Karen Bakke Haga

Maximal almost disjoint families, determinacy, and forcing

PhD Thesis

Department of Mathematical Sciences
University of Copenhagen

PhD thesis in mathematics

This thesis has been submitted to the PhD School of the Faculty of Science, University of Copenhagen, Denmark in March 2019.

Academic advisor:

Asger Törnquist
Associate Professor
University of Copenhagen,
Denmark

Assessment committee:

William Weiss
Professor
University of Toronto,
Canada

Mirna Džamonja
Professor
University of East Anglia,
UK

Ryszard Nest (chair)
Professor
University of Copenhagen,
Denmark

This work was partially supported by Lars Hesselholts Niels Bohr Professorship.

Karen Bakke Haga
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5
DK-2100 Copenhagen
Denmark
karenbhaga@math.ku.dk

ISBN 978-87-7078-889-2

Abstract

This thesis is based on [5], which is joint work with David Schrittesser and Asger Törnquist. We study the notion of $\mathcal{J}$-MAD families where $\mathcal{J}$ is a Borel ideal on ω. We show that if $\mathcal{J}$ is an arbitrary F_σ ideal, or is any finite or countably iterated Fubini product of F_σ ideals, then there are no analytic infinite $\mathcal{J}$-MAD families; and assuming Projective Determinacy there are no infinite projective $\mathcal{J}$-MAD families; and under the full Axiom of Determinacy + $V = \mathbf{L}(\mathbb{R})$ there are no infinite $\mathcal{J}$-mad families. These results apply in particular when $\mathcal{J}$ is the ideal of finite sets Fin, which corresponds to the classical notion of MAD families. The proofs combine ideas from invariant descriptive set theory and forcing.

Resumé

Denne afhandling er baseret på [5], som er lavet i samarbejde med David Schrittesser og Asger Törnquist. Vi arbejder med $\mathcal{J}$-MAD familier, hvor $\mathcal{J}$ er et Borel ideal på ω. Vi viser at hvis $\mathcal{J}$ er et vilkårligt F_σ ideal, eller et endelig eller tælleligt itereret Fubini produkt af F_σ idealer, da er der ingen analytiske uendelige $\mathcal{J}$-MAD familier, og under antagelse af Projective Determinacy er der ingen projektive $\mathcal{J}$-MAD familier, og under det fulde Axiom of Determinacy + $V = \mathbf{L}(\mathbb{R})$ er der ingen uendelige $\mathcal{J}$-MAD familier. Disse resultater gælder specielt når $\mathcal{J}$ er idealet bestående af endelige mængder Fin, hvilket svarer til den klassiske definition af MAD-familier. Beviserne kombinerer idéer fra invariant deskriptiv mængdelære og forcing.

Acknowledgements

I sincerely thank my advisor Asger Törnquist for all his help and support throughout the years, and for always believing in me. I also thank David Schrittesser, who acted as an unofficial co-advisor for this thesis, for a great collaboration and for taking the time to explain many a complicated thing to me. Thanks to Vibeke Quorning and Martin Christensen for proof-reading the thesis, and to Kristian Olesen for letting me exploit his LaTeX-skills. And last but not least, I thank my husband Niels and my children Molly and Elmer for making me happy every day.

Contents

Chapter 1

Introduction

Consider a family $\mathcal{A} \subseteq [\omega]^\omega$ of infinite sets of natural numbers. We say that $\mathcal{A}$ is almost disjoint (short: AD) if for every two distinct $A, B \in \mathcal{A}$ it holds that $|A \cap B| < \infty$. Whenever we have an AD family $\mathcal{A}$, we can ask if it is possible to add another infinite set and maintain the almost disjointness of $\mathcal{A}$. We say that the AD family $\mathcal{A}$ is maximal (short: MAD) if this is not possible, i.e. if for every $x \in [\omega]^\omega$ the family $\mathcal{A} \cup \{x\}$ is not almost disjoint.

We can easily find MAD families of any finite cardinality. The family $\{\{2n \mid n \in \omega\}, \{2n+1 \mid n \in \omega\}\}$ is one example. On the other hand there are no countably infinite MAD families: indeed, let $\{A_n \mid n \in \omega\}$ be an AD family. Construct $x = \{x_n \mid n \in \omega\} \subseteq \omega$ by letting $x_n \in A_n \setminus \bigcup_{i<n} A_i$ be the least such that $x_n > x_{n-1}$. Then $\{A_n \mid n \in \omega\} \cup \{x\}$ is AD, and thus $\{A_n \mid n \in \omega\}$ is not maximal.

However, infinite MAD families do exist. This is a consequence of Zorn's lemma. Indeed, let $\mathbb{P}$ be the set of all infinite, almost disjoint families partially ordered by $\subseteq$. Note that $\mathbb{P}$ is non-empty; if $(p_k)_{k \in \omega}$ is an enumeration of the primes and $P_k := \{p_k^n \mid n \in \omega\}$, then $\{P_k \mid k \in \omega\} \in \mathbb{P}$. For any chain $\mathbb{C}$ in $\mathbb{P}$, consider the union $\bigcup \mathbb{C}$. If $A, B \in \bigcup \mathbb{C}$, then since $\mathbb{C}$ is totally ordered, there is some $\mathcal{A} \in \mathbb{C}$ such that $A, B \in \mathcal{A}$. Since $\mathcal{A} \in \mathbb{P}$ is almost disjoint, we have $|A \cap B| < \infty$, thus proving that $\bigcup \mathbb{C}$ is almost disjoint. By Zorn's lemma, there is a maximal element in $\mathbb{P}$, i.e. an infinite maximal almost disjoint family.

The non-constructive proof above gives us no information about what an infinite MAD family looks like – all we know is that it has to be uncountable. There are several perspectives one might take on infinite MAD families, and from a descriptive set theoretic point of view it is natural to ask how complex the definition of such a family would be. For this purpose, we identify the power set $\mathcal{P}(\omega)$ of ω with 2^ω equipped with the product topology by identifying each subset with its characteristic function. An infinite MAD family will then be a subset of the Polish space 2^ω, and we will in the following investigate the possible complexity of its definition with respect to this topology.

The starting point of this area of research is Mathias' famous result from 1969 (which was published in 1976) that there are no analytic infinite MAD families [16]. Furthermore, he proved that assuming the existence of a Mahlo cardinal, there is a model of ZFC in which there are no projective infinite MAD families, and there is a model of ZF + Dependent Choice in which there are no infinite MAD families at all. The weaker assumption that "there is some inaccessible cardinal" gives rise to Solovay's model, which is a model of ZF in which all sets of real numbers are Lebesgue measurable. In 2015, Törnquist answered negatively the longstanding question, posed by Mathias, about existence of infinite MAD families in Solovay's model [21]. Horowitz and Shelah removed the cardinal assumption altogether in 2016 by proving that ZF + Dependent Choice + "There are no infinite MAD families" is equiconsistent with ZFC [7].

The proofs in the present thesis are based on a Mathias-like forcing together with ideas from invariant descriptive set theory, which is the theory of definable equivalence relations and invariance properties. This turned out to be a fruitful approach, which led to a variety of results. Let us start by presenting the most fundamental theorem, which was also showed independently by Neeman and Norwood [20] using different methods:

Theorem 1.0.1. *1. Under* ZF *+ Dependent Choice + Projective Determinacy, there are no projective infinite MAD families.*

2. Under ZF *+ Axiom of Determinacy +* $V = \mathbf{L}(\mathbb{R})$*, there are no infinite MAD families.*

The notion of almost disjointness easily generalizes to other ideals, which we think of as collections of sets considered to be "small" in some sense. Let towards this end Fin denote the ideal on ω consisting of finite sets and let Fin^+ denote the co-ideal. Then an AD family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is a family that satisfies $\mathcal{A} \subseteq \mathrm{Fin}^+$ and for every two distinct $A, B \in \mathcal{A}$, the intersection $A \cap B \in \mathrm{Fin}$. Consider now a countable set S, and let $\mathcal{J}$ be any ideal on S with corresponding co-ideal $\mathcal{J}^+$. We say that a family $\mathcal{A} \subseteq \mathcal{P}(S)$ is $\mathcal{J}$-almost disjoint (short: $\mathcal{J}$-AD) if $\mathcal{A} \subseteq \mathcal{J}^+$ and for every distinct $A, B \in \mathcal{A}$ we have $A \cap B \in \mathcal{J}$, and we say that $\mathcal{A}$ is maximal $\mathcal{J}$-almost disjoint (short: $\mathcal{J}$-MAD) if it is maximal among $\mathcal{J}$-AD families. The natural question to ask, which has also been the motivation for the rest of the present thesis, is this:

Question 1.0.2. For which ideals $\mathcal{J}$ do the non-existence results about MAD families also apply to $\mathcal{J}$-MAD families?

We will investigate a large class of Borel ideals, namely iterated Fubini products of F_σ ideals. First, we study F_σ ideals, which are ideals that are countable unions of closed sets. By a result of Mazur, any F_σ ideal is given as the finite part of a lower semicontinuous submeasure, in the sense that any F_σ ideal is of the form $\mathrm{Fin}(\phi) = \{X \subseteq \omega \mid \phi(X) < \infty\}$ where ϕ is a lower semicontinuous submeasure [17]. Note that Fin is F_σ, and given as the finite part of the counting measure. As in the case of Fin, it is not hard to see that for any F_σ ideal $\mathcal{J}$ there are $\mathcal{J}$-MAD families of any finite cardinality, but no countably infinite $\mathcal{J}$-MAD family. Moreover, Zorn's lemma also yields the existence of an infinite $\mathcal{J}$-MAD family. We prove the following:

Theorem 1.0.3. *Let $\mathcal{J} \subseteq \mathcal{P}(\omega)$ be an F_σ ideal on ω.*

1. *There are no analytic infinite $\mathcal{J}$-MAD families.*
2. *Under ZF + Dependent Choice + Projective Determinacy, there are no projective infinite $\mathcal{J}$-MAD families.*
3. *Under ZF + Determinacy + $V = \mathbf{L}(\mathbb{R})$, there are no infinite $\mathcal{J}$-MAD families.*

In order to generalize this result even further, let $(S_k, \mathcal{I}_k)_{k\in\omega}$ be a sequence of countable sets together with ideals, and let $\mathcal{I}$ be an ideal on ω. We construct an ideal $\oplus_{\mathcal{I}}\mathcal{I}_k$ on $S = \bigsqcup_{k\in\omega} S_k$ called the *Fubini sum of $(\mathcal{I}_k)_{k\in\omega}$ over $\mathcal{I}$* in the following way:

$$\oplus_{\mathcal{I}}\mathcal{I}_k = \{I \subseteq S \mid \{k \in \omega \mid I \cap S_k \notin \mathcal{I}_k\} \in \mathcal{I}\}.$$

Note that if $S_k = \omega$ and $\mathcal{I}_k = \mathcal{I} = \mathrm{Fin}$ for every $k \in \omega$, then the Fubini sum equals the well-known ideal

$$\mathrm{Fin} \otimes \mathrm{Fin} = \{I \subseteq \omega \times \omega \mid \{n \in \omega \mid \{m \in \omega \mid (n,m) \in I\} \text{ is infinite }\} \text{ is finite}\}.$$

We prove the following:

Theorem 1.0.4. *Let $\mathcal{J} = \oplus_{\mathcal{I}}\mathcal{I}_k$, where $\mathcal{I}$, $\mathcal{I}_k$ are F_σ ideals on ω for every $k \in \omega$.*

1. *There are no analytic infinite $\mathcal{J}$-MAD families.*
2. *Under ZF + Dependent Choice + Projective Determinacy, there are no projective infinite $\mathcal{J}$-MAD families.*

3. *Under ZF + Determinacy + $V = \mathbf{L}(\mathbb{R})$, there are no infinite $\mathcal{J}$-MAD families.*

This notion of Fubini sums will be iterated into the transfinite, as will be defined in Section 3.3. This way we obtain Borel ideals of aribitrarily high complexity [22, Chapter 2], and we generalize the previous result even further:

Theorem 1.0.5. *Let $\mathcal{J} = \mathrm{Fin}(\vec{\phi})$ be a Borel ideal as defined in Section 3.3.*

1. *There are no analytic infinite $\mathcal{J}$-MAD families.*
2. *Under ZF + Dependent Choice + Projective Determinacy, there are no projective infinite $\mathcal{J}$-MAD families.*
3. *Under ZF + Determinacy + $V = \mathbf{L}(\mathbb{R})$, there are no infinite $\mathcal{J}$-MAD families.*

This means that the Borel ideals which satisfy these results concerning non-definability of corresponding MAD families lie cofinally in the Borel hierarchy.

The proofs of the theorems above all follow the same pattern. In order to give an idea of this pattern we will now sketch a proof of the most basic case, namely when the family $\mathcal{A}$ is analytic Fin-almost disjoint. The following is therefore a sketch of a new proof of Mathias' classical result that such a family cannot be maximal.

Given an analytic AD-family $\mathcal{A}$, we define a forcing notion, closely related to Mathias forcing, which allows us to build a generic element which is infinite and almost disjoint to everything in $\mathcal{A}$. We denote by x_G the generic real added by this forcing. Since $\mathcal{A}$ is analytic, we can represent $\mathcal{A}$ by the projection of the infinite branches through some countable tree T, i.e. $\mathcal{A} = \pi[T]$. We show the **Main Proposition** (3.1.5), stating that for any generic G the set $\pi[T] \cup \{x_G\}$ is almost disjoint also in the forcing extension. In order to prove the Main Proposition we define for $x \subseteq \omega$ a pruned subtree $T^x \subseteq T$ that consists of the nodes in T that have an extension whose projection intersects x in an infinite set. In other words, T^x is the subtree of T consisting of the nodes that have infinite extensions that would witness non-almost disjointness of $\pi[T]$ and x. Then $\{x\} \cup \pi[T]$ is almost disjoint if and only if $T^x = \varnothing$. It is not that hard to see that the function $x \mapsto T^x$ is invariant under E_0-equivalence, i.e. that for xE_0z we have $T^x = T^z$. For a generic G, the tree T^{x_G} is therefore definable from the equivalence class $[x_G]_{E_0}$. Now we use a diagonalization property of the forcing to prove that the sets that are hereditary definable using parameteres from $V \cup \{[x_G]_{E_0}\}$ are in fact in V. The next crucial ingredient in the proof is the **Branch Lemma** (3.1.14), which ensures that

T^{x_G} does not split in the first coordinate, and that $\pi[T^{x_G}]$ therefore has at most one element. Now we see that if $\pi[T^{x_G}] = \{y\}$, then y is definable from $[x_G]$ and thus $y \in V$. Moreover, since $T^{x_G} \subseteq T$, we obtain $y \in \mathcal{A}$. By definition of T^{x_G}, we have $x_G \cap y \notin \mathrm{Fin}$, contradicting the definition of x_G which made sure that x_G was almost disjoint from everything in $\mathcal{A}$. Thus $\pi[T^{x_G}] = T^{x_G} = \varnothing$ for any generic G, and this proves the Main Proposition. The fact that $\mathcal{A}$ cannot be maximal now follows from an absoluteness lemma.

In the proofs of all the other cases, where both the complexity of $\mathcal{A}$ and the complexity of the Borel ideal in addition to the background theory varies, we exploit the tree structure of $\mathcal{A}$ in the given setting to prove an analogue of the Main Proposition by means of an invariant subtree and an analogue of the Branch Lemma. The theorems above follow from the Main Proposition by the already mentioned absoluteness lemma. The least straightforward part is the proof of the Branch Lemma, which demands more work in the higher dimensional cases. We will need to define an intermediary object $U^{x_{\dot{G}}}$ with a tree-like structure, and a well-founded partially ordered set Γ to keep track of which elements that are forced into $U^{x_{\dot{G}}}$. The well-foundedness of Γ together with a couple of lemmas that allow us to alter both the finite and the infinite part of a forcing condition while maintaining that something is forced about $U^{x_{\dot{G}}}$ will then yield the wanted contradiction if we assume that the invariant subtree has more than one branch.

The thesis is structured as follows:

In Chapter 2 we explain the background theory used in the thesis. The chapter is divided into the following sections:

In Section 2.1 we give a few basic definitions and establish some notation.

In Section 2.2 we introduce the subject of definability in descriptive set theory. We describe the Borel hierarchy and the projective hierarchy, and we give several equivalent definitions of being analytic. We also briefly explain the idea of invariant descriptive set theory.

In Section 2.3 we define determinacy and discuss a few crucial consequences of the determinacy axioms.

In Section 2.4 we give an informal introduction to forcing, and give an example of how forcing is used to prove independency of the continuum hypothesis. We also define classical Mathias forcing. Moreover, we introduce the subject of inner model theory and define the inner model $\mathbf{L}$. We also state lemmas which allows us to assume that our $\mathcal{J}$-MAD families are κ-Suslin witnessed by a tree from a model where $\mathcal{P}(\mathcal{P}(\omega))$ is countable. Finally, we discuss the notion of absoluteness between models, and prove an absoluteness lemma which will be used throughout the thesis.

In Chapter 3 we are ready to see the proofs of the main results of the present thesis.

In Section 3.1 we prove Theorem 1.0.3 for any F_σ ideal $\mathcal{I}$. We define the Mathias forcing $\mathbf{M}^{\mathcal{I}}$ relative to an ideal $\mathcal{I}$. We claim the **Main Proposition** 3.1.5, saying that in this forcing extension, the AD-family in question is not maximal, and use absoluteness to see that this holds in general. In order to prove this proposition, we first collect some facts about the forcing. Then we move on to the actual proof, which to great extent relies on the definition of an invariant tree and the purely combinatorial **Branch Lemma** 3.1.14.

In Section 3.2 we prove Theorem 1.0.4. The structure is similar to that of Section 3.1, however the proof of the corresponding **Branch Lemma** 3.2.11 is much more involved.

In Section 3.3 we prove Theorem 1.0.5. We introduce the α-dimensional Fubini product $\mathrm{Fin}(\vec{\phi})$, and define an α-dimensional version $\mathbf{M}^{\mathcal{I}}_\alpha$ of $\mathbf{M}^{\mathcal{I}}$. The structure is again similar to that of Section 3.2, but the proofs are even more intricate.

In Chapter 4 we discuss a few questions that could be interesting to pursue, and also the general open problem of for which Borel ideals on ω one can hope to achieve analogues of Theorem 1.0.5.

Chapter 2

Set theory

2.1 Preliminaries and notation

Sets and sequences

We will denote the natural numbers including zero by ω. By a natural number $n \in \omega$ we understand the ordinal $n = \{0, \ldots, n-1\}$.

Let X be a set. We denote the powerset of X by $\mathcal{P}(X)$. The set of finite subsets of X is denoted $[X]^{<\omega}$, while $[X]^{\omega}$ denotes the set of countably infinite subsets of X. For $a \in [X]^{<\omega}$ and $b \in [X]^{<\omega} \cup [X]^{\omega}$, we write $a \sqsubseteq b$ if and only if $a \subseteq b$ and $n \in b \backslash a \Rightarrow n > \max(a)$.

We let $X^{<\omega}$ denote the set of finite sequences in X, i.e.

$$X^{<\omega} = \{(s_0, \ldots, s_{n-1}) \mid n \in \omega \wedge (\forall i < n)\ s_i \in X\}.$$

For a finite sequence $s = (s_0, \ldots, s_{n-1}) \in X^{<\omega}$, we let $\mathrm{lh}(s) = n$ denote the length of s. For $m \leqslant n$, we let $s{\restriction}m = (s_0, \ldots, s_{m-1})$. If $s, t \in X^{<\omega}$, we say that s is an *initial segment* of t and t *extends* s if $s = t{\restriction}m$ for some $m \leqslant \mathrm{lh}(t)$, and we write $s \sqsubseteq t$. If $s \sqsubseteq t$ and $\mathrm{lh}(t) > \mathrm{lh}(s)$, then we write $s \sqsubsetneq t$ and say that t is a *proper extension* of s. Two finite sequences that satisfies that one is an initial segment of the other are called *compatible*. If this is not the case, the sequences are *incompatible*. The *concatenation* of $s = (s_0, \ldots, s_{n-1}) \in X^{<\omega}$ and $t = (t_0, \ldots, t_{m-1}) \in X^{<\omega}$ is denoted $s^\frown t$ and defined by $s^\frown t = (s_0, \ldots, s_{n-1}, t_0, \ldots, t_{m-1})$. If $a \in X$ we write $s^\frown a$ for the sequence $(s_0, \ldots, s_{n-1}, a)$.

The set of infinite sequences in X is denoted by

$$X^{\omega} = \{(x_k)_{k \in \omega} \mid (\forall k \in \omega)\ x_k \in X\}.$$

For $x \in X^\omega$ and $m \in \omega$ we let $x \upharpoonright m = (x_0, \dots, x_{m-1})$. We say that $s \in X^{<\omega}$ and $x \in X^\omega$ are *compatible* if s is an *initial segment* of x, i.e. if there exists $n \in \omega$ such that $s = x \upharpoonright n$. The *concatenation* of a finite sequence $s = (s_0, \dots, s_n) \in X^{<\omega}$ and an infinite sequence $x = (x_k)_{k\in\omega} \in X^\omega$ is the infinite sequence given by $s^\frown x = (s_0, \dots, s_{n-1}, x_0, x_1, \dots)$.

Note that X^δ in general can be viewed as a set of functions $f \colon \delta \to X$. For $X = 2 = \{0, 1\}$, a function $f \colon \delta \to 2$ can be identified with the subset of δ for which f is the characteristic function, so 2^δ can be identified with $\mathcal{P}(\delta)$.

Trees

Let X be a set. A *tree* on X is a set $T \subseteq X^{<\omega}$ of finite sequences, which is closed under initial segments, i.e. if $t \in T$, then for any $m \leqslant \mathrm{lh}(t)$ the restriction $t \upharpoonright m \in T$. Given a tree T, we can consider the set $[T]$ of *branches through* T;

$$[T] = \{x \in X^\omega \mid (\forall n \in \omega)\ x \upharpoonright n \in T\}.$$

For $t \in T$ we can also define trees

$$T_t = \{u \in X^{<\omega} \mid t^\frown u \in T\}$$

and

$$T_{[t]} = \{u \in T \mid u \text{ is compatible with } t\}.$$

We say that a tree T is *well-founded* if $[T] = \varnothing$. For a well-founded tree T we can recursively define a *rank function* $\rho_T : T \to \omega_1$ in the following way:

$$\rho_T(t) = \sup\{\rho_T(u) + 1 \mid u \in T, t \subsetneq u\}.$$

The rank of T is defined by $\rho(T) = \sup\{\rho_T(t) + 1 \mid t \in T\}$. We say that $t \in T$ is *terminal* if for every $s \in S$ we have $t^\frown s \notin T$, i.e. if $\rho_T(t) = 0$. Note furthermore that $\rho_T(t) = \rho_{T_t}(\varnothing)$.

Trees can also be defined on finite product spaces. Let X_0, X_1 be a sets. We follow established descriptive set theoretic conventions and call a *tree T on $X_0 \times X_1$* a subset of $X_0^{<\omega} \times X_1^{<\omega}$ which is closed under initial segments and such that $(t_0, t_1) \in T \Rightarrow \mathrm{lh}(t_0) = \mathrm{lh}(t_1)$ (compare [10, 2.C]). Given $t = (t_0, t_1) \in T$, let $\pi(t) = t_0$ be the projection onto the first coordinate. For any $s \in T$, $T_{[s]} = \{t \in T \mid t \text{ is compatible with } s\}$, and $T_s = \{t \in \bigcup_{n\in\omega} X_0^n \times X_1^n \mid s^\frown t \in T\}$. Naturally we set

$$[T] = \{(x_0, x_1) \in X_0^\omega \times X_1^\omega \mid (\forall n \in \omega)\ (x_0 \upharpoonright n, x_1 \upharpoonright n) \in T\},$$

and for $w = (x_0, x_1) \in [T]$, we let $\pi(w) = x_0$ denote the projection onto the first coordinate. Finally we write

$$\pi[T] = \{x_0 \in X_0^\omega \mid (\exists x_1 \in X_1^\omega)\ (x_0, x_1) \in [T]\}.$$

Ideals

Fix a countable set S. An *ideal* on S is a family $\mathcal{J} \subseteq \mathcal{P}(S)$ satisfying

1. $\varnothing \in \mathcal{J}$;
2. if $A \in \mathcal{J}$, then for any subset $B \subseteq A$ we have $B \in \mathcal{J}$;
3. if $A \in \mathcal{J}$ and $B \in \mathcal{J}$, then $A \cup B \in \mathcal{J}$.

We denote by Fin the ideal of finite sets.

Given an ideal $\mathcal{J}$, we write $\mathcal{J}^+$ to denote the co-ideal, i.e.,

$$\mathcal{J}^+ = \{A \subseteq S \mid A \notin \mathcal{J}\}.$$

For A, $B \in \mathcal{P}(S)$, we write

$$A \subseteq^*_{\mathcal{J}} B \Leftrightarrow (\exists I \in \mathcal{J})\ A \subseteq B \cup I.$$

We write $A \subseteq^* B$ for $A \subseteq^*_{\mathrm{Fin}} B$.

We say that a family $\mathcal{A} \subseteq \mathcal{P}(S)$ is $\mathcal{J}$-*almost disjoint* (short: $\mathcal{J}$-*AD*) if $\mathcal{A} \subseteq \mathcal{J}^+$ and for any A, $B \in \mathcal{A}$ we have $A \cap B \in \mathcal{J}$. A set $\mathcal{A} \subseteq \mathcal{P}(S)$ is said to be a $\mathcal{J}$-*MAD family* if $\mathcal{A}$ is a $\mathcal{J}$-AD family which is maximal with respect to inclusion among $\mathcal{J}$-AD families.

Definition 2.1.1. Let $\mathcal{A} \subseteq \mathcal{P}(S)$. By the *ideal generated by* $\mathcal{A}$ we mean the ideal $\mathcal{I}$ on S defined as follows:

$$\mathcal{I} = \{I \subseteq S \mid (\exists n \in \omega)(\exists A_0, \ldots, A_n \in \mathcal{A})\ I \subseteq \bigcup_{i \leqslant n} A_i\},$$

i.e., the smallest (under $\subseteq$) ideal on S containing each set from $\mathcal{A}$.

Suppose $\mathcal{A} \subseteq \mathcal{P}(S)$ and $\mathcal{J}$ is an ideal on S. Then note that the ideal generated by $\mathcal{A} \cup \mathcal{J}$ is

$$\{I \subseteq S \mid I \in \mathcal{J} \vee (\exists n \in \omega)(\exists A_0, \ldots, A_n \in \mathcal{A})\ I \subseteq^*_{\mathcal{J}} \bigcup_{i \leqslant n} A_i\}.$$

We point out that if $\mathcal{A}$ is an infinite $\mathcal{J}$-AD family then $\mathcal{J}$ is proper (i.e., $S \notin \mathcal{J}$; otherwise there are no non-empty, let alone infinite, $\mathcal{J}$-AD families).

