm{H}^2(G;A)$ for the extension class of E_i. From the assumption that the λ_i agree in $\mathrm{Hom}(\mathrm{H}_2(G), A)$ together with the naturality of the universal coefficient theorem and the fact that multiplication by k annihilates $\mathrm{Ext}(\mathrm{H}_1(G), A)$ since it annihilates $\mathrm{H}_1(G)$ we get that $k \cdot \lambda_0 = k \cdot \lambda_1 \in \mathrm{H}^2(G;A)$. The latter implies that there is a commutative diagram of groups with exact rows of the form

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\iota_0"] \arrow[d, "k\cdot(-)"'] & E_0 \arrow[r] \arrow[d, "\varphi_0"] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\bar{\iota}"] & \bar{E} \arrow[r] & G \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "\iota_1"] \arrow[u, "k\cdot(-)"] & E_1 \arrow[r] \arrow[u, "\varphi_1"'] & G \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

whose middle row is by definition the central extension classified by $k \cdot \lambda_0 = k \cdot \lambda_1$. Since A is torsion free, the leftmost vertical morphisms are monomorphisms and their images have finite index, so the same holds for the middle vertical morphisms. From (ii) applied to φ_i we thus get $\mathrm{fr}(E_i) = \varphi_i^{-1}(\mathrm{fr}(\bar{E}))$ and thus $\iota_i^{-1}(\mathrm{fr}(E_i)) = \iota_i^{-1}(\varphi_i^{-1}(\mathrm{fr}(\bar{E})))$ for $i = 0, 1$. But $\varphi_i \circ \iota_i = \bar{\iota} \circ (k \cdot (-))$ does not depend on i, so $\iota_0^{-1}(\mathrm{fr}(E_0)) = \iota_1^{-1}(\mathrm{fr}(E_1))$ as claimed.

To show (iv), note that it follows from the universal coefficient theorem that the assumption on the extension class in the statement is equivalent to $0 \to A \to E \to G \to 0$ being pulled back from an abelian extension $0 \to A \to E' \to \mathrm{H}_1(G) \to 0$ along the abelianisation $G \to \mathrm{H}_1(G)$. In particular, we have a monomorphism $E \hookrightarrow E' \times G$. Since A and $\mathrm{H}_1(G)$ are finitely generated abelian by assumption, the same holds for E', so $\mathrm{fr}(E') = 0$ and thus

$\mathrm{fr}(E' \times G) = \mathrm{fr}(E') \times \mathrm{fr}(G) = 0$, since we assumed $\mathrm{fr}(G) = 0$. The claim then follows from an application of (i) to $E \hookrightarrow E' \times G$. □

Before turning to the computation of the finite residual $\mathrm{fr}(\Gamma_g)$, we will discuss another preparatory lemma. It involves the *signature class* $\mathrm{sgn} \in \mathrm{H}^2(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z})$ which is a certain second cohomology class of the symplectic group over the integers, constructed in terms of signatures of certain symmetric bilinear forms that one can associate to a bundle of symplectic forms over surfaces (see e.g. [KRW20, p. 471] for a definition for $g \geq 3$; for $g \leq 2$ the class is defined via pullback along the standard inclusions $\mathrm{Sp}_{2g}(\mathbf{Z}) \leq \mathrm{Sp}_{2g'}(\mathbf{Z})$ for $g \leq g'$).

Lemma 5.2. *Fix $g \geq 2$, a subgroup $\Lambda \leq \mathrm{Sp}_{2g}(\mathbf{Z})$, and a cohomology class $\lambda \in \mathrm{H}^2(\Lambda; \mathbf{Z})$ such that the following conditions are satisfied:*

(i) $\Lambda \leq \mathrm{Sp}_{2g}(\mathbf{Z})$ has finite index.
(ii) The abelianisation of Λ is finite.
(iii) The class $8 \cdot \lambda \in \mathrm{H}^2(\Lambda; \mathbf{Z})$ and the pullback of the signature class $\mathrm{sgn} \in \mathrm{H}^2(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z})$ have the same image in $\mathrm{Hom}(\mathrm{H}_2(\Lambda; \mathbf{Z}), \mathbf{Z})$.

Then the extension $0 \to \mathbf{Z} \xrightarrow{\iota} E(\lambda) \to \Lambda \to 0$ classified by λ satisfies $\mathrm{fr}(E(\lambda)) = \iota(\mathbf{Z})$.

