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Boundaries, injective envelopes, and reduced crossed products

PhD Thesis

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Abstract

In this dissertation, we study boundary actions, equivariant injective envelopes, as well as the ideal structure of reduced crossed products. These topics have recently been linked to the study of C^*-simple groups, that is, groups with simple reduced group C^*-algebras.

In joint work with Matthew Kennedy, we consider reduced twisted crossed products over C^*-simple groups. For any twisted C^*-dynamical system over a C^*-simple group, we prove that there is a one-to-one correspondence between maximal invariant ideals in the underlying C^*-algebra and maximal ideals in the reduced crossed product. When the amenable radical of the underlying group is trivial, we verify a one-to-one correspondence between invariant tracial states on the underlying C^*-algebra and tracial states on the reduced crossed product.

In subsequent joint work with Tron Omland, we give criteria ensuring C^*-simplicity and the unique trace property for a non-ascending countable HNN extension. This is done by both purely algebraic and dynamical methods. Moreover, we also characterize C^*-simplicity of an HNN extension in terms of its boundary action on its Bass-Serre tree.

We finally consider equivariant injective envelopes of unital C^*-algebras, and relate the intersection property for group actions on unital C^*-algebras to the intersection property for the equivariant injective envelope. Moreover, we also prove that the equivariant injective envelope of the centre of the injective envelope of a unital C^*-algebra can be regarded as a C^*-subalgebra of the centre of the equivariant injective envelope of the original C^*-algebra.

Resumé

Denne afhandling omhandler randvirkninger, ækvivariante injektive hylstre, samt idealstrukturen af reducerede krydsprodukter. Disse emner er for nylig blevet knyttet til studiet af C^*-simple grupper, dvs. grupper hvis reducerede C^*-algebra er simpel.

I samarbejde med Matthew Kennedy betragter vi reducerede snoede krydsprodukter over C^*-simple grupper. For alle snoede C^*-dynamiske systemer over C^*-simple grupper viser vi, at der er en bijektiv korrespondence mellem maksimale invariante idealer i den underliggende C^*-algebra og maksimalidealer i det reducerede krydsprodukt. Under antagelse af at det amenable radikal af den underliggende gruppe er trivielt, påviser vi endvidere en bijektiv korrespondence mellem invariante sportilstande på den underliggende C^*-algebra og sportilstande på det reducerede krydsprodukt.

I efterfølgende samarbejde med Tron Omland giver vi kriterier for C^*-simplicitet og entydighed af spor for en ikke-opadgående tællelig HNN-udvidelse. Dette gøres både ved rent algebraiske og ved dynamiske metoder. Endvidere karakteriserer vi også C^*-simplicitet af HNN-udvidelser ved randvirkningen på deres Bass-Serre-træer.

Til sidst undersøger vi ækvivariante injektive hylstre af C^*-algebraer. Vi viser, at en virkning af en diskret gruppe på en C^*-algebra med enhed har skæringsegenskaben, hvis og kun hvis virkningen på dens ækvivariante injektive hylster har skæringsegenskaben. Ydermere viser vi, at det ækvivariante injektive hylster af centret af det injektive hylster af en C^*-algebra kan ses som en C^*-delalgebra af centret af det ækvivariante injektive hylster af den givne C^*-algebra.

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The past three years have been intensely interesting (or was that interestingly intense?), to say the least!

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Contents

Chapter 1

Introduction

"Über die Grenzen des All blicktest du sinnend hinaus;"

Richard Engländer (1859–1919), as Peter Altenberg

A discrete group is said to be C^**-simple* when the C^*-algebra associated to its left regular representation is simple. This property for discrete groups found its primus motor in a paper by Powers, who proved in 1975 that the non-abelian free group on two generators is C^*-simple. Since then, many other examples of C^*-simple groups of all sizes and shapes have been found, one of the chief C^*-simple group detectives being de la Harpe.

A common denominator for all C^*-simple groups is that they are always highly non-amenable, in the sense that they have *trivial amenable radical*, i.e., they admit no normal non-trivial amenable subgroups. Another property related to simplicity of the reduced group C^*-algebra is the *unique trace property*, i.e., that the reduced group C^*-algebra admits a unique tracial state. The unique trace property also implies triviality of the amenable radical. An early question of de la Harpe [41, p. 239] was whether there was any connection between the aforementioned properties. Until recently, no characterizations of C^*-simplicity nor the unique trace property were known, nor did there exist examples of groups that only satisfied one of these two properties. Only in 2014 did Kalantar and Kennedy obtain the first known characterization of C^*-simplicity [90], and later that year, Breuillard, Kalantar, Kennedy and Ozawa gave a characterization of the unique trace property in terms of its amenable radical [23]. By means of the result of Kalantar and Kennedy, de la Harpe's question was finally completely settled in 2015, when Le Boudec found examples of non-C^*-simple groups with the unique trace property, by examining actions of countable groups on trees [99].

The results of Kalantar and Kennedy were achieved by comparing two constructions with completely different backgrounds, namely the equivariant injective envelope of a C^*-algebra and the Furstenberg boundary of a discrete group. In 1979, Hamana proved that any unital C^*-algebra is contained in an injective C^*-algebra that is minimal with respect to this containment, called the *injective envelope* of the original C^*-algebra [73]. Six years later, Hamana generalized the result to G-C^*-algebras, meaning unital C^*-algebras with an action of a discrete group G by automorphisms. By requiring that the maps determining the injectivity of a G-C^*-algebra should also be G-equivariant, or that the G-C^*-algebra is G-injective, Hamana obtained the existence of a G-C^*-algebra, containing the original G-C^*-algebra as an invariant C^*-subalgebra, which is G-injective and minimal in the same sense as above [79]. The second type of construction originated with Furstenberg in the 1960's. He considered what came to be known as boundary actions, in order to investigate the irreducible unitary representations of a group. A boundary action is a minimal action of a group on a compact Hausdorff space that is *strongly proximal*, i.e., each orbit in the space of probability measures on the space contains a point mass. Furstenberg proved that any locally compact group G always admits a universal boundary action [58], meaning a compact Hausdorff space $\partial_F G$ that maps uniquely G-equivariantly onto any other compact Hausdorff space with a boundary action of the group G.

The realization of Kalantar and Kennedy was that the G-injective envelope of the smallest possible G-C^*-algebra $\mathbb{C}$, or equivalently, the smallest G-injective G-C^*-algebra, was in fact isomorphic to the commutative C^*-algebra $C(\partial_F G)$ for any discrete group G. By means of Hamana's results on G-injective envelopes as well as a result due to Archbold and Spielberg [5], they were able to show that a group G is C^*-simple if and only if the action of G on the Furstenberg boundary $\partial_F G$ is topologically free. Later that year, Breuillard, Kalantar, Kennedy and Ozawa used the relationship between equivariant injective envelopes and the Furstenberg boundary to show that the unique trace property of a group is equivalent to the group having trivial amenable radical. Because of the above characterizations, as well as their close affiliation with non-amenability, C^*-simplicity and the unique trace property have since witnessed a spike in interest.

The aim of the present thesis is to widen the understanding of boundary actions, C^*-simplicity *and* injective envelopes, as well as to indicate that the connection between these topics, at the base of the Kalantar-Kennedy theorem, perhaps goes much deeper. In an effort to increase the readability and comprehensilibity of our results, as well as inspire future research, we have elected to give a full treatment of all the theory needed in the original 2014 proof, as well as a parade of examples of varying degrees of generality.

Let us now give a brief review of our main results.

- In Sections 3.5 and 3.6, we consider non-ascending countable HNN extensions and give criteria for these to be C^*-simple and have trivial amenable radical (Theorem 3.5.7 and Proposition 3.6.4). These criteria improve upon previously known ones given by de la Harpe and Préaux [43]. Furthermore, we prove that a non-ascending countable HNN extension Γ is C^*-simple if and only if the subgroup of elements $g \in \Gamma$ whose fixed point sets have non-empty interior is C^*-simple (Theorem 3.6.8). This is joint work with Tron Omland.
- In Sections 4.2, 4.3 and 4.4, we investigate twisted C^*-dynamical systems over C^*-simple groups and generalize results of Bédos and Conti [12] on the ideal structure of reduced twisted crossed products. More precisely, we prove that any reduced twisted crossed product of a unital C^*-algebra A over a C^*-simple group G admits a one-to-one correspondence from its set of maximal ideals to the set of maximal G-invariant ideals of A (Theorem 4.3.4). Furthermore, if we only assume that G has trivial amenable radical, there is a one-to-one correspondence between tracial states on the reduced twisted crossed product and G-invariant tracial states on A (Corollary 4.4.2). Therefore reduced twisted group C^*-algebras of C^*-simple groups are simple and have a unique tracial state. This is joint work with Matthew Kennedy [27].

- In Section 5.4, we prove that the action of a discrete group G on a unital C^*-algebra A has the intersection property if and only if the G-action on the G-injective envelope $I_G(A)$ has the intersection property (Theorem 5.4.3). This generalizes a recent result by Kawabe [92, Theorem 3.4]. We also prove for all unital C-C^*-algebras A that there exists a G-equivariant injective *-homomorphism from $I_G(Z(I(A)))$, the G-injective envelope of the centre of the injective envelope $I(A)$, into $Z(I_G(A))$, the centre of $I_G(A)$ (Theorem 5.4.8) [26].

We finally give an overview of what is contained in the present thesis.

Chapter 2. Here, we give an introduction to proximal actions and boundary actions, both notions due to Furstenberg [58], and we give proofs of the existence and uniqueness of the universal minimal compact space of a group, as well as for the universal minimal compact proximal space and the Furstenberg boundary. We then give examples of boundary actions; this includes detailed introductions to the Gromov boundary of a hyperbolic metric space, as well as boundary actions on countable trees. We explain the relation between amenability and boundary actions, by means of a thorough discussion of group actions on compact convex spaces, as well as an important theorem of Furman [56]. Almost every result in this chapter is stated in the fullest possible generality, meaning that the groups considered need not be discrete.

In Section 2.5, we consider the question of whether the action of a discrete group G on its universal minimal compact space is proximal or strongly proximal. Our conjecture is that proximality and strongly proximality only holds if $G = \{1\}$, though we have not been able to prove why that is. Nonetheless, we give a strategy of how an eventual proof might proceed, along with many examples of groups for which the answer to the question is negative, by means of known properties of proximal and strongly proximal spaces.

Chapter 3. This chapter contains an introduction to *and* a brief history of C^*-simplicity and the unique trace property for discrete groups, as well as a proof of Kalantar and Kennedy's theorem by means of the theory established in the previous chapter. We also give half a proof of the equivalence of the unique trace property and having trivial amenable radical, due to Breuillard, Kalantar, Kennedy and Ozawa. Finally, we give an account of the stability properties of C^*-simplicity and having trivial amenable radical, as well as a selection of examples of C^*-simple discrete groups. The chapter is concluded by our study of when HNN extensions are C^*-simple, in which we take both a purely algebraic point of view, and a geometric point of view, the latter made possible by every HNN extension admitting an action on its Bass-Serre tree.

Chapter 4. In this chapter, we give a thorough introduction to the notion of a twisted C^*-dynamical system and its crossed products. We apply another characterization of C^*-simplicity (due to Breuillard, Kalantar, Kennedy and Ozawa [23, Section 3]) to obtain approximation and ideal structure results for reduced twisted crossed products over C^*-simple groups. Aside from our main results mentioned above, we are also able to determine the ideals of certain reduced group C^*-algebras that are close to being C^*-simple, by means of a structure theorem due to Bédos [9], as well as select properties of reduced crossed products in terms of properties of the reduced crossed product by a normal icc subgroup.

Chapter 5. In this chapter, we give an introduction to the G-equivariant injective envelope of a G-C^*-algebra for a discrete group G, a construction due to Hamana. We give a self-contained proof of its existence and uniqueness, as well as an account of some its most basic properties. Furthermore, we explain in detail Hamana's results on the relation between the injective envelope of a reduced crossed product of a C^*-algebra A by G and the reduced crossed product of the G-injective envelope of A by G. In our own work (Section 5.4), we consider the relation between ideals in reduced crossed products and ideals in reduced crossed products over the G-injective envelope. Furthermore, in an attempt to generalize techniques of Breuillard, Kalantar, Kennedy and Ozawa from [23], we obtain the inclusion of centres of equivariant injective envelopes mentioned above, and derive some notable applications from relating this inclusion to our ideal structure results.

1.1 Notation

Before delving into the main part of the thesis at hand, we take the opportunity to introduce some notation and terminology.

All topological groups considered throughout are assumed to be Hausdorff, but will be discrete from Chapter 3 onward. All compact spaces are also assumed to be Hausdorff, unless otherwise stated. The complex numbers, real numbers, integers and positive integers (with 0) are denoted by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Z}_+$, respectively.

For any subset S of a set X, $\chi_S\colon X \to \{0,1\}$ denotes the characteristic function mapping each $x \in S$ to 1 and each $x \in X \setminus S$ to 0. For any $x \in X$, the *point mass* δ_x is the function $\chi_{\{x\}}\colon X \to \{0,1\}$. The set X will always be clear from the context. The identity map on X is denoted by id_X.

A group will usually be denoted by G, H or N, and elements in groups are typically denoted by g, h, s, or t. The identity element of a group is always denoted by 1. Now let G be a group. For any $g \in G$, the subgroup generated by g is denoted by $\langle g \rangle$. The automorphism group of a group G is denoted by $\mathrm{Aut}(G)$, and the inner automorphism $G \to G$ given by $s \mapsto gsg^{-1}$, is denoted by $\mathrm{Ad}(g) \in \mathrm{Aut}(G)$. The *centralizer* $C_G(S)$ of a subset S of a group G is the subset of all elements of G commuting with every element of S. If every non-trivial conjugacy class in a group is infinite, we say that the group is *icc*.

If a group G acts on a set X, we will usually suppress the symbol of the map from G into the symmetric group of X. Thus the image of (g,x) will usually be written gx for all $g \in G$ and $x \in X$.

C^*-algebras are usually named A and B, or $\mathscr{M}$ and $\mathscr{N}$ if they are von Neumann algebras, and operator subsystems of a C^*-algebra will often be denoted by E or S. The identity element of a unital C^*-algebra A is denoted by 1_A, or 1, if A is clear from the context. A general operator in a C^*-algebra is usually denoted by a, b, x, or y. The centre of a C^*-algebra A is written $Z(A)$. The automorphism group of a C^*-algebra A is denoted by $\mathrm{Aut}(A)$. The unitary group of a unital C^*-algebra A is denoted by $\mathcal{U}(A)$, and if $u \in \mathcal{U}(A)$ is a unitary operator, the inner automorphism $x \mapsto uxu^*$ is denoted by $\mathrm{Ad}(u) \in \mathrm{Aut}(A)$. Every closed ideal in a C^*-algebra is assumed to be two-sided.

For any Hilbert space H, the C^*-algebra of bounded linear operators on H is denoted by $B(H)$, and the unitary group of $B(H)$ is denoted by $\mathcal{U}(H)$. The identity operator on H is denoted by 1_H, or 1 if H is clear from the context.

The dual space of complex-valued continuous linear functionals on a topological vector space X is denoted by X^*. The convex hull of a subset X of a linear space is denoted by $\mathrm{conv}X$, and if a topology is specified, the closure of the convex hull of X is denoted by $\overline{\mathrm{conv}}X$. If A is a C^*-algebra, the state space on A is denoted by $S(A)$.

The minimal tensor product of two C^*-algebras A and B is denoted by $A \otimes B$ [25, Section 3.3]. The C^*-algebra of n-by-n complex matrices is denoted by $M_n(\mathbb{C})$, and the n-by-n matrix algebra over an operator system is either denoted by $M_n(A)$, or $A \otimes M_n(\mathbb{C})$ or A is a C^*-algebra. The n-fold amplification of a linear map $\psi\colon A \to B$ to the n-by-n matrix algebras is analoguously denoted by either $\psi^{(n)}\colon M_n(A) \to M_n(B)$, or by means of the tensor product map $\psi \otimes \mathrm{id}_n\colon A \otimes M_n(\mathbb{C}) \to B \otimes M_n(\mathbb{C})$, where $\mathrm{id}_n\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is the identity map.

Finally, we will often consider completely positive maps between operator systems; to save space, we write *c.p.* whenever we mean completely positive. A *c.c.p.* map is a contractive c.p. map, and a *u.c.p.* map of operator systems is a c.p. map that is also *unital* or *identity-preserving*, meaning that it preserves identity elements.

Chapter 2

Topological dynamics

Consider for a moment the extended real line $\mathbb{R}^*$. Applying a fixed non-trivial translation of $\mathbb{R}$ to any bounded subset repeatedly, be it a point or a compact interval, this subset is moved closer to one of the two "end points" in the boundary of $\mathbb{R}$ in $\mathbb{R}^*$, namely $-\infty$ and $+\infty$. The concept of a boundary action was originally introduced by Furstenberg in 1970, the basic idea being to describe to what degree a fixed group of homeomorphisms of a space can treat any (or at least some) point in the space as if it were "infinity" in the sense of the above example.

The aim of this chapter is twofold: to give a thorough introduction to Furstenberg's [58] and Glasner's [63] notion of a boundary action, as well as to argue why the definitions are suitable by providing both concrete and general examples. In describing these dynamical phenomena, we will draw connections to objects associated to the space – in our case, the natural object to consider is the C^*-algebra of continuous complex-valued functions on the space. We also consider what properties a space or group must have in order to admit a boundary action.

2.1 Boundary actions

In this section we give the definition of a proximal and a strongly proximal action, as well as Furstenberg's proof that any Hausdorff topological group possesses a universal boundary.

Definition 2.1.1. Let X be a locally compact Hausdorff space and let G be a topological group. If X is equipped with a continuous action $G \times X \to X$, we say that X is a *G-space*.

A continuous map $f\colon X \to Y$ between G-spaces X and Y is said to be *G-equivariant* if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$.

Definition 2.1.2. Let G be a topological group and let X be a G-space. If every G-orbit is dense in X, we say that the action of G on X is *minimal*. If the action is understood, we say that X is a *minimal G-space*.

A *minimal subset* of a G-space X is a non-empty closed G-invariant subset M of X for which the action of G on M is minimal.

Remark 2.1.3. As in the above definition of minimality, if G is a topological group, X is a G-space and the action of G on X has some property, we will allow ourselves to say that X has the same property *as a G-space* if the action is understood.

If a G-space is compact (so that the collection of closed subsets has the finite intersection property), the following well-known result is an easy consequence of Zorn's lemma.

Proposition 2.1.4. *Let G be a topological group and let X be a compact G-space. Then X has a minimal subset.*

The first item on our agenda is to realize the following result, which constitutes one half of a 1960 theorem (Theorem 2.5.1) due to Ellis [50].

Theorem 2.1.5. *Every Hausdorff topological group G has a universal minimal compact G-space M_G, i.e., for every minimal compact G-space X there exists a continuous G-equivariant surjection $M_G \to X$.*

In Section 2.5 we will give the rest of Ellis' result, namely that a universal minimal compact G-space is unique up to G-equivariant homeomorphism, and give a more detailed account of the properties of this space (or, what properties it may not have).

The proof of Theorem 2.1.5 requires our allowing C^*-algebras into the frame; let us first comment on the relationship between actions on compact spaces X and actions on the C^*-algebra $C(X)$ by automorphisms. For any compact Hausdorff space X, we let $\mathcal{P}(X)$ denote the space of all Radon probability measures on X. By means of the Riesz representation theorem we identify $\mathcal{P}(X)$ with the state space of the unital C^*-algebra $C(X)$, equipped with the weak* topology.

For any compact G-space X, we may define an action of G by automorphisms on the C^*-algebra $C(X)$, by

$$gf := f \circ g^{-1}, \quad g \in G,\ f \in C(X).$$

We observe for any $f \in C(X)$ that the map $G \to C(X)$ given by $g \mapsto gf$ is norm-continuous. Defining the *dual action* of G by surjective isometries on the dual space $C(X)^*$ by $g\varphi = \varphi \circ g^{-1}$ for $g \in G$ and $\varphi \in C(X)^*$, then $\mathcal{P}(X)$ is a G-invariant, weak*-compact and convex subset of $C(X)^*$, making it a compact G-space.

In general, for any continuous linear map $\varphi\colon A \to B$ of normed spaces, the *dual map* is the map $\varphi^*\colon B^* \to A^*$ given by $\varphi^*(\psi)(x) = \psi(\varphi(x))$ for $\psi \in B^*$ and $x \in A$; thus $\psi \mapsto g\psi$ for $\psi \in C(X)^*$ is the dual map of the automorphism $f \mapsto g^{-1}f$ of $C(X)$.

The map $X \to \mathcal{P}(X)$ given by $x \mapsto \delta_x$, where δ_x denotes the point mass at x, is continuous and G-equivariant, and we denote its image by δ_X.

If A is a unital commutative C^*-algebra, then A is *-isomorphic to the C^*-algebra of continuous complex-valued functions on the compact Hausdorff space of characters $\hat{A}$. In this case, $\hat{A}$ can be identified with the *maximal ideal space* of A, which is the *nom de guerre* we will employ for this space. If G is a Hausdorff topological group and A is equipped with a continuous G-action by automorphisms, $\hat{A}$ is a weak*-closed subset of the weak*-compact unit ball of A^* and it is G-invariant with respect to the dual action of G on $C(\hat{A}) \cong A$ defined above, thus making it a compact G-space.

For any Hausdorff topological group G, we denote by $C_{\mathrm{b}}(G)$ the unital C^*-algebra of continuous, bounded complex-valued functions $G \to \mathbb{C}$ equipped with the uniform norm. An action of G on $C_{\mathrm{b}}(G)$ is then given by the action of G on itself by left translation, i.e.,

$$(gf)(s) = f(g^{-1}s), \quad g, s \in G,\ f \in C_{\mathrm{b}}(G).$$

Let $C_{\mathrm{b}}^{\mathrm{lu}}(G) \subseteq C_{\mathrm{b}}(G)$ denote the unital translation-invariant C^*-subalgebra of all bounded *left uniformly continuous* complex-valued functions on G, so that

$$C_{\mathrm{b}}^{\mathrm{lu}}(G) = \{f \in C_{\mathrm{b}}(G) \mid g \mapsto gf, \text{ is a norm-continuous map } G \to C_{\mathrm{b}}(G)\}.$$

In other words, a function $f\colon G \to \mathbb{C}$ is left uniformly continuous if for all $\varepsilon > 0$ there exists a neighbourhood U of 1 such that $\|gf - f\| < \varepsilon$ for all $g \in U$.

The following is based on the construction given in [138]. Let S_G be the maximal ideal space of $C_{\mathrm{b}}^{\mathrm{lu}}(G)$. The action of G on $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ by left translation then passes to a continuous G-action on S_G, so that S_G is a compact G-space. The map $G \to S_G$ given by $g \mapsto \delta_g$, δ_g denoting the evaluation map at g, defines an open embedding of G into S_G, as G is completely regular and points of G are separated by bounded left uniformly continuous functions [85, Theorem I.13]. Moreover, since S_G is the space of extreme points of the weak*-compact, convex space $\mathcal{P}(S_G) = S(C_{\mathrm{b}}^{\mathrm{lu}}(G))$ and the convex hull of $\delta_G = \{\delta_g \,|\, g \in G\}$ is weak*-dense in $S(C_{\mathrm{b}}^{\mathrm{lu}}(G))$, Milman's converse to the Krein-Milman theorem implies that δ_G is weak*-dense in S_G.

Definition 2.1.6. The compact G-space S_G constructed above is called the *greatest ambit of* G.

We note that the same considerations as above, but for $C_{\mathrm{b}}(G)$, defines the well-known *Stone-Čech compactification* βG of G, and that if G is discrete, then the greatest ambit and Stone-Čech compactification are identical.

The greatest ambit S_G of G has the following universal property: for any compact G-space X and any point $x_0 \in X$, there exists a unique continuous G-equivariant map $\omega\colon S_G \to X$ satisfying $\omega(1) = x_0$. Indeed, if $F\colon C(X) \to C_{\mathrm{b}}^{\mathrm{lu}}(G)$ is the *-homomorphism given by $F(f)(g) = f(gx_0)$, let ω be the restriction of the dual map of F to S_G. Uniqueness follows from density of G in S_G.

Proof of Theorem 2.1.5. If M_G is a minimal subset of S_G (Proposition 2.1.4), suppose that X is a minimal G-space and let $x_0 \in X$. The universal property of S_G yields the existence of a continuous G-equivariant map $f\colon S_G \to X$ such that $f(1) = x_0$. Restricting f to M_G finishes the proof. $\square$

In the above discussion we used the important property that a group action on a compact Hausdorff space can be seen as a group action on a C^*-algebra by automorphisms; let us quickly remind ourselves what we require of such an action.

Definition 2.1.7. Let G be a topological group and let A be a C^*-algebra. An *action* of G on A by automorphisms is a group homomorphism $\alpha\colon G \to \mathrm{Aut}(A)$ such that, writing $\alpha_g := \alpha(g)$ for $g \in G$, the map $G \to A$ given by $g \mapsto \alpha_g(a)$ is continuous for all $a \in A$. We say that A is a *G-C^*-algebra*, that α is the *G-action* on A and that the triple (A, G, α) is a *C^*-dynamical system*. If the action is understood, we will often suppress the name of the action and simply write (A, G) instead of (A, G, α), and ga instead of $\alpha_g(a)$ for all $g \in G$ and $a \in A$.

If A and B are G-C^*-algebras, a linear map $\varphi\colon A \to B$ is said to be *G-equivariant* if $\varphi(ga) = g\varphi(a)$ for all $g \in G$ and $a \in A$.

In the following we will restrict our attention to compact minimal G-spaces. The following notions are due to Furstenberg [58] and were later elaborated upon by Glasner in [63].

Definition 2.1.8. The action of a topological group G on a compact G-space X is said to be

- *proximal* if for any two points $x, y \in X$ there exists a net (g_i) in G such that $\lim_i g_i x = \lim_i g_i y$;
- *strongly proximal* if the weak* closure of every G-orbit in $\mathcal{P}(X)$ intersects δ_X.

A *G-boundary* is a compact G-space X on which the G-action is minimal and strongly proximal. In this case, we say that the action of G on X is a *boundary action*.

Notice that strong proximality combined with minimality yields that in a G-boundary X, δ_X is contained in the weak* closure of the G-orbit of any $\mu \in \mathcal{P}(X)$.

The next couple of observations are mainly due to Glasner [62, 63] (see also [58, Lemma 4.1]).

Lemma 2.1.9. *Let G be a Hausdorff topological group and let X be a compact G-space. Then $\mathcal{P}(X)$ is a proximal G-space if and only if X is a strongly proximal G-space. In particular, strong proximality implies proximality.*

Proof. If X is strongly proximal, then for $\mu, \nu \in \mathcal{P}(X)$ there is a net (g_i) in G such that $\frac{1}{2}(g_i\mu + g_i\nu) \to \delta_x$ for some $x \in X$ in the weak* topology. By compactness, we may assume that $(g_i\mu)$ and $(g_i\nu)$ both converge in $\mathcal{P}(X)$, in which case their limit is δ_x since δ_x is an extreme point of $\mathcal{P}(X)$. This proves that $\mathcal{P}(X)$ is a proximal G-space. Since δ_X is G-invariant and weak*-closed, we see that X is a proximal G-space whenever $\mathcal{P}(X)$ is. Finally, if $\mathcal{P}(X)$ is proximal, then for $\mu \in \mathcal{P}(X)$ and $x \in X$ there is a net (g_i) such that $\lim g_i\mu = \lim g_i\delta_x \in \delta_X$, meaning that X is strongly proximal. □

Remark 2.1.10. Let G be a Hausdorff topological group and let X be a proximal compact G-space. Then X has a unique minimal subset. Indeed, suppose that M_1 and M_2 are minimal subsets of X. Then for $x_i \in M_i$, $i = 1, 2$, there is a net (g_j) in G such that $\lim_j g_j x_i = y$ for some $y \in X$, meaning that $y \in M_1 \cap M_2$ and $M_1 = \overline{Gy} = M_2$.

The following proposition and its proof are extremely basic, but its descriptory value should definitely not be underestimated.

Proposition 2.1.11. *Let G be a Hausdorff topological group and let X be a minimal, proximal compact G-space. If X has an isolated point, then X is a one-point space.*

Proof. If $\{x\}$ were open for some $x \in X$, then for any $y \in X \setminus \{x\}$ there would exist a net (g_i) in G such that $\lim g_i x = x = \lim g_i y$. It follows that $g_i x = g_i y$ for some i, a contradiction. □

The above proposition has some interesting consequences. Suppose, for instance, that X is a minimal proximal compact G-space, and that X is disconnected. *Then X has uncountably many connected components.*

To see why, recall that the quasi-component of X containing $x \in X$ is the intersection of all clopen sets containing x. Since X is disconnected, X has at least two quasi-components. Defining an equivalence relation by saying that two points in X are equivalent if they belong to the same quasi-component, we obtain a quotient space X_0 of quasi-components, and the G-action on X passes to a continuous G-action on X_0. Moreover, X_0 is a compact Hausdorff G-space: to verify that X_0 is Hausdorff, simply observe for quasi-components $A \neq B$ that there are disjoint clopen sets U and $V = X \setminus U$ containing A and B, and that any quasi-component of X is either contained in U or V, so that their images in X_0 are disjoint and open. Via the quotient map from X, X_0 now has the structure of a minimal, proximal compact G-space, so it has no isolated points. Hence it is uncountable, and since any connected component is contained in a unique quasi-component, the claim follows. In fact, the quasi-components and connected components coincide, since X is a compact Hausdorff space (cf. [51, Theorem 6.1.23]). It is possible that any disconnected minimal proximal compact G-space X is totally disconnected, though we have not been able to prove this.

Both of the following lemmata are due to Glasner [62]:

Lemma 2.1.12. *Let X be a compact G-space.*

(i) *The action of G on X is proximal if and only if any minimal subset $M \subseteq X \times X$ is contained in the diagonal, when $X \times X$ is equipped with the diagonal G-action.*
(ii) *The action of G on X is strongly proximal if and only if any minimal subset $M \subseteq \mathcal{P}(X)$ is contained in δ_X.*

Proof. (i) The action of G on X is proximal if and only if the closure of any G-orbit in $X \times X$ intersects the diagonal Δ. The latter property implies that any minimal subset $M \subseteq X \times X$ contains (x, x) for some $x \in X$, so that $M = \overline{G(x, x)} \subseteq \Delta$. Conversely, if $x, y \in X$, then $\overline{G(x, y)} \subseteq X \times X$ contains a minimal subset M, so that $\overline{G(x, y)} \cap \Delta$ is non-empty if $M \cap \Delta$ is.

(ii) If the action of G on X is strongly proximal and $M \subseteq \mathcal{P}(X)$ is minimal, then $\delta_x \in M$ for some $x \in X$. Since $G\delta_x$ is weak*-dense in M, every element in M is a point mass. The converse is similar to what we saw in the proof of (i). □

One of the most important consequences of how we define a proximal action is that there is no room for equivariant homogeneity. We formulate this property as follows.

Lemma 2.1.13. *Let X be a minimal compact G-space. If the action of G on X is proximal, then the only G-equivariant continuous map $X \to X$ is the identity map.*

Proof. Let $\varphi\colon X \to X$ be a G-equivariant and continuous map. For $x \in X$, take a net (g_i) in G such that $x = \lim_i g_i x = \lim g_i \varphi(x)$. Then

$$\varphi(x) = \lim_i \varphi(g_i x) = \lim_i g_i \varphi(x) = x. \qquad \square$$

To get to the main result of this section, we now consider the question of whether proximality and strong proximality are preserved when forming products. For a continuous map $\rho\colon X \to Y$ between compact Hausdorff spaces, we define the *pushforward map* $\rho_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ to be the dual map of the induced *-homomorphism $\tilde{\rho}\colon C(Y) \to C(X)$, i.e.,

$$\rho_*(\mu)(f) = \mu(\tilde{\rho}(f)), \quad \mu \in \mathcal{P}(X), \ f \in C(Y).$$

Moreover, ρ_* is surjective (resp. injective) if and only if ρ is surjective (resp. injective).

Lemma 2.1.14. *Let $(X_i)_{i\in I}$ be a family of compact Hausdorff spaces, let $\mathcal{F}$ be the collection of finite subsets of I, let $X = \prod_{i\in I} X_i$ and for each $F \in \mathcal{F}$, let $\pi_F\colon X \to \prod_{i\in F} X_i$ be the projection onto the coordinates determined by F. If two subsets $A, B \subseteq \mathcal{P}(X)$ satisfy $(\pi_F)_*(A) \subseteq (\pi_F)_*(B)$ for all $F \in \mathcal{F}$ and B is weak*-closed, then $A \subseteq B$.*

Proof. Let $\mathscr{C}$ be the *-subalgebra of $C(X)$ consisting of functions $f \in C(X)$ of the form $f = \prod_{i\in F} f_i \circ \pi_i$ for $F \in \mathcal{F}$ and functions $f_i \in C(X_i)$, $i \in F$. Then $\mathscr{C}$ is norm-dense in $C(X)$ by the Stone-Weierstrass theorem. If $\mu \in A$ and $F \in \mathcal{F}$, let $\nu_F \in B$ such that $(\pi_F)_*(\mu) = (\pi_F)_*(\nu_F)$. Since B is weak*-compact, we may let $h\colon \Lambda \to \mathcal{F}$ be a monotone, final function such that $(\nu_{h(\lambda)})_{\lambda\in\Lambda}$ converges to some $\nu \in B$. For any $F \in \mathcal{F}$ and functions $f_i \in C(X_i)$ for $i \in F$, let $f = \prod_{i\in F} f_i \circ \pi_i$. For any $\lambda \in \Lambda$ such that $F \subseteq h(\lambda)$, define $f_\lambda \in C(\prod_{i\in h(\lambda)} X_i)$ by $f_\lambda((x_i)_{i\in h(\lambda)}) = \prod_{i\in F} f_i(x_i)$. Since $f = f_\lambda \circ \pi_{h(\lambda)}$, we see that

$$\int f \,\mathrm{d}\mu = \int f_\lambda \,\mathrm{d}(\pi_{h(\lambda)})_*(\mu) = \int f_\lambda \,\mathrm{d}(\pi_{h(\lambda)})_*(\nu_{h(\lambda)}) = \int f \,\mathrm{d}\nu_{h(\lambda)}.$$

Hence $\int f \,\mathrm{d}\mu = \int f \,\mathrm{d}\nu$, so $\mu = \nu$ since $\mathscr{C}$ is dense in $C(X)$. □

Lemma 2.1.15. *Let G be a topological group, let $(X_i)_{i\in I}$ be a family of compact G-spaces and let $X = \prod_{i\in I} X_i$ be the product space equipped with the diagonal G-action. Then the action of G on X is proximal (resp. strongly proximal) if and only if the action of G on X_i is proximal (resp. strongly proximal) for all $i \in I$.*

Proof. Since each X_i is a G-equivariant quotient of X, "only if" is easy in both the proximal and strongly proximal case. If the action of G on each X_i is proximal, then let $M \subseteq X \times X$ be a minimal subset with respect to the diagonal G-action. If $\pi_i\colon X \to X_i$ is the canonical projection, then since $\pi_i \times \pi_i\colon X \times X \to X_i \times X_i$ is G-equivariant, it follows that $(\pi_i \times \pi_i)(M)$ is a minimal subset of $X_i \times X_i$. Thus $(\pi_i \times \pi_i)(M)$ is contained in the diagonal of $X_i \times X_i$, so M is contained in the diagonal of $X \times X$.

For the strongly proximal case, first assume that I is finite, in which case it suffices to prove the case $|I| = 2$. Therefore let X and Y be strongly proximal compact G-spaces and let $\mu \in \mathcal{P}(X \times Y)$. If $\pi\colon X \times Y \to X$ is the projection onto X, then because $\overline{G\pi_*(\mu)} = \pi_*(\overline{G\mu})$, there exist $x \in X$ and $\mu' \in \overline{G\mu}$ such that $(\pi_1)_*(\mu') = \delta_x$. As μ' is concentrated on $\pi_1^{-1}(\{x\})$, there exists $\nu \in \mathcal{P}(Y)$ such that $\mu' = \delta_x \otimes \nu$. By hypothesis there exist a net (g_k) in G and $y \in Y$ such that $g_k\nu \to \delta_y$. By compactness we may assume that $g_k x \to x'$ for some $x' \in X$, in turn implying $\delta_{(x',y)} = \lim_k g_k\mu' \in \overline{G\mu}$.

If I is infinite, let $\mathcal{F}$ be the collection of finite subsets of I, let $F \in \mathcal{F}$ and let $\pi_F\colon X \to \prod_{i\in F} X_i$ be the projection of Lemma 2.1.14. If M is a minimal subset of $\mathcal{P}(X)$, then for all $F \in \mathcal{F}$ we see that $(\pi_F)_*(M)$ is minimal in $\mathcal{P}(X_F)$. By Lemma 2.1.12 it follows that $(\pi_F)_*(M) \subseteq \delta_{\pi_F(X)} = (\pi_F)_*(\delta_X)$, so since F was arbitrary, $M \subseteq \delta_X$ by Lemma 2.1.14. Hence the action of G on X is strongly proximal. □

Our first application of the above result, and an observation due to Rørdam, states that proximal actions on compact spaces are in fact much more flexible than the definitions immediately suggest. The proof essentially follows from the Alaoglu theorem.

Proposition 2.1.16. *Let G be a topological group, let X be a minimal compact G-space and let $Y \subseteq \mathcal{P}(X)$ be a weak*-closed G-invariant subset such that $Y \cap \{\delta_x \mid x \in X\} \neq \emptyset$*

and the G-action on Y is proximal. Then for all $x \in X$, there exists a net (g_i) in G such that $g_i\mu \to \delta_x$ for all $\mu \in Y$.

Proof. For all $\mu \in Y$, let $Y_\mu = Y$ and consider the compact space

$$\mathscr{Y} = \prod_{\mu \in Y} Y_\mu$$

with the diagonal G-action. For any $x \in X$ such that $\delta_x \in Y$, we will let x' denote the point $(\delta_x)_{\mu \in Y}$ of $\mathscr{Y}$. Define $\xi = (\xi_\mu)_{\mu \in Y} \in \mathscr{Y}$ by $\xi_\mu = \mu$ for $\mu \in Y$. Picking $x \in X$ such that $\delta_x \in Y$, then since Y is proximal, $\mathscr{Y}$ is too, by Proposition 2.1.15. Therefore there is a net (t_i) in G such that $\lim t_i\xi = \lim t_i x'$. In particular, $(t_i x)$ converges in X, so there is $y \in X$ such that $y' \in \mathscr{Y}$ belongs to the weak*-closure of the G-orbit $G\xi$. By minimality, we may take a net (s_j) in G such that $s_j y \to x$. Then $s_j y' \to x'$, so x' belongs to the weak* closure of $G\xi$ as well. Hence there is a net (g_m) such that $g_m\xi \to x'$ in the weak* topology which in turn implies $g_m\mu \to \delta_x$ for all $\mu \in Y$. □

Corollary 2.1.17. *If X is a minimal proximal compact G-space (resp. a G-boundary), then for all $x \in X$ there exists a net (g_i) in G such that $g_i y \to x$ for all $y \in X$ (resp. $g_i\mu \to \delta_x$ for all $\mu \in \mathcal{P}(X)$).*

We finally arrive at the pivotal result(s) of this section, due to Furstenberg.

Theorem 2.1.18. *For any Hausdorff topological group G, there exists a universal minimal proximal compact G-space $\Pi(G)$, and a universal G-boundary $\partial_F G$, universality meaning that for any minimal proximal compact G-space (resp. G-boundary X), there exists a G-equivariant continuous surjection $\Pi(G) \to X$ (resp. $\partial_F G \to X$). Both spaces are unique up to G-equivariant homeomorphism.*

Proof. If X_1 and X_2 are universal minimal proximal compact G-spaces or boundaries of G, there exist continuous surjective G-equivariant maps $\varphi_1 \colon X_1 \to X_2$ and $\varphi_2 \colon X_2 \to X_1$. Composing these maps, it follows from Proposition 2.1.13 that φ_1 and φ_2 are inverses of one another, so X_1 and X_2 are G-equivariantly homeomorphic.

We will only prove existence and uniqueness in the proximal case – the other case adapts *mutatis mutandis*. Let M_G be a universal minimal compact G-space. Then any minimal compact G-space X is the surjective image of a G-equivariant continuous map from M_G, so that

$$|X| \leq |M_G| \leq 2^{2^{|G|}}.$$

We may therefore index all minimal proximal G-spaces up to G-equivariant homeomorphism classes. Forming their product Z, it is a proximal, compact G-space by Lemma 2.1.15. Now let $\Pi(G)$ be the unique minimal subset of Z (Remark 2.1.10). As any minimal proximal compact G-space is an image of Z, we conclude that $\Pi(G)$ is a universal minimal proximal compact G-space. □

The uniqueness part of the above theorem ensures that we may speak of *the* universal minimal proximal G-space $\Pi(G)$ and *the* universal G-boundary $\partial_F G$. We will add one caveat here: the word "universality" as used in this chapter for the spaces M_G, $\Pi(G)$ and $\partial_F G$ need not imply in itself that the G-equivariant surjections onto other G-spaces of the same kind are *uniquely* determined. Indeed, M_G may possess more than one

G-equivariant continuous self-map – we discuss this matter in more detail in Section 2.5. However, in Section 2.4 we shall see that for any G-boundary X, there is one and only one G-equivariant continuous surjection $\partial_F G \to X$. We are not sure whether a similar result holds for $\Pi(G)$.

Definition 2.1.19. The universal G-boundary $\partial_F G$ of a topological group G is called the *Furstenberg boundary*.

A different approach to constructing the Furstenberg boundary, observed by Hamana [72], as well as Kalantar and Kennedy [90], that is more operator-algebraic in nature, will be described in Section 5.1.

We finally discuss another way of constructing the universal minimal proximal compact G-space for a Hausdorff topological group, first devised by Melleray, Nguyen van Thé and Tsankov [103, Theorem 4.1].

Remark 2.1.20. If G is a Hausdorff topological group, let M_G be a universal minimal G-space and let Γ be the group of G-equivariant homeomorphisms of M_G. Let $A \subseteq C(M_G)$ be the unital C^*-subalgebra of Γ-invariant functions on M_G, and let P denote the maximal ideal space of A. Since A is G-invariant, P is a compact G-space equipped with a continuous G-equivariant map $\sigma\colon M_G \to P$ such that $\sigma(\gamma(x)) = \sigma(x)$ for all $x \in M_G$ and $\gamma \in \Gamma$. Notice also that minimality of M_G implies that of P. In fact, P is the universal minimal proximal G-space.

To verify that P is proximal, let $x, y \in M_G$ and let X be a minimal subset of $\overline{G(x,y)} \subseteq M_G \times M_G$ with respect to the diagonal G-action. Letting $\pi_1, \pi_2\colon X \to M_G$ be the projections onto the first and second factors, then π_1 and π_2 are both G-equivariant, and if $\varphi\colon M_G \to X$ is a G-equivariant surjection, then $\pi_i \circ \varphi$ is a homeomorphism for $i = 1, 2$ by Theorem 2.5.4 (allowing ourselves to run ahead just a bit). Hence π_1 and π_2 are homeomorphisms, so $\gamma = \pi_2\pi_1^{-1} \in \Gamma$. For $(x', y') \in X$, we now let (g_i) be a net in G such that $g_i(x, y) \to (x', y')$. As $\pi_2\pi_1^{-1}(x') = \pi_2(x', y') = y'$, it follows that $\lim_i g_i\sigma(y) = \lim_i g_i\sigma(x)$. Hence P is proximal.

Finally, if X is a minimal proximal compact G-space, let $\rho\colon M_G \to X$ be a G-equivariant surjection. For all $x \in M_G$ and $\gamma \in \Gamma$, there is a net (g_i) in G and $w \in X$ such that $w = \lim_i \rho(g_i x) = \lim_i \rho(g_i\gamma(x))$. Passing to a subnet, we may assume that $g_i x \to x'$ for some $x' \in M_G$. Then $\rho(x') = w = \rho(\gamma(x'))$. Now let (h_j) be a net in G such that $h_j x' \to x$, so that $\rho(\gamma(h_j x')) = h_j\rho(x') \to \rho(x)$ and $\rho(x) = \rho(\gamma(x))$. Therefore the image of the injective *-homomorphism $\tilde{\rho}\colon C(X) \to C(M_G)$ induced by ρ is contained in A, meaning that ρ factors through P.

2.2 Examples of boundary actions

This section contains exactly what the title hints at. We first give some concrete examples of boundary actions and non-boundary actions, and then proceed to give more general constructions (organized in subsections) that provide larger classes of examples.

We first give criteria ensuring strong proximality, due to Ozawa [111, Example 2].

Lemma 2.2.1. *Let X be a compact G-space and assume that $(g_n)_{n\geq 1}$ and $(h_n)_{n\geq 1}$ are sequences of G and that $x_g, y_g, x_h, y_h \in X$ are points such that $g_n x \to y_g$ for all*

$x \in X \setminus \{x_g\}$, $h_n x \to y_h$ for all $x \in X \setminus \{x_h\}$ and $g_n x_g = x_g$ and $h_n x_h = x_h$ for all $n \geq 1$. If $x_h \notin \{x_g, y_g\}$, then the action of G on X is strongly proximal.

Proof. Let $\mu \in \mathcal{P}(X)$ and write $\mu = \lambda\delta_{x_g} + (1-\lambda)\eta$ where $\eta \in \mathcal{P}(X)$ satisfies $\eta(\{x_g\}) = 0$. By Lebesgue's dominated convergence theorem, then for all $f \in C(X)$,

$$\int f \,\mathrm{d}(g_n\mu) = \lambda f(x_g) + (1-\lambda)\int f(g_n x)\,\mathrm{d}\mu(x) \to \lambda f(x_g) + (1-\lambda) f(y_g).$$

Thus $\lambda\delta_{x_g} + (1-\lambda)\delta_{y_g} = \lim_n g_n\mu \in \overline{G\mu}$, so $\delta_{y_h} = \lim_{n\to\infty} h_n(\lambda\delta_{x_g} + (1-\lambda)\delta_{y_g}) \in \overline{G\mu}$. $\square$

Evidently one can allow for more than two sets of "attractors and repellers" in order for the proof above still to hold (see also the subsection on weakly hyperbolic actions), but we will stick with this set-up for now.

This first example of a boundary action motivates the subsections on boundaries arising from hyperbolic metric spaces and countable trees; we will realize it by more general means shortly.

I. The canonical boundary of a non-abelian free group (Furstenberg [58, Example 4.B])**.** Let $\mathbb{F}$ be a non-abelian free group of finite rank, with $A \subseteq \mathbb{F}$ a free generating set. Let $Y = (A \sqcup A^{-1})^{\mathbb{N}}$ be the compact space of one-sided infinite words with alphabet $A \sqcup A^{-1}$. In order to obtain a well-defined action of $\mathbb{F}$ on a set of one-sided infinite words, let Y_n be the clopen subset $\{x \in Y \mid x_n x_{n+1} = 1 \text{ in } \mathbb{F}\}$ for $n \geq 1$ and define $X = Y \setminus \bigcup_{n=1}^{\infty} Y_n$. Representing each sequence $x = (x_n)_{n\geq 1} \in X$ by the infinite word $x = x_1 x_2 x_3 \cdots$, we say that $x \in Y$ is *reduced* if $x \in X$. Therefore X is a compact Hausdorff space, consisting of all infinite one-sided reduced words in Y. For all $n \geq 1$, define

$$\pi'_n(x)_i = \begin{cases} x_i & \text{if } i < n \\ x_{i+2} & \text{if } i \geq n, \end{cases} \quad \text{and } \pi_n(x) = \begin{cases} x & \text{if } x \notin Y_n \\ \pi'_n(x) & \text{if } x \in Y_n, \end{cases}$$

Then π_n is a map cancelling the n'th and $n+1$'st letter in a word in X *if* said letters can be cancelled, and it is continuous. For any reduced word $\omega \in \mathbb{F}$, let x_ω be the concatenation of ω and x without reduction. If n is the length of ω, we define $\omega x = (\pi_n^n \circ \cdots \circ \pi_1^n)(x_\omega)$. This yields a continuous action of $\mathbb{F}$ on X which is then given by concatenation of words in $\mathbb{F}$ and X with subsequent cancelling, should need be.

We claim that X is an $\mathbb{F}$-boundary. For $x, y \in X$ and $n \geq 1$ there exists $\omega \in \mathbb{F}$ such that ωx and y have the same first n letters, so that X is a minimal $\mathbb{F}$-space. Indeed, if y starts with the word ω' of length n, let $f \in A \cup A^{-1}$ be the last letter of ω'. Taking $g \in A \cup A^{-1} \cup \{1\}$ such that fgx_1 is a reduced word, we define $\omega = \omega' g$.