To avoid trivialities we always assume that $[S]^{<\omega} \subseteq \mathcal{J}$ (otherwise discard all singletons which are not in J). Moreover we could assume $\bigcup \mathcal{A} = S$ (although we shall never need this).

We also point out that enlarging an ideal $\mathcal{J}$ by an infinite $\mathcal{J}$-AD family yields a proper ideal:

Lemma 2.1.2. *Let S be arbitrary, $\mathcal{J}$ an ideal on S and $\mathcal{A} \subseteq \mathcal{P}(S)$ a $\mathcal{J}$-AD family. If $\mathcal{A}$ is infinite, the ideal $\mathcal{I}$ generated by $\mathcal{A} \cup \mathcal{J}$ is proper. (The other implication holds if $\bigcup \mathcal{A} = S$.)*

Proof. We show the contrapositive. Suppose $S \in \mathcal{I}$. Then there exist $A_0, \ldots, A_n \in \mathcal{A}$, $J \in \mathcal{J}$ such that $S \subseteq \bigcup_{i \leqslant n} A_i \cup J$. Since $\mathcal{A}$ is $\mathcal{J}$-almost disjoint $\mathcal{A} = \{A_0, \ldots, A_n\}$ is finite; indeed, if there were $A \in \mathcal{A} \backslash \{A_0, \ldots, A_n\}$ then

$$A = S \cap A = (\cup_{i \leqslant n} A_i \cap A) \cup (J \cap A) \in \mathcal{J}.$$

For the last claim, suppose $\bigcup \mathcal{A} = S$; show the contrapositive. If $\mathcal{A}$ is finite, then $S \subseteq \bigcup_{A \in \mathcal{A}} A$, and thus $S \in \mathcal{I}$. $\square$

A *submeasure* on ω is a function $\phi \colon \mathcal{P}(\omega) \to [0, \infty]$ which satisfies

- $\phi(\varnothing) = 0$;
- $\phi(X) \leqslant \phi(Y)$ for $X \subseteq Y$;
- $\phi(X \cup Y) \leqslant \phi(X) + \phi(Y)$ for $X, Y \in \mathcal{P}(\omega)$;
- $\phi(\{n\}) < \infty$ for every $n \in \omega$.

We say that ϕ is *lower semi-continuous* (lsc) if identifying $\mathcal{P}(\omega)$ with 2^ω carrying product topology, it is lower semi-continuous as a function $\phi \colon 2^\omega \to [0, \infty]$, i.e., if $X_n \to X$ implies $\liminf_{n \to \infty} \phi(X_n) \geqslant \phi(X)$. For submeasures, this is equivalent to saying that $\phi(X) = \lim_{n \to \infty} \phi(X \cap n)$.

As mentioned already in the introduction, an F_σ set is a countable union of closed sets. Given a submeasure ϕ on ω, the family

$$\mathrm{Fin}(\phi) = \{X \in \mathcal{P}(\omega) \mid \phi(X) < \infty\}$$

is an F_σ ideal on ω and every F_σ ideal $\mathcal{J} \supseteq \mathrm{Fin}$ arises in this way [17, 1.2].

2.2 Descriptive set theory: A study of definability

Descriptive set theory is the study of "definable" subsets of certain well-behaved topological spaces called *Polish spaces*, where the notion of definability is referring to the topological definition of the set. In this section we will review some of the most basic definitions and results. For a more thorough exposition, the reader is referred to [10].

Definition 2.2.1. Let $(X, \mathcal{T})$ be a topological space. If $(X, \mathcal{T})$ is separable and completely metrizable, then we say that $(X, \mathcal{T})$ is Polish.

An important example of a Polish space is the real line $\mathbb{R}$ with the usual topology. Moreover, any countable set X equipped with the discrete toplogy is Polish, and so is the product X^ω of countably infinitely many copies of the countable, discrete space X. The topology of a space X^ω has as a basis the sets

$$N_s = \{x \in X^\omega \mid x{\restriction}\mathrm{lh}(s) = s\},$$

where $s \in X^{<\omega}$. In general, any finite or countably infinite product of Polish spaces is Polish.

The following characterization of closed subsets in Polish spaces of the form X^ω will turn out to be crucial in determining the definability of the various $\mathcal{I}$-MAD families that we will be looking into:

Proposition 2.2.2. *Let X be a discrete, countable set. Any closed subset of the Polish space X^ω is the set of branches through some tree. Conversely, any set of branches is closed.*

Proof. Let $F \subseteq X^\omega$ be a closed set. Consider the map $F \mapsto T_F$, where T_F is the following tree:

$$T_F = \{s \in X^n \mid n \in \omega \wedge (\exists x \in F)\ s = x{\restriction}n\}.$$

Then $F = [T_F]$. Conversely, if T is a tree, then $[T]$ is a closed set: suppose $[T] \neq X^\omega$ and let $x \in X^\omega \backslash [T]$. Then there is $n \in \omega$ such that $x{\restriction}n \notin T$, so $N_{x\restriction n} \subseteq X^\omega \backslash [T]$. Since x was arbitrary, the set $X^\omega \backslash [T]$ is open. $\square$

There are two Polish spaces constructed in this fashion that are of particular interest. These are the *Cantor space* $2^\omega = \{0,1\}^\omega$ and the *Baire space* ω^ω. Before we look into them, we need to establish two important hierarchies describing the complexity of subsets of Polish spaces, namely the Borel hierarchy and the projective hierarchy.

In descriptive set theory we study sets of reals (or, more generally, subsets of Polish spaces) that in some sense have a simple description in terms of topology. In general, we are mostly interested in sets that are definable using certain basic operations from a baseline of some collection of sets, and continuous images of such sets.

Let $(X, \mathcal{T})$ be a Polish space. By alternating between taking complements and countable unions, we obtain a hierarchy of subsets of X definable from the open sets using only these operations. This hierarcy is called the *Borel hierarchy*, and is recursively defined as follows:

- $\mathbf{\Sigma}^0_1(X)$ = open subsets of X;
- $\mathbf{\Pi}^0_1(X)$ = closed subsets of X, i.e. complements of open sets;
- For $1 < \alpha < \omega_1$, $\mathbf{\Sigma}^0_\alpha(X) = \{\bigcup_{n\in\omega} B_n \mid B_n \in \mathbf{\Pi}^0_{\beta_n}(X) \text{ for } \beta_n < \alpha\}$;
- For $1 < \alpha < \omega_1$, $\mathbf{\Pi}^0_\alpha(X)$ = complements of sets in $\mathbf{\Sigma}^0_\alpha(X)$.
- $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$.

It follows immediately from the definitions that $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha(X)$. This class of sets is called the *Borel sets*, and it is denoted by $\mathbb{B}(X)$. In other words, a set B is Borel if and only if $B \in \mathbf{\Sigma}^0_\alpha(X)$ for some $\alpha < \omega_1$ (or, equivalently, $B \in \mathbf{\Pi}^0_\alpha(X)$ for some $\alpha < \omega_1$).

Note that $\mathbf{\Sigma}^0_2(X)$ are exactly the F_σ subsets of X.

Once we have defined the Borel hierarcy, we can consider projections of Borel sets, which are called *analytic* sets. More precisely, a set $A \subseteq X$ of a Polish space is analytic if there exists a Polish space Y and a Borel set $B \subseteq X \times Y$ such that $A = \mathrm{proj}_X(B)$. The *co-analytic* sets are complements of analytic sets. By continuing to take projections and complements, we obtain another hierarchy of subsets of X definable with the analytic sets as baseline, the *projective hierarchy*:

- $\mathbf{\Sigma}^1_1(X)$ = analytic subsets of X;
- $\mathbf{\Pi}^1_1(X)$ = co-analytic subsets of X;
- $A \in \mathbf{\Sigma}^1_{n+1}(X) \Leftrightarrow (\exists Y \text{ Polish})(\exists B \in \mathbf{\Pi}^1_n(X \times Y))\ A = \mathrm{proj}_X(B)$.
- $\mathbf{\Pi}^1_{n+1}(X)$ = complements of sets in $\mathbf{\Sigma}^1_{n+1}(X)$.
- $\mathbf{\Delta}^1_n(X) = \mathbf{\Sigma}^1_n(X) \cap \mathbf{\Pi}^1_n(X)$.

As with the Borel hierarchy, we note that $\bigcup_{n\in\omega} \boldsymbol{\Sigma}^1_n(X) = \bigcup_{n\in\omega} \boldsymbol{\Pi}^1_n(X)$. We call this the class of *projective sets*, and denote it by $\mathbb{P}(X)$.

The collections $\boldsymbol{\Sigma}^0_\alpha(X)$ and $\boldsymbol{\Pi}^0_\alpha(X)$ for $\alpha < \omega_1$ and $\boldsymbol{\Sigma}^1_n(X)$ and $\boldsymbol{\Pi}^1_n(X)$ for $n \in \omega$ are called *pointclasses*.

The Borel hierarchy and the projective hierarchy are two distinct entities, since projections of Borel sets need not be Borel. Indeed, the following theorem due to Souslin ensures this (a proof can be found in [10, Theorem 14.2]):

Proposition 2.2.3. *Let X be an uncountable Polish space. Then there are analytic subsets of X that are not Borel.*

Furthermore, it turns out that the following characterization holds (for a proof, see [10, Theorem 14.11]):

Proposition 2.2.4. *A set is Borel if and only if it is both analytic and co-analytic.*

The following two results indicates the importance of the already mentioned Cantor space and Baire space. For proofs, see [10, Theorem 6.4 and Theorem 13.7]. Remember that a *perfect space* is a space whose points are all limit points. A subset of a space is called *perfect* if it is closed and perfect in the subspace topology. Any non-empty perfect Polish space contains a copy of the Cantor space [10, Theorem 6.2], ensuring that the cardinality of a space satisfying these conditions is at least $2^{\aleph_0}$.

Theorem 2.2.5 (Cantor-Bendixson). *Let X be a Polish space. Then X can be uniquely written as $X = P \cup C$, where P is a perfect subset of X and C is countable.*

This means that an uncountable Polish space contains a homeomorphic copy of the Cantor space, thus any Polish space satisfies the Continuum Hypothesis; if it is not countable, it has cardinality at least $2^{\aleph_0}$.

Theorem 2.2.6 (Lusin-Souslin). *Let X be Polish and $A \subseteq X$ Borel. If $A \neq \varnothing$, then there is a continuous surjection $f : \omega^\omega \to A$ from the Baire space onto A.*

This implies in particular that any Polish space is the continuous image of the Baire space. In many cases this allows us to focus on the Baire space when studying properties of Polish spaces, as long as we can ensure that the properties in question are preserved under continuous images.

Note that since the cardinality of the Baire space is $2^{\aleph_0}$, the two theorems imply that any uncountable Polish space has cardinality precisely $2^{\aleph_0}$.

There are several equivalent characterizations of a set being analytic. One that we will use is the following:

Proposition 2.2.7. *Let X be a Polish space, and let $A \subseteq X$ be a subset. Then A is analytic if and only if there is a closed set $F \subseteq X \times \omega^\omega$ such that $A = \mathrm{proj}_X(F)$.*

Proof. Let $A \subseteq X$ be analytic, and let Y be a Polish space and $B \subseteq X \times Y$ a Borel subset such that $\mathrm{proj}_X(B) = A$. We may assume that A, and therefore also B, is non-empty, and by Theorem 2.2.6 there is a continuous surjection $g : \omega^\omega \to B$. Then $F = \mathrm{graph}(\mathrm{proj}_X \circ g)$ is a closed subset of $\omega^\omega \times X$ satisfying $\mathrm{proj}_X(F) = A$.

The other direction holds since closed sets are Borel. □

This can be generalized in the following way: Let X be a Polish space and κ an ordinal with the discrete topology. If $A = \mathrm{proj}_X(F)$, where $F \subseteq X \times \kappa^\omega$ is closed, then we say that A is *κ-Suslin*. Note that the analytic sets are exactly the ω-Suslin sets.

If we let X be a countable, discrete set, we obtain the following characterization of the analytic subsets of X^ω:

Proposition 2.2.8. *A set $A \subseteq X^\omega$ is analytic if and only if there is a tree T on $X \times \omega$ with $A = \pi[T]$, where π denotes the projection onto the first coordinate.*

Proof. By Proposition 2.2.7, A is analytic if and only if it is the projection of a closed set $F \subseteq X^\omega \times \omega^\omega$. We may view $X^\omega \times \omega^\omega$ as $(X \times \omega)^\omega$, and by Proposition 2.2.2 the closed set F equals the branch set of a tree T_F. □

Also the κ-Suslin subsets of ω^ω has this tree structure:

Proposition 2.2.9. *A set $A \subseteq \omega^\omega$ is κ-Suslin if and only if there is some tree on $\omega \times \kappa$ such that $A = \pi[T]$.*

There is another reason why the Cantor space is significant in this context. As already noted, it is natural to identify the power set $\mathcal{P}(S)$ of a countably infinite set S with the set of infinite sequences 2^S. However, in order to talk about the topological complexity of a family of subsets of S, we need to view the power set $\mathcal{P}(S)$ as a topological space. To this end, we shall therefore rather identify $\mathcal{P}(S)$ with the Polish space 2^ω under some fixed bijection $\phi : S \to \omega$. This means that any subset $A \subseteq S$ will be identified with its characteristic function $\chi_A \in 2^S$, and then via ϕ with an element in 2^ω. Moreover, this means that $\mathcal{A} \subseteq \mathcal{P}(S)$ is κ-Suslin if and only if there is a tree

T on $2 \times \kappa$ such that $\mathcal{A} = \{A \in \mathcal{P}(S) \mid \chi_A \circ \phi \in \pi[T]\}$. We shall (sloppily and through the identifications of S with ω and χ_A with A) also write $\mathcal{A} = \pi[T]$ in such a case.

Equivalence relations and invariant descriptive set theory

Invariant descriptive set theory studies the complexity of equivalence relations on Polish spaces. In general, one is often interested in partitioning a class of objects by assigning invariants to them and thus defining an equivalence relation. We would then want to decide the complexity of such an equivalence relation, the goal being that it is less complex than the identity relation while also conserving a substantial amount of information about the objects within each equivalence class.

Let X and Y be Polish spaces. We say that an equivalence relation $E \subseteq X \times X$ is *Borel-reducible* to an equivalence relation $F \subseteq Y \times Y$ if there is a Borel map (i.e. a map satisfying that the preimage of a Borel set is Borel) $f : X \to Y$ such that $x_0 E x_1 \Leftrightarrow f(x_0) F f(x_1)$. If this is the case, then we say that E is less complex than F. If they are both Borel-reducible to each other, we say that they have the same complexity.

The *tail equivalence relation* E_0 on ω^ω will play an important role in this thesis. It is defined by

$$x E_0 y \Leftrightarrow (\exists n \in \omega)(\forall m \geqslant n)\ x_m = y_m,$$

and it holds that for any Borel equivalence relation E that is not Borel reducible to equality on ω^ω, the tail equivalence relation E_0 is Borel reducible to E [6].

For a more solid introduction, the reader is referred to [3].

2.3 Determinacy

In order to talk about determinacy, we need to introduce the concept of infinite games. For this, let $X \subseteq \omega$ be a non-empty set. Two players, player I and player II, take turns playing one element in X. The same element may be chosen several times, and both players have full information about the moves already made. The game is infinite, so in the end we obtain an infinite sequence $(x_n)_{n\in\omega}$ of elements of X. Before the game, a payoff set $A \subseteq X^\omega$ is defined. If $(x_n)_{n\in\omega} \in A$, then player I wins. If not, player II wins.

A *strategy* for a player is a function $f : X^{<\omega} \to X$ taking finite sequences in X (of even length for player I, and odd for player II) as input, and producing an element of X. The strategy is *winning* if whenever the player follows the strategy, she wins the game. We say that the game is *determined* if one of the players have a winning strategy.

The question is for which payoff sets $A \subseteq X^\omega$ the game is determined. The Axiom of Choice (AC) ensures that not all games are determined. AC allows, for instance, the existence of a *Bernstein set*, i.e. a subset $A \subseteq \{0,1\}^\omega$ such that neither A nor $\{0,1\}^\omega \backslash A$ contains a non-empty perfect set (see [10, Example 8.24]). For any strategy $\sigma : \bigcup_{n \in \omega} \{0,1\}^{\phi(n)} \to \{0,1\}$, where $\phi(n)$ is either $2n$ or $2n+1$, the set of possible outcomes, $\{x \in \{0,1\}^\omega \mid (\forall n \in \omega)\ x_{\phi(n)} = \sigma(x \restriction \phi(n))\}$ is a non-empty perfect set. Since for a strategy for player I or player II resp. to be winning, we need the payoff set or the complement of the payoff set resp. to contain the set of possible outcomes, neither player can have a winning strategy. On the other hand, Martin proved Borel determinacy in 1975, i.e. that any such game where the payoff set is Borel is determined [13].

If we instead consider the case where the payoff set is projective, the situation is more complex. In 1964, Davis proved that if all projective games are determined, then this implies that the *perfect set property* holds for all projective sets, i.e. that every projective set is either countable or contains a non-empty perfect set [1]. Alas, this property is known to be unprovable in ZFC. Indeed, by results of Martin and Steele [15], Neeman [19] and Woodin [18] it is characterized by the existence of suitable large cardinals, which also implies consistency of ZFC. By Gödel's incompleteness theorem we cannot prove this within ZFC itself. However, since the assumption that every projective game is determined is not known to be inconsistent with ZFC, it can therefore be considered as an additional axiom. The axiom is called the *Axiom of Projective Determinacy*, and is abbreviated **PD**.

From **PD**, one can derive many basic structural properties of the projective sets. Many of the proofs of such regularity properties rely on the fact that **PD** implies that certain projective point classes are *scaled*. We shall therefore look into what this means.

A *rank* on a set A is a function $\phi : A \to \mathbb{ON}$. A *prewellordering* is a relation which is reflexive, transitive and connected (i.e. any two elements are related). A rank ϕ on A gives rise to a prewellordering $\leqslant_\phi$ on A defined by

$$x \leqslant_\phi y \Leftrightarrow \phi(x) \leqslant \phi(y).$$

Let $A \subseteq \omega^\omega$, and let Γ be a pointclass. Let Γ^C denote the pointclass consisting of complements of sets in Γ. A rank $\phi : A \to \mathbb{ON}$ is called a Γ-rank if there are relations $\leqslant_\phi^\Gamma, \leqslant_\phi^{\Gamma^C} \subseteq \omega^\omega \times \omega^\omega$ in Γ and Γ^C respectively, such that for $y \in A$:

$$\phi(x) \leqslant \phi(y) \Leftrightarrow x \in A \wedge \phi(x) \leqslant \phi(y) \Leftrightarrow x \leqslant_{\phi}^{\Gamma} y \Leftrightarrow x \leqslant_{\phi}^{\Gamma^C} y.$$

A *scale* on A is a sequence $\phi_n \colon A \to \mathbb{ON}$ of ranks such that if $(x_i)_{i\in\omega} \in A$ and $x_i \to x$ and furthermore $\phi_n(x_i) \to \alpha_n$ for every $n \in \omega$ (in the sense that there exists $j \in \omega$ such that $\phi_n(x_i) = \alpha_n$ for every $i > j$), then $x \in A$ and $\phi_n(x) \leqslant \alpha_n$ for every $n \in \omega$. If $\phi_n : A \to \kappa$, we say that $(\phi_n)_{n\in\omega}$ is a κ-scale.

We say that a scale $(\phi_n)_{n\in\omega}$ is a Γ-scale if every rank ϕ_n is a Γ-rank. The pointclass Γ is called *scaled* if every $A \in \Gamma$ admits a Γ-scale. A proof of the following (which requires only ZF + Dependent Choice) can be found in [10, Chapter 39]:

Theorem 2.3.1. *(***PD***) The pointclasses $\mathbf{\Pi}^1_{2n+1}$ and $\mathbf{\Sigma}^1_{2n+2}$ are scaled for every $n \in \omega$.*

For our purpose, the most important consequence of this scale property is the fact that it can be used to give projective sets a certain tree structure. To see this, let A be projective. We may assume that A is $\mathbf{\Pi}^1_{2m+1}$ for some $m \in \omega$. Let $(\phi_n)_{n\in\omega}$ be a $\mathbf{\Pi}^1_{2m+1}$-scale which is also a κ-scale for some ordinal κ. Define a tree $T_{\vec{\phi}}$ on $\omega \times \kappa$ in the following way:

$$\begin{aligned} &((k_0, \ldots, k_{n-1}), (\alpha_0, \ldots, \alpha_{n-1})) \in T_{\vec{\phi}} \Leftrightarrow \\ &(\exists x \in A)\ x{\restriction}n = (k_0, \ldots, k_{n-1}) \wedge (\forall i < n)\ \alpha_i = \phi_i(x). \end{aligned}$$

Then $A = \pi[T_{\vec{\phi}}]$. This means that under the assumption of **PD**, any projective set can be represented as the projection of a canonical tree.

The *Axiom of Determinacy* (**AD**) states that for any payoff set the above game is determined. This is, as noted above, inconsistent with the Axiom of Choice. Furthermore, we note that the tree structure of projective sets that are obtained by **PD** and the scale property relies on the hierarchic structure of the projective sets. Due to the lack of hierarchic structure of all sets, **AD** does not provide a canonical tree structure representation of every set. However, **AD** is still a powerful assumption which implies that sets of reals are well-behaved in the sense that they are Lebesgue measurable, they have the Baire property and the perfect set property. For a proof, see [8, Theorem 33.3].

2.4 Advanced set theory: Forcing, inner models, and absoluteness

Suppose we have models M, N of ZFC. Any formula which is satisfiable by either M or N is *consistent* with ZFC. If there is a formula ϕ satisfied in N

such that $\neg\phi$ is satisfied in M, then we say that ϕ is *independent* of ZFC. Moreover, the notion of *absoluteness* allows us to prove a theorem in general by considering it in a specific model.

In order to prove consistency results or to make use of absoluteness, we need to be able to expand or shrink a given model in a controlled manner. This is the motivation behind the theory of forcing and inner model theory. We will give a short introduction to both forcing and inner model theory before we look into some of the most fundamental absoluteness results. For a more elaborate introduction, see [12] or [8].

Forcing

Forcing is an efficient technique in set theory which is used to extend models in order to obtain new ones. In a bit more detail, we do the following: let M be a model of ZFC. This will be referred to as the *ground model*. We would like to extend the model M by adding some element G and construct the smallest model $M[G]$ of ZFC which contains both M and G.

In order to add such an element while keeping full control of the extended model, we work with posets and generics. Let $(\mathbb{P}, \leqslant)$ be a set $\mathbb{P}$ together with a partial order, i.e. a reflexive, antisymmetric and transitive relation. Such a pair is called a *poset*. Elements of $\mathbb{P}$ are called *conditions*, and if $q \leqslant p$ we say that q *extends* p or that q is *stronger* than p. We say that two conditions $p, q \in \mathbb{P}$ are *compatible* if there exists some $r \in \mathbb{P}$ such that $r \leqslant p$ and $r \leqslant q$. If no such r exists, we say that p and q are incompatible, and we write $p \perp q$. A *forcing poset* for the model M is a countably infinite set $\mathbb{P}$ together with a partial order $\leqslant$ and a largest element 1 such that $(\mathbb{P}, \leqslant, 1) \in M$ and where the partial order $\leqslant$ satisfies the following:

$$(\forall p \in \mathbb{P})(\exists q, r \in \mathbb{P})\ q, r \leqslant p \wedge q \perp r. \tag{2.4.1}$$

The intuition is that if we have a sequence of conditions in $\mathbb{P}$ that extend each other, then the stronger conditions provide more information about the "limit", in the same way that smaller and smaller intervals around a real provide more and more information about the real in question. However, we do not know whether this "limit" exists in M and this is exactly the point; we want to find such a "limit" which does not exist in the ground model, and then add this element in order to obtain a new model which properly extends the ground model. To detect such elements, we need to talk about dense sets and generic filters.

Let $D \subseteq \mathbb{P}$ be a subset. We say that D is *dense* if for every $p \in \mathbb{P}$ there is some $q \in D$ such that $q \leqslant p$, i.e. for every condition p there is an extension q

which is in D. We say that a set $G \subseteq \mathbb{P}$ is a *filter* if for every $p, q \in G$ there is $r \in G$ such that $r \leqslant p$ and $r \leqslant q$, and for every $p \in G$ and $q \in \mathbb{P}$, if $p \leqslant q$ then $q \in G$. So in other words, for every two conditions in G there is a condition in G extending both of them, and every weakening of a condition in G is also in G. We say that the filter G is $\mathbb{P}$-*generic over* M if for every dense set $D \subseteq \mathbb{P}$ such that $D \in M$, the intersection $G \cap D \neq \varnothing$ is non-empty.

In the following, we will assume that the ground model M is countable and transitive. Note that this is okay to assume for our purpose, since the countability assumption follows from the Löwenheim-Skolem theorem [8, Theorem 12.1] if we just assume the existence of an infinite model, while the transitivity assumption is enabled by the Mostowski's Collapsing Theorem (see [8, Theorem 6.15]) as long as the membership relation is well-founded. Abusing notation, we will often write $\mathbb{P}$ for the triplet $(\mathbb{P}, \leqslant, 1)$. We will also often say generic (or $\mathbb{P}$-generic) when we mean $(\mathbb{P}, \leqslant, 1)$-generic.