Proof. Consider the maps of extensions

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Z} \arrow[r, "\iota_\lambda"] \arrow[d, hook, "8\cdot(-)"'] & E(\lambda) \arrow[r] \arrow[d, hook, "\alpha"] & \Lambda \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbf{Z} \arrow[r, "\iota_{8\cdot\lambda}"] & E(8\cdot\lambda) \arrow[r] & \Lambda \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] & \mathbf{Z} \arrow[r, "\iota_{\mathrm{sgn}'}"] \arrow[d, equal] & E(\mathrm{sgn}') \arrow[r] \arrow[d, hook, "\beta"] & \Lambda \arrow[r] \arrow[d, hook, "\subset"] & 0 \\ 0 \arrow[r] & \mathbf{Z} \arrow[r, "\iota_{\mathrm{sgn}}"] & E(\mathrm{sgn}) \arrow[r] & \mathrm{Sp}_{2g}(\mathbf{Z}) \arrow[r] & 0 \end{tikzcd}$$

where the left-hand diagram arises from pushing out the top extension along $8 \cdot (-) \colon \mathbf{Z} \hookrightarrow \mathbf{Z}$ and the right-hand diagram arises from pulling back the bottom extension classified by the signature class along the inclusion $\Lambda \leq \mathrm{Sp}_{2g}(\mathbf{Z})$. By the discussion in [KRW20, p. 472], we have $\mathrm{fr}(E(\mathrm{sgn})) = \iota_{\mathrm{sgn}}(8 \cdot \mathbf{Z})$. Applying Lemma 5.1 (ii) to β using assumption (i) yields $\mathrm{fr}(E(\mathrm{sgn}')) = \beta^{-1}(\iota_{\mathrm{sgn}}(8 \cdot \mathbf{Z})) = \iota_{\mathrm{sgn}'}(8 \cdot \mathbf{Z})$. Moreover, using the assumptions (ii) and (iii) as well as the fact that $\mathrm{fr}(\Lambda) = 0$ by Lemma 5.1 (i) applied to the inclusion $\Lambda \leq \mathrm{Sp}_{2g}(\mathbf{Z})$, an application of Lemma 5.1 (iii) to $E(8\cdot\lambda)$ and $E(\mathrm{sgn}')$ yields $\iota_{8\cdot\lambda}^{-1}(\mathrm{fr}(E(8 \cdot \lambda))) = \iota_{\mathrm{sgn}'}^{-1}(\mathrm{fr}(E(\mathrm{sgn}'))) = 8 \cdot \mathbf{Z}$ and $\mathrm{fr}(E(8 \cdot \lambda)) \subset \iota_{8\cdot\lambda}(\mathbf{Z})$, so we conclude the equality $\mathrm{fr}(E(8 \cdot \lambda)) = \iota_{8\cdot\lambda}(8 \cdot \mathbf{Z})$. Finally, from an application of Lemma 5.1 (ii) to α, we get $\mathrm{fr}(E(\lambda)) = \alpha^{-1}(\iota_{8\cdot\lambda}(8 \cdot \mathbf{Z})) = \iota_\lambda(\mathbf{Z})$ as claimed. □

5.1. **Proof of Theorem 2.3.** We fix $g \geq 0$ and an odd integer $n \geq 3$ throughout and adopt the notation from [Kra20, Section 1]. In particular:

(a) $G_g \leq \mathrm{Sp}_{2g}(\mathbf{Z})$ denotes the image of the morphism $p\colon \Gamma_g \to \mathrm{Sp}_{2g}(\mathbf{Z})$ given acting on $\mathrm{H}_n(W_g; \mathbf{Z}) \cong \mathbf{Z}^{2g}$. The latter isomorphism involves the choice of a hyperbolic basis with respect to the intersection form on $\mathrm{H}_n(W_{g,1}; \mathbf{Z})$, which we fix once and for all. For $n = 3, 7$, we have $G_g = \mathrm{Sp}_{2g}(\mathbf{Z})$ and for $n \neq 3, 7$ the subgroup $G_g \leq \mathrm{Sp}_{2g}(\mathbf{Z})$ has finite index and agrees with a certain subgroup $G_g = \mathrm{Sp}_{2g}^q(\mathbf{Z}) \leq \mathrm{Sp}_{2g}(\mathbf{Z})$ whose definition involves a quadratic refinement (see e.g. Section 1.2 loc.cit.).