For strong proximality, take $a \in A$ and let a^∞ and $a^{-\infty}$ denote the words $aaa\cdots$ and $a^{-1}a^{-1}a^{-1}\cdots$ respectively. For any $x \in X \setminus \{a^{-\infty}\}$, let $k \geq 1$ be the smallest integer such that $x_{k+1} \neq a^{-1}$, so that the first $n-k$ letters of $a^n x$ all equal a. Hence $a^n x \to a^\infty$. If we let $b \in A \setminus \{a\}$ and construct b^∞ and $b^{-\infty}$ as above, then similarly we have $b^n x \to b^\infty$ for all $x \in X \setminus \{b^{-\infty}\}$. Since $a^n a^{-\infty} = a^{-\infty}$, $b^n b^{-\infty} = b^{-\infty}$ and $b^{-\infty} \notin \{a^{-\infty}, a^\infty\}$, Lemma 2.2.1 applies.

II. Actions on the circle (Glasner [63, Example III.7.2])**.** Let $\mathbb{T}$ denote the unit circle in the complex plane. If g is an irrational rotation, i.e., if we define $g(e^{2\pi i\theta}) = e^{2\pi i(\theta+r)}$ for $r \in \mathbb{R}\setminus\mathbb{Q}$ and $\theta \in \mathbb{R}$, then the orbit of any $x \in \mathbb{T}$ under g is dense in $\mathbb{T}$. If we let h be the homeomorphism of $\mathbb{T}$ given by $h(e^{2\pi i\theta}) = e^{2\pi i\theta^2}$ for $\theta \in [0,1]$, then $\lim_{n\to\infty} h^n x = 1$ for all $x \in \mathbb{T}$. Applying the Lebesgue dominated convergence theorem as in the proof of Lemma 2.2.1, it follows that $h^n\mu \to \delta_1$ for all $\mu \in \mathcal{P}(\mathbb{T})$, so if we let G denote the group of homeomorphisms generated by g and h, it follows that $\mathbb{T}$ is a G-boundary. (Actually, G is the free group on two generators in disguise.)

III. Boundaries of connected Lie groups (Furstenberg-Moore [57]). Let G be a connected, semisimple Lie group with finite centre. Then there exist a maximal compact subgroup K, a simply connected abelian subgroup A and a simply connected, normal, nilpotent subgroup N of G, such that every $g \in G$ has a unique factorization $g = kan$ for $k \in K$, $a \in A$ and $n \in N$; the identity $G = KAN$ is called an *Iwasawa decomposition* of G. If M is the centralizer of A in K, then $P = MAN$ is the normalizer of N in G and is called the *minimal parabolic subgroup* (see also [63, Section IV.3].

If H denotes the normalizer of AN in G, then G/H is a compact homogeneous G-space. What's more is that H actually equals P, and that the following holds ([58, Theorem 1.4], [105]).

Theorem 2.2.2. *For G and P as above, G/P is the Furstenberg boundary of G. Moreover, P is a maximal amenable subgroup of G and any boundary of G is of the form G/H where H is a closed subgroup containing a conjugate of P.*

In fact, Furstenberg devised the concept of a boundary action as a means to study harmonic functions on Lie groups. We refer to [63, Chapter IV] for a more thorough discussion of boundaries of Lie groups, but will briefly mention that a *parabolic subgroup* of G is defined to be a closed subgroup H of G such that G/H is a G-boundary.

IV. Projective special linear groups acting on projective space (Furstenberg [58, Example 4.A]). In the following, let $n \geq 2$ and define $(\mathbb{R}^n)_o = \mathbb{R}^n \setminus \{(0,\dots,0)\}$. Recall that *real projective n-space* $\mathbb{P}^{n-1}(\mathbb{R})$ is the space of equivalence classes of $(\mathbb{R}^n)_o$ where $x, y \in (\mathbb{R}^n)_o$ are equivalent if there exists $\lambda \in \mathbb{R}$ such that $x = \lambda y$. When equipped with the quotient topology $\mathbb{P}^{n-1}(\mathbb{R})$ becomes a compact Hausdorff space. Let π denote the quotient map $(\mathbb{R}^n)_o \to \mathbb{P}^{n-1}(\mathbb{R})$.

The projective special linear group $\mathrm{PSL}(n,\mathbb{R})$ is the quotient group of the special linear group $\mathrm{SL}(n,\mathbb{R})$ by its center, consisting of the real scalar multiples of the identity with determinant 1. For any subring R of $\mathbb{R}$ containing $\mathbb{Z}$, let $\mathrm{SL}(n,R)$ be the subgroup of $\mathrm{SL}(n,\mathbb{R})$ of R-valued matrices with determinant 1, and let $\mathrm{PSL}(n,R)$ be the image of $\mathrm{SL}(n,\mathbb{R})$ under the quotient map $\mathrm{SL}(n,\mathbb{R}) \to \mathrm{PSL}(n,\mathbb{R})$.

There is a natural continuous action of $\mathrm{PSL}(n,\mathbb{R})$ on $\Omega = \mathbb{P}^{n-1}(\mathbb{R})$. We will give an elementary proof (of our own design) that the action of any subgroup G containing $\mathrm{PSL}(n,\mathbb{Z})$ on Ω is in fact strongly proximal. For minimality, we offer the following short explanation: it is easy to show that Ω is a homogeneous space of $H = \mathrm{SL}(n,\mathbb{R})$, and the proof below yields that Ω is then a H-boundary. Using the classical result that $\mathrm{SL}(n,\mathbb{Z})$ is a lattice in H, it acts minimally on H/P for any parabolic subgroup P of H [106, Lemma 8.5] and thus on any boundary of H (see III).

If n is even, write $n = 2m$ and define $P_n(x) = \prod_{i=1}^{m}(x^2 - (i+2)x + 1)$. If n is odd, then for $n = 2m+1$ we define $P_n(x) = (x-1)P_{2m}(x)$. Let A be the companion matrix of P_n, that is, if we write $P_n(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_0$, then

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix}.$$

Then A is integer-valued with determinant 1, since the roots of $x^2 - (i+2)x + 1$ have product 1, and it has n distinct positive eigenvalues $\lambda_1 > \ldots > \lambda_n > 0$. Moreover, no eigenvector of A has 0 in the n'th coordinate.

Let $x_1, \ldots, x_n \in \mathbb{R}^n$ be a basis of eigenvectors for A, with each x_i contained in the eigenline of λ_i. For $x = \sum_{i=1}^n \alpha_i x_i \in (\mathbb{R}^n)_o$, then for the smallest $1 \leq j \leq n$ such that $\alpha_j \neq 0$ we see that

$$\pi(A^k x) = \pi\left(\sum_{i=1}^n \alpha_i \lambda_i^k x_i\right) = \pi\left(\alpha_j x_1 + \sum_{i=j+1}^n \alpha_i (\tfrac{\lambda_i}{\lambda_j})^k x_i\right) \to \pi(\alpha_j x_j) = \pi(x_j)$$

for $k \to \infty$. Let a be the image of A in $\mathrm{PSL}(n, \mathbb{Z}) \subseteq G$. Now define Borel sets

$$U_i = \left\{ \pi(x) \;\middle|\; x = \sum_{j=i}^n \alpha_j x_j \in (\mathbb{R}^n)_o \text{ for } \alpha_i, \cdots, \alpha_n \in \mathbb{R},\ \alpha_i \neq 0 \right\} \subseteq \Omega, \quad 1 \leq i \leq n.$$

Then $U_i \cap U_j = \emptyset$ for $1 \leq i < j \leq n$, $\bigcup_{i=1}^n U_i = \Omega$, and $a^k x \to \pi(x_i)$ whenever $x \in U_i$ for $1 \leq i \leq n$. For any $\mu \in \mathcal{P}(\Omega)$, write $\mu = \sum_{i=1}^n \mu(U_i)\mu_i$ for $\mu_i \in \mathcal{P}(\Omega)$, so that $\mu(U_i)\mu_i(E) = \mu(E \cap U_i)$ for all Borel sets $E \subseteq \Omega$ and $1 \leq i \leq n$. Now, for all $f \in C(\Omega)$,

$$\int f \,\mathrm{d}(a^k \mu) = \sum_{i=1}^n \mu(U_i) \int_{U_i} f(a^k x)\,\mathrm{d}\mu_i(x) \to \sum_{i=1}^n \mu(U_i) f(\pi(x_i)),$$

meaning that

$$a^k \mu \to \sum_{i=1}^n \mu(U_i)\delta_{\pi(x_i)} \in \overline{G\mu}$$

in the weak* topology. Write $x_j = \sum_{i=1}^n \beta_i^j e_i \in \mathbb{R}^n$ where $e_1, \ldots, e_n$ is the standard basis and for $m \in \mathbb{Z}$, let B_m be the matrix in $\mathrm{SL}(n, \mathbb{Z})$ with ones in the diagonal, m in the $(1, n)$ entry and zeros everywhere else. Since $\beta_n^j \neq 0$ for all $1 \leq j \leq n$, it follows that $\pi(B_m x_j) \to \pi(e_1)$ for $m \to \infty$. Consequently, if we let $b_m \in G$ be the image of B_m, we conclude that the action of G on Ω is strongly proximal, since

$$\delta_{\pi(e_1)} = \lim_{m \to \infty} b_m \sum_{i=1}^n \mu(U_i)\delta_{\pi(x_i)} \in \overline{G\mu}.$$

V. A proximal action which is not strongly proximal (Glasner [62, p. 162])**.** First let Γ be a discrete group, $\Gamma \neq \{1\}$, and let $X = \{0,1\}^\Gamma$. For each $g \in \Gamma$, define a homeomorphism g on X by $(g\xi)_s = \xi_{g^{-1}s}$. This defines an injective homomorphism of Γ into the homeomorphism group of X.

Additionally, let $g_0 \in \Gamma \setminus \{1\}$ and define homeomorphisms

$$(c\xi)_s = \begin{cases} 1 - \xi_{g_0} & \text{if } s = g_0, \\ \xi_s & \text{otherwise,} \end{cases} \qquad h_0\xi = \begin{cases} \xi & \text{if } \xi_1 = 0, \\ c\xi & \text{if } \xi_1 = 1, \end{cases} \qquad h_1\xi = \begin{cases} \xi & \text{if } \xi_1 = 1, \\ c\xi & \text{if } \xi_1 = 0 \end{cases}$$

for $\xi \in X$. We immediately observe that $h_0^2 = h_1^2 = c^2 = \mathrm{id}_X$ and $h_0h_1 = h_1h_0 = c$ since $(c\xi)_1 = \xi_1$.

If $m = \frac{1}{2}(\delta_0 + \delta_1)$, let μ be the product measure given by $(m_t)_{t\in\Gamma}$ on X where $m_t = m$ for all $g \in \Gamma$. For every finite subset $F \subseteq \Gamma$, we let $\pi_F \colon X \to X_F = \prod_{i\in F}\{0,1\}$ be the projection with respect to the coordinates determined by F. Now μ is the unique Radon probability measure on X satisfying $\mu(\pi_F^{-1}(\{x\})) = 2^{-|F|}$ for all finite subsets $F \subseteq \Gamma$ and $x \in X_F$, and one may check that μ is invariant under Γ, h_0 and h_1.

A. Let G be the group of homeomorphisms of X generated by Γ and c. Then the action of G on X is minimal, but not strongly proximal. Indeed, for $\xi \in X$ we may shift by any $g \in \Gamma$ and apply c, to change any coordinate of ξ. Hence the G-orbit of ξ is dense in X. However, since the measure $\mu \in \mathcal{P}(X)$ constructed above is G-invariant, the action of G on X is not strongly proximal.

B. We next consider the case when $\Gamma = \mathbb{Z}$, and let g_0 be the integer 1 in the above definition of c. Note that $1 \in \Gamma$ translates to $0 \in \mathbb{Z}$ in the definitions of h_0 and h_1. Let G be the group of homeomorphisms generated by g, h_0 and h_1 where g is the shift given by $(g\xi)_i = \xi_{i-1}$ (so that g generates all of $\mathbb{Z}$). Then the action of G on X is minimal, but not strongly proximal, by the same argument as above. However, the action *is* proximal.

To see this, let $A \subseteq \mathbb{Z}$ be a finite subinterval, meaning that A contains all numbers in between its smallest and largest element. We need to prove for all $\xi, \eta, \alpha \in X$ that there exists $s \in G$ such that $(s\xi)_i = (s\eta)_i = \alpha_i$ for all $i \in A$. Due to G containing the shift g, we may assume that $\min A = 1$.

We now proceed by induction on $|A|$. Assume that $\eta_1 \neq \xi_1$. If $\eta_0 \neq \xi_0$, then $(h_j\xi)_1 = (h_j\eta)_1$ for some $j \in \{0,1\}$, and if $\eta_0 = \xi_0$ then $(g\eta)_1 = (g\xi)_1$. Applying c if necessary, we obtain the case $|A| = 1$.

Now assume for all $\xi, \eta, \alpha \in X$ we can find $s \in G$ such that $(s\xi)_i = (s\eta)_i = \alpha_i$ for $i = 1, \ldots, n$. Let $\xi, \eta, \alpha \in X$ and take $s \in G$ such that $(s\xi)_i = (s\eta)_i = (g^{-1}\alpha)_i$ for all $i = 1, \ldots, n$. Defining $t = gs \in G$, then $(t\xi)_i = (t\eta)_i = \alpha_i$ for all $i = 2, \ldots, n+1$. If $(t\xi)_1 = (t\eta)_1$, then we are done, as we may then apply c if need be. Therefore, suppose that $(t\xi)_1 \neq (t\xi)_1$.

If $(t\xi)_0 \neq (t\eta)_0$ then we may apply h_0 or h_1 as in the case $|A| = 1$ to get equality in the first coordinate. If not, let $N = \max\{k \mid k < 0,\ (t\xi)_k \neq (t\eta)_k\}$, so that $(t\xi)_k = (t\eta)_k$ for all $N < k \leq 0$. Observe that $(g^{-N}t\xi)_0 \neq (g^{-N}t\eta)_0$ and $(g^{-N}t\xi)_1 = (g^{-N}t\eta)_1$; thus there is $j \in \{0,1\}$ such that $(h_jg^{-N}t\xi)_0 \neq (h_jg^{-N}t\eta)_0$ and $(h_jg^{-N}t\xi)_1 \neq (h_jg^{-N}t\eta)_1$, so that for $r_N = g^Nh_jg^{-N}$ we have

$$(r_Nt\xi)_N \neq (r_Nt\eta)_N, \quad (r_Nt\xi)_{N+1} \neq (r_Nt\eta)_{N+1}.$$

Furthermore, for $2 \leq i \leq n+1$, then $i - N \neq 1$ so that

$$(r_Nt\xi)_i = (h_jg^{-N}t\xi)_{i-N} = (g^{-N}t\xi)_{i-N} = (t\xi)_i$$

and $(r_Nt\eta)_i = (t\eta)_i$. We may now repeat this process to produce $r = r_{-1}\cdots r_N \in G$ such that $(rt\xi)_k \neq (rt\eta)_k$ for all $N \leq k \leq 0$ and $(rt\xi)_i = (rt\eta)_i$ for all $2 \leq i \leq n+1$. In particular, since $(rt\xi)_0 \neq (rt\eta)_0$ we can apply either h_0 or h_1, followed by c if necessary, to find $r' \in G$ such that $(r't\xi)_i = (r'\eta)_i = \alpha_i$ for all $1 \leq i \leq n+1$. This ends the proof.

Weakly hyperbolic actions.

In discussing the above examples, then with the exception of example II we do not really have a indication at the outset that the boundaries uncovered should actually *be* boundaries for the group – we just show that they are (in example II, we are deliberately forcing the circle to be a boundary by constructing an appropriate group of homeomorphisms). We will now discuss a property for homeomorphisms that often produces a boundary action from a given group of homeomorphisms of a compact Hausdorff space.

Recalling the discussion of the extended real line $\mathbb{R}$ from the introduction to this chapter, observe that repeated applications of any non-trivial translation of $\mathbb{R}$ with positive derivative moves any point in $\mathbb{R} \cup \{\infty\}$ closer to $+\infty$, and that $\{\pm\infty\}$ are the only fixed points. This picture should, hopefully, motivate the following definition.

Definition 2.2.3. Let X be a Hausdorff space and let γ be a homeomorphism of X with two fixed points $x_\gamma^-, x_\gamma^+ \in X$. We say that γ is *weakly hyperbolic* if $\lim_{n\to\infty} \gamma^n(x) = x_\gamma^+$ for all $x \in X \setminus \{x_\gamma^-\}$ and $\lim_{n\to\infty} \gamma^{-n}(x) = x_\gamma^-$ for all $x \in X \setminus \{x_\gamma^+\}$. The points x_γ^+ and x_γ^- are called the *attracting point* and *repelling point* of γ, respectively.

If for any neighbourhoods U of x_γ^- and V of x_γ^+ there exists $N \geq 1$ such that $\gamma^n(X \setminus U) \subseteq V$ for all $n \geq N$, we say that γ is *hyperbolic*.

If X is a G-space, the *weakly hyperbolic limit set* of G in X is given by

$$L_G = \{x_g^\pm \mid g \in G \text{ weakly hyperbolic}\}.$$

If γ and γ' are weakly hyperbolic homeomorphisms of X, they are said to be *transverse* if their fixed point sets are disjoint. If $|L_G| \leq 2$, the action of G on X is said to be *elementary.*

Notice that a homeomorphism γ is (weakly) hyperbolic if and only if γ^{-1} is, and that the attracting point of γ is the repelling point of γ^{-1} and vice versa.

Actions that are non-elementary are also called "strongly hyperbolic" [39, 43]. Many of the actions we will consider in this section are defined by means of weakly hyperbolic homeomorphisms. We include the stronger (or rather, non-weak) notion of hyperbolicity mainly for historical purposes: in perhaps the first comprehensive survey on C^*-simplicity (to be discussed in the next chapter), courtesy of de la Harpe, an abundance of C^*-simple groups were given by means of group actions by hyperbolic homeomorphisms on Hausdorff spaces [39]. We shall soon review a few of these, but not before we provide a picture of how we may ensure that the two notions actually coincide (even though the weak version of hyperbolicity does seem much weaker at the outset).

The following result is an adaptation of [54, Lemma 2.2]; recall that a subset of a topological space is *precompact* if it has compact closure.

Lemma 2.2.4. *Let X be a locally compact Hausdorff space, let $x_0 \in X$ be a fixed point for a homeomorphism γ of X and assume that $X_\gamma = \{x \in X \mid \gamma^n(x) \to x_0\}$ is a neighbourhood of x_0. Then the following are equivalent:*

(i) *There exists a compact neighbourhood W of x_0 such that $\bigcap_{n=1}^\infty \gamma^n(W) = \{x_0\}$ and $\gamma(W) \subseteq W$.*

(ii) *For any compact subset C of X_γ and any open neighbourhood V of x_0, there exists $N \geq 1$ such that $\gamma^n(C) \subseteq V$ for all $n \geq N$.*

Proof. (ii) $\Rightarrow$ (i): Let $V \subseteq X_\gamma$ be a compact neighbourhood of x_0. Taking $N \geq 1$ such that $\gamma^n(V) \subseteq V$ for all $n \geq N$, then $W = \bigcup_{n=0}^{N} \gamma^n(V)$ satisfies $\gamma(W) \subseteq W$ and $\bigcap_{n=1}^{\infty} \gamma^n(W) = \{x_0\}$.

(i) $\Rightarrow$ (ii): Let W be a compact neighbourhood of x_0 such that $\bigcap_{n=1}^{\infty} \gamma^n(W) = \{x_0\}$ and $\gamma(W) \subseteq W$. Then $(\gamma^n(W))_{n\in\mathbb{Z}}$ is decreasing. Taking complements and applying compactness, we see that for all neighbourhoods U of x_0 there exists $N \geq 1$ such that $\gamma^n(W) \subseteq U$ for $n \geq N$. In particular, $W \subseteq X_\gamma$.

Assume that $C \subseteq X_\gamma$ is a compact subset. As $X_\gamma = \bigcup_{k=1}^{\infty} \gamma^{-k}(\mathrm{int}(W))$, there is $K \geq 1$ such that $C \subseteq \bigcup_{k=1}^{K} \gamma^{-k}(\mathrm{int}(W)) \subseteq \gamma^{-n}(\mathrm{int}(W))$ for all $n \geq K$. Hence for $n \geq N + K$ we find that $\gamma^n(C) \subseteq \gamma^N(\mathrm{int}(W'')) \subseteq V$. $\square$

In particular, if a homeomorphism γ of a compact Hausdorff space X is weakly hyperbolic and there exists a closed neighbourhood W of x_γ^+ such that $\bigcap_{n=1}^{\infty} \gamma^n(W) = \{x_\gamma^+\}$ and $\gamma(W) \subseteq W$, then γ is hyperbolic.

The following result, relating weak hyperbolicity to boundary actions, was observed by Ozawa in [111]. We consider two large-scale applications in the next two subsections.

Proposition 2.2.5. *Let G be a Hausdorff topological group, and let X be a non-elementary compact G-space. Then $\overline{L_G}$ is the unique minimal subset of X. In fact, $\overline{L_G}$ is a G-boundary.*

Proof. Let M be a minimal subset of X, and let $g \in G$ be weakly hyperbolic. Let $h \in G$ be a weakly hyperbolic homeomorphism such that $x_h^- \notin \{x_g^\pm\}$ and let $x \in M$. If $x \neq x_g^-$, then $x = \lim_{n\to\infty} g^n x \in M$. Otherwise, $x_h^+ = \lim_{n\to\infty} h^n x \in M$ and $x_g^+ = \lim_{n\to\infty} g^n x_h^+ \in M$. Hence $L_G \subseteq M$, so that $\overline{L_G} = M$ by L_G being G-invariant.

To verify that $\overline{L_G}$ is a G-boundary, notice for weakly hyperbolic homeomorphisms $g, h \in G$ with $x_h^- \notin \{x_g^\pm\}$ that the sequences (g^n) and (h^n) and the attractors and repellers $x_g^\pm, x_h^\pm \in \overline{L_G}$ together satisfy the conditions of Lemma 2.2.1. $\square$

Let us finally show that any non-elementary action admits a large amount of transverse weakly hyperbolic homeomorphisms. The result generalizes [43, Lemma 6] in the weakly hyperbolic case. We quickly observe that any conjugate of a weakly hyperbolic homeomorphism is also weakly hyperbolic.

Lemma 2.2.6. *Let G be a topological group and let X be a non-elementary compact G-space. Then for each fixed weakly hyperbolic automorphism $g_0 \in G$, there exists a sequence $(g_n)_{n\geq 1}$ of pairwise transverse conjugates of g_0.*

Proof. We first show that G contains at least two transverse weakly hyperbolic homeomorphisms of X. Since $\overline{L_G}$ is a non-trivial G-boundary, L_G is infinite by Proposition 2.1.11. Supposing that any two weakly hyperbolic $g, h \in G$ have at least one common fixed point, let $G_{\mathrm{hyp}} \subseteq G$ be the subset of all weakly hyperbolic homeomorphisms of X in G, and let X^g be the fixed points of each $g \in G_{\mathrm{hyp}}$. For $g_1, g_2, g_3 \in G_{\mathrm{hyp}}$, take $x \in X^{g_1} \cap X^{g_2}$. If $X^{g_1} \cap X^{g_2} \cap X^{g_3} = \emptyset$, then $x \notin X^{g_3}$, so that $X^{g_1} = \{x, x_{g_3}^+\}$ and $X^{g_2} = \{x, x_{g_3}^-\}$, say. Since L_G is infinite, we may take a weakly hyperbolic $h \in G$ such that $x_h^+ \notin \{x, x_{g_3}^\pm\}$, but then $x_h^- \in X^{g_1} \cap X^{g_2} \cap X^{g_3} = \emptyset$. Hence $x \in X^{g_3}$.

For $n \geq 3$ assume that $g_1, \ldots, g_n \in G_{\mathrm{hyp}}$ satisfy $\bigcap_{i=1}^{n} X^{g_i} = \{x\}$ for some $x \in X$ and let $h \in G$ be weakly hyperbolic. After removing equal fixed point sets, we may assume that $X^{g_i} \cap X^{g_j} = \{x\}$ for all $i \neq j$. By what we saw above, $X^{g_1} \cap X^{g_2} \cap X^h \neq \emptyset$, so that $x \in X^h$. Thus $\bigcap_{g \in F} X^g \neq \emptyset$ for all finite $F \subseteq G_{\text{hyp}}$, so by compactness there exists $x \in \bigcap_{g \in G_{\text{hyp}}} X^g \subseteq L_G$. Now, for any $g \in G_{\text{hyp}}$ and $h \in G$, $h^{-1}gh$ is weakly hyperbolic. Therefore $ghx = hx$, so g fixes all of $\overline{L_G}$ by minimality. In particular, g fixes more than 2 points, which is clearly a contradiction.

Now let $g, h \in G_{\text{hyp}}$ be transverse. Let $N \geq 1$ such that $h^n(\{x_g^{\pm}\}) \subseteq X \setminus \{x_g^{\pm}\}$ for all $n \geq N$, note that $h^n x_g^{\pm}$ are the attracting points of $h^n g h^{-n}$. If (n_k) is any sequence of positive integers such that $n_{k+1} \geq n_k + N$ for all $k \geq 1$, then for $k > j$ we have $\{h^{n_k} x_g^{\pm}\} \cap \{h^{n_j} x_g^{\pm}\} = h^{n_k - n_j}(\{x_g^{\pm}\}) \cap \{x_g^{\pm}\} = \emptyset$ since $n_k - n_j \geq N$. Hence $(h^n g h^{-n})$ is a sequence of pairwise transverse weakly hyperbolic homeomorphisms.

Finally, for any $g_0 \in G_{\text{hyp}}$, then by replacing h with $g^n h g^{-n}$ for some n we may assume that g_0 and h are transverse. We may then repeat the above procedure for g_0 and h. $\square$

We finally mention that the boundaries considered in the remainder of this section, which are all constructed by means of weakly hyperbolic limit sets, will actually consist of fixed points of not just weakly hyperbolic, but full-on hyperbolic homeomorphisms (see [59, Lemme 8.18] and [43, Section 3]).

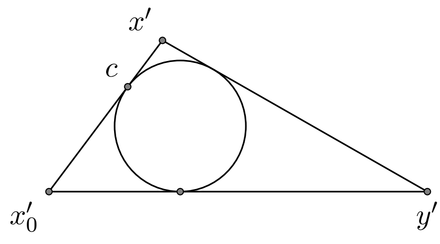
The Gromov boundary.

Our first general example of a boundary action comes from the Gromov boundary of a hyperbolic metric space. For a general hyperbolic metric space, we will only construct the boundary and draw conclusions relating to boundary actions in the case of the space being *proper*, meaning that closed bounded sets are compact. Except for the last two propositions (and their proofs), this subsection is a summary of the introduction to hyperbolicity given in [35]; we suggest [59] for another approach.

Let (X, d, x_0) be a pointed metric space X with metric d and base point x_0. We then define the *Gromov product*

$$(x.y)_{x_0} := \frac{1}{2}(d(x, x_0) + d(y, x_0) - d(x, y)), \quad x, y \in X.$$

It is possible to realize $(x.y)_{x_0}$ geometrically as follows. Let $\{x_0', x', y'\}$ be the corners of a triangle Δ in the plane such that $d(x_0', x') = d(x_0, x)$, $d(x_0', y') = d(x_0, y)$ and $d(x', y') = d(x, y)$. If we let c be the intersection between the inscribed circle in Δ (to which the three sides are tangents) and the line between x_0' and x' (or x_0' and y'), then $(x.y)_{x_0}$ is the distance between x_0' and c.



For $\delta \geq 0$, we say that a pointed metric space (X, d, x_0) is *δ-hyperbolic* if

$$(x.y)_{x_0} \geq \min\{(x.z)_{x_0}, (y.z)_{x_0}\} - \delta$$

for all $x, y, z \in X$. If (X, x_0) is δ-hyperbolic, then (X, x_1) is 2δ-hyperbolic for all $x_1 \in X$. This inspires the following notion.

Definition 2.2.7. A metric space X is said to be *δ-hyperbolic* if (X, x_0) is δ-hyperbolic for all $x_0 \in X$. We say that X is *hyperbolic* if there exists $\delta \geq 0$ such that X is δ-hyperbolic.

If X is a metric space with base point x_0, we say that a sequence $(x_n)_{n\geq 1}$ in X *converges to infinity* if $(x_m.x_n)_{x_0} \to \infty$ for $m, n \to \infty$. Convergence to infinity of a sequence does not depend on the base point x_0, and so we may unambiguously define the set $S_\infty(X)$ of sequences in X converging to infinity. We then define a relation $\mathfrak{R}$ on $S_\infty(X)$ by writing $(x_n)\,\mathfrak{R}\,(y_n)$ if $(x_n.y_n) \to \infty$ for $n \to \infty$. Clearly, $\mathfrak{R}$ is reflexive and symmetric, and if X is hyperbolic, $\mathfrak{R}$ is transitive as well.

Definition 2.2.8. If X is a hyperbolic metric space, we define the *hyperbolic boundary* or *Gromov boundary* ∂X of X to be the set of equivalence classes in $S_\infty(X)$ under the relation $\mathfrak{R}$. Moreover, we say that the sequence $(x_n)_{n\geq 1}$ in X converges to the point $x \in \partial X$ if $(x_n)_{n\geq 1}$ converges to infinity and x is the equivalence class of $(x_n)_{n\geq 1}$.

The following definition can easily be generalized to arbitrary ordered abelian groups (see [30]), but we restrict ourselves to considering closed subgroups of $\mathbb{R}$.

Definition 2.2.9. Let X be a metric space and let Λ be a closed additive subgroup of $\mathbb{R}$. We say that X is a *Λ-metric space* if its metric takes values in Λ. A *closed interval* in Λ is the intersection of Λ with a closed interval in $\mathbb{R}$ and we say that λ_1, λ_2 are the endpoints of $[\lambda_1, \lambda_2]$. We allow $\lambda_1 = -\infty$ and $\lambda_2 = \infty$.

If I is a closed interval in Λ with endpoints $\lambda_1 < \lambda_2$ and $\gamma\colon I \to X$ is an isometry, we say that γ is a *geodesic*. Additionally, if $-\infty < \lambda_1 < \lambda_2 < \infty$, $\gamma(I)$ is a *geodesic segment* in X with *endpoints* $x = \gamma(\lambda_1)$ and $y = \gamma(\lambda_2)$, and we write $\gamma(I) = [\gamma(\lambda_1), \gamma(\lambda_2)]$. Notice that a geodesic segment connecting two points need not be unique.

If any two points in X are endpoints of a geodesic segment, we say that X is a *geodesic space*, and if it also holds that any geodesic segment in X is uniquely determined by its endpoints, X is said to be *uniquely geodesic*.

Definition 2.2.10. Let Λ be either $\mathbb{Z}$ or $\mathbb{R}$ with $\Lambda_+ = \Lambda \cap \mathbb{R}_+$, let X be a geodesic Λ-metric space and fix $p \in X$.

(i) A *ray* emanating from a point $p \in X$ is a geodesic $r\colon \Lambda_+ \to X$ such that $r(0) = p$. Let $R = R(X)$ be the set of rays $r\colon \Lambda_+ \to X$ and for $p \in X$, let $R_p \subseteq R$ be the subset of rays emanating from p. We say that $r_1, r_2 \in R_p$ are *asymptotic* if the map $\Lambda_+ \to X$ given by $t \mapsto d(r_1(t), r_2(t))$ is bounded. This defines an equivalence relation on R_p, and we let $\partial_p X$ be the space of equivalence classes of R_p with respect to being asymptotic, called the *visual boundary of X at p*. For any $r \in R_p$ we let $[r] \in \partial_p X$ denote the asymptotic equivalence class containing it.

(ii) Let X_p be the set of maps $f\colon \Lambda_+ \to X$ such that $f(0) = p$ and there exists $\lambda \in \Lambda_+ \setminus \{0\}$ such that f is a geodesic on $[0, \lambda]$ and $f(t) = f(\lambda)$ for $t \geq \lambda$. We say that $f, g \in X_p$ are *equivalent* if $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t)$ (the limits taken relative to Λ_+).

(iii) We define the *strong* topology ([104, 4.1], [35, Chapitre 2]) on $X \cup \partial_p X$ as follows. Define $\pi_p \colon X_p \cup R_p \to X \cup \partial_p X$ by $\pi_p(f) = \lim_{t\to\infty} f(t)$ for $f \in X_p$ and $\pi_p(r) = [r]$ for $r \in R_p$. We equip $X \cup \partial_p X$ with the quotient topology induced by π_p, when $X_p \cup R_p$ is viewed as a closed subspace of $C(\Lambda_+, X)$, the latter equipped with the topology of compact convergence.

With the respect to the strong topology, one may check that X is an open dense subset of $X \cup \partial_p X$, and that it coincides with the original topology on X.

Recall that a metric space X is *proper* if closed bounded subsets of X are compact. If X is a proper, geodesic hyperbolic metric space, we may endow ∂X with a compact Hausdorff topology as follows. Note that $X_p \cup R_p$ is compact (due to Ascoli's theorem [113, Theorem 1.4.9]) and that the map π_p is perfect, i.e., pre-images of singletons are compact, implying that $X \cup \partial_p X$ is a Hausdorff space. If X is separable, then $X_p \cup R_p$ is second-countable, and $X \cup \partial_p X$ is metrizable. For fixed $p \in X$ and $r \in R_p$, let $r(\infty) \in \partial X$ be the equivalence class of $(r(\lambda_n))$ for any sequence (λ_n) such that $\lambda_n \to \infty$. Then the map $\partial_p X \to \partial X$ given by $[r] \mapsto r(\infty)$ is a bijection due to X being proper, geodesic and hyperbolic. Thus we obtain a compact topology on $X \cup \partial X$, and it does not depend on the choice of p. In this way, the Gromov boundary ∂X of X becomes a compact Hausdorff space.

If (X, d_X) and (Y, d_Y) are metric spaces, $\lambda > 0$ and $c \geq 0$, an (λ, c)-*quasi-isometric embedding* is a map $f \colon X \to Y$ such that that

$$\lambda^{-1} d_X(x, x') - c \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + c$$

for all $x, x' \in X$. If Y is hyperbolic and $f \colon X \to Y$ is a (λ, c)-quasi-isometric embedding for some $\lambda > 0$ and $c \geq 0$, then X is hyperbolic as well, and f induces a continuous injection $f \colon \partial X \to \partial Y$, given by mapping the equivalence class of a sequence (x_n) in $S_\infty(X)$ to the equivalence class of $(f(x_n))$ in $S_\infty(Y)$. If the image of f is also *co-bounded* in Y, i.e., if the function $x \mapsto \mathrm{dist}(x, f(X))$ is bounded on Y, we say that f is a *quasi-isometry* and that X and Y are quasi-isometric. In this case, $f \colon \partial X \to \partial Y$ is also surjective and thus a homeomorphism.

Theorem 2.2.11. *Let X be a proper, geodesic, hyperbolic Λ-metric space. Then a surjective isometry $\sigma \colon X \to X$ satisfies one and only one of the following conditions:*

(i) *σ is* elliptic*: there is a point $x \in X$ for which the sequence $(\sigma^n(x))_{n\geq 1}$ is bounded.*
(ii) *σ is* parabolic*: there is a point $x \in X$ for which the sequence $(\sigma^n(x))_{n\geq 1}$ has exactly one accumulation point in ∂X.*
(iii) *σ is* hyperbolic*: there is a point $x \in X$ such that the map $\mathbb{Z} \to X$ given by $n \mapsto \sigma^n(x)$ is a quasi-isometric embedding.*

If σ is a hyperbolic isometry of a proper, geodesic and hyperbolic Λ-metric space X, then σ has exactly two fixed points $x_\sigma^+, x_\sigma^- \in \partial X$ which satisfy $\sigma^n(x) \to x_\sigma^+$ for all $x \in (X \cup \partial X) \setminus \{x_\sigma^-\}$, and $\sigma^{-n}(x) \to x_\sigma^-$ for all $x \in (X \cup \partial X) \setminus \{x_\sigma^+\}$.

Combining the above discussion with Proposition 2.2.5 we obtain the following result:

Proposition 2.2.12. *Let G be a group acting by isometries on a proper, geodesic, hyperbolic Λ-metric space X. If the action of G on ∂X is non-elementary, $\overline{L_G} \subseteq \partial X$ is a G-boundary.*

We next consider the case of a finitely generated discrete group G acting on itself by left translation. Letting $S \subseteq G$ be a finite generating set, then we associate to S the word metric

$$d_S(g,h) = \min\{n \mid g^{-1}h = s_1 \cdots s_n \text{ for } s_1, \ldots, s_n \in S \cup S^{-1}\},$$

so that G becomes a proper, geodesic $\mathbb{Z}$-metric space. Notice that d_S is invariant under the action of G on itself by left translation. Moreover, for any two finite generating sets S, T of G, the metric spaces (G, d_S) and (G, d_T) are quasi-isometric.

We say that a finitely generated group G is *hyperbolic* if (G, d_S) is hyperbolic for some finite generating set $S \subseteq G$ (so that all (G, d_S) is hyperbolic for all generating sets $S \subseteq G$) If G is hyperbolic, the boundary ∂G is unique up to G-equivariant homeomorphism. As G acts on itself by isometries, the *type* of an element g in a hyperbolic group G is the type of the isometry $h \mapsto gh$. One can show that a hyperbolic group contains no parabolic elements, and that a group element is elliptic if and only if it has finite order.

A hyperbolic group is *non-elementary* if the action of G on ∂G is non-elementary. One can show that a hyperbolic group is elementary if and only if it is either finite or has an infinite cyclic subgroup of finite index.

Proposition 2.2.13. *Let G be a non-elementary hyperbolic group. Then the action of G on itself by left translation induces a boundary action of G on ∂G.*

Proof. Any element g of infinite order is hyperbolic, so we may let $g^+ = \lim_{n\to\infty} g^n \in \partial G$. We will give a proof of a important result of Gromov [65, 8.2.D] that

$$\{g^+ \mid g \in G \text{ has infinite order}\}$$

is dense in ∂G. We follow [22], but we will translate their proof to a setting that does not require knowledge of *automata*. For details on automata, as well as the proof technique used, we refer to [52, Chapter 3].

Choose some finite generating set $S \subseteq G$ and let $d = d_S$ be the corresponding word metric. Since (G, d) is hyperbolic, there exists $\delta_0 > 0$ such that for any geodesic triangle Δ in G (i.e., Δ is the union of three geodesic segments $[x, y]$, $[x, z]$ and $[y, z]$ in G), any of the segments in Δ is contained in the δ_0-neighbourhood of the two other segments (indeed, this is a property equivalent to hyperbolicity [35, Proposition 1.3.6]). Let $\delta = \delta_0 + 1$ and assume that $d(y, z) \leq 1$ for $y, z \in G$. Then for any $x \in G$ and $x_1 \in [x, y]$ there exists $x' \in [x, z] \cup [y, z]$ such that $d(x_1, x') < \delta$. If $x' \in [y, z]$, then $d(x_1, z) \leq d(x_1, x') + d(x', z) < \delta_0 + 1 = \delta$, so $[x, y]$ is contained in the δ-neighbourhood of $[x, z]$.

Let S^* be the collection of finite strings of letters in $S \cup S^{-1}$ including the empty string ε. The length of a string $\omega = s_1 \cdots s_n \in S^*$ is $|\omega| = n$, and the length of ε is defined to be 0. Next define a map $S^* \to G$, $\omega \mapsto \overline{\omega}$ by $\overline{\omega} = s_1 \cdots s_n \in G$ for $\omega = \omega_1 \cdots \omega_n$ and $\overline{\varepsilon} = 1$. For ω as above, let us say that ω is *geodesic* if the map $f_\omega \colon \{0, \ldots, |\omega|\} \to G$ given by $f_\omega(0) = 1$ and $f_\omega(n) = \prod_{i=1}^n \overline{s_i}$ for all $n = 1, \ldots, |\omega|$ defines a geodesic in G. We then define $f_\omega(n) = \overline{\omega}$ for all $n \geq |\omega|$. Out of pity, we will allow ε to be a geodesic string. For any $\omega \in S^*$ define the *cone type* $C_\omega = \{\gamma \in S^* \mid \omega\gamma \text{ geodesic}\}$ of ω where $\omega\gamma \in S^*$ is the concatenation of ω and γ. Note that $\omega \in S^*$ satisfies $C_\omega = \emptyset$ if and only if ω is not geodesic.

Note that the above map $S^* \to G$ maps the set of geodesic strings onto G. If ω, ω' are geodesic strings and $d(\overline{\omega}, \overline{\omega}') \leq 1$, then for $0 \leq n \leq |\omega|$, take $0 \leq m \leq |\omega'|$ such that $d(f_\omega(n), f_{\omega'}(m)) < \delta$. As $|n-m| = |d(f_\omega(n),1) - d(f_{\omega'}(m),1)| < \delta$, we find that $d(f_\omega(n), f_{\omega'}(n)) < 2\delta$, since ω' is geodesic.

For all $g \in G$, let $B_g = \{h \in G \mid d_S(1,h) < 2\delta + 1,\ d(gh,1) < d(g,1)\}$. Suppose for two geodesic strings $\omega, \omega' \in G$ that $B_{\overline{\omega}} = B_{\overline{\omega}'}$. We claim that $C_\omega = C_{\omega'}$ (a result of Cannon [52, Lemma 3.2.4]), which we will prove by induction on the length of strings in S^*. Since $\{B_g \mid g \in G\}$ is finite, this proves that there are only finitely many cone types of strings in S^*.

Evidently, $\varepsilon \in C_\omega \cap C_{\omega'}$. If $\omega' = \varepsilon$, then $B_{\overline{\omega}} = \emptyset$ and $\overline{\omega} = 1$, meaning that $\omega = \varepsilon$ since ω is geodesic. Therefore we assume $|\omega'| \geq 1$. Now let $\gamma \in S^*$ be a string such that $\omega\gamma$, $\omega'\gamma$ and that $\omega\gamma s$ are geodesic strings, where $s \in S \cup S^{-1}$. If there were a geodesic string $\nu \in S^*$ with length less than that of $\omega'\gamma s$ such that $\overline{\nu} = \overline{\omega'\gamma s}$, notice that $d(\overline{\nu}, \overline{\omega'\gamma}) = d(\overline{s}, 1) \leq 1$. Write $\nu = \alpha_1\alpha_2$ for geodesic strings $\alpha_1, \alpha_2 \in S^*$ where $|\alpha_1| = |\omega'| - 1$. As $|\nu| \leq |\omega'| + |\gamma|$, we see that $|\alpha_2| = |\nu| - |\alpha_1| \leq |\omega'| - |\alpha_1| + |\gamma| = |\gamma| + 1$. Now,

$$d(\overline{\omega'}, \overline{\alpha_1}) \leq d(f_{\omega'\gamma}(|\omega'|), f_\nu(|\omega'|)) + d(f_\nu(|\omega'|), f_\nu(|\alpha_1|)) < 2\delta + 1,$$

so $\overline{\omega'}^{-1}\overline{\alpha_1} \in B_{\overline{\omega'}} = B_{\overline{\omega}}$. Therefore $d(\overline{\omega}\overline{\omega'}^{-1}\overline{\alpha_1}, 1) < d(\overline{\omega}, 1) = |\omega|$, yielding

$$\begin{aligned} d(\overline{\omega\gamma s}, 1) - |\omega| &< d(\overline{\omega\gamma s}, 1) - d(\overline{\omega}\overline{\omega'}^{-1}\overline{\alpha_1}, 1) \\ &\leq d(\overline{\omega\gamma s}, \overline{\omega}\overline{\omega'}^{-1}\overline{\alpha_1}) \\ &= d(\overline{\gamma s}, \overline{\omega'}^{-1}\overline{\alpha_1}) \\ &= d(\overline{\nu}, \overline{\alpha_1}) = |\alpha_2| \leq |\gamma| + 1. \end{aligned}$$

We conclude that $d(\overline{\omega\gamma s}, 1) < |\omega| + |\gamma| + 1$, but $\omega\gamma s$ was assumed to be geodesic. Hence $\omega'\gamma s$ is geodesic, so $C_\omega = C_{\omega'}$.

If $r\colon \mathbb{Z}_+ \to G$ is a ray with $r(0) = 1$, choose a sequence $(s_n)_{n\geq 1}$ in $S \cup S^{-1}$ such that $\omega_n = s_1 \cdots s_n \in S^*$ is geodesic and $\overline{\omega_n} = r(n)$ for all $n \geq 1$. There now exists a cone type C such that $C_{\omega_n} = C$ for infinitely many $n \geq 1$. Let $M \geq 0$ be the smallest integer such that $C = C_{\omega_M}$. If $n \geq 0$ is fixed, there exists $N \geq n$ such that $C_{\omega_{M+N}} = C$. Then for all $p \geq 0$, the string $\omega_M(s_{M+1} \cdots s_{M+N})^p \in S^*$ is geodesic. Indeed, since $s_{M+1} \cdots s_{M+N} \in C_{\omega_M} = C_{\omega_{M+N}}$ it follows that $\omega_M(s_{M+1} \cdots s_{M+N})^2$ is geodesic. Thus $(s_{M+1} \cdots s_{M+N})^2 \in C_{\omega_M} = C_{\omega_{M+N}}$, so by continuing inductively it follows that $(s_{M+1} \cdots s_{M+N})^p \in C$ for all $p \geq 0$.

It follows from the above observation (proving a variant of the "pumping lemma") that there is a strictly increasing sequence $(N(i))_{i\geq 1}$ such that $\omega_M(s_{M+1} \cdots s_{M+N(i)})^p$ is geodesic for all $i, p \geq 0$. Define $\omega = \omega_M$, $\gamma_i = s_{M+1} \cdots s_{M+N(i)}$ and $g_i = \overline{\omega\gamma_i}(\overline{\omega}^{-1})$ for $i \geq 1$. Then g_i has infinite order, and the sequence $(\omega\gamma_i^p)_{p\geq 1}$ in S^* defines a ray r_i in G with $r_i(0) = 1$ and $r_i(\infty) = g_i^+$, since $d(\overline{\omega\gamma_i}^p, g_i^p) = M$ for all $p \geq 1$.

Finally, notice that $r_i(n) = r(n)$ for all $0 \leq n \leq M + N(i)$. Therefore $r_i \to r$ in $C(\mathbb{Z}_+, G)$ with respect to the topology of compact convergence, so that $r_i(\infty) \to r(\infty)$ in ∂G. As r was arbitrary, the proof is complete. $\square$

We finally remark that example I of a boundary in this section (see p. 14) is actually the Gromov boundary $\partial\mathbb{F}$ of a non-abelian free group $\mathbb{F}$ of finite rank; the Cayley graph of $\mathbb{F}$ with respect to any free generating set is a regular $\mathbb{Z}$-tree, and any $\mathbb{Z}$-tree is 0-hyperbolic [35, Proposition 1.2.2].

Countable trees.

In the following, we will consider a boundary action obtained by means of an action of a group on a countable tree. First and foremost, a *graph* $Y = (V, E)$ consists of a vertex set V and a set E of edges, respectively. All graphs will be unoriented, meaning that we employ the convention of Serre [129] that Y is also equipped with origin and terminus maps $o, t \colon E \to V$ and an inversion map $E \to E$, $e \mapsto \overline{e}$, such that $\overline{\overline{e}} = e$ and $o(\overline{e}) = t(e)$ for any edge $e \in E$. The *degree* of a vertex $v \in V$ is the cardinality of $o^{-1}(v)$ or $t^{-1}(v)$ (the numbers coincide), and a vertex is a *leaf* if it has degree 1. We say that Y is *locally finite* if every vertex in V has finite degree.

A *path* p in a graph $Y = (V, E)$ is a finite sequence $v_0, e_1, v_1, \ldots, e_n, v_n$ of vertices $v_0, \ldots, v_n \in V$ and $e_1, \ldots, e_n \in V$, such that $o(e_i) = v_{i-1}$ and $t(e_i) = v_i$ for all $1 \leq i \leq n$. The length of the path p is the number of edges in the path, and the *endpoints* of p are the initial and terminal vertices of the path. If the endpoints of p are equal, p is called a *circuit*.

A graph $Y = (V, E)$ is *connected* if any two vertices are endpoints of a path in the graph, and a *tree* is a connected graph without circuits. If $Y = (V, E)$ is connected, we define the *path metric* d on V by letting $d(v, w)$ be the length of the shortest path between v and w for $v, w \in V$. We take the liberty of viewing d as a metric on T that only measures distances between vertices.

In the following, let $T = (V, E)$ be a countable tree (i.e., V and E are countable). Letting d be the path metric on V, then (T, d) is a uniquely geodesic $\mathbb{Z}$-metric space. In fact, (T, d) is also 0-hyperbolic [35, Proposition 1.2.2].