Lemma 2.4.1. *If $p \in \mathbb{P}$, then there is a filter G which is generic over M such that $p \in G$.*

Proof. Since M is countable, let $(D_n)_{n\in\omega}$ be an enumeration of all the dense sets $D_n \subseteq \mathbb{P}$ in M. Set $q_0 = p$, and recursively choose $q_{n+1} \in D_n$ such that $q_{n+1} \leqslant q_n$. Let G be the filter generated by $\{q_n : n \in \omega\}$. □

However, the generic filter G will not be in M:

Lemma 2.4.2. *If G is $\mathbb{P}$-generic over M, then $G \notin M$.*

Proof. Suppose towards a contradiction that $G \in M$, and set $D = \mathbb{P} \backslash G$. Since M is transitive, we have $\mathbb{P} \backslash G = (\mathbb{P} \cap M) \backslash (G \cap M)$, which means that $D \in M$. Since $\mathbb{P}$ is a forcing poset, for any $p \in \mathbb{P}$ there are incomparable $q, r \in \mathbb{P}$ such that $q \leqslant p$ and $r \leqslant p$. Since G is a filter, they cannot both be in G, so one is in D, proving that D is dense. Since G is generic, we should then have $G \cap D \neq \varnothing$, which contradicts the definition of D. □

When we have such a generic filter G which is not in M, then there is a transitive set $M[G]$ called the *forcing extension* which is a model of ZFC containing the same ordinals as M and that contains $M \cup \{G\}$ as a subset [8, Theorem 14.5]. Moreover, $M[G]$ is minimal with these properties. To construct $M[G]$, we add the generic filter G and close under set operations definable in M. In this process, we give each element of $M[G]$ a $\mathbb{P}$-*name* in M, which describes how the element is constructed in $M[G]$. Even though the name can be understood within M, the element itself will not be decidable in M, only in $M[G]$. For any element $x \in M$, there is a canonical way of representing x by a $\mathbb{P}$-name, $\check{x}$, decorated by a check. $\mathbb{P}$-names of elements in $M[G]\backslash M$ are on the other hand decorated with dots, $\dot{x}$. Since the $\mathbb{P}$-names are expressible inside M, then given any particular generic filter G, one can also define how a $\mathbb{P}$-name should be interpreted as a set in the specific model $M[G]$. This is done such that $\mathbb{P}$-names of elements of M are interpreted as themselves. Abusing notation, we will therefore sometimes confuse an element $x \in M$ with its name $\check{x}$.

We are now able to make first-order logic statements about $M[G]$ expressible inside M by use of membership as binary relation and $\mathbb{P}$-names as constant symbols, and this language is called the *forcing language*. Such statements, though expressible in M, are not necessarily decidable in M since their truth value in general depend on the generic G. However, finite information about G might be enough to determine the truth of a statement in the forcing language. We define the *forcing relation* in the following way: if ϕ is a formula in the forcing language and $p \in \mathbb{P}$, then $p \Vdash \phi$ if and only if $M[G]$ satisfies ϕ for every generic filter G containing p. In other words, this property is "forced" to hold in the extension $M[G]$, even though we only have partial information about G (namely that $p \in G$), and we say that *p forces* ϕ.

The powerful forcing theorem now states that for any formula ϕ in the forcing lanuguage and any generic filter G, we have $M[G] \models \phi$ if and only if there is some $p \in \mathbb{P}$ such that $p \Vdash \phi$ [8, Theorem 14.6]. In other words, for any theorem of $M[G]$ there is a finite $p \in \mathbb{P}$ that forces it, meaning that truth in $M[G]$ of formulas in the forcing language can be decided within M.

The art of forcing is to find the suitable poset given a formula that we would like to hold in the forcing extension. We want to build the poset in such a way that each condition consists of a small fraction of sets of the kind we need in order to prove the theorem in question. The generic filter is then able to single out the object we want to add. However, we need to ensure that the building blocks satisfies the poset axioms. If a proper poset $\mathbb{P}$ can be defined, a sufficiently strong $p \in \mathbb{P}$ will force the wanted statement.

A classical example is the independency result of the Continuum Hypothesis (CH). We will sketch the proofs, in order to give the unexperienced reader an overall idea of how forcing works [8, Theorem 14.32], [2]:

Example 2.4.3 (The Continuum Hypothesis)**.** To prove that ¬CH is consistent with ZFC, let M be a countable, transitive model of ZFC and let κ be some ordinal such that $M \models |\kappa| = \aleph_2$, and let $\mathbb{P}$ be the poset of finite partial functions from $\kappa \times \omega$ to $\{0, 1\}$. A generic filter G through this poset will then correspond to a total function $f_G : \kappa \times \omega \to \{0, 1\}$. For each $x \in \kappa$, we can now define a function $f_x : \omega \to \{0, 1\}$ by $f_x(n) = f_G(x, n)$. Using genericity of

G, it is not hard to prove that if $x, y \in \kappa$ and $x \neq y$, then $f_x \neq f_y$; indeed, the set

$$D = \{p \in \mathbb{P} \mid p(x,n) \neq p(y,n) \text{ for some } n\}$$

is dense in $\mathbb{P}$ and thus $G \cap D \neq \varnothing$. Furthermore, each function f_x corresponds to a subset of ω, and thus the power set of ω has cardinality at least $|\kappa|$. Moreover, it is one of the most fundamental results in forcing that if $\mathbb{P}$ satisfies the *countable chain condition*, i.e. that any antichain in $\mathbb{P}$ is at most countable, then $\mathbb{P}$ preserves cardinals [8, Theorem 13.34]. Since it can be showed that $\mathbb{P}$ above has the countable chain condition, this means that $|\kappa| = \aleph_2^{M[G]}$, and this proves that $\neg$CH holds in the extension and is therefore consistent with ZFC.

To see that CH is consistent with ZFC, let in stead $\mathbb{P}$ be the poset of countable (in M) partial functions from $\aleph_1^M$ to $\mathcal{P}(\omega)^M$. Since the generic intersects all dense subsets of $\mathbb{P}$, for instance the sets $D_x = \{p \in \mathbb{P} \mid p(x) \text{ is defined}\}$ for $x \in \aleph_1^M$ and $\{p \in \mathbb{P} \mid (\exists x)\ p(x) = r\}$ for any $r \in \mathcal{P}(\omega)$, the generic is a totally defined surjection. So $|\mathcal{P}(\omega)^M| \leqslant |\aleph_1^M|$ in the extension. We only need to show that $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^{M[G]}$, and to see this we show that a function $f : \omega \to \{0,1\}$ in $M[G]$ is actually contained in M. We can define a descending sequence of elements of $\mathbb{P}$ forcing more and more of $\check{f}$. The union of this is itself a countable function, and knows all values of f.

The forcing notion used in this thesis is based on Mathias forcing. Let us introduce this, and establish some general facts:

Definition 2.4.4 (Mathias forcing)**.** An element of the poset $\mathbf{M}$ in Mathias forcing is a pair (a, A) where $a \in [\omega]^{<\omega}$ and $A \in [\omega]^{\omega}$ are such that $\max(a) < \min(A)$. The poset is ordered by

$$(a', A') \leqslant (a, A) \Leftrightarrow a \sqsubseteq a' \wedge a' \subseteq a \cup A \wedge A' \subseteq A.$$

The maximal element is $(\varnothing, \omega)$. Note how the order ensures that stronger conditions fixes larger and larger finite sequences, while at the same time narrowing the infinite set in which the sequence is allowed to grow. Note also that this poset satisfies the condition 2.4.1: Let $(a, A) \in \mathbf{M}$. Let $b_0 \subseteq A$ and $b_1 \subseteq A$ be finite and disjoint subsets. Then $(a, A \backslash b_0), (a, A \backslash b_1) \leqslant (a, A)$ and $(a, A \backslash b_0) \perp (a, A \backslash b_1)$.

Let G be a generic for the forcing, and let $x_G = \bigcup\{a \mid (\exists A)\ (a, A) \in G\}$ denote the union of the finite parts of the conditions in G. Note that for any $p \in \mathbf{M}$ we have $p \Vdash a(p) \sqsubseteq \dot{G}$. Furthermore x_G is infinite; indeed, suppose towards a contradiction that this does not hold. Then by the forcing theorem there is a forcing condition $p \in \mathbf{M}$ and some $n \in \omega$ such that $p \Vdash x_{\dot{G}} \subseteq n$. Let now $d \in [\omega]^{<\omega}$ be a finite set such that $a(p) \sqsubseteq d \subseteq a(p) \cup A(p)$ and $\max(d) > n$, and set $D = \{n \in A(p) \mid n > \max(d)\}$. Then $(d, D) \leqslant p$, so $(d, D) \Vdash x_{\dot{G}} \subseteq n$, contradicting the fact that $(d, D) \Vdash d \sqsubseteq x_{\dot{G}}$ and $\max(d) > n$.

Furthermore, x_G is not in the ground model. If this was not true, then the dense set $D_{x_G} = \{q \mid a(q) \not\sqsubseteq x_G\}$ would be in the ground model as well, but D_{x_G} does not intersect the generic.

The set x_G is called a *Mathias real*. One can put further restrictions on the infinite sets, in order to steer the Mathias real in the direction that we want. As we will see, this can for instance be done such that the generic almost avoids every element in an AD-family.

Inner Model Theory

An *inner model of* ZF (or ZFC) is a transitive class that contains all the ordinals and satisfies the axioms of ZF (or ZFC, respectively). It is called an inner model because it can be viewed as an inner universe of the von Neumann universe V.

The first non-trivial example of an inner model of ZF is *Gödel's constructible universe* $\mathbf{L}$. Before we give the formal definition, we remember that a set X is definable over a model $(M, \in)$ if there exists a formula ϕ in the language of the model and some $a_0, \ldots, a_n \in M$ such that $X = \{x \in M \mid (M, \in) \models \phi[x, a_0, \ldots, a_n]\}$. Now let

$$def(M) = \{X \subseteq M \mid X \text{ is definable over } (M, \in)\},$$

where M is a set.

Definition 2.4.5. The building blocks of Gödel's constructible universe $\mathbf{L}$ are defined by transfinite recursion:

1. $L_0 = \varnothing$;
2. $L_{\alpha+1} = def(L_\alpha)$;
3. $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ if α is a limit ordinal.

Now set $\mathbf{L} = \bigcup_{\alpha \in \mathbb{ON}} L_\alpha$.

In other words, Gödels constructible universe $\mathbf{L}$ is a class of sets such that each set can be described entirely in terms of simpler sets. We say that a set x is *constructible* if $x \in L_\alpha$ for some α. It turns out that $\mathbf{L}$ is a model of ZF [8, Theorem 13.3].

The *Axiom of constructibility* is the axiom stating that every set is constructible, and is often denoted $V = \mathbf{L}$. The axiom of constructibility is satisfied in $\mathbf{L}$ and therefore consistent with ZF. Note that the fact that $V = \mathbf{L}$ holds in $\mathbf{L}$ is not as trivial as one could suspect at first sight; one needs to prove that the statement "x is constructible" holds *relative to* $\mathbf{L}$, meaning that if there exists some ordinal α such that $x \in L_\alpha$, then there exists $\alpha \in \mathbf{L}$ such that $x \in L_\alpha$ when interpreted in $\mathbf{L}$. It can furthermore be showed that $V = \mathbf{L}$ implies both Axiom of Choice and the Generalized Continuum Hypothesis (GCH), which means that $\mathbf{L}$ is a model of both AC and GCH and proves their relative consistency [4].

However, the constructible universe is too narrow for many purposes. In stead, we often consider the universe $\mathbf{L}(\mathbb{R})$ which uses the reals as base set for the recursive definition of constructible sets. i.e. with $L_0 = \mathbb{R}$ in the definition of $\mathbf{L}$ above. The class $\mathbf{L}(\mathbb{R})$ is therefore the smallest transitive inner model of ZF containing the reals. We will be considering the theory ZF + **AD** + $V = \mathbf{L}(\mathbb{R})$. Note that since **AD** contradicts AC, the latter does not hold in this theory. However, axiom of Dependent Choice (DC) does [9].

In stead of changing the base set in order to obtain various constructible universes, we can also alter the definition of being definable. For a given set A, we can talk about the sets that are *constructible relative to* A in the following way:

$$def_A(M) = \{X \subseteq M \mid X \text{ is definable over } (M, \in, A \cap M)\},$$

where $A \cap M$ is considered a unary predicate. The class of sets constructible from A are now defined as follows:

1. $L_0[A] = \varnothing$;

2. $L_{\alpha+1}[A] = def_A(L_\alpha[A])$;

3. $L_\alpha[A] = \bigcup_{\beta<\alpha} L_\beta[A]$ if α is a limit ordinal.

Now set $\mathbf{L}[A] = \bigcup_{\alpha\in\mathbb{ON}} L_\alpha[A]$. Then $\mathbf{L}[A]$ is a model of ZFC [8, Theorem 13.22 (i)].

Remember that assuming Dependent Choice and Projective Determinacy, the pointclasses $\mathbf{\Pi}^1_{2n+1}$ and $\mathbf{\Sigma}^1_{2n+2}$ are known to be scaled, and that these scales provide us with tree representations for projective set. At the same time, each scale can be captured by a 'small' model, namely the model consisting of sets that are constructible relative to this tree. We have the following lemma, whose proof can be found in [5, Lemma 2.4]:

Lemma 2.4.6. *Assume* **PD** *and* DC. *Suppose A is projective. There exists an inner model M of* ZFC *and a tree $T \in M$ on $\omega \times \kappa$ (for some ordinal κ) such that $\pi[T] = A$ and $\mathcal{P}(\mathcal{P}(\omega))^M$ is countable in V.*

There is a version of this based on the full Axiom of Determinacy, which we shall also use. The proof can be found in [5, Lemma 2.5].

Lemma 2.4.7. *Assume* **AD** *holds and $V = \mathbf{L}(\mathbb{R})$. Suppose A is $\mathbf{\Sigma}^2_1$. There exists an inner model M of* ZFC *and a tree $T \in M$ on $\omega \times \kappa$ (for some ordinal κ) such that $\pi[T] = A$ and $\mathcal{P}(\mathcal{P}(\omega))^M$ is countable in V.*

Finally we shall need a result (due to Woodin) known as Solovay's Basis Theorem (see [11, Remark 2.29(3)]). Note that a $\mathbf{\Sigma}^2_1$ statement is equivalent to a statement of the form $(\exists X \subseteq \mathcal{P}(\omega))\ \phi(X, r)$ where r is a fixed real parameter and the quantifiers occuring in ϕ are ranging over $\mathcal{P}(\omega)$ or ω.

Theorem 2.4.8 (Solovay's Basis Theorem)**.** *Assume* **AD** *holds and $V = \mathbf{L}(\mathbb{R})$. Then every $\mathbf{\Sigma}^2_1$ statement is witnessed by a set $A \subseteq \mathbb{R}$ which is itself $\mathbf{\Delta}^2_1$.*

Absoluteness

Let M and N be models of a theory T, and $\phi(x_0, \ldots, x_{n-1})$ a first order formula with all free variables listed. Then we say that ϕ is *absolute* for M, N iff

$$(\forall x_0, \ldots, x_{n-1})\ \phi^M(x_0, \ldots, x_{n-1}) \leftrightarrow \phi^N(x_0, \ldots, x_{n-1}).$$

In other words, ϕ is absolute for M, N if it is true when interpreted in M if and only if it is true when interpreted in N. We say that ϕ is absolute for T if it is absolute for any two models of T.

A formula ϕ is said to be *upwards absolute* for a theory T if for any two models M and N such that $M \subseteq N$ and ϕ is true in M, the formula ϕ is also true in N. Similarly, we say that ϕ is *downwards absolute* for a theory T if for any two models M and N such that $M \subseteq N$ and ϕ is true in N, the formula ϕ is also true in M.

It follows immediately from the definitions that formulas with only existensial quantifiers ranging over M, N are upwards absolute, while formulas with only universal quantifiers ranging over M, N are downwards absolute.

Proposition 2.4.9. *The notion of the empty set is absolute for transitive models of* ZF*. In other words, the formula $x = \varnothing$ is absolute.*

Proof. Let M, N be transitive models of ZF and $x \in M \cap N$. Note that $x = \varnothing \leftrightarrow (\forall w \in x)\ w \neq w$. Thus

$$(x = \varnothing)^M \leftrightarrow (\forall w \in x \cap M)\ w \neq w \leftrightarrow (\forall w \in x)\ w \neq w \leftrightarrow (\forall w \in x \cap N)\ w \neq w \\ \leftrightarrow (x = \varnothing)^N,$$

where the first and last arrows are just the definition of relativization of formulas, and the two middles ones follow from transitivity of M and N. $\square$

Recall how we noted above that the statement "x is constructible" is true relative to $\mathbf{L}$, in the sense that if $x \in L_\alpha$ holds when interpreted in V, then it also holds when interpreted in $\mathbf{L}$. What is actually shown is that the sets L_α are absolute for inner models M, N of ZF, i.e. that $L_\alpha^M = L_\alpha^N$ for every ordinal α. Thus we obtain

$$(x \text{ is constructible})^N \leftrightarrow (\exists \alpha \in N)\ x \in L_\alpha^N \leftrightarrow (\exists \alpha \in M)\ x \in L_\alpha^M \\ \leftrightarrow (x \text{ is constructible})^M.$$

Note that this implies that $\mathbf{L}$ is the smallest inner model of ZF. Indeed, for any inner model M we can consider $\mathbf{L}^M \subseteq M$, the constructible universe defined in M. Since $L_\alpha^M = L_\alpha$ for every ordinal α by absoluteness of L_α, and since M contains all the ordinals, we have $\mathbf{L}^M = \mathbf{L}$ and we conclude that $\mathbf{L} \subseteq M$.

We have the following easy and well-known result:

Proposition 2.4.10. *Let M, N be inner models of* ZFC*, and $T \in M \cap N$ a countable tree. Then "T is well-founded" (equivalently: "$[T] = \varnothing$") is absolute for M and N.*

Proof. Suppose T^M is well-founded. Then the rank function ρ_T^M is well-defined. This rank function also works for T^N, ensuring well-foundedness in N. $\square$

This gives rise to many deeper absoluteness results, such as Lévy-Shoenfield Absoluteness, and it is also the basic idea behind the following absoluteness lemma [5, Lemma 2.3]:

Lemma 2.4.11. *Let T be a tree on $2 \times \kappa$ and let $\mathcal{J}$ be a Borel ideal on a countable set S. Then the following properties are absolute between inner models of* ZFC*:*

1. *"$\pi[T] \subseteq \mathcal{J}^+$".*

2. *"$\pi[T]$ is an $\mathcal{J}$-almost disjoint family".*

3. "There exists y such that y is $\mathcal{J}$-almost disjoint from every set in $\pi[T]$".

In the above, we mean by $\mathcal{J}$ the ideal obtained by interpreting the Borel definition *in the current model.*

Proof. (1) Let U be a tree on $2 \times \omega$ such that $\mathcal{J} = \pi[U]$. Consider the tree T_+ on $2 \times \kappa \times \omega$ defined by

$$T_+ = \{(a, s, u) \mid (a, s) \in T, (a, u) \in U\}.$$

Then $\pi[T] \subseteq \mathcal{J}^+$ if and only if $[T_+] = \varnothing$, which is absolute.

(2) By the previous item is suffices to show that "$\forall x, y \in \pi[T]\ x \neq y \Rightarrow x \cap y \in \mathcal{J}$" is absolute. Let U be a tree on $2 \times \omega$ such that $\mathcal{J}^+ = \pi[U]$. Consider the tree $T_\cap$ on $(2 \times \kappa)^2 \times \omega$ defined by

$$\begin{aligned} T_\cap = &\{(a_0, t_0, a_1, t_1, u) \mid \\ &\bigwedge_{i \in \{0,1\}} (a_i, t_i) \in T, \mathrm{lh}(a_0) = \mathrm{lh}(a_1), \text{ and } (a_0 \cdot a_1, u) \in U\} \end{aligned}$$

where we momentarily write $a_0 \cdot a_1$ for the characteristic function of $a_0 \cap a_1$ on $\mathrm{lh}(a_0)$. Then the statement in question holds if and only if $[T_\cap] = \varnothing$, which is absolute.

(3) As in the previous item, let U be a tree on $2 \times \omega$ such that $\mathcal{J}^+ = \pi[U]$, and for $y \in 2^\omega$ consider the tree $T_\cap^y$ on $(2 \times \kappa \times 2 \times \omega)$ defined by

$$T_\cap^y = \{(a, t, u) \mid (a, t) \in T, \text{ and } (a \cdot y{\restriction}\mathrm{lh}(a), u) \in U\}.$$

Then y is $\mathcal{J}$-almost disjoint to everything in $\pi[T]$ if and only if $T_\cap^y$ is well-founded.

For $s \in 2^{<\omega}$, define a similar tree $T_\cap^s$ by

$$T_\cap^s = \{(a, t, u) \mid (a, t) \in T, \mathrm{lh}(a) \leqslant \mathrm{lh}(s) \text{ and } (a \cdot s{\restriction}\mathrm{lh}(a), u) \in U\}.$$

Define a tree $\hat{T}$ on $2 \times \kappa^+$ by

$$\hat{T} = \{(s, \rho) \mid \rho : T_\cap^s \to \kappa^+ \text{ is a rank function}\}.$$

Then $\hat{T}$ has an infinite branch if and only if there exists some $y \in 2^\omega$ such that $T_\cap^y$ is well-founded.

$\square$

Chapter 3

Definable $\mathcal{J}$-MAD families

We are now ready to prove the main results of this thesis. This chapter is an amended version of [5, Section 2, 3 and 4], with only minor changes made. All of the results in this chapter have been obtained in joint work with David Schrittesser and Asger Törnquist.

3.1 Classical MAD families (and a bit more)

In this section we prove the following:

Theorem 3.1.1. *Let* $\mathcal{J} = \mathrm{Fin}$*, or more generally* $\mathcal{J} = \mathrm{Fin}(\phi)$ *where* ϕ *is an lsc submeasure on* ω*.*

1. *There are no analytic infinite* $\mathcal{J}$*-MAD families.*
2. *Under ZF + Dependent Choice + Projective Determinacy, there are no projective infinite* $\mathcal{J}$*-MAD families.*
3. *Under ZF + Determinacy +* $V = L(\mathbb{R})$ *there are no infinite* $\mathcal{J}$*-MAD families.*

The first item was first shown by Mathias [16] (at least in the case of Fin). The next two items are independently, and by a somewhat different method shown by Neeman and Norwood [20] (also in the case of Fin).

We use the following close relative of Mathias forcing:

Definition 3.1.2. Suppose that $\mathcal{I} \supseteq \mathrm{Fin}$ is a (proper) ideal on ω, and $\mathcal{I}^+$ its co-ideal. Define

$$\mathbf{M}^{\mathcal{I}} = \{(a, A) \mid a \in [\omega]^{<\omega}, A \in \mathcal{I}^+, \max(a) < \min(A)\}$$

ordered by

$$(a', A') \leqslant (a, A) \text{ if and only if } a \sqsubseteq a' \subseteq a \cup A \text{ and } A' \subseteq A.$$

We write $\mathbf{M}$ for $\mathbf{M}^{\mathrm{Fin}}$.

We use the following notation:

Notation 3.1.3.

1. Given a filter G on $\mathbf{M}^{\mathcal{I}}$, let
$$x_G = \bigcup \{a \mid (\exists A \in \mathcal{I}^+)\ (a, A) \in G\}.$$
2. For $(a, A) \in \mathbf{M}^{\mathcal{I}}$, and $b \subseteq A$ finite, let $A/b = \{n \in A \mid n > \max(b)\}$.
3. For $p \in \mathbf{M}^{\mathcal{I}}$, we write $p = (a(p), A(p))$ when we want to refer to its components.
4. For $p \in \mathbf{M}^{\mathcal{I}}$, we let $\mathbf{M}^{\mathcal{I}}(\leqslant p) = \{q \in \mathbf{M}^{\mathcal{I}} \mid q \leqslant p\}$.

Assumption 3.1.4. Until the end of Section 3.1 let $\mathcal{J} = \mathrm{Fin}$ or more generally $\mathcal{J} = \mathrm{Fin}(\phi)$ and assume $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is an infinite $\mathcal{J}$-AD family which is κ-Suslin. Fix a tree T on $2 \times \kappa$ such that $\pi[T] = \mathcal{A}$. Let $\mathcal{I}$ be the ideal generated by $\mathcal{A} \cup \mathcal{J}$.

To avoid overly cumbersome notation, we shall phrase our presentation in terms of the ideal Fin. However this section is written so that whenever relevant, the reader may replace Fin (but *not* the word "finite" or the expression $[\omega]^{<\omega}$) with $\mathrm{Fin}(\phi)$, for any lsc submeasure ϕ on ω, in which case she must also replace "almost disjoint" by "$\mathrm{Fin}(\phi)$-AD", etc. We will point out how to modify proofs when these trivial substitutions do not suffice.

The main workload in the proof of Theorem 3.1.1 is carried by the following Main Proposition. The proof of this depends to a great extent on the Branch Lemma, and we will prove both the Branch Lemma and how this leads to the Main Proposition after we have collected some properties of the forcing $\mathbf{M}^{\mathcal{I}}$.

Main Proposition 3.1.5. $\Vdash_{\mathbf{M}^{\mathcal{I}}} (\forall y \in \pi[T])\ y \cap x_{\dot{G}} \in \mathrm{Fin}$. *In other words,* $\Vdash_{\mathbf{M}^{\mathcal{I}}} x_{\dot{G}} \notin \pi[T]$ *and* $\{x_{\dot{G}}\} \cup \pi[T]$ *is an almost disjoint family.*

Before we prove the Main Proposition, we show how easily it leads to Theorem 3.1.1. Firstly, we give a very short proof of the classical result that there are no analytic MAD families:

Corollary 3.1.6 ([16]). *There are no analytic MAD families.*

Proof. Suppose $\mathcal{A}$ is an analytic almost disjoint family, and fix a tree T on $2\times\omega$ such that $\mathcal{A} = \pi[T]$ (identifying $\mathcal{P}(\omega)$ with 2^ω). By Main Proposition 3.1.5 there is a forcing extension $V[G]$ containing a real which is almost disjoint from any set in $\pi[T]^{V[G]}$, and by Item 3 of Lemma 2.4.11 the existence of such a real is absolute for inner models of ZFC. Thus $\mathcal{A}$ is not maximal. □

We likewise obtain an easy and transparent proof that under projective determinacy, there are no projective MAD families.

Corollary 3.1.7. *Under* **PD** *there are no projective MAD families.*

Proof. Assume **PD** holds and suppose $\mathcal{A}$ is an infinite almost disjoint family which is projective. Fix a tree T so that $\mathcal{A} = \pi[T]$ and a model M as in Lemma 2.4.6. Note that $M \vDash \pi[T]$ is an infinite almost disjoint family. Working inside M let $\mathcal{I}$ be the ideal generated by $\mathrm{Fin} \cup \pi[T]$ and let $\mathbb{P}$ denote $\mathbf{M}^{\mathcal{I}}$ in M. As $\mathcal{P}(\mathcal{P}(\omega))^M$ is countable in V we may find $r \in [\omega]^\omega$ which is $\mathbb{P}$-generic. By Main Proposition 3.1.5

$$M[r] \vDash (\forall y \in \pi[T])\ y \text{ is almost disjoint from } r,$$

and then Item 3 of Lemma 2.4.11 ensures that $\mathcal{A}$ is not maximal in any inner model of ZFC. □

A similar proof can be given of the **AD** analogue:

Corollary 3.1.8. *If* $\mathbf{L}(\mathbb{R}) \vDash \mathbf{AD}$*, there are no MAD families in* $\mathbf{L}(\mathbb{R})$*.*

Proof. Suppose towards a contradiction that $V = \mathbf{L}(\mathbb{R})$, **AD** holds, and there is a MAD family. The existence of a MAD family is a $\mathbf{\Sigma}^2_1$ statement:

$$\begin{aligned}&(\exists X \subseteq \mathcal{P}(\omega))(\forall x, y \in X)(\forall r \in \mathcal{P}(\omega))\\ &[x \neq y \to (\exists n \in \omega)\ x \cap y \subseteq n] \wedge [r \notin X \to (\exists z \in X)(\forall n \in \omega)\ z \cap r \nsubseteq n].\end{aligned}$$

By Fact 2.4.8, there is a $\mathbf{\Sigma}^2_1$ MAD family $\mathcal{A}$. By Lemma 2.4.7 we may pick an ordinal κ and a tree T on $\kappa \times \omega$ such that $\pi[T] = \mathcal{A}$. Moreover, there is a model M such that $T \in M$ and $\mathcal{P}(\mathcal{P}(\omega))^M$ is countable. Proceed precisely as in Corollary 3.1.7 above to show that $\mathcal{A}$ is not maximal, reaching a contradiction. □

Properties of Mathias forcing relative to an ideal

For the proof of the Main Proposition 3.1.5 we need to explore the immediate properties of the forcing notion $\mathbf{M}^{\mathcal{I}}$.