(b) Fixing an embedded codimension 0 disc $D^{2n} \subset W_g$, the groups $\Gamma_{g,1} \coloneqq \pi_0(\mathrm{Diff}_\partial(W_{g,1}))$ and $\Gamma_{g,1/2} \coloneqq \pi_0(\mathrm{Diff}_{1/2\partial}(W_{g,1}))$ are the groups of isotopy classes of diffeomorphisms of $W_{g,1} \coloneqq (\sharp^g(S^n \times S^n)) \backslash \mathrm{int}(D^{2n})$ that fix pointwise a neighbourhood of the boundary or a fixed embedded codimension 0 disc in the boundary, respectively. We need a few facts on these groups: Firstly, the morphism $\Gamma_{g,1} \to \Gamma_g$ induced by extending diffeomorphisms along $W_{g,1} \subset W_g$ by the identity is an isomorphism (see Lemma 1.1 loc.cit.). Secondly, the forgetful morphism $\Gamma_{g,1} \to \Gamma_{g,1/2}$ is surjective and its kernel can be identified with Θ_{2n+1} via the morphism $\Theta_{2n+1} \cong \pi_0(\mathrm{Diff}_\partial(D^{2n})) \to \Gamma_{g,1}$ induced by extension diffeomorphisms of a disc $D^{2n} \subset \mathrm{int}(W_{g,1})$ by the identity (see (1.6) loc.cit.). Thirdly, the morphism $p\colon \Gamma_{g,1} \to G_g$ factors over $\Gamma_{g,1/2}$ (see (1.7) loc.cit.).

(c) Fixing a stable framing F of $W_{g,1}$, acting on F yields a morphism $(s_F, p)\colon \Gamma_{g,1/2} \to (\mathbf{Z}^{2g} \otimes \pi_n(\mathrm{SO})) \rtimes G_g$ which is an isomorphism for $n \neq 3,7$, and is for $n = 3,7$ a monomorphism onto a subgroup of finite index (see (2.2) and Lemma 2.1 loc.cit.).

The main ingredient in the proof of Theorem 2.3 is the identification from loc.cit. of the extension of groups resulting from (b) of the form

$$0 \longrightarrow \Theta_{2n+1} \longrightarrow \Gamma_{g,1} \longrightarrow \Gamma_{g,1/2} \longrightarrow 0. \tag{27}$$

This identification involves two morphisms

$$\mathrm{sgn}\colon \mathrm{H}_2(\mathrm{Sp}_{2g}(\mathbf{Z});\mathbf{Z}) \longrightarrow \mathbf{Z} \quad \text{and} \quad \chi^2\colon \mathrm{H}_2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z});\mathbf{Z}) \longrightarrow \mathbf{Z},$$

defined in Sections 3.4-3.5 loc.cit., which enjoy the following properties:

(i) When pulled back along $\mathrm{Sp}_{2g}^q(\mathbf{Z}) \le \mathrm{Sp}_{2g}(\mathbf{Z})$, the first morphism becomes divisible by 8 and there is a preferred lift of the resulting morphism $\mathrm{sgn}/8\colon \mathrm{H}_2(\mathrm{Sp}_{2g}^q(\mathbf{Z});\mathbf{Z}) \to \mathbf{Z}$ to a cohomology class (see Definition 3.17 (i) loc.cit.)
$$\tfrac{\mathrm{sgn}}{8} \in \mathrm{H}^2(\mathrm{Sp}_{2g}^q(\mathbf{Z});\mathbf{Z}). \tag{28}$$

(ii) When pulled back along the inclusion $\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}^q(\mathbf{Z}) \longrightarrow \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z})$, the second morphism becomes divisible by 2 and there is a preferred lift of the resulting morphism $\chi^2/2\colon \mathrm{H}_2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}^q(\mathbf{Z});\mathbf{Z}) \to \mathbf{Z}$ to a class (see Definition 3.20 (i) loc.cit.)
$$\tfrac{\chi^2}{2} \in \mathrm{H}^2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}^q(\mathbf{Z});\mathbf{Z}). \tag{29}$$