A *morphism* σ of two trees $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$ is a tuple of maps $\sigma_V \colon V_1 \to V_2$ and $\sigma_E \colon E_1 \to E_2$ such that $o(\sigma_E(e)) = \sigma_V(o(e))$ and $\sigma_E(\overline{e}) = \overline{\sigma_E(e)}$ for all $e \in E_1$. If σ_V and σ_E are bijections, σ is called an *isomorphism*, and if $T_1 = T_2$ we say that σ is an *automorphism*. All automorphisms of trees will be assumed here to *act without inversion*, meaning that no $e \in E$ satisfies $\sigma_E(e) = \overline{e}$, and the group of automorphisms of T acting without inversion is denoted by $\mathrm{Aut}(T)$. This ensures that $\mathrm{Aut}(T)$ contains no automorphisms that fix an edge set $\{e, \overline{e}\}$, but not the endpoints. Notice that $\mathrm{Aut}(T)$ is just the isometry group of T with respect to the path metric, so that we may easily extend $\sigma \in \mathrm{Aut}(T)$ to $T \cup \partial T$ by defining $\sigma([r]) = [\sigma \circ r]$ for any ray r in T.

We say that an automorphism $\sigma \in \mathrm{Aut}(T)$ is *hyperbolic* if it has no fixed points in T. If T is locally finite, so that (T, d) is proper, this coincides with the definition given in Theorem 2.2.11.

For any automorphism $\sigma \in \mathrm{Aut}(T)$, the *amplitude* of σ is $\ell(\sigma) = \min_{v \in V} d(v, \sigma(v))$. The *characteristic* set of σ is the σ-invariant set

$$T^\sigma = \{v \in V \mid d(v, \sigma(v)) = \ell(\sigma)\}.$$

A *bi-infinite path* in a graph is a subgraph isomorphic to the graph with vertex set $\mathbb{Z}$ and edge set $\{e_n \mid n \in \mathbb{Z}\} \cup \{\overline{e_n} \mid n \in \mathbb{Z}\}$, with $o(e_n) = t(\overline{e_n}) = n$ and $t(e_n) = o(\overline{e_n}) = n + 1$ for all $n \in \mathbb{Z}$.

A fundamental result of J. Tits states that for a hyperbolic automorphism $\sigma \in \mathrm{Aut}(T)$, T^σ is always the vertex set of a bi-infinite path L in T, called the *axis* of σ, and any non-empty subtree of T which is invariant under σ and σ^{-1} always contains L. We refer to [129, Proposition 6.4.24] for details.

To build a "compactification" of T, fix a base vertex v in T. We consider the strong topology on the closed subspace $T_v \cup R_v$ of $C(\mathbb{Z}_+, T)$ in the topology of compact convergence. Two rays $r_1, r_2 \colon \mathbb{Z}_+ \to T$ are said to be *cofinal* if there exist integers $k, m \geq 0$ such that $r_1(n+m) = r_2(n+k)$ for all $n \geq 0$, we let ∂T be the space of equivalence classes of rays in T with respect to being cofinal. Let $\partial_v T$ be the visual boundary of T at v (cf. Definition 2.2.10).

As two rays in T emanating from the same vertex v are asymptotic if and only if they are cofinal, we have a natural map $\partial_v T \to \partial T$ for all v, and it is a bijection. Due to T being uniquely geodesic, the map $\pi_v \colon T_v \cup R_v \to T \cup \partial_v T$ is also a bijection. Hence the strong topology on $T \cup \partial_v T$ translates to a topology on $T \cup \partial T$, also called the *strong topology*, and it is independent of the choice of base vertex v [24, II.8]. Now, in order to eventually obtain a boundary from $T \cup \partial T$, we need this space to be compact. This is hindered if T has a vertex of infinite degree.

The solution is to define a weaker topology. For each edge $e \in E$, we define its *shadow*

$$S_e = \{v \in V \mid d(t(e), v) < d(o(e), v)\},$$

i.e., the set of vertices closer to the terminus of e than to the origin. One may also describe S_e as the component containing $t(e)$ of the graph obtained by removing e from T, and appropriately, S_e is also called a *half-tree* of T. We let τ be the *shadow topology* on $T \cup \partial T$ generated by the *extended shadows* $\overline{S_e} \subseteq T \cup \partial T$ for $e \in E$. Then τ is weaker than the strong topology on $T \cup \partial T$: indeed, for each $e \in E$ the shadow $S_e \subseteq V$ corresponds to the subset $\{f \in T_{t(e)} \mid d(f(1), o(e)) \geq 1\}$ of $T_{t(e)} \cup R_{t(e)}$. Hence

$$\overline{S_e} = \{f \in T_{t(e)} \cup R_{t(e)} \mid d(f(1), o(e)) \geq 1\}$$

is clopen in the strong topology. A theorem due to Monod and Shalom [104, 4.1] states that τ is a *compact* metrizable, totally disconnected Hausdorff topology on $T \cup \partial T$, and that τ and the strong topology coincide on ∂T. However, ∂T need not be closed – indeed, ∂T is closed (and hence compact) if and only if T is locally finite. To remedy this, instead we will consider the compact closure $\overline{\partial T}$ in $(T \cup \partial T, \tau)$.

In the same 2004 paper [104], Monod and Shalom prove that $\overline{\partial T}$ is the union of ∂T and all vertices in T of infinite degree, *as long as T has no leaves.* The proof goes as follows. Notice that the set $F \subseteq T$ of vertices of finite degree contains only isolated points, meaning in particular that F is open in τ. Moreover, if we index the edges $\{e_n\}_{n \geq 1}$ for which $o(e_i) = v$, where v is a vertex of infinite degree, then each $\overline{S_{e_n}}$ contains some point $x_n \in \partial T$ (because T has no leaves, so that there is a ray contained with image in S_{e_n}). By the compactness of $(T \cup \partial T, \tau)$, the sequence $(x_n)_{n \geq 1}$ has an accumulation point. As the sets $\overline{S_{e_n}} \setminus \{v\}$ constitute a partition of $(T \cup \partial T) \setminus \{v\}$, this accumulation point is seen to be our good old friend v, proving that $v \in \overline{\partial T}$.

Next observe that the extension of an automorphism $\sigma \in \mathrm{Aut}(T)$ to $T \cup \partial T$ is a homeomorphism in both the strong topology and the shadow topology τ. Moreover, if σ is a hyperbolic automorphism of T, then the axis of σ admits two fixed points $x_\sigma^\pm$ of the extension $\sigma \colon T \cup \partial T \to T \cup \partial T$ in ∂T, such that $\lim_{n \to \infty} \sigma^n(x) = x_\sigma^+$ for all $x \in (T \cup \partial T) \setminus \{x_\sigma^-\}$ and $\lim_{n \to \infty} \sigma^{-n}(x) = x_\sigma^-$ for all $x \in (T \cup \partial T) \setminus \{x_\sigma^+\}$. Hence σ is weakly hyperbolic as a homeomorphism of $T \cup \partial T$ in both of the aforementioned topologies.

For any automorphism $\sigma \in \operatorname{Aut}(T)$, we have $\sigma(\overline{\partial T}) = \overline{\partial T}$. Saying that the action of a discrete group G on a countable tree T is *minimal* if T contains no proper G-invariant subtrees, our desire is now to explain why certain actions of discrete groups on trees are in fact boundary actions. The proposition below, due to Le Boudec [99, Proposition 3.1], appears in a much stronger version in [100, Proposition 4.26].

Proposition 2.2.14. *Let T be a countable, leafless tree and let G be a discrete group acting minimally on T by automorphisms without inversion. If the action of G on ∂T is non-elementary, then $\overline{\partial T}$ is a G-boundary in the shadow topology.*

Proof. The subset $\overline{\partial T}$ is closed and G-invariant, and by hypothesis the action of G on $T \cup \partial T$ is non-elementary, so that the closure of the weakly hyperbolic limit set L_G of G in $(T \cup \partial T, \tau)$ is a G-boundary by Proposition 2.2.5. Our aim is to prove that $\partial T \subseteq \overline{L_G}$, which will complete the proof.

We require the "bridge lemma" [129, Lemma 6.4.9], formulated as follows. If T_1 and T_2 are subtrees of a tree T, with vertex sets V_1 and V_2, respectively, and $V_1 \cap V_2$ contains at most one vertex, there are unique vertices $v_1 \in V_1$, $v_2 \in V_2$ with $d(v_1, v_2) = d(T_1, T_2)$. Moreover, we have $d(w_1, w_2) = d(w_1, v_1) + d(v_1, v_2) + d(v_2, w_2)$ for all $w_1 \in V_1$ and $w_2 \in V_2$.

Let G_{hyp} be the subset of hyperbolic automorphisms of T in G and for each $g \in G_{\text{hyp}}$ let $L_g \subseteq T$ be the axis of g. Let $T_0 \subseteq T$ be the union of all L_g for $g \in G_{\text{hyp}}$. Then T_0 is a subtree of T. Supposing that $g, h \in G$ have disjoint characteristic sets, then gh is hyperbolic and its axis intersects L_g and L_h [2, Proposition 8.1] – in fact, if $x_1 \in L_g$ and $x_2 \in L_h$ are the unique vertices such that $d(x_1, x_2) = d(L_g, L_h)$, then

$$\ell(gh) = d(x_1, ghx_1) = \ell(g) + \ell(h) + 2d(x_1, x_2).$$

Hence T_0 is connected and therefore a subtree. Moreover, if $g \in G$ is hyperbolic and $x \in T^g$, then hgh^{-1} is hyperbolic and $hx \in T^{hgh^{-1}}$ for all $h \in G$. By minimality, it follows that $T_0 = T$.

Letting $e_1, \dots, e_n$ be edges of T, assume that $\bigcap_{i=1}^n \overline{S_{e_i}} \cap \partial T \neq \emptyset$ and let

$$r = 2 \max\{d(o(e_i), o(e_j)) \mid i, j = 1, \dots, n\}.$$

If $S = \bigcap_{i=1}^n S_{e_i} \subseteq T$ contains the axis of a hyperbolic automorphism in G, we are done, so assume that this is not the case. Let $g \in G_{\text{hyp}}$ such that $L_g \cap S \neq \emptyset$. Since L_g is not contained in S, L_g must intersect $T \setminus S_{e_{i_0}}$ for some $1 \leq i_0 \leq n$ and so $o(e_{i_0}), t(e_{i_0}) \in L_g$ for that i_0, since L_g also intersects $S_{e_{i_0}}$ and is the image of a geodesic.

If $L_g \cap S$ is finite, then L_g is not contained in S_{e_j} for all $j \neq i_0$, as that would imply that $L_g \cap S_{e_{i_0}} = L_g \cap S$ – but $L_g \cap S_{e_{i_0}}$ is either empty or infinite! Hence pick $j_0 \neq i_0$ such that $o(e_{j_0}), t(e_{j_0}) \in L_g$ by the same argument as above. Now, any $x \in L_g \cap S$ is necessarily contained in the unique geodesic segment connecting $o(e_{j_0})$ and $o(e_{i_0})$. For $i \notin \{j_0, i_0\}$, let $p \in [o(e_{j_0}), o(e_{i_0})]$ such that $d(o(e_i), p) = d(o(e_i), [o(e_{j_0}), o(e_{i_0})])$ by the bridge lemma. Then

$$d(o(e_i), x) = d(o(e_i), p) + d(p, x) \leq d(o(e_i), o(e_{i_0})) + d(o(e_{j_0}), o(e_{i_0})) \leq r.$$

Therefore $d(o(e_i), x) \leq r$ for all $x \in L_g \cap S$. It follows that there exists $g \in G_{\text{hyp}}$ such that $L_g \cap S$ is infinite; otherwise any vertex in

$$\bigcup_{g \in G_{\text{hyp}}} (L_g \cap S) = T \cap S = S$$

would have distance at most r to any $o(e_i)$, but S is unbounded by assumption. Hence $\overline{S}$ contains a fixed point of some $g \in G_{\mathrm{hyp}}$. This proves that $\partial T \subseteq \overline{L_G}$, so $\overline{\partial T} = \overline{L_G}$. $\square$

2.3 Amenability

In this section we give an introduction to the concept of amenability of a group which we shall soon see is closely related to boundary actions. The notion of amenability harks back to von Neumann, who in 1929 devised the concept motivated by the then recent proof of the Banach-Tarski paradox, and essentially describes groups who do *not* allow for analoguously paradoxical anomalies. We will mainly follow the exposition on amenability given in [64].

If G is a locally compact group, we consider the unital C^*-algebra $L^\infty(G)$ of measurable, complex-valued functions on G that are essentially bounded with respect to a left Haar measure on G. There is a well-defined action of G on $L^\infty(G)$ given by left translation:

$$(gf)(s) = f(g^{-1}s), \quad g, s \in G, \ f \in L^\infty(G).$$

If A is a G-invariant unital C^*-subalgebra of $L^\infty(G)$, a *mean* on A is a state of A as a C^*-algebra. If a mean $\mathfrak{m}\colon A \to \mathbb{C}$ satisfies

$$\mathfrak{m}(gf) = \mathfrak{m}(f), \quad g \in G, \ f \in A,$$

the mean is said to be *(left-)invariant*.

Definition 2.3.1. A locally compact group G is said to be *amenable* if $L^\infty(G)$ admits an invariant mean.

Examples of amenable groups include the class of compact groups (indeed, the normalized left Haar measure immediately defines an invariant mean on $C_{\mathrm{b}}(G)$ which suffices by the next theorem), as well as abelian groups. The most well-known example of a non-amenable group is a non-abelian free group – we will establish this fact, once we start looking into C^*-simple groups in Chapter 3.

The C^*-algebra $C_{\mathrm{b}}(G)$ of continuous bounded functions on G naturally embeds into $L^\infty(G)$ as a G-invariant unital C^*-subalgebra. If $A \subseteq C_{\mathrm{b}}(G)$ is a G-invariant C^*-subalgebra, then for any $g \in G$, the point mass $\delta_g\colon A \to \mathbb{C}$ defines a mean on A. A mean $\mathfrak{m}$ on A is *finite* if it belongs to the convex hull of the set of point masses $\delta_G = \{\delta_g \,|\, g \in G\}$. Letting $\Delta(A)$ be the subset of finite means in the space $S(A)$ of means on A, a standard Hahn-Banach separation argument shows that $\Delta(A)$ is in fact weak*-dense in the weak*-compact state space $S(A)$ of A.

There is a smorgasbord of alternate characterizations of amenability, and needless to say, we will not cover all of them. The following ones, of a more dynamical variety, are due to Hulanicki [84], Day [37] and Rickert [127]. We give references to other characterizations in what follows, when needed.

Theorem 2.3.2. *Let G be a locally compact group. Then the following are equivalent:*

(i) *G is amenable.*

(ii) *$C_{\mathrm{b}}(G)$ admits an invariant mean.*

(iii) *$C_{\mathrm{b}}^{\mathrm{lu}}(G)$ admits an invariant mean.*

(iv) *If G acts continuously on a compact convex subset Y of a locally convex space by affine homeomorphisms, then Y contains a point fixed by all $g \in G$.*

(v) *Every compact G-space carries a G-invariant Radon probability measure.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are trivial. If $\mathfrak{m}\colon C_{\mathrm{b}}^{\mathrm{lu}}(G) \to \mathbb{C}$ is an invariant mean, then let $h \in L^1(G)$ be a positive function of norm 1 such that $h(g^{-1}) = h(g)$ for all $g \in G$. Then the convolution $(f * h)(s) = \int f(t)h(t^{-1}s)\,\mathrm{d}t$ is bounded and left uniformly continuous for all $f \in L^\infty(G)$ [81, Theorem 20.16]. Therefore we may define an invariant mean $\varphi\colon L^\infty(G) \to \mathbb{C}$ by $\varphi(f) = \mathfrak{m}(f * h)$, so (i), (ii) and (iii) are equivalent.

If (ii) holds, let Y be a compact convex subset of a locally convex space E, equipped with a continuous affine G-action. For fixed $y \in Y$, define $f\colon G \to Y$ by $f(g) = gy$. Then for any $\varphi \in E^*$, $\varphi \circ f$ is continuous and bounded, since Y is compact. If $\mathfrak{m}\colon C_{\mathrm{b}}(G) \to \mathbb{C}$ is an invariant mean, let $\mathfrak{m}_i$ be a net in $\Delta(C_{\mathrm{b}}(G))$ such that $\mathfrak{m}_i \to \mathfrak{m}$ in the weak* topology. Fixing $i \in I$, write $\mathfrak{m}_i = \sum_{j=1}^n \lambda_j \delta_{s_j}$ for $s_1, \ldots, s_n \in G$ and define $x_i = \sum_{j=1}^n \lambda_j s_j y$. Then

$$\varphi(gx_i) = \mathfrak{m}_i(g^{-1}(\varphi \circ f)), \quad g \in G,\ \varphi \in E^*.$$

Since Y is compact, we may assume that (x_i) converges in Y to some $x \in X$, which now satisfies $\varphi(gx) = \mathfrak{m}(g^{-1}(\varphi \circ f))$ for all $g \in G$ and $\varphi \in E^*$. Since $\mathfrak{m}$ is invariant and φ was arbitrary, it follows that $gx = x$ for all $g \in G$. Thus (iii) implies (iv).

Finally, (iv) implies (v) since $\mathcal{P}(X)$ is a weak*-compact convex G-space for any G-space X, and if (v) holds, then in particular, S_G admits a G-invariant Radon probability measure. Since $\mathcal{P}(S_G)$ is the state space of $C_{\mathrm{b}}^{\mathrm{lu}}(G)$, $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ has an invariant mean, meaning that (v) implies (iii). ☐

We remark that the class of amenable groups possesses good stability properties: indeed, amenability is preserved under taking subgroups, quotients, extensions or direct limits, to name the ones usually coveted, as long as the subgroups considered are closed. Therefore solvable groups are also amenable. Moreover, since non-abelian free groups are non-amenable, we see that a discrete group is non-amenable if it contains a non-abelian free subgroup. The so-called *von Neumann conjecture*, arising from von Neumann's defining work on amenable groups, asked whether a group was amenable if and only if it did not contain a free subgroup of rank 2; the conjecture turned out to be false, when Ol'šanskiĭ proved in 1980 that the so-called *Tarski monster groups*, whose only proper subgroups are finite and of prime order p, exist for all primes p larger than 10^{75} and are non-amenable [109].

Let us end the section by stating a result due to Day [36, Lemma 4.1].

Lemma 2.3.3. *Every locally compact group G has a largest closed, normal, amenable subgroup N, i.e., all closed, normal, amenable subgroups of G are contained in N.*

Definition 2.3.4. The largest closed, normal, amenable subgroup of a locally compact group G is called the *amenable radical* of G and is denoted by $R(G)$.

The amenable radical of a group has proven to be an essential constituent in the study of C^*-simple groups and the unique trace property, and we shall return to it repeatedly.

2.4 Boundaries in compact convex spaces

In this section we resume providing important facts about boundary actions, the main point of focus being their relationship with actions on compact convex subsets of locally convex spaces, as well as amenability of the group in consideration. Along the way, we will relate a result of Glasner concerning the ubiquity of boundaries to injectivity for C^*-algebras. We will also briefly consider the notion of strong amenability, which arises from reformulating the relation between boundary actions and amenability in terms of minimal, proximal actions.

The first three results are due to Furstenberg [58].

Proposition 2.4.1. *Let G be a Hausdorff topological group. If X' is a compact minimal G-space and X is a G-boundary, then any continuous G-equivariant map $\varphi\colon X' \to \mathcal{P}(X)$ has image δ_X. It follows that there is at most one continuous G-equivariant map $X' \to X$, and it is surjective.*

Proof. Since X' is compact, $\varphi(X')$ is closed and contains a G-orbit, so it contains a point mass. Therefore $\varphi^{-1}(\delta_X) \neq \emptyset$, so $Gx \subseteq \varphi^{-1}(\delta_X)$ for some $x \in X'$. Since δ_X is weak*-closed, minimality yields $\varphi^{-1}(\delta_X) = X'$, and thus $\varphi(X') = \delta_X$.

If $\varphi_1, \varphi_2\colon X' \to X$ are continuous G-equivariant maps, then $\alpha(x) = \frac{1}{2}(\delta_{\varphi_1(x)} + \delta_{\varphi_2(x)})$ is a continuous G-equivariant map $X' \to \mathcal{P}(X)$. Therefore $\alpha(X') = \delta_X$ and it follows that $\varphi_1 = \varphi_2$. Since X is a minimal G-space, φ_1 is surjective. ☐

The above result has the interpretation that a G-boundary resembles a one-point space in the category of minimal compact G-spaces and G-equivariant continuous maps, strengthening the rigidity result for minimal proximal actions given in Lemma 2.1.13.

We translate the above result to a C^*-algebra setting as follows. Recall for compact Hausdorff spaces X and X' that there is a duality between continuous maps $\alpha\colon X' \to \mathcal{P}(X)$ and u.c.p. maps $\alpha^*\colon C(X) \to C(X')$ given by

$$\alpha^*(f)(x') = \alpha(x')(f), \quad f \in C(X),\ x' \in X'. \tag{2.4.1}$$

If X and X' are G-spaces, then α is G-equivariant if and only if α^* is G-equivariant.

Corollary 2.4.2. *Let G be a Hausdorff topological group. If X' is a compact minimal G-space and X is a G-boundary, then there is at most one G-equivariant u.c.p. map $C(X) \to C(X')$ and it is an injective *-homomorphism.*

Proof. If $\varphi\colon C(X) \to C(X')$ is a G-equivariant u.c.p. map, the map $\alpha\colon X' \to \mathcal{P}(X)$ given by $\alpha(x) = \delta_x \circ \varphi$ is G-equivariant and continuous, so it has image δ_X. Hence we obtain a surjective G-equivariant continuous map $\rho\colon X' \to X$ such that $\alpha(x) = \delta_{\rho(x)}$. As $\varphi(f)(x) = \alpha(x)(f) = f(\rho(x))$, φ is a *-homomorphism and it is injective because ρ is surjective. Finally, since ρ is the unique G-equivariant continuous map $X' \to X$, φ is also unique. ☐

We shall now consider how these uniqueness results translate to the setting of state spaces of C^*-algebras. For any compact convex subset Y of a locally convex space E, there exists a unique weak*-continuous, affine *barycenter map* $\beta\colon \mathcal{P}(Y) \to Y$ such that

$$f(\beta(\mu)) = \int_Y f \,\mathrm{d}\mu$$

for all continuous functions $f\colon Y \to \mathbb{R}$. In other words, any probability measure on Y is represented by a unique point in Y. This result, known as the *Choquet-Bishop-de Leeuw theorem*, is an immense asset to the study of compact convex spaces, and notably provides an alternative proof of the Krein-Milman theorem. We refer to [117, Chapter 1] for details.

Remark 2.4.3. With Y as above, then the barycenter map clearly satisfies $\beta(\delta_y) = y$ for all $y \in Y$. Conversely, if $y \in Y$ is an extreme point and $\beta(\mu) = y$ for some $\mu \in \mathcal{P}(Y)$, then $\mu = \delta_y$. Indeed, suppose that $\frac{1}{2}(\nu_1 + \nu_2) = \mu$ for $\nu_1, \nu_2 \in \mathcal{P}(Y)$. Then $f(\frac{1}{2}(\beta(\nu_1) + \beta(\nu_2))) = f(y)$ for any continuous real-valued linear functional $f \in E^*$. By the Hahn-Banach theorem and the assumption that y is an extreme point, $\beta(\nu_i) = y$ and $\int_Y f \,\mathrm{d}\mu = \int_Y f \,\mathrm{d}\nu_i$ for $i = 1, 2$ and all continuous functions $f\colon Y \to \mathbb{R}$, so that μ is an extreme point. Therefore $\mu \in \delta_Y$, implying $\mu = \delta_y$.

Definition 2.4.4. Let E be a locally convex space. If Y is a compact, convex subset of E, equipped with an action of a topological group G by affine homeomorphisms, we say that Y is a *compact convex G-space.* If Y contains no non-empty, proper, convex and closed G-invariant subsets, the action of G on Y is *irreducible.*

Similar to Proposition 2.1.4, an application of Zorn's lemma yields that any compact convex G-space contains an irreducible subset, i.e., a minimal non-empty, convex, closed and G-invariant subset. Note, moreover, that if Y is a compact convex G-space and $\beta\colon \mathcal{P}(Y) \to Y$ is the barycenter map, then β satisfies $f(g\beta(\mu)) = f(\beta(g\mu))$ for $f \in C(Y)$, $g \in G$ and $\mu \in C(Y)^*$, and is therefore G-equivariant.

The next result is due to Furstenberg, but first appeared in a seminal paper by Glasner on proximal actions [62].

Theorem 2.4.5. *Let E be a locally convex space and let $Y \subseteq E$ be a compact convex G-space. Then Y contains a G-boundary. In fact, if Y is irreducible, the closure of the set K of extreme points of Y is the unique minimal subset of Y, and it is a G-boundary. Conversely, if Y is the closed convex hull of a G-boundary inside E, then Y is irreducible.*

Proof. Any compact convex G-space contains an irreducible subset, so we may assume first that Y is irreducible. If $X \subseteq Y$ is a minimal subset, then Y is the closed convex hull of X. By Milman's converse to the Krein-Milman theorem, $K \subseteq X$. Thus $\overline{K} \subseteq X$, and since K is G-invariant we have equality. Therefore, if we prove that $X = \overline{K}$ is a strongly proximal G-space, then the first two assertions follow.

Let $M \subseteq \mathcal{P}(X)$ be a minimal subset, and let $\iota\colon X \to Y$ be the inclusion map with pushforward map $\iota_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$. If $\beta\colon \mathcal{P}(Y) \to Y$ is the G-equivariant affine barycenter map, then $\overline{\mathrm{conv}}\beta(\iota_*(M)) = Y$ since Y is irreducible. Therefore $K \subseteq \beta(\iota_*(M))$ by Milman's converse, so that $\delta_K \subseteq \iota_*(M)$ by the above remark. Therefore $\delta_K \subseteq M$ and $M = \overline{\delta_K} = \delta_X$ by minimality, so the claim follows from Lemma 2.1.12 (ii).

Finally, if $X \subseteq E$ is a G-boundary and Y is the closed convex hull of X, then the continuous G-equivariant map $\partial_F G \to X \subseteq Y$ extends to an affine weak*-continuous map $\mathcal{P}(\partial_F G) \to \mathcal{P}(Y)$. Composing this map with the barycenter map $\beta\colon \mathcal{P}(Y) \to Y$ yields an affine, continuous and G-equivariant map $\Phi\colon \mathcal{P}(\partial_F G) \to Y$. It is surjective since $X \subseteq \Phi(\delta_{\partial_F G})$ by construction. Finally, since $\partial_F G$ is a G-boundary, $\mathcal{P}(\partial_F G)$ is evidently irreducible, and therefore Y is also irreducible. $\square$

The above theorem has the effect that the Furstenberg boundary $\partial_F G$ admits an equivariant continuous map into any compact convex G-space, which we will immediately take advantage of to the fullest. The following proposition was first proved by Kalantar and Kennedy [90] and describes the so-called *G-injectivity* trait of the unital G-C^*-algebra $C(\partial_F G)$. One should regard it as a G-equivariant Hahn-Banach theorem: seeing as one may freely extend the domain of a $\mathbb{C}$-valued positive map from a C^*-algebra and retain the positivity, swapping $\mathbb{C}$ for $C(\partial_F G)$ allows us to retain G-equivariance of the map as well. We shall give a more detailed explanation of this aspect of the Furstenberg boundary in Section 5.1. The following proof, written in the language of compact convex G-spaces, is in the vein of [23, Proposition 2.5]. Notice that we require the C^*-algebras to be unital, as weak*-compactness of the state spaces cannot be done without.

Proposition 2.4.6. *Let G be a Hausdorff topological group and let A be a unital G-invariant C^*-subalgebra of a unital G-C^*-algebra B. Then any G-equivariant u.c.p. map $A \to C(\partial_F G)$ extends to a G-equivariant u.c.p. map $B \to C(\partial_F G)$.*

Proof. Let $r\colon S(B) \to S(A)$ be the G-equivariant, continuous and affine restriction map. If $\Phi\colon A \to C(\partial_F G)$ is G-equivariant and u.c.p., then we consider the G-equivariant continuous dual map $\phi\colon \partial_F G \to S(A)$ given by $\phi(x)(a) = \Phi(a)(x)$. If $Y \subseteq S(A)$ is the closed convex hull of the G-boundary $X = \phi(\partial_F G)$ by Theorem 2.4.5, then Y is irreducible and X is the unique minimal subset of Y.

Since $r^{-1}(Y) \subseteq S(B)$ is a compact convex G-space, there exists a G-boundary $Y' \subseteq r^{-1}(Y)$ by Theorem 2.4.5. Therefore, universality yields a G-equivariant continuous map $\tilde{\phi}\colon \partial_F G \to Y'$, so that we obtain a G-equivariant continuous map $r \circ \tilde{\phi}\colon \partial_F G \to Y$. As $(r \circ \tilde{\phi})(\partial_F G)$ is a minimal subset of Y, we have $(r \circ \tilde{\phi})(\partial_F G) = X$. By uniqueness of the G-equivariant surjection $\partial_F G \to X$ (Proposition 2.4.1), $r \circ \tilde{\phi} = \phi$, so we may define

$$\tilde{\Phi}(b)(x) = \tilde{\phi}(x)(b), \quad b \in B,\ x \in \partial_F G.$$

It is now easy to see that $\tilde{\Phi}$ extends Φ and is G-equivariant, unital and positive; by [115, Theorem 3.9] it is then completely positive. ☐

Lemma 2.4.7. *For any Hausdorff topological group G, there exists an embedding of $C(\partial_F G)$ into $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ as a G-invariant unital C^*-subalgebra. With respect to this embedding, there exists a G-equivariant conditional expectation $\rho\colon C_{\mathrm{b}}^{\mathrm{lu}}(G) \to C(\partial_F G)$.*

Proof. For $x \in \partial_F G$, define a G-equivariant *-homomorphism $P\colon C(\partial_F G) \to C_{\mathrm{b}}^{\mathrm{lu}}(G)$ by $P(f)(g) = f(gx)$. By Proposition 2.4.6 there is a G-equivariant u.c.p. map $\rho\colon C_{\mathrm{b}}^{\mathrm{lu}}(G) \to C(\partial_F G)$ extending the inclusion $\mathbb{C}1 \subseteq C_{\mathrm{b}}^{\mathrm{lu}}(G)$. Corollary 2.4.2 now yields that $\rho \circ P$ is the identity map on $C(\partial_F G)$, so that P is injective and $P \circ \rho$ is a conditional expectation of $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ onto $P(C(\partial_F G))$. ☐

Remark 2.4.8. A C^*-algebra A is said to be *monotone complete* if every increasing net of self-adjoint operators in A that is norm-bounded has a least upper bound in A. It is well-known that a commutative unital C^*-algebra $C(X)$ is monotone complete if and only if X is *extremally disconnected*, meaning that closures of open subsets of X are open [134, Proposition III.1.7]. A compact Hausdorff space that is extremally connected is also called a *Stonean* space.

For any non-empty set X, the C^*-algebra $\ell^\infty(X)$ is evidently monotone complete. If G is a discrete group, then the existence of a conditional expectation $\ell^\infty(G) \to C(\partial_F G)$, as verified above, now implies that $C(\partial_F G)$ is monotone complete. One may see this by means of an argument used in [33, Theorem 3.1], due to Tomiyama and Fakhoury. Indeed, if $(f_i)_{i\in I}$ is a bounded increasing net of real-valued functions in $C(\partial_F G) \subseteq \ell^\infty(G)$, then $(f_i)_{i\in I}$ has a least upper bound f in $\ell^\infty(G)$. Then $\rho(f) \geq \rho(f_i) = f_i$ for all i, and if $g \in C(\partial_F G)$ is real-valued and satisfies $g \geq f_i$ for all i, then $g \geq f$ and $g = \rho(g) \geq \rho(f)$, so that $\rho(f) \in C(\partial_F G)$ is a least upper bound for $(f_i)_{i\in I}$.

We note that this argument applies equally well to show that monotone completeness is preserved by positive projections; we will return to the topic of monotone complete C^*-algebras in Chapter 5.

Recalling our discussion of amenability in the previous section, we will now give a characterization of amenability in terms of boundary actions.

Proposition 2.4.9. *Let N be a closed, normal, amenable subgroup of a locally compact group G and let X be a G-boundary. Then N acts trivially on X.*

Proof. By Theorem 2.3.2, X has an N-invariant probability measure μ. Now let $n \in N$. We then have $n(g\mu) = g(g^{-1}ng\mu) = g\mu$ for all $g \in G$, so because δ_X is contained in the weak* closure of $G\mu$ it follows that n fixes every point in X. □

The corollary below is due to Furstenberg [58, Proposition 4.3] and Glasner [62, Theorem 7.1].

Corollary 2.4.10. *Let G be a locally compact group. Then $\partial_F G$ is a one-point space if and only if G is amenable.*

Proof. If G is amenable, then G acts trivially on the minimal G-space $\partial_F G$ by the previous proposition, so that $\partial_F G$ is a one-point space. Conversely, if X is a compact convex G-space, then let Y be an irreducible subset of X and let $x \in Y$. As the unique minimal subset of Y is a G-boundary by Theorem 2.4.5, it is trivial if $\partial_F G$ is a one-point space, so $gx = x$ for all $g \in G$. By Theorem 2.3.2, G is amenable. □

A generalization of the above corollary due to Furman [56, Proposition 7], which we will now prove (Theorem 2.4.12), has turned out to be an essential component of the study of C^*-simplicity and the unique trace property for discrete groups – in particular it links the latter property to the amenable radical, as seen in the paper [23] by Breuillard, Kalantar, Kennedy and Ozawa. We provide further details in the next chapter.

Definition 2.4.11. Let G be a group acting on a set X. For any $x \in X$, the *stabilizer subgroup* G_x is the subgroup $\{g \in G \mid gx = x\}$ of G.

Notice that if G is a topological group acting continuously on a Hausdorff space X, then for each $x \in X$ the stabilizer subgroup G_x is closed.

Theorem 2.4.12. *Let G be a locally compact group. Then $R(G) = \bigcap_{x\in\partial_F G} G_x$. In particular, $R(G) = \{1\}$ if and only if G admits a faithful boundary action. Moreover, $\partial_F G$ is G-equivariantly homeomorphic to $\partial_F(G/R(G))$.*

Contemplating why this is true, notice that Lemma 2.4.7 yields a G-equivariant u.c.p. extension $\rho\colon C_{\mathrm{b}}^{\mathrm{lu}}(G) \to C(\partial_F G)$ of the inclusion $\mathbb{C} \subseteq C_{\mathrm{b}}^{\mathrm{lu}}(G)$. Then for any $x \in \partial_F G$, the linear functional $\delta_x \circ \rho\colon C_{\mathrm{b}}^{\mathrm{lu}}(G) \to \mathbb{C}$ defines a mean on $C_{\mathrm{b}}^{\mathrm{lu}}(G)$ which is G_x-invariant. If it were possible to define a G_x-equivariant embedding $C_{\mathrm{b}}^{\mathrm{lu}}(G_x) \to C_{\mathrm{b}}^{\mathrm{lu}}(G)$, then G_x would be amenable, so that $\bigcap_{x\in\partial_F G} G_x$ would be a closed, normal, amenable subgroup of G. However, it is not clear why this would hold for any locally compact group G.

To remedy this, we require a notion first defined by Caprace and Monod in [29].

Definition 2.4.13. A subgroup H of a locally compact group G is said to be *relatively amenable* in G if there exists an H-invariant mean on $C_{\mathrm{b}}^{\mathrm{lu}}(G)$.

It is evident that an amenable subgroup of a locally compact group is relatively amenable, but it is still unknown whether the converse is true. In [29], Caprace and Monod prove that the class of locally compact groups such that each relatively amenable subgroup is in fact amenable is very large and has a wealth of stability properties. For instance, if G is a discrete group, then as $C_{\mathrm{b}}^{\mathrm{lu}}(H) = \ell^\infty(H)$ is contained in $C_{\mathrm{b}}^{\mathrm{lu}}(G) = \ell^\infty(G)$ as an H-invariant C^*-subalgebra for every subgroup $H \subseteq G$. This inclusion is obtained by identifying $f \in \ell^\infty(H)$ with the function $\tilde{f} \in \ell^\infty(G)$ given by $\tilde{f}(hr) = f(h)$, for $h \in H$ and r belonging to a subset of right coset representatives of H in G. Therefore any H-invariant mean on $\ell^\infty(G)$ restricts to an invariant mean on $\ell^\infty(H)$, so that any discrete group is contained in the aforementioned class. This proves Furman's result (Theorem 2.4.12) in the discrete case, as well as the following important observation.

Lemma 2.4.14. *For any discrete group G and any $x \in \partial_F G$, the stabilizer G_x is an amenable subgroup of G.*

We will need one of the stability properties of relative amenability to complete the proof of Theorem 2.4.12. The result itself follows from combining an observation of Derighetti [45] with a result of Caprace and Monod [29, Proposition 3]: if N is a closed, normal and relatively amenable subgroup of a locally compact group G, then N is amenable. A more self-contained proof of this fact is given in [111, Theorem 11].

Proof of Theorem 2.4.12. By Proposition 2.4.9, $R(G) \subseteq \bigcap_{x\in\partial_F G} G_x$. Since $\bigcap_{x\in\partial_F G} G_x$ is closed, normal and relatively amenable, it is amenable, so the reverse inclusion holds. Thus $R(G) = \{1\}$ if and only if the action of G on $\partial_F G$ is faithful, and if G admits a faithful boundary action, then universality of the Furstenberg boundary implies that the action of G on $\partial_F G$ is faithful too.

Now let $N = R(G)$. The surjective product map $G \times \partial_F G \to G/N \times \partial_F G$ is open and thus a quotient map. Since the G-action $G \times \partial_F G \to \partial_F G$ is constant on sets of the form $gN \times \{x\}$ for $g \in G$ and $x \in \partial_F G$, it passes to a continuous G/N-action $G/N \times \partial_F G \to \partial_F G$, with respect to which $\partial_F G$ is a G/N-boundary. Hence there exists a G/N-equivariant continuous surjection $\rho\colon \partial_F(G/N) \to \partial_F G$. Defining a G-action on $\partial_F(G/N)$ by means of the quotient map $G \to G/N$, ρ is G-equivariant, and since $\partial_F(G/N)$ is a G-boundary, there is a G-equivariant continuous surjection $\varphi\colon \partial_F G \to \partial_F(G/N)$. Due to Lemma 2.1.13 $\rho \circ \varphi$ is the identity map, so that φ is injective. $\square$

We will finally discuss the notion of strong amenability. Since a group is amenable if and only if it has trivial Furstenberg boundary, it is natural to ask when its universal minimal proximal compact space is trivial.

Definition 2.4.15. A Hausdorff topological group G is said to be *strongly amenable* if $\Pi(G)$ is a one-point space.

The class of strongly amenable groups contains many well-known classes of groups. To name one, if G is abelian and X is a minimal proximal G-space, then any $g \in G$ defines a G-equivariant continuous map $x \mapsto gx$, meaning that it must be the identity map by Lemma 2.1.13. By minimality X is a one-point space, so that all abelian groups are strongly amenable. In fact, Glasner proved that any compact extension of a nilpotent group is strongly amenable [62, Theorem 7.5]. Since there is a G-equivariant continuous surjection $\Pi(G) \to \partial_F G$ for any locally compact group G, strong amenability implies amenability, but the converse is not true: indeed, there exists a *solvable*, non-strongly amenable group, in an example due to Furstenberg [63, Example 5.5]. Recently, it has been shown by Hartman, Juschenko, Tamuz and Vahidi Ferdowsi [80] that the Thompson group F is not strongly amenable.

In the discrete case at the very least, we suspect that an analogue of the amenability of stabilizer subgroups for the action of a group G on $\partial_F G$ holds for strong amenability, i.e., that the stabilizer subgroups of a discrete group G acting on $\Pi(G)$ are strongly amenable.

2.5 The universal minimal compact G-space

In this section, we take a closer look at the properties of the universal minimal compact space of a discrete group, and we consider whether this space may be universal in other respects than minimality. At the end, we include some original observations to shed some light on this matter.

We gave a proof of the first half of this theorem, due to Ellis [49], in Section 2.1:

Theorem 2.5.1. *Let G be a Hausdorff topological group. There exists a universal minimal compact G-space M_G, i.e., for every minimal G-space X there exists a G-equivariant continuous surjection $M_G \to X$. Moreover, M_G is unique up to G-equivariant homeomorphism.*

A proof of the second half will, as we shall now see, give a good look into what additional structure one may find in M_G.

First, we observe that by using the universal property of the greatest ambit S_G, we may extend the action of G on S_G to a product $S_G \times S_G \to S_G$, $(x, y) \mapsto xy$, that extends the topological group structure of G to a *right-topological semigroup structure* on S_G, i.e., the binary operation $S_G \times S_G \to S_G$ is continuous in the first variable. Indeed, for fixed $y \in S_G$ there exists a unique G-equivariant continuous map $R_y\colon S_G \to S_G$ such that $R_y(1) = y$. Defining $xy := R_y(x)$ for $x, y \in S_G$, it is easy to see that this binary operation is associative and continuous in the first variable, its restriction to $G \times G$ is the original binary operation on G and its restriction to $G \times S_G$ is the G-action on S_G. Moreover, $1 \in G$ is the identity element of S_G and if $g \in G$ is fixed, then the self-map $x \mapsto gx$ of S_G is continuous by construction.

We will now take a closer look at the right-topological semigroup structure of S_G. A non-empty subset I of a semigroup E is called a *left ideal* of E if $EI \subseteq I$. Since G is dense in S_G, a subset $Z \subseteq S_G$ is then closed and G-invariant if and only if it is a closed left ideal in S_G. Moreover, if $Z \subseteq S_G$ is minimal among left ideals of S_G, then $Z = S_G z = \overline{Gz}$ for all $z \in Z$, so that Z is closed. It follows that $Z \subseteq S_G$ is a minimal left ideal if and only if Z is a minimal subset with respect to the G-action on S_G.

To make the semigroup structure of S_G a bit more tangible, we need the following lemma which, in the formulation below, is also due to Ellis [48, Lemma 1]. It requires only one-sided continuity, as opposed to earlier versions by Numakura and Wallace.

Lemma 2.5.2. *Let E be a compact right-topological semigroup.*

(i) *E contains an idempotent element.*

(ii) *If $S \subseteq E$ is a closed subset and $x \in E$ satisfies $x \in Sx$, then there is an idempotent $e \in S$ such that $ex = x$.*

Proof. (i) Consider the collection of closed non-empty subsets S of E satisfying $S^2 \subseteq S$. This collection is non-empty, as $E^2 \subseteq E$, and by the finite intersection property and Zorn's lemma, this collection has a minimal element S. Now, for any $x \in S$, Sx is closed and non-empty, and since $(Sx)(Sx) \subseteq S^3 x \subseteq Sx$ and $Sx \subseteq S$ it follows from minimality that $Sx = S$. Therefore $H = \{y \in S \mid yx = x\}$ is non-empty as well as closed, and since $H^2 \subseteq H \subseteq S$ we have $H = S$. Hence $x^2 = x \in E$.

(ii) Define $H = \{y \in S \mid yx = x\}$. Then H is closed and non-empty since $x \in Sx$. Moreover, $H^2 \subseteq H$ so that H is a compact right-topological subsemigroup of E. We then apply (i). $\square$

The above result, in particular (ii), now helps establish an array of algebraic properties for a minimal left ideal of the greatest ambit. The following two results are also due to Ellis [49].

Proposition 2.5.3. *Let Z be a minimal left ideal of a compact right-topological semigroup E and let $x \in Z$.*

(i) *Z contains an idempotent.*

(ii) *$Zx = Z$.*

(iii) *For all idempotents $e \in Z$, $xe = x$.*

(iv) *There is $z \in Z$ and an idempotent $e \in Z$ such that $ez = z$ and $xz = zx = e$.*

(v) *If $a \in E$ and $z \in Z$ satisfy $xa = za$, then $x = z$.*

Proof. (i) follows from Lemma 2.5.2. Moreover, Zx is a left ideal with $Zx \subseteq Z$, so $Zx = Z$ by minimality, yielding (ii). For (iii), let $e \in Z$ be an idempotent element. Then by (ii) there is $y \in Z$ with $ye = x$, so that $xe = ye^2 = ye = x$.

For (iv), note that by Lemma 2.5.2 there is an idempotent e with $ex = x$ since $x \in Z = Zx$. Since $e \in Z = Zx$, let $z \in Z$ with $e = zx$, and then take $y \in Z$ with $e = yz$. We now have

$$x = ex = yzx = ye = y,$$

so that $e = xz = zx$, proving (iv). Finally, for (v) let $y = ax \in Z$, so that $xy = xax = zax = zy$. Due to (iv) there is $r \in Z$ and an idempotent element $e \in Z$ such that $yr = e$, yielding $x = xe = xyr = zyr = ze = z$. $\square$

Theorem 2.5.4. *If $Z \subseteq S_G$ is a minimal left ideal and $\eta\colon Z \to Z$ is a continuous G-equivariant map, then there exists $x \in Z$ such that $\eta(z) = zx$ for all $z \in Z$. Moreover, η is a homeomorphism.*

Proof. Let $e \in Z$ be an idempotent. Then $\nu(x) = \eta(xe)$, $x \in S_G$, defines a G-equivariant continuous map $S_G \to S_G$. As $\nu(g) = g\nu(1) = R_{\nu(1)}(g)$ for all $g \in G$, it follows that $\nu = R_{\nu(1)}$. For $z \in Z$, $\eta(z) = \eta(ze) = \nu(z) = z\nu(1) = z\eta(e)$ by Proposition 2.5.3 (iii). Since $Z\eta(e) = Z$, η is surjective, and if $\eta(z_1) = \eta(z_2)$ for $z_1, z_2 \in Z$, then $z_1\eta(e) = z_2\eta(e)$. Therefore $z_1 = z_2$ by Proposition 2.5.3 (v). □

We can now prove the second half of Theorem 2.5.1. Indeed, let M_G be a minimal subset of S_G, i.e., M_G is a minimal left ideal of S_G. If Z is another universal minimal compact G-space, then there exist continuous G-equivariant surjections $\varphi_1\colon M_G \to Z$ and $\varphi_2\colon Z \to M_G$. Composing the two yields a continuous G-equivariant surjection $\varphi_2 \circ \varphi_1\colon M_G \to M_G$ which is a bijection due to Theorem 2.5.4. Thus φ_1 is injective, which completes the proof. We can therefore speak of the unique universal minimal G-space:

Definition 2.5.5. For any Hausdorff topological group G, let M_G denote the universal minimal compact G-space.

In view of the types of group actions considered in Section 2.1, it is natural to ask the following: for a group G, what separates the universal minimal compact G-space M_G from the other universal compact G-spaces constructed, namely $\Pi(G)$ and $\partial_F G$? It turns out that this is a very hard question to answer in full generality. We suspect, at least for discrete groups G, that M_G is only a proximal G-space when $G = \{1\}$, although we have not been able to verify why this should be so.

Of course there are simple ways of finding counterexamples to the statement that M_G is always homeomorphic to $\Pi(G)$, as indicated by the following remark.

Remark 2.5.6. Let G be a Hausdorff topological group containing a proper, closed subgroup H of finite index. Then G/H is a compact Hausdorff G-space on which the action is transitive. Since M_G surjects G-equivariantly onto G/H, proximality of M_G would imply proximality of G/H, but this contradicts (ii) of Proposition 2.1.11. Hence $M_{G/H}$ is not a proximal G-space.

We can give a more complete description of what is required in order for M_G to be a proximal G-space.

Proposition 2.5.7. *Let G be a Hausdorff topological group and let $M_G \subseteq S_G$ be the universal minimal compact G-space. Then the following are equivalent:*

(i) *M_G is a proximal G-space.*
(ii) *Every G-equivariant continuous surjection $M_G \to \Pi(G)$ is a homeomorphism.*
(iii) *Every compact minimal G-space is proximal.*
(iv) *Every element in $M_G \subseteq S_G$ is idempotent.*
(v) *Every element in any minimal left ideal of S_G is idempotent.*

Proof. The implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (v) $\Rightarrow$ (iv) are clear. If M_G contains z for which $z^2 \neq z$, then the map $x \mapsto xz$ defines a G-equivariant continuous map $Z \to Z$ (by Lemma 2.5.3 (ii)) which is not the identity map. Therefore by Lemma 2.1.13 M_G is not proximal, proving (i) $\Rightarrow$ (iv). Conversely, if every element in M_G is idempotent, then the only G-equivariant homeomorphism of M_G is the identity map due to Theorem 2.5.4 and Lemma 2.5.3 (iii). By Remark 2.1.20, M_G is G-equivariantly homeomorphic to $\Pi(G)$. That (iv) implies (v) follows from the fact that any two minimal left ideals in S_G are isomorphic [83, Theorem 2.11]. $\square$

One tactic for trying to prove that M_G is not proximal is thus to find a non-idempotent element in a minimal left ideal. We only give an idea of how this could be done in the case when G is a discrete group.