The following lemma holds not just under our Assumption 3.1.4 but for any proper ideal $\mathcal{I} \supseteq \mathrm{Fin}$ or more generally $\mathcal{I} \supseteq \mathrm{Fin}(\phi)$.

Lemma 3.1.9.

1. *$\Vdash_{\mathbf{M}^{\mathcal{I}}} (\forall y \in \check{\mathcal{I}})\ x_{\dot{G}} \cap y \in \mathrm{Fin}$.*
2. *$\Vdash_{\mathbf{M}^{\mathcal{I}}} x_{\dot{G}} \in \mathrm{Fin}^+$.*
3. *Fix $A \in \mathcal{I}^+$ and $a_0, a_1 \in [\omega]^{<\omega}$ with $\max(a_i) < \min(A)$ for each $i \in \{0,1\}$. Let $p_i = (a_i, A)$. Then $h\colon \mathbf{M}^{\mathcal{I}}(\leqslant p_0) \to \mathbf{M}^{\mathcal{I}}(\leqslant p_1)$ given by*
$$h(a_0 \cup b, B) = (a_1 \cup b, B),$$
where $b \subseteq A$ is finite and $B \subseteq A/b$, is an isomorphism of partial orders.
4. *For p_0, p_1 as above, θ a formula in the language of set theory, and $v \in V$ it holds that*
$$p_0 \Vdash \theta(v, [x_{\dot{G}}]_{E_0}) \text{ if and only if } p_1 \Vdash \theta(v, [x_{\dot{G}}]_{E_0}).$$

Proof. (1) For any $y \in \mathcal{I}$, the set
$$D_y = \{p \in \mathbf{M}^{\mathcal{I}} \mid A(p) \cap y = \varnothing\}$$
is dense in $\mathbf{M}^{\mathcal{I}}$, which implies that for any generic G we have $x_G \cap y \in \mathrm{Fin}$.

(2) We verify the general case where $\mathcal{J} = \mathrm{Fin}(\phi)$. Supposing $p \Vdash x_{\dot{G}} \in \mathrm{Fin}(\phi)$ we can find $p' \leqslant p$ and $n \in \omega$ so that $p' \Vdash \phi(x_{\dot{G}}) < \check{n}$. Since ϕ is lower semi-continuous and $\phi(A(p')) = \infty$ we can find a finite set a such that $a(p') \sqsubseteq a \subseteq a(p') \cup A(p')$ and $\phi(a) > n$. Since $(a, A(p')/a) \Vdash a \subseteq x_{\dot{G}}$ we reach a contradiction.

(3) Immediate from the definitions.

(4) Suppose $p_1 \Vdash \theta(v, [x_{\dot{G}}]_{E_0})$. Let G be a generic such that $p_0 \in G$. Use $h\colon \mathbf{M}^{\mathcal{I}}(\leqslant p_0) \to \mathbf{M}^{\mathcal{I}}(\leqslant p_1)$ from (3) to obtain a generic $h(G)$ containing p_1. Since $x_G E_0 x_{h(G)}$, we conclude $\theta(v, [x_{h(G)}]_{E_0})$, proving that "if" holds. The proof of "only if" is analogous. $\square$

Furthermore, we have the following diagonalization result.

Lemma 3.1.10. *Let $(A_k)_{k\in\omega}$ be a sequence from $\mathcal{I}^+$ satisfying that $A_{k+1} \subseteq A_k$ for every $k \in \omega$. Then there is $A_\infty \in \mathcal{I}^+$ such that $A_\infty \subseteq^* A_k$ for every $k \in \omega$.*

For the reader who wishes to verify the lemma in the general case where $\mathcal{J} = \mathrm{Fin}(\phi)$, we point out that since finite sets have finite measure for any lower semicontinuous measure ϕ there is no need to substitute $\subseteq^*_{\mathrm{Fin}(\phi)}$ for $\subseteq^*$ and the lemma as well as its proof go through without this change.

Proof. We construct two sequences $(B_n)_{n\in\alpha}$ and $(C_n)_{n\in\alpha}$ of length $\alpha \leqslant \omega$ such that for each $n < \alpha$,

- $B_n \subseteq A_n$;
- for each m, $B_n \subseteq^* A_m$;
- $C_n \in \mathcal{A}\backslash\{A_i \mid i < n\}$;
- $B_n \cap C_n \notin \mathrm{Fin}$ and $B_n \cap C_i \in \mathrm{Fin}$ for $i < n$.

Suppose we have found B_i and C_i as above for $i < n$. Define a sequence $m_0, m_1, \ldots$ from ω by recursion on k as follows:

$$m_k = \min\Big(A_{n+k}\backslash\big(\{m_i \mid i < k\} \cup \bigcup_{i<n} C_i\big)\Big)$$

and let $B = \{m_k \mid k \in \omega\}$.

In the case of $\mathrm{Fin}(\phi)$, instead chose finite sets $M_0, M_1, \ldots$ such that $M_k \subseteq A_{n+k}\backslash\big(\bigcup_{i<n} C_i \cup \bigcup_{i<k} M_i\big)$ and $\phi(M_k) > 0$ for each $k \in \omega$. This is possible since for each k, $A_{n+k}\backslash\big(\bigcup_{i<n} C_i \cup \bigcup_{i<k} M_i\big) \in \mathrm{Fin}(\phi)^+$. Then let $B = \bigcup_{k\in\omega} M_k$.

If $B \in \mathcal{I}^+$, we let $A_\infty = B$ and we are done since $B \subseteq^* A_i$ for every $i \in \omega$. If on the other hand $B \notin \mathcal{I}^+$, we let $B_n = B$; since $B \in \mathrm{Fin}^+$ we can pick $C_n \in \mathcal{A}\backslash\{C_i \mid i < n\}$ such that $B_n \cap C_n \notin \mathrm{Fin}$.

Supposing that the construction does not end at a finite stage, let $A_\infty = \bigcup_{n\in\omega} B_n \cap C_n$. It is clear by construction that $A_\infty \subseteq^* A_m$ for every $m \in \omega$. Furthermore, since A_∞ is an infinite union of sets in Fin^+ which are also subsets of distinct elements in the AD-family $\mathcal{A}$, we conclude that $A_\infty \in \mathcal{I}^+$. □

The following lemma indicates how invariant descriptive set theory will help us achieve the desired results. We say that a set S is *hereditary definable using parameters from V and X* if S is definable by a first order formula using parameters from V and X, and the same holds for every set in the transitive closure of S. We will now see how we can use the previous diagonalization lemma to prove that any sequence of ordinals that is hereditary definable using parameters from V and from the E_0-equivalence class $[x_G]$ for a generic G actually belongs to V.

Lemma 3.1.11. *Let $HVD(X)$ denote the sets which are hereditary definable using parameters from $V \cup \{X\}$. Then the following holds:*

$$\Vdash_{\mathbf{M}^{\mathcal{I}}} (\mathbb{ON}^{\omega})^{HVD([x_{\dot{G}}]_{E_0})} \subseteq V.$$

Proof. Suppose $\theta(x_1, x_2, x_3, x_4)$ is a formula with all free variables shown, $p_0 \in \mathbf{M}^{\mathcal{I}}$, a is arbitrary, and $\dot{x}$ is a $\mathbf{M}^{\mathcal{I}}$-name such that

$$p_0 \Vdash \dot{x}\colon \omega \to \mathbb{ON} \wedge (\forall n \in \omega)(\forall \alpha \in \mathbb{ON})\ \dot{x}(n) = \alpha \Leftrightarrow \theta(n, \alpha, \check{a}, [x_{\dot{G}}]_{E_0}).$$

Let $A_0 = A(p_0)$, and build $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ and $\alpha_0, \alpha_1, \alpha_2, \ldots$ a sequence of ordinals as follows: given A_n, find $(b, A_{n+1}) \leqslant (a(p_0), A_n)$ and α_n such that

$$(b, A_{n+1}) \Vdash \theta(n, \check{\alpha}_n, \check{a}, [x_{\dot{G}}]_{E_0}).$$

Finally, find A_∞ such that $A_\infty \subseteq^* A_n$ for every $n \in \omega$.

W claim that $(a(p_0), A_\infty) \Vdash (\forall n \in \omega)\ \dot{x}(n) = \check{\alpha}_n$, and thus $\dot{x} \in V$. To prove this, suppose towards a contradiction that there is $n \in \omega$ such that $(a(p_0), A_\infty) \nVdash \dot{x}(n) = \check{\alpha}_n$, and find $(b, B) \leqslant (a(p_0), A_\infty)$ such that $(b, B) \Vdash \dot{x}(n) \neq \check{\alpha}_n$. That is, $(b, B) \Vdash \neg\theta(n, \check{\alpha}_n, \check{a}, [x_{\dot{G}}]_{E_0})$. By Lemma 3.1.9(4), also $(a(p_0), B) \Vdash \neg\theta(n, \check{\alpha}_n, \check{a}, [x_{\dot{G}}]_{E_0})$. However, since $B \subseteq A_\infty \subseteq^* A_{n+1}$ we know that $(a(p_0), B \cap A_{n+1}) \leqslant (a(p_0), B), (a(p_0), A_{n+1})$. This contradicts the fact that $(a(p_0), A_{n+1}) \Vdash \theta(n, \check{\alpha}_n, \check{a}, [x_{\dot{G}}]_{E_0})$.

□

The Branch Lemma

We will now finally prove the Main Proposition 3.1.5. We make a crucial definition (imported from [21]), followed by some fairly straightforward observations:

Definition 3.1.12. For $x \subseteq \omega$, let

$$T^x = \{t \in T \mid (\exists w \in [T_{[t]}])\ \pi(w) \cap x \notin \mathrm{Fin}\}.$$

Facts 3.1.13.

1. If $x\ E_0\ z$, then $T^x = T^z$. This means that for a generic G, the tree T^{x_G} is definable from $[x_G]_{E_0}$.

2. T^x is a pruned tree on $2 \times \kappa$.

3. $t \in T^x$ if and only if there is some $y \in \pi[T^x_{[t]}]$ such that $y \cap x \notin \mathrm{Fin}$.

4. $\varnothing \notin T^x$ is equivalent to $T^x = \varnothing$, as well as to $[T^x] = \varnothing$, as well as to that $\{x\} \cup \mathcal{A}$ is an AD family.

5. Since T^x is a subtree of T, $\pi[T^x] \subseteq \mathcal{A}$.

The proof of the Main Proposition is based on the following Branch Lemma.

The Branch Lemma 3.1.14. $\Vdash_{\mathbf{M}^{\mathcal{I}}} |\pi[T^{x_{\dot{G}}}]| \leqslant 1$.

Momentarily assuming the Branch lemma, we can very quickly show the Main Proposition 3.1.5, i.e., that

$$\Vdash_{\mathbf{M}^{\mathcal{I}}} (\forall y \in \pi[T])\ y \cap x_{\dot{G}} \in \mathrm{Fin}$$

as follows.

Proof of the Main Proposition 3.1.5. Towards a contradiction, suppose G is $\mathbf{M}^{\mathcal{I}}$-generic and we have $y \in \pi[T]^{V[G]}$ such that $y \cap x_G \notin \mathrm{Fin}$. By the Branch Lemma $\pi[T^{x_G}] = \{y\}$. Thus, since y is definable from $[x_G]_{E_0}$, we have $y \in \pi[T] \cap V \subseteq \mathcal{I}$ by Lemma 3.1.11. But then by 3.1.9(1), $x_G \cap y \in \mathrm{Fin}$, contradiction. Main Proposition 3.1.5. $\square$

For the proof of Theorem 3.1.1, it remains but to prove the Branch Lemma.

Proof of the Branch Lemma 3.1.14. Towards a contradiction, suppose G is $\mathbf{M}^{\mathcal{I}}$-generic and we have distinct $x_0, x_1 \in \pi[T^{x_G}]$. Fix n such that $x_0 \restriction n \neq x_1 \restriction n$, and let $s_i = w_i \restriction n$ where $x_i = \pi(w_i)$ and $w_i \in [T]$.

Claim 3.1.15. *There exists $t_0, t_1 \in T^{x_G}$ such that*

1. *$s_i \sqsubseteq t_i$ for $i \in \{0, 1\}$;*

2. *for every x_0^*, x_1^* such that $x_i^* \in \pi[T^{x_G}_{[t_i]}]$ it holds that $x_0^* \cap x_1^* \subseteq \pi(t_0) \cap \pi(t_1)$.*

Proof of claim. Suppose otherwise. Then for all $t_0, t_1 \in T^{x_G}$ extending s_0, s_1 respectively, there exists x_0^*, x_1^* such that $x_i^* \in \pi[T^{x_G}_{[t_i]}]$ and $\pi(t_0) \cap \pi(t_1) \subsetneq x_0^* \cap x_1^*$. We may build branches $w_0^*, w_1^* \in [T^{x_G}]$ such that $s_i \sqsubseteq w_i^*$ and $\pi(w_0^*) \cap \pi(w_1^*) \notin \mathrm{Fin}$. This however contradicts the fact that $\pi[T^{x_G}] \subseteq \pi[T]$, which is an almost disjoint family. Claim 3.1.15. $\square$

Thus, pick $t_0, t_1 \in T^{x_G}$ as in the claim, and let

$$y_i = \bigcup \pi[T^{x_G}_{[t_i]}], \quad i \in \{0,1\}.$$

It must be the case that $y_0 \in V$ since y_0 is definable from $[x_G]_{E_0}$ (the same is true of y_1). Noting $y_0 \in \mathrm{Fin}^+$, one of the two following cases occurs:

Case 1: $x_G \cap y_0 \in \mathrm{Fin}$. This, however, is a contradiction; indeed, since $y_0 = \bigcup \pi[T^{x_G}_{[t_0]}]$ where $t_0 \in T^{x_G}$, Facts 3.1.13 yields the existence of a set $y \in \pi[T^{x_G}_{[t_0]}]$ such that $y \cap x_G \notin \mathrm{Fin}$.

Case 2: If the first case fails, since $\{p \in \mathbf{M}^{\mathcal{I}} \mid A(p) \subseteq^* y_0 \vee A(p) \cap y_0 \in \mathrm{Fin}\}$ is dense in $\mathbf{M}^{\mathcal{I}}$ we have $x_G \subseteq^* y_0$. But then $x_G \cap y_1 \in \mathrm{Fin}$. This is also a contradiction, for the same reasons as above. Lemma 3.1.14. $\square$

3.2 Simple Fubini products

The ideas from the previous section can be used to prove similar results about ideals that are further up the Borel hierarchy. In this section, we will take one step up the ladder, whilst in the following section we see that we can go all the way.

Recall from Chapter 1 that given ideals $\mathcal{J}_*$, $\mathcal{J}_k$ on ω (for each $k \in \omega$) we can form the ideal $\bigoplus_{\mathcal{J}_*} \mathcal{J}_k$ on $\omega \times \omega$. If $\mathcal{J}_k = \mathcal{J}'$ for each $k \in \omega$, one writes $\mathcal{J}_* \otimes \mathcal{J}'$ for $\bigoplus_{\mathcal{J}_*} \mathcal{J}_k$ (called the *Fubini product of $\mathcal{J}_*$ with $\mathcal{J}'$*).

We will study ideals of the form $\mathcal{J} = \bigoplus_{\mathrm{Fin}(\phi)} \mathrm{Fin}(\phi_k)$, where ϕ and ϕ_k for each $k \in \omega$ are lsc submeasures on ω. Clearly this includes $\mathrm{Fin} \otimes \mathrm{Fin}$, which is $\mathrm{Fin}(\phi) \otimes \mathrm{Fin}(\phi)$ where ϕ is the counting measure. For $X \subseteq \omega \times \omega$ we write

$$\begin{aligned} X(n) &= \{k \in \omega \mid (n,k) \in X\}, \\ \mathrm{dom}(X) &= \{n \in \omega \mid X(n) \neq \varnothing\}, \\ \mathrm{dom}^{\mathcal{J}}_{\infty}(X) &= \{n \in \omega \mid X(n) \notin \mathrm{Fin}(\phi_n)\}. \end{aligned}$$

We write dom_∞ for $\mathrm{dom}^{\mathrm{Fin}\otimes\mathrm{Fin}}_\infty$, and note that

$$\mathrm{Fin} \otimes \mathrm{Fin} = \{X \subseteq \omega \times \omega \mid \mathrm{dom}_\infty(X) \in \mathrm{Fin}\}.$$

We will use the two following orderings on $\mathcal{P}(\omega \times \omega)$. For $X \subseteq \omega \times \omega$ finite and $Y \subseteq \omega \times \omega$ we say

$$X \sqsubseteq_2 Y \Leftrightarrow \mathrm{dom}(X) \sqsubseteq \mathrm{dom}(Y) \wedge (\forall n \in \mathrm{dom}(X))\ X(n) \sqsubseteq Y(n),$$

and

$$X \sqsubset_2 Y \Leftrightarrow \mathrm{dom}(X) \subsetneq \mathrm{dom}(Y) \wedge (\forall n \in \mathrm{dom}(X))\ X(n) \subsetneq Y(n)$$

In the general case where $\mathcal{J} = \bigoplus_{\mathrm{Fin}(\phi)} \mathrm{Fin}(\phi_k)$, let

$$\begin{aligned} X \sqsubset_2 Y \Leftrightarrow X \sqsubseteq_2 Y \wedge \phi(\mathrm{dom}(X)) < \phi(\mathrm{dom}(Y)) \wedge \\ (\forall n \in \mathrm{dom}(X))\ \phi_n(X(n)) < \phi_n(Y(n)). \end{aligned}$$

This section was written so that most proofs generalize almost mechanically from $\mathrm{Fin} \otimes \mathrm{Fin}$ to the above more general case; often this is made possible by the definition of $\sqsubset_2$ given above.

We let as usual $(\mathrm{Fin} \otimes \mathrm{Fin})^+$ (resp., $\mathcal{J}^+$) denote the co-ideal.

Definition 3.2.1. Let $(\mathrm{Fin} \otimes \mathrm{Fin})^{++}$ denote the set of $A \in (\mathrm{Fin} \otimes \mathrm{Fin})^+$ such that for all $k \in \mathrm{dom}(A)$, $A(k) \notin \mathrm{Fin}$.

Conditions of the forcing notion $\mathbf{M}_2$ are pairs (a, A) where

(a) $a \subseteq \omega \times \omega$ and is finite;

(b) $A \in (\mathrm{Fin} \otimes \mathrm{Fin})^{++}$;

(c) $\max(a(k)) < \min(A(k))$ for every $k \in \mathrm{dom}(a)$;

(d) $\mathrm{dom}(a) \sqsubseteq \mathrm{dom}(A)$.

We let $(a', A') \leqslant (a, A)$ just in case $A' \subseteq A$, and $a \sqsubseteq_2 a' \subseteq a \cup A$.

In the general case when $\mathcal{J} = \bigoplus_{\mathrm{Fin}(\phi)} \mathrm{Fin}(\phi_k)$, $\mathcal{J}^{++}$ denotes the set of $A \in \mathcal{J}^+$ such that for all $k \in \mathrm{dom}(A)$, $A(k) \notin \mathrm{Fin}(\phi_k)$. Moreover, replace (b) in the definition[1] of $\mathbf{M}_2$ by $A \in \mathcal{J}^{++}$.

Note that if (a, A) is a condition in $\mathbf{M}_2$ then for every $k \in \mathrm{dom}(a)$, the pair $(a(k), A(k))$ is a Mathias forcing condition (resp., a condition in $\mathbf{M}^{\mathrm{Fin}(\phi_k)}$). Moreover, the pair $(\mathrm{dom}(a), \mathrm{dom}(A))$ is a Mathias forcing condition (resp., a condition in $\mathbf{M}^{\mathrm{Fin}(\phi)}$) as well.

As in the 1-dimensional case, a relativized forcing notion is needed.

Definition 3.2.2. If $\mathcal{I}^+$ is a co-ideal of an ideal $\mathcal{I} \supseteq \mathrm{Fin} \otimes \mathrm{Fin}$, then we write $\mathcal{I}^{++}$ for $\mathcal{I}^+ \cap (\mathrm{Fin} \otimes \mathrm{Fin})^{++}$. We let

$$\mathbf{M}_2^{\mathcal{I}} = \{(a, A) \in \mathbf{M}_2 : A \in \mathcal{I}^{++}\}$$

equipped with the ordering inherited from $\mathbf{M}_2$.

[1] This designation is left ambiguous in that $\mathbf{M}_2$ implicitly depends on the ideal $\mathcal{J}$—or rather, on the set $\mathcal{J}^{++}$. The same will be true of $\mathbf{M}_2^{\mathcal{I}}$ introduced below.

Note that if $A \in \mathcal{I}^+$, then we can always find a subset $B \subseteq A$ such that $B \in \mathcal{I}^{++}$. We need to establish some notation:

Notation 3.2.3.

1. Given a filter G on $\mathbf{M}_2^{\mathcal{I}}$, let
$$x_G = \bigcup \{a : (\exists A)(a, A) \in G\}.$$
2. For $p \in \mathbf{M}_2^{\mathcal{I}}$, we write $p = (a(p), A(p))$ when we want to refer to the components of p.
3. For $(a, A) \in \mathbf{M}_2^{\mathcal{I}}$ and $b \subseteq a \cup A$ finite, let
$$A/b = \bigcup_{n \in N} A(n) \setminus \{m \in \omega \mid m \leqslant \max(b(n))\}$$
where $N = \mathrm{dom}(b) \cup [\max(\mathrm{dom}(b)) + 1, \infty)$.
4. For $p \in \mathbf{M}_2^{\mathcal{I}}$, let $\mathbf{M}_2^{\mathcal{I}}(\leqslant p) = \{q \in \mathbf{M}_2^{\mathcal{I}} \mid q \leqslant p\}$.

Remark 3.2.4. Note that in order to meaningfully talk about κ-Suslin sets in $\mathcal{P}(\omega \times \omega)$, we identify $\omega \times \omega$ with ω (via some fixed bijection), sets with their characteristic functions, and in effect, $\mathcal{P}(\omega \times \omega)$ with 2^ω.

Assumption 3.2.5. Until the end of Section 3.2 let $\mathcal{J} = \mathrm{Fin} \otimes \mathrm{Fin}$, or more generally let $\mathcal{J} = \bigoplus_{\mathrm{Fin}(\phi)} \mathrm{Fin}(\phi_k)$ as above. Moreover suppose $\mathcal{A} \subseteq \mathcal{P}(\omega \times \omega)$ to be a $\mathcal{J}$-almost disjoint family which is κ-Suslin and fix a tree T on $2 \times \kappa$ such that $\pi[T] = \mathcal{A}$. Finally, let $\mathcal{I}$ be the ideal generated by $\mathcal{A} \cup \mathcal{J}$.

To ease the notation, we will focus our attention on $\mathcal{J} = \mathrm{Fin} \otimes \mathrm{Fin}$. However, our proofs work for $\mathcal{J} = \bigoplus_{\mathrm{Fin}(\phi)} \mathrm{Fin}(\phi_k)$ as above. For the general case, substitute $\mathrm{Fin} \otimes \mathrm{Fin}$ (but *not* the word finite or the expression $[\omega^2]^{<\omega}$) by $\bigoplus_{\mathrm{Fin}(\phi)} \mathrm{Fin}(\phi_k)$, dom_∞ by $\mathrm{dom}_\infty^{\mathcal{J}}$, etc. wherever relevant, unless we provide commentary.

Now we are ready to state the Main Proposition regarding $\mathbf{M}_2^{\mathcal{I}}$ from which Theorem 1.0.4 follows as a corollary, precisely analogous to the previous section. The proof of the Main Proposition will again rely on a Branch Lemma and will be postponed for now.

Main Proposition 3.2.6. $\Vdash_{\mathbf{M}_2^{\mathcal{I}}} (\forall y \in \pi[T])\ y \cap x_{\dot{G}} \in \mathrm{Fin} \otimes \mathrm{Fin}$.

As in the one-dimensional case, our main result about $\mathrm{Fin} \otimes \mathrm{Fin}$ also follows directly from the Main Proposition.

Corollary 3.2.7. *Theorem 1.0.4 holds.*

Proof. The proofs are essentially identical to those of Corollary 3.1.6, Corollary 3.1.7, and Corollary 3.1.8, simply substituting $\mathbf{M}_2^{\mathcal{I}}$ for $\mathbf{M}^{\mathcal{I}}$. $\square$

Properties of the two-dimensional forcing

Before we can prove the Main Proposition, we shall collect some of the necessary facts about the forcing $\mathbf{M}_2^{\mathcal{I}}$.

Lemma 3.2.8.

1. *For any $y \in \mathcal{I}$, $\Vdash_{\mathbf{M}_2^{\mathcal{I}}} x_{\dot{G}} \cap \check{y} \in \mathrm{Fin} \otimes \mathrm{Fin}$.*
2. *For any $k \in \omega$ the partial order $\mathbf{M}_2^{\mathcal{I}}$ is isomorphic to the product $\mathbf{M}^k \times \mathbf{M}_2^{\mathcal{I}}(\leqslant (\varnothing, (\omega \backslash k) \times \omega))$, where $\mathbf{M}^k$ is the set of k-tuples of classical (1-dimensional) Mathias forcing conditions. In the general case where $\mathcal{J} = \bigoplus_{\mathrm{Fin}(\phi)} \mathrm{Fin}(\phi_i)$ we have*
$$\mathbf{M}_2^{\mathcal{I}} \cong \Big(\prod_{i<k} \mathbf{M}^{\mathrm{Fin}(\phi_i)}\Big) \times \mathbf{M}_2^{\mathcal{I}}\Big(\leqslant \big(\varnothing, (\omega \backslash k) \times \omega\big)\Big).$$
3. $\Vdash_{\mathbf{M}_2^{\mathcal{I}}} x_{\dot{G}} \in (\mathrm{Fin} \otimes \mathrm{Fin})^{++}$.