(iii) When pulled back along the inclusion $(s_F, p)\colon \Gamma_{g,1/2} \hookrightarrow \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z})$, the morphism $\chi^2 - \mathrm{sgn}\colon \mathrm{H}_2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z});\mathbf{Z}) \to \mathbf{Z}$ becomes for $n = 3,7$ divisible by 8 and there is a preferred lift to the resulting morphism $(\chi^2 - \mathrm{sgn})/8\colon \mathrm{H}_2(\Gamma_{g,1/2};\mathbf{Z}) \to \mathbf{Z}$ to a class
$$\tfrac{\chi^2 - \mathrm{sgn}}{8} \in \mathrm{H}^2(\Gamma_{g,1/2};\mathbf{Z}).$$

(iv) The morphism sgn is induced by the same-named signature class $\mathrm{sgn} \in \mathrm{H}^2(\mathrm{Sp}_{2g}(\mathbf{Z});\mathbf{Z})$ featuring in Lemma 5.2 above. For $g = 1$, the morphism $\mathrm{sgn}\colon \mathrm{H}_2(\mathrm{Sp}_{2g}(\mathbf{Z});\mathbf{Z}) \to \mathbf{Z}$ is trivial (see Lemma 3.15 loc.cit.) and the lift (28) vanishes too (see Lemma 3.21 loc.cit.).

(v) For $k \in \mathbf{Z}$, we have $\chi^2 \circ (k,\mathrm{id})_* = k^2 \cdot \chi^2$ as morphisms $\mathrm{H}_2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z});\mathbf{Z}) \to \mathbf{Z}$ where $(k,\mathrm{id})\colon \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}) \to \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z})$ is given by multiplication by k in $\mathbf{Z}^{2g}$. The claimed identity follows directly from the definition. In particular, for $k = 0$, this implies that the precomposition of χ^2 with the map induced by the inclusion $\mathrm{Sp}_{2g}(\mathbf{Z}) \le \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z})$ vanishes. Moreover, going through the construction of (29), one sees that $(k,\mathrm{id})^* \frac{\chi^2}{2} = k^2 \cdot \frac{\chi^2}{2}$ in $\mathrm{H}^2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}^q(\mathbf{Z});\mathbf{Z})$.

The identification of the extension (27) also involves two exotic spheres $\Sigma_P, \Sigma_Q \in \Theta_{2n+1}$, which are defined in Section 3.2.1 loc.cit.. All we need to know about them is that Σ_P generates the subgroup $\mathrm{bP}_{2n+2} \le \Theta_{2n+1}$. In terms of the pullbacks of the cohomology classes in (i)-(iii) above to $\Gamma_{g,1/2}$ and the two exotic spheres, the extension (27) is classified by the class (see Theorem 3.22 and Lemma 3.4 loc.cit.)

$$\begin{cases} \frac{\mathrm{sgn}}{8} \cdot \Sigma_P & n \equiv 1(4) \\ \frac{\mathrm{sgn}}{8} \cdot \Sigma_P + \frac{\chi^2}{2} \cdot \Sigma_Q & n \equiv 3(4) \text{ and } n \neq 3,7 \\ -\frac{\chi^2 - \mathrm{sgn}}{8} \cdot \Sigma_P & n = 3,7 \end{cases} \in \mathrm{H}^2(\Gamma_{g,1/2};\Theta_{2n+1}). \tag{30}$$

We now turn to the proof of Theorem 2.3, which is divided into four sub-claims. We write

$$\mu \coloneqq \begin{cases} \frac{\mathrm{sgn}}{8} & n \neq 3,7 \\ -\frac{\chi^2 - \mathrm{sgn}}{8} & n = 3,7 \end{cases} \in \mathrm{H}^2(\Gamma_{g,1/2};\mathbf{Z})$$

and write $0 \to \Theta_{2n+1} \to E(\mu \cdot \Sigma_P) \to \Gamma_{g,1/2} \to 0$ for the extension classified by $\mu \cdot \Sigma_P$.

Claim ①: We have $\mathrm{fr}(\Gamma_{g,1}) \cap \Theta_{2n+1} = \mathrm{fr}(E(\mu \cdot \Sigma_P)) \cap \Theta_{2n+1}$.