For a discrete group G, we identify S_G with the Stone-Čech compactification βG. It is well-known that βG can be identified with the space of ultrafilters on G. To give a short explanation of why this is so, we first recall that any subset of G is open in βG, that G is dense in βG and that βG has the following universal property: for any compact Hausdorff space X, any map $G \to X$ extends to a unique continuous map $\beta G \to X$. Using this property, one can then show that βG is an extremally disconnected topological space, i.e., every open set in βG has open closure. To see this, one may first note that any open set $S \subseteq \beta G$ satisfies $\overline{S} = \overline{S \cap G}$ since G is dense in βG, so that it suffices to show that $\overline{S}$ is open for $S \subseteq G$. It follows that $\beta G \setminus \overline{S} \subseteq \overline{G \setminus S}$, and that $\overline{S}$ and $\overline{G \setminus S}$ are disjoint by extending the characteristic function of S on G to a continuous map $\beta G \to \{0, 1\}$.

Recall that a *filter* on G is a collection $\mathcal{U}$ of subsets of G that does not contain the empty set and is stable under taking intersections and supersets. An *ultrafilter* on G is a filter $\mathcal{U}$ on G such that any subset $S \subseteq G$ satisfying $G \setminus S \notin \mathcal{U}$ is contained in $\mathcal{U}$. Equivalently formulated, ultrafilters are maximal filters on G. Finally, an ultrafilter $\mathcal{U}$ is *principal* if $\bigcap_{S\in\mathcal{U}} S \neq \emptyset$ – in which case $\bigcap_{S\in\mathcal{U}} S$ consists of one point – and *free* otherwise. We refer to [83] for more information on filters and ultrafilters.

One may now show for any $x \in \beta G$ that the collection $\mathcal{U}$ of subsets $G \cap U$, where $U \subseteq \beta G$ is an open neighbourhood of x, is an ultrafilter on G, and that it is principal if and only if $x \in G$. We identify any $x \in \beta G$ with the corresponding ultrafilter $\mathcal{U}$ on G, and the G-action on βG is given by left translation of the sets contained in the ultrafilter.

If we are to explain how to interpret an idempotent element in M_G in the language of ultrafilters, let us first see what a product of two ultrafilters in βG looks like. The following result can be found in [83, Theorem 4.12].

Theorem 2.5.8. *For all $x, y \in \beta G$ and $F \subseteq G$, the following are equivalent:*

(i) $F \in xy$.

(ii) *There exists $P \in x$ such that for all $s \in P$ there is some $Q \in y$ such that $sQ \subseteq F$.*

Proof. (i) $\Rightarrow$ (ii): Let U be an open neighbourhood of $xy \in \beta G$ such that $F = U \cap G \in xy$. By continuity there is an open neighbourhood V of $x \in \beta G$ such that $Vy \subseteq U$. Let $P = G \cap V \in x$ and let $s \in P$. As $sy \in Vy \subseteq U$, we can find an open neighbourhood W of $y \in \beta G$ such that $sW \subseteq U$ by continuity of the G-action. In particular, by letting $Q = W \cap G \in y$ we see that $sQ \subseteq sW \cap G \subseteq F$.

(ii) $\Rightarrow$ (i): Let $P \in x$ as in (ii) and for all $s \in P$, let $Q_s \in y$ with $sQ_s \subseteq F$. Let V_s be an open neighbourhood of y such that $Q_s = G \cap V_s$. Then

$$sy \in s\overline{V_s} = s\overline{Q_s} = \overline{sQ_s} \subseteq \overline{F}$$

for all $s \in P$, so that $Py \subseteq \overline{F}$. Letting U be an open neighbourhood of x such that $P = G \cap U$, we have $xy \in \overline{U}y = \overline{P}y = \overline{Py} \subseteq \overline{F}$. Thus $F \in xy$. ☐

Let us now show that $\beta G \setminus G$ contains a non-idempotent ultrafilter whenever G is an infinite discrete group. Suppose that $e \in \beta G \setminus G$ is an idempotent element. By the above result, for all $F \in e$ there exists $P \in e$ such that for all $s \in P$ there is a set $Q_s \in e$ satisfying $sQ_s \subseteq F$. Fixing some $s \in F \cap P$, then $Q = Q_s \cap F \cap (G \setminus \{s\})$ satisfies $Q \subseteq F \setminus \{s\}$ and $sQ \subseteq F$. Therefore, to find non-idempotent elements in $\beta G \setminus G$ for infinite G it suffices to find a countably infinite subset F such that

$$\{gs \mid g, s \in F,\ g \neq s\} \cap F = \emptyset,$$

i.e., an infinite *"product-free"* subset. This can be done as follows – we thank André Henriques (of Utrecht University) for providing us with an idea for the following proof. Let g_1 be a non-identity element in G, and let $g_2 \in G \setminus \{1, g_1, g_1^{-1}\}$. Then no product of g_1 and g_2 equals g_1, g_2 or 1. Let $F_2 = \{g_1, g_2\}$, define

$$G_2 = \{gs \mid g, s \in \{1\} \cup F_2 \cup F_2^{-1}\},$$

and take $g_3 \in G \setminus G_2$. Then no two-factor product of g_1, g_2 and g_3 equals either g_1, g_2 or g_3. Letting $F_3 = \{g_1, g_2, g_3\}$, we define G_3 by replacing F_2 by F_3, and proceed inductively. The set $F = \{g_n\}_{n \geq 1}$ then has the desired properties.

The question is now whether one can ensure that a product-free subset F belongs to an ultrafilter x contained in a minimal left ideal of βG, and it is possible to give a precise formulation of what is needed. We refer to [83, Theorem 4.40] for a proof of the following result.

Theorem 2.5.9. *Let A be a non-empty subset of a discrete group G. Then the following are equivalent:*

(i) *$A \in x$ for some ultrafilter x inside a minimal left ideal of βG.*
(ii) *There exists a finite subset $F \subseteq G$ such that the family $\{\bigcup_{t \in F} gtA \mid g \in G\}$ has the finite intersection property.*

We say that a subset $A \subseteq G$ is *piecewise syndetic* if it satisfies condition (ii). In a sense, a piecewise syndetic subset of a group is a subset to which all elements in the group have "short distance". A good visual to keep in mind here is the odd integers in $\mathbb{Z}$ – incidentally, it is clearly also product-free (the existence of a non-idempotent ultrafilter in $M_{\mathbb{Z}}$ can also be concluded by Proposition 2.5.7, since $\mathbb{Z}$ is strongly amenable).

Our conclusion is therefore that if a discrete group G contains a piecewise syndetic, product-free subset, then M_G is not a proximal G-space. However, we have not verified whether *every* non-trivial discrete group has this property (it may be easy!).

For the rest of this section, we will concentrate solely on finding criteria for a discrete group G *not* to satisfy $M_G \cong \Pi(G)$ or $M_G \cong \partial_F G$.

Definition 2.5.10. Let G be a group acting on a set X. The action of G on X is said to be *free* if $G_x = \{1\}$ for all $x \in X$, and *faithful* if $\bigcap_{x\in X} G_x = \{1\}$.

Using the structure of βG as the space of ultrafilters on G, we have the following theorem, due to Ellis [50]. Veech gave a proof of the locally compact case in [139].

Theorem 2.5.11. *The action of a discrete group G on βG is free. In particular, the action of G on the universal minimal compact G-space M_G is free.*

Proof. Let $g \in G \setminus \{1\}$. Then $\mathcal{F} = \{F \mid F \subseteq G \text{ and } gF \cap F = \emptyset\}$ is a non-empty collection of subsets of G, partially ordered by inclusion. If $(F_i)_{i\in I}$ is an increasing chain of subsets of $\mathcal{F}$, then for any distinct $i, j \in I$ we may assume $i \leq j$ and note that $gF_i \cap F_j \subseteq gF_j \cap F_j = \emptyset$, implying that $\bigcup_{i\in I} F_i$ is an upper bound. Hence $\mathcal{F}$ has a maximal subset F by Zorn's lemma. If $s \in G \setminus (gF \cup F)$, then by maximality we have $F \cup \{s\} \notin \mathcal{F}$, so that $(gF \cup \{gs\}) \cap (F \cup \{s\}) \neq \emptyset$. Therefore $gs \in F$ and $s \in g^{-1}F$, so that $G = F \cup gF \cup g^{-1}F$.

Assuming now that $x \in \beta G$ satisfies $gx = x$, then $g^{-1}x = x$ too, and

$$x = \{gS \mid S \in x\} = \{g^{-1}S \mid S \in x\}.$$

As x contains at least one of the sets $F, gF, g^{-1}F$, it must contain all of them, but then $\emptyset = F \cap gF \in x$, a contradiction. Hence G acts freely on βG. ☐

Remark 2.5.12. Let G be a discrete group with universal minimal compact G-space M_G and suppose that M_G is G-equivariantly homeomorphic to $\Pi(G)$. If N is a normal subgroup of G, then the quotient map $G \to G/N$ defines a minimal G-action on the universal compact minimal G/N-space $M_{G/N}$. By assumption, this action is now proximal, but due to how we defined the G-action on G/N it follows that the G/N-action on $M_{G/N}$ is also proximal. In particular the action of G/N on $\Pi(G/N)$ is free by the above theorem.

We do not know whether freeness of the action of a discrete group G on $\Pi(G)$ can be characterized in terms of the algebraic structure of the group. What we *can* say is that G is icc whenever G acts *faithfully* on a minimal proximal compact G-space X. Indeed, if $g \in G$ has finite conjugacy class, then the centralizer H of $g \in G$ has finite index in G. The action of H on X is minimal and proximal [63, Lemma III.3.2], so since the map $X \to X$ given by $x \mapsto gx$ is H-equivariant and continuous, Lemma 2.1.13 yields $gx = x$ for all $x \in X$. By faithfulness, $g = 1$.

Remark 2.5.13. Remark 2.5.12 still holds if we replace $\Pi(G)$ with the Furstenberg boundary $\partial_F G$. In this case we can say a bit more, thanks to a theorem by Breuillard, Kalantar, Kennedy and Ozawa to be elaborated upon in the next chapter (Theorem 3.2.6): the action of a discrete group G on $\partial_F G$ is free if and only if G is C^*-simple. Hence if G has a non-C^*-simple quotient, the universal minimal compact G-space is not strongly proximal.

We conclude the chapter in humble fashion by combining the pieces of information from the above discussion. Note that the existence of a finite index subgroup implies the existence of a finite index normal subgroup and thus a finite quotient (see Proposition 3.3.1), so that Remark 2.5.6 is also covered.

Proposition 2.5.14. *Let G be a discrete group and let M_G be the universal minimal compact G-space.*

(i) *If G has a non-icc quotient or contains a piecewise syndetic, product-free subset, then M_G is not a proximal G-space, nor a strongly proximal G-space.*

(ii) *If G has a non-C^*-simple quotient, then M_G is not a strongly proximal G-space.*

In particular, if a discrete group G has non-trivial abelianization, then M_G is not proximal. Since the abelianization of any non-abelian free group $\mathbb{F}$ is the free abelian group of the same set of generators, $M_{\mathbb{F}}$ is not a proximal $\mathbb{F}$-space. Many other examples can be found this way.

Chapter 3

C^*-simple discrete groups

In this chapter, we discuss important applications of the theory of boundary actions to reduced group C^*-algebras, the chief situation of interest being when the reduced group C^*-algebra is simple and/or has a unique tracial state. We give a runthrough of the recent characterizations of these two situations made by Kalantar and Kennedy, and Breuillard, Kalantar, Kennedy and Ozawa, and finally we give examples of groups with these properties. The last two sections of this chapter constitutes our own work with Tron Omland on the topic of C^*-simplicity of non-ascending HNN extensions.

3.1 Prologue

In this section, we give definitions of the two properties for discrete groups to be investigated in this chapter, and we use the opportunity to give some historical remarks, explaining how these properties came to be considered important, in order to set the stage properly for the next section. We follow [25] and [140].

For a discrete group G, consider the group algebra $\ell^1(G)$ equipped with the following product and involution:

$$(xy)(s) = \sum_{g \in G} x(g)y(g^{-1}s), \quad x^*(s) = \overline{x(s^{-1})}, \quad x, y \in \ell^1(G),\ s \in G.$$

The product is also known as the *convolution* of two functions $x, y\colon G \to \mathbb{C}$. With respect to these operations and the usual 1-norm, $\ell^1(G)$ is a Banach *-algebra with identity δ_1. The characteristic functions $\delta_g \in \ell^1(G)$ satisfy $\delta_g\delta_g^* = \delta_g^*\delta_g = \delta_1$, and the self-adjoint subalgebra $C_c(G)$ of finitely supported functions on G constitute a dense subset of $\ell^1(G)$.

Recalling that any *-homomorphism from a Banach *-algebra to a C^*-algebra is contractive [46, Proposition 1.3.7], we define a norm $\|\cdot\|_u$ on $\ell^1(G)$ by setting $\|x\|_u = \sup\|\pi(x)\|$ for $x \in \ell^1(G)$, where π runs through all non-degenerate representations of $\ell^1(G)$ on a Hilbert space. Completing $\ell^1(G)$ with respect to $\|\cdot\|_u$, we obtain the unital C^*-algebra known as the *full group C^*-algebra*, and it is denoted by $C^*(G)$. By definition, any non-degenerate representation of $\ell^1(G)$ on a Hilbert space H extends to a non-degenerate representation of $C^*(G)$ on H, and this correspondence of representations is one-to-one.

A *unitary representation* of G is a group homomorphism of G into the group $\mathcal{U}(H)$ of unitary operators on some Hilbert space H. There is a one-to-one correspondence between unitary representations of G and non-degenerate representations of $\ell^1(G)$, given by mapping $f = \sum_{g\in G} f(g)\delta_g \in \ell^1(G)$ to the operator $\sum_{g\in G} f(g)\pi_g$ in $B(H)$, where $\pi\colon g \mapsto \pi_g$ is a unitary representation of G on the Hilbert space H. In particular, any unitary representation of G can be used to construct a C^*-algebra. Indeed, if $\pi\colon C^*(G) \to B(H)$ is the non-degenerate representation induced by a unitary representation $\pi\colon G \to \mathcal{U}(H)$, then the C^*-algebra associated to π is given by $C^*_\pi(G) = \pi(C^*(G))$.

We next consider the *(left) regular representation* λ in the unitary group of $\ell^2(G)$, given by left translation:

$$[\lambda_g\xi](s) = \xi(g^{-1}s), \quad g, s \in G, \ \xi \in \ell^2(G).$$

With respect to the canonical orthonormal basis $\{\delta_s \mid s \in G\}$ of $\ell^2(G)$, λ uniquely satisfies

$$\lambda_g\delta_s = \delta_{gs}, \quad g, s \in G.$$

The *reduced group C^*-algebra* $C^*_r(G)$ is the C^*-algebra $C^*_\lambda(G)$ associated to λ, and $C^*_r(G)$ is therefore the norm-closure in $B(\ell^2(G))$ of the set of operators of the form $\sum_{g\in G} \eta_g\lambda_g$, where $\eta_g \in \mathbb{C}$ is non-zero for at most finitely many $g \in G$. Moreover, $C^*_r(G)$ is equipped with a faithful tracial state τ, given by $\tau(x) = \langle x\delta_1, \delta_1\rangle$. We refer to τ as the *canonical tracial state* on $C^*_r(G)$.

Definition 3.1.1. A discrete group G is said to be *C^*-simple* if $C^*_r(G)$ is a simple C^*-algebra, and it is said to have the *unique trace property* if $C^*_r(G)$ admits a unique tracial state.

We briefly remark that Bédos was the first to use the word "C^*-simple" in an article [9]; nonetheless, this name did not gain ground until de la Harpe's 2007 survey [41].

In the seminal four-paper series *On rings of operators* (1936–43) by Murray and von Neumann, laying the foundation for the topic of classification for von Neumann algebras, group von Neumann algebras were among the first concrete examples given [107, §5.3]. In their study, they gave a sufficient and necessary criterion for a discrete group G to generate a *factor* (that is, a von Neumann algebra with trivial center) from its left regular representation: G had to be *icc.*

With the advent of C^*-algebras in the 1940's came a desire to translate the classification results of Murray and von Neumann into C^*-algebraic versions. The observation that the kernel of a *-homomorphism between two von Neumann algebras that also respects the topologies (nowadays known as a *normal* *-homomorphism), always has a maximal projection which is central in the domain, implies that a normal *-homomorphism from a factor will necessarily be one-to-one. As a consequence, simplicity of a C^*-algebra naturally arose as the C^*-analogue of being a factor.

In a 1958 survey on problems and results in functional analysis [91], Kaplansky posed the question whether a simple C^*-algebra always contained a non-trivial projection; he had first discussed the question with Kadison nine years earlier. At a conference in Baton Rouge in 1967, Dixmier again raised the question. Kadison suggested to Robert T. Powers, a then newly-hatched PhD student from Princeton University, that the reduced group C^*-algebra of the free group $\mathbb{F}_2$ of rank two might provide a counterexample. Within a week, Powers had proved that this C^*-algebra was in fact simple [41, p. 13], finally publishing the result in 1975 [120]. In 1982, Pimsner and Voiculescu gave a proof that this C^*-algebra did indeed contain only trivial projections.

After the result of Powers, the question of when the reduced group C^*-algebra of a discrete group is simple slowly gained interest. Powers had also proved that the aforementioned reduced group C^*-algebra had a unique tracial state. Generalizing both the result of Powers and an analoguous result due to Choi for the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ [32], Paschke and Salinas proved in 1979 that simplicity and uniqueness of the tracial state held equally true for the reduced C^*-algebra of a non-degenerate free product, using the techniques of Powers developed in his proof. Moreover, they also proved that any group satisfying either of these two properties must also have trivial amenable radical [114].

In 1985, Pierre de la Harpe published a survey [39] on groups with simple reduced group C^*-algebras admitting a unique tracial state, asking several questions some of which would continue to be open problems until 2015. Among these were the question whether simplicity of the reduced group C^*-algebra would imply uniqueness of the tracial state, or vice versa. Harpe also coined the notion of a *Powers group*, meaning a group that had combinatorial traits similar to those of $\mathbb{F}_2$ that yielded Powers' 1975 result, and he found many new examples of Powers groups, most of which were obtained from investigating hyperbolic actions. Most importantly, Powers groups were shown to be C^*-simple and have the unique trace property. The year after, de la Harpe and Skandalis proved that Powers groups had the remarkable property that reduced crossed products over Powers groups were simple if and only if the trivial ideals in the underlying C^*-algebra were the only ones to be invariant under the group action [44].

Following the work of de la Harpe, C^*-simplicity has really become a topic in its own right, especially when viewed as a sort of polar opposite to amenability. Whereas amenable groups are quite "benevolent" in their behaviour towards dynamical systems in particular (as already evidenced in Section 2.4), C^*-simple groups constitute a class of *very* non-amenable groups that nonetheless behave quite nicely, especially with respect to reduced crossed products.

3.2 C^*-simplicity and boundary actions

In this section, we present some of the recent advances in C^*-simplicity and the unique trace property, including both the dynamical characterization of C^*-simplicity in terms of boundary actions, due to Kalantar and Kennedy [90], and the fact that C^*-simple groups always have the unique trace property, due to Breuillard, Kalantar, Kennedy and Ozawa [23]. To obtain the most efficient proofs, we combine arguments from the original paper [23] with techniques found in [27], [68] and [92]. We also briefly mention other recent results on the same topic.

For an action α of a discrete group G on a C^*-algebra A, let us first recall the definition and properties of the reduced crossed product $A \rtimes_r G$ (cf. [25, Chapter 4.1]). We will define the more general case of a reduced twisted crossed product in Chapter 4, but it is worth treating this simpler set-up first. We keep in mind that the properties to ensure for this C^*-algebra is that its multiplier algebra contains a copy of the group G as a subgroup of unitary operators, *and* a copy of A in such a way that the action of G on A is implemented by the unitary multipliers of G.

Let $\ell^1(G,A)$ denote the space of functions $x\colon G\to A$ satisfying $\sum_{g\in G}\|x(g)\|<\infty$. We will often use the notation $x=\sum_{g\in G}x_g\delta_g$ for a function $x\in\ell^1(G,A)$, where $x_g=x(g)\in A$ for $g\in G$. We next equip $\ell^1(G,A)$ with a product and involution by defining

$$(xy)(s)=\sum_{g\in G}x(g)(gy(g^{-1}s)),\quad x^*(s)=sx(s^{-1})^*,$$

so that $\ell^1(G,A)$ becomes a Banach *-algebra in the 1-norm. We identify A with the image of A under the *-homomorphism $a\mapsto a\delta_1$. Notice also that the subset $C_c(G,A)$ of finitely supported functions $G\to A$ is a dense *-subalgebra of $\ell^1(G,A)$, and that an approximate identity (e_i) in A yields an approximate identity $(e_i\delta_1)$ in $\ell^1(G,A)$.

A *covariant representation* of the C^*-dynamical system (A,G,α) is a triple (π,u,H), where H is a Hilbert space, $\pi\colon A\to B(H)$ is a non-degenerate representation and $u\colon G\to\mathcal{U}(H)$ is a unitary representation, such that $\pi(ga)=u_gau_g^*$ for $g\in G$ and $a\in A$. We often suppress the Hilbert space H from the notation if it is clear from the context. The associated *integrated form* of a covariant representation (π,u) is the map $\pi\times u\colon\ell^1(G,A)\to B(H)$ defined by

$$(\pi\times u)(x)=\sum_{g\in G}\pi(x_g)u_g,\quad x\in\ell^1(G,A).$$

The *full crossed product* of (A,G,α), denoted by $A\rtimes_\alpha G=A\rtimes G$, is the completion of $\ell^1(G,A)$ or $C_c(G,A)$ with respect to the norm

$$\|x\|_u=\sup\|(\pi\times u)(x)\|,\quad x\in\ell^1(G,A),$$

the supremum taken over all (cyclic) covariant representations (π,u,H) of (A,G,α).

To define the reduced crossed product, we assume that $A\subseteq B(H)$ is faithfully represented and define a faithful representation $\pi\colon A\to B(H\otimes\ell^2(G))$ and a unitary representation $\lambda\colon G\to B(H\otimes\ell^2(G))$ by

$$\pi(a)(\xi\otimes\delta_s)=(s^{-1}a)\xi\otimes\delta_s,\quad\lambda_g(\xi\otimes\delta_s)=\xi\otimes\delta_{gs},\quad a\in A,\ \xi\in H,\ g,s\in G.$$

It is easily verified that $(\pi,\lambda,H\otimes\ell^2(G))$ is a covariant representation of (A,G,α), and we call it a *regular* representation of this C^*-dynamical system. Notice also that λ is actually an amplification of the left regular representation of G on $\ell^2(G)$. The associated form $\pi\times\lambda\colon\ell^1(G,A)\to B(H\otimes\ell^2(G))$ is faithful, and the *reduced crossed product* $A\rtimes_{\alpha,r}G=A\rtimes_r G$ is the completion of $\ell^1(G,A)$ or $C_c(G,A)$ in the *reduced norm*

$$\|x\|_r=\|(\pi\times\lambda)(x)\|_{B(H\otimes\ell^2(G))},\quad x\in\ell^1(G,A).$$

Equivalently, $A\rtimes_r G$ can be taken to be the norm closure of the image of $\pi\times\lambda$ or $\pi\times\lambda|_{C_c(G,A)}$. It is well-known that $A\rtimes_r G$ does not depend on the choice of faithful representation $A\subseteq B(H)$ (see, e.g., [25, Proposition 4.1.5]). We define a G-action on $A\rtimes_r G$ by means of the inner automorphisms $g\mapsto\mathrm{Ad}(\lambda_g)$, so that the inclusion $A\subseteq A\rtimes_r G$ is G-equivariant.

Identifying A via its image under π, then the reduced crossed product also has the nifty property of admitting a faithful conditional expectation $E_A\colon A\rtimes_r G\to A$ that is G-equivariant and uniquely satisfies $E_A(x)=x_1$ for all $x=\sum_{g\in G}x_g\lambda_g\in\ell^1(G,A)\subseteq A\rtimes_r G$. This will be referred to as the *canonical conditional expectation*, and we will write E instead of E_A, if the dynamical system is clear from the context. The existence of a faithful conditional expectation of $A \rtimes_r G$ onto A also characterizes the reduced crossed product among all C^*-algebras generated by the image of (the integrated form of) a covariant representation of (A, G, α) [140, Théorème 4.22].

In fact, something more general holds. If $H \subseteq G$ is a subgroup, then there exists an injective *-homomorphism $A \rtimes_r H \to A \rtimes_r G$ that extends the inclusion $C_c(H, A) \to C_c(G, A)$, and if we identify $A \rtimes_r H$ with its image under this *-homomorphism, there exists a faithful conditional expectation $E_H\colon A \rtimes_r G \to A \rtimes_r H$ that uniquely satisfies $E_H(\lambda_g) = 0$ for all $g \notin H$. We give a proof in the more general case of reduced *twisted* crossed products in Proposition 4.1.2.

Finally we mention that if A, B are G-C^*-algebras and $\varphi\colon A \to B$ is a G-equivariant c.c.p. map, then the map $\tilde{\varphi}\colon \ell^1(G, A) \to \ell^1(G, B)$ given by

$$\varphi(x)_g = \varphi(x_g), \quad x \in \ell^1(G, A),\ g \in G,$$

extends to a c.c.p. map $\tilde{\varphi}\colon A \rtimes_r G \to B \rtimes_r G$ [130, Lemma 1.2.1]. Notice that $\tilde{\varphi}$ uniquely satisfies

$$\tilde{\varphi}(a\lambda_g) = \varphi(a)\lambda_g, \quad a \in A,\ g \in G.$$

More often than not, it happens that a property of φ is inherited by $\tilde{\varphi}$. It is easy to show that this includes faithfulness, surjectivity and being a *-homomorphism.

In order to state the results of Kalantar and Kennedy, we need first to introduce the notion of a topologically free action.

Definition 3.2.1. For an action of a discrete group G on a topological space X, define

$$X^g = \{x \in X \mid gx = x\}, \quad g \in G.$$

We say that the action of G on X is *topologically free* if X^g has empty interior for all $g \in G \setminus \{1\}$.

There is another way of describing topological freeness, in terms of subgroups of the group G. If we define $G_x^\circ = \{g \in G \mid gx' = x' \text{ for all } x' \text{ in a neighbourhood of } x\}$ for all $x \in X$, then it is easy to see that the action of G on X is topologically free if and only if $G_x^\circ = \{1\}$ for all $x \in X$. Notice also that G_x° is always a normal subgroup of the stabilizer G_x, and that the action of G on X is free if X^g is empty for all $g \in G \setminus \{1\}$.

Another notion that will be most helpful in our own results is the following.

Definition 3.2.2. Let A and B be C^*-algebras and let φ be a c.c.p. map $\varphi\colon A \to B$. The *multiplicative domain* $\mathrm{mult}(\varphi)$ of φ is the subset of A given by

$$\mathrm{mult}(\varphi) = \{a \in A \mid \varphi(a^*a) = \varphi(a)^*\varphi(a),\ \varphi(aa^*) = \varphi(a)\varphi(a)^*\}.$$

A result of Choi [31, Theorem 3.1] (and indeed, the reason for naming the above subset so) is that for φ as above,

$$\mathrm{mult}(\varphi) = \{a \in A \mid \varphi(ax) = \varphi(a)\varphi(x),\ \varphi(xa) = \varphi(x)\varphi(a) \text{ for all } x \in A\}.$$

In particular, if $B \subseteq A$ is a C^*-subalgebra and $\varphi\colon A \to B$ is a c.c.p. map that restricts to the identity map on B, then φ is in fact a conditional expectation of A onto B.

We now prove a commutative version of a theorem of Archbold and Spielberg [5, Theorem 1].

Proposition 3.2.3. *Let X be a compact G-space on which the action of G is topologically free. If $I \subseteq C(X) \rtimes_r G$ is a closed ideal such that $I \cap C(X) = \{0\}$, then $I = \{0\}$.*

Proof. Let $E\colon C(X) \rtimes_r G \to C(X)$ be the canonical conditional expectation. Since E is faithful, it is enough to show that $E(I) = \{0\}$. Assume for a contradiction that there exists $a \in I$ with $E(a) \neq 0$. Then there exists $b \in C_c(G, C(X))$ such that $2\|a - b\| < \|E(a)\|$. Letting $S = \{g \in G \setminus \{1\} \mid b_g \neq 0\}$, then $X \setminus X^g$ is open and dense in X for all $g \in S$ by hypothesis, so that $Y = \bigcap_{g \in S\setminus\{1\}}(X \setminus X^g)$ is open and dense in X as well.

Now fix $y \in Y$. Since $I \cap C(X) = \{0\}$, we obtain a *-homomorphism

$$\varphi_0\colon C(X) + I \to (C(X) + I)/I \cong C(X) \xrightarrow{\delta_y} \mathbb{C}$$

satisfying $\varphi_0|_{C(X)} = \delta_y$ and $\varphi_0|_I = 0$. Letting $\varphi\colon C(X) \rtimes_r G \to \mathbb{C}$ be a Hahn-Banach extension of φ_0, then φ is a state. For $g \in S \setminus \{1\}$ there exists $f \in C(X)$ with $f(y) \neq f(g^{-1}y)$. As $C(X) + I \subseteq \mathrm{mult}(\varphi_0) \subseteq \mathrm{mult}(\varphi)$, we now see that

$$\varphi(\lambda_g)f(y) = \varphi(\lambda_g)\varphi(f) = \varphi(\lambda_g f) = \varphi(gf\lambda_g) = \varphi(gf)\varphi(\lambda_g) = f(g^{-1}y)\varphi(\lambda_g).$$

Therefore $\varphi(\lambda_g) = 0$, so that

$$\varphi(b) = \sum_{g\in S}\varphi(b_g\lambda_g) = \sum_{g\in S}\varphi(b_g)\varphi(\lambda_g) = \varphi(b_1) = b_1(y) = E(b)(y).$$

Since $a \in I$ we have $\varphi(a) = 0$, so that $|E(b)(y)| = |\varphi(b)| = |\varphi(b - a)| \leq \|b - a\|$ for all $y \in Y$. Therefore $\|E(b)\| \leq \|b - a\|$ since Y is dense in X, but then

$$\|E(a)\| \leq \|E(a - b)\| + \|E(b)\| \leq 2\|a - b\| < \|E(a)\|,$$

a contradiction. Hence $E(I) = \{0\}$. $\square$

The above result also holds true for topologically free actions on C^*-algebras that are possibly non-unital and non-commutative; we refer to the original article [5] for details.

If X is a compact G-space and $x \in X$, then by composing the faithful conditional expectation $E_{G_x^\circ}\colon C(X) \rtimes_r G \to C(X) \rtimes_r G_x^\circ$ and the *-homomorphism $C(X) \rtimes_r G_x^\circ \to C_r^*(G_x^\circ)$ extending the G_x°-equivariant *-homomorphism $\delta_x\colon C(X) \to \mathbb{C}$, we obtain a u.c.p. map $E_x\colon C(X) \rtimes_r G \to C_r^*(G_x^\circ)$, satisfying

$$E_x(f\lambda_g) = f(x)E_{G_x^\circ}(\lambda_g), \quad f \in C(X),\ g \in G.$$

The next result is a reformulation of [92, Lemma 2.4], the chief inspiration for our proof being a result by Kawamura and Tomiyama [93, Theorem 4.1].

Proposition 3.2.4. *Let X be a compact G-space for which $\{x \in X \mid G_x^\circ \text{ is amenable}\}$ is dense in X. If the action of G on X is not topologically free, then there exists a non-zero closed ideal $I \subseteq C(X) \rtimes_r G$ for which $I \cap C(X) = \{0\}$.*

Proof. Let $X_{\mathrm{am}} = \{x \in X \mid G_x^\circ \text{ is amenable}\}$. For every $x \in X_{\mathrm{am}}$ there exists a *-homomorphism $\tau\colon C_r^*(G_x^\circ) \to \mathbb{C}$ such that $\tau(\lambda_g) = 1$ for all $g \in G_x^\circ$ [25, Theorem 2.6.8]. We then define $\varphi_x = \tau \circ E_x$. Notice that $C(X) \subseteq \mathrm{mult}(\varphi_x)$ by construction.

If the action of G on X is not topologically free, then there exists $g \in G \setminus \{1\}$ and a non-zero function $\xi \in C(X)$ with support contained in the interior of X^g. Then $\xi - \xi\lambda_g \neq 0$, so the closed ideal $I \subseteq C(X) \rtimes_r G$ generated by $\xi - \xi\lambda_g \in C_c(G, C(X))$ is non-zero. We now claim that $C(X) \cap I = \{0\}$. To see this, we first observe that for $x \in X_{\mathrm{am}}$, $f_1, f_2 \in C(X)$ and $h, s \in G$,

$$\begin{aligned}\varphi_x(f_1\lambda_h(\xi - \xi\lambda_g)\lambda_s f_2) &= f_1(x)\varphi_x(\lambda_h\xi\lambda_s - \lambda_h\xi\lambda_{gs})f_2(x)\\ &= f_1(x)\xi(h^{-1}x)(\varphi_x(\lambda_{hs}) - \varphi_x(\lambda_{hgs}))f_2(x) = 0.\end{aligned}$$

Indeed, if $\xi(h^{-1}x) \neq 0$, then $h^{-1}x$ is contained in the interior U of X^g, so that $hg^{-1}h^{-1}$ acts as the identity on hU. In particular, $(hs)(hgs)^{-1} = hg^{-1}h^{-1} \in G_x^\circ$, so that $hs \in G_x^\circ$ if and only if $hgs \in G_x^\circ$, so that $\varphi_x(\lambda_{hs}) = \varphi_x(\lambda_{hgs})$. Since the linear span of all elements of the form $f_1\lambda_h(\xi - \xi\lambda_g)\lambda_s f_2$ is dense in I, we have $\varphi_x(I) = \{0\}$ for all $x \in X_{\mathrm{am}}$. Finally, if $f \in C(X) \cap I$ then $f(x) = \varphi_x(f) = 0$ for all $x \in X_{\mathrm{am}}$, so $f = 0$ by X_{am} being dense in X. □

For later reference we include a theorem of Frolík on homeomorphisms of Stonean spaces [55, Theorem 3.1]. A more self-contained proof is given in [118, Proposition 2.7].

Theorem 3.2.5. *Let X be a Stonean space (see Remark 2.4.8). If $f\colon X \to X$ is a homeomorphism, then the fixed point set of f is clopen. In particular, a group action on X is topologically free if and only if it is free.*

We are now ready to prove the theorem of Kalantar and Kennedy [90, Theorem 6.2] that characterizes C^*-simplicity in terms of boundary actions. The equivalence of topological freeness and freeness of the action in (vi) was first observed by Kalantar, Kennedy, Breuillard and Ozawa [23, Theorem 3.1].

Theorem 3.2.6. *Let G be a discrete group. Then the following are equivalent:*

(i) *G is C^*-simple.*
(ii) *$C(X) \rtimes_r G$ is simple for some G-boundary X.*
(iii) *$C(X) \rtimes_r G$ is simple for all G-boundaries X.*
(iv) *$C(\partial_F G) \rtimes_r G$ is simple.*
(v) *The action of G on some G-boundary is topologically free.*
(vi) *The action of G on $\partial_F G$ is free, or equivalently, topologically free.*

Proof. The implications (iii) ⇒ (ii), (iii) ⇒ (iv) and (vi) ⇒ (v) are trivial. Moreover, by the Frolík theorem, the action of G on $\partial_F G$ is topologically free if and only if it is free, since $\partial_F G$ is Stonean (Remark 2.4.8). (v) ⇒ (ii) and (vi) ⇒ (iv) follow from Theorem 3.2.3. If $C(\partial_F G) \rtimes_r G$ is simple, then since all stabilizer subgroups for the G-action on $\partial_F G$ are amenable by Lemma 2.4.14, the action of G on $\partial_F G$ is topologically free by Proposition 3.2.4, proving (iv) ⇒ (vi).

We next prove (i) ⇒ (iii). Let X be a G-boundary. By Corollary 2.4.2 we may assume that there is a G-equivariant unital C^*-subalgebra inclusion $C(X) \subseteq C(\partial_F G)$.

Let $\pi\colon C(X) \rtimes_r G \to B$ be a unital *-homomorphism. We define an action of G on B by means of the inner automorphisms $\mathrm{Ad}(\pi(\lambda_g))$ of B, so that π becomes G-equivariant. Via the inclusion $\mathbb{C} \subseteq C(X)$, we view $C_r^*(G)$ as a unital G-invariant C^*-subalgebra of $C(X) \rtimes_r G$. If $C_r^*(G)$ is simple, then $\pi|_{C_r^*(G)}$ is injective, so the canonical tracial state $\tau\colon C_r^*(G) \to \mathbb{C} \subseteq C(\partial_F G)$ extends to a G-equivariant u.c.p. map $\tilde{\tau}\colon B \to C(\partial_F G)$ such that $\tilde{\tau} \circ \pi|_{C_r^*(G)} = \tau$ by Proposition 2.4.6. By Corollary 2.4.2 the map $\tilde{\tau} \circ \pi|_{C(X)}\colon C(X) \to C(\partial_F G)$ is the inclusion map $C(X) \hookrightarrow C(\partial_F G)$. In particular, $C(X) \subseteq \mathrm{mult}(\tilde{\tau} \circ \pi)$. If $E\colon C(X) \rtimes_r G \to C(X)$ is the canonical faithful conditional expectation, then $\tilde{\tau}(\pi(f\lambda_g)) = f\tau(\lambda_g) = E(f\lambda_g)$ in $C(\partial_F G)$ for all $f \in C(X)$ and $g \in G$. Hence $\tilde{\tau} \circ \pi = E$, meaning that π is faithful and therefore injective, so $C(X) \rtimes_r G$ is simple.

To complete the proof, we need to prove (ii) $\Rightarrow$ (i). If $C(X) \rtimes_r G$ is simple for some G-boundary X, let $I \subseteq C_r^*(G)$ be a proper closed ideal. If $\varphi\colon C_r^*(G) \to \mathbb{C}$ is a state such that $\varphi(I) = \{0\}$, extend φ to a state on $C(X) \rtimes_r G$ and let (g_i) be a net in G such that $g_i\mu \to \delta_x$ for some $x \in X$ where $\mu = \varphi|_{C(X)}$. By weak*-compactness we may assume that $(\varphi \circ \mathrm{Ad}(\lambda_{g_i}))$ converges to some state ψ on $C(X) \rtimes_r G$, so that $\psi|_{C(X)} = \delta_x$ and $\psi|_I = 0$. In particular, $C(X) \subseteq \mathrm{mult}(\psi)$. Now for any $b \in I$, $f_1, f_2 \in C(\partial_F G)$ and $g_1, g_2 \in G$,

$$\psi((f_1\lambda_{g_1})b(f_2\lambda_{g_2})) = f_1(x)\psi(\lambda_{g_1}b\lambda_{g_2})f_2(g_2x) = 0,$$

since $\lambda_{g_1}b\lambda_{g_2} \in I$. Hence the ideal generated by I is proper, so $I = \{0\}$ because $C(\partial_F G) \rtimes_r G$ was assumed to simple. Therefore $C_r^*(G)$ is simple. ☐

Remark 3.2.7. From the above theorem, it follows that any C^*-simple discrete group G has trivial amenable radical. Indeed, if the action of G on $\partial_F G$ is free, then $R(G) = \bigcap_{x \in \partial_F G} G_x = \{1\}$ by Theorem 2.4.12.

Remark 3.2.8. Since the result of Kalantar and Kennedy, other characterizations of C^*-simplicity have been obtained, a few of which we will list here.

A. Simplicity of reduced crossed products. Breuillard, Kalantar, Kennedy and Ozawa proved that C^*-simple discrete groups have the property that a reduced crossed product $A \rtimes_r G$ of a unital G-C^*-algebra by G is simple if and only if A is G-*simple*, meaning that A has no non-trivial G-invariant closed ideals [23, Theorem 7.1]. This settled a question of de la Harpe and Skandalis [44] in the affirmative; we will generalize this result in Chapter 4.

B. An averaging property. Independently of one another, Haagerup [68] and Kennedy [94] proved that a discrete group G is C^*-simple if and only if for all $t_1, t_2, \dots, t_m \in G \setminus \{1\}$ and $\varepsilon > 0$ there exist $s_1, \dots, s_n \in G$ such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} \lambda_{s_k t_j s_k^{-1}} \right\| < \varepsilon.$$

This characterization was significant, because many previously studied classes of C^*-simple groups were always shown to satisfy an at most minor variant of the latter property – in fact, it was part of the original proof of Powers that $\mathbb{F}_2$ is C^*-simple. We will prove in Chapter 4 that reduced crossed products over C^*-simple groups satisfy a similar property.

We also record that the above property is a group C^*-algebra variant of the Dixmier property. A unital C^*-algebra A is said to satisfy the *Dixmier property* if the closed convex hull of $\{uau^* \mid u \in \mathcal{U}(A)\}$ intersects the centre of A for all $a \in A$. Haagerup and Zsidó proved that a unital, simple C^*-algebra A always satisfies the Dixmier property, and that the intersection of the aforementioned closed convex hull and the centre always reduces to a point, if the C^*-algebra has a unique tracial state [69].

C. Recurrent subgroups. In [94], Kennedy obtained an algebraic characterization of C^*-simplicity by means of the notion of recurrence for subgroups. Recurrence is a group-theoretical version of the topological-dynamical notion of a uniformly recurrent subgroup. If we say that a subgroup H of a group G is *recurrent* if there exists a finite subset $F \subseteq G \setminus \{1\}$ such that $F \cap gHg^{-1} \neq \emptyset$ for all $g \in G$, a discrete group is C^*-simple if and only if it has no amenable, recurrent subgroups.

We finally state the infamous result by Breuillard, Kalantar, Kennedy and Ozawa [23, Corollary 4.3] that settles one half of the question of de la Harpe: whether there exist C^*-simple groups without the unique trace property.

Theorem 3.2.9. *Let G be a discrete group with amenable radical $R(G)$. Then $g \in G$ satisfies $\tau(\lambda_g) = 0$ for all tracial states τ on $C^*_r(G)$ if and only if $g \notin R(G)$. In particular, G has the unique trace property if and only if $R(G) = \{1\}$.*

Proof. We will generalize "if" in Chapter 4 (specifically, Theorem 4.4.2), so we omit the proof of that implication for now. For the converse, we may compose the conditional expectation $C^*_r(G) \to C^*_r(R(G))$ with the trivial representation $C^*_r(R(G)) \to \mathbb{C}$ (the existence of which follows from amenability of $R(G)$ [25, Theorem 2.6.8]), yielding a state $\tau\colon C^*_r(G) \to \mathbb{C}$ such that $\tau(\lambda_g) = 1$ for all $g \in R(G)$. As $R(G)$ is normal, then for any two $g, h \in G$ we have $gh \in R(G)$ if and only if $hg \in R(G)$, implying $\tau(\lambda_g\lambda_h) = \tau(\lambda_h\lambda_g)$. Hence τ is a tracial state on $C^*_r(R(G))$. □

The original proof makes use of Furman's theorem (Theorem 2.4.12) as well as the fact that $C(\partial_F G)$ is G-injective, as formulated in Proposition 2.4.6. A later proof due to Haagerup, which will serve as the main inspiration for our generalization in Chapter 4, also makes use of Furman's result, but reduces the ingredients of the proof to the fact that $\partial_F G$ is a G-boundary [68, Theorem 3.3].

Let us end the section by mentioning that Kennedy and Raum have recently generalized the above theorem to the setting of locally compact groups [95], their main result being the reduced group C^*-algebra of a locally compact group G admits a non-zero tracial positive linear functional τ if and only if the amenable radical $R(G)$ is an open subgroup. In this case, every such τ is concentrated on $R(G)$.

3.3 Stability properties

In this section, we give some examples of what stability properties that the classes of C^*-simple groups and groups with trivial amenable radical satisfy. In all known cases these properties overlap, and we will use them in the next section to give additional examples of C^*-simple groups other than using the criteria for C^*-simplicity in the previous section.

We first establish stability criteria to ensure that a lot of other stability properties are automatically satisfied for any class of groups.

Proposition 3.3.1. *Let P be a property for discrete groups such that*

(a) *the trivial group $\{1\}$ has property P;*

(b) *if G has property P, then G is icc;*

(c) *if N is a normal subgroup of G, then G has property P if and only if N and $C_G(N)$ have property P.*

Then the following hold:

(i) *A direct product $G_1 \times G_2$ has property P if and only if G_1 and G_2 have property P.*

(ii) *G has property P if and only if $\mathrm{Aut}(G)$ has property P.*

(iii) *If N is a normal subgroup of G such that N and G/N have property P, then G has property P.*

(iv) *If H is a finite index subgroup of G, then G has property P if and only if G is icc and H has property P.*

Proof. (i) is clear from (c). For (ii), we can identify G with the normal subgroup of $\mathrm{Aut}(G)$ of inner automorphisms, since G can be assumed to be icc by (b) and (c). As $C_{\mathrm{Aut}(G)}(G) = \{1\}$, (ii) is immediate.

For (iii), we need to show that $C_G(N)$ has property P. Since $NC_G(N)$ is normal in G, $NC_G(N)/N$ is normal in G/N and therefore has property P. Because N is centerless (being icc due to (b)), $NC_G(N)/N$ is isomorphic to $C_G(N)/(N \cap C_G(N)) = C_G(N)$.

Finally, for (iv) we can assume as before that G is icc, and let $N = \bigcap_{g \in G} gHg^{-1}$. Then N is the kernel of the canonical action of G on the finite coset space G/H (commonly called the *normal core* of H in G), so it is a normal finite-index subgroup of G. It follows that any element in $C_G(N)$ has finite conjugacy class in G, so $C_G(N) = \{1\}$ by hypothesis. As N has property P if either H or G has property P, and $C_H(N) \subseteq C_G(N) = \{1\}$, the proof is complete. □

Notice that a discrete group G containing a C^*-simple finite-index subgroup H is not necessarily icc, as G could be isomorphic to the direct product of H and a finite, cyclic group.

We now show that the classes of C^*-simple groups and of groups with trivial amenable radical satisfy the properties (a), (b) and (c) of the above proposition. One would expect that (c) is the harder of the three to obtain, and one would be right: (a) is immediate, and if a group G has trivial amenable radical, then G admits a faithful boundary action by Theorem 2.4.12, so that G is icc by the discussion after Remark 2.5.12.

The following proof is due to Tucker-Drob [137, Lemma B.6].

Proposition 3.3.2. *Let G be a discrete group with a normal subgroup N. Then G has trivial amenable radical if and only if N and $C_G(N)$ have trivial amenable radical.*

Proof. First observe that the amenable radical $R(H)$ of a group H is *characteristic*, i.e., $\alpha(H) = H$ for any automorphism $\alpha \in \mathrm{Aut}(H)$. If $H \subseteq G$ is a normal subgroup, then conjugation by $g \in G$ is an automorphism of H, implying $gR(H)g^{-1} = R(H)$. Therefore $R(H)$ is normal in G and amenable, so that $R(H) \subseteq R(G) \cap H$. Since $R(G) \cap H$ is amenable and also normal in H, $R(H) = R(G) \cap H$.

Now we prove the claim. Since both N and $C_G(N)$ are normal, $R(G) = \{1\}$ implies $R(N) = R(C_G(N)) = \{1\}$ by the above observation. Conversely, assume that $R(N) = R(C_G(N)) = \{1\}$. Then $R(G) \cap N = R(N) = \{1\}$, so normality of $R(G)$ and N implies $g(ng^{-1}n^{-1}) = (gng^{-1})n^{-1} \in R(G) \cap N = \{1\}$ for all $g \in R(G)$ and $n \in N$. Therefore $R(G)$ and N commute, meaning that

$$R(G) = R(G) \cap C_G(N) = R(C_G(N)) = \{1\}. \qquad \square$$

That C^*-simplicity propagates to direct products may have been known since the proof of Takesaki [133] that the minimal tensor product of two simple C^*-algebras is simple, as $C_r^*(G_1 \times G_2) \cong C_r^*(G_1) \otimes C_r^*(G_2)$ for discrete groups G_1 and G_2. The more general question of whether (c) was satisfied for C^*-simple groups was finally settled in 2014 by Breuillard, Kalantar, Kennedy and Ozawa [23, Theorem 1.4]. This finally yielded a proof that C^*-simplicity is also stable under extensions. We give their proof here, and we give a partial generalization of their result in Chapter 4 (Theorem 4.3.15).

It merits mention that several permanence properties for C^*-simplicity and the unique trace property were known before the aforementioned 2014 result. Bekka and de la Harpe gave the first proof that C^*-simplicity and the unique trace property passes to subgroups and supergroups of finite index [16]; one year later, Popa's work on conditional expectations of C^*-algebras satisfying a finite index condition yielded another proof of these results as a corollary [119, Remark 4.7.1]. In line with the stability under direct products, one can show that C^*-simplicity and the uniqueness trace property are stable under direct limits, by means of more general results about direct limits of C^*-algebras: this was first observed by Bekka and de la Harpe in [16, Corollary 11], but may have been known before then.