Proof. (1) Follows from the fact that for any $y \in \mathcal{I}$, the set

$$D_y = \{p \in \mathbf{M} \mid A(p) \cap y = \varnothing\}$$

is dense.

(2) Define a map $\phi\colon \mathbf{M}^k \times \mathbf{M}_2^{\mathcal{I}}(\leqslant (\varnothing, (\omega \backslash k) \times \omega)) \to \mathbf{M}_2^{\mathcal{I}}$ by

$$((c_i, C_i)_{i<k}, (a, A)) \mapsto (\bigcup_{i<k} \{i\} \times c_i \cup a, \bigcup_{i<k} \{i\} \times C_i \cup A)$$

This map is easily seen to be bijective and order preserving. The same definition works in the general case.

(3) We prove this in general for $\bigoplus_{\mathrm{Fin}(\phi)} \mathrm{Fin}(\phi_i)$. First note that $\Vdash_{\mathbf{M}_2^{\mathcal{I}}} \mathrm{dom}(x_{\dot{G}}) = \mathrm{dom}_\infty(x_{\dot{G}})$: For let n and p be such that $p \Vdash \check{n} \in \mathrm{dom}(x_{\dot{G}})$. It must hold that $n \in \mathrm{dom}(a(p))$. By Lemma 3.1.9(2),

$$\big(a(p)(n), A(p)(n)\big) \Vdash_{\mathbf{M}} x_{\dot{G}} \notin \mathrm{Fin}(\phi_n)$$

so by item (2) of the present lemma, $p \Vdash_{\mathbf{M}_2^{\mathcal{I}}} \check{n} \in \mathrm{dom}_\infty(x_{\dot{G}})$.

It remains to show $\Vdash_{\mathbf{M}_2^{\mathcal{I}}} \operatorname{dom}_\infty(x_{\dot{G}}) \notin \operatorname{Fin}(\phi)$. Towards a contradiction, suppose there is n and p so that $p \Vdash \phi(\operatorname{dom}(x_{\dot{G}})) < \check{n}$. Find a finite set d such that $\operatorname{dom}(a(p)) \sqsubseteq d \subseteq \operatorname{dom}(a(p)) \cup \operatorname{dom}(A(p))$ and $\phi(d) > n$, and a such that $a(p) \sqsubseteq_2 a \subseteq a(p) \cup A(p)$ and $\operatorname{dom}(a) = d$. We reach a contradiction since $(d, A(p)/d) \Vdash d \subseteq \operatorname{dom}(x_{\dot{G}})$ and $\phi(d) > n$. □

We prove a general diagonalization result (which shall be put to use in Lemma 3.2.14 below):

Lemma 3.2.9. *Let $(A_k)_{k\in\omega}$ be a sequence from $\mathcal{I}^{++}$ satisfying that $A_{k+1} \subseteq A_k$ for every $k \in \omega$. Then there is $A_\infty \in \mathcal{I}^{++}$ such that $A_\infty \subseteq^*_{\operatorname{Fin}\otimes\operatorname{Fin}} A_k$ for every $k \in \omega$.*

Just as Lemma 3.1.10 for the one-dimension case, Lemma 3.2.9 holds verbatim for $\mathcal{J} = \bigoplus_{\operatorname{Fin}(\phi)} \operatorname{Fin}(\phi_i)$ (i.e., with $\subseteq^*_{\operatorname{Fin}\otimes\operatorname{Fin}}$ and not just with $\subseteq^*_{\mathcal{J}}$).

Proof. As in the previous section, we construct two sequences $(B_n)_{n\in\alpha}$ and $(C_n)_{n\in\alpha}$ of length $\alpha \leqslant \omega$ such that for each $n < \alpha$,

- $B_n \in (\operatorname{Fin}\otimes\operatorname{Fin})^{++}$;
- $B_n \subseteq A_n$ and $(\forall k \in \omega)\ B_n \subseteq^*_{\operatorname{Fin}\otimes\operatorname{Fin}} A_k$;
- $C_n \in \mathcal{A}\backslash\{C_i \mid i < n\}$;
- $B_n \cap C_n \in (\operatorname{Fin}\otimes\operatorname{Fin})^+$ and $B_n \cap C_i \in \operatorname{Fin}\otimes\operatorname{Fin}$ for $i < n$.

Suppose we have found B_i and C_i as above for $i < n$. Define a sequence $m_0^n, m_1^n, \ldots$ from ω by recursion on k as follows:

$$m_k^n = \min\Big(\operatorname{dom}\big(A_{n+k}\backslash(\{m_i \mid i < k\} \cup \bigcup_{i<n} C_i)\big)\Big)$$

and let $B = \bigcup_{k\in\omega} A_{n+k}(m_k^n)$.

In the case of $\operatorname{Fin}(\phi)$, instead chose finite sets $M_0^n, M_1^n, \ldots$ such that $M_k^n \subseteq \operatorname{dom}\big(A_{n+k}\backslash\bigcup_{i<n}(C_i \cup M_i^n)\big)$ and $\phi(M_k^n) > 0$ for each $k \in \omega$. Then let

$$B = \bigcup_{k\in\omega} \bigcup_{m\in M_k^n} A_{n+k}(m).$$

The remainder of the proof is essentially identical to the 1-dimensional case, i.e., Lemma 3.1.10, simply replacing Fin by $\operatorname{Fin}\otimes\operatorname{Fin}$ everywhere. We leave this to the reader.

□

The two-dimensional Branch Lemma

The crucial definition is again that of an invariant tree, analogous to Definition 3.1.12.

Definition 3.2.10. For $x \subseteq \omega \times \omega$, let

$$T^x = \{t \in T \mid (\exists y \in \pi[T_{[t]}]) \; y \cap x \notin \mathrm{Fin} \otimes \mathrm{Fin}\}$$

As in Fact 3.1.13(1), it is easy to see that the map $x \mapsto T^x$ is invariant to small changes in x, in the sense that whenever $x \Delta x' \in \mathrm{Fin} \otimes \mathrm{Fin}$, $T^x = T^{x'}$. Moreover Facts 3.1.13(2)–(5) hold here as well.

We are now ready to state the main lemma of this section.

The Branch Lemma 3.2.11. $\Vdash_{\mathbf{M}_2^{\mathcal{I}}} |\pi[T^{x_{\dot{G}}}]| \leqslant 1$.

We postpone the proof of the Branch Lemma and first give the proof of the Main Proposition 3.2.6, assuming the lemma. The proof is not quite as straightforward as in the previous section, but the idea remains the same. Claim 3.2.12 will play the role as a less general analogue of Lemma 3.1.11.

Proof of the Main Proposition 3.2.6. Suppose towards a contradiction there is $p_0 \in \mathbf{M}_2^{\mathcal{I}}$ such that $p_0 \Vdash (\exists A \in \pi[T]^{V[\dot{G}]}) \; A \cap x_{\dot{G}} \notin \mathrm{Fin} \otimes \mathrm{Fin}$. By the Branch Lemma 3.2.11, p_0 forces that $\pi[T^{x_{\dot{G}}}]$ has precisely one element; let $\dot{A}$ be a name for it.

Claim 3.2.12. *There is $q \in \mathbf{M}_2^{\mathcal{I}}$ and $A' \in V$ such that $q \Vdash \dot{A} = \check{A}'$.*

Proof of Claim. It suffices to show that if $p \leqslant p_0$ and p decides $(n, m) \in \dot{A}$ then in fact $(a(p_0), A(p))$ decides $(n, m) \in \dot{A}$: For then we may pick $A_0 \supseteq A_1 \supseteq \dots$ such that for each pair $(n, m) \in \omega \times \omega$, some $(a(p_0), A_k)$ decides $(n, m) \in \dot{A}$; by Lemma 3.2.9 we can find A_∞ diagonalizing $(A_k)_{k \in \omega}$. Any condition below $q = (a(p_0), A_\infty)$ is compatible with each $(a(p_0), A_k)$, and so q decides all of $\dot{A}$.

So suppose $p \leqslant q$ decides $(n, m) \in \dot{A}$; we must show $(a(p_0), A(p))$ decides $(n, m) \in \dot{A}$. Let us suppose that $p \Vdash (n, m) \in \dot{A}$; the proof is similar in case $p \Vdash (n, m) \notin \dot{A}$ and we leave this case to the reader.

Fix any $\mathbf{M}_2^{\mathcal{I}}$-generic G such that $(a(p_0), A(p)) \in G$. By Lemma 3.2.8(2) we can decompose G as $G_0 \times G_1$ where G_1 is $\mathbf{M}_2^{\mathcal{I}}$-generic and G_0 is $\mathbf{M}^k$-generic for k large enough so that $\mathrm{dom}(a(p)) \subseteq k$. Note that as $x_G \Delta x_{G_1} \in \mathrm{Fin} \otimes \mathrm{Fin}$, $T^{x_G} = T^{x_{G_1}} \in V[G_1]$. Since $V[G] \vDash \pi[T^{x_G}] = \{\dot{A}^G\}$, by a simple absoluteness argument the same must hold in $V[G_1]$, i.e., $\dot{A}^G \in \dot{V}[G_1]$ and $V[G_1] \vDash \pi[T^{x_G}] = \{\dot{A}^G\}$.

Since $\mathrm{dom}(a(p)) \subseteq k$ we can find G' which is $\mathbf{M}_2^{\mathcal{I}}$-generic over V such that $G' = G'_0 \times G_1$ and $p \in G'$. Clearly $(n,m) \in \dot{A}^{G'}$ (since $p \Vdash (n,m) \in \dot{A}$). Arguing as before using absoluteness, this time between $V[G']$ and $V[G_1]$, $\dot{A}^{G'}$ must equal the unique element of $\pi[T^{x_{G_1}}]$, i.e., $\dot{A}^{G'} = \dot{A}^G$ and so $(n,m) \in \dot{A}^G$. Since G was arbitrary, $(a(p_0), A(p)) \Vdash (n,m) \in \dot{A}$. Claim 3.2.12. $\square$

Now $A' \in \pi[T] \cap V$ and thus $A' \in \mathcal{I}$, but also $q \Vdash x_{\dot{G}} \cap \check{A}' \notin \mathrm{Fin} \otimes \mathrm{Fin}$, contradicting Lemma 3.2.8(1). Main Proposition 3.2.6. $\square$

We now gradually work towards the proof of the Branch Lemma, for which it is necessary to introduce some notation. Firstly, write

$$U = [\omega \times \omega]^{<\omega} \times T.$$

Given a pair $\vec{u} \in U$, we write it as $(a(\vec{u}), t(\vec{u}))$ if we want to refer to the components of $\vec{u}$. We define a partial order $\leqslant_U$ on U as follows:

$$\vec{u}_1 \leqslant_U \vec{u}_0 \Leftrightarrow a(\vec{u}_1) \sqsupseteq_2 a(\vec{u}_0) \wedge t(\vec{u}_1) \sqsupseteq t(\vec{u}_0).$$

Now secondly assume G is $\mathbf{M}_2^{\mathcal{I}}$-generic over V; working in $V[G]$ for the moment and for a fixed $x \in \mathcal{P}(\omega \times \omega)$, define the set $U^x \subseteq U$ consisting of those pairs $(a,t) \in U$ such that there is $w \in [T_{[t]}]$ with

1. $\pi(w) \cap x \notin \mathrm{Fin} \otimes \mathrm{Fin}$;
2. $\mathrm{dom}(a) \subseteq \mathrm{dom}_\infty(\pi(w) \cap x)$;
3. for each $k \in \mathrm{dom}(a)$, $a(k) \subseteq \pi(w)(k) \cap x(k)$.

Intuitively, U^x searches for a branch through T whose projection has large intersection with x and a subset of this intersection in $(\mathrm{Fin} \otimes \mathrm{Fin})^{++}$ to witness its largeness.

In analogy to the tree T^x, when $\vec{u}_0 \in U$ write $U^x_{[\vec{u}_0]}$ for $\{\vec{u} \in U \mid \vec{u} \leqslant_U \vec{u}_0\}$.

The following three lemmas gather some observations concerning U^{x_G} which will be important in the proof of the Branch Lemma.

Lemma 3.2.13. *Suppose $(a, A) \Vdash \vec{u} \in U^{x_{\dot{G}}}$.*

1. *It holds that $a \supseteq a(\vec{u})$ and moreover if $a' \subseteq a(\vec{u})$ also*

$$(a(\vec{u}), A) \Vdash (a', t(\vec{u})) \in U^{x_{\dot{G}}}.$$

2. *The set* $A' \subseteq \omega \times \omega$ *defined by*

$$A' = \{(k,l) \mid (\exists p' \leqslant (a,A))(\exists \vec{u}' \leqslant_U \vec{u})\ (k,l) \in a(\vec{u}') \wedge p' \Vdash \vec{u}' \in U^{x_{\dot{G}}}\}$$

is not in $\mathcal{I}$.

3. *For any* $k \in \operatorname{dom}(a(\vec{u}))$, *the set* $A_k \subseteq \omega$ *defined by*

$$\{l \mid (\exists p' \leqslant (a,A))(\exists \vec{u}' \leqslant_U \vec{u})\ l \in a(\vec{u}')(k) \wedge p' \Vdash \vec{u}' \in U^{x_{\dot{G}}}\}$$

is not in Fin *(resp., not in* $\operatorname{Fin}(\phi_k)$*).*

Proof. (1) Immediate from the definition of $U^{x_{\dot{G}}}$.

(2) Assume to the contrary that $A' \in \mathcal{I}$. Then $A \backslash A' \in \mathcal{I}^+$, so take $B \subseteq A \backslash A'$ such that $B \in \mathcal{I}^{++}$ and set $p = (a,B) \in \mathbf{M}_2^{\mathcal{I}}$. Since $p \Vdash \vec{u} \in U^{x_{\dot{G}}}$ we can find a name $\dot{w}$ such that

$$p \Vdash \dot{w} \in \pi[T_{[t(\vec{u})]}] \wedge \dot{w} \cap x_{\dot{G}} \notin \mathrm{Fin} \otimes \mathrm{Fin}\,.$$

(In fact, all we need here is that $p \Vdash T^{x_{\dot{G}}} \neq \varnothing$). Thus we can extend p to p' to force a pair (k,l) into $\dot{w} \cap x_{\dot{G}} \backslash a(p)$. But it has to be the case that $(k,l) \in a(p')$, whence $(k,l) \in A'$ by definition of A', contradicting that also $(k,l) \in B$ which is disjoint from A'.

(3) Assume to the contrary that $k \in \operatorname{dom}(a(\vec{u}))$ and $A_k \in \mathrm{Fin}$. Take $B \subseteq A \backslash (\{k\} \times A_k)$ such that $B \in \mathcal{I}^{++}$, and set $p = (a,B) \in \mathbf{M}_2^{\mathcal{I}}$. Since $p \Vdash \vec{u} \in U^{x_{\dot{G}}}$ we can find a name $\dot{w}$ such that

$$p \Vdash \dot{w} \in \pi[T_{[t(\vec{u})]}] \wedge \dot{w} \cap x_{\dot{G}} \in (\mathrm{Fin} \otimes \mathrm{Fin})^+$$

and

$$p \Vdash \operatorname{dom}(a(\vec{u})) \subseteq \operatorname{dom}_\infty(\dot{w} \cap x_{\dot{G}}).$$

As $k \in \operatorname{dom}_\infty(\dot{w} \cap x_{\dot{G}})$, we can extend p to p' to force a pair (k,l) into $\dot{w} \cap x_{\dot{G}} \backslash a(p)$. But as in the proof of the previous item, it has to be the case that $(k,l) \in a(p')$, whence $l \in A_k$ by definition of A_k, contradicting that also $l \in B(k)$ which is disjoint from A_k. □

In order to prove the two-dimensional Branch Lemma, we also need to introduce the partially ordered set Γ defined as follows:

$$\Gamma = \{(p, \vec{u}^0, \vec{u}^1) \in \mathbf{M}_2^{\mathcal{I}} \times U \times U \mid (\forall i \in \{0,1\})\ p \Vdash \vec{u}^i \in U^{x_{\dot{G}}}\}.$$

This set carries a weak and a strict order, defined as follows:

$$(p_1, \vec{u}_1^0, \vec{u}_1^1) \leqslant_\Gamma (p_0, \vec{u}_0^0, \vec{u}_0^1)$$

if and only if $p_1 \leqslant p_0$, and for each $i \in \{0,1\}$, $a(u_1^i) \sqsupseteq_2 a(u_0^i)$ and $t(u_1^i) \sqsupseteq t(u_0^i)$ (that is, $u_1^i \leqslant_U u_0^i$); and

$$(p_1, \vec{u}_1^0, \vec{u}_1^1) \prec_\Gamma (p_0, \vec{u}_0^0, \vec{u}_0^1)$$

if and only in addition, $a(\vec{u}_0^0) \cap a(\vec{u}_0^1) \sqsubset_2 a(\vec{u}_1^0) \cap a(\vec{u}_1^1)$.

Note that Γ is well-founded with respect to the second, strict ordering $\prec_\Gamma$; indeed, suppose towards a contradiction that there is an infinite $\prec_\Gamma$-descending sequence

$$\ldots \prec_\Gamma (p_3, \vec{u}_3^0, \vec{u}_3^1) \prec_\Gamma (p_2, \vec{u}_2^0, \vec{u}_2^1) \prec_\Gamma (p_1, \vec{u}_1^0, \vec{u}_1^1)$$

from Γ. Define

$$y^i = \bigcup_{n>1} t(\vec{u}_n^i)$$

for $i \in \{0,1\}$ and

$$A = \bigcup_{n>1} a(\vec{u}_n^0) \cap a(\vec{u}_n^1).$$

Since the sequence is $\prec_\Gamma$-decreasing and from Γ, $A \in (\mathrm{Fin} \otimes \mathrm{Fin})^{++}$ and $A \subseteq \pi(y^0) \cap \pi(y^1)$, contradicting that $\pi[T]$ is $\mathrm{Fin} \otimes \mathrm{Fin}$-almost disjoint.

The proof of the Branch Lemma will crucially depend on the two following lemmas, which in combination will allow us to manipulate both the infinite and the finite part of a forcing condition while maintaining that something is forced about $U^{x_{\dot{G}}}$. Note that the second of these lemmas plays the same role as Lemma 3.1.9(4).

Lemma 3.2.14. *For each $\vec{u}_0 \in U$ the set $D(\vec{u}_0)$ is dense and open in $\mathbf{M}_2^{\mathcal{I}}$, where we define $D(\vec{u}_0)$ to be the set of $p \in \mathbf{M}_2^{\mathcal{I}}$ such that for all $p' \leqslant p$ and any $\vec{u} \in U$,*

$$\left[p' \Vdash \vec{u} \in U^{x_{\dot{G}}}_{[\vec{u}_0]}\right] \Rightarrow (a(p'), A(p)/a(p')) \Vdash (\exists t \in T)\ (a(\vec{u})), t) \in U^{x_{\dot{G}}}_{[\vec{u}_0]}.$$

The proof follows the same strategy as Lemma 3.2.9 (the diagonalization lemma) to build a set in $\mathcal{I}^{++}$. While we build this set, we carefully anticipate each of its finite subsets a to see if there is some $t \in T$ and some forcing condition $q \in \mathbf{M}_2^{\mathcal{I}}$ which forces (a,t) to be in $U^{x_{\dot{G}}}$. If so, we make sure that our final set is contained in $a \cup A(q)$. We succeed as there are only countably many finite $a \subseteq \omega \times \omega$ to consider. Note though that due to the nature of the proof of Lemma 3.2.9, we have to consider each finite a again and again, and the construction potentially takes $\omega \times \omega$ stages.

Proof. Fix $q_0 \in \mathbf{M}_2^{\mathcal{I}}$ and $\vec{u}_0 \in U$. If $q_0 \not\Vdash \vec{u}_0 \in U^{x_{\dot{G}}}$, then find $q \leqslant q_0$ such that $q_0 \Vdash \vec{u}_0 \notin U^{x_{\dot{G}}}$ and note that $q \in D(\vec{u}_0)$. Suppose therefore that $q_0 \Vdash \vec{u}_0 \in U^{x_{\dot{G}}}$. We construct $q \leqslant q_0$ such that $q \in D(\vec{u}_0)$.

As in the proof of Lemma 3.2.9, we construct sequences $B_0, B_1, \dots,$ and $C_0, C_1, \dots$ both of which are possibly finite, such that whenever defined

- $B_n \in (\mathrm{Fin} \otimes \mathrm{Fin})^{++}$;
- $C_n \in \mathcal{A} \backslash \{C_i \mid i < n\}$;
- $B_n \cap C_n \in (\mathrm{Fin} \otimes \mathrm{Fin})^{+}$ while for $i < n$, $B_n \cap C_i \in \mathrm{Fin} \otimes \mathrm{Fin}$.

Suppose B_i and C_i have been defined for $i < n$ (this includes the case $n = 0$). In ω-many steps we define a descending sequence of conditions $(b_n^k, B_n^k)_k$ from $\mathbf{M}_2^{\mathcal{I}}$ and at the end let

$$B_n = \bigcup_{k \in \omega} b_n^k. \tag{3.2.1}$$

If $n = 0$, let $b_0^0 = a(q_0)$ and $B_0^0 = A(q_0)$. Otherwise, let

$$b_n^0 = \bigcup_{i,j<n} b_j^i$$

and

$$B_n^0 = \big(B_{n-1}^{n-1} \cap (\mathrm{dom}(b_n^0) \times \omega)\big) \cup \big(B_{n-1}^{n-1} \backslash \bigcup \{C_i \mid i < n\}\big)$$

noting that $B_n^0 \in (\mathrm{Fin} \otimes \mathrm{Fin})^{+}$ since $B_{n-1}^{n-1} \in \mathcal{I}^{++}$ by induction hypothesis. So $(b_n^0, B_n^0) \in \mathbf{M}_2^{\mathcal{I}}$ and $B_n^0 \cap C_i \in \mathrm{Fin} \otimes \mathrm{Fin}$ for $i < n$.

Supposing we have already defined $(b_n^k, B_n^k) \in \mathbf{M}_2^{\mathcal{I}}$ make finitely many extensions to reach $(b_n^k, B^*) \leqslant (b_n^k, B_n^k)$ so that whenever $a \subseteq b_n^k$ and

$$(\exists p' \leqslant (a(q_0), B_n^k))(\exists \vec{u} \in U)\ a(\vec{u}) = a \wedge a(p') \subseteq b_n^k \wedge p' \Vdash \vec{u} \in U_{[\vec{u}_0]}^{x_{\dot{G}}} \tag{3.2.2}$$

then for some $t' \in T$

$$\big(a(p'), B^*/a(p')\big) \Vdash (a(\vec{u}), t') \in U_{[\vec{u}_0]}^{x_{\dot{G}}}. \tag{3.2.3}$$

Extend b_n^k to some finite (we mean finite also in the general case!) set $b_n^{k+1} \subseteq \omega \times \omega$ satisfying

$$b_n^k \sqsubset_2 b_n^{k+1} \subseteq b_n^k \cup B^* \tag{3.2.4}$$

and let

$$B_n^{k+1} = B^*/b_n^{k+1}.$$

Assuming we have defined b_n^k for each $k \in \omega$ and letting B_n be defined by (3.2.1), note that (3.2.4) ensures that $B_n \in (\mathrm{Fin} \otimes \mathrm{Fin})^{++}$. Should it be the case that $B_n \in \mathcal{I}^+$ the construction terminates and we let

$$q = (a(q_0), B_n).$$

Otherwise, we may chose $C_n \in \mathcal{A} \backslash \{C_i \mid i < n\}$ such that $C_n \cap B_n \in (\mathrm{Fin} \otimes \mathrm{Fin})^+$ as in Lemma 3.2.9 and continue the construction.

If the construction does not terminate at any stage $n < \omega$, let

$$B_\infty = \bigcup_{n \in \omega} b_n^n.$$

Note that $B_\infty = \bigcup_{k \in \omega} B_k$ and thus since $B_\infty \cap C_k \in (\mathrm{Fin} \times \mathrm{Fin})^+$ for each $k \in \omega$, it must be the case that $B_\infty \in \mathcal{I}^{++}$ (as in the proof of Lemma 3.2.9). So we obtain a condition in $\mathbf{M}_2^{\mathcal{I}}$ by letting

$$q = (a(q_0), B_\infty).$$

To see that $q \in D(\vec{u}_0)$, let $p' \leqslant q$, $\vec{u} \in U$ such that $p' \Vdash \vec{u} \in U^{x_{\dot{G}}}_{[\vec{u}_0]}$ be given. Let us first assume that the construction did not stop at any stage $n < \omega$ and that B_∞ is defined. We can find $n > 0$ so that $a(p') \subseteq b_{n-1}^{n-1}$. Thus, at stage $k = n$ in the construction of B_n, (3.2.2) was satisfied for $a = \vec{u}$, and so (3.2.3) is also satisfied. By construction $B_\infty \backslash b_{n-1}^{n-1} \subseteq B_n^n$. Thus any condition below $(a(p'), B_\infty) = (a(p'), A(q))$ is compatible with $(a(p'), B_n^n)$, and so we may replace B^* by $A(q)$ in (3.2.3), obtaining

$$(\exists t \in T)\ (a(p'), A(q)) \Vdash (a(\vec{u}), t) \in U^{x_{\dot{G}}}_{[\vec{u}_0]}$$

and showing that $q \in D(\vec{u})$.