Proof. For $n \equiv 1(4)$ or $n = 3, 7$, we have $\Gamma_{g,1} = E(\mu \cdot \Sigma_P)$ by (30), so there is nothing to show. We thus assume $n \equiv 3(4)$ and $n \neq 3, 7$. Fixing an isomorphism $\pi_n(\mathrm{O}) \cong \mathbf{Z}$, we identify $\Gamma_{g,1/2}$ with $\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}^q(\mathbf{Z})$ via (s_F, p) from (c), and we consider the map of extensions

$$\begin{tikzcd} 0 \arrow[r] & \Theta_{2n+1} \arrow[r] \arrow[d, equal] & (k,\mathrm{id})^*\Gamma_{g,1} \arrow[r] \arrow[d, hook] & \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}^q(\mathbf{Z}) \arrow[r] \arrow[d, hook, "(k,\mathrm{id})"] & 0 \\ 0 \arrow[r] & \Theta_{2n+1} \arrow[r] & \Gamma_{g,1} \arrow[r] & \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}^q(\mathbf{Z}) \arrow[r] & 0 \end{tikzcd} \tag{31}$$

obtained by pulling back the extension (27) along the self-monomorphism (k, id) from (v) for some $k \neq 0$. From Lemma 5.1 (ii), we get $\mathrm{fr}(\Gamma_{g,1}) \cap \Theta_{2n+1} = \mathrm{fr}((k, \mathrm{id})^*\Gamma_{g,1}) \cap \Theta_{2n+1}$, so it suffices to show that there exists $k \neq 0$ such that the top extension in (31) is classified by $\frac{\mathrm{sgn}}{8} \cdot \Sigma_P$, which in view of (30) is equivalent to showing that

$$(k, \mathrm{id})^*(\tfrac{\mathrm{sgn}}{8}) \cdot \Sigma_P + (k, \mathrm{id})^*(\tfrac{\chi^2}{2}) \cdot \Sigma_Q = \tfrac{\mathrm{sgn}}{8} \in \mathrm{H}^2(\Gamma_{g,1/2}; \Theta_{2n+1}). \tag{32}$$

Since the endomorphism (k, id) commutes with the projection to $\mathrm{Sp}_{2g}^q(\mathbf{Z})$ and $\frac{\mathrm{sgn}}{8}$ is pulled back from $\mathrm{Sp}_{2g}^q(\mathbf{Z})$, we have $(k, \mathrm{id})^*(\frac{\mathrm{sgn}}{8}) = \frac{\mathrm{sgn}}{8}$. Moreover, as explained in (v), we have $(k, \mathrm{id})^*(\frac{\chi^2}{2}) = k^2 \cdot \frac{\chi^2}{2}$, so if we choose k to be any multiple of the order of Σ_Q in Θ_{2n+1}, then $(k, \mathrm{id})^*(\frac{\chi^2}{2}) \cdot \Sigma_Q = 0$, and thus the identity (32) holds and the claim follows. □

Claim ②: We have $\mathrm{fr}(\Gamma_{g,1}) \subset \mathrm{bP}_{2n+2}$.

Proof. Since $\Gamma_{g,1/2}$ is residually finite since it arises as a subgroup of the residually finite group $(\mathbf{Z}^{2g} \otimes \pi_n(\mathrm{O})) \rtimes \mathrm{Sp}_{2g}(\mathbf{Z})$ by (c), we have $\mathrm{fr}(\Gamma_{g,1}) \subset \Theta_{2n+1}$ by applying Lemma 5.1 (i) to the morphism $\Gamma_{g,1} \to \Gamma_{g,1/2}$ with kernel Θ_{2n+1}. Together with Claim ①, we get $\mathrm{fr}(\Gamma_{g,1}) = \mathrm{fr}(E(\mu \cdot \Sigma_P)) \cap \Theta_{2n+1}$. As the extension involving $E(\mu \cdot \Sigma_P)$ becomes trivial after taking quotients by the subgroup $\mathrm{bP}_{2n+2} \leq \Theta_{2n+1}$ since $\Sigma_P \in \mathrm{bP}_{2n+2}$, it follows from Lemma 5.1 (i) applied to $E(\mu \cdot \Sigma_P) \to E(\mu \cdot \Sigma_P)/\mathrm{bP}_{2n+2} \cong \Theta_{2n+1}/\mathrm{bP}_{2n+2} \times \Gamma_{g,1/2}$ that $\mathrm{fr}(\mu \cdot \Sigma_P) \subset \mathrm{bP}_{2n+2}$, so we get that $\mathrm{fr}(\Gamma_{g,1}) = \mathrm{fr}(E(\mu \cdot \Sigma_P)) \cap \Theta_{2n+1} \subset \mathrm{bP}_{2n+2}$, as claimed. □

Claim ③: For $g \leq 1$, we have $\mathrm{fr}(\Gamma_{g,1}) = 0$.