The extension and centralizer problems were far more delicate matters. Inspired by the 1985 exposition of de la Harpe, Boca and Niţică modified the definition of de la Harpe's Powers groups, giving name to the so-called *weak Powers groups* that were also shown to be C^*-simple with the unique trace property. This way, they gave the first positive result on stability of C^*-simplicity under group extensions [20]: if the normal subgroup was a Powers group, and the quotient group was a *weak* Powers group, the group itself would be a weak Powers group.

Bédos was the first to achieve positive results in the case when at least one of the groups was assumed to "just" be C^*-simple. By means of a structure theorem for reduced twisted crossed products (Theorem 4.1.3) he proved in 1990 [9] that whenever a group G contains a normal C^*-simple subgroup N, then G is C^*-simple itself, if N has trivial centralizer in G. Defining *ultraweak Powers groups* to be groups containing a normal weak Powers subgroup with trivial centralizer, Bédos also proved that an extension G of a C^*-simple group N by an ultraweak Powers group is C^*-simple. He later proved under the same hypotheses that if N satisfies the unique trace property instead of being C^*-simple, so does G [10, 11].

To prove the theorem of Breuillard, Kalantar, Kennedy and Ozawa, we will need a result of Furstenberg that first appeared in a monograph by Glasner [63, Theorems 4.3-4.4].

Observe that the results hold equally true if we replace the Furstenberg boundary $\partial_F G$ by the universal minimal proximal compact G-space $\Pi(G)$.

Proposition 3.3.3. *Let G be a discrete group. Then there exists a group homomorphism $\mathrm{Aut}(G) \to \mathrm{Homeo}(\partial_F G)$, $\alpha \mapsto \alpha_\star$, such that the inner automorphism $\sigma_g \colon t \mapsto gtg^{-1}$ satisfies $(\sigma_g)_*(x) = gx$ for all $g \in G$ and $x \in X$.*

In particular, if $H \subseteq \mathrm{Aut}(G)$ contains all σ_g for $g \in G$, then the G-action on $\partial_F G$ extends to an H-action, with respect to which $\partial_F G$ is an H-boundary.

Proof. For $\alpha \in \mathrm{Aut}(G)$, define a new G-action on $\partial_F G$ by

$$g \star x = \alpha(g)x.$$

This action is clearly minimal and strongly proximal. By universality, there exists a continuous surjection $\alpha_\star \colon \partial_F G \to \partial_F G$ such that

$$\alpha_\star(gx) = g \star \alpha_\star(x) = \alpha(g)\alpha_\star(x), \quad g \in G,\ x \in \partial_F G.$$

In the same way, we obtain a continuous surjection $(\alpha^{-1})_\star \colon \partial_F G \to \partial_F G$ such that

$$(\alpha^{-1})_\star(gx) = \alpha^{-1}(g)(\alpha^{-1})_\star(x), \quad g \in G,\ x \in \partial_F G.$$

Thus $(\alpha^{-1})_\star \circ \alpha_\star \colon \partial_F G \to \partial_F G$ is continuous and G-equivariant with respect to the original action, so $(\alpha^{-1})_\star \circ \alpha_\star = \mathrm{id}_{\partial_F G}$ by Lemma 2.1.13. Hence $\alpha_\star$ is a homeomorphism, and $(\alpha_\star)^{-1} = (\alpha^{-1})_\star$. Similarly, for $\alpha, \beta \in \mathrm{Aut}(G)$, one can easily show that that $\alpha_\star \circ \beta_\star \circ ((\alpha \circ \beta)^{-1})_\star$ is continuous and G-equivariant, so that the map $\alpha \mapsto \alpha_\star$ is a homomorphism, and that $x \mapsto (\sigma_g)_\star(g^{-1}x)$ is G-equivariant for all $g \in G$, so that σ_g is mapped to $x \mapsto gx$. The last assertion follows immediately. □

The lemma below first appeared in the article by Breuillard, Kalantar, Kennedy and Ozawa [23, Lemma 5.1], and it is more or less an extension of the fact that a minimal, proximal space has no isolated points.

Lemma 3.3.4. *Let G be a discrete group, let X be a minimal and proximal compact G-space and let $U \subseteq X$ be a non-empty open subset. Then the set*

$$G' = \{t \in G \mid tU \cap U \neq \emptyset\}$$

generates G.

Proof. Let H be the subgroup of G generated by the set G'. Then $HU = \bigcup_{t \in H} tU$ is non-empty and open. For $t \in G \setminus H$ and $r, s \in H$, we have $s^{-1}tr \in G \setminus H$, so that $(s^{-1}tr)U \cap U = \emptyset$ and $trU \cap sU = \emptyset$. Hence $tHU \cap HU = \emptyset$ for all $t \in G \setminus H$. Now consider the family $(tHU)_{tH \in G/H}$ of non-empty open subsets of X. This is clearly an open cover of X. Furthermore, if $tH \neq sH$ then $s^{-1}t \notin H$ and $tHU \cap sHU = \emptyset$, so that $(tHU)_{tH \in G/H}$ is in fact a partition of X. Since X is compact, the family is finite. Now define a map $X \to G/H$ by mapping $x \in X$ to the unique element $tH \in G/H$ such that $x \in tHU$. This map is a continuous, G-equivariant surjection, so because X is proximal, then G/H is proximal and finite, meaning that it is a one-point space by Proposition 2.1.11. □

It might be possible for the above result to uncover easily interpretable criteria for the universal minimal compact G-space not to be proximal.

We end this section with the devilishly ingenious proof that property (c) of Proposition 3.3.1 holds for the class of C^*-simple groups.

Theorem 3.3.5. *Let N be a normal subgroup of a discrete group G. Then G is C^*-simple if and only if both N and $C_G(N)$ are C^*-simple.*

Proof. Let $C = C_G(N)$ and $D = NC$ in the following. By Proposition 3.3.3, the N-action on $\partial_F N$ (resp. the C-action on $\partial_F C$) extends to a G-boundary action on $\partial_F N$ (resp. $\partial_F C$). Further, by means of the quotient homomorphism $G \to G/D$ we obtain a G-action on $\partial_F(G/D)$. With respect to these actions, N acts trivially on $\partial_F C$ and $\partial_F(G/D)$, since conjugation by n on C is the identity map for all $n \in N$, and C acts trivially on $\partial_F N$ and $\partial_F(G/D)$ by the same argument.

We now equip

$$X = \partial_F N \times \partial_F C \times \partial_F(G/D).$$

with the diagonal G-action. By Lemma 2.1.15, this action is strongly proximal, and it is also minimal: for $(x, y, z) \in X$ and non-empty open subsets U_1, U_2 and U_3 of $\partial_F N$, $\partial_F C$ and $\partial_F(G/D)$ respectively, let $g \in G$ such that $gz \in U_3$, and let $n \in N$ and $c \in C$ such that $nx \in g^{-1}U_1$ and $cy \in g^{-1}U_2$. Then $gnc(x, y, z) = (gnx, gcy, gz) \in U_1 \times U_2 \times U_3$, so that X is a G-boundary.

We next show that every point in X has an amenable stabilizer in G. For $(x, y, z) \in X$ the stabilizer $G_{(x,y,z)}$ satisfies $G_{(x,y,z)} \cap D = N_x C_y$. Since N_x and C_y are amenable by Proposition 2.4.14, $N_x C_y$ is amenable since it is a quotient of the group $N_x \times C_y$. As we may view $G_{(x,y,z)}/(G_{(x,y,z)} \cap D)$ as a subgroup of the amenable group $(G/D)_z$, it is amenable itself. Due to amenability being stable under group extensions, $G_{(x,y,z)}$ is amenable.

Suppose that N and C are both C^*-simple. If $g \in G$ satisfies $X^g \neq \emptyset$, then $U = (\partial_F N)^g \neq \emptyset$. Let $N' = \{n \in N \mid nU \cap U \neq \emptyset\}$. For all $n \in N'$, let $x \in U \cap n^{-1}U$. Then $gng^{-1}x = gnx = nx$. Since the N-action on $\partial_F N$ is free by Theorem 3.2.6, we have $gng^{-1} = n$ so that g commutes with n. Since N is generated by N', g commutes with all of N, meaning that $g \in C$. As $(\partial_F C)^g \neq \emptyset$ and the action of C on $\partial_F C$ is free, $g = 1$. Therefore G acts freely on X, so G is C^*-simple by Theorem 3.2.6.

If G is C^*-simple, then $C(X) \rtimes_r G$ is simple by Theorem 3.2.6. Therefore, since the stabilizers of G on X are amenable, it follows from Proposition 3.2.4 that the G-action on X is topologically free. Since

$$(\partial_F N)^n \times \partial_F C \times \partial_F(G/D) \subseteq X^n, \quad \partial_F N \times (\partial_F C)^c \times \partial_F(G/D) \subseteq X^c$$

for all $n \in N$ and $c \in C$, the N-action on $\partial_F N$ and the C-action on $\partial_F C$ are both topologically free, so N and C are C^*-simple. ☐

3.4 Examples of C^*-simple discrete groups

We next give some examples of C^*-simple groups, mainly using the characterization of C^*-simplicity obtained by Kalantar and Kennedy (Theorem 3.2.6). As we will only be discussing discrete groups in the following, let us briefly mention that Raum [126] has recently given the first examples of non-discrete *locally compact* groups G such that the reduced group C^*-algebra $C_r^*(G) \subseteq B(L^2(G))$ is simple. This answered a question of de la Harpe [41, Question 5]. Furthermore, Raum proved that any C^*-simple locally compact group is totally disconnected.

In all of the following examples, we consider boundaries X that are not one-point spaces, so that X has no isolated points by Proposition 2.1.11. In particular, finite subsets of X have empty interior.

Having given examples of boundary actions in Section 2.2, we will first consider three of these and argue why these yield C^*-simple groups.

Example 3.4.1 (I). *Non-abelian free groups of finite rank.* For $n \geq 2$, the action of the non-abelian free group $\mathbb{F}_n$ on its boundary $\partial\mathbb{F}_n$ of one-sided reduced infinite words is topologically free, meaning that $\mathbb{F}_n$ is C^*-simple (a result due to Powers [120]). Indeed, if A is a free generating set for $\mathbb{F}_n$, let $\omega = g_1 \cdots g_n \in \mathbb{F}_n \setminus \{1\}$ be a word in reduced form, where $g_1, \cdots, g_n \in A \cup A^{-1}$. We claim that X^ω is finite, so that it has empty interior. Performing conjugations if necessary, we may assume that $g_1 g_n \neq 1$ since $\omega \neq 1$ and

$$gX^\omega = X^{g\omega g^{-1}}, \quad g \in \mathbb{F}_n,$$

so that X^ω has empty interior if and only if $X^{g\omega g^{-1}}$ has empty interior.

If $\omega x = x$ for some $x \in \partial\mathbb{F}_n$, assume first that the concatenation ωx is reduced. Then the first n letters of x are the n letters of ω. Since $g_1 g_n \neq 1$, $\omega^2 x$ is also reduced, so the next n letters of x are those of ω, and by iterating this process, we see that $x = \omega\omega\omega\cdots$. If ωx is not reduced, then let $1 \leq k \leq n$ be largest such that the first k letters of x are $g_n^{-1} \cdots g_{n-k+1}^{-1}$. Since the first letter of x is g_n^{-1} and the first letter of ωx is g_1 if $k < n$, we must have $k = n$. Thus the first n letters of x are the n letters of ω^{-1}, so as above, we conclude that $x = \omega^{-1}\omega^{-1}\omega^{-1}\cdots$. All in all, X^ω consists of two points, so $\mathbb{F}_n$ is C^*-simple.

Example 3.4.2. *Torsion-free hyperbolic groups* [40, Proposition]. We generalize Example 3.4.1 immediately, in this result due to de la Harpe. Indeed, the Gromov boundary ∂G of a non-elementary hyperbolic group G is a G-boundary by Proposition 2.2.13. If G is torsion-free, any non-identity element in G is hyperbolic and therefore has exactly two fixed points, so the action of G on ∂G is topologically free.

Example 3.4.3 (IV). *Projective special linear groups.* For $n \geq 2$, the action of $G = \mathrm{PSL}(n, \mathbb{R})$ on real projective $n-1$-space $\Omega = \mathbb{P}^{n-1}(\mathbb{R})$ is topologically free. To realize this, let $\pi\colon (\mathbb{R}^n)_o \to \Omega$ be the quotient map and assume that $g \in \mathrm{SL}(n, \mathbb{R})$ fixes a non-empty open subset $U \subseteq \mathbb{P}^{n-1}(\mathbb{R})$ pointwise. Let $V \subseteq \pi^{-1}(U)$ be a non-empty open ball in $\mathbb{R}^n$. For $v, w \in V$ such that $\pi(v) \neq \pi(w)$ and $\lambda, \lambda' \in \mathbb{R}$ such that $gv = \lambda v$ and $gw = \lambda' w$, then by convexity there exists $\lambda'' \in \mathbb{R}$ such that

$$\lambda''(v + w) = g(v + w) = \lambda v + \lambda' w.$$

Since $v \notin \mathbb{R}w$, it follows that $\lambda = \lambda' = \lambda''$. Therefore $g = \lambda 1$ on $V \cup \{0\}$. Letting $x \in V$, then for all $y \in \mathbb{R}^n$ there exists $c \neq 0$ such that $cy + x \in V$. By linearity, it follows that $gy = \lambda y$. Since g factors to the identity element in G, we see that any discrete subgroup $\Gamma \subseteq G$ for which Ω is a Γ-boundary is C^*-simple; in particular, $\mathrm{PSL}(n, \mathbb{Z})$ is C^*-simple, which was originally shown by Bekka, Cowling and de la Harpe [14].

In 1994, Bekka, Cowling and de la Harpe proved the following more general result: if Γ is a Zariski-dense subgroup of a connected semi-simple Lie group G without compact factors, and Γ has trivial centre, then Γ is C^*-simple [15, Theorem 2]. They did so by proving that the action of Γ on the Furstenberg boundary $\partial_F G$ of G satisfies a hyperbolicity property ($\mathrm{P}_{\mathrm{geo}}$) which, on closer inspection, in fact implies that the action is topologically free. We suspect that $\partial_F G$ is also a Γ-boundary.

In line with this result, Poznansky extended it considerably in 2009, proving that a discrete linear group is C^*-simple if and only if it has trivial amenable radical ([121], see also [23, Section 6.2]).

We follow up on the above verifications of C^*-simplicity by listing a few more general examples of groups that are also C^*-simple with unique trace, without proofs. A wealth of other examples can be found in de la Harpe's 2007 survey on C^*-simplicity [41] (see also [23, Section 6]).

Example 3.4.4. *Free products with amalgamation.* For two groups G_1 and G_2, we consider words of the form $s = s_1 \cdots s_n$ where each s_i belongs either to G_1 or G_2. After removing occurrences of the identity element in s, we say that s is *reduced* if $s_i \in G_j$ implies $s_{i+1} \in G_{3-j}$ for all $1 \leq i \leq n-1$ and $j \in \{1,2\}$. The *free product* $G_1 * G_2$ is the group of all reduced words s with letters in G_1 and G_2, with the composition given by concatenation and subsequent reduction. Paschke and Salinas proved in 1979 that $G_1 * G_2$ is C^*-simple whenever $(|G_1|-1)(|G_2|-1) \geq 2$ [114].

For groups G_1, G_2 and H, for which there exists an injective homomorphism $f_i \colon H \to G_i$ for $i = 1,2$, let N be the normal subgroup of $G_1 * G_2$ generated by the elements $f_1(n)f_2(n)^{-1}$. Then $G_1 *_H G_2 = (G_1 * G_2)/N$ is the *free product of G_1 and G_2 with amalgamated subgroup H*. Bédos was the first to find C^*-simplicity criteria for non-degenerate free amalgamated products (i.e., both f_i are not isomorphisms) [8]; they were also considered in de la Harpe's 1985 survey [39, Proposition 10].

In recent developments, Ivanov and Omland [87] have improved upon previously known criteria ensuring C^*-simplicity for a non-degenerate free amalgamated product. Moreover, they have characterized C^*-simplicity of a non-degenerate free amalgamated product by means of the action on its Bass-Serre tree. We will do the same for HNN extensions in Sections 3.5 and 3.6.

Example 3.4.5. *Burnside groups and Tarski monster groups.* The *free Burnside group* $B(m,n)$ of rank m and exponent n is the quotient of $\mathbb{F}_m$ by the normal subgroup generated by $\{g^n \mid g \in \mathbb{F}_m\}$. In 1982, Adyan gave a proof that $B(m,n)$ is non-amenable for $m \geq 2$ and odd $n \geq 665$, which was notable as $B(m,n)$ does not contain non-abelian free subgroups, thus yielding one of the earliest known counterexamples to von Neumann's conjecture. Olshanskii and Osin gave a proof in 2014 that under the above hypotheses, $B(m,n)$ is in fact also C^*-simple [108, Theorem 1.2], thus answering a question of de la Harpe [41, Question 15] whether there exist C^*-simple groups without free subgroups.

As mentioned at the end of Section 2.3, the first known examples of non-amenable groups without non-abelian free subgroups were the *Tarski monster groups.* By means of their dynamical characterization of C^*-simplicity, Kalantar and Kennedy showed in [90, Theorem 6.5] that every Tarski monster group is C^*-simple.

Example 3.4.6. *Braid groups.* For $2 \leq n \leq \infty$, the *Artin braid group* B_n is the group with presentation

$$B_n = \left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i s_j s_i = s_j s_i s_j & \text{if } |i-j| = 1 \\ s_i s_j = s_j s_i & \text{if } |i-j| > 1 \end{array} \right\rangle.$$

Roughly speaking, the braid group B_n is the group of configurations of two copies of the set $\{1, \ldots, n\}$ where strands are drawn between items in each copy in a one-to-one correspondence. Each configuration thus determines a permutation in the symmetric group S_n, and the *pure braid group* P_n is the kernel of the homomorphism $B_n \to S_n$.

For $n \geq 3$, let C_n denote the centre of B_n. Then C_n coincides with the centre of P_n [60, Lemme 5]. Letting $\mathscr{B}_n = B_n/C_n$ and $\mathscr{P}_n = P_n/C_n$ for $3 \leq n < \infty$, Giordano and de la Harpe proved in 1989 that $\mathscr{P}_n$ is C^*-simple [60, Proposition 12] (see also Example 4.3.13). Bédos proved in 1991 [9, p. 536] that $\mathscr{B}_n$ is C^*-simple for $3 \leq n < \infty$, by means of an observation by Dyer and Grossmann that $\mathscr{B}_n$ contains $\mathbb{F}_{n-1}$ as a normal subgroup with trivial centralizer (so that we may apply Theorem 3.3.5). Recently, Omland has shown that $P_\infty \cong \mathscr{P}_\infty$ and $B_\infty \cong \mathscr{B}_\infty$ are in fact also C^*-simple [110, Proposition 7.2].

Example 3.4.7. *Outer automorphism groups of non-abelian free groups.* For $3 \leq n < \infty$, the outer automorphism group

$$\mathrm{Out}(\mathbb{F}_n) = \mathrm{Aut}(\mathbb{F}_n)/\mathrm{Inn}(\mathbb{F}_n)$$

is C^*-simple [42, Theorem 2.6] (here $\mathrm{Inn}(\mathbb{F}_n)$ denotes the normal subgroup of inner automorphisms of $\mathbb{F}_n$).

We now explain how one may search for non-C^*-simple groups with trivial amenable radical, as done in the first examples of such groups given by Adrien Le Boudec [99]. If G is a discrete group and X is a G-boundary such that G_x° is amenable for some $x \in X$, then the action of G on X is topologically free if G is C^*-simple. Indeed, $C(X) \rtimes_r G$ is simple by Theorem 3.2.6, so that we may then apply Proposition 3.2.4.

In [99, Theorem A], Le Boudec outlines a general strategy to find such groups, by means of minimal actions of discrete groups on trees. Suppose that T is a leafless tree and that $G \subseteq \mathrm{Aut}(T)$ is a subgroup, such that the action of G on T is minimal and the action of G on ∂T is non-elementary. If G_x° is non-trivial and amenable for some $x \in \partial T$, then G has trivial amenable radical, but is not C^*-simple. We have already seen that $\overline{\partial T}$ is a G-boundary in the shadow topology (Proposition 2.2.14), so that the above remark implies that G is not C^*-simple.

We will not delve into the particular examples that Le Boudec gives of groups and actions with the above properties. Nonetheless, let us quickly argue why $R(G) = \{1\}$. Notice that $N = R(G)$ acts trivially on ∂T by Proposition 2.4.9. Therefore N contains no hyperbolic automorphisms of T, since ∂T is infinite and every element of N fixes more than two points. Now let d be the path metric on T, let v be a vertex in T and let $g \in N$. For any p fixed by g, let $f\colon \mathbb{Z}_+ \to T$ be a ray such that $f(0) = p$ and $f(m) = v$ for some $m \geq 0$ (using that T is leafless). Since g fixes all $x \in \partial T$, $g \circ f$ and f are cofinal, meaning that there exist $j, k > m$ such that $gf(j) = f(k)$. Because f is a geodesic and $gf(0) = f(0)$, we see that $j = k$; therefore, v is fixed by g, since T is uniquely geodesic. We conclude that $N = \{1\}$.

We finally give de la Harpe's original definition of a Powers group [39, 41], originally devised to ensure C^*-simplicity and the unique trace property of a group.

Definition 3.4.8. A group G is called a *Powers group* if for any finite subset $F \subseteq G \setminus \{1\}$ and $n \geq 1$ there is a partition $G = C \sqcup D$ and $g_1, \ldots, g_n \in G$ such that

(i) $fC \cap C = \emptyset$ for all $f \in F$, and

(ii) $g_i D \cap g_j D = \emptyset$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$.

Many known examples of C^*-simple groups are Powers groups, including many non-degenerate free amalgamated products, non-solvable subgroups of $\mathrm{PSL}(2, \mathbb{R})$ and also all torsion-free non-elementary hyperbolic groups. In fact, if a group G admits a non-elementary action on a locally compact Hausdorff space X such that for any finite subset $F \subseteq G$ there is x in the weakly hyperbolic limit set for which $x \notin Fx$, then G is a Powers group [41, Proposition 11].

3.5 Graphs of groups and HNN extensions

Joint work with Tron Omland (University of Oslo)

In this section we give some of the preliminaries for the branch of geometric group theory now known as *Bass-Serre theory*. The basic premise for this theory is that one can build a group action on a tree from a connected graph of groups. We then zoom in on the case when the acting group is an HNN extension, for which we provide some structure results and we define subgroups that we shall interpret geometrically in the next section. Much of our exposition is based on the original source [129, §1] for this topic, but is also inspired by the approach taken in [43].

A *graph of groups* (G, Y) consists of a non-empty connected graph $Y = (V, E, o, t)$, families of groups $(G_v)_{v \in V}$ and $(G_e)_{e \in E}$ such that $G_{\overline{e}} = G_e$ for all $e \in E$, and a family of injective group homomorphisms $\varphi_e \colon G_e \to G_{t(e)}$, $e \in E$. We pick an *orientation* $E_+ \subseteq E$ of Y, meaning that $E_+ \cap \{e, \overline{e}\}$ contains only one edge for all $e \in E$, and define $E_- = E \setminus E_+$.

Now, let (G, Y) be a graph of groups as above and let $M = (V(M), E(M))$ be a maximal subtree of Y. We define the *fundamental group* $\Gamma = \pi_1(G, Y, M)$ of (G, Y) by

$$\Gamma = \left\langle \{G_v\}_{v \in V},\ \{\tau_e\}_{e \in E} \;\middle|\; \begin{array}{cl} \tau_{\overline{e}} = \tau_e^{-1} & \text{for all } e \in E, \\ \tau_e \varphi_e(g) \tau_e^{-1} = \varphi_{\overline{e}}(g) & \text{for all } e \in E,\ g \in G_e \\ \tau_e = 1 & \text{for all } e \in E(M) \end{array} \right\rangle$$

We have natural group homomorphisms $G_y \to \Gamma$ for all $y \in V$, and they are all *injective.* Moreover, $\tau_e \in \Gamma$ has infinite order for all $e \in E \setminus E(M)$.

We next define a graph $T = (V(T), E(T), o, t)$ as follows. For any edge $e \in E$, let $|e|$ be the unique edge in $\{e, \overline{e}\} \cap E^+$ and define $\Gamma_e = \varphi_{|e|}(G_e)$. We define $V(T)$ and $E(T)$ by means of left coset spaces in the following way:

$$V(T) = \bigsqcup_{v \in V} \Gamma / G_v, \quad E(T) = \bigsqcup_{e \in E} \Gamma / \Gamma_e.$$

The origin, terminus and inversion maps are given by

$$o(g\Gamma_e) = \begin{cases} gG_{o(e)} & \text{for } e \in E^+ \\ g\tau_e^{-1}G_{o(e)} & \text{for } e \in E^-, \end{cases} \qquad t(g\Gamma_e) = \begin{cases} g\tau_e G_{t(e)} & \text{for } e \in E^+ \\ gG_{t(e)} & \text{for } e \in E^-, \end{cases}$$

$$\overline{g\Gamma_e} = g\Gamma_{\overline{e}}$$

for all $g \in \Gamma$ and $e \in E$. One would not immediately deduce the following pivotal result [129, §I.5], due to Bass and Serre, from the set-up.

Theorem 3.5.1. *For (G, Y), M and E^+ as above, the graph T constructed above is a tree. Up to isomorphisms, Γ and T are independent of the choice of maximal subtree M and orientation E^+.*

The tree T is the so-called *Bass-Serre tree* of the graph of groups (G, Y); we will also say that T is the Bass-Serre tree of the fundamental group Γ of (G, Y).

The definition of T incites us to define an action of Γ on T by left translation of cosets, and Γ acts on T by automorphisms without inversions, i.e., $ge \neq \overline{e}$ for all $g \in \Gamma$ and $e \in E(T)$ (cf. Section 2.2).

Suppose now that (F, Y) is a graph of groups, where Y is a loop with one vertex v and one pair of edges $\{e, \overline{e}\}$. We let $\{e\}$ be the orientation of Y and $G = F_v$. As the homomorphisms $\varphi_e \colon F_e \to G$ and $\varphi_{\overline{e}} \colon F_e \to G$ of F_e are both injective, we may define $H = \varphi_e(F_e)$ and a injective group homomorphism $\theta = \varphi_{\overline{e}}\varphi_e^{-1} \colon H \to G$. Defining the *stable letter* $\tau = \tau_{\overline{e}}$, the fundamental group Γ is the well-known *HNN extension* $\mathrm{HNN}(G, H, \theta)$:

$$\Gamma = \mathrm{HNN}(G, H, \theta) = \langle G, \tau \mid \tau^{-1}h\tau = \theta(h) \text{ for all } h \in H \rangle.$$

The HNN extensions were devised in 1949 by Higman, Neumann and Neumann – after whom they are named – as a means of establishing embedding results for (mainly) countable, torsion-free groups [82].

For the remainder of this section, let G, H, θ and Γ be as above. Let S'_{-1} and S'_1 be systems of representatives for the *right* cosets of H and $\theta(H)$ in G, respectively, such that $1 \in S'_{-1} \cap S'_1$. Therefore $G = HS'_1 = \theta(H)S'_{-1}$. In most literature about HNN extensions (e.g., [21, Theorem 2.14.3]) one encounters the fact that any $g \in \Gamma$ has a unique *normal form*, meaning that we may uniquely decompose g as the product

$$g = g_0\tau^{\varepsilon_1}g_1 \cdots \tau^{\varepsilon_n}g_n,$$

where $g_0 \in G$, $\varepsilon_i \in \{\pm 1\}$, $g_i \in S'_{\varepsilon_i}$ for all $1 \leq i \leq n$, and that $\varepsilon_i = \varepsilon_{i+1}$ whenever $g_i = 1$ for $1 \leq i \leq n-1$. This also entails that the natural map $G \to \Gamma$ is actually an injection.

Taking the inverse of the above expression of g, we obtain a unique normal form with respect to *left* cosets, and it is this approach (and normal form) that we will use. Therefore, if we let S_{-1} and S_1 be systems of representatives for the *left* cosets of H and $\theta(H)$ in G such that $1 \in S_{-1} \cap S_1$, the unique normal form of $g \in \Gamma$ can be written

$$g = g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2} \cdots g_n\tau^{\varepsilon_n}g_{n+1},$$

where

(i) $g_{n+1} \in G$ and $\varepsilon_i \in \{\pm 1\}$ for $1 \leq i \leq n$,
(ii) $g_i \in S_{-\varepsilon_i}$ for all $1 \leq i \leq n$, and
(iii) $g_i = 1$ implies $\varepsilon_{i-1} = \varepsilon_i$ for $2 \leq i \leq n$.

With g as above, $n = |g|$ is the *length* of g, and if $n \geq 1$, we say that ε_1 and ε_n is the *type* and *direction* of g, respectively. The *initial letter* of g is $g_1 \in S_{-\varepsilon_1}$ and the *end letter* of g is $g_{n+1} \in G$. For $g \in G$, we define the length of g to be 0, and the initial letter and end letter of g as an element in Γ are given by 1 and g, respectively.

Remark 3.5.2. For $n \geq 1$, $g_1, \ldots, g_{n+1} \in G$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$, the word

$$g_1 \tau^{\varepsilon_1} \cdots g_n \tau^{\varepsilon_n} g_{n+1}$$

is said to be *reduced* (in Γ) if for all $1 \leq i \leq n-1$ we have

(a) $g_{i+1} \notin H$ whenever $\varepsilon_i = -1$ and $\varepsilon_{i+1} = 1$, and
(b) $g_{i+1} \notin \theta(H)$ whenever $\varepsilon_i = 1$ and $\varepsilon_{i+1} = -1$.

If we define

$$H_{-1} = H, \quad H_1 = \theta(H),$$

the conditions (a) and (b) can be rephrased as follows: for all $1 \leq i \leq n-1$, $g_{i+1} \notin H_{\varepsilon_i}$ whenever $\varepsilon_{i+1} = -\varepsilon_i$. Notice that

$$\tau^{-\varepsilon} H_{-\varepsilon} \tau^{\varepsilon} = H_{\varepsilon}. \tag{3.5.1}$$

We say that $g_1 \in G$ is *reduced* if $g_1 \neq 1$. A fundamental result for HNN extensions, also known as *Britton's lemma*, is that reduced words always define non-identity elements. The result itself can be derived from the uniqueness of the normal form. Indeed, if for $n \geq 1$ the word $g = g_1 \tau^{\varepsilon_1} \cdots g_n \tau^{\varepsilon_n} g_{n+1} \in \Gamma$ is reduced, let $s_1 \in S_{-\varepsilon_1}$ and $h_1 \in H_{-\varepsilon_1}$ such that $g_1 = s_1 h_1$ and rewrite

$$g = s_1 \tau^{\varepsilon_1} (\tau^{-\varepsilon_1} h_1 \tau^{\varepsilon_1}) g_2 \tau^{\varepsilon_2} \cdots g_n \tau^{\varepsilon_n} g_{n+1}.$$

The remainder of the proof divides into two possible situations, depending on whether consecutive powers of τ in the word coincide. Indeed, define $g_2' = (\tau^{-\varepsilon_1} h_1 \tau^{\varepsilon_1}) g_2$. If $\varepsilon_2 = \varepsilon_1$, then write $g_2' = s_2 h_2$ for $s_2 \in S_{-\varepsilon_2}$ and $h_2 \in H_{-\varepsilon_2}$ as above, and write

$$g = s_1 \tau^{\varepsilon_1} g_2' \tau^{\varepsilon_2} \cdots g_n \tau^{\varepsilon_n} g_{n+1} = s_1 \tau^{\varepsilon_1} s_2 \tau^{\varepsilon_2} (\tau^{-\varepsilon_2} h_2 \tau^{\varepsilon_2}) g_3 \cdots g_n \tau^{\varepsilon_n} g_{n+1}.$$

If $\varepsilon_2 = -\varepsilon_1$, then $g_2 \notin H_{-\varepsilon_2} = H_{\varepsilon_1}$ by assumption, so that due to (3.5.1), $g_2' = (\tau^{-\varepsilon_1} h_1 \tau^{\varepsilon_1}) g_2 \in H_{\varepsilon_1} g_2$ and $g_2' \notin H_{\varepsilon_1} = H_{-\varepsilon_2}$. We then proceed as for g_1, noting that the coset representative of g_2' with respect to $H_{-\varepsilon_2}$ is not 1. Iterating the process yields the normal form of g, which contains at least $2n-1$ terms. If $n \geq 2$ then $g \neq 1$ due to uniqueness of the normal form, and if $n = 1$, then $g = g_1 \tau^{\varepsilon_1} g_2 = 1$ would imply $\tau \in G$, a contradiction.

The above proof of Britton's lemma also proves the following lemma.

Lemma 3.5.3. *Let $n \geq 1$ and let $g = g_1 \tau^{\varepsilon_1} \cdots g_n \tau^{\varepsilon_n} g_{n+1} \in \Gamma$ be a reduced word. Then*

(i) *$g \notin G$ and $|g| = n$;*

(ii) *the type of g is ε_1;*

(iii) *the direction of g is ε_n;*

(iv) *the initial letter of g is 1 if and only if $g_1 \in H_{-\varepsilon_1}$;*

(v) *if $g_{n+1} \in H_{\varepsilon_n}$, the end letter of the normal form is contained in H_{ε_n} as well.*

The *kernel* of the HNN extension Γ is the normal subgroup

$$\ker \Gamma = \bigcap_{r \in \Gamma} rHr^{-1}.$$

For $\varepsilon \in \{\pm 1\}$, let T_ε be the collection of elements $g \in \Gamma$ of length $n \geq 1$ and type ε. Let $T_\varepsilon^\dagger$ be the subset of $g \in T_\varepsilon$ of length $n \geq 1$ with initial letter 1. We then define the *quasi-kernel*

$$K_\varepsilon = \bigcap_{r \in \Gamma \setminus T_\varepsilon^\dagger} rHr^{-1}.$$

Notice that $(\Gamma \setminus T_{-1}^\dagger) \cup (\Gamma \setminus T_1^\dagger) = \Gamma$, so that $\ker \Gamma = K_{-1} \cap K_1$.

We will consider criteria for HNN extensions to be C^*-simple and to have the unique trace property in the following. In order to make the most of Britton's lemma, we will mostly consider *non-ascending* HNN extensions, which luckily is not too restrictive of a property.

Definition 3.5.4. An HNN extension $\Gamma = \mathrm{HNN}(G, H, \theta)$ is *ascending* if either $H = G$ or $\theta(H) = G$.

We next use the opportunity offered by having recently discussed Britton's lemma to prove this next lemma, essential for the proof of Theorem 3.6.8 in the next section. Recall that the *normal closure* in Γ of a subset $S \subseteq \Gamma$ is the smallest normal subgroup of Γ containing S.

Lemma 3.5.5. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be a non-ascending HNN extension. Then for $g \in \Gamma \setminus H$ and $\varepsilon \in \{\pm 1\}$ there exists $s \in \Gamma$ such that*

(i) *$sgs^{-1} \notin G$;*

(ii) *sgs^{-1} has type $-\varepsilon$ and direction ε;*

(iii) *the initial letter of sgs^{-1} is 1, and the end letter of sgs^{-1} is contained in H_ε.*

Moreover, if $s \in \Gamma$ satisfies the above conditions, $sgs^{-1}r \in \Gamma \setminus T_\varepsilon^\dagger$ for all $r \in \Gamma \setminus T_{-\varepsilon}^\dagger$. In particular, the normal closures of K_1 and K_{-1} in Γ coincide.

Proof. Once we have found $s \in \Gamma$ satisfying (i), (ii) and (iii), write

$$sgs^{-1} = h_1 \tau^{-\varepsilon} h_2 \tau^{\varepsilon_2} \cdots h_n \tau^{\varepsilon} h_{n+1}$$

in the normal form. Then for any $r \in \Gamma \setminus T_{-\varepsilon}^\dagger$ with normal form $r_1 \tau^{f_1} \cdots r_m \tau^{f_m} r_{m+1}$, we have

$$sgs^{-1} r = (h_1 \tau^{-\varepsilon} h_2 \tau^{\varepsilon_2} \cdots h_n \tau^{\varepsilon} h_{n+1})(r_1 \tau^{f_1} \cdots r_m \tau^{f_m} r_{m+1}).$$

If $f_1 = \varepsilon$, the word is reduced; if $f_1 = -\varepsilon$, then $r_1 \notin H_{-f_1} = H_\varepsilon = h_{n+1}^{-1}H_\varepsilon$, so that the above word is still reduced. Here we use that Γ is non-ascending. Hence $sgs^{-1}r \in \Gamma \setminus T_\varepsilon^\dagger$ for all $r \in \Gamma \setminus T_{-\varepsilon}^\dagger$. In particular, there exists $t \in \Gamma$ such that $t(\Gamma \setminus T_{-\varepsilon}^\dagger) \subseteq \Gamma \setminus T_\varepsilon^\dagger$, so that

$$K_\varepsilon = \bigcap_{r\in\Gamma\setminus T_\varepsilon^\dagger} rHr^{-1} \subseteq s\left(\bigcap_{r\in\Gamma\setminus T_{-\varepsilon}^\dagger} rHr^{-1}\right)s^{-1} \subseteq \langle\!\langle K_{-\varepsilon}\rangle\!\rangle.$$

Hence $\langle\!\langle K_\varepsilon\rangle\!\rangle \subseteq \langle\!\langle K_{-\varepsilon}\rangle\!\rangle$.

It remains to prove that there exists an $s \in \Gamma$ satisfying (i), (ii) and (iii). Let $g = g_1\tau^{\delta_1}\cdots g_r\tau^{\delta_r}g_{r+1}$ in the normal form, and first assume that $r \geq 1$. Let $t > r$. For $g_1 \neq 1$, define $m_t = -\delta_1$; if $g_1 = 1$, define $m_t = \delta_1$. Since Γ is non-ascending, we may choose an element $s \in \Gamma \setminus G$ of length t and with normal form $s = \tau^{-\varepsilon}s_2\tau^{m_2}\cdots s_t\tau^{m_t}$ such that $m_t = -m_{t-r}$. Then

$$sgs^{-1} = (\tau^{-\varepsilon}s_2\tau^{m_2}\cdots s_t\tau^{m_t})(g_1\tau^{\delta_1}\cdots g_r\tau^{\delta_r}g_{r+1})(\tau^{-m_t}s_t^{-1}\cdots\tau^{-m_2}s_2^{-1}\tau^{\varepsilon}).$$

If this word is not reduced, let $0 \leq j \leq r-1$ be largest with the property that

$$h = (g_{r-j}\tau^{\delta_{r-j}}\cdots g_r\tau^{\delta_r}g_{r+1})(\tau^{-m_t}s_t^{-1}\cdots\tau^{-m_{t-j}}s_{t-j}^{-1}) \in G.$$

If $j < r-1$, then

$$sgs^{-1} = (\tau^{-\varepsilon}s_2\tau^{m_2}\cdots s_t\tau^{m_t})(g_1\tau^{\delta_1}\cdots g_{r-j-1}\tau^{\delta_{r-j-1}})h(\tau^{-m_{t-j-1}}s_{t-j-1}^{-1}\cdots\tau^{-m_2}s_2^{-1}\tau^{\varepsilon}).$$

As $\tau^{\delta_{j-1}}h\tau^{-m_{t-r-j}} \in G$ would contradict maximality of j, the above word is reduced. If $j = r-1$, then $sgs^{-1} = (\tau^{-\varepsilon}s_2\tau^{m_2}\cdots s_t\tau^{m_t})h(\tau^{-m_{t-r}}s_{t-r}^{-1}\cdots\tau^{-m_2}s_2^{-1}\tau^{\varepsilon})$ which is also reduced since $m_t = -m_{t-r}$ by assumption.

If $r = 0$, then $g \in G \setminus H$ by hypothesis. We then see that $\tau^{-1}g\tau r \in \Gamma \setminus T_1^\dagger$ for all $r \in \Gamma \setminus T_{-1}^\dagger$, and for all $s \in S_1 \setminus \{1\}$ and $r \in \Gamma \setminus T_1^\dagger$,

$$(\tau s\tau^{-1})g(\tau s^{-1}\tau^{-1})r \in \Gamma \setminus T_{-1}^\dagger.$$

In either of the cases $r = 0$ and $r \geq 1$, (i), (ii) and (iii) now follow from Lemma 3.5.3. $\square$

In some cases when working with large subsets of an HNN extension, it will prove helpful to be able to uncover properties of elements in these subsets without having to reduce. We introduce a simple lemma to remedy this situation, the proof modeled after the proof of the preceding lemma.

Lemma 3.5.6. *For an HNN extension $\Gamma = \mathrm{HNN}(G, H, \theta)$, $n \geq 1$, $g_1, \ldots, g_{n+1} \in G$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$, define*

$$g = g_1\tau^{\varepsilon_1}\cdots g_n\tau^{\varepsilon_n}g_{n+1} \in \Gamma.$$

If n is odd, then $g \notin G$, and if $\varepsilon_1 = \ldots = \varepsilon_k$ for some $k > \frac{n}{2}$, the type of g is ε_1.

Proof. Notice that the above expression of g is reduced if and only if $\tau^{\varepsilon_j}g_{j+1}\tau^{\varepsilon_{j+1}} \notin G$ for all $1 \leq j \leq n-1$. We may therefore assume that there is $1 \leq j \leq n-1$ such that $\tau^{\varepsilon_j}g_{j+1}\tau^{\varepsilon_{j+1}} \in G$. Let $1 \leq i \leq \min\{j, n-j\}$ be largest such that

$$h = g_{j-(i-1)}\tau^{\varepsilon_{j-(i-1)}}g_{j-(i-1)+1}\cdots g_{j+i}\tau^{\varepsilon_{j+i}}g_{j+i+1} \in G.$$

We may now write $g = g_1\tau^{\varepsilon_1}\cdots g_{j-i}\tau^{\varepsilon_{j-i}}h\tau^{\varepsilon_{j+i+1}}\cdots g_n\tau^{\varepsilon_n}g_{n+1}$. If this word is not reduced, we can continue this process for this new expression of g. After a finite number of iterations, the word must be reduced, so because this process always removes an even number of powers of τ from the preceding expression of g, n being odd implies that $g \notin G$ by Lemma 3.5.3 (i). If $\varepsilon_1 = \ldots = \varepsilon_k$ for some $k > \frac{n}{2}$, then reduction removes at most $n - k < k$ of the identical first k powers of τ in the original expression of g. Therefore the type of g is ε_1 by Lemma 3.5.3 (ii). $\square$

The following theorem (and the main result of this section) is motivated by condition (SF') in [39, Proposition 11] and [87, Theorem 3.2].

Theorem 3.5.7. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be a non-ascending HNN extension. The following are equivalent:*

(i) *the quasi-kernel K_ε is trivial for some or both ε;*
(ii) *for every finite subset $F \subseteq \Gamma \setminus \{1\}$, there exists $g \in \Gamma$ such that $gFg^{-1} \cap H = \emptyset$;*
(iii) *for every finite subset $F \subseteq G \setminus \{1\}$, there exists $g \in \Gamma$ such that $gFg^{-1} \cap H = \emptyset$;*
(iv) *for every finite subset $F \subseteq H \setminus \{1\}$, there exists $g \in \Gamma$ such that $gFg^{-1} \cap H = \emptyset$.*

Moreover, if any of these equivalent conditions holds, then Γ is a Powers group.

Proof. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i): If K_ε is trivial for some $\varepsilon \in \{\pm 1\}$, then $K_{-\varepsilon}$ is trivial by Lemma 3.5.5. Therefore suppose that K_{-1} and K_1 are both non-trivial. First, if $\ker\Gamma = K_{-1} \cap K_1$ is non-trivial, we pick $f \in \ker\Gamma \setminus \{1\}$. Set $F = \{f\}$. Clearly $F \subseteq H$ since $\ker\Gamma \subseteq H$. Then $gfg^{-1} \in \ker\Gamma \subseteq H$ for all $g \in \Gamma$.

If $\ker\Gamma$ is trivial, pick $f_\varepsilon \in K_\varepsilon \setminus \{1\}$ for $\varepsilon \in \{\pm 1\}$ and set $F = \{f_{-1}, f_1\}$. For an arbitrary $g \in \Gamma$, then $g \in \Gamma \setminus T_\varepsilon^\dagger$ for some ε. Then $gf_\varepsilon g^{-1} \in H$, i.e., $gf_\varepsilon g^{-1} \in gFg^{-1} \cap H$.

(i) $\Rightarrow$ (ii): Choose a finite $F \subset \Gamma \setminus \{1\}$. Assume first there is an element $f_1 \in F \cap H$ (otherwise, there is nothing to show). Since $f_1 \neq 1$, we may pick $g_1 \in \Gamma \setminus T_1^\dagger$ such that $g_1^{-1}f_1g_1 \notin H$. We may now assume that $g_1^{-1}f_1g_1 \notin G$; if $g_1^{-1}f_1g_1 \in G$, we can freely replace g_1 by $g_1\tau$. In particular, $g_1 \notin G$, and so we may let ε_1 be the direction of g_1. We then see that the type and direction of $g_1^{-1}f_1g_1$ are $-\varepsilon_1$ and ε_1, respectively, since we can write $g_1^{-1}f_1g_1$ as a reduced word by means of the normal form of g_1 and then apply Lemma 3.5.3. In this way, we also see that replacing g_1 by g_1h^{-1} where h is the end letter of g_1 does not change this conclusion, so we may assume that $g_1^{-1}f_1g_1$ has initial letter 1 and end letter contained in H_{ε_1}.

We now assume that there is an element $f_2 \in F$ such that $g_1^{-1}f_2g_1 \in H$ (otherwise, we are done). Pick $g_2 \in \Gamma \setminus T_{-\varepsilon_1}^\dagger$ such that $g_2^{-1}g_1^{-1}f_2g_1g_2 \notin H$. In the same manner as above, we may assume that $g_2^{-1}g_1^{-1}f_2g_1g_2 \notin G$, $g_2 \notin G$ and g_2 has end letter 1, and if ε_2 is the direction of g_2, then $g_2^{-1}g_1^{-1}f_2g_1g_2$ has type $-\varepsilon_2$ and direction ε_2. We now claim that $g_2^{-1}g_1^{-1}f_1g_1g_2 \notin G$ as well. Indeed, since $g_1^{-1}f_1g_1 \notin G$ has type $-\varepsilon_1$ and direction ε_1 with initial letter 1 and end letter in H_{ε_1}, then $g_2^{-1}g_1^{-1}f_1g_1g_2$ has type $-\varepsilon_2$ and direction ε_2, with initial letter 1 and end letter contained in H_{ε_2}. To realize this, we consider the normal forms of $g_1^{-1}f_1g_1$ and of g_2, say, $g_2 = h_1\tau^\varepsilon\cdots h_m\tau^{\varepsilon_2}$. Then

$$g_2^{-1}(g_1^{-1}f_1g_1)g_2 = \tau^{-\varepsilon_2}h_m^{-1}\cdots\tau^{-\varepsilon}h_1^{-1}(g_1^{-1}f_1g_1)h_1\tau^\varepsilon\cdots h_m\tau^{\varepsilon_2}.$$

Hence reduction is only possible if $\varepsilon = -\varepsilon_1$, but $h_1 \notin H_{-\varepsilon} = H_{\varepsilon_1}$ by assumption since $g_2 \in \Gamma \setminus T^{\dagger}_{-\varepsilon_1}$. Therefore the above word is always reduced, so Lemma 3.5.3 applies.

It should be clear how this process continues, choosing g_i from the set $\Gamma \setminus T^{\dagger}_{\varepsilon}$, depending on the direction of g_{i-1}, and since F is finite, we take g to be the product of the g_i's, and then $g^{-1}fg \notin H$ for every $f \in F$.

We refer to [39, Proposition 11] for a proof that any of the four conditions implies that Γ is a Powers group, if Γ is non-ascending. □

In the next section, we will give another proof (Proposition 3.6.4) that a non-ascending, countable HNN extension is C^*-simple if it has trivial quasi-kernels.

Remark 3.5.8. In the 2011 article [43, Theorem 2 (ii)], a sufficient criterion to ensure C^*-simplicity of a non-ascending, countable HNN extension was given by de la Harpe and Préaux, formulated as follows. For $\Gamma = \mathrm{HNN}(G, H, \theta)$ non-ascending and G countable, define $H_0 = H$, and recursively define a descending sequence of subgroups $(H_k)_{k \geq 1}$ of $H_0 = H$ by $H'_k = H_k \cap \tau^{-1} H_k \tau$ and

$$H_k = \left(\bigcap_{g \in G} g H'_k g^{-1} \right) \cap \tau \left(\bigcap_{g \in G} g H'_k g^{-1} \right) \tau^{-1}.$$

The criterion ensuring C^*-simplicity of Γ was that $H_k = \{1\}$ for some $k \geq 0$ (in fact, Γ is a Powers group).