If the construction of $B_0, B_1, \ldots$ terminated with $B_n \in \mathcal{I}^{++}$, we may find k such that $a(p') \subseteq b_n^{k-1}$ and argue similarly with B_n in place of B_∞. $\square$

Lemma 3.2.15. *For any $p \in \mathbf{M}_2^{\mathcal{I}}$, $\vec{u} \in U$ such that $p \Vdash \vec{u} \in U^{x_{\dot{G}}}$ and any $a \subseteq a(p)$ it holds that $(a, A(p)/a) \Vdash (a, t(\vec{u})) \in U^{x_{\dot{G}}}$.*

Proof. Let G be a generic over V with $(a, A(p)/a) \in G$, and let

$$I = \mathrm{dom}(a(p)) \backslash \mathrm{dom}(a).$$

Suppose H is $\prod_{j \in I} \mathbf{M}$-generic over $V[G]$ such that $(a(p)(j), A(p)(j))_{j \in I} \in H$. Then $G \times H$ is generic over V for

$$\mathbf{M}_2^{\mathcal{I}} \times \prod_{j \in I} \mathbf{M}.$$

We define a bijection

$$\phi\colon \mathbf{M}_2^{\mathcal{I}}\Big(\leqslant \big(a, A(p)/a\big)\Big) \times \prod_{j\in I} \mathbf{M}\Big(\leqslant \big(a(p)(j), A(p)(j)\big)\Big) \to \mathbf{M}_2^{\mathcal{I}}(\leqslant p)$$

by

$$\Phi\big((b,B),(c_j,C_j)_{j\in I}\big) = \Big(a(p)\cup b\cup \bigcup_{k\in I} c_k, B\cup \bigcup_{k\in I} C_k\Big).$$

Note that $p\in\phi(G\times H)$, so $\vec{u}\in U^{x_{\phi(G\times H)}}$ in $V[G][H]$. By definition of U^x this means that in $V[G][H]$ we can find $w\in[T_{[t(\vec{u})]}]$ so that

$$(\exists u\in(\mathrm{Fin}\otimes\mathrm{Fin})^{++})\ a\sqsubseteq_2 u\subseteq\pi(w)\cap x_{\phi(G\times H)}. \tag{3.2.5}$$

Since $x_G\Delta x_{\phi(G\times H)}\in\mathrm{Fin}\otimes\mathrm{Fin}$ we may replace $x_{\phi(G\times H)}$ by x_G in (3.2.5), and thus in $V[G][H]$,

$$(\exists w\in[T_{[t(\vec{u})]}])(\exists u\in(\mathrm{Fin}\otimes\mathrm{Fin})^{++})\ a\sqsubseteq_2 u\subseteq\pi(w)\cap x_G. \tag{3.2.6}$$

It is easy to find a tree $S\in V[G]$ such that $[S]$ consists of the pairs (w,u) witnessing the two existential quantifiers in (3.2.6). Since being well-founded is absolute between models of ZFC, we conclude (3.2.6) holds in $V[G]$. But (3.2.6) implies (in fact, is equivalent to) $(a,t(\vec{u}))\in U^{x_G}$, so since G was arbitrary, we have shown that $(a,A/a)\Vdash(a,t(\vec{u}))\in U^{x_{\dot G}}$. □

With this notation and the lemmas at our disposal, we are ready to prove

$$\Vdash_{\mathbf{M}_2^{\mathcal{I}}} |\pi[T^{x_{\dot G}}]|\leqslant 1,$$

i.e., the Branch Lemma 3.2.11.

Proof of the Branch Lemma 3.2.11. Assume towards a contradiction that the lemma is false, whence we may find $p\in\mathbf{M}_2^{\mathcal{I}}$ and a pair of $\mathbf{M}_2^{\mathcal{I}}$-names $\dot w^0$ and $\dot w^1$ so that

$$p\Vdash(\forall i\in\{0,1\})\ \dot w^i\in[T^{x_{\dot G}}]\wedge x_{\dot G}\cap\pi(\dot w^i)\notin\mathrm{Fin}\otimes\mathrm{Fin}$$

and $p\Vdash\pi(\dot w^0)\neq\pi(\dot w^1)$. Then clearly we may also find $(p_0,\vec{u}_0^0,\vec{u}_0^1)\in\Gamma$ such that $\pi(t(\vec{u}_0^0))\neq\pi(t(\vec{u}_0^1))$ ($a(\vec{u}_0^i)$ plays no role here).

Claim 3.2.16. *One of the following holds:*

1. *There is $n^*\in\omega$ and $(p_1,\vec{u}_1^0,\vec{u}_1^1)\leqslant_\Gamma(p_0,\vec{u}_0^0,\vec{u}_0^1)$ in Γ such that for any $(p_2,\vec{u}_2^0,\vec{u}_2^1)\leqslant_\Gamma(p_1,\vec{u}_1^0,\vec{u}_1^1)$ from Γ, $\mathrm{dom}(a(\vec{u}_2^0))\cap\mathrm{dom}(a(\vec{u}_2^1))\subseteq n^*$; or*

2. *There is* $(p_1, \vec{u}_1^0, \vec{u}_1^1) \leqslant_\Gamma (p_0, \vec{u}_0^0, \vec{u}_0^1)$, $l^* \in \omega$ *and* $k^* \in \mathrm{dom}(a(\vec{u}_1^0)) \cap \mathrm{dom}(a(\vec{u}_1^1))$ *such that for any* $(p_2, \vec{u}_2^0, \vec{u}_2^1) \leqslant_\Gamma (p_1, \vec{u}_1^0, \vec{u}_1^1)$ *from* Γ,
$$a(\vec{u}_2^0)(k^*) \cap a(\vec{u}_2^1)(k^*) \subseteq l^*.$$

Proof of Claim. Suppose that both Items 1 and 2 above fail; we show that there is a $\prec_\Gamma$-descending sequence in Γ, which contradicts the wellfoundedness of $(\Gamma, \prec_\Gamma)$.

It suffices to show that any $(p, \vec{u}^0, \vec{u}^1) \leqslant_\Gamma (p_0, \vec{u}_0^0, \vec{u}_0^1)$ has a $\prec_\Gamma$-extension. That Items 1 and 2 above fail means precisely that

(1') For each $n^* \in \omega$ and $(p_1, \vec{u}_1^0, \vec{u}_1^1) \leqslant_\Gamma (p_0, \vec{u}_0^0, \vec{u}_0^1)$ in Γ there is $n > n^*$ and $(p_2, \vec{u}_2^0, \vec{u}_2^1) \leqslant_\Gamma (p_1, \vec{u}_1^0, \vec{u}_1^1)$ such that $n \in \mathrm{dom}(a(\vec{u}_2^0)) \cap \mathrm{dom}(a(\vec{u}_2^1))$; and

(2') For each $(p_1, \vec{u}_1^0, \vec{u}_1^1) \leqslant_\Gamma (p_0, \vec{u}_0^0, \vec{u}_0^1)$, $k^* \in \mathrm{dom}(a(\vec{u}_1^0)) \cap \mathrm{dom}(a(\vec{u}_1^1))$ and $l^* \in \omega$ there is $l > l^*$ and $(p_2, \vec{u}_2^0, \vec{u}_2^1) \leqslant_\Gamma (p_1, \vec{u}_1^0, \vec{u}_1^1)$ such that
$$l \in a(\vec{u}_2^0)(k^*) \cap a(\vec{u}_2^1)(k^*).$$

This means that in finitely many steps, we can extend any $(p, \vec{u}^0, \vec{u}^1) \leqslant_\Gamma (p_0, \vec{u}_0^0, \vec{u}_0^1)$ to some $(q, \vec{v}^0, \vec{v}^1) \leqslant_\Gamma (p, \vec{u}^0, \vec{u}^1)$ so that
$$a(\vec{u}^0) \cap a(\vec{u}^1) \sqsubset_2 a(\vec{v}^0) \cap a(\vec{v}^1)$$
by applying (2') once for each vertical in $a(\vec{u}^0) \cap a(\vec{u}^1)$ and (1') once for the domain. Thus $(q, \vec{v}^0, \vec{v}^1) \prec_\Gamma (p, \vec{u}^0, \vec{u}^1)$. □

Finally, having established that one of Items 1 and 2 above must hold, we use Lemmas 3.2.15 and 3.2.14 to finish the proof of Lemma 3.2.11 by case distinction.

Case 1: If Item 2 holds, we may fix $(p_1, \vec{u}_1^0, \vec{u}_1^1) \in \Gamma$, $l^* \in \omega$ and $k^* \in \mathrm{dom}(a(\vec{u}_1^0)) \cap \mathrm{dom}(a(\vec{u}_1^1))$ such that for any $(p_2, \vec{u}_2^0, \vec{u}_2^1) \leqslant_\Gamma (p_1, \vec{u}_1^0, \vec{u}_1^1)$ from Γ,
$$a(\vec{u}_2^0)(k^*) \cap a(\vec{u}_2^1)(k^*) \subseteq l^*.$$

We may also assume that $p_1 \in D(\vec{u}_1^0) \cap D(\vec{u}_1^1)$ (see Lemma 3.2.14). We now reach a contradiction: Define $A \subseteq \omega \times \omega$ by letting $A(k) = A(p_1)(k)$ for each $k \neq k^*$, and letting
$$A(k^*) = \{l \in \omega \mid (\exists p \leqslant p_1)(\exists \vec{u} \leqslant_U \vec{u}_1^0)\ l \in a(\vec{u})(k^*) \wedge p \Vdash \vec{u} \in U^{x_{\dot{G}}}\}.$$

Lemma 3.2.13 ensures that $A \in \mathcal{I}^{++}$ and that $A \subseteq A(p_1)$. Let
$$p^* = (a(p_1), A).$$

Since $p^* \Vdash \vec{u}_1^1 \in U^{x_{\dot{G}}}$, we can find $p \leqslant p^*$, $l \in \omega \backslash l^*$ and $\vec{u}$ such that

$$l \in a(\vec{u})(k^*) \wedge p \Vdash \vec{u} \in U^{x_{\dot{G}}}_{[\vec{u}_1^1]}.$$

It follows that $l \in A(k^*)$ and so by definition of $A(k^*)$ we can find $p' \leqslant p_1$ and $\vec{u}'$ such that

$$l \in a(\vec{u}')(k^*) \wedge p' \Vdash \vec{u}' \in U^{x_{\dot{G}}}_{[\vec{u}_1^0]}.$$

Then, as $p, p' \leqslant p_1$ and $p_1 \in D(\vec{u}_1^0) \cap D(\vec{u}_1^1)$, we can find $t_0 \in T$ and $t_1 \in T$ and set $\vec{u}^0 = (a(\vec{u}), t_0)$ and $\vec{u}^1 = (a(\vec{u}'), t_1)$ such that

$$l \in a(\vec{u}^0)(k^*) \wedge (a(p), A(p_1)/a(p)) \Vdash \vec{u}^0 \in U^{x_{\dot{G}}}_{[\vec{u}_1^0]}$$

and

$$l \in a(\vec{u}^1)(k^*) \wedge (a(p'), A(p_1)/a(p')) \Vdash \vec{u}^1 \in U^{x_{\dot{G}}}_{[\vec{u}_1^1]},$$

in order to uniformize the infinte parts of the forcing conditions. As we also want to alter the finite part, note that $\{(k^*, l)\} \cup a(\vec{u}_1^i) \subseteq a(p_1) \sqsubseteq a(p) \cap a(p')$ for each $i \in \{0, 1\}$. Let $a = a(p) \cap a(p')$, let $p_2 = (a, A(p_1)/a)$ and let $a^i = a(\vec{u}_1^i) \cup \{(k^*, l)\}$ for each $i \in \{0, 1\}$. By Lemma 3.2.15 we conclude

$$p_2 \Vdash (a^i, t(\vec{u}^i)) \in U^{x_{\dot{G}}}_{[\vec{u}_1^i]},$$

which contradicts the choice of $(p_1, \vec{u}_1^0, \vec{u}_1^1)$ and l^*.

Case 2: Otherwise, Item 1 holds and we may fix $n^* \in \omega$ and $(p_1, \vec{u}_1^0, \vec{u}_1^1) \in \Gamma$ such that for any $(p_2, \vec{u}_2^0, \vec{u}_2^1) \leqslant_\Gamma (p_1, \vec{u}_1^0, \vec{u}_1^1)$ from Γ,

$$\operatorname{dom}(a(\vec{u}_2^0)) \cap \operatorname{dom}(a(\vec{u}_2^1)) \subseteq n^*.$$

We now argue entirely analogously to the previous case, but in the domain instead of in one of the verticals. To this end, set

$$A' = \{(k, l) \mid (\exists p \leqslant p_1)(\exists \vec{u} \leqslant_U \vec{u}_1^0)\ (k, l) \in a(\vec{u}') \wedge p \Vdash \vec{u} \in U^{x_{\dot{G}}}\}.$$

Note that $A' \subseteq A(p_1)$ and $A \in \mathcal{I}^+$ by Lemma 3.2.13 (2). Let $A \subseteq A'$ be the largest subset satisfying $A \in \mathcal{I}^{++}$. Letting $p^* = (a(p_1), A)$ we reach a contradiction almost exactly as in the previous case; details are left to the reader. Lemma 3.2.11. □

3.3 Iterated Fubini products

In this section we will look at iterated Fubini products of $\mathrm{Fin}(\phi)$-ideals. In order to study these, we will first recursively define sets M^α:

Definition 3.3.1. Set $M^1 = \omega$. For a successor ordinal, set $M^{\alpha+1} = \omega \times M^\alpha$. For α limit ordinal, fix once and for all a sequence $(\alpha_n)_{n\in\omega} \subseteq \alpha$ which is cofinal in α, and set $M^\alpha = \bigcup_{n\in\omega}\{n\} \times M^{\alpha_n}$.

We will fix some notation concerning the sets M^α:

Notation 3.3.2. We often write elements of M^α as vectors $\vec{n} = (n_0, \ldots, n_k)$, but also treat them as sequences via the obvious identification of k-tuples and sequences of length k, writing $\vec{n}(i)$ for n_i, $\vec{n} \upharpoonright l$ for $(n_0, \ldots, n_{l-1})$ when $1 \leqslant l \leqslant k+1$ and $\mathrm{lh}(\vec{n})$ for $k+1$. Of course $\vec{n} \upharpoonright 0 = \varnothing$ and $\mathrm{lh}(\varnothing) = 0$.

Any proper initial segment of a sequence from M^α is called a *domain sequence*. Note that we allow a domain sequence to be empty. Elements of M^α are in contrast called *terminal sequences*.

Let $X \subseteq M^\alpha$. Viewing X as a relation, we write

$$X(n) = \{x \in \bigcup_{\beta<\alpha} M_\beta \mid (n, x) \in X\}.$$

For $\alpha > 1$, we let as usual $\mathrm{dom}(X) = \{n \in \omega \mid X(n) \neq \varnothing\}$. Given a domain sequence $\vec{n}$ write $X(\vec{n})$ for $X(n_0)\cdots(n_k)$, setting $X(\varnothing) = X$. We also set

$$\mathrm{dom}_\alpha(X) = \{\vec{n} \upharpoonright l \mid l < \mathrm{lh}(\vec{n}) \wedge \vec{n} \in X\}$$

and refer to the elements of this set as the domain sequences in X.

We denote by $\delta_\alpha(\vec{n})$ the ordinal $\delta \leqslant \alpha$ such that $X(\vec{n}) \subseteq M^\delta$ (for any $X \subseteq M^\alpha$). If the origin of the domain sequence $\vec{n}$ is unambiguous, we will often just write $\delta(\vec{n})$.

We now define a hierarchy of ideals which complexity-wise lies cofinally in the Borel hierarchy:

Definition 3.3.3. We define an ideal Fin^α on M^α for $\alpha \in \omega_1 \backslash \{0\}$ by recursion as follows:

- $\mathrm{Fin}^1 = \mathrm{Fin}$.
- For a successor ordinal $\alpha + 1 > 1$, let

$$A \in \mathrm{Fin}^{\alpha+1} \Leftrightarrow \{n \in \omega \mid A(n) \notin \mathrm{Fin}^\alpha\} \in \mathrm{Fin}\,.$$

- For a limit ordinal α with cofinal sequence $(\alpha_n)_{n\in\omega}$, let

$$A \in \mathrm{Fin}^\alpha \Leftrightarrow \{n \in \omega \mid A(n) \notin \mathrm{Fin}^{\alpha_n}\} \in \mathrm{Fin}\,.$$

Generalizing the previous definition, we also define iterated Fubini products of a sequence of F_σ ideals on ω (given as the finite part of a submeasure):

Definition 3.3.4. We define an ideal $\mathrm{Fin}(\vec{\phi})$ on M^α, where $\vec{\phi} = (\phi_\beta)_{0<\beta\leqslant\alpha}$ is a sequence of lsc submeasures on ω and $\alpha \in \omega_1\backslash\{0\}$. The definition is again by recursion on α:

- For $\alpha = 1$ and $A \subseteq M^1$ set

$$A \in \mathrm{Fin}(\vec{\phi}) \Leftrightarrow A \in \mathrm{Fin}(\phi_1).$$

- For a successor ordinal $\alpha > 1$ and $A \subseteq M^\alpha$ set

$$A \in \mathrm{Fin}(\vec{\phi}) \Leftrightarrow \{n \in \omega \mid A(n) \notin \mathrm{Fin}(\vec{\phi} \upharpoonright \alpha)\} \in \mathrm{Fin}(\phi_\alpha).$$

- For a limit ordinal α with cofinal sequence $(\alpha_n)_{n\in\omega}$ and $A \subseteq M^\alpha$ set

$$A \in \mathrm{Fin}(\vec{\phi}) \Leftrightarrow \{n \in \omega \mid A(n) \notin \mathrm{Fin}(\phi \upharpoonright \alpha_n + 1)\} \in \mathrm{Fin}(\phi_\alpha).$$

Clearly $\mathrm{Fin}^\alpha = \mathrm{Fin}(\vec{\phi})$ where for each β, ϕ_β is just the counting measure.

One could think of defining yet more general ideals of the form $\mathrm{Fin}(\vec{\phi})$ on M^α where $\vec{\phi} = (\phi_s)_{s\in D(\alpha)}$ is an assignment of submeasures on ω to the set $D(\alpha)$ of domain sequences in M^α, i.e. to the set

$$D(\alpha) = \mathrm{dom}_\alpha(M^\alpha) = \{\vec{n} \upharpoonright l \mid l < \mathrm{lh}(\vec{n}) \wedge \vec{n} \in M^\alpha\}.$$

By convention $D(1) = \{\varnothing\}$. Write

$$\vec{\phi}^*(n) = (\phi_{n^\frown t})_{t\in D(\alpha_n)},$$

where if α is a limit ordinal, $(\alpha_n)_{n\in\omega}$ is its cofinal sequence, and if α is a successor we let $\alpha_n = \alpha - 1$. Now define $\mathrm{Fin}(\vec{\phi})$ by recursion on α as follows: For $\alpha = 1$ (when $D(\alpha) = \varnothing$ and $M^1 = \omega$) let

$$\mathrm{Fin}(\vec{\phi}) = \mathrm{Fin}(\phi_\varnothing).$$

For $\alpha > 1$, let

$$\mathrm{Fin}(\vec{\phi}) = \bigoplus_{\mathrm{Fin}(\phi_\varnothing)} \mathrm{Fin}(\vec{\phi}^*(n)),$$

using the Fubini sum notation defined on p. 9 and noting that by induction $\mathrm{Fin}(\vec{\phi}^*(n))$ is an ideal on M^{α_n} with α_n as above. We conjecture all our proofs go through for ideals of the form $\mathrm{Fin}(\vec{\phi})$ as well.

Since we can view any element in M^α as a finite sequence from ω, the set M^α can be identified with a subset of $\omega^{<\omega}$ – essentially, the set of terminal sequences in M^α. Note that a set $a \subseteq M^\alpha$ is finite if and only if there are finite sets $K_0, \ldots K_{n-1}$ with $K_i \subseteq \omega$ such that $a \subseteq K_0 \times K_1 \times \cdots \times K_{n-1}$ under this identification. Furthermore, there is a natural ordering on M^α, namely the lexicographical ordering, $\leqslant_{lex}$, inherited from $\omega^{<\omega}$. We will also consider several other orderings on M^α:

Definition 3.3.5. We recursively define $\sqsubseteq_\alpha$ on M^α as follows:

- Set $X \sqsubseteq_1 Y$ if and only if $X \sqsubseteq Y$, i.e. if X is an initial segment of Y.
- Set $X \sqsubseteq_{\alpha+1} Y$ if and only if $\mathrm{dom}(X) \sqsubseteq \mathrm{dom}(Y)$ and for every $i \in \mathrm{dom}(Y)$ we have $X(i) \sqsubseteq_\alpha Y(i)$;
- For α a limit ordinal with cofinal sequence $(\alpha_n)_{n\in\omega}$, we set $X \sqsubseteq_\alpha Y$ if and only if $\mathrm{dom}(X) \sqsubseteq \mathrm{dom}(Y)$ and for every $i \in \mathrm{dom}(Y)$ we have $X(i) \sqsubseteq_{\alpha_i} Y(i)$.

In order to determine if a set properly extends another set, we need a strict ordering $\sqsubset_\alpha$ on M^α to be a version of $\sqsubseteq_\alpha$ which is strict at every level. For the case $\mathcal{J} = \mathrm{Fin}^\alpha$ we make the following definition:

- Set $X \sqsubset_1 Y$ if and only if $X \sqsubsetneq Y$, i.e. if X is a proper initial segment of Y.
- Set $X \sqsubset_{\alpha+1} Y$ if and only if $\mathrm{dom}(X) \sqsubsetneq \mathrm{dom}(Y)$ and for every $i \in \mathrm{dom}(X)$ we have $X(i) \sqsubset_\alpha Y(i)$;
- For α a limit ordinal with cofinal sequence $(\alpha_n)_{n\in\omega}$, we set $X \sqsubset_\alpha Y$ if and only if $\mathrm{dom}(X) \sqsubsetneq \mathrm{dom}(Y)$ and for every $i \in \mathrm{dom}(Y)$ we have $X(i) \sqsubset_{\alpha_i} Y(i)$.

In the general case of an ideal $\mathcal{J} = \mathrm{Fin}(\vec{\phi})$ on M^α, we define $\sqsubset_\alpha$ on M^α by recursion on α as follows:

- Set $X \sqsubset_1 Y$ if and only if $X \sqsubseteq Y$ and $\phi_1(X) < \phi_1(Y)$.
- Set $X \sqsubset_{\alpha+1} Y$ if and only if $\mathrm{dom}(X) \sqsubseteq \mathrm{dom}(Y)$,
$$\phi_{\alpha+1}(\mathrm{dom}(X)) < \phi_{\alpha+1}(\mathrm{dom}(Y)),$$
and for every $i \in \mathrm{dom}(X)$ we have $X(i) \sqsubset_\alpha Y(i)$;

- For α a limit ordinal with cofinal sequence $(\alpha_n)_{n\in\omega}$, we set $X \sqsubset_\alpha Y$ if and only if $\mathrm{dom}(X) \sqsubseteq \mathrm{dom}(Y)$, $\phi_\alpha(\mathrm{dom}(X)) < \phi_\alpha(\mathrm{dom}(Y))$, and for every $i \in \mathrm{dom}(Y)$ we have $X(i) \sqsubset_{\alpha_i} Y(i)$.

As was the case for the previous section, the material of the present section generalizes almost mechanically from Fin^α to $\mathrm{Fin}(\vec{\phi})$. Often this is made possible by the above definition of $\sqsubset_\alpha$.

When defining the α-dimensional Mathias forcing notion, we will need an ordering $<_\alpha$ on M^α defined as follows:

- Set $X <_1 Y$ if and only if $\max(X) < \min(Y)$.
- Set $X <_{\alpha+1} Y$ if and only if $\mathrm{dom}(X) \subsetneq \mathrm{dom}(Y)$, and for every $i \in \mathrm{dom}(X)$ we have $X(i) <_\alpha Y(i)$.
- For α a limit ordinal with cofinal sequence $(\alpha_n)_{n\in\omega}$, we set $X <_\alpha Y$ if and only if $\mathrm{dom}(X) \subsetneq \mathrm{dom}(Y)$ and for every $i \in \mathrm{dom}(Y)$ we have $X(i) <_{\alpha_i} Y(i)$.

We let as usual $(\mathrm{Fin}^\alpha)^+$ denote the co-ideal.

The α-dimensional forcing notion is now defined as follows:

Definition 3.3.6. Let $(\mathrm{Fin}^\alpha)^{++}$ denote the set of $A \subseteq M^\alpha$ such that for every $\vec{n} \in \mathrm{dom}_\alpha(A)$ we have $A(\vec{n}) \notin \mathrm{Fin}^{\delta_\alpha(\vec{n})}$. Conditions of $\mathbf{M}_\alpha$ are pairs (a, A) where

(a) $a \subseteq M^\alpha$ is finite;

(b) $A \in (\mathrm{Fin}^\alpha)^{++}$;

(c) $a <_\alpha A$.

We let $(a', A') \leqslant (a, A)$ if and only if $A' \subseteq A$ and $a \sqsubseteq_\alpha a' \subseteq a \cup A$.

For the general case, define $\mathrm{Fin}(\vec{\phi})^{++}$ to be the set of $A \subseteq M^\alpha$ such that for every $\vec{n} \in \mathrm{dom}_\alpha(A)$ we have $A(\vec{n}) \notin \mathrm{Fin}(\vec{\phi} \restriction \delta(\vec{n}) + 1)$, and replace (b) by $A \in \mathrm{Fin}(\vec{\phi})^{++}$ in the definition of $\mathbf{M}_\alpha$.

Note that for any $\vec{n} \in \mathrm{dom}_\alpha(a)$, the pair $(a(\vec{n}), A(\vec{n}))$ is a forcing condition in $\mathbf{M}_{\delta_\alpha(\vec{n})}$. The pair $(\mathrm{dom}(a), \mathrm{dom}(A))$ is a classical (1-dimensional) Mathias forcing condition. As before, we need a relativized forcing notion:

Definition 3.3.7. If $\mathcal{I}^+$ is the co-ideal of an ideal $\mathcal{I} \supseteq \mathrm{Fin}^\alpha$, then we write $\mathcal{I}^{++}$ for $\mathcal{I}^+ \cap (\mathrm{Fin}^\alpha)^{++}$ (or more generally, for $\mathcal{I}^+ \cap \mathrm{Fin}(\vec{\phi})^{++}$) and we let

$$\mathbf{M}_\alpha^{\mathcal{I}} = \{(a, A) \in \mathbf{M}_\alpha \mid A \in \mathcal{I}^{++}\}.$$

Note that if $\mathcal{I} = \mathrm{Fin}^\alpha$ then $\mathbf{M}^{\mathcal{I}}_\alpha = \mathbf{M}_\alpha$. Note furthermore that if $A \in \mathcal{I}^+$, then we can always find $B \subseteq A$ such that $B \in \mathcal{I}^{++}$.

Notation 3.3.8.

1. For any $X \in M^\alpha$, we define the *generalized infinity domain* by
$$\mathrm{dom}^\infty_\alpha(X) = \{\vec{n} \in \mathrm{dom}_\alpha(X) \mid X(\vec{n}) \notin \mathrm{Fin}^{\delta_\alpha(\vec{n})}\},$$
and note that $A \in (\mathrm{Fin}^\alpha)^{++}$ if and only if $\mathrm{dom}_\alpha(A) = \mathrm{dom}^\infty_\alpha(A)$.

2. Given a filter G on $\mathbf{M}^{\mathcal{I}}_\alpha$, let
$$x_G = \bigcup\{a \mid (\exists A)(a, A) \in G\}.$$

3. For a condition $p \in \mathbf{M}^{\mathcal{I}}_\alpha$, we write $(a(p), A(p))$ when we want to refer to its components.