Proof. For $g = 0$ we have $\Gamma_{g,1} \cong \Theta_{2n+1}$, so since Θ_{2n+1} is a finite group, we have $\mathrm{fr}(\Gamma_{g,1}) = 0$ as claimed. For $g = 1$ and $n \neq 3, 7$, this follows from Claim ① together with the fact from (iv) that the class $\frac{\mathrm{sgn}}{8}$ is trivial for $g = 1$, which implies that $E(\frac{\mathrm{sgn}}{8} \cdot \Sigma_P) \cong \Gamma_{g,1/2} \times \Theta_{2n+1}$ is residually finite. In the case $n = 3, 7$ and $g = 1$, there is an isomorphism $\Gamma_{g,1/2} \cong \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z})$ with respect to which the monomorphism $(s_F, p)\colon \Gamma_{g,1/2} \hookrightarrow \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z})$ from (c) is given by $(2, \mathrm{id})\colon \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}) \hookrightarrow \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z})$ (combine Theorem 2.2 and Lemma 2.1 in loc.cit.). Using this isomorphism, we obtain a map of extensions as in (31) with $\mathrm{Sp}_{2g}^q(\mathbf{Z})$ replaced by $\mathrm{Sp}_{2g}(\mathbf{Z})$. Arguing as in the proof of Claim ①, we have $\mathrm{fr}(\Gamma_{g,1}) \cap \Theta_{2n+1} = \mathrm{fr}((k, \mathrm{id})^*\Gamma_{g,1}) \cap \Theta_{2n+1}$ for any $k \neq 0$. Moreover, by (30), the extension $0 \to \Theta_{2n+1} \to (k, \mathrm{id})^*\Gamma_{g,1} \to \mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}) \to 0$ is classified by the class $(k, \mathrm{id})^*(-\frac{\chi^2 - \mathrm{sgn}}{8}) \cdot \Sigma_P$. Combining (iv) and (v), the induced morphism $\mathrm{H}_2(\mathbf{Z}^{2g} \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}) \to \Theta_{2n+1}$ of the class $(k, \mathrm{id})^*(-\frac{\chi^2 - \mathrm{sgn}}{8}) \cdot \Sigma_P$ is given by $-k^2 \cdot (\chi^2(-)/8) \cdot \Sigma_P$, so it vanishes if we choose k to be a multiple of the order of Σ_P in Θ_{2n+1}. An application of Lemma 5.1 (iv) then shows $\mathrm{fr}((k, \mathrm{id})^*\Gamma_{g,1}) = 0$, so together with Claim ② we conclude $\mathrm{fr}(\Gamma_{g,1}) = 0$ as claimed. □

Claim ④: For $g \geq 2$, we have $\mathrm{bP}_{2n+2} \subset \mathrm{fr}(\Gamma_{g,1})$, and thus $\mathrm{bP}_{2n+2} = \mathrm{fr}(\Gamma_{g,1})$ by Claim ②.

Proof. We first form the pullback involving the monomorphism (s_F, p) from (c)

$$\begin{tikzcd} \Gamma_{g,1/2}^F \arrow[r, hook, "p_F"'] \arrow[d, hook, "\iota_F"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Sp}_{2g}(\mathbf{Z}) \arrow[d, hook, "\mathrm{inc}"] \\ \Gamma_{g,1/2} \arrow[r, hook, "(s_F, p)"'] & (\mathbf{Z}^{2g} \otimes \pi_n(\mathrm{O})) \rtimes \mathrm{Sp}_{2g}(\mathbf{Z}) \end{tikzcd} \tag{33}$$