We claim that Theorem 3.5.7 is a stronger result. Indeed, for $k \geq 1$, let C_k be the set of elements in Γ of length $\leq k+1$. Then $H_k = \{1\}$ implies $\bigcap_{r \in C_k} rHr^{-1} = \{1\}$, since each H_k is obtained by taking intersections of sets of the form rHr^{-1}, r running through a subset of C_k. For $\varepsilon \in \{\pm 1\}$ and $s \in S_{-\varepsilon} \setminus \{1\}$, then $s\tau^{(k+2)\varepsilon} C_k \subseteq \Gamma \setminus T^{\dagger}_{\varepsilon}$ due to Lemma 3.5.6. Therefore

$$K_\varepsilon = \bigcap_{r \in \Gamma \setminus T^{\dagger}_{\varepsilon}} rHr^{-1} \subseteq s\tau^{(k+2)\varepsilon} \left(\bigcap_{r \in C_k} rHr^{-1} \right) \tau^{-(k+2)\varepsilon} s^{-1} = \{1\}.$$

The following corollary is a rather simple criterion for when a quasi-kernel coincides with the kernel of an HNN extension.

Corollary 3.5.9. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be a non-ascending HNN extension. If there exists $g_\varepsilon \in G \setminus H_{-\varepsilon}$ such that $g_\varepsilon K_\varepsilon g_\varepsilon^{-1} = K_\varepsilon$ for $\varepsilon \in \{\pm 1\}$, then $K_\varepsilon = \ker(\Gamma)$. If Γ also satisfies $\ker \Gamma = \{1\}$, Γ is a Powers group.*

Proof. For $\varepsilon \in \{\pm 1\}$ and all $g_\varepsilon \in G \setminus H_{-\varepsilon}$ we have

$$\Gamma \setminus T^{\dagger}_{-\varepsilon} \subseteq (\Gamma \setminus T^{\dagger}_{\varepsilon}) \cup g_\varepsilon (\Gamma \setminus T^{\dagger}_{\varepsilon}).$$

Indeed, if $x \in \Gamma \setminus T^{\dagger}_{-\varepsilon}$ is of length ≥ 1 and has initial letter distinct from 1, this is clear; if not, then x has type ε and initial letter 1, in which case $g_\varepsilon^{-1} x$ has type ε and initial letter different from 1. Therefore $\Gamma = (\Gamma \setminus T^{\dagger}_{\varepsilon}) \cup g_\varepsilon (\Gamma \setminus T^{\dagger}_{\varepsilon})$, so that

$$\ker \Gamma = \bigcap_{r \in \Gamma \setminus T^{\dagger}_{\varepsilon}} rHr^{-1} \cap \bigcap_{r \in \Gamma \setminus T^{\dagger}_{\varepsilon}} g_\varepsilon r H r^{-1} g_\varepsilon^{-1} = K_\varepsilon \cap g_\varepsilon K_\varepsilon g_\varepsilon^{-1}.$$

If $g_\varepsilon K_\varepsilon g_\varepsilon^{-1} = K_\varepsilon$, then $\ker \Gamma = K_\varepsilon$. If $\ker \Gamma = \{1\}$, it then follows from Theorem 3.5.7 that Γ is a Powers group. □

We next discuss the amenable radical $R(\Gamma)$ of a non-ascending HNN extension Γ. Let $FC(\Gamma)$ denote the normal subgroup of elements in Γ of finite conjugacy class and let $NF(\Gamma)$ denote the largest normal subgroup of Γ that does not contain any non-abelian free subgroup. By [38, Proposition 8], these groups generally fit into a sequence of subgroups:

$$FC(\Gamma) \subseteq R(\Gamma) \subseteq NF(\Gamma) \subseteq \ker \Gamma \subseteq H \cap \theta(H).$$

Notice that Γ always contains a non-abelian free subgroup. In fact, there is an even longer sequence

$$FC(\Gamma) \subseteq FC(\ker \Gamma) \subseteq R(\ker \Gamma) = R(\Gamma) \subseteq NF(\Gamma) = NF(\ker \Gamma) \subseteq \ker \Gamma. \tag{3.5.2}$$

The first containment is clear, the second holds since $FC(\ker \Gamma)$ is an amenable normal subgroup of $\ker \Gamma$, and the two equalities follow from [87, Examples 6.4, 6.6, and Lemma 6.7].

In particular, since $R(\ker \Gamma) = R(\Gamma)$, the following is immediate:

Proposition 3.5.10. *Let Γ be a non-ascending HNN extension. Then Γ has the unique trace property if and only if $\ker \Gamma$ has the unique trace property.*

Moreover, if $H \cap \theta(H)$ is finite, then $FC(\Gamma) = \ker \Gamma$, and the sequence (3.5.2) collapses. To see why, note that for $h \in \ker \Gamma$, then $h \in g(H \cap \theta(H))g^{-1}$ for all $g \in \Gamma$ by definition, so $g^{-1}hg \in H \cap \theta(H)$ for all $g \in \Gamma$. Therefore the conjugacy class of h is contained in $H \cap \theta(H)$, which is finite, so $h \in FC(\Gamma)$.

The following result, similar to [87, Theorem 3.7], now holds.

Proposition 3.5.11. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be a non-ascending HNN extension such that $H \cap \theta(H)$ is finite. Then the following are equivalent:*

(i) Γ *is icc;*
(ii) $\ker \Gamma = \{1\}$*;*
(iii) $K_{-1} = K_1 = \{1\}$*;*
(iv) Γ *is a Powers group;*
(v) Γ *is C^*-simple;*
(vi) Γ *has the unique trace property.*

Proof. It is clear that (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (i) $\Rightarrow$ (ii) from the above discussion and what we have seen in Sections 3.2 and 3.3, so only (ii) $\Rightarrow$ (iii) remains. Suppose that $\ker \Gamma = \{1\}$. Note that $\ker \Gamma$ coincides with the intersection of the decreasing sequence

$$H_0 \supseteq H_1 \supseteq \cdots H_k \supseteq H_{k+1} \supseteq \cdots$$

of Remark 3.5.8. For $k \geq 1$, H_k is a subgroup of $H \cap \theta(H)$ and therefore finite, meaning that H_k must be trivial for some $k \geq 1$. As in Remark 3.5.8, we conclude that K_ε is trivial for $\varepsilon \in \{\pm 1\}$. □

The above result indicates that in order to search for examples of non-ascending HNN extensions $\mathrm{HNN}(G, H, \theta)$ that are not C^*-simple but do satisfy the unique trace property, we have to ensure that the image of θ inside G is not too far away from H.

3.6 Boundary actions of non-ascending HNN extensions

Joint work with Tron Omland (University of Oslo)

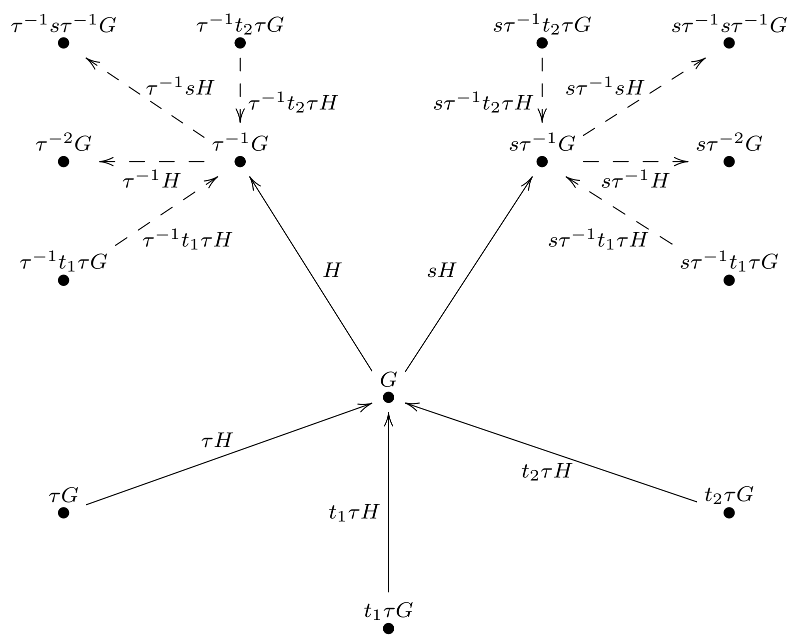
In this section, we investigate the action of a non-ascending HNN extension Γ on its Bass-Serre tree and give a characterization of C^*-simplicity of Γ in terms of this action. Our work should be compared with similar results by Ivanov and Omland for free products with amalgamation [87] – whereas HNN extensions are fundamental groups of loops, free amalgamated products are fundamental groups of a segment of length 1.

Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be an HNN extension and let T be the Bass-Serre tree of Γ. The vertices in T are left cosets of Γ/G and the edge set of T consists of two disjoint copies of Γ/H, say, $\Gamma/H \sqcup \overline{\Gamma/H}$, where the inversion map sends gH to $\overline{gH}$ and vice versa. The origin and terminus maps satisfy

$$o(gH) = gG = t(\overline{gH}), \quad t(gH) = g\tau^{-1}G = o(\overline{gH}), \quad g \in \Gamma.$$

The action of Γ on T by left translation is transitive. Moreover, T is regular, i.e., all vertices have the same degree $[G : H] + [G : \theta(H)]$, so that in particular, T is leafless, and T is countable if and only if G/H and $G/\theta(H)$ are of at most countably infinite cardinality.

The picture below illustrates part of the Bass-Serre tree of the HNN extension of a group G, and subgroups $H \cong \theta(H)$ such that $[G : H] = 2$ and $[G : \theta(H)] = 3$. We let $S_{-1} = \{1, s\}$ and $S_1 = \{1, t_1, t_2\}$ be sets of left coset representatives for H and $\theta(H)$, respectively. Observe that for $g \in S_1$, if we want to add $g\tau^{-1}$ to the right in one of the vertex cosets mG (e.g., going from mG to $mg\tau^{-1}G$), we traverse an edge *emanating* from mG, and when adding $g\tau$ to the right for $g \in S_{-1}$, one traverses an edge *ending* in mG.



Let $g \in \Gamma$ be an element with normal form $g = g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2}\cdots g_n\tau^{\varepsilon_n}$. Defining $s_0 = 1$ and

$$s_k = \prod_{i=1}^{k} g_i\tau^{\varepsilon_i}, \quad e_k = \begin{cases} s_k\tau H = s_{k-1}g_kH & \text{if } \varepsilon_k = -1, \\ \overline{s_kH} & \text{if } \varepsilon_k = 1 \end{cases} \tag{3.6.1}$$

for $1 \le k \le n$, the unique path from G to gG is given by

$$G \xrightarrow{e_1} s_1G \xrightarrow{e_2} \cdots \xrightarrow{e_n} gG$$

Indeed, if $\varepsilon_k = -1$, then $o(s_{k-1}g_kH) = s_{k-1}G$ and $t(s_{k-1}g_kH) = s_{k-1}g_k\tau^{-1}G = s_kG$, and if $\varepsilon_k = 1$ we notice that $o(\overline{s_kH}) = s_k\tau^{-1}G = s_{k-1}G$ and $t(\overline{s_kH}) = s_kG$.

We now recall the discussion of the shadow topology on the space $T \cup \partial T$, in Section 2.2. We first note that even though all results in this section are stated for countable HNN extensions Γ, they in fact hold under the weaker assumption that *the Bass-Serre tree of* Γ *is countable*, as we only need to be able to apply Proposition 2.2.14.

It was remarked previously in this section that the action of Γ on T is transitive, so it is minimal. Moreover, we either have $\overline{\partial T} = \partial T$ or $\overline{\partial T} = T \cup \partial T$ in $T \cup \partial T$ in the shadow topology, since T is regular. If $|\partial T| \le 2$, every vertex in T has degree 2, so that $H = \theta(H) = G$ and $\Gamma = G \rtimes_\theta \mathbb{Z}$. Furthermore, we have the following fact, due to de la Harpe and Préaux [43, Proposition 19].

Proposition 3.6.1. *Let* $\Gamma = \mathrm{HNN}(G, H, \theta)$ *be a countable HNN extension, and let* T *be the Bass-Serre tree of* Γ*. If* $|\partial T| \ge 3$*, then the following are equivalent:*

(i) *The weakly hyperbolic limit set of* Γ *in* $\overline{\partial T}$ *contains at least three points.*

(ii) Γ *is non-ascending.*

If any of these two conditions is satisfied, $\overline{\partial T}$ *is a* Γ*-boundary in the shadow topology.*

Proof. We will only prove that (ii) implies (i); the other implication is proved in [43]. Taking $s \in G \setminus H$ and $t \in G \setminus \theta(H)$, then $s\tau$ and $t\tau^{-1}$ define hyperbolic automorphisms of T due to Lemma 3.5.6, since they fix no vertices in T. Moreover, the intersection of their axes is $\{G\}$, as one may easily see that $(t\tau^{-1})^{-m}(s\tau)^n \in G$ for $m, n \in \mathbb{Z}$ implies $m = n = 0$. Hence $s\tau$ and $t\tau^{-1}$ are transverse. Finally, since the action of G on T is minimal, it follows from (i) and Proposition 2.2.14 that $\overline{\partial T}$ is indeed a Γ-boundary. □

A basis of clopen sets for the shadow topology on $T \cup \partial T$ is defined as follows. If g has normal form $g = g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2}\cdots g_n\tau^{\varepsilon_n}g_{n+1}$, we let $U(g)$ be the subset of all elements hG where the normal form of h begins with $gg_{n+1}^{-1} = g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2}\cdots g_n\tau^{\varepsilon_n}$, as well as all equivalence classes of rays identifiable with a ray beginning with $g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2}\cdots g_n\tau^{\varepsilon_n}$. Under the identification $T \cup \partial T = T_G \cup R_G$ (cf. the subsection on countable trees in Section 2.2), $U(g)$ is therefore the sets of maps $\mathbb{Z}_+ \to T$ whose images of $0 \le i \le n$ are determined by the path described in (3.6.1). The collection of all $U(g)$, $g \in \Gamma$, is the desired basis of clopen sets. For any $g \in \Gamma$, let $K(g)$ be the subgroup of G of elements fixing every vertex in $U(g)$.

To specify the properties of the subgroups $K(g)$, we require two technical lemmata.

Lemma 3.6.2. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be a non-ascending countable HNN extension with Bass-Serre tree T. For any $g \in \Gamma \setminus G$ with direction ε, $K(g) = gK_{-\varepsilon}g^{-1}$. Moreover, $g \in \Gamma$ fixes all $x \in T$ if and only if $g \in \ker \Gamma$.*

Proof. Suppose that $g' \in \Gamma$ is a fixator of all elements in $U(g)$ where

$$g = g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2}\cdots g_n\tau^{\varepsilon_n}g_{n+1} \in \Gamma$$

in the normal form, and let ε be the direction of g. Then g' fixes all edges in the subtree spanned by $U(g)$, i.e., $g'(mH) = mH$ for all $m \in \Gamma$ with normal form beginning with $g_1\tau^{\varepsilon_1}g_2\tau^{\varepsilon_2}\cdots g_n\tau^{\varepsilon}$. Therefore $g' \in K(g)$ if and only if $g'(grH) = grH$ for all $r \in \Gamma$ with the normal form $r = r_1\tau^{f_1}r_2\tau^{f_2}\cdots r_m\tau^{f_m}r_{m+1}$ where either

(1) $m = 0$, or
(2) $m \geq 1$ and either (2a) $f_1 = \varepsilon$, or (2b) $f_1 = -\varepsilon$ and $r_1 \notin H_\varepsilon$.

In case (2b), the fact $r_1 \in S_{-f_1} = S_\varepsilon$ implies that $r_1 \neq 1$. Therefore the $r \in \Gamma$ of the above form constitute the set $\Gamma \setminus T^{\dagger}_{-\varepsilon}$, so

$$g' \in g\left(\bigcap_{r\in\Gamma\setminus T^{\dagger}_{-\varepsilon}} rHr^{-1}\right)g^{-1} = gK_{-\varepsilon}g^{-1}.$$

Hence $K(g) = gK_{-\varepsilon}g^{-1}$. □

Lemma 3.6.3. *Let Γ be a non-ascending countable HNN extension with Bass-Serre tree T. For all $g, s \in \Gamma$, $g \in K(s)$ whenever $U(s) \cap \partial T \subseteq (\partial T)^g$.*

Proof. Let $g \in \Gamma \setminus \{1\}$ and assume that g fixes all $x \in U(s) \cap \partial T$. Let $\varepsilon \in \{\pm 1\}$ be the type of s, if $|s| \geq 1$, and arbitrary otherwise. If $g \notin sGs^{-1}$ write

$$s^{-1}gs = g_1\tau^{\varepsilon_1}\cdots g_n\tau^{\varepsilon_n}g_{n+1}$$

in the normal form for $n \geq 1$. Let us also assume, without loss of generality, that $\varepsilon_1 = 1$. For all $j, m \geq 0$, consider the word

$$\tau^{-m\varepsilon}r^{-1}(s^{-1}gs)r\tau^{j\varepsilon} = \tau^{-m\varepsilon}r^{-1}(g_1\tau\cdots g_n\tau^{\varepsilon_n}g_{n+1})r\tau^{j\varepsilon},$$

where $r \in G$. Then this word is reduced whenever (1) $\varepsilon = \varepsilon_n = -1$; (2) $\varepsilon = \varepsilon_n = 1$ and $r \in G$ satisfies $r^{-1}g_1 \notin H$; (3) $\varepsilon = -1$, $\varepsilon_n = 1$ and $r \in G$ satisfies $g_{n+1}r \notin \theta(H)$. This is always possible, since Γ is non-ascending. In these cases, g cannot fix the point $x \in U(s) \cap \partial T$ determined by the ray passing through the vertices $(sr\tau^{j\varepsilon})_{j\geq 0}$.

We now assume that $\varepsilon = 1$ and $\varepsilon_n = -1$, so that $n \geq 2$. If r can be chosen such that $g_{n+1}r \notin H$ and $r^{-1}g_1 \notin H$, we are done. Otherwise, we have $g_{n+1}g_1H \cup H = G$. Let $p \in G$ such that $p^{-1}g_2 \notin H$. For all $j, m \geq 0$,

$$\begin{aligned}\tau^{-m}p^{-1}\tau^{-1}g_1^{-1}(s^{-1}gs)g_1\tau p\tau^j &= \tau^{-m}(p^{-1}\tau^{-1}g_1^{-1}g_1\tau g_2)\tau^{\varepsilon_2}g_3\tau^{\varepsilon_3}\cdots g_n\tau^{-1}g_{n+1}g_1\tau p\tau^j \\ &= \tau^{-m}u\tau^{\varepsilon_2}g_3\tau^{\varepsilon_3}\cdots g_n\tau^{-1}v\tau p\tau^j\end{aligned}$$

for $u, v \in G \setminus H$, and this word is reduced. Hence g does not fix the point $x \in U(s) \cap \partial T$ determined by the sequence of vertices given by $(sg_1\tau p\tau^j)_{j\geq 0}$. Therefore $g \in sGs^{-1}$.

We next assume that $s \notin G$ and let $\varepsilon \in \{\pm 1\}$ be the direction of s. For any $r \in \Gamma \setminus T^{\dagger}_{-\varepsilon}$, write $r = r_1\tau^{f_1}\cdots r_n\tau^{f_n}r_{n+1}$ in the normal form. Then let $(e_i)_{i\geq 1}$ be a sequence of edges in T constituting a ray, where $e_1,\dots,e_n$ describes the unique path in T from G to srG. By hypothesis, g fixes all points $x \in \partial T$ arising from rays of the above form. Let $m = |s|$. For any sequence $(e_i)_{i\geq 1}$ as above, there exist positive integers $j, k > m+n$ such that $ge_j = e_k$. Since $o(e_m) = sG = s(s^{-1}gs)G = o(ge_m)$, we have

$$j - m = d(t(e_j), o(e_m)) = d(t(ge_j), o(ge_m)) = d(t(e_k), o(e_m)) = k - m,$$

so $j = k$. As $(e_i)_{i\geq 1}$ defines a geodesic, $ge_{m+n+1} = e_{m+n+1}$ as well. Now the fact $e_1,\dots,e_{m+n}$ uniquely connects G and srG implies that $gf = f$ for any edge f in T with $o(f) = srG$ and $t(f) \neq (sr)(r_{n+1}^{-1}\tau^{-f_n})G$, since x was arbitrary. Due to g being a non-inversion and necessarily also fixing the remaining edge set, that is, the set of edges joining srG and $(sr)(r_{n+1}^{-1}\tau^{-f_n})G$, g must fix *any* edge f in T with $o(f) = srG$. In particular we have $gsrH = srH$, so that $s^{-1}gs \in rHr^{-1}$. As $r \in \Gamma \setminus T^{\dagger}_{-\varepsilon}$ was arbitrary, $s^{-1}gs \in K_{-\varepsilon}$.

If $s = 1$, then $g \in G$ and all of ∂T is fixed by g, which implies $g \in \ker\Gamma$. Indeed, the above argument applies verbatim if $r \notin G$, and if $r \in G$, we still see that g fixes all edges f in T with $o(f) = G$, so that $grH = rH$ in any case. □

In short, whenever a group element g in a non-ascending HNN extension Γ acts as the identity on some open subset of the boundary of the Bass-Serre tree ∂T, then g fixes a large chunk of any ray whose image is eventually in the open subset.

For a continuous action of a group G on a topological space X, define

$$\ker(G \curvearrowright X) = \{g \in G \mid X^g = X\}$$

and

$$\mathrm{int}(G \curvearrowright X) = \{g \in G \mid X^g \text{ has non-empty interior}\}.$$

If D is a G-invariant subset of X, then so is $\overline{D}$, and we have

$$\ker(G \curvearrowright D) = \ker(G \curvearrowright \overline{D}), \quad \mathrm{int}(G \curvearrowright D) = \mathrm{int}(G \curvearrowright \overline{D}).$$

Here the first equality follows from continuity of the action. If D^g has non-empty interior for some $g \in G$, there is a non-empty open subset V of $\overline{D}$ such that $V \cap D \subseteq D^g$ which ensures that $V \subseteq \overline{V \cap D} \subseteq \overline{D^g} \subseteq \overline{D}^g$. Conversely, if $\overline{D}^g$ has non-empty interior, then there is a non-empty open subset V of $\overline{D}$ such that $V \subseteq \overline{D}^g$, meaning that $V \cap D \subseteq D^g$. As V is non-empty, so is $V \cap D$.

Proposition 3.6.4. *Let Γ be a non-ascending countable HNN extension with Bass-Serre tree T, and consider $T \cup \partial T$ equipped with the shadow topology. The action of Γ on T satisfies*

$$\ker\Gamma = \ker(\Gamma \curvearrowright T) = \ker(\Gamma \curvearrowright \partial T) = \ker(\Gamma \curvearrowright \overline{\partial T})$$

and

$$\langle\!\langle K_{-1}\rangle\!\rangle = \langle\!\langle K_1\rangle\!\rangle = \langle\!\langle K_{-1} \cup K_1\rangle\!\rangle = \mathrm{int}(\Gamma \curvearrowright \partial T) = \mathrm{int}(\Gamma \curvearrowright \overline{\partial T}),$$

where $\langle\!\langle S\rangle\!\rangle$ denotes the normal closure in Γ of a subset $S \subseteq \Gamma$.

Proof. By definition we have $\ker \Gamma = \ker(\Gamma \curvearrowright T)$, and $\ker(\Gamma \curvearrowright T) \subseteq \ker(\Gamma \curvearrowright \partial T)$ by continuity. Moreover, $\ker(\Gamma \curvearrowright \partial T) \subseteq \ker \Gamma$ by Lemma 3.6.3.

Any $g \in K_{-1}$ is contained in $K(\tau)$. Indeed, if $r^{-1}gr \in H$ for all $r \in \Gamma \setminus T^{\dagger}_{-1}$, then all rays emanating in G and passing through τG are fixed by g. Hence $K_{-1} \subseteq \operatorname{int}(G \curvearrowright \partial T)$ and $\langle\!\langle K_{-1} \rangle\!\rangle \subseteq \operatorname{int}(G \curvearrowright \partial T)$ by normality of the latter. Conversely, suppose that $(\partial T)^g$ has non-empty interior for $g \in \Gamma \setminus \{1\}$ in the shadow topology. Then $(\partial T)^g$ contains $U(s) \cap \partial T$ for some $s \in \Gamma$, so that g belongs to $K(s) = sK_{-\varepsilon}s^{-1}$ for some $\varepsilon \in \{\pm 1\}$ by Lemmas 3.6.2 and 3.6.3, depending on the direction of s. Thus $g \in \langle\!\langle K_{-1} \cup K_1 \rangle\!\rangle$. The remaining identities follow from Lemma 3.5.5. $\square$

Remark 3.6.5. In [43, p. 11], de la Harpe and Préaux call the action of an HNN extension Γ on its Bass-Serre tree T *slender* if $\partial T \neq \emptyset$ and the action of G on ∂T is topologically free. They prove that this implies that the action of Γ on ∂T is *strongly faithful* [43, Corollary 10], i.e., for all finite subsets $F \subseteq \Gamma \setminus \{1\}$ there is $x \in \partial T$ such that $fx \neq x$ for all $f \in F$.

Our results show that the converse also holds for non-ascending HNN extensions: indeed, if $F \subseteq \Gamma \setminus \{1\}$ is a finite subset and $x \in \partial T$ satisfies $fx \neq x$ for all $f \in F$, let $s \in G$ such that $fU(s) \cap U(s) = \emptyset$ for all $f \in F$ (and $x \in U(s)$). In particular, no vertex gG in $U(s)$ satisfies $gG \in fU(s)$ for any $f \in F$, so there is an edge gH for which $fgH \neq gH$ for all $f \in F$. Therefore $g^{-1}Fg \cap H = \emptyset$, so by Theorem 3.5.7 and Proposition 3.6.4, the action of G on ∂T is topologically free.

Let us now give another example – a known one – of a collection of C^*-simple groups.

Example 3.6.6. Let $G = \mathbb{Z}$ and let $g \in G$ be a generator of G. For $m, n \in \mathbb{Z}\setminus\{0\}$, define $H = \langle g^m \rangle$ (thus corresponding to $m\mathbb{Z}$) and an injective homomorphism $\theta \colon H \to G$ by $\theta(g^{km}) = g^{kn}$ for $k \in \mathbb{Z}$. Then the HNN extension $\mathrm{HNN}(G, H, \theta) = \mathrm{HNN}(\mathbb{Z}, m\mathbb{Z}, km \mapsto kn)$ is the *Baumslag-Solitar group*

$$\Gamma = \mathrm{BS}(m, n) = \langle g, \tau \mid \tau^{-1} g^m \tau = g^n \rangle.$$

A 2007 result due to Ivanov [86, Theorem 4.9] states that $\mathrm{BS}(m,n)$ is C^*-simple if and only if $\min\{|m|, |n|\} \geq 2$ and $|m| \neq |n|$.

We give a new proof using the C^*-simplicity criterion for HNN extensions given above. Notice first that if $|m| = |n|$, then $H = \langle g^m \rangle$ is a normal abelian subgroup of Γ. Furthermore, $\mathrm{BS}(\pm 1, n)$ and $\mathrm{BS}(m, \pm 1)$ are solvable. Indeed, in the case $m = 1$, $N = \langle \{\tau^k g \tau^{-k} \mid k \in \mathbb{Z}\} \rangle$ is a normal abelian subgroup of Γ, with the corresponding quotient group being infinite and cyclic.

If $\min\{|m|, |n|\} \geq 2$ and $|m| \neq |n|$, let us assume that m and n are coprime for the sake of keeping the proof on the elegant side. Now $\tau^{-1} H \tau = \langle g^n \rangle$. For $k \in \mathbb{Z}$, write $kn = qm + r$ for $q \in \mathbb{Z}$ and $0 \leq r < m$. If

$$G \ni \tau^{-1} g^{kn} \tau = g^{qn} \tau^{-1} g^r \tau,$$

then $r = 0$, so m divides kn and hence also k. In turn n divides q, so $\tau^{-1} g^{kn} \tau \in \langle g^{n^2} \rangle$. Hence $\tau^{-1} \langle g^n \rangle \tau \cap G = \langle g^{n^2} \rangle$. Continuing this way, one may show that $\tau^{-i} H \tau^i \cap G = \langle g^{n^i} \rangle$ for $i \geq 1$, meaning that $K_1 = \{1\}$. By Lemma 3.5.5, $K_{-1} = \{1\}$ as well, so that Γ is C^*-simple by Theorem 3.5.7 or Proposition 3.6.4.

We next investigate what properties are forced onto the action of a non-ascending HNN extension Γ on its Bass-Serre tree when Γ is C^*-simple.

Lemma 3.6.7. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be an HNN extension, and let K be a normal subgroup of Γ contained in H. Then*

$$\Gamma/K \cong \mathrm{HNN}(G/K, H/K, \tilde{\theta}),$$

if we define $\tilde{\theta}\colon H/K \to G/K$ by $\tilde{\theta}(hK) = \theta(h)K$ for $h \in H$. The quotient map $\pi\colon G \to G/K$ extends to a surjective homomorphism $\tilde{\pi}\colon \Gamma \to \mathrm{HNN}(G/K, H/K, \tilde{\theta})$ with kernel K such that $\tilde{\pi}|_G = \pi$ and $\tilde{\pi}$ maps the stable letter of Γ to the stable letter of $\mathrm{HNN}(G/K, H/K, \tilde{\theta})$.

Proof. Let $\Gamma' = \mathrm{HNN}(G/K, H/K, \tilde{\theta})$ and let τ' be the stable letter of Γ'. If $\varphi\colon G * \langle\tau\rangle \to \Gamma$ and $\varphi'\colon G/K * \langle\tau\rangle \to \Gamma'$ are the natural quotient maps, then there exists a group homomorphism $\tilde{\pi}\colon \Gamma \to \Gamma'$ such that the diagram

$$\begin{tikzcd} G * \langle\tau\rangle \arrow[r, "\varphi"] \arrow[d, "\pi*\mathrm{id}"] & \Gamma \arrow[d, "\tilde{\pi}"] \\ G/K * \langle\tau\rangle \arrow[r, "\varphi'"] & \Gamma' \end{tikzcd}$$

commutes, since $\pi \circ \theta = \tilde{\theta} \circ \pi$, and it is evident that $\tilde{\pi}$ satisfies all of the desired properties bar having kernel K. If $H_{-1} = H$ and $H_1 = \theta(H)$, then if $g \in \Gamma \setminus K$ has normal form $g = g_1\tau^{\varepsilon_1} \cdots g_n\tau^{\varepsilon_n} g_{n+1}$ and $1 \leq i \leq n-1$, we observe that $\pi(g_{i+1}) \notin H_{\varepsilon_i}/K$ whenever $\varepsilon_{i+1} = -\varepsilon_i$, since $K \subseteq H \cap \theta(H)$ by assumption. Hence $\tilde{\pi}(g) \neq 1$ by Britton's lemma, so that $\ker\tilde{\pi} \subseteq K$. The reverse inclusion is clear. ☐

Theorem 3.6.8. *Let Γ be a non-ascending countable HNN extension with Bass-Serre tree T. Then Γ is C^*-simple if and only if $\mathrm{int}(G \curvearrowright \partial T)$ is C^*-simple.*

Proof. Since $\mathrm{int}(\Gamma \curvearrowright \partial T)$ is a normal subgroup of Γ, "only if" follows from Theorem 3.3.5. Therefore suppose that $I = \mathrm{int}(\Gamma \curvearrowright \partial T)$ is C^*-simple, and define $K = \ker\Gamma$.

We first let $K_\varepsilon(\Gamma)$ and $K_\varepsilon(\Gamma/K)$ denote the K_ε's of the HNN extensions Γ and Γ/K, respectively. Let $\tilde{\pi}$ be the surjective homomorphism of Γ onto $\mathrm{HNN}(G/K, H/K, \tilde{\theta})$ as described in the above lemma. Choosing the left coset representatives $\pi(S_{-1})$ and $\pi(S_1)$ of H/K and $\tilde{\theta}(H/K)$ in G/K, then for $g = g_1\tau^{\varepsilon_1} \cdots g_n\tau^{\varepsilon_n} g_{n+1} \in \Gamma$ written in normal form, we see that $\tilde{\pi}(g)$ has initial letter $1 = K$ and type ε in the normal form in Γ/K, if and only if g has initial letter 1 and type ε. Therefore $\tilde{\pi}$ maps $K_\varepsilon(\Gamma)$ onto $K_\varepsilon(\Gamma/K)$, and since $\ker\tilde{\pi} = K$ it follows that $K_\varepsilon(\Gamma)/K$ is isomorphic to $K_\varepsilon(\Gamma/K)$ for $\varepsilon \in \{\pm 1\}$.

If $K = I$, then $K_\varepsilon(\Gamma) = K$ for $\varepsilon \in \{\pm 1\}$. Hence $K_\varepsilon(\Gamma/K) = \{1\}$ for $\varepsilon \in \{\pm 1\}$, so that the action of Γ/K on $\partial T'$ is topologically free, T' denoting the Bass-Serre tree of Γ/K. We conclude that $\Gamma/I = \Gamma/K$ is C^*-simple, so Γ is C^*-simple by Proposition 3.3.1 and Theorem 3.3.5.

Let C be the centralizer of I in Γ. Suppose first that $C \setminus H$ is non-empty and let $g \in C \setminus H$. For all $s \in \Gamma$ and $\varepsilon \in \{\pm 1\}$, $s^{-1}K_\varepsilon s \subseteq I$, so that $sgs^{-1}K_\varepsilon(sgs^{-1})^{-1} = K_\varepsilon$ for all $\varepsilon \in \{\pm 1\}$ and $s \in \Gamma$. For $\varepsilon \in \{\pm 1\}$, then by Lemma 3.5.5 there exists $s \in \Gamma$ such that $sgs^{-1}r \in \Gamma \setminus T_\varepsilon^\dagger$ whenever $r \in \Gamma \setminus T_{-\varepsilon}^\dagger$. Hence $t \in K_\varepsilon$ implies

$$r^{-1}(sgs^{-1})^{-1}t(sgs^{-1})r = (sgs^{-1}r)^{-1}t(sgs^{-1}r) \in H$$

for all $r \in \Gamma \setminus T^{\dagger}_{-\varepsilon}$ and $(sgs^{-1})^{-1}t(sgs^{-1}) \in K_{-\varepsilon}$. Therefore $t \in (sgs^{-1})K_{-\varepsilon}(sgs^{-1})^{-1} = K_{-\varepsilon}$. This proves that $K_{-1} = K_1$ and $I = K$, so that Γ is C^*-simple due to the discussion above.

If $C \subseteq H$, then normality of C in Γ implies that $C \subseteq \bigcap_{r\in\Gamma} rHr^{-1} = K \subseteq I$. Hence C is the centre of I, so by C^*-simplicity of I it follows that $C \subseteq R(I) = \{1\}$. In particular, C is C^*-simple, so that Γ is C^*-simple by Theorem 3.3.5. ☐

The above result therefore states that C^*-simplicity of an HNN extension imposes some stern demands on the quasi-kernels, also exemplified by the next result.

Proposition 3.6.9. *Let Γ be a non-ascending countable HNN extension, with quasi-kernels K_{-1} and K_1. If either K_{-1} or K_1 is amenable, then Γ is C^*-simple if and only if K_{-1} and K_1 are trivial.*

Proof. Suppose that Γ is C^*-simple and let T be the Bass-Serre tree of Γ. Let $\varepsilon \in \{\pm 1\}$ such that $K_{-\varepsilon}$ is amenable and let $x = \lim_{k\to\infty} \tau^{\varepsilon k} G \in \partial T$. We observe that Γ_x° is the direct limit of the increasing sequence $(K(\tau^{\varepsilon k}))_{k\geq 1}$ of subgroups of Γ, since if $g \in \Gamma$ is the identity on a neighbourhood of x, then g is the identity on $U(\tau^{\varepsilon k})$ for some $k \geq 1$, so that $g \in K(\tau^{\varepsilon k})$. As $K(\tau^{\varepsilon k}) = \tau^{\varepsilon k} K_{-\varepsilon} \tau^{-\varepsilon k}$ is amenable for all $k \geq 1$ and amenability is preserved under direct limits, Γ_x° is amenable.

As Γ is C^*-simple and $\overline{\partial T}$ is a Γ-boundary, $C(\overline{\partial T}) \rtimes_r \Gamma$ is simple by Theorem 3.2.6. Minimality of the Γ-action on $\overline{\partial T}$ implies that $\{x \in \overline{\partial T} \mid \Gamma_x^\circ \text{ is amenable}\}$ is dense in $\overline{\partial T}$, so the action of Γ on $\overline{\partial T}$ is topologically free due to Proposition 3.2.4. Therefore $K_{-1} = K_1 = \{1\}$, and the converse follows from Theorem 3.2.6 and Proposition 3.6.4. ☐

Chapter 4

Twisted C^*-dynamical systems

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Parts of the article have been rewritten to fit the thesis, with auxiliary proofs and explanations. Moreover, Theorems 4.3.15 and 4.4.4 are new.

The relation between boundary actions and reduced group C^*-algebras, as reviewed in the preceding chapter, has inarguably clarified how strong of a property C^*-simplicity really is. In this chapter, we show that C^*-simplicity of a group G is in fact so strong that it can shed light on the ideal structure of a reduced crossed product over G. We can even go as far as to not require that the action yielding the crossed product is genuine, meaning that the map of G into the automorphism group of the C^*-algebra need not be a homomorphism, as long as we can measure the non-genuineness of the action by a unitary-valued 2-cocycle.

The above type of action is known as a *twisted action*, and the greater generality in this construction has some great benefits, many of which have previously been considered with respect to C^*-simple groups by Bédos and Conti [9, 10, 12]. The equivalence of a group being C^*-simple and admitting a free boundary action allows us to generalize many of their results.

4.1 Reduced twisted crossed products

In this section, we generalize the notion of a C^*-dynamical system, the resultant notion being that of a *twisted C^*-dynamical system.*

Recall first that the *multiplier algebra* $M(A)$ of a C^*-algebra A is the set of operators $T \in B(A)$ for which there is a map $T^*\colon A \to A$ such that $T(x)^*y = x^*T^*(y)$ for all $x, y \in A$. With the operator norm, the usual product on $B(A)$ by composition and the involution $T \mapsto T^*$, $M(A)$ is a unital C^*-algebra, and it is the largest unital C^*-algebra containing A as an essential ideal, i.e., any C^*-algebra B containing A as an essential ideal admits a unique embedding into $M(A)$ that maps the inclusion $A \subseteq B$ to the inclusion $A \subseteq M(A)$. We embed A into $M(A)$ as a C^*-subalgebra by identifying $a \in A$ with the left multiplication map $A \to A$ by a, and

$$xa = x(a) \in A, \quad ax = x^*(a^*)^* \in A$$

for $a \in A$ and $x \in M(A)$, whenever x is viewed as a map $A \to A$. Moreover, it is easy to see that A is unital if and only if $M(A) = A$. For more on multiplier algebras, we refer to [125, Section 2.3].

Definition 4.1.1. A *(Busby-Smith) twisted C^*-dynamical system* [28] is a quadruple (A, G, α, σ), where A is a C^*-algebra, G is a discrete group and $\alpha\colon G \to \mathrm{Aut}(A)$ and $\sigma\colon G \times G \to \mathcal{U}(M(A))$ are maps satisfying the identities

$$\begin{gathered}\alpha_g \circ \alpha_h = \mathrm{Ad}(\sigma(g,h)) \circ \alpha_{gh},\\ \sigma(g,h)\sigma(gh,s) = \alpha_g(\sigma(h,s))\sigma(g,hs),\\ \sigma(g,1) = \sigma(1,g) = 1,\end{gathered}$$

for all $g, h, s \in G$. Here $\alpha_g = \alpha(g) \in \mathrm{Aut}(A)$ for all $g \in G$. The tuple (α, σ) is referred to as a *twisted action* of G on A, and σ is called the *normalized 2-cocycle* (or just the *cocycle*) of the twisted action.

If A is unital, we say that (A, G, α, σ) is *unital*. Whenever $A = \mathbb{C}$ (so that α is trivial), we say that $\sigma\colon G \times G \to \mathbb{T}$ is a *multiplier*.

Notice that if σ is *trivial*, i.e., $\sigma(g,h)=1$ for all $g,h\in G$, then α is a group homomorphism and (A,G,α) is a C^*-dynamical system in the usual sense.

Twisted C^*-dynamical systems generalize genuine C^*-dynamical systems in the same way group extensions generalize semidirect products. First, recall that the semidirect product of a group N by another group H is a group G for which there exists a short exact sequence

$$1 \longrightarrow N \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1$$

and a map $k\colon H\to G$ such that $\pi\circ k=\mathrm{id}_H$ that is also a group homomorphism, i.e., the short exact sequence *splits*. A good question is now what it takes for the above exact sequence even to exist, and a twist quickly enters the picture. If G is a group fitting into the sequence, let $k\colon H\to G$ be a *cross-section*, i.e., k satisfies $\pi\circ k=\mathrm{id}_H$, such that $k(1)=1$. For all $s,t\in H$, we observe that $\pi(k(gh))=\pi(k(g)k(h))$ and $k(g)k(h)k(gh)^{-1}\in N$, if we regard N as a normal subgroup of G. Letting $\sigma(g,h)\in N$ such that $k(g)k(h)=\sigma(g,h)k(gh)$, we obtain a map $\sigma\colon H\times H\to N$ that satisfies the *cocycle identity*

$$\sigma(g,h)\sigma(gh,s)=k(g)\sigma(h,s)k(g)^{-1}\sigma(g,hs),\quad g,h,s\in H,$$

since $(k(g)k(h))k(s)=k(g)(k(h)k(s))$. If we define $\alpha\colon H\to \mathrm{Aut}(N)$ by $\alpha_g=\mathrm{Ad}(k(g))$ and assume that $k(1)=1$, then (α,σ) can be regarded as a twisted action of G on N, in the sense that it satisfies the axioms of Definition 4.1.1. We refer to [124] for a very detailed discussion of how twisted actions arise in various guises, both in group theory and operator algebras.

Let (A,G,α,σ) be a twisted C^*-dynamical system; the following exposition on twisted C^*-algebraic crossed products is based on [112, 123, 140]. Just like what we saw for genuine C^*-dynamical systems (Section 3.2), the Banach space $\ell^1(G,A)$ is equipped with the structure of a Banach *-algebra with an approximate identity, if one defines a product and involution by

$$(xy)(s)=\sum_{g\in G}x(g)\alpha_g(y(g^{-1}s))\sigma(g,g^{-1}s),\quad x^*(s)=\sigma(s,s^{-1})\alpha_s(x(s^{-1}))^*.$$

For any $x\in\ell^1(G,A)$, we will often write $x=\sum_{g\in G}x_g\delta_g$, where $x_g=x(g)$ for $g\in G$, and we identify any $a\in A$ with $a\delta_1\in C_c(G,A)$.

A *covariant representation* of (A,G,α,σ) is a triple (π,u,H) where H is a Hilbert space, $\pi\colon A\to B(H)$ is a non-degenerate representation and $u\colon G\to\mathcal{U}(H)$ is a map such that

$$u_g\pi(a)u_g^*=\pi(\alpha_g(a)),\quad u_gu_h=\pi(\sigma(g,h))u_{gh},\quad g,h\in G,\ a\in A,$$

where π also denotes the extension of π to $M(A)$ [140, p. 135]. The associated integrated form of a covariant representation and the full reduced crossed product $A\rtimes_\alpha^\sigma G$ of (A,G,α,σ) are defined in exactly the same way as in Section 3.2.

For the remainder of this chapter, we will only concern ourselves with the reduced crossed product of a twisted C^*-dynamical system, which is defined as follows. Assuming once again that A is faithfully represented on some Hilbert space H, we define a faithful representation $\pi_\alpha\colon A\to B(H\otimes\ell^2(G))$ by

$$\pi_\alpha(a)(\xi\otimes\delta_t)=\alpha_{t^{-1}}(a)\xi\otimes\delta_t,\quad a\in A,\ \xi\in H,\ t\in G.$$

For each $g \in G$, we define a unitary operator $\lambda_\sigma(g) \in \mathcal{U}(H \otimes \ell^2(G))$ by

$$\lambda_\sigma(g)(\xi \otimes \delta_t) = \sigma(t^{-1}g^{-1}, g)\xi \otimes \delta_{gt}, \quad t \in G,\ \xi \in H.$$

Then $(\pi_\alpha, \lambda_\sigma, H \otimes \ell^2(G))$ defines a *regular* covariant representation of (A, G, α, σ). One can show that the integrated form $\pi_\alpha \times \lambda_\sigma \colon \ell^1(G, A) \to B(H \otimes \ell^2(G))$ is faithful [140, Théorème 4.11].

The *reduced twisted crossed product* $A \rtimes^\sigma_{\alpha,r} G$ of (A, G, α, σ) is the completion of $\ell^1(G, A)$ (or the dense *-subalgebra $C_c(G, A)$) in the *reduced norm* $x \mapsto \|(\pi_\alpha \times \lambda_\sigma)(x)\|$. However, it will often be convenient to view $A \rtimes^\sigma_{\alpha,r} G$ as a C^*-subalgebra of $B(H \otimes \ell^2(G))$, which is achieved by (equivalently) defining $A \rtimes^\sigma_{\alpha,r} G$ to be the norm-closure of the image of $\pi_\alpha \times \lambda_\sigma$ in $B(H \otimes \ell^2(G))$. As one would hope, $A \rtimes^\sigma_{\alpha,r} G$ does not depend on the choice of faithful representation $A \subseteq B(H)$ [123, p. 552]. We identify A with its image under π_α. If $A = \mathbb{C}$, the reduced twisted crossed product $\mathbb{C} \rtimes^\sigma_r G$ is the *twisted reduced group C^*-algebra* $C^*_r(G, \sigma)$.

The conjugation map $\alpha'(s) = \mathrm{Ad}(\lambda_\sigma(s)) \in \mathrm{Aut}(B(H \otimes \ell^2(G)))$ maps $A \rtimes^\sigma_{\alpha,r} G$ onto itself for all $s \in G$. Viewing $\sigma(g, h)$ as a unitary multiplier of $A \rtimes^\sigma_{\alpha,r} G$ for $r, s \in G$ by extending π_α to $M(A)$, we see that $(A \rtimes^\sigma_{\alpha,r} G, G, \alpha', \sigma)$ is a twisted C^*-dynamical system such that the inclusion $A \to A \rtimes^\sigma_{\alpha,r} G$ is G-equivariant, i.e., $\alpha'_g|_A = \alpha_g$ for all $g \in G$.

As in the non-twisted case, there exists a faithful conditional expectation E_A of $A \rtimes^\sigma_{\alpha,r} G$ onto A, uniquely satisfying $E_A(x) = x_1$ for all $x \in \ell^1(G, A)$. Equivalently, E_A satisfies $E_A(\lambda_\sigma(g)) = 0$ for all $g \in G \setminus \{1\}$. We refer to E_A as the *canonical conditional expectation* of $A \rtimes^\sigma_{\alpha,r} G$ onto A. As noted in Section 3.2 the existence of this particular conditional expectation can be deduced from the existence of a conditional expectation of $A \rtimes^\sigma_{\alpha,r} G$ onto the reduced crossed product with respect to a subgroup. We have not been able to locate a proof of the twisted case, so we include one for completeness, inspired by [9, Theorem 2.2].

Proposition 4.1.2. *Let (A, G, α, σ) be a twisted C^*-dynamical system and let H be a subgroup of G. Denote by (A, H, α, σ) the twisted C^*-dynamical system obtained by restricting α and σ to H. There exists an injective *-homomorphism $J \colon A \rtimes^\sigma_{\alpha,r} H \to A \rtimes^\sigma_{\alpha,r} G$ such that $J(a\lambda_\sigma(h)) = a\lambda_\sigma(h)$ for all $a \in A$ and $h \in H$ and a faithful conditional expectation $E \colon A \rtimes^\sigma_{\alpha,r} G \to J(A \rtimes^\sigma_{\alpha,r} H)$ satisfying $E(\lambda_\sigma(g)) = 0$ for $g \in G \setminus H$.*

Proof. Since the reduced crossed product $A \rtimes^\sigma_{\alpha,r} G$ is independent of the choice of faithful representation of A, we may identify A with its image under π_α and then repeat the construction of the reduced twisted crossed product, allowing us to assume that A is faithfully represented on some Hilbert space K admitting unitary operators $u_g \in \mathcal{U}(K)$ satisfying $u_g a u_g^* = \alpha_g(a)$ for all $g \in G$ and $a \in A$.