4. For $(a, A) \in \mathbf{M}^{\mathcal{I}}_\alpha$ and $b \subseteq a \cup A$ finite, let
$$A/b = \bigcup_{\vec{n} \in N} A(\vec{n}) \backslash \big\{x \in M^{\delta_\alpha(\vec{n})} \mid \big(\exists \vec{m} \in b(\vec{n})\big)\ x \leqslant_{lex} \vec{m}\big\},$$
where $N = \mathrm{dom}_\alpha(b) \cup \{\vec{n} \mid \vec{n} >_{lex} \max(\mathrm{dom}_\alpha(b))\}$.

5. For $p \in \mathbf{M}^{\mathcal{I}}_\alpha$, we let $\mathbf{M}^{\mathcal{I}}_\alpha(\leqslant p) = \{q \in \mathbf{M}^{\mathcal{I}}_\alpha \mid q \leqslant p\}$.

Remark 3.3.9. The definition of A/b was made to guarantee $b <_\alpha A/b$. Note that $\vec{n} \in A/b$ if and only if $\vec{n} \notin b$ and letting $\vec{n} \restriction l$ be the longest common initial segment of $\vec{n}$ with some element of b, then there is no $\vec{m} \in b$ with $\vec{n} \restriction l \sqsubseteq \vec{m}$ and $\vec{n}(l) < \vec{m}(l)$.

Following the same strategy as in previous sections, our main pursuit will be a generalization of the Main Proposition 3.2.6.

Remark 3.3.10. Recall that in order to meaningfully talk about κ-Suslin sets in $\mathcal{P}(M^\alpha)$, we identify M^α with ω (via some fixed arbitrary bijection), sets with their characteristic functions, and in effect, $\mathcal{P}(M^\alpha)$ with 2^ω.

Assumption 3.3.11. For the remainder of this article, let $\mathcal{J} = \mathrm{Fin}^\alpha$ where $\alpha \geqslant 2$ (or more generally, $\mathcal{J} = \mathrm{Fin}(\vec{\phi})$). Suppose $\mathcal{A} \subseteq \mathcal{P}(M^\alpha)$ is a $\mathcal{J}$-almost disjoint family which is κ-Suslin. Moreover, fix a tree T on $2 \times \kappa$ such that $\pi[T] = \mathcal{A}$. Finally, let $\mathcal{I}$ be the ideal generated by $\mathcal{A} \cup \mathcal{J}$.

Although the proofs in this section work for $\mathrm{Fin}(\vec{\phi})$ as above we will of notational concern only consider the case where ϕ_β is the counting measure for $0 < \beta \leqslant \alpha$, i.e., where $\mathcal{J} = \mathrm{Fin}^\alpha$. Whenever relevant, we either make an explicit comment or the reader can substitute Fin^α by $\mathrm{Fin}(\vec{\phi})$ (perhaps needless to mention at this point, do not substitute for the word finite).

Main Proposition 3.3.12. $\Vdash_{\mathbf{M}^{\mathcal{I}}_\alpha} (\forall y \in \pi[T])\ y \cap x_{\dot{G}} \in \mathrm{Fin}^\alpha$.

The Main Proposition will be proved in Section 3.3 below. All our results about Fin^α from Theorem 1.0.5 follow from the Main Proposition 3.3.12 as a corollary:

Corollary 3.3.13. *Assuming the Main Proposition 3.3.12, Theorem 1.0.5 holds.*

Proof. It suffices to replace $\mathbf{M}^{\mathcal{I}}$ by $\mathbf{M}^{\mathcal{I}}_\alpha$ in the proofs of Corollaries 3.1.6, 3.1.7, and 3.1.8 (just as we did in Corollary 3.2.7 in the two-dimensional case). $\square$

Properties of the general higher-dimensional forcing

Before we prove the Main Proposition 3.3.12 we collect the necessary facts about $\mathbf{M}^{\mathcal{I}}_\alpha$.

Lemma 3.3.14.

1. *For any $y \in \mathcal{I}$, $\Vdash_{\mathbf{M}^{\mathcal{I}}_\alpha} x_{\dot{G}} \cap \check{y} \in \mathrm{Fin}^\alpha$.*
2. *Let $k \in \omega$. The partial order $\mathbf{M}^{\mathcal{I}}_{\alpha+1}$ is isomorphic to the product $\mathbf{M}^{\mathcal{I}}_{\alpha+1}(\leqslant (\varnothing, A)) \times (\mathbf{M}_\alpha)^k$, where $A = \{\vec{n} \in M^\alpha \mid \vec{n}(0) \geqslant k\}$, and by $(\mathbf{M}_\alpha)^k$ we mean k-fold (side-by-side) product of α-dimensional Mathias forcing $\mathbf{M}_\alpha$. If α is a limit ordinal, $\mathbf{M}^{\mathcal{I}}_\alpha$ is isomorphic to $\mathbf{M}^{\mathcal{I}}_\alpha(\leqslant (\varnothing, A)) \times (\Pi_{i<k}\mathbf{M}_{\alpha_i})$.*
3. *$\Vdash_{\mathbf{M}^{\mathcal{I}}_\alpha} x_{\dot{G}} \in (\mathrm{Fin}^\alpha)^{++}$.*

Proof. (1) Since $D_y = \{p \in \mathbf{M}^{\mathcal{I}}_\alpha \mid A(p) \cap y = \varnothing\}$ is dense for any $y \in \mathcal{I}$.

(2) First we consider the successor case. Define a map

$$\phi\colon \mathbf{M}^{\mathcal{I}}_{\alpha+1}(\leqslant (\varnothing, A)) \times (\mathbf{M}_\alpha)^k \to \mathbf{M}^{\mathcal{I}}_{\alpha+1}$$

by

$$\big((b, B), (c_i, C_i)_{i<k}\big) \mapsto \Big(b \cup \bigcup_{i<k} \{i\} \times c_i, B \cup \bigcup_{i<k} \{i\} \times C_i\Big).$$

For a limit ordinal α the map can be defined in exactly the same way. Both of these maps are easily seen to be bijective and order preserving.

(3) This is shown easily by induction, slightly adapting the general case of the proof of 3.2.8(3). We leave this to the reader. $\square$

We shall need a more sophisticated way of decomposing the forcing as a product.

Towards this, let $(a, A) \in \mathbf{M}^{\mathcal{I}}_{\alpha}$. Let us regard M^{α} and a as trees, ordered by the initial segment relation $\sqsubseteq$. Given $\vec{n} \in A$, let us see how we can characterize the "type" of $\vec{n}$ in relation to a with respect to $\leqslant_{lex}$.

First note that since $\vec{n} \in A$ and $a <_{\alpha} A$ it is enough to characterize the type of $\vec{n} \upharpoonright (\mathrm{lh}(\vec{n}) - 1)$ relative to the following set of domain sequences

$$a^* = \left\{\vec{n}' \upharpoonright \left(\mathrm{lh}(\vec{n}') - 1\right) \mid \vec{n}' \in a\right\}$$

(for if $\vec{n}$ extends $\vec{n}^* \in a^*$, $\vec{n}' <_{lex} \vec{n}$ for every $\vec{n}' \in a$ which extends $\vec{n}^*$).

Let $\vec{n}_0, \ldots, \vec{n}_k$ enumerate a^* in lexicographically increasing order, and momentarily fix i such that $\vec{n}_i$ is the lexicographically maximal in a^* with $\vec{n}_i \leqslant_{lex} \vec{n}$. We then know by $a <_{\alpha} A$ that $\vec{n}$ must have a longer initial segment in common with $\vec{n}_i$ than it does with $\vec{n}_{i+1}$, provided $i < k$.

Let therefore $\vec{m}_j$ be the shortest initial segment of $\vec{n}_j$ such that $\vec{m}_j <_{lex} \vec{n}_{j+1}$ for $j < k$, and let $\vec{m}_k = \varnothing$. We have just seen that $\vec{m}_i \sqsubseteq \vec{n}$ (for i and $\vec{n}$ as above). Moreover if $j < i$, $\vec{m}_j \not\sqsubseteq \vec{n}$ (for $\vec{m}_j <_{lex} \vec{n}_{j+1} \leqslant_{lex} \vec{n}_i$ and so $\vec{m}_j <_{lex} \vec{n}$).

We have thus shown the following lemma:

Lemma 3.3.15. *Suppose $(a, A) \in \mathbf{M}^{\mathcal{I}}_{\alpha}$. Let $\vec{n}_0, \ldots, \vec{n}_k$ enumerate*

$$a^* = \left\{\vec{n} \upharpoonright \left(\mathrm{lh}(\vec{n}) - 1\right) \mid \vec{n} \in a\right\}$$

in lexicographically ascending order, let $\vec{m}_k = \varnothing$ and for $i < k$ let $\vec{m}_i$ be the shortest initial segment of n_i such that $\vec{m}_i <_{lex} \vec{n}_{i+1}$ (just as above).

Then for each $\vec{n} \in A$ there is precisely one i such that $\vec{m}_i \sqsubseteq \vec{n}$ and $\vec{n}_i \leqslant_{lex} \vec{n}$ (namely the maximal i such that $\vec{n}_i \leqslant_{lex} \vec{n}$).

Technical as the previous lemma may be, it allows us to decompose the forcing as a product in a very useful manner.

Lemma 3.3.16. *Suppose $(a, A) \in \mathbf{M}^{\mathcal{I}}_{\alpha}$, and $\vec{m}_0, \ldots, \vec{m}_k$ and $\vec{n}_0, \ldots, \vec{n}_k$ are defined as in the previous lemma. Then $\mathbf{M}^{\mathcal{I}}_{\alpha}\big(\leqslant (a, A)\big)$ is isomorphic to*

$$\left(\prod_{i<k} \mathbf{M}_{\delta(\vec{m}_i)}\Big(\leqslant \big(a_i(\vec{m}_i), A_i(\vec{m}_i)\big)\Big)\right) \times \mathbf{M}^{\mathcal{I}}_{\alpha}\Big(\leqslant \big(a_k, A_k\big)\Big) \tag{3.3.1}$$

where we let

$$A_i = A \cap \{\vec{n} \mid \vec{m}_i \sqsubseteq \vec{n} \wedge \vec{n}_i \leqslant_{lex} \vec{n}\},$$
$$a_i = a \cap \{\vec{n} \mid \vec{n}_i \sqsubseteq \vec{n}\},$$

for each $i \leqslant k$.

Proof. The crucial observation is that by Lemma 3.3.15, $a \cup A$ may be written as a disjoint union

$$a \cup A = \bigcup_{i \leqslant k} a_i \cup A_i \tag{3.3.2}$$

Define a map ϕ from $\mathbf{M}^{\mathcal{I}}_\alpha\big(\leqslant (a, A)\big)$ to the forcing in (3.3.1) as follows: For $(b, B) \leqslant (a, A)$ define

$$\phi(b, B) = \big(a_i(\vec{m}_i) \cup \big((b \cap A_i)(\vec{m}_i)\big), (B \cap A_i)(\vec{m}_i)\big)_{i \leqslant k}.$$

Using the partition from (3.3.2), it is straightforward to verify that this map is an isomorphism of partial orders. □

Of course we also have a diagonalization lemma for $\mathbf{M}^{\mathcal{I}}_\alpha$ (compare Lemmas 3.1.10 resp. 3.2.9). Just as these two lemmas, Lemma 3.3.17 holds verbatim with $\subseteq^*_{\mathrm{Fin}^\alpha}$ and not just with $\subseteq^*_{\mathcal{J}}$ even for $\mathcal{J} = \mathrm{Fin}(\vec{\phi})$.

Lemma 3.3.17. *Let $(A_k)_{k \in \omega}$ be a sequence from $\mathcal{I}^{++}$ satisfying $A_{k+1} \subseteq A_k$ for every $k \in \omega$. Then there is $A_\infty \in \mathcal{I}^{++}$ such that $A_\infty \subseteq^*_{\mathrm{Fin}^\alpha} A_k$ for every $k \in \omega$.*

Proof. The proof of Lemma 3.2.9 can be transcribed completely mechanically by replacing $\mathrm{Fin} \otimes \mathrm{Fin}$ by Fin^α everywhere; we leave this to the reader. □

The Branch Lemma for general higher dimensions

The reader will find that our line of argumentation in this section is remarkably close to that of the previous section; of course this is only true since the proofs there were written with the general case in mind.

Yet again, the crucial definition is that of an invariant tree, analogous to Definitions 3.1.12 and 3.2.10.

Definition 3.3.18. For $x \subseteq M^\alpha$, let

$$T^x = \{t \in T \mid (\exists w \in \pi[T_{[t]}])\ w \cap x \notin \mathrm{Fin}^\alpha\}$$

As in Sections 3.1 and 3.2, it is easy to see that whenever $x\Delta x' \in \text{Fin}^\alpha$, $T^x = T^{x'}$. Moreover Facts 3.1.13(2)–(5) hold here as well.

We are now ready to state the main lemma of this section.

The Branch Lemma 3.3.19. $\Vdash_{\mathbf{M}_\alpha^{\mathcal{I}}} |\pi[T^{x_{\dot G}}]| \leqslant 1$.

In keeping with the pattern established in previous sections, we postpone the proof of the Branch Lemma and first give the proof of the Main Proposition 3.3.19, assuming the lemma. The proof is verbatim the proof of Main Proposition 3.2.6 except that we use Lemma 3.3.14(2)to decompose the forcing; we repeat it for the incredulous reader.

Proof of Main Proposition 3.3.12. Suppose towards a contradiction that some $p_0 \in \mathbf{M}_\alpha^{\mathcal{I}}$ forces that there is $A \in \pi[T]^{V[\dot G]}$ with $A \cap x_{\dot G} \notin \text{Fin}_\alpha$. The Branch Lemma 3.3.19 lets us choose a name $\dot A$ so that $p_0 \Vdash \pi[T^{x_{\dot G}}] = \{\dot A\}$.

We show the following generalization of Claim 3.2.12:

Claim 3.3.20. *There is $q \in \mathbf{M}_\alpha^{\mathcal{I}}$ and $A' \in \pi[T]$ such that $q \Vdash \dot A = \check A'$.*

Proof of Claim. By the generalized diagonalization lemma (Lemma 3.3.17), it suffices to show that if $p \leqslant p_0$ and p decides $\vec n \in \dot A$ then in fact $(a(p_0), A(p))$ decides $\vec n \in \dot A$.

So let us assume $p \Vdash \vec n \in \dot A$ (if $p \Vdash \vec n \notin \dot A$ the proof is similar). We must show that for an arbitrary $\mathbf{M}_\alpha^{\mathcal{I}}$-generic G with $(a(p_0), A(p)) \in G$, it holds that $\vec n \in \dot A^G$.

Fix k large enough so that $\text{dom}(a(p)) \subseteq k$. By Lemma 3.3.14(2) we can decompose G as $G_0 \times G_1$ where G_0 is generic for $\prod_{i<k} \mathbf{M}_{\alpha_i}^{\text{Fin}^{\alpha_i}}$ and G_1 is $\mathbf{M}_\alpha^{\mathcal{I}}$-generic. As $x_G \Delta x_{G_1} \in \text{Fin}^\alpha$, $T^{x_G} = T^{x_{G_1}} \in V[G_1]$.

By absoluteness, $\dot A^G \in V[G_1]$ and $\pi[T^{x_G}] = \{\dot A^G\}$ holds in both $V[G]$ and $V[G_1]$.

Since $a(p) \subseteq \prod_{i<k}\{i\} \times M^{\alpha_i}$ we can find G_0' which is $(\prod_{i<k} \mathbf{M}_{\alpha_i}, V[G_1])$-generic over $V[G_1]$ so that letting $G' = G_0' \times G_1$, $p \in G'$. Again by Fin^α-invariance of T^x and by absoluteness, $\dot A^{G'} \in V[G_1]$ and $\pi[T^{x_{G_1}}] = \{\dot A^{G'}\}$ and so $\dot A^{G'} = \dot A^G$ and $\vec n \in \dot A^G$. Claim 3.3.20. $\square$

Just as in the proof of Main Proposition 3.2.6 we conclude that $A' \in \mathcal{I}$ by absoluteness while $q \Vdash x_{\dot G} \cap \check A' \notin \text{Fin}_\alpha$, contradicting Lemma 3.3.14(1).

Main Proposition 3.3.12. $\square$

Gradually working towards a proof of the Branch Lemma 3.3.19, we start by introducing some notation. Set

$$U = \{(a, t) \in \mathcal{P}(M^\alpha) \times T \mid a \text{ is finite}\}.$$

For $\vec{u} \in U$, we will often write $\vec{u} = (a(\vec{u}), t(\vec{u}))$. Define an ordering $\leqslant_U$ on U by

$$\vec{u}_1 \leqslant_U \vec{u}_0 \Leftrightarrow a(\vec{u}_1) \sqsupseteq_\alpha a(\vec{u}_0) \wedge t(\vec{u}_1) \sqsupseteq t(\vec{u}_0).$$

Assume for a moment that G is $\mathbf{M}^{\mathcal{I}}_\alpha$-generic over V and work in $V[G]$. For a fixed $x \in \mathcal{P}(M^\alpha)$, define a set $U^x \subseteq U$ consisting of those pairs $(a, t) \in U$ such that there is $w \in [T_{[t]}]$ with

1. $\pi(w) \cap x_G \notin \mathrm{Fin}^\alpha$;
2. $\mathrm{dom}_\alpha(a) \subseteq \mathrm{dom}^\infty_\alpha(x \cap \pi(w))$;
3. $(\forall \vec{n} \in \mathrm{dom}_\alpha(a))\ a(\vec{n}) \subseteq x(\vec{n}) \cap \pi(w)(\vec{n})$.

Note that U^x is closed under initial segments with respect to $\leqslant_U$, and that an infinite chain through U^x will give a set $A \in \pi[T]$ with a large intersection with x, and a $(\mathrm{Fin}_\alpha)^{++}$-subset of this intersection to witness its largeness in a useful manner.

In analogy to trees, when $\vec{u}_0 \in U^x$ we again write

$$U^x_{[\vec{u}_0]} = \{\vec{u} \in U^x \mid \vec{u}_0 \leqslant_U \vec{u}\}.$$

Finally working in V again, we note the following about $U^{x_{\dot{G}}}$:

Lemma 3.3.21. *Suppose* $(a, A) \Vdash \vec{u} \in U^{x_{\dot{G}}}$.

1. *It holds that* $a(\vec{u}) \subseteq a$. *Moreover if* $a' \subseteq a(\vec{u})$ *also*
$$(a, A) \Vdash (a', t(\vec{u})) \in U^{x_{\dot{G}}}.$$
2. *If* $A' \subseteq^*_{\mathcal{I}} A$ *such that* $(a, A') \in \mathbf{M}^{\mathcal{I}}_\alpha$, *then also* $(a, A') \Vdash \vec{u} \in U^{x_{\dot{G}}}$.
3. *The set* $A' \subseteq M^\alpha$ *defined by*
$$A' = \{\vec{n} \mid (\exists p' \leqslant (a, A))(\exists \vec{u}' \leqslant_U \vec{u})\ \vec{n} \in a(\vec{u}') \wedge p' \Vdash \vec{u}' \in U^{x_{\dot{G}}}\}$$
is not in $\mathcal{I}$.
4. *For a non-empty domain sequence* $\vec{n} \in \mathrm{dom}_\alpha(a(\vec{u}))$, *the set* $A_{\vec{n}} \subseteq M^{\delta_\alpha(\vec{n})}$ *defined by*
$$A_{\vec{n}} = \{\vec{m} \mid (\exists p' \leqslant (a, A))(\exists \vec{u}' \leqslant_U \vec{u})\ \vec{m} \in a(\vec{u}')(\vec{n}) \wedge p' \Vdash \vec{u}' \in U^{x_{\dot{G}}}\}$$
is in $(\mathrm{Fin}^{\delta_\alpha(\vec{n})})^+$.

Proof. (1) Immediate from the definition of $U^{x_{\dot{G}}}$.

(2) Suppose that $(a, A') \not\Vdash \vec{u} \in U^{x_{\dot{G}}}$. Then there is some $(b, B) \leqslant (a, A')$ such that $(b, B) \Vdash \vec{u} \notin U^{x_{\dot{G}}}$. Since $A' \backslash A \in \mathcal{I}$, there is some $B' \subseteq B \cap A$ such that $B' \in \mathcal{I}^{++}$. However, $(b, B') \leqslant (b, B)$ and $(b, B') \leqslant (a, A)$, which is a contradiction.

(3) Although the proof is practically identical to that of Lemma 3.2.13(2), we give the details for the reader's convenience. Assume to the contrary that $A' \in \mathcal{I}$. Then $A \backslash A' \in \mathcal{I}^{+}$, so take $B \subseteq A \backslash A'$ such that $B \in \mathcal{I}^{++}$ and set $p = (a, B) \in \mathbf{M}^{\mathcal{I}}_{\alpha}$. Since $p \Vdash \vec{u} \in U^{x_{\dot{G}}}$ we can find a name $\dot{w}$ such that

$$p \Vdash \dot{w} \in \pi[T_{[t(\vec{u})]}] \wedge \dot{w} \cap x_{\dot{G}} \notin \mathrm{Fin}^{\alpha}.$$

(As in Lemma 3.2.13(2) it would suffice if $p \Vdash T^{x_{\dot{G}}} \neq \varnothing$). Thus we can extend p to p' to force some terminal sequence $\vec{n}$ into $\dot{w} \cap x_{\dot{G}} \backslash a(p)$. But it has to be the case that $\vec{n} \in a(p')$. Whence $\vec{n} \in A'$ by definition of A', contradicting that also $\vec{n} \in B$ which is disjoint from A'.

(4) The proof is identical to that of Lemma 3.2.13(3) in essence, but differs substantially in notation. Assume to the contrary that $A_{\vec{n}} \in \mathrm{Fin}^{\delta(\vec{n})}$. Then we can find $p \leqslant (a, A)$ such that $A(p)(\vec{n})$ is disjoint from $A_{\vec{n}}$. Since $p \Vdash \vec{u} \in U^{x_{\dot{G}}}$ we can find a name $\dot{w}$ such that

$$p \Vdash \dot{w} \in \pi[T_{[t(\vec{u})]}] \wedge \dot{w} \cap x_{\dot{G}} \in (\mathrm{Fin}^{\alpha})^{+}$$

and

$$p \Vdash \mathrm{dom}_{\alpha}(a(\vec{u})) \subseteq \mathrm{dom}^{\infty}_{\alpha}(\dot{w} \cap x_{\dot{G}}).$$

Therefore $\vec{n} \in \mathrm{dom}^{\infty}_{\alpha}(\dot{w} \cap x_{\dot{G}})$ and we can extend p to p' to force a terminal sequence $\vec{n} \frown \vec{n}'$ into $\dot{w} \cap x_{\dot{G}} \backslash a(p)$. But as in the proof of the previous item, it has to be the case that $\vec{n} \frown \vec{n}' \in a(p')$, whence $\vec{n}' \in A_{\vec{n}}$ by definition of $A_{\vec{n}}$, contradicting that also $\vec{n}' \in A(p')(\vec{n})$ which is disjoint from $A_{\vec{n}}$. □

Define a set Γ as follows:

$$\Gamma = \{(p, \vec{u}^{0}, \vec{u}^{1}) \in \mathbf{M}^{\mathcal{I}}_{\alpha} \mid (\forall i \in \{0,1\}) p \Vdash \vec{u}^{i} \in U^{x_{\dot{G}}}\}.$$

Define two orderings on Γ:

$$(p_1, \vec{u}^{0}_{1}, \vec{u}^{1}_{1}) \leqslant_{\Gamma} (p_0, \vec{u}^{0}_{0}, \vec{u}^{1}_{0}) \Leftrightarrow p_1 \leqslant p_0 \wedge \vec{u}^{i}_{1} \leqslant_{U} \vec{u}^{i}_{0}$$

for $i \in \{0, 1\}$, and

$$\begin{aligned} &(p_1, \vec{u}^{0}_{1}, \vec{u}^{1}_{1}) <_{\Gamma} (p_0, \vec{u}^{0}_{0}, \vec{u}^{1}_{0}) \Leftrightarrow \\ &(p_1, \vec{u}^{0}_{1}, \vec{u}^{1}_{1}) \leqslant_{\Gamma} (p_0, \vec{u}^{0}_{0}, \vec{u}^{1}_{0}) \wedge \left[a(\vec{u}^{0}_{0}) \cap a(\vec{u}^{1}_{0}) \sqsubset_{\alpha} a(\vec{u}^{0}_{1}) \cap a(\vec{u}^{1}_{1})\right]. \end{aligned}$$

Note that Γ is well-founded with respect to the second ordering, $\prec_\Gamma$. Indeed, suppose towards a contradiction that there is an infinite sequence $(p_0, \vec{u}_0^0, u_0^1) \prec_\Gamma (p_1, \vec{u}_1^0, u_1^1) \prec_\Gamma \cdots$. Set

$$y^i = \bigcup_{n\in\omega} t(\vec{u}_n^i)$$

for $i \in \{0,1\}$, and

$$A = \bigcup_{n\in\omega} a(\vec{u}_n^0) \cap a(\vec{u}_n^1).$$

The sequence is $\prec_\Gamma$-decreasing and from Γ, hence $A \in (\mathrm{Fin}^\alpha)^{++}$ and $A \subseteq \pi(y^0) \cap \pi(y^1)$, contradicting Fin^α-almost disjointness of $\pi[T]$.

The following lemmas are analogues of Lemma 3.2.14 and Lemma 3.2.15, giving us some freedom in tampering with the both the infinite and the finite parts of conditions while maintaining that something is forced about $U^{x_{\dot G}}$.