and consider the diagram of horizontal of extensions

$$\begin{tikzcd}
 & 0 \arrow[r] & \Theta_{2n+1} \arrow[rr] \arrow[dd, equal] & & E(\iota_F^*\mu \cdot \Sigma_P) \arrow[rr] \arrow[dd, hook] & & \Gamma^F_{g,1/2} \arrow[rr] \arrow[dl, equal] \arrow[dd, hook, "\iota_F"] & & 0 \\
0 \arrow[r] & \mathbf{Z} \arrow[rr] \arrow[ur, "\Sigma_P"'] \arrow[dd, equal] & & E(\iota_F^*\mu) \arrow[rr] \arrow[ur] \arrow[dd, hook] & & \Gamma^F_{g,1/2} \arrow[rr] \arrow[dd, hook, "\iota_F"] & & 0 & \\
 & 0 \arrow[r] & \Theta_{2n+1} \arrow[rr] & & E(\mu \cdot \Sigma_P) \arrow[rr] & & \Gamma_{g,1/2} \arrow[rr] \arrow[dl, equal] & & 0 \\
0 \arrow[r] & \mathbf{Z} \arrow[rr, "\Sigma_P"] \arrow[ur] & & E(\mu) \arrow[rr] \arrow[ur] & & \Gamma_{g,1/2} \arrow[rr] & & 0 &
\end{tikzcd} \tag{34}$$

obtained from the front bottom row classified by $\mu \in \mathrm{H}^2(\Gamma_{g,1/2}; \mathbf{Z})$ by pulling it back along the inclusion ι_F to obtain the front part of the diagram, and then pushing out along $\Sigma_P \colon \mathbf{Z} \to \Theta_{2n+1}$ to obtain the back part. In a moment, we will show that $\mathrm{fr}(E(\iota_F^*\mu)) = \mathbf{Z}$. Assuming this for now, the proof concludes as follows: Applying Lemma 5.1 (i) to the map $E(\iota_F^*\mu) \to E(\mu \cdot \Sigma_P)$ and using commutativity of the diagram as well as that Σ_P generated the subgroup $\mathrm{bP}_{2n+2} \subset \Theta_{2n+1}$, it follows that $\mathrm{bP}_{2n+2} \subset \mathrm{fr}(\mu \cdot \Sigma_P)$, which yields the claim when combined with Claim ①.

We are thus left to show that $\mathrm{fr}(E(\iota_F^*\mu)) = \mathbf{Z}$ which we do by verifying the assumptions of Lemma 5.2 for the class $\iota_F^*\mu \in \mathrm{H}^2(\Gamma^F_{g,1/2}; \mathbf{Z})$, viewing $\Gamma^F_{g,1/2}$ as a subgroup of $\mathrm{Sp}_{2g}(\mathbf{Z})$ via the inclusion p_F in (33). The first condition holds since the inclusion (s_F, p) in (33) has finite index by Item (c), so the same holds for p_F. To show the second condition, i.e. that $\mathrm{H}_1(\Gamma^F_{g,1/2})$ is finite, note that since the right vertical inclusion in (33) is a split injection, the same applies for the left vertical inclusion, so $\mathrm{H}_1(\Gamma^F_{g,1/2})$ is a summand of $\mathrm{H}_1(\Gamma_{g,1/2})$. The latter is finite for $g \geq 2$ by a combination of Corollary 2.4 and Table 2 in loc.cit., so the former is too. This leaves us with establishing the third condition, i.e. that $8 \cdot \mu \circ (\iota_F)_* = \mathrm{sgn} \circ (p_F)_*$ as morphisms $\mathrm{H}_2(\Gamma^F_{g,1/2}) \to \mathbf{Z}$. From the definition of μ and commutativity of (33), we see that the left hand side of the claimed equation is given by the composition

$$\mathrm{H}_2(\Gamma^F_{g,1/2}) \xrightarrow{(p_F)_*} \mathrm{H}_2(\mathrm{Sp}_{2g}(\mathbf{Z})) \xrightarrow{\mathrm{inc}_*} \mathrm{H}_2(\mathbf{Z}^{2g} \otimes \pi_n(\mathrm{O}) \rtimes \mathrm{Sp}_{2g}(\mathbf{Z})) \longrightarrow \mathbf{Z}$$

where the final arrow is $\mathrm{sgn} \circ (\mathrm{pr}_2)_*$ if $n \neq 3, 7$ and $-\chi^2 + (\mathrm{sgn} \circ (\mathrm{pr}_2)_*)$ if $n = 3, 7$. as $\chi^2 \circ \mathrm{inc}_*$ vanishes by Item (v) and we clearly have $\mathrm{pr}_2 \circ \mathrm{inc} = \mathrm{id}_{\mathrm{Sp}_{2g}(\mathbf{Z})}$, the composition indeed agrees with $\mathrm{sgn} \circ (p_F)_*$ in both cases, so the claim follows. □

Combining Claims ③ and ④, we conclude Theorem 2.3.

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