Let Q be a system of representatives of the right cosets of H in G. We may then define unitary operators $U \colon \ell^2(H) \otimes \ell^2(Q) \to \ell^2(G)$ and $V \in \mathcal{U}(K \otimes \ell^2(H) \otimes \ell^2(Q))$ by

$$U(\delta_h \otimes \delta_q) = \delta_{hq}, \quad V(\xi \otimes \delta_h \otimes \delta_q) = \sigma(q^{-1}, h^{-1})^* u_{q^{-1}} \xi \otimes \delta_h \otimes \delta_q$$

for $\xi \in K$, $h \in H$ and $q \in Q$. Define $J \colon B(K \otimes \ell^2(H)) \to B(K \otimes \ell^2(G))$ by

$$J(x) = (1_K \otimes U)V(x \otimes 1_{\ell^2(Q)})V^*(1_K \otimes U^*).$$

Then $J(a\lambda_\sigma(h)) = a\lambda_\sigma(h)$ for $a \in A$ and $h \in H$, so that $J(A \rtimes^\sigma_{\alpha,r} H) \subseteq A \rtimes^\sigma_{\alpha,r} G$. If we now define $E\colon B(K \otimes \ell^2(G)) \to B(K \otimes \ell^2(G))$ by

$$E(x) = \sum_{q\in Q} P_{Hq} x P_{Hq},$$

where P_{Hq} is the projection onto $K \otimes \ell^2(Hq)$ for all $q \in Q$, then E is a contractive, faithful, idempotent linear map. As $E(a\lambda_\sigma(g)) = \chi_H(g)a\lambda_\sigma(g)$ for $a \in A$ and $g \in G$, we have $E(A \rtimes^\sigma_{\alpha,r} G) \subseteq J(A \rtimes^\sigma_{\alpha,r} H)$ since E is contractive. Furthermore, as the image of E is closed, it coincides with the image of J. Hence J and E are the desired maps. $\square$

If (A, G, α, σ) and (B, G, β, τ) are twisted C^*-dynamical systems and $\varphi\colon A \to B$ is a *-homomorphism satisfying

$$\varphi \circ \alpha_g = \beta_g \circ \varphi, \quad \varphi(\sigma(g, h)) = \tau(g, h), \quad g, h \in G,$$

then φ extends to a *-homomorphism $\tilde{\varphi}\colon A \rtimes^\sigma_{\alpha,r} G \to B \rtimes^\tau_{\beta,r} G$ [140, Théorème 4.22] that uniquely satisfies

$$\tilde{\varphi}(a\lambda_\sigma(g)) = \varphi(a)\lambda_\tau(g), \quad a \in A,\ g \in G.$$

Moreover, $\tilde{\varphi}$ is injective (resp. surjective) if and only if φ is injective (resp. surjective).

We will finally mention one of the premier reasons for working with twisted C^*-dynamical systems: their crossed products can be realized as iterated crossed products with respect to a normal subgroup and the corresponding quotient group. We refer to [9, Theorem 2.1] for a proof of the following result, due to Bédos. A similar statement holds for full twisted crossed products (see [112, Theorem 4.1]).

Theorem 4.1.3. *Let (A, G, α, σ) be a twisted C^*-dynamical system, let N be a normal subgroup of G, and let $j\colon G \to Q = G/N$ be the quotient map. Denoting the restriction of the twisted action (α, σ) to N also by (α, σ), then for all $g \in G$ there is an automorphism $\gamma_g \in \mathrm{Aut}(A \rtimes^\sigma_{\alpha,r} N)$ satisfying*

$$\begin{aligned}\gamma_g(a\lambda_\sigma(n)) &= \alpha_g(a)\sigma(g, n)\sigma(gng^{-1}, g)^*\lambda_\sigma(gng^{-1}) \\ &= \alpha_g(a)\lambda_\sigma(g)\lambda_\sigma(n)\lambda_\sigma(g)^*,\end{aligned}$$

for all $n \in N$ and $a \in A$, the latter identity holding in $A \rtimes^\sigma_{\alpha,r} G$ (see Proposition 4.1.2). If $k\colon Q \to G$ is a cross-section for which $k(1) = 1$, and we let

$$\beta = \gamma \circ k\colon Q \to \mathrm{Aut}(A \rtimes^\sigma_{\alpha,r} N),$$

then there exists a normalized 2-cocycle $\nu\colon Q \times Q \to \mathcal{U}(M(A \rtimes^\sigma_{\alpha,r} N))$ such that (β, ν) is a twisted action of Q on $A \rtimes^\sigma_{\alpha,r} N$, and

$$A \rtimes^\sigma_{\alpha,r} G \cong (A \rtimes^\sigma_{\alpha,r} N) \rtimes^\nu_{\beta,r} Q.$$

In particular, the reduced group C^*-algebra of a group extension can be expressed as a reduced twisted crossed product, in the same way the reduced group C^*-algebra of a semidirect product group can be expressed as a reduced crossed product of a genuine C^*-dynamical system.

4.2 Powers' averaging property

In this section we devise a stratagem for the possible boundary actions of a group to reveal information about the reduced twisted crossed products that it admits, and prove that C^*-simplicity of a group forces a powerful approximation property on all reduced twisted crossed products over this group. All twisted C^*-dynamical systems in this section are assumed to be unital.

Let G be a discrete group and let X be a G-boundary. If (A, G, α, σ) is a unital twisted C^*-dynamical system over G, we will frequently consider the twisted C^*-dynamical system $(A \otimes C(X), G, \beta, \tau)$ obtained by defining $\beta\colon G \to \mathrm{Aut}(A \otimes C(X))$ and $\tau\colon G \times G \to \mathcal{U}(A \otimes C(X))$ by

$$\beta_g(a \otimes f) = \alpha_g(a) \otimes (gf), \quad \tau(g,h) = \sigma(g,h) \otimes 1$$

for $g, h \in G$, $a \in A$ and $f \in C(X)$. Identifying $A \rtimes^\sigma_{\alpha,r} G$ with the image of the injective *-homomorphism $A \rtimes^\sigma_{\alpha,r} G \to (A \otimes C(X)) \rtimes^\tau_{\beta,r} G$ induced by the map $a \mapsto a \otimes 1$, the unitaries $\lambda_\sigma(g)$ and $\lambda_\tau(g)$ are identified in $(A \otimes C(X)) \rtimes^\tau_{\beta,r} G$ for all $g \in G$.

Definition 4.2.1. For (A, G, α, σ) as above and a G-boundary X, then we refer to the twisted C^*-dynamical system $(A \otimes C(X), G, \beta, \tau)$ constructed above as a *natural extension* of (A, G, α, σ).

The inclusion of a boundary in a natural extension allows us to conjugate states by inner automorphisms to achieve some degree of multiplicativity in the limit, as conveyed by the following lemma which will be useful throughout this chapter.

Lemma 4.2.2. *Let (A, G, α, σ) be a unital twisted C^*-dynamical system and let X be a G-boundary. The natural extension $(A \otimes C(X), G, \beta, \tau)$ has the following property. For any point $x \in X$, there exists a net (g_i) in G such that for any state ϕ on the reduced crossed product $A \rtimes^\sigma_{\alpha,r} G$, there is a state ψ on the reduced crossed product $(A \otimes C(X)) \rtimes^\tau_{\beta,r} G$ for which*

$$\psi|_{A \rtimes^\sigma_{\alpha,r} G} = \lim_i \phi \circ \mathrm{Ad}(\lambda_\sigma(g_i)) \quad \textit{and} \quad \psi|_{C(X)} = \delta_x,$$

the limit being taken in the weak* topology.

Proof. By Corollary 2.1.17 there exists a net (g_i) in G such that $g_i\mu \to \delta_x$ for all $\mu \in \mathcal{P}(X)$ in the weak*-topology. If ϕ is a state on $A \rtimes^\sigma_{\alpha,r} G$, extend ϕ to a state $\hat{\phi}$ on $(A \otimes C(X)) \rtimes^\tau_{\beta,r} G$ and define $\mu = \hat{\phi}|_{C(X)}$. The twisted action (β, τ) restricted to $C(X)$ is simply the G-action on $C(X)$, so $g_i\mu \to \delta_x$ in the weak*-topology. Compactness of the state space of $(A \otimes C(X)) \rtimes^\tau_{\beta,r} G$ then allows us to assume that $(\hat{\phi} \circ \mathrm{Ad}(\lambda_\tau(g_i)))$ converges to a state ψ on $(A \otimes C(X)) \rtimes^\tau_{\beta,r} G$ in the weak* topology. By construction, $\psi|_{C(X)} = \delta_x$ and

$$\psi|_{A \rtimes^\sigma_{\alpha,r} G} = \lim_i (\hat{\phi} \circ \mathrm{Ad}(\lambda_\tau(g_i)))|_{A \rtimes^\sigma_{\alpha,r} G} = \lim_i \phi \circ \mathrm{Ad}(\lambda_\sigma(g_i)). \qquad \square$$

The following definition can be seen as a generalization of [94, Definition 5.2] (see also B in Remark 3.2.8).

Definition 4.2.3. A unital twisted C^*-dynamical system (A, G, α, σ) is said to have *Powers' averaging property* if for every element b in the reduced twisted crossed product $A \rtimes_{\alpha,r}^{\sigma} G$ satisfying $E(b) = 0$ and every $\epsilon > 0$, there are $g_1, \ldots, g_n \in G$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \lambda_\sigma(g_i) b \lambda_\sigma(g_i)^* \right\| < \epsilon.$$

Here $E\colon A \rtimes_{\alpha,r}^{\sigma} G \to A$ is the canonical conditional expectation of $A \rtimes_{\alpha,r}^{\sigma} G$ onto A.

A key result of Bédos and Conti [12, Theorem 3.8] is that reduced crossed products of twisted C^*-dynamical systems over $(\mathrm{P}_{\mathrm{com}})$ and PH groups, which are subclasses of the class of C^*-simple groups, respectively, satisfy an averaging property that is similar to Definition 4.2.3. Groups with the $(\mathrm{P}_{\mathrm{com}})$ and PH properties were first analyzed in [15] and [122], respectively.

We will now prove that twisted C^*-dynamical systems over C^*-simple groups always satisfy Powers' averaging property. With the initial requirement that we consider only C^*-simple groups, we will first prove some preliminary results on linear functionals of reduced twisted crossed products, by means of natural extensions.

Lemma 4.2.4. *Let (A, G, α, σ) be a unital twisted C^*-dynamical system where G is C^*-simple, and let ϕ be a bounded linear functional on $A \rtimes_{\alpha,r}^{\sigma} G$. If E denotes the canonical conditional expectation of $A \rtimes_{\alpha,r}^{\sigma} G$ onto A, then there exists ψ in the weak* closure of $\{\phi \circ \mathrm{Ad}(\lambda_\sigma(g)) \mid g \in G\}$ such that $\psi = \psi \circ E$.*

Proof. By the Hahn-Jordan decomposition for bounded linear functionals on C^*-algebras, we can write $\phi = c_1\phi_1 - c_2\phi_2 + i(c_3\phi_3 - c_4\phi_4)$ for real numbers $c_1, \ldots, c_4 \geq 0$ and states $\phi_1, \ldots, \phi_4$ on $A \rtimes_{\alpha,r}^{\sigma} G$.

Consider the natural extension $(A \otimes C(\partial_F G), G, \beta, \tau)$ of (A, G, α, σ) and fix $x \in \partial_F G$. By Lemma 4.2.2, we can find states $\psi_1, \psi_2, \psi_3, \psi_4$ on $(A \otimes C(\partial_F G)) \rtimes_{\beta,r}^{\tau} G$ and a net (g_j) in G such that

$$\psi_i|_{A \rtimes_{\alpha,r}^{\tau} G} = \lim_j \phi_i \circ \mathrm{Ad}(\lambda_\sigma(g_j)) \quad \text{and} \quad \psi_i|_{C(\partial_F G)} = \delta_x, \quad i = 1, 2, 3, 4.$$

In particular, $C(\partial_F G) \subseteq \mathrm{mult}(\psi_i)$ for each i. Due to Theorem 3.2.6, the G-action on $\partial_F G$ is free, so for any $g \in G \setminus \{1\}$ there exists $f \in C(\partial_F G)$ such that $f(x) = 1$ and $f(g^{-1}x) = 0$. It follows that for each i and every $a \in A$,

$$\begin{aligned} \psi_i|_{A \rtimes_{\alpha,r}^{\sigma} G}(a\lambda_\sigma(g)) &= \psi_i(a\lambda_\tau(g)) f(x) \\ &= \psi_i((a \otimes 1)\lambda_\tau(g)(1 \otimes f)) \\ &= \psi_i((a \otimes 1)(1 \otimes (gf))\lambda_\tau(g)) \\ &= \psi_i((1 \otimes (gf))(a \otimes 1)\lambda_\tau(g)) \\ &= f(g^{-1}x)\psi_i((a \otimes 1)\lambda_\tau(g)) \\ &= 0. \end{aligned}$$

It follows from continuity that $\psi_i|_{A \rtimes_{\alpha,r}^{\sigma} G} = \psi_i|_{A \rtimes_{\alpha,r}^{\sigma} G} \circ E$ for each i. Hence the restriction of $\psi = c_1\psi_1 - c_2\psi_2 + i(c_3\psi_3 - c_4\psi_4)$ to $A \rtimes_{\alpha,r}^{\sigma} G$ satisfies the desired property. $\square$

Theorem 4.2.5. *If (A, G, α, σ) is a unital twisted C^*-dynamical system where G is C^*-simple, and E denotes the canonical conditional expectation of $A \rtimes_{\alpha,r}^{\sigma} G$ onto A, then*

$$0 \in \overline{\mathrm{conv}}\{\lambda_\sigma(g) x \lambda_\sigma(g)^* \mid g \in G\}$$

for all $x \in A \rtimes_{\alpha,r}^{\sigma} G$ satisfying $E(x) = 0$, the closure being in the norm. In particular, (A, G, α, σ) has Powers' averaging property.

Proof. Fix $x \in A \rtimes_{\alpha,r}^{\sigma} G$ satisfying $E(x) = 0$, and suppose for the sake of contradiction that the claim does not hold for this x. Let

$$K = \overline{\mathrm{conv}}\{\lambda_\sigma(g) x \lambda_\sigma(g)^* \mid g \in G\}.$$

By the Hahn-Banach theorem, there exists a bounded linear functional ϕ on $A \rtimes_{\alpha,r}^{\sigma} G$ such that $\inf_{y \in K} \mathrm{Re}(\phi(y)) > 0$. In particular, this implies that

$$\inf_{g \in G} |\phi \circ \mathrm{Ad}(\lambda_\sigma(g))(x)| > 0.$$

By Lemma 4.2.4, there exists a state ψ in the weak* closure of $\{\phi \circ \mathrm{Ad}(\lambda_\sigma(g)) \mid g \in G\}$ such that $\psi = \psi \circ E$, but then ψ satisfies $|\psi(E(x))| = |\psi(x)| > 0$, a contradiction. □

4.3 The ideal structure of reduced crossed products

In this section, we consider the ideal structure of reduced crossed products of unital twisted C^*-dynamical systems, with particular focus on when a reduced twisted crossed product is simple. In general, it is not possible to relate the ideal structure of a reduced crossed product to the ideal structure of its underlying C^*-algebra, even when the group is C^*-simple. Indeed, de la Harpe and Skandalis [44] constructed examples of C^*-dynamical systems over Powers groups (which are C^*-simple) with the property that the reduced crossed product has many non-trivial ideals, but the underlying C^*-algebra has only a single non-trivial invariant ideal. These examples will be elaborated upon in Remark 4.3.7, as well as in Section 5.4.

Let (A, G, α, σ) be a twisted C^*-dynamical system, and let I be a G-invariant ideal of A. Then (α, σ) restricts to a twisted action of G on I, and the image of the injective *-homomorphism $I \rtimes_{\alpha,r}^{\sigma} G \to A \rtimes_{\alpha,r}^{\sigma} G$ is the ideal in $A \rtimes_{\alpha,r}^{\sigma} G$ generated by I. If $\pi \colon A \to A/I$ denotes the quotient map, the twisted action (α, σ) also induces a twisted action $(\dot{\alpha}, \dot{\sigma})$ of G on A/I such that $\dot{\alpha}_g \circ \pi = \pi \circ \alpha_g$ for all $g \in G$ and $\dot{\sigma} = \pi \circ \sigma$. Therefore π induces a surjective *-homomorphism $\tilde{\pi} \colon A \rtimes_{\alpha,r}^{\sigma} G \to A/I \rtimes_{\dot{\alpha},r}^{\dot{\sigma}} G$ at the level of crossed products, giving rise to the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \rtimes_{\alpha,r}^{\sigma} G \arrow[r] \arrow[d, "E_I"] & A \rtimes_{\alpha,r}^{\sigma} G \arrow[r, "\tilde{\pi}"] \arrow[d, "E_A"] & A/I \rtimes_{\dot{\alpha},r}^{\dot{\sigma}} G \arrow[r] \arrow[d, "E_{A/I}"] & 0 \\ 0 \arrow[r] & I \arrow[r] & A \arrow[r, "\pi"] & A/I \arrow[r] & 0 \end{tikzcd} \tag{4.3.1}$$

Here E_I, E_A and $E_{A/I}$ denote the canonical conditional expectations. In the following, we will use the notation

$$I \bar{\rtimes}_{\alpha,r}^{\sigma} G = \ker \tilde{\pi}.$$

Now observe that the upper sequence in (4.3.1) is exact precisely when $I \rtimes^\sigma_{\alpha,r} G = I \bar{\rtimes}^\sigma_{\alpha,r} G$. It is clear that the inclusion $I \rtimes^\sigma_{\alpha,r} G \subset I \bar{\rtimes}^\sigma_{\alpha,r} G$ always holds, but equality does not necessarily hold in general (see, e.g., [131, Remark 1.17]). If equality does hold for every G-invariant ideal I in A, then the C^*-dynamical system (A, G, α, σ) is said to be *exact*. If G is an exact group (i.e., the reduced group C^*-algebra $C^*_r(G)$ is exact), then a result of Exel [53] implies that every twisted C^*-dynamical system over G is exact; this generalizes a deep theorem, due to Kirchberg and Wassermann, that this holds for genuine C^*-dynamical systems [96].

Bédos and Conti showed that for exact unital twisted C^*-dynamical systems over the aforementioned ($\mathrm{P_{com}}$) and PH groups, there is a bijective correspondence between maximal ideals of the reduced crossed product and maximal invariant ideals of the underlying C^*-algebra [12]. We will show that this bijective correspondence between maximal ideals in fact holds for all unital twisted C^*-dynamical systems over C^*-simple groups. In particular, we will not require the system to be exact.

The first lemma of this section generalizes [23, Lemma 7.2].

Lemma 4.3.1. *Let (A, G, α, σ) be a unital twisted C^*-dynamical system, let X be a G-boundary and let $(A \otimes C(X), G, \beta, \tau)$ denote the associated natural extension. Let I be a proper ideal in $A \rtimes^\sigma_{\alpha,r} G$ and let J denote the ideal in $(A \otimes C(X)) \rtimes^\tau_{\beta,r} G$ generated by I. Then J is proper.*

Proof. Let ϕ be a state on $A \rtimes^\sigma_{\alpha,r} G$ such that $\phi(I) = 0$. By Lemma 4.2.2 there is a state ψ on $(A \otimes C(X)) \rtimes^\tau_{\beta,r} G$, a net (g_i) in G and $x \in X$ such that $\psi|_{A \rtimes^\sigma_{\alpha,r} G} = \lim_j \phi \circ \mathrm{Ad}(\lambda_\sigma(g_j))$ and $\psi|_{C(X)} = \delta_x$.

Note that $\psi|_{A \rtimes^\tau_{\alpha,r} G}(I) = 0$ and that $C(X)$ is contained in the multiplicative domain of ψ. Hence for $b \in I$, $a_1, a_2 \in A$, $f_1, f_2 \in C(X)$ and $s_1, s_2 \in G$ we have

$$
\begin{aligned}
&\psi((a_1 \otimes f_1)\lambda_\tau(s_1) b (a_2 \otimes f_2)\lambda_\tau(s_2)) \\
&= f_1(x)\psi(a_1 \lambda_\sigma(s_1) b a_2 \lambda_\sigma(s_2)) f_2(s_2 x) \\
&= 0.
\end{aligned}
$$

It follows that $\psi(J) = 0$. Hence J is proper. $\square$

The next lemma is a special case of [5, Theorem 1], also proved for transformation groups in Proposition 3.2.3, but we state it for twisted C^*-dynamical systems. We will need it in this form in the next chapter as well.

Lemma 4.3.2. *Let (A, G, α, σ) be a unital twisted C^*-dynamical system and let X be the maximal ideal space of the centre $Z(A)$ of A. Assume that the action of G on X is free. If J is a closed ideal in $A \rtimes^\sigma_{\alpha,r} G$, then for $J_A = J \cap A$ we have*

$$J_A \rtimes^\sigma_{\alpha,r} G \subseteq J \subseteq J_A \bar{\rtimes}^\sigma_{\alpha,r} G.$$

Observe that the above formulation makes sense also in the twisted case, since the twisted action of G on A restricts to a genuine action of G on the centre of A.

Proof. Let $I_A = I \cap A$ and let $\pi \colon A \to A/I_A$ be the quotient map. Now let $\gamma \colon A/I_A \to B(H)$ be an irreducible representation of A/I_A and consider the representation

$$A + I \to (A + I)/I \cong A/I_A \xrightarrow{\gamma} B(H).$$

By Arveson's extension theorem, this map extends to a u.c.p. map $\varphi\colon A \rtimes^{\sigma}_{\alpha,r} G \to B(H)$ such that $\varphi(I) = 0$ and $A \subseteq \mathrm{mult}(\varphi)$, since $\varphi|_A = \gamma \circ \pi$. By irreducibility, the restriction of φ to $Z(A) \cong C(X)$ is a point mass on X, i.e., $\varphi|_{Z(A)} = \delta_x$ for some $x \in X$. Letting $g \in G \setminus \{1\}$, then there exists $f \in C(X)$ such that $f(g^{-1}x) \neq f(x)$, implying

$$\varphi(\lambda_\sigma(g))(f(x)1_H) = \varphi(\lambda_\sigma(g)f) = \varphi(gf\lambda_\sigma(g)) = f(g^{-1}x)\varphi(\lambda_\sigma(g)).$$

Therefore $\varphi(\lambda_\sigma(g)) = 0$. Letting $E_A\colon A \rtimes^{\sigma}_{\alpha,r} G \to A$ be the canonical conditional expectation, it follows that $\varphi = \varphi \circ E_A$. Hence

$$\gamma(\pi(E_A(I))) = \varphi(E_A(I)) = \varphi(I) = \{0\}.$$

Since γ was arbitrary, $\pi(E_A(I)) = \{0\}$, so that $E_A(I) \subseteq I$.

For any positive element $x \in I$, let y be the image of x under $\tilde{\pi}\colon A \rtimes^{\sigma}_{\alpha,r} G \to (A/I_A) \rtimes^{\dot\sigma}_{\dot\alpha,r} G$, and let $E_{A/I}\colon (A/I_A) \rtimes^{\dot\sigma}_{\dot\alpha,r} G \to A/I_A$ be the canonical faithful conditional expectation. Since $E_{A/I} \circ \tilde{\pi} = \pi \circ E_A$, it follows that $E_{A/I}(y) = 0$ since $E_A(x) \in I \cap A$. As $E_{A/I}$ is faithful, $y = 0$ and $x \in I_A \bar{\rtimes}^{\sigma}_{\alpha,r} G$. ☐

For any twisted C^*-dynamical system (A, G, α, σ) and any G-invariant, closed ideal I in $A \bar{\rtimes}^{\sigma}_{\alpha,r} G$, the commutative diagram (4.3.1) yields the identity

$$(I \bar{\rtimes}^{\sigma}_{\alpha,r} G) \cap A = I. \tag{4.3.2}$$

Moreover, let X be a G-boundary. If A is unital, then for the natural extension $(A \otimes C(X), G, \beta, \tau)$ we observe that if $K \subseteq (A \otimes C(X)) \rtimes^{\tau}_{\beta,r} G$ is a closed ideal and $K_A = K \cap (A \otimes C(X))$, then there is a commutative diagram of *-homomorphisms

$$\begin{array}{ccc}
A \rtimes^{\sigma}_{\alpha,r} G & \longrightarrow & (A \otimes C(X)) \rtimes^{\tau}_{\beta,r} G \\
\downarrow & & \downarrow \\
A/(K \cap A) \rtimes^{\dot\sigma}_{\dot\alpha,r} G & \longrightarrow & (A \otimes C(X))/K_A \rtimes^{\dot\tau}_{\dot\beta,r} G
\end{array}$$

where the horizontal arrows are injective. It follows that

$$(K_A \bar{\rtimes}^{\tau}_{\beta,r} G) \cap (A \rtimes^{\sigma}_{\alpha,r} G) = (K \cap A) \bar{\rtimes}^{\sigma}_{\alpha,r} G. \tag{4.3.3}$$

Definition 4.3.3. Let (A, G, α, σ) be a twisted C^*-dynamical system. A closed ideal I of A is said to be *maximal G-invariant* if it is a proper, G-invariant ideal, which is maximal among proper G-invariant ideals in A.

We say that A is *G-simple* if the only maximal G-invariant closed ideal in A is $\{0\}$, or equivalently, that the only G-invariant closed ideals of A are $\{0\}$ and A.

Note for (A, G, α, σ) as above that a maximal G-invariant ideal of A need *not* be a maximal ideal; for instance, for non-trivial G one may consider $A = C(X)$ where X is a minimal compact G-space with more than one point.

Theorem 4.3.4. *Let (A, G, α, σ) be a unital twisted C^*-dynamical system where G is C^*-simple. For a maximal ideal I of $A \rtimes^{\sigma}_{\alpha,r} G$, $I \cap A$ is a maximal G-invariant ideal of A. Conversely, for a maximal G-invariant ideal Y of A, the ideal $Y \bar{\rtimes}^{\sigma}_{\alpha,r} G$ of $A \rtimes^{\sigma}_{\alpha,r} G$ is maximal. Moreover, this correspondence is bijective.*

Proof. Let Y be a maximal G-invariant ideal in A. We claim that the ideal $Y \bar{\rtimes}^{\sigma}_{\alpha,r} G$ in $A \rtimes^{\sigma}_{\alpha,r} G$ is maximal; suppose that J is a proper ideal in $A \rtimes^{\sigma}_{\alpha,r} G$ such that $Y \bar{\rtimes}^{\sigma}_{\alpha,r} G \subseteq J$.

Consider the natural extension $(A \otimes C(\partial_F G), G, \beta, \tau)$ of (A, G, α, σ). Let K denote the ideal in $(A \otimes C(\partial_F G)) \rtimes^{\tau}_{\beta,r} G$ generated by J. By Lemma 4.3.2, $K \subseteq K_A \bar{\rtimes}^{\tau}_{\beta,r} G$, where $K_A = K \cap (A \otimes C(\partial_F G))$. By (4.3.3),

$$J \subseteq K \cap (A \rtimes^{\sigma}_{\alpha,r} G) \subseteq (K_A \bar{\rtimes}^{\tau}_{\beta,r} G) \cap (A \rtimes^{\sigma}_{\alpha,r} G) = (K \cap A) \bar{\rtimes}^{\sigma}_{\alpha,r} G,$$

and applying (4.3.2) to Y and $K \cap A$ gives

$$Y \subseteq J \cap A \subseteq K \cap A.$$

Since J is proper, Lemma 4.3.1 implies that K is proper, so the maximality of Y implies that $Y = K \cap A$ since $K \cap A$ is G-invariant. From above, $J \subseteq Y \bar{\rtimes}^{\sigma}_{\alpha,r} G$, and it follows that $Y \bar{\rtimes}^{\sigma}_{\alpha,r} G$ is maximal.

Now let I be a maximal ideal in $A \rtimes^{\sigma}_{\alpha,r} G$. We must show that the ideal $I \cap A$ is maximal among proper G-invariant ideals in A.

Let J denote the ideal in $(A \otimes C(\partial_F G)) \rtimes^{\tau}_{\beta,r} G$ generated by I. By Lemma 4.3.2, $J \subset J_A \bar{\rtimes}^{\tau}_{\beta,r} G$, where $J_A = J \cap (A \otimes C(\partial_F G))$. Hence by (4.3.3)

$$I \subset J \cap (A \rtimes^{\sigma}_{\alpha,r} G) \subset (J_A \bar{\rtimes}^{\tau}_{\beta,r} G) \cap (A \rtimes^{\sigma}_{\alpha,r} G) = (J \cap A) \bar{\rtimes}^{\sigma}_{\alpha,r} G.$$

Since I is proper, Lemma 4.3.1 implies that $J \cap A$ is proper in A, so the maximality of I implies that $I = (J \cap A) \bar{\rtimes}^{\sigma}_{\alpha,r} G$. Hence $I \cap A = J \cap A$ by (4.3.2), and it follows that

$$I = (I \cap A) \bar{\rtimes}^{\sigma}_{\alpha,r} G. \tag{4.3.4}$$

Now suppose that Z is a proper G-invariant ideal in A such that $I \cap A \subset Z$. Then $Z \bar{\rtimes}^{\sigma}_{\alpha,r} G$ is a proper ideal in $A \rtimes^{\sigma}_{\alpha,r} G$ and

$$I = (I \cap A) \bar{\rtimes}^{\sigma}_{\alpha,r} G \subset Z \bar{\rtimes}^{\sigma}_{\alpha,r} G,$$

so the maximality of I implies that $I = Z \bar{\rtimes}^{\sigma}_{\alpha,r} G$. Hence

$$I \cap A = (Z \bar{\rtimes}^{\sigma}_{\alpha,r} G) \cap A = Z,$$

and it follows that $I \cap A$ is maximal. Finally it follows from the identities (4.3.2) and (4.3.4) that the correspondence is bijective. □

As a corollary we obtain the following generalization of [23, Theorem 7.1] (see also Remark 3.2.8, A).

Corollary 4.3.5. *Let (A, G, α, σ) be a unital twisted C^*-dynamical system where G is C^*-simple. Then $A \rtimes^{\sigma}_{\alpha,r} G$ is simple if and only if A is G-simple.*

Corollary 4.3.6. *If G is C^*-simple, then the reduced twisted group C^*-algebra $C^*_r(G, \sigma)$ is simple for every multiplier $\sigma\colon G \times G \to \mathbb{T}$.*

Remark 4.3.7. A small modification of an example constructed by de la Harpe and Skandalis [44] shows that Theorem 4.3.4 does not generalize to prime ideals. This particular counterexample is immensely useful, and we will return to it in Chapter 5.

Let $G \neq \{1\}$ be a C^*-simple discrete group. We say that an ideal $I \subseteq A$ in a G-C^*-algebra is *G-prime* if it is G-invariant and $J \cap J' \subseteq I$ for G-invariant ideals J and J' implies $J \subseteq I$ or $J' \subseteq I$. If H is a non-trivial amenable subgroup of G, let $X = G/H$ be the left coset space. Then X is an infinite discrete space, since a group is amenable if it contains an amenable finite-index subgroup. We consider the unitization A of the non-unital commutative C^*-algebra $c_0(X)$. Then A becomes a unital G-C^*-algebra with exactly two proper G-invariant ideals, namely $\{0\}$ and $c_0(X)$, both of which are G-prime.

By Green's imprimitivity theorem, the ideal $J = c_0(X) \rtimes_r G$ of $A \rtimes_r G$ is Morita equivalent to $C^*_r(H) \cong C^*(H)$ (see, e.g., [47, Theorem 6.4]), so the prime ideals of J are in one-to-one correspondence with the prime ideals of $C^*(H)$. If H is cyclic, then $C^*(H)$ has at least two prime ideals. Since $A = \mathbb{C}1 + c_0(X)$, it is easy to see that J is maximal in $A \rtimes_r G$ and therefore prime. Furthermore, it is easily checked that every prime ideal of J is a prime ideal of $A \rtimes_r G$. We may then ensure that $A \rtimes_r G$ has at least three prime ideals if H is cyclic.

Notice also that the conclusion of Theorem 4.3.4 is not true if we allow the underlying C^*-algebra to be non-unital. Indeed, $c_0(X)$ is always G-simple, even though $c_0(X) \rtimes_r G$ may contain many ideals.

By appealing to the previously mentioned structure theorem for twisted reduced crossed products (Theorem 4.1.3), it is possible to say something about twisted C^*-dynamical systems whenever the underlying group has a C^*-simple quotient. This also gives another proof that C^*-simplicity is stable under extensions (see Section 3.3).

Corollary 4.3.8. *Let (A, G, α, σ) be a unital twisted C^*-dynamical system and let N be a normal subgroup of G. Continue to write (α, σ) for the restriction of (α, σ) to N. If G/N is C^*-simple, then $A \rtimes^\sigma_{\alpha,r} G$ is simple whenever $A \rtimes^\sigma_{\alpha,r} N$ is simple.*

Proof. By Theorem 4.1.3 there exists a twisted action (γ, ν) of G/N on $A \rtimes^\sigma_{\alpha,r} N$ such that

$$A \rtimes^\sigma_{\alpha,r} G \cong (A \rtimes^\sigma_{\alpha,r} N) \rtimes^\nu_{\gamma,r} (G/N).$$

The desired conclusion now follows from Corollary 4.3.5. □

We now give a generalization of [12, Proposition 3.13] of Bédos and Conti, to describe the situation where the action α is trivial; the proof is just the same as theirs.

Proposition 4.3.9. *Let (A, G, α, σ) be an exact, unital twisted C^*-dynamical system, where α is trivial (that is, $\alpha_g = \mathrm{id}_A$ for all $g \in G$). If (A, G, α, σ) has Powers' averaging property, then there is a bijective correspondence between the set of closed ideals in A and the set of closed ideals in $A \rtimes^\sigma_r G$ given by $Y \mapsto Y \rtimes^\sigma_r G = Y \bar{\rtimes}^\sigma_r G$.*

Proof. We already know that the map $Y \mapsto Y \rtimes^\sigma_r G$ is injective due to (4.3.2), so let I be a closed ideal in $A \rtimes^\sigma_r G$. For any $x \in I$ and $\varepsilon > 0$ there exist $g_1, \ldots, g_n \in G$ such that

$$\left\| \frac{1}{n} \sum_{i=1}^{n} \lambda_\sigma(g_i)(x - E(x))\lambda_\sigma(g_i)^* \right\| < \varepsilon,$$

where $E\colon A \rtimes_r^\sigma G \to A$ is the canonical conditional expectation.

Since α is trivial, $\frac{1}{n}\sum_{i=1}^n \lambda_\sigma(g_i)E(x)\lambda_\sigma(g_i)^* = E(x)$, so that $\|E(x)-y\| < \varepsilon$ for some $y \in I$. Thus $E(x) \in \overline{I} = I$, meaning that $E(I) \subseteq I$. As in the proof of Lemma 4.3.2, it follows that $I \subseteq (I \cap A)\bar{\rtimes}_r^\sigma G$. By exactness of (A, G, α, σ) we have $I = (I \cap A) \rtimes_r^\sigma G$. □

The above result has a very neat application if we restrict our attention to extensions of abelian groups.

Corollary 4.3.10. *Let (A, G, α, σ) be an exact, unital twisted C^*-dynamical system and let Z be the centre of G. If α is trivial, G/Z is C^*-simple and $\sigma(g, z) = \sigma(z, g)$ for all $z \in Z$ and $g \in G$, then there is a bijective correspondence between closed ideals of $A \rtimes_r^\sigma Z$ and closed ideals of $A \rtimes_r^\sigma G$.*

Indeed, under the above hypotheses, the structure theorem ensures that we can define a twisted action (β, ν) of G/Z on $A \rtimes_r^\sigma Z$ such that

$$A \rtimes_r^\sigma G \cong (A \rtimes_r^\sigma Z) \rtimes_{\beta,r}^\nu (G/Z),$$

and $\beta_g = \mathrm{id}_{A \rtimes_r^\sigma Z}$ for all $g \in G/Z$. Since $(A \rtimes_r^\sigma Z, G/Z, \beta, \nu)$ has Powers' averaging property by Theorem 4.2.5, Proposition 4.3.9 applies.

In particular, to count the ideals of certain group C^*-algebras, one may "disregard the C^*-simple part" of the group in question.

Corollary 4.3.11. *Let G be a discrete group with centre Z such that G/Z is exact and C^*-simple. Then the ideals of $C_r^*(G)$ are in one-to-one correspondence with the open (or closed) subsets of the dual group $\hat{Z}$ of Z.*

In the next example, we will consider some applications of the above results, adapted from [12, Example 4.4].

Example 4.3.12. Let $n \geq 2$, let $G_n = \mathrm{GL}(n, \mathbb{Z}) \subseteq \mathrm{GL}(n, \mathbb{R})$ be the group of integer-valued matrices with determinant ± 1. The centre of $\mathrm{GL}(n, \mathbb{R})$ is the subgroup of real-valued non-zero scalar matrices, and $Z_n = \{\pm I\}$ is the center of G_n.

If n is odd, then $\mathrm{PSL}(n, \mathbb{Z}) \cong \mathrm{SL}(n, \mathbb{Z})$ is C^*-simple (Example 3.4.3) and exact (by a theorem of Guentner, Higson and Weinberger [67]). Since G_n decomposes as the product $\mathrm{SL}(n, \mathbb{Z})Z_n$ and $C_r^*(Z_n) \cong \mathbb{C} \oplus \mathbb{C}$, it follows from Corollary 4.3.11 that $C_r^*(G_n)$ has two non-trivial closed ideals.

If n is even, then $\mathrm{PSL}(n, \mathbb{Z})$ is C^*-simple still. Furthermore, G_n/Z_n (also known as the *projective general linear group* $\mathrm{PGL}(n, \mathbb{Z})$) is C^*-simple by Theorem 3.3.5, since it contains $\mathrm{PSL}(n, \mathbb{Z})$ as an index two (hence normal) subgroup with trivial centralizer. Moreover, it is exact, since G_n is exact [67], Z_n is amenable, and quotients of exact groups by amenable normal subgroups are exact. Therefore $C_r^*(G_n)$ also has two non-trivial closed ideals in this case. An similar argument shows that $C_r^*(\mathrm{SL}(n, \mathbb{Z}))$ also has two non-trivial closed ideals.

Example 4.3.13. For $n \geq 3$, the braid group B_n with n generators has centre $Z_n \cong \mathbb{Z}$, and Bédos proved in [9, p. 536] that B_n/Z_n is C^*-simple (see also Example 3.4.6). Further, we have short exact sequences

$$1 \to \mathbb{F}_{n-1} \to P_n/Z_n \to P_{n-1}/Z_{n-1} \to 1,$$
$$1 \to P_n/Z_n \to B_n/Z_n \to S_n \to 1$$

where $\mathbb{F}_{n-1}$ is the free non-abelian group of $n-1$ generators, P_n is the pure braid group on n strands and S_n is the finite symmetric group on n generators [60, Proposition 6]. We recall that free groups and finite groups are exact, and that extensions of exact groups are exact [25, Proposition 5.1.11]. As $P_2 = Z_2$, we may thus conclude by induction that P_n/Z_n is exact, so that B_n/Z_n is exact. By Corollary 4.3.11, the ideals of $C_r^*(B_n)$ are in one-to-one correspondence with the open (or closed) subsets of $\mathbb{T}$.

Theorem 4.3.4 has other interesting applications. As mentioned in Section 3.3, Bédos proved in 1990 that a discrete group is C^*-simple if it contains a normal, C^*-simple subgroup with trivial centralizer. This stability result was a consequence of the following theorem [9, Theorem 3.5].

Theorem 4.3.14. *Let (A, G, α, σ) be a unital twisted C^*-dynamical system, and assume that A admits a faithful G-invariant state. If N is a normal icc subgroup of G with trivial centralizer and $A \rtimes^\sigma_{\alpha,r} N$ is simple, then $A \rtimes^\sigma_{\alpha,r} G$ is simple.*

Let us quickly sketch Bédos' proof. By first observing that conjugation by $g \in G \setminus N$ defines an outer isomorphism of N due to N having trivial centralizer, Bédos proved that any automorphism σ of N is outer if and only if $\{\sigma(n)pn^{-1} \mid n \in N\}$ is infinite, due to N being icc. This latter condition forces the automorphism $\gamma_g \in \mathrm{Aut}(A \rtimes^\sigma_{\alpha,r} N)$ defined in Theorem 4.1.3 to be outer whenever $g \in G \setminus N$ *if* A has a faithful G-invariant state. Therefore the twisted action of G/N on $A \rtimes^\sigma_{\alpha,r} N$ is by outer automorphisms, and an adaptation of a simplicity criterion of Kishimoto to the twisted case (cf. [97, Theorem 3.1]) then yielded the above theorem.

With this in mind, we bring this section to a close by proving a partial generalization of Theorem 3.3.5.

Theorem 4.3.15. *Let (A, G, α, σ) be a unital twisted C^*-dynamical system, and assume that A admits a faithful G-invariant state. If N is a normal icc subgroup of G with C^*-simple centralizer and $A \rtimes^\sigma_{\alpha,r} N$ is simple, then $A \rtimes^\sigma_{\alpha,r} G$ is simple.*

Proof. Let $C_G(N)$ be the centralizer of N in G and let $H = NC_G(N)$. Then H is a normal subgroup of G, and it is isomorphic to $N \times C_G(N)$ since $N \cap C_G(N) = \{1\}$ by N being icc. Since $C_G(N)$ is C^*-simple, it is also icc, so H is icc, and

$$C_G(H) = C_G(N) \cap C_G(C_G(N)) = \{1\}.$$

As $H/N \cong C_G(N)$ is C^*-simple and $A \rtimes^\sigma_{\alpha,r} N$ is simple, $A \rtimes^\sigma_{\alpha,r} H$ is simple by Corollary 4.3.8. Applying Theorem 4.3.14 to H, it follows that $A \rtimes^\sigma_{\alpha,r} G$ is simple. ☐

4.4 Tracial states on reduced twisted crossed products

In this section we will relate tracial states on reduced crossed products of twisted C^*-dynamical systems to tracial states on the underlying C^*-algebra, inspired by the characterization of the unique trace property due to Breuillard, Kalantar, Kennedy and Ozawa (Theorem 3.2.9).

Lemma 4.4.1. *Let (A, G, α, σ) be a twisted C^*-dynamical system and let E denote the canonical conditional expectation of $A \rtimes^\sigma_{\alpha,r} G$ onto A. If τ is a G-invariant tracial state on A, then $\tau \circ E$ is a tracial state on $A \rtimes^\sigma_{\alpha,r} G$.*

Proof. For $x, y \in \ell^1(G, A)$ we have

$$\begin{aligned}
\tau((xy)(1)) &= \sum_{g\in G} \tau(x(g)\alpha_g(y(g^{-1}))\sigma(g, g^{-1})) \\
&= \sum_{g\in G} (\tau \circ \alpha_{g^{-1}})(x(g)\alpha_g(y(g^{-1}))\sigma(g, g^{-1})) \\
&= \sum_{g\in G} \tau(\alpha_{g^{-1}}(x(g)) \left[\sigma(g^{-1}, g)y(g^{-1})\sigma(g^{-1}, g)^*\right] \alpha_{g^{-1}}(\sigma(g, g^{-1}))) \\
&= \sum_{g\in G} \tau(y(g^{-1})\alpha_{g^{-1}}(x(g))\sigma(g^{-1}, g)) \\
&= \sum_{g\in G} \tau(y(g)\alpha_g(x(g^{-1}))\sigma(g, g^{-1})) \\
&= \tau((yx)(1)),
\end{aligned}$$

since $\sigma(g^{-1}, g) = \alpha_{g^{-1}}(\sigma(g, g^{-1}))$ for all $g \in G$. □

The above lemma gives a means of translating tracial states on a C^*-algebra to tracial states on the reduced crossed product. We now add the missing ingredient in the proof of Theorem 3.2.9 – the theorem of Furman (Theorem 2.4.12) – to a generalization for reduced twisted crossed products.

Theorem 4.4.2. *Let (A, G, α, σ) be a unital twisted C^*-dynamical system over a discrete group G with amenable radical $R(G)$. For every tracial state τ on the reduced crossed product $A \rtimes^\sigma_{\alpha,r} G$, $\tau = \tau \circ E_{R(G)}$, where $E_{R(G)}$ denotes the canonical conditional expectation from $A \rtimes^\sigma_{\alpha,r} G$ to $A \rtimes^\sigma_{\alpha,r} R(G)$.*

Proof. We must show that for any tracial state τ on $A \rtimes^\sigma_{\alpha,r} G$ and any $a \in A$, we have $\tau(a\lambda_\sigma(g)) = 0$ for all $a \in A$ and $g \notin R(G)$. We consider the natural extension $(A \otimes C(\partial_F G), G, \beta, \rho)$. By Lemma 4.2.2 and the G-invariance of τ, there is a state ψ on $(A \otimes C(\partial_F G)) \rtimes^\rho_{\beta,r} G$, a net (g_i) in G and $x \in \partial_F G$ such that

$$\psi|_{A\rtimes^\sigma_{\alpha,r}G} = \lim_j \tau \circ \mathrm{Ad}(\lambda_\sigma(g_j)) = \tau$$

and $\psi|_{C(\partial_F G)} = \delta_x$.

For $g \notin R(G)$, then by Theorem 2.4.12 there exists $y \in \partial_F G$ such that $g^{-1}y \neq y$. By minimality of the G-action on $\partial_F G$, there is a net (h_i) in G such that $h_i x \to y$. Due to weak*-compactness, we may assume that $(\psi \circ \mathrm{Ad}\lambda_\rho(h_i))$ converges to a state η on $(A \otimes C(\partial_F G)) \rtimes^\rho_{\beta,r} G$ which, by the G-invariance of τ, then satisfies

$$\eta|_{A\rtimes^\sigma_{\alpha,r}G} = \tau, \quad \eta|_{C(\partial_F G)} = \delta_y.$$

As $C(\partial_F G)$ is contained in the multiplicative domain of η, taking a function $f \in C(\partial_F G)$ such that $f(g^{-1}y) = 0$ and $f(y) = 1$ now yields $\tau(a\lambda_\sigma(g)) = 0$ for any $a \in A$, just as in the proof of Lemma 4.2.4. □

The next result was previously shown in [12, Corollary 3.9] for the case when G is a $(\mathrm{P}_{\mathrm{com}})$ or PH group.

Corollary 4.4.3. *Let (A, G, α, σ) be a twisted C^*-dynamical system over a discrete group G. Suppose that G is C^*-simple or, more generally, that the amenable radical of G is trivial. For every G-invariant tracial state τ on A, $\tau \circ E$ is a tracial state on the reduced twisted crossed product $A \rtimes_{\alpha,r}^{\sigma} G$, where E denotes the canonical conditional expectation of $A \rtimes_{\alpha,r}^{\sigma} G$ onto A. Conversely, every tracial state on $A \rtimes_{\alpha,r}^{\sigma} G$ arises in this way from a G-invariant tracial state on A. Moreover, this correspondence is bijective. Thus $A \rtimes_{\alpha,r}^{\sigma} G$ has a unique tracial state if and only if A has a unique G-invariant tracial state.*

Proof. By Theorem 4.4.2, it follows that $\tau = \tau \circ E$ for any tracial state τ on $A \rtimes_{\alpha,r}^{\sigma} G$. Any tracial state τ on $A \rtimes_{\alpha,r}^{\sigma} G$ is therefore uniquely determined by its restriction to A. The claim now follows. $\square$

We finally state an analogue of Theorem 4.3.15 for the unique trace property. In 1993, Bédos proved [11, Proposition 15] that if the unital twisted C^*-dynamical system (A, G, α, σ) and the normal subgroup N are as in Theorem 4.3.14 but one assumes that the C^*-algebra A has a faithful G-invariant *tracial state*, then $A \rtimes_{\alpha,r}^{\sigma} G$ has a unique tracial state whenever $A \rtimes_{\alpha,r}^{\sigma} N$ has a unique tracial state. If we replace the reference to Corollary 4.3.8 by one to the above corollary, the proof of Theorem 4.3.15 applies almost word for word to yield the following partial generalization of Proposition 3.3.2.

Theorem 4.4.4. *Let (A, G, α, σ) be a unital twisted C^*-dynamical system, and assume that A admits a faithful G-invariant tracial state. If N is a normal icc subgroup of G such that the centralizer of N in G has trivial amenable radical, then $A \rtimes_{\alpha,r}^{\sigma} G$ has a unique tracial state whenever $A \rtimes_{\alpha,r}^{\sigma} N$ has a unique tracial state.*

Chapter 5

Equivariant injective envelopes

In this chapter, we consider the G-injective envelopes of unital C^*-algebras equipped with an action by a discrete group G, and how these objects may uncover a lot of information not just about the original C^*-algebras, but also about the associated reduced crossed products. Injective envelopes have recently experienced a resurgence in interest, instigated by the 2014 paper by Kalantar and Kennedy [90], in which C^*-simplicity of a discrete group G was described by means of the action of G on the G-injective envelope of the complex numbers $\mathbb{C}$.

For C^*-algebras the topic of injectivity harks back to 1969, the year of Arveson's extension theorem. The theorem states for any Hilbert space H that the space $B(H)$ of bounded operators on H has the following Hahn-Banach-like property: for any inclusion $S \subseteq A$ of an operator system in a unital C^*-algebra such that S contains the identity of A, then any u.c.p. map $S \to B(H)$ extends to a u.c.p. map $A \to B(H)$ [7, Theorem 1.2.3]. Due to a theorem of Loebl [101] applying the famous 1959 result of Tomiyama that norm-one projections of C^*-algebras onto C^*-subalgebras are always completely positive [135, 136], studies of von Neumann algebras $\mathscr{M} \subseteq B(H)$ admitting a norm-one projection $B(H) \to \mathscr{M}$ (i.e., von Neumann algebras having "the extension property", see [71]) consequently uncovered a wealth of injective von Neumann algebras.