Lemma 3.3.22. *For each $\vec{u}_0 \in U$ the set $D(\vec{u}_0)$ is dense and open in $\mathbf{M}_\alpha^{\mathcal{I}}$, where we define $D(\vec{u}_0)$ to be the set of $p \in \mathbf{M}_\alpha^{\mathcal{I}}$ such that for all $p' \leqslant p$ and any $\vec{u} \leqslant_U \vec{u}_0 \in U$,*

$$\left[p' \Vdash \vec{u} \in U^{x_{\dot G}}_{[\vec{u}_0]}\right] \Rightarrow (a(p'), A(p)/a(p')) \Vdash (\exists t \in T)(a(\vec{u}), t) \in U^{x_{\dot G}}_{[\vec{u}_0]}.$$

Proof. The proof from the two-dimensional case, i.e., of Lemma 3.2.14 applies exactly as written once we make the following adaptations: Firstly replace $\mathrm{Fin} \otimes \mathrm{Fin}$ by Fin^α. Secondly, replace $\sqsubset_2$ by $\sqsubset_\alpha$. Thirdly, adapt the definition of B_n^0 as follows:

$$B_n^0 = \{\vec{n} \in B_{n-1}^{n-1} \mid (\exists \vec{n}' \in \mathrm{dom}_\alpha(b_n^0))\ \vec{n}' \subseteq \vec{n}\} \cup B_{n-1}^{n-1} \setminus \bigcup \{C_i \mid i < n\}.$$

Then $(b_n^0, B_n^0) \in \mathbf{M}_\alpha^{\mathcal{I}}$, and $B_n^0 \cap C_i \in \mathrm{Fin}^\alpha$ for each $i < n$. With these changes, the remainder of the argument for Lemma 3.2.14 applies verbatim. $\square$

Lemma 3.3.23. *Suppose we are given $\vec{u} \in U$, $(a, A) \in \mathbf{M}_\alpha^{\mathcal{I}}$, and $a' \subseteq a$ so that $(a', A/a') \in \mathbf{M}_\alpha^{\mathcal{I}}$, and so that the lexicographically maximal element of a' is also the lexicographically maximal element of a. Suppose further $(a, A) \Vdash \vec{u} \in U^{x_{\dot G}}$. Then*

$$(a', A/a') \Vdash (a', t(\vec{u})) \in U^{x_{\dot G}}.$$

We remark that the lemma is true without the assumption that a' and a have the same lexicographic maximum. But this is easy to arrange when we apply the lemma, and simplifies notation in its proof.

Proof. We will decompose the forcing as a product. Let $\vec{n}_0, \ldots, \vec{n}_k$ and $\vec{m}_0, \ldots, \vec{m}_k$ be defined as in Lemma 3.3.15. Then by Lemma 3.3.16,

$$\mathbf{M}^{\mathcal{I}}_\alpha\big(\leqslant (a, A)\big) \cong \prod_{i<k} \mathbf{M}_{\delta(\vec{m}_i)}\Big(\leqslant \big(a_i(\vec{m}_i), A_i(\vec{m}_i)\big)\Big) \times \mathbf{M}^{\mathcal{I}}_\alpha\Big(\leqslant \big(a_k, A_k\big)\Big) \tag{3.3.3}$$

with A_i and a_i defined as in the lemma.

Let D consist of those $i < k$ such that some element of a' extends $\vec{n}_i$. Then writing $A' = A/a'$, Lemma 3.3.16 also gives us an isomorphism

$$\mathbf{M}^{\mathcal{I}}_\alpha\big(\leqslant (a', A')\big) \cong \prod_{i\in D} \mathbf{M}_{\delta(\vec{m}_i)}\Big(\leqslant \big(a'_i(\vec{m}_i), A'_i(\vec{m}_i)\big)\Big) \times \mathbf{M}^{\mathcal{I}}_\alpha\Big(\leqslant \big(a'_k, A'_k\big)\Big) \tag{3.3.4}$$

with A'_i and a'_i defined analogously as in the lemma. We have $A'_i = A_i$ for each $i \in D \cup \{k\}$ and so it is easy to see—e.g., using Lemma 3.3.14(2), a finite induction, and Lemma 3.1.9(3)—that

$$\mathbf{M}_{\delta(\vec{m}_i)}\big(\leqslant (a_i(\vec{m}_i), A_i(\vec{m}_i))\big) \cong \mathbf{M}_{\delta(\vec{m}_i)}\big(\leqslant (a'_i(\vec{m}_i), A'_i(\vec{m}_i))\big)$$

for $i \in D$ and

$$\mathbf{M}^{\mathcal{I}}_\alpha\Big(\leqslant \big(a_i(\vec{m}_k), A_k\big)\Big) \cong \mathbf{M}^{\mathcal{I}}_\alpha\Big(\leqslant \big(a'_k, A'_k\big)\Big).$$

Write

$$\begin{aligned}
\mathbb{P}_+ &= \prod_{i\notin D} \mathbf{M}_{\delta(\vec{m}_i)}\Big(\leqslant \big(a_i(\vec{m}_i), A_i(\vec{m}_i)\big)\Big),\\
\mathbb{P}_- &= \prod_{i\in D} \mathbf{M}_{\delta(\vec{m}_i)}\Big(\leqslant \big(a_i(\vec{m}_i), A_i(\vec{m}_i)\big)\Big),\\
\mathbb{P}'_- &= \prod_{i\in D} \mathbf{M}_{\delta(\vec{m}_i)}\Big(\leqslant \big(a'_i(\vec{m}_i), A'_i(\vec{m}_i)\big)\Big),\\
\mathbb{P}_\infty &= \mathbf{M}^{\mathcal{I}}_\alpha\Big(\leqslant \big(a_k, A_k\big)\Big),\\
\mathbb{P}'_\infty &= \mathbf{M}^{\mathcal{I}}_\alpha\Big(\leqslant \big(a'_k, A'_k\big)\Big).
\end{aligned}$$

noting that we have established

$$\mathbf{M}^{\mathcal{I}}_\alpha\big(\leqslant (a', A')\big) \cong \mathbb{P}'_- \times \mathbb{P}'_\infty \cong \mathbb{P}_- \times \mathbb{P}_\infty \tag{3.3.5}$$

and

$$\mathbf{M}^{\mathcal{I}}_\alpha\big(\leqslant (a, A)\big) \cong \mathbb{P}_+ \times \mathbb{P}_- \times \mathbb{P}_\infty \tag{3.3.6}$$

Now finally, let G' be $\mathbf{M}^{\mathcal{I}}_\alpha$-generic over V with $(a', A') \in G'$. We must show $(a', t(\vec{u})) \in U^{x_{G'}}$. Using (3.3.5) transform G' into a $\mathbb{P}_- \times \mathbb{P}_\infty$ generic $H_- \times H_\infty$. Find a $\mathbb{P}_+$-generic H_+ over $V[H_-][H_\infty]$ and let G be the $\mathbf{M}^{\mathcal{I}}_\alpha$-generic given by $H_+ \times H_- \times H_\infty$ using (3.3.6). By construction $(a, A) \in G$, whence $(a(\vec{u}), t(\vec{u})) \in U^{x_G}$.

By definition of U^{x_G} this means that in $V[G]$ we can find $w \in \pi[T_{[t(\vec{u})]}]$ so that

$$(\exists u \in (\mathrm{Fin}^\alpha)^{++})\; a' \sqsubseteq_2 u \subseteq \pi(w) \cap x_G. \tag{3.3.7}$$

Since $x_G \Delta x_{G'} \in \mathrm{Fin}^\alpha$ we may replace x_G by $x_{G'}$ in (3.3.7), and thus

$$(\exists x \in [T_{[t(\vec{u})]}])(\exists u \in (\mathrm{Fin}^\alpha)^{++})\; a' \subseteq u \subseteq \pi(x) \cap x_{G'}. \tag{3.3.8}$$

Just as in the two-dimensional case (i.e., the proof of Lemma 3.2.15) an absoluteness argument easily shows that (3.3.8) and hence $(a', t(\vec{u})) \in U^{x_{G'}}$ must hold in $V[G']$, proving $(a', A/a') \Vdash (a', t(\vec{u})) \in U^{x_{\dot{G}}}$. □

After all these preparations, we are finally ready to prove our last and most general instance of the Branch Lemma.[2]

Proof of the Branch Lemma 3.3.19. Suppose towards a contradiction we have $p \in \mathbf{M}^{\mathcal{I}}_\alpha$ and a pair of $\mathbf{M}^{\mathcal{I}}_\alpha$-names $\dot{w}^0$, $\dot{w}^1$ such that

$$p \Vdash (\forall i \in \{0,1\})\; \dot{w}^i \in \pi[T] \wedge x_{\dot{G}} \cap \dot{w}^i \notin \mathrm{Fin}^\alpha$$

and $p \Vdash \dot{w}^0 \neq \dot{w}^1$. By definition of Γ we may find $(p_0, \vec{u}^0_0, \vec{u}^1_0) \in \Gamma$ such that $\pi(t(\vec{u}^0_0)) \neq \pi(t(\vec{u}^1_0))$.

Claim 3.3.24. *There is $(p_1, \vec{u}^0_1, \vec{u}^1_1) \leqslant_\Gamma (p_0, \vec{u}^0_0, \vec{u}^1_0)$, a terminal sequence $\vec{n}^* \in a(\vec{u}^0_1) \cap a(\vec{u}^1_1)$, and numbers $l < \mathrm{lh}(\vec{n}^*)$ and and $k^* \in \omega$ such that for any $(p_2, \vec{u}^0_2, \vec{u}^1_2) \leqslant_\Gamma (p_1, \vec{u}^0_1, \vec{u}^1_1)$ and any*

$$\vec{n} \in a(\vec{u}^0_2) \cap a(\vec{u}^1_2)$$

such that $\vec{n}^* \restriction l \sqsubseteq \vec{n}$, we have $\vec{n}(l) \leqslant k^*$.

Proof of Claim. We show that if the claim fails, there is a $<_\Gamma$-descending sequence in Γ. It suffices to show that any $(p_1, \vec{u}^0_1, \vec{u}^1_1) \leqslant_\Gamma (p_0, \vec{u}^0_0, \vec{u}^1_0)$ has a $<_\Gamma$-extension. So let $(p_1, \vec{u}^0_1, \vec{u}^1_1) \leqslant_\Gamma (p_0, \vec{u}^0_0, \vec{u}^1_0)$ be given.

[2] We point out that the previous series of lemmas can also be used to show that every sequence of ordinals in an $\mathbf{M}^{\mathcal{I}}_\alpha$-generic extension of V which is definable by a Σ_1 formula with parameters from $V \cup \{[x_G]_{\mathcal{J}}\}$ is already in V, where $[x]_{\mathcal{J}}$ denotes $\{x' \mid x \Delta x' \in \mathcal{J}\}$.

Since the claim fails, given any terminal sequence $\vec{n} \in a(u_1^0) \cap a(u_1^1)$, any $k < \mathrm{lh}(\vec{n})$, and

$$(p, \vec{u}^0, \vec{u}^1) \leqslant_\Gamma (p_1, \vec{u}_1^0, \vec{u}_1^1)$$

we can form an extension

$$(q, \vec{v}^0, \vec{v}^1) \leqslant_\Gamma (p, \vec{u}^0, \vec{u}^1)$$

such that there is $\vec{n}' \in a(v_1) \cap a(v_1)$ with $\vec{n}' \upharpoonright k = \vec{n} \upharpoonright k$ and $\vec{n}'(k) > \vec{n}(k)$.

In finitely many steps, construct a (finite) descending sequence

$$(p_1, \vec{u}_1^0, \vec{u}_1^1) \geqslant_\Gamma (p_2, \vec{u}_2^0, \vec{u}_2^1) \geqslant_\Gamma \ldots \geqslant_\Gamma (p_m, \vec{u}_m^0, \vec{u}_m^1),$$

at each step taking an extension of the previous element as just described. We can deal with each $\vec{n} \in a(\vec{u}_1^0) \cap a(\vec{u}_1^1)$ and each $k < \mathrm{lh}(\vec{n})$, so that at the end

$$a(\vec{u}_1^1) \cap a(\vec{u}_1^1) \sqsubset_\alpha a(\vec{u}_1^m) \cap a(\vec{u}_1^m).$$

Thus we have found $(p_m, \vec{u}_m^0, \vec{u}_m^1) \prec_\Gamma (p_1, \vec{u}_1^0, \vec{u}_1^1)$. Claim 3.3.24. $\square$

Let $(p_1, \vec{u}_1^0, \vec{u}_1^1) \leqslant_\Gamma (p_0, \vec{u}_0^0, \vec{u}_0^1)$, $\vec{n}^* \in a(\vec{u}_1^0) \cap a(\vec{u}_1^1)$, $l < \mathrm{lh}(\vec{n}^*)$ and $k^* \in \omega$ be as in the claim. By Lemma 3.3.22 and by replacing p_1 by a stronger condition if necessary, we may assume that $p_1 \in D(\vec{u}_1^0) \cap D(\vec{u}_1^1)$.

Case 1: Assume first that $l = 0$. Let $A' \subseteq M^\alpha$ be defined as in Lemma 3.3.21(3), namely

$$A' = \{\vec{n} \mid (\exists p' \leqslant p_1)(\exists \vec{u}' \leqslant_U \vec{u}_1^0)\ \vec{n} \in a(\vec{u}') \wedge p' \Vdash \vec{u}' \in U^{x_{\dot{G}}}\}$$

By Lemma 3.3.21(3), $A' \in \mathcal{I}^+$. Find $A \subseteq A(p_1)$ such that $A \in \mathcal{I}^{++}$ and letting $k^{**} = \max(\mathrm{dom}(a(p_1)))$ we have

$$A(p_1)(i) = A(i),$$

for $i \leqslant k^{**}$ while

$$A \cap \Big(\bigcup_{i > k^{**}} \{i\} \times M^{\delta(i)} \Big) \subseteq A'.$$

Letting $p^* = (a(p_1), A)$ we obtain a condition in $\mathbf{M}_\alpha^{\mathcal{I}}$ such that $p^* \leqslant p_1$. Since $p^* \Vdash \vec{u}_1^1 \in U^{x_{\dot{G}}}$, and we can find $p \leqslant p^*$, $\vec{u}$, and $\vec{n}$ with $\vec{n}(0) > k^*, k^{**}$ such that

$$\vec{n} \in a(\vec{u}) \wedge p \Vdash \vec{u} \in U^{x_{\dot{G}}}_{[\vec{u}_1^1]},$$

it follows that $\vec{n} \in A'$. Hence by definition of A' we can find $p' \leqslant p_1$ and $\vec{u}'$ such that

$$\vec{n} \in a(\vec{u}') \wedge p' \Vdash \vec{u}' \in U^{x_{\dot{G}}}_{[\vec{u}_1^0]}.$$

By extending p, p' if necessary, we can assume that $a(p)$ and $a(p')$ have the same lexicographically maximal element. As $p_1 \in D(\vec{u}_1^0) \cap D(\vec{u}_1^1)$ and $p, p' \leqslant p_1$, we can find $\vec{u}^0$ and $\vec{u}^1$ such that

$$\vec{n} \in a(\vec{u}^0) \wedge (a(p), A(p_1)/a(p)) \Vdash \vec{u}^0 \in U^{x_{\dot{G}}}_{[\vec{u}_1^0]}$$

and

$$\vec{n} \in a(\vec{u}^1) \wedge (a(p'), A(p_1)/a(p')) \Vdash \vec{u}^1 \in U^{x_{\dot{G}}}_{[\vec{u}_1^1]}.$$

Let $a = a(p) \cap a(p')$ (whose lexicographically maximal element is also that of $a(p)$ as well as that of $a(p')$). For each $i \in \{0, 1\}$ we have

$$a(\vec{u}_1^i) \cup \{\vec{n}\} \subseteq a \subseteq a(p), a(p')$$

and so by Lemma 3.3.23

$$\big(a, A(p_1)/a\big) \Vdash (a(\vec{u}_1^i) \cup \{\vec{n}\}, t(\vec{u}^i)) \in U^{x_{\dot{G}}}_{[\vec{u}_1^i]}$$

for each $i \in \{0, 1\}$. Letting

$$p_2 = \big(a, A(p_1)/a\big)$$

and

$$u_2^i = (a(\vec{u}_1^i) \cup \{\vec{n}\}, t(\vec{u}^i))$$

for $i \in \{0, 1\}$ we obtain $(p_2, \vec{u}_2^0, \vec{u}_2^1) \leqslant_\Gamma (p_1, \vec{u}_1^0, \vec{u}_1^1)$ with $\vec{n} \in a(\vec{u}_2^0) \cap a(\vec{u}_2^1)$ and $\vec{n}(0) > k^*$, which contradicts the choice of $(p_1, \vec{u}_1^0, \vec{u}_1^1)$, $\vec{n}^*$, and k^*.

Case 2: In case $l > 0$, let $\vec{n} = \vec{n}^* \restriction l$ and consider the set $A_{\vec{n}}$ defined as in Lemma 3.3.21(4), namely

$$A_{\vec{n}} = \{\vec{m} \mid (\exists p \leqslant p_1)(\exists \vec{u} \leqslant_U \vec{u}_1^0)\ \vec{m} \in a(\vec{u})(\vec{n}) \wedge p \Vdash \vec{u} \in U^{x_{\dot{G}}}\}.$$

Lemma 3.3.21(4) ensures $A_{\vec{n}} \in (\mathrm{Fin}^{\delta_\alpha(\vec{n})})^+$.

Let $k^{**} = \max(\{\vec{n}(l) \mid \vec{n} \in a(p_1)\})$. Find $A \subseteq A(p_1)$ such that $A \in (\mathrm{Fin}^{\delta(\vec{n})})^{++}$,

$$A(p_1)(\vec{n}) \cap \{\vec{m} \in \omega^{<\omega} \mid \vec{m}(0) \leqslant k^{**}\} \subseteq A$$

and

$$A(\vec{n}) \cap \{\vec{m} \in \omega^{<\omega} \mid \vec{m}(0) > k^{**}\} \subseteq A_{\vec{n}}.$$

By choice of k^{**}, letting $p^* = (a(p_1), A)$ we obtain a condition in $\mathbf{M}^{\mathcal{I}}_\alpha$, $p^* \leqslant p_1$.

Since $p^* \Vdash \vec{u}_1^1 \in U^{x_{\dot{G}}}$ we can find $p \leqslant p^*$ and $\vec{u} \leqslant_U \vec{u}_1^1$ such that there exists a terminal sequence $\vec{m} \in a(\vec{u})$ with $\vec{m}(l) > k^*, k^{**}$ and such that

$$p \Vdash \vec{u} \in T^{x_{\dot{G}}}_{[\vec{u}_1^1]}.$$

By definition of A we infer $\vec{m} = \vec{n} ^\frown \vec{m}'$ for some $\vec{m}' \in A_{\vec{n}}$, and so we can find $p' \leqslant p_1$ and $\vec{u}' \leqslant_U \vec{u}_1^0$ such that

$$\vec{m} \in a(\vec{u}') \wedge p' \Vdash \vec{u}' \in T^{x_{\dot{G}}}_{[\vec{u}_1^0]}.$$

Using that $p_1 \in D(\vec{u}_1^0) \cap D(\vec{u}_1^1)$ and Lemma 3.3.23, argue verbatim as in the previous case to construct $(p_2, \vec{u}_2^0, \vec{u}_2 1) \leqslant_\Gamma$ with $\vec{m} \in a(\vec{u}_2^0) \cap a(\vec{u}_2 1)$. Since $\vec{m}(l) > k^*$, this contradicts the choice of $(p_1, \vec{u}_1^0, \vec{u}_1^1)$, $\vec{n}^*$, l, and k^*.

Branch Lemma 3.3.19. $\square$

Chapter 4

Related questions

There are several questions related to the non-existence results from Chapter 3 that could be interesting to investigate further. We will discuss some of them in this section, and start with a question that was asked in [5].

The results of this thesis show that for $\mathcal{J}$ in a rather vast class of Borel ideals in ω, one can prove that there are no definable $\mathcal{J}$-MAD families, under suitable assumptions on either what definable means, or what background theory is adopted.

It is worth noting that it is *not* the case that such a theorem is true for *every* Borel ideal on ω. Indeed, the ideal on $\omega \times \omega$, defined by

$$\mathcal{J} = \{x \subseteq \omega \times \omega : (\forall n \in \omega)\{m : (n, m) \in x\} \text{ is finite}\}$$

clearly admits the $\mathcal{J}$-MAD family, namely $\{\{i\} \times \omega : i \in \omega\}$.

It remains an interesting open problem if it is possible to characterize the Borel ideals for which an analogue of Theorem 1.0.5 is true, and for which that type of theorem fails. In other words:

Question 4.0.1. Is there a dichotomy for Borel ideals on ω which characterizes when there are/are no definable MAD families with respect to a given Borel ideal?

Secondly, we could ask the following question, which is related to Törnquist's result that there are no infinite MAD families in Solovay's model:

Question 4.0.2. For which Borel ideals $\mathcal{J}$ are there no infinite $\mathcal{J}$-MAD families in Solovay's model?

Törnquist's proof was based on his new proof that there are no analytic MAD families [21]. In this proof, an ordinal analysis of a tree representation of the MAD families is used to, for an analytic family $\mathcal{A}$ of subsets of ω, construct a sequence $(A_n)_{n\in\omega}$ of subsets of ω such that any $x \in \mathcal{A}$ is almost contained in the union of finitely many A_n, and ω without any finite union of A_n is always infinite. If such a sequence exists, it is not hard to construct an element which can be added to $\mathcal{A}$ without affecting the almost disjointness. In other words, $\mathcal{A}$ is not maximal. The sequence is constructed by defining a subtree of the tree representation of $\mathcal{A}$ called a *diagonal sequence* and proving that the tree representation admits an infinite such diagonal sequence. The leaves of this diagonal sequence give rise to a sequence $(A^0_n)_{n\in\omega}$. Then a tree derivative argument allows us to only consider the nodes with some extension that is not almost covered by a finite union of this sequence, and repeat the procedure at most countably many times. The proof that there are no infinite MAD families in Solovay's model mimics this, replacing the tree with a poset $\mathbb{P}$ and the real with a $\mathbb{P}$-name.

It would be interesting to investigate whether there is a corresponding sequence $(A_n)_{n\in\omega}$ of subsets of $\omega \times \omega$ in the case $\mathcal{I} = \mathrm{Fin}\otimes\mathrm{Fin}$, and if so, whether this sequence can be produced by ordinal analysis of the tree representation in a similar way. If this is the case, then we could hope that the proof could be mimicked in a similar way replacing the tree with a poset, in order to answer Question 4.0.2 positively when $\mathcal{I} = \mathrm{Fin}\otimes\mathrm{Fin}$.

In the case of Fin, it is not so hard to see that everytime the tree splits, it is possible to extend the nodes so that their extensions are almost disjoint. This ensures that the sets arising from the diagonal sequence are in fact almost disjoint. This does not happen in the more complex case of a $\mathrm{Fin}\otimes\mathrm{Fin}$-AD family. In the earlier mentioned recent proof that there are no analytic $\mathrm{Fin}\otimes\mathrm{Fin}$-MAD families (which was a proof by forcing) a countably infinite process was needed in order to approximate the extended tree in the ground model, and this could pose a challenge with respect to finding a tree derivative proof. However, we might be able to reuse the idea of considering pairs (s, t) of nodes and add information about the intersection of extensions x and y of s and t respectively, and perform a tree derivative argument on this expanded tree instead.

If we succeed in performing this kind of tree analysis for some Borel ideal $\mathcal{J}$, for instance $\mathcal{J} = \mathrm{Fin}\otimes\mathrm{Fin}$, then there is also another direction in which this result could be generalized. In order to look into this, we need to know a little bit about Martin's axiom.

Martin's axiom is an axiom introduced by Martin and Solovay in 1970 [14]. It is independent of ZFC, consistent with the negation of CH, and implied by CH. It has many interesting comsequences, but is by many mathematicians considered less intuitive than many other possible axioms. First we define the following statement:

MA(κ): If $(\mathbb{P}, \leqslant, 1)$ is a non-empty poset which is c.c.c., and $\mathcal{D}$ is a family of dense subsets of $\mathbb{P}$ such that $|\mathcal{D}| \leqslant \kappa$, then there is a filter G in $\mathbb{P}$ such that $(\forall D \in \mathcal{D})\ G \cap D \neq \varnothing$.

Now Martins' axiom says: $(\forall \kappa < 2^{\omega})$ MA(κ).

In the case of Fin, the tree analysis allowed Törnquist to, with small alterations, conclude that if MA(κ) holds for $\kappa < 2^{\aleph_0}$, then there are no infinite κ-Suslin MAD families. A natural question to ask is therefore the following:

Question 4.0.3. Suppose MA(κ) holds for some $\kappa < 2^{\aleph_0}$. For which Borel ideals $\mathcal{I}$ can we conclude that there are no infinite κ-Suslin $\mathcal{I}$-MAD families?

Bibliography

[1] Morton Davis, *Infinite games of perfect information*, Annals of Mathematical Studies **52** (1964), 85–101.

[2] Kenny Easwaran, *A cheerful introduction to forcing and the continuum hypothesis*, arXiv:0712.2279., 2007.

[3] Su Gao, *Invariant descriptive set theory*, Chapman and Hall/CRC Pure and Applied Mathematics, 2008.

[4] Kurt Gödel, *The consistency of the axiom of choice and of the generalized continuum-hypothesis*, Proceedings of the National Academy og Sciences on the United States of America **12** (1938), 556–557.

[5] Karen Haga, David Schrittesser, and Asger Törnquist, *Maximal almost disjoint families, determinacy, and forcing*, arXiv:1810.03016., October 2018.

[6] Leo A. Harrington, Alexander S. Kechris, and Alain Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, J. Amer. Math. Soc. **3** (1990), no. 4, 903–928.

[7] Haim Horowitz and Saharon Shelah, *Can you take Törnquist's inaccessible away?*, arXiv:1605.02419., May 2016.

[8] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513

[9] Alexander S. Kechris, *The axiom of determinacy implied dependent choices in* $L(\mathbb{R})$, The Journal of Symbolic Logic **49** (1984), no. 1, 161–173.

[10] ———, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR 1321597

[11] Peter Koellner and W. Hugh Woodin, *Large cardinals from determinacy*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1951–2119. MR 2768702

[12] Kenneth Kunen, *Set theory.*

[13] Donald A. Martin, *Borel determinacy*, Annals of Mathematics **102** (1975), no. 2, 363–371, Second Series.

[14] Donald A. Martin and Robert M. Solovay, *Internal cohen extensions*, Annals of Mathematical Logic **2** (1970), 143–178.

[15] Donald A. Martin and John R. Steele, *A proof of projective determinacy*, Journal of the American Mathematical Society **2** (1989), no. 1, 71–125.

[16] A. R. D. Mathias, *Happy families*, Ann. Math. Logic **12** (1977), no. 1, 59–111. MR 0491197

[17] Krzysztof Mazur, *F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$*, Fund. Math. **138** (1991), no. 2, 103–111. MR 1124539

[18] Sandra Müller, Ralf Schindler, and W. Hugh Woodin, *Mice with finitely many Woodin cardinals from optimal determinacy hypotheses*, Submitted.

[19] Itay Neeman, *Optimal proofs of determinacy*, Bull. Symbolic Logic **1** (1995), no. 3, 327–339.

[20] Itay Neeman and Zach Norwood, *Happy and mad families in $L(\mathbb{R})$*, J. Symb. Log. **83** (2018), no. 2, 572–597. MR 3835078

[21] Asger Törnquist, *Definability and almost disjoint families*, Adv. Math. **330** (2018), 61–73. MR 3787540

[22] Samy Zafrany, *Borel ideals vs. Borel sets of countable relations and trees*, Annals of Pure and Applied Logic **43** (1989), 161 – 195.