The idea that any C^*-algebra might be embedded in a smallest possible injective C^*-algebra was due to Hamana, who was inspired by a 1964 theorem of Cohen [34], stated as follows. Saying that a Banach space X is *injective* if for any isometric, linear inclusion of Banach spaces $E \subseteq F$, any linear contraction $E \to X$ extends to a linear contraction $F \to X$, then any Banach space E embeds isometrically into a injective Banach space X such that the only injective closed subspace of X containing E is X itself, and X is unique up to isometric isomorphism. Cohen then called the Banach space X the *injective envelope of* E. Replacing Banach spaces by unital C^*-algebras and linear contractions by u.c.p. maps, Hamana proved in 1979 that analogue statements held true for C^*-algebras [73] and operator systems [74], and finally for discrete C^*-dynamical systems in 1985 [79].

The first three sections constitute an exposition of Hamana's original results: we construct the equivariant injective envelope and relate properties of injective envelopes to notions considered in the previous chapters. Finally, the last section lines up our own original results on equivariant injective envelopes. Throughout the chapter, G will always denote a discrete group.

5.1 Extensions of operator systems

In this section, based on [73, 74, 79], we consider equivariant embeddings of operator systems into injective C^*-algebras, and we define all notions required in order to perceive an injective C^*-algebra containing a given operator system as "minimal" with respect to this containment.

Definition 5.1.1. Let S and T be operator systems, i.e., S and T are self-adjoint subspaces of unital C^*-algebras, containing the identity elements of the latter. A *complete order isomorphism* of S onto T is a u.c.p. linear isomorphism $S \to T$ with completely positive inverse. A complete order isomorphism of S onto itself is called an *automorphism* of S, and the group of automorphisms of S is denoted by $\mathrm{Aut}(S)$.

If G is a discrete group, we say that the operator system S is a *G-operator system* if there is an action of G on S by automorphisms. A linear map $\varphi\colon S \to T$ between G-operator systems is *G-equivariant* if $\varphi(gx) = g\varphi(x)$ for all $g \in G$ and $x \in S$.

Remark 5.1.2. A complete order isomorphism is automatically completely isometric [115, Proposition 3.2]. Furthermore, for unital C^*-algebras, the above definition of an automorphism coincides with the usual notion of an automorphism of a C^*-algebra. Indeed, if $\varphi\colon A \to B$ is a complete order isomorphism of unital C^*-algebras, then by the Schwarz inequality for 2-positive maps we have

$$a^*a = \varphi^{-1}(\varphi(a))^*\varphi^{-1}(\varphi(a)) \leq \varphi^{-1}(\varphi(a)^*\varphi(a)) \leq \varphi^{-1}(\varphi(a^*a)) = a^*a$$

for all $a \in A$, so that $a \in \mathrm{mult}(\varphi)$. Hence φ is a *-isomorphism.

Definition 5.1.3. Let G be a discrete group and let S be a G-operator system. We say that S is *G-injective* if the following holds. Given two G-operator systems E and F, a G-equivariant unital, complete isometry $\kappa\colon E \to F$ and a G-equivariant u.c.p. map $\varphi\colon E \to S$, there exists a G-equivariant u.c.p. map $\tilde{\varphi}\colon F \to S$ such that $\tilde{\varphi} \circ \kappa = \varphi$, i.e., the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, hook, "\kappa"] \arrow[dr, "\varphi"'] & F \arrow[d, dashed, "\tilde{\varphi}"] \\ & S \end{tikzcd}$$

Stated differently, a G-injective G-operator system is therefore an injective object in the category of G-operator systems and G-equivariant u.c.p. maps. For $G = \{1\}$ we get the well-studied notion of an *injective operator system* (cf. [33]).

We finally introduce three properties of embeddings of operator systems, the study of which will provide the bulk of the next section. Recall that a unital contraction of operator systems is completely positive if it is completely isometric [115, Proposition 3.2].

Definition 5.1.4. Let S be a G-operator system. An *extension* of S is a pair (W, κ) consisting of a G-operator system W and a G-equivariant complete isometry $\kappa\colon S \to W$. Furthermore, the extension (W, κ) is said to be

- *G-injective* if W is G-injective;

- *G-essential* if it holds for any G-equivariant u.c.p. map $\varphi\colon W \to Z$ that φ is completely isometric whenever $\varphi \circ \kappa$ is;
- *G-rigid* if it holds for any G-equivariant u.c.p. map $\varphi\colon W \to W$ that $\varphi = \mathrm{id}_W$ whenever $\varphi \circ \kappa = \kappa$.

A G-injective and G-essential extension of S is called a *G-injective envelope.*

If $G = \{1\}$ in the preceding two definitions, we obtain the notion of an *injective* (resp. *essential*, *rigid*) extension of an operator system.

For any extension (M, κ) of a G-operator system S, we will often suppress the complete isometry κ and simply assume that S is a G-invariant operator subsystem of M. Further, an extension (M, κ) of a unital C^*-algebra S is said to be a *C^*-algebra extension* if M is a unital G-C^*-algebra and $\kappa\colon S \to M$ is a G-equivariant, unital, injective *-homomorphism.

For any operator subsystem $S \subseteq B(H)$ for some Hilbert space H, we will consider the space $\ell^\infty(G, S)$ of bounded maps $G \to S$. Then $\ell^\infty(G, S)$ is an operator subsystem of $B(H \otimes \ell^2(G))$, if we let $\ell^\infty(G, S)$ act on $H \otimes \ell^2(G)$ by defining

$$f(\xi \otimes \delta_g) = f(g)\xi \otimes \delta_g, \quad g \in G,\ \xi \in H,\ f \in \ell^\infty(G, S).$$

Moreover, $\ell^\infty(G, S)$ may be given a natural G-action by

$$(sf)(t) = f(s^{-1}t), \quad s, t \in G,\ f \in \ell^\infty(G, S),$$

making it a G-operator system.

In the proof of the following lemma, recall for any unital C^*-algebra A and $n \geq 1$ that the map $M_n(\ell^\infty(G, A)) \to \ell^\infty(G, M_n(A))$ given by $[x_{i,j}] \mapsto (g \mapsto [x_{i,j}(g)])$ is a *-isomorphism. Hence its restriction to an operator subsystem $S \subseteq A$ is a u.c.p. map.

Lemma 5.1.5. *If the operator system S is injective, then $\ell^\infty(G, S)$ is a G-injective G-operator system.*

Proof. Assume that $S \subseteq B(H)$ for some Hilbert space H. Let $\kappa\colon E \to F$ and $\varphi\colon E \to \ell^\infty(G, S)$ be G-equivariant u.c.p. maps, and assume that κ is completely isometric. The map given $\pi\colon \ell^\infty(G, B(H)) \to B(H)$ given by $\pi(f) = f(1)$ for $f \in \ell^\infty(G, B(H))$ is then a unital *-homomorphism, so its restriction $\pi\colon \ell^\infty(G, S) \to S$ is a u.c.p. map. Hence $\psi = \pi \circ \varphi\colon E \to S$ is also a u.c.p. map. Since S is injective, there exists a u.c.p. map $\tilde{\psi}\colon F \to S$ such that $\tilde{\psi} \circ \kappa = \psi$.

We now define $\tilde{\varphi}\colon F \to \ell^\infty(G, S)$ by

$$\tilde{\varphi}(x)(g) = \tilde{\psi}(g^{-1}x), \quad x \in F,\ s \in G.$$

Then $\|\tilde{\varphi}(x)\| \leq \|\tilde{\varphi}\|\|x\|$, so that $\tilde{\varphi}$ is well-defined, and it is clearly linear and unital. Moreover, under the identification $M_n(\ell^\infty(G, S)) \cong \ell^\infty(G, M_n(S))$, the image of $g \in G$ under the amplification $\tilde{\varphi}^{(n)}$ is just multiplication by g^{-1} and $\tilde{\psi}^{(n)}$, both of which are positive. Hence $\tilde{\varphi}$ is a G-equivariant u.c.p. map. Since

$$\tilde{\varphi}(\kappa(x))(g) = \tilde{\psi}(g^{-1}\kappa(x)) = \tilde{\psi}(\kappa(g^{-1}x)) = \varphi(g^{-1}x)(1) = (g^{-1}\varphi(x))(1) = \varphi(x)(g)$$

for all $x \in E$ and $g \in G$, it follows that $\tilde{\varphi} \circ \kappa = \varphi$, and thus $\ell^\infty(G, S)$ is G-injective. $\square$

Lemma 5.1.6. *Let S be a G-operator subsystem of an injective C^*-algebra A and let $\kappa\colon S \to \mathscr{M} = \ell^\infty(G, A)$ be the map given by*

$$\kappa(x)(g) = g^{-1}x, \quad x \in S, \ g \in G.$$

Then $(\mathscr{M}, \kappa)$ is a G-injective extension of S and κ is a complete order isomorphism onto its image.

Moreover, S is G-injective if and only if S is injective as an operator system and there exists a G-equivariant u.c.p. map $\omega\colon \ell^\infty(G, S) \to S$ such that $\omega \circ \kappa = \mathrm{id}_S$.

Proof. Evidently κ is completely isometric, unital and G-equivariant, so the first assertion follows from Lemma 5.1.5. Note that $\mathscr{M} \cong \bigoplus_{g\in G} A$ is injective, since injectivity of C^*-algebras is preserved under direct sums. Assuming that S is G-injective, then there exists a G-equivariant u.c.p. map $\omega\colon \mathscr{M} \to S$ such that $\omega \circ \kappa = \mathrm{id}_S$. If $\rho\colon E \to F$ is a unital complete isometry and $\varphi\colon E \to S$ is a u.c.p. map, then injectivity of $\mathscr{M}$ yields a u.c.p. map $\tilde{\varphi}\colon F \to \mathscr{M}$ such that $\tilde{\varphi} \circ \rho = \kappa \circ \varphi$. Hence $\omega \circ \tilde{\varphi}\colon F \to S$ is the map desired in order to obtain injectivity of S. The converse follows from noting that if S is injective, then $\ell^\infty(G, S)$ is G-injective, so if there also exists a G-equivariant u.c.p. map $\omega\colon \ell^\infty(G, S) \to S$ such that $\omega \circ \kappa = \mathrm{id}_S$, then S is G-injective as well. □

First and foremost, the above lemma states that the category of G-operator systems and G-equivariant u.c.p. map contains sufficiently many injectives, and it also provides the background scenery for the construction of the injective envelope, since any operator system embeds into the injective C^*-algebra $B(H)$ for some Hilbert space H. Indeed, the way we embed a G-operator system into a G-injective G-C^*-algebra as done above provides perhaps the most easily understandable G-injective extension available to us.

Definition 5.1.7. For any G-operator subsystem S of an injective C^*-algebra A, the map $\kappa\colon S \to \ell^\infty(G, A)$ defined in Lemma 5.1.6 will be referred to as the *canonical inclusion map.*

Finally, we address the case of what it takes to be a G-essential extension of the simplest possible G-operator system, $\mathbb{C}$. Notice first that since $\mathbb{C}$ is an injective operator system by the Hahn-Banach theorem, then $\mathbb{C}$ is G-injective if and only if there exists a G-equivariant u.c.p. map $\ell^\infty(G) \to \mathbb{C}$ by Lemma 5.1.6. Therefore $\mathbb{C}$ is G-injective if and only if G is amenable, and this motivates a translation of statements in Chapter 2 about the Furstenberg boundary $\partial_F G$ into statements about equivariant injective envelopes.

In order to do so, we first show that $C(\partial_F G)$ is in fact G-injective, and this requires a translation of Proposition 2.4.6 to an analoguous statement for G-operator systems.

Proposition 5.1.8. *Let A be a unital G-C^*-algebra. If A is injective in the category of unital G-C^*-algebras and G-equivariant u.c.p. maps, then A is G-injective (in the sense of Definition 5.1.3).*

Proof. Assuming that A is faithfully represented on a Hilbert space H, then consider the canonical inclusion map $\kappa\colon A \to \ell^\infty(G, B(H))$. By hypothesis, there exists a G-equivariant u.c.p. map $\varphi\colon \ell^\infty(G, B(H)) \to A$ such that $\varphi \circ \kappa = \mathrm{id}_A$. Since $\ell^\infty(G, B(H))$ is G-injective, so is A. □

We end this section with the following result, due to Kalantar and Kennedy [90, Section 3.2], which provides the first step toward a description of C^*-simplicity via equivariant injective envelopes. We note that Hamana himself was in fact the first to observe the next result [72, Remark 4], albeit without proof.

Proposition 5.1.9. *Let X be a compact G-space. Then $C(X)$ is a G-essential extension of $\mathbb{C}$ if and only if X is a G-boundary. Consequently, $C(\partial_F G)$ is a G-injective envelope of $\mathbb{C}$.*

Proof. If X is a G-boundary, let $\varphi\colon C(X) \to B$ be a G-equivariant u.c.p. map into a G-operator system B. Since $C(\partial_F G)$ is G-injective by Propositions 2.4.6 and 5.1.8, there exists a G-equivariant u.c.p. map $\psi\colon B \to C(\partial_F G)$. By Corollary 2.4.2, $\psi \circ \varphi\colon C(X) \to C(\partial_F G)$ is an injective *-homomorphism, and thus completely isometric. In particular, φ is also completely isometric.

Next we assume that $C(X)$ is a G-essential extension of $\mathbb{C}$ and let $\mu \in \mathcal{P}(X)$. Defining a G-equivariant u.c.p. map $P\colon C(X) \to \ell^\infty(G)$ by $P(f)(g) = (g\mu)(f)$, then P is a complete isometry by hypothesis, so all real-valued $f \in C(X)$ satisfy

$$\|f\| = \sup_{g\in G} |(g\mu)(f)|.$$

From a standard Hahn-Banach separation argument (cf. [89, Theorem 4.3.9]) it now follows that the convex hull of $G\mu$ is weak*-dense in $\mathcal{P}(X)$. By Milman's converse applied to $\mathcal{P}(X)$ and $G\mu$, we see that δ_X is contained in the weak* closure of $G\mu$; in particular, by letting $\mu \in \delta_X$ we see that the action of G on X is minimal. ☐

We will see in the next section (as its title indicates) that the G-injective envelope of a G-operator system is unique up to G-equivariant complete order isomorphism, and therefore $C(\partial_F G)$ is *the* G-injective envelope of $\mathbb{C}$.

The above observation that $I_G(\mathbb{C}) = \mathbb{C} = I(\mathbb{C})$ if and only if G is amenable, has a generalization to von Neumann algebras, which was also remarked by Hamana [79, Remark 3.8]. We give an argument due to Anantharaman-Delaroche [3]. Since the space of normal states on $\ell^\infty(G)$ is weak*-dense in the state space of $\ell^\infty(G)$ (seen by means of the Hahn-Banach separation theorem), amenability of G translates to the existence of a net of normal states (φ_i) on $\ell^\infty(G)$ converging to a G-invariant state (an invariant mean) in the weak* topology. For any injective von Neumann algebra $\mathscr{M}$ equipped with a G-action, let $\mathscr{N} = \ell^\infty(G, \mathscr{M})$ and $\kappa\colon \mathscr{M} \to \mathscr{N}$ be the canonical inclusion map. We may then extend the action on $\mathscr{M}$ to a G-action on the von Neumann algebra tensor product $\ell^\infty(G) \mathbin{\overline{\otimes}} \mathscr{M}$ by defining

$$g.(f \otimes x) = gf \otimes gx, \quad g \in G,\ f \in \ell^\infty(G),\ x \in \mathscr{M}. \tag{5.1.1}$$

This yields a net of normal u.c.p. maps $\varphi_i \otimes \mathrm{id}\colon \ell^\infty(G) \mathbin{\overline{\otimes}} \mathscr{M} \to \mathscr{M}$ that has a convergent subnet in the point-ultraweak topology (see the discussion before Theorem 5.2.2 below), and the limit $\varphi\colon \ell^\infty(G) \mathbin{\overline{\otimes}} \mathscr{M} \to \mathscr{M}$ of this subnet is u.c.p. and G-equivariant with respect to the action (5.1.1). Finally, for $f \in \mathscr{N}$, one may check that $(g.f)(h) = gf(g^{-1}h)$ when $\mathscr{N}$ and $\ell^\infty(G) \mathbin{\overline{\otimes}} \mathscr{M}$ are naturally identified. Defining an automorphism β of $\mathscr{N}$ by $\beta(f)(h) = h^{-1}f(h)$, then $\beta(g.f) = g\beta(f)$ for all $f \in \mathscr{N}$ and $g \in G$, so that composing φ with β defines a G-equivariant u.c.p. map $\Phi\colon \mathscr{N} \to \mathscr{M}$ such that $\Phi \circ \kappa = \mathrm{id}_{\mathscr{M}}$. Therefore $\mathscr{M}$ is G-injective by Lemma 5.1.6.

We are unsure whether any injective, unital G-C^*-algebra is G-injective whenever G is amenable, but have been unable to find a counterexample.

5.2 Existence and uniqueness of the injective envelope

In this section, we give a self-contained explanation of how the G-injective envelope of a G-operator system comes about, and we give an account of its basic properties. The idea of the construction is to apply Zorn's lemma to a collection of projections on the canonical G-injective G-operator system, $\ell^\infty(G, B(H))$, which also fix the original operator system, and then show that a minimal projection must necessarily have a G-injective envelope of the latter as its image.

Definition 5.2.1. Let (S, ι) be an extension of a G-operator system E.

An *E-projection* on S is an idempotent G-equivariant u.c.p. map $\varphi\colon S \to S$ such that $\varphi \circ \iota = \iota$.

An *E-seminorm* on S is a seminorm $p\colon S \to \mathbb{R}$ such that $p(x) = \|\varphi(x)\|$ for some G-equivariant u.c.p. map $\varphi\colon S \to S$ with $\varphi \circ \iota = \iota$.

With E and (S, ι) as above, we now define a partial ordering $\prec$ on the set of E-projections on S by writing $\varphi \prec \psi$ if

$$\varphi \circ \psi = \psi \circ \varphi = \varphi.$$

Moreover, we define a partial ordering $\leq$ on the set of E-seminorms on S by writing $p \leq q$ if $p(x) \leq q(x)$ for all $x \in S$.

The proof below is due to Hamana (cf. [74, §3]). Before we set sail, we record the following facts. If $\mathscr{M} \subseteq B(H)$ is a von Neumann algebra and $x_i \to x$ in $\mathscr{M}$ with respect to the weak operator topology, then

$$|\langle x\xi, \eta\rangle| = \lim_i |\langle x_i\xi, \eta\rangle| \leq \limsup_i \|x_i\|$$

for all $\xi, \eta \in H$ of norm less than or equal to 1, proving that $\|x\| \leq \limsup_i \|x_i\|$.

Moreover, the Banach space $B(S, \mathscr{M})$ of bounded operators $S \to \mathscr{M}$ can equipped with a locally convex Hausdorff topology as follows. It is easy to show that $B(S, \mathscr{M})$ can be identified with the dual space of the closed subspace X of $B(S, \mathscr{M})^*$, generated by the linear functionals $x \otimes \omega$ for $x \in S$ and $\omega \in \mathscr{M}_*$, where $x \otimes \omega$ is defined by

$$(x \otimes \omega)(T) = \omega(Tx), \quad T \in B(S, \mathscr{M}).$$

The identification is via evaluation, i.e., the map $\varphi\colon B(S, \mathscr{M}) \to X^*$ given by $\varphi(f)(\psi) = \psi(f)$ for $f \in B(S, \mathscr{M})$ and $\psi \in X$. The weak* topology on X^* induces the *point-ultra-weak topology* on $B(S, \mathscr{M})$, and a *bounded* net (T_i) in $B(S, \mathscr{M})$ converges to T in this topology if and only if $T_i x \to Tx$ in $\mathscr{M}$ ultraweakly for all $x \in S$.

Theorem 5.2.2. *For any G-injective extension (S, ι) of a G-operator system E, there is an E-projection $\varphi\colon S \to S$ such that $(\varphi(S), \iota)$ is a G-injective, G-essential and G-rigid extension of E.*

Proof. We may assume that $S \subseteq B(H)$ for a Hilbert space H. Let $\mathscr{M} = \ell^\infty(G, B(H))$ and $\kappa\colon S \to \mathscr{M}$ be the canonical inclusion map. Since S is G-injective, there exists a G-equivariant u.c.p. map $\psi\colon \mathscr{M} \to S$ such that $\psi \circ \kappa = \mathrm{id}_S$.

We first prove that any chain $(p_i)_{i\in I}$ of E-seminorms on S has a lower bound, so that there exists a minimal E-seminorm on S by Zorn's lemma. For each $i \in I$, let $\varphi_i\colon S \to S$ be a G-morphism with $\varphi_i \circ \iota = \iota$ such that

$$p_i(x) = \|\varphi_i(x)\|, \quad x \in S,$$

and define $\tilde\varphi_i = \kappa \circ \varphi_i\colon S \to \mathscr{M}$. Then each $\tilde\varphi_i$ is in the unit ball of $B(S, \mathscr{M})$. We have the following diagram of the maps:

$$\begin{tikzcd} S \arrow[r, "\varphi_i"] \arrow[dr, "\tilde\varphi_i"'] & S \arrow[d, bend right, "\kappa"'] \\ & \mathscr{M} \arrow[u, bend right, "\psi"'] \end{tikzcd}$$

By compactness of the unit ball of $B(S, \mathscr{M})$ in the point-ultraweak topology (due to the Banach-Alaoglu theorem), there is a subnet $(\tilde\varphi_j)_{j\in J}$ of $(\tilde\varphi_i)_{i\in I}$ and an operator $\tilde\varphi \in B(S, \mathscr{M})$ such that $\tilde\varphi_j \to \tilde\varphi$ in the point-ultraweak topology. Since the G-action on $\mathscr{M}$ is ultraweakly continuous, it is easily verified that the point-ultraweak limit of any net of G-equivariant u.c.p. maps in $B(S, \mathscr{M})$ in the point-ultraweak topology is itself a G-equivariant u.c.p. map. Clearly $\psi \circ \tilde\varphi \circ \iota = \iota$, so $x \mapsto \|\psi(\tilde\varphi(x))\|$ is an E-seminorm on S. Moreover, for any $i \in I$ we can take $j_0 \in J$ with $j_0 \geq i$, so that for all $x \in S$,

$$\|\psi \circ \tilde\varphi(x)\| \leq \|\tilde\varphi(x)\| \leq \limsup_{j\in J} \|\tilde\varphi_j(x)\| = \limsup_{j\in J} p_j(x) \leq \sup_{j \geq j_0} p_j(x) \leq p_i(x).$$

Hence we have found a lower bound for $(p_i)_{i\in I}$.

We now fix a G-equivariant u.c.p. map $\varphi\colon S \to S$ such that $\varphi \circ \iota = \iota$ and $p(x) = \|\varphi(x)\|$ is a minimal E-seminorm on S. We claim that $\varphi = \varphi \circ \beta \circ \varphi$ for any G-equivariant u.c.p. map $\beta\colon S \to S$ such that $\beta \circ \iota = \iota$. Once this has been shown, we obtain the desired properties of $(\varphi(S), \iota)$ as follows:

(a) *$(\varphi(S), \iota)$ is G-injective.* By letting $\beta = \mathrm{id}_S$ we see that φ is an E-projection on S.

(b) *$(\varphi(S), \iota)$ is G-essential.* If $\omega\colon \varphi(S) \to F$ is a G-equivariant u.c.p. map such that $\omega \circ \iota$ is completely isometric, then by G-injectivity there is a G-equivariant u.c.p. map $\xi\colon F \to \varphi(S)$ such that $\xi \circ \omega \circ \iota = \iota$. Now $\xi \circ \omega \circ \varphi = \varphi \circ (\xi \circ \omega \circ \varphi) \circ \varphi = \varphi$, so $\xi \circ \omega = \mathrm{id}_{\varphi(S)}$. Hence ω is completely isometric.

(c) *$(\varphi(S), \iota)$ is G-rigid.* If $\alpha\colon \varphi(S) \to \varphi(S)$ is a G-equivariant u.c.p. map such that $\alpha \circ \iota = \iota$, then $\alpha \circ \varphi = \varphi \circ (\alpha \circ \varphi) \circ \varphi = \varphi$, so that $\alpha = \mathrm{id}_{\varphi(S)}$.

We finally prove the claim. For β as above, define $\alpha = \beta \circ \varphi\colon S \to S$ and a bounded sequence $(\tilde\varphi_n)$ in $B(S, \mathscr{M})$ by $\tilde\varphi_n = \frac{1}{n}\sum_{i=1}^n \kappa \circ \alpha^i\colon S \to \mathscr{M}$ where α^i is the i-fold composite of $\beta\circ\varphi$ with itself for all $1 \leq i \leq n$. Take a subnet $(\tilde\varphi_{n_i})$ in $B(S, \mathscr{M})$ converging point-ultraweakly to some G-equivariant u.c.p. map $\tilde\varphi\colon S \to \mathscr{M}$. Now $\psi \circ \tilde\varphi\colon S \to S$ is a G-equivariant u.c.p. map such that $\psi \circ \tilde\varphi \circ \iota = \iota$, and

$$\begin{aligned} \|\psi(\tilde\varphi(x))\| &\leq \|\tilde\varphi(x)\| \leq \limsup_i \|\tilde\varphi_{n_i}(x)\| \\ &\leq \limsup_i \frac{1}{n_i} \sum_{i=1}^{n_i} \|\kappa \circ (\beta \circ \varphi)^i(x)\| \leq \|\varphi(x)\| = p(x) \end{aligned}$$

for all $x \in S$. Since p is minimal, it follows that $\limsup_i \|\tilde{\varphi}_{n_i}(x)\| = \|\varphi(x)\|$ and

$$\begin{aligned}\|\varphi(\alpha(x) - x)\| &= \limsup_i \|\tilde{\varphi}_{n_i}(\alpha(x) - x)\| = \limsup_i \frac{1}{n_i}\|\kappa(\alpha^{n_i+1}(x)) - \kappa(\alpha(x))\| \\ &\leq \limsup_i \frac{2}{n_i}\|\alpha(x)\| = 0\end{aligned}$$

for all $x \in S$, so $\varphi = \varphi \circ \alpha = \varphi \circ \beta \circ \varphi$. □

We next note how this leads to the main result of this section, first proved by Hamana in 1979 for C^*-algebras [73, Theorem 4.1], then later in 1979 for operator systems [74, Theorem 4.1] and finally in 1985 for G-operator systems [79, Theorem 2.5].

Theorem 5.2.3. *Any G-operator system S has a G-injective envelope $(I_G(S), \kappa)$. It is unique in the sense that for any G-injective envelope (W, ι) of S, there exists a G-equivariant complete order isomorphism $\alpha\colon I_G(S) \to W$ with $\alpha \circ \kappa = \iota$.*

Proof. By Lemma 5.1.6, S has a G-injective extension $(\mathscr{M}, \kappa)$. By Theorem 5.2.2 there is an S-projection $\varphi\colon \mathscr{M} \to \mathscr{M}$ for which the extension $(\varphi(\mathscr{M}), \kappa)$ is a G-injective, G-essential and G-rigid extension of S. We then let $I_G(S) = \varphi(\mathscr{M})$, yielding a G-injective envelope $(I_G(S), \kappa)$ of S.

If (W, ι) is another G-injective envelope of S, there exist G-equivariant u.c.p. maps $\alpha\colon I_G(S) \to W$ and $\psi\colon W \to I_G(S)$ such that $\alpha \circ \kappa = \iota$ and $\psi \circ \iota = \kappa$. By G-essentiality, ψ is completely isometric. Since $\psi \circ \alpha = \mathrm{id}_{I_G(S)}$ by G-rigidity, ψ is also surjective and $\psi^{-1} = \alpha$. Hence α is a complete order isomorphism. □

The above theorem then allows us to make the following definition.

Definition 5.2.4. For any G-operator system S, we denote its G-injective envelope by $(I_G(S), \kappa_S)$. If $G = \{1\}$, then $I(S) = I_G(S)$ is the injective envelope.

As explained after Definition 5.1.4, we will often suppress the inclusion and assume that S is a G-invariant operator subsystem of its G-injective envelope $I_G(S)$.

Remark 5.2.5. (i) If (W, ρ) had been assumed to be a G-injective and G-rigid extension of S in the above proof, we would also get a complete order isomorphism $\psi\colon I_G(S) \to W$ such that $\psi \circ \kappa = \rho$. Hence $(I_G(S), \kappa)$ is also the unique G-rigid and G-injective extension of S ("unique" in the sense of Theorem 5.2.3).

(ii) In the proof of Theorem 5.2.2 we established that the G-injective envelope $I_G(E)$ satisfies a stronger property: for any G-injective G-operator system S containing E, then we may embed the G-injective envelope $I_G(E)$ in S in such a way that if $\varphi\colon S \to S$ is an E-projection onto $I_G(E)$, then $\varphi = \varphi \circ \beta \circ \varphi$ for any G-equivariant u.c.p. map $\beta\colon S \to S$ satisfying $\beta|_E = \mathrm{id}_E$.

We next make a minor observation that essentially follows from the uniqueness of the G-injective envelope.

Lemma 5.2.6. *Let S be a G-operator system and let M be a G-injective G-operator system such that $S \subseteq M$. If $\varphi\colon M \to M$ is an S-projection, there exists an S-projection $\psi\colon M \to M$ such that $\psi \prec \varphi$ and $\psi(M) = I_G(S)$.*

Proof. Since $\varphi(M)$ is a G-injective G-operator system containing $S = \varphi(S)$, we may let $\Psi\colon \varphi(M) \to \varphi(M)$ be an S-projection such that $\Psi(\varphi(M))$ is the G-injective envelope of S, due to Theorems 5.2.2 and 5.2.3. Define $\psi = \Psi \circ \varphi$. $\square$

By the existence alone of the G-injective envelope, a lot of information can be uncovered on the properties of extensions we have considered, G-essentiality in particular. One example is the following result.

Corollary 5.2.7. *If $E \subseteq S$ is a G-essential extension, then $E \subseteq S$ is G-rigid.*

Proof. Let $\varphi\colon S \to S$ be a G-equivariant u.c.p. map such that $\varphi|_E = \mathrm{id}_E$. If $(I_G(E), \kappa_E)$ is the G-injective envelope of E, there is a G-equivariant complete isometry $\psi\colon S \to I_G(E)$ such that $\psi|_E = \kappa$. There is now a G-equivariant u.c.p. map $\xi\colon I_G(E) \to I_G(E)$ such that $\xi \circ \psi = \psi \circ \varphi$, but then $\xi \circ \kappa = \kappa$ so that $\xi = \mathrm{id}_{I_G(E)}$. Thus $\psi = \psi \circ \varphi$ and $\varphi = \mathrm{id}_S$, due to ψ being an isometry. $\square$

The following lemma is a G-equivariant version of [73, Lemma 4.6].

Lemma 5.2.8. *Let (M, ι) be an extension of a G-operator system S and let $(I_G(S), \kappa)$ be the G-injective envelope of S. Then (M, ι) is G-essential if and only if there exists a unital G-equivariant complete isometry $\varphi\colon M \to I_G(S)$ such that $\varphi \circ \iota = \kappa$. In this case, there is a G-equivariant complete order isomorphism $\varphi\colon I_G(S) \to I_G(M)$ such that $\varphi \circ \kappa = \iota$.*

Proof. If (M, ι) is a G-essential extension of S, then G-injectivity of $I_G(S)$ yields a G-equivariant u.c.p. map $\varphi\colon M \to I_G(S)$ satisfying $\varphi \circ \iota = \kappa$. Since κ is a complete isometry, so is φ.

If there exists a completely isometric G-equivariant map $\varphi\colon M \to I_G(S)$ with $\varphi \circ \iota = \kappa$, assume that $\psi\colon M \to N$ is a G-equivariant u.c.p. map such that $\psi \circ \iota\colon S \to N$ is completely isometric. By G-injectivity we obtain G-equivariant u.c.p. maps $\alpha\colon N \to I_G(S)$ and $\varphi'\colon I_G(S) \to I_G(S)$ such that the following diagram commutes:

$$
\begin{tikzcd}
S \arrow[r, hook, "\iota"] \arrow[dr, "\kappa"'] \arrow[rrr, bend left, "\kappa"] & M \arrow[r, "\psi"] \arrow[d, "\varphi"] & N \arrow[r, dashed, "\alpha"] & I_G(S) \\
& I_G(S) \arrow[urr, dashed, "\varphi'"'] & &
\end{tikzcd}
$$

Since $\varphi' \circ \kappa = \kappa$, G-rigidity yields $\varphi' = \mathrm{id}_{I_G(S)}$ and $\alpha \circ \psi = \varphi$, so because φ is completely isometric, ψ must be as well.

Whenever $\varphi\colon M \to I_G(S)$ is a G-equivariant complete isometry such that $\varphi \circ \iota = \kappa$, the inclusions $\kappa\colon S \to I_G(S)$ and $\kappa'\colon M \to I_G(M)$ allow us to construct G-equivariant u.c.p. maps $\tilde{\varphi}\colon I_G(M) \to I_G(S)$ and $\psi\colon I_G(S) \to I_G(M)$ such that $\tilde{\varphi} \circ \kappa' = \varphi$ and $\psi \circ \varphi = \kappa'$. By G-rigidity, ψ is then a G-equivariant complete order isomorphism with inverse $\tilde{\varphi}$. $\square$

One of the most important facts about injective operator systems is a 1977 result by Choi and Effros [33, Theorem 3.1]: every injective operator system can be given the structure of a unital, monotone complete C^*-algebra (cf. Remark 2.4.8) to which it is completely order isomorphic. We give a proof here, by combining parts of the original argument with observations of Hamana.

Consider first the following lemma [73, Lemma 2.4].

Lemma 5.2.9. *Let A be a unital C^*-algebra and let $\varphi\colon A \to A$ be an idempotent, unital contraction. If φ satisfies the Schwarz inequality, i.e.,*

$$\varphi(x)^*\varphi(x) \leq \varphi(x^*x), \quad x \in A,$$

then

$$\varphi(\varphi(x)\varphi(y)) = \varphi(x\varphi(y)) = \varphi(\varphi(x)y), \quad x, y \in A.$$

Proof. Fixing a state ψ on A, $\Psi = \psi \circ \varphi$ is also a state on A. Letting $(\pi_\Psi, H_\Psi, \xi_\Psi)$ be the GNS triple associated to Ψ, we can define an operator $P_\Psi \in B(H_\Psi)$ by

$$P_\Psi(\pi_\Psi(x)\xi_\Psi) = \pi_\Psi(\varphi(x))\xi_\Psi, \quad x \in A.$$

For any $x \in A$,

$$\|\pi_\Psi(\varphi(x))\xi_\Psi\|^2 = \Psi(\varphi(x)^*\varphi(x)) \leq \psi(\varphi(x^*x)) = \Psi(x^*x) = \|\pi_\Psi(x)\xi_\Psi\|^2$$

due to the Schwarz inequality, so P_Ψ is well-defined and bounded. Clearly $P_\Psi^2 = P_\Psi$, as φ is idempotent, and P_Ψ is also a contraction. Therefore P_Ψ is a projection. If $x, y \in A$, then

$$\begin{aligned}
\psi(\varphi(\varphi(x)\varphi(y))) &= \langle \pi_\Psi(\varphi(x))\pi_\Psi(\varphi(y))\xi_\Psi, \xi_\Psi\rangle \\
&= \langle P_\Psi(\pi_\Psi(y)\xi_\Psi), P_\Psi(\pi_\Psi(x^*)\xi_\Psi)\rangle \\
&= \langle P_\Psi(\pi_\Psi(y)\xi_\Psi), \pi_\Psi(x^*)\xi_\Psi\rangle = \psi(\varphi(x\varphi(y))),
\end{aligned}$$

and similarly, $\psi(\varphi(\varphi(x)\varphi(y))) = \psi(\varphi(\varphi(x)y))$. As ψ was arbitrary, the claim follows. $\square$

The next result is due to Hamana in its form below [73, Theorem 2.3], but is more or less a restatement of the theorem of Choi and Effros.

Theorem 5.2.10. *If $\varphi\colon A \to A$ is an idempotent, unital contraction satisfying the Schwarz inequality, define*

$$x \circ y = \varphi(xy), \quad x, y \in \varphi(A).$$

Then $\circ$ defines a product on A, and with respect to the norm and involution of A, $(\varphi(A), \circ)$ is in fact a unital C^*-algebra.

If φ is completely positive, then $\mathrm{id}_{\varphi(A)}\colon \varphi(A) \to (\varphi(A), \circ)$ is a complete order isomorphism of operator systems, and the C^*-algebra structure on $\varphi(A)$ is unique.

Proof. It follows from Lemma 5.2.9 that $\circ$ defines an associative binary operation on $B = \varphi(A)$, as

$$x \circ (y \circ z) = \varphi(x\varphi(yz)) = \varphi(\varphi(x)yz) = \varphi(xyz) = \varphi(xy\varphi(z)) = \varphi(\varphi(xy)z) = (x \circ y) \circ z$$

for all $x, y, z \in B$. Moreover, we see that the C^*-axiom is satisfied for this product, since $x^*x = \varphi(x)^*\varphi(x) \leq \varphi(x^*x) = x^* \circ x$ for all $x \in B$, so that

$$\|x\|^2 = \|x^*x\| \leq \|x^* \circ x\| = \|\varphi(x^*x)\| \leq \|x^*x\| = \|x\|^2.$$

The rest of the axioms are easy to verify.

We next assume that φ is completely positive. Uniqueness of the C^*-algebra structure on $(B, \circ)$ then follows from Remark 5.1.2, once we prove that the identity map from the operator system B and the new C^*-algebra $(B, \circ)$ is a complete order isomorphism. We follow the original argument of [33, Theorem 3.1]. Observe that

$$[r_{ij}] \circ [s_{ij}] = \left[\sum_k r_{ik} \circ s_{kj} \right] = \varphi^{(n)}([r_{ij}][s_{ij}]) \tag{5.2.1}$$

for all elements $[r_{ij}], [s_{ij}] \in M_n(B)$. Since $M_n(B)$ is also a unital C^*-algebra with respect to the product $\circ_n$ defined by means of the completely positive projection $\varphi^{(n)} \colon M_n(A) \to M_n(A)$, it follows from the above identity that $M_n(B, \circ)$ and $(M_n(B), \circ_n)$ are *-isomorphic, and thus $M_n(B, \circ)$ has the subspace norm relative to $M_n(A)$. Evidently, if $x \in M_n(S, \circ)$ is positive, then x is positive in $M_n(A)$ by (5.2.1); for the converse, if $x \in M_n(B)$ is positive in $M_n(A)$, then x is self-adjoint in $M_n(B, \circ)$ and $\| \|x\| 1 - x \| \leq \|x\|$ in $M_n(A)$. By the above observation and [89, Lemma 4.2.1], this implies that x is positive in the C^*-algebra $M_n(B, \circ)$. □

We are not sure whether complete positivity of φ is necessary in order to prove the second half of the above theorem.

Definition 5.2.11. If $\varphi \colon A \to A$ is an idempotent, unital and positive map satisfying the identities of Lemma 5.2.9, the product $\circ$ on $\varphi(A)$ given by $x \circ y = \varphi(xy)$ for $x, y \in \varphi(A)$ is known as the *Choi-Effros product*.

We can now prove the result of Choi and Effros in the equivariant case very easily:

Corollary 5.2.12. *Let S be a G-injective operator system. Then S is completely order isomorphic to a unital, monotone complete G-C^*-algebra, unique up to *-isomorphism.*

Proof. By Lemma 5.1.6, S is an injective operator system. Assuming that $S \subseteq B(H)$, then the existence of a u.c.p. projection $B(H) \to S$ yields a C^*-algebra structure on S by Theorem 5.2.10, with respect to which the complete order isomorphisms of G can be regarded as C^*-algebra automorphisms (due to Remark 5.1.2). Moreover, the argument given in Remark 2.4.8 yields that S is monotone complete, since $B(H)$ is monotone complete [89, Lemma 5.1.4]. □

We remark that a monotone complete C^*-algebra is an example of an *AW^*-algebra*, i.e., a C^*-algebra A in which every maximal commutative C^*-subalgebra is generated by projections, and the projections of A form a complete lattice. These were originally devised by Kaplansky as a attempt to give an abstract, algebraic description of the C^*-algebras that were isomorphic to von Neumann algebras, i.e., W^*-algebras. Dixmier gave an example in 1951 of a commutative AW^*-algebra that is not W^* (see [18, III.1.8]); nonetheless, many structure results that hold for von Neumann algebras also hold for AW^*-algebras. As of now, however, it is still unknown whether AW^*-algebras are always monotone complete.

A noteworthy characterization of AW^*-algebras was given by Kaplansky in 1951: a C^*-algebra is an AW^*-algebra if and only the right annihilator

$$\{x \in A \,|\, sx = 0 \text{ for all } s \in S\}$$

of any non-empty subset $S \subseteq A$ is of the form pA for some projection $p \in A$. We refer to [17] and [128, Chapter 8] for more information on AW^*-algebras and their relation to monotone complete C^*-algebras.

As the G-injective envelope $I_G(S)$ of a G-operator system S admits a C^*-algebra structure, we may say a bit more about the inclusion $S \to I_G(S)$. The following proposition is an adaptation of [115, Proposition 15.10] to the equivariant case.

Proposition 5.2.13. *Let $S \subseteq B(H)$ be a G-operator system containing an operator subsystem A which is a unital C^*-subalgebra of $B(H)$. Then the restriction of the inclusion $S \to I_G(S)$ to A is an injective *-homomorphism.*

Proof. Let $\mathscr{M} = \ell^\infty(G, B(H))$, let $\kappa\colon S \to \mathscr{M}$ be the canonical inclusion map and let $\varphi\colon \mathscr{M} \to \mathscr{M}$ be a projection such that $\varphi \circ \kappa = \kappa$ and $\varphi(\mathscr{M}) = I_G(S)$. Then $\kappa|_A$ is a *-homomorphism, and due to the multiplication on $I_G(S)$ being given by $x \circ y = \varphi(xy)$ for $x, y \in I_G(S)$, we have $\kappa(x) \circ \kappa(y) = \varphi(\kappa(xy)) = \kappa(xy)$ for all $x, y \in A$, meaning that $\varphi \circ \kappa\colon S \to I_G(S)$ restricts to a *-homomorphism on A. ☐

Remark 5.2.14. In the case when S is a unital G-C^*-algebra, the above lemma allows us to regard S as a G-invariant unital C^*-subalgebra of its G-injective envelope $I_G(S)$.

Moreover, if M is a G-essential C^*-algebra extension of S, then the G-equivariant complete isometry $\varphi\colon M \to I_G(S)$ of Lemma 5.2.8 can be chosen to be a *-homomorphism, since M is a G-invariant C^*-subalgebra of $I_G(M)$ (which is completely order isomorphic and hence *-isomorphic to $I_G(S)$).

Remark 5.2.15. (i) For any unital C^*-algebra A, viewed as a C^*-subalgebra of $I(A)$, we have $Z(I(A)) = A' \cap I(A)$ [73, Corollary 4.3]. Indeed, one inclusion is obvious, and if $u \in I(A)$ is a unitary operator that commutes with all elements in A, then $x \mapsto uxu^*$, $x \in I(A)$, defines an automorphism of $I(A)$ that fixes all of A. Hence it fixes all of $I(A)$ by rigidity, so that $u \in Z(I(A))$. Since any unital C^*-algebra is the linear span of its unitary operators, the other inclusion follows.

(ii) Let A be a unital commutative G-C^*-algebra. Then the G-injective envelope $I_G(A)$ is also commutative. We present a slightly different argument than the one given in [70, Theorem 4.2]. Since $A \subseteq A' \cap I(A) = Z(I(A))$, then $uxu^* = x$ for all $x \in A$ and all unitaries $u \in I(A)$, so that u commutes with all of $I(A)$ by rigidity. Therefore $I(A)$ is commutative. If $\varphi\colon \ell^\infty(G, I(A)) \to \ell^\infty(G, I(A))$ is a minimal $\kappa(A)$-projection, where κ is the canonical inclusion map, then $I_G(A)$ is the image of φ and the Choi-Effros product on $I_G(A)$ is evidently commutative.

In the case when we just consider the injective envelope $I(A)$ of a given C^*-algebra A, it will prove helpful in our own work to determine when an ideal of $I(A)$ is actually the zero ideal. The proof of the following lemma is due to Hamana [76, Lemma 1.2].

Lemma 5.2.16. *If A is a unital C^*-algebra, regarded as a C^*-subalgebra of $I(A)$, and B is a C^*-subalgebra of $I(A)$ such that $xB + By \subseteq B$ for all $x, y \in A$ and $A \cap B = \{0\}$, then $B = \{0\}$.*

Proof. Note that $A + B$ is a C^*-subalgebra of $I(A)$ with a closed two-sided ideal B. As $A \cap B = \{0\}$, the quotient homomorphism $A + B \to (A + B)/B$ is injective when restricted to A. As $A + B$ is an essential extension of A by Lemma 5.2.8, the quotient homomorphism must be injective, so that $B = \{0\}$. ☐

Remark 5.2.17. If A is a unital, prime C^*-algebra, then $I(A)$ has trivial centre [75, Theorem 7.1]. Indeed, any non-zero central projection $p \in I(A)$ defines closed ideals $A \cap pA$ and $A \cap (1-p)A$ of A, one of which must be zero, say, p. It then follows from Lemma 5.2.16 that $pA = \{0\}$. Since the map $a \mapsto (1-p)a$ is the identity map on A, $(1-p)x = x$ for all $x \in I(A)$ by rigidity, i.e., $p = 1$. In fact, the converse also holds; we omit the proof.

We end this section by relating the injective envelope and the G-injective envelope of a G-operator system to one another, an observation from [79, Remark 2.6].

Proposition 5.2.18. *For any G-operator system S, let $(I(S), \kappa)$ be the injective envelope of S. Then there is a unique unital G-C^*-algebra structure on $I(S)$ such that κ is G-equivariant, and $I(S)$ is a G-essential extension of S.*

Proof. By Theorem 5.2.10, we can endow $I(S)$ with the structure of a unital C^*-algebra. For each $g \in G$, the map $x \mapsto gx$ of S extends to a unital, completely positive map $\alpha_g \colon I(S) \to I(S)$ such that $\alpha_g(\kappa(x)) = \kappa(gx)$ for all $x \in S$. By rigidity of the extension $S \subseteq I(S)$, each α_g is a complete order isomorphism and thus an automorphism of the C^*-algebra $I(S)$, so that $g \mapsto \alpha_g$ defines an action of G on $I(S)$ by automorphisms for which κ is G-equivariant.

By G-injectivity of $I_G(S)$, there exists a G-equivariant u.c.p. map $\varphi \colon I(S) \to I_G(S)$ such that $\varphi|_S$ is the inclusion map $S \to I_G(S)$, so by essentiality of $I(S)$, φ is a complete isometry. Due to Lemma 5.2.8, the inclusion $S \subseteq I(S)$ is G-essential. ☐

5.3 Reduced crossed products of injective envelopes

Let G be a discrete group. In the previous section, we saw that the G-injective envelope of a unital G-C^*-algebra contains a copy of the original C^*-algebra as a G-invariant G-C^*-subalgebra. We now review Hamana's original results ([78, §3] and [79, Section 3]) on the connection between the G-injective envelope of a G-C^*-algebra and the injective envelope of the reduced crossed product of the associated C^*-dynamical systems; this may have been the incitement for even constructing the G-injective envelope in the first place.

Indeed, the main result (Theorem 5.3.4) of this section can be motivated as follows. Suppose that A is a unital G-C^*-algebra and let $A \rtimes_r G$ be the associated reduced crossed product. Even though the injective envelope $I(A \rtimes_r G)$ may have a lot of the properties of the original crossed product and possibly nicer algebraic properties (we have seen that it is monotone complete), it is nonetheless a rather intangible structure. The question is thus whether there is a more easily understood C^*-algebra, with injectivity features, that carries the same amount of information that $I(A \rtimes_r G)$ does about $A \rtimes_r G$. This is where the G-injective envelope comes into play: $I_G(A)$ is the largest G-C^*-algebra B containing a G-invariant copy of A such that the inclusion $A \rtimes_r G \to I(A \rtimes_r G)$ extends to an injective *-homomorphism $B \rtimes_r G \to I(A \rtimes_r G)$. Put differently, $I_G(A) \rtimes_r G$ is the largest essential extension of $A \rtimes_r G$ that takes the form of a reduced crossed product.

We first argue how rigidity of an C^*-algebra extension passes to a larger C^*-algebra when the latter admits a faithful conditional expectation [79, Lemma 3.3].

Lemma 5.3.1. *Let B be a G-C^*-algebra, let A_1 be a unital G-C^*-algebra and let A_2 be a G-rigid C^*-algebra extension of A_1. Let $\iota\colon A_2 \to B$ be a G-equivariant u.c.p. map, and assume that there is a faithful idempotent G-equivariant u.c.p. map $\rho\colon B \to A_2$ such that $\rho \circ \iota = \mathrm{id}_{A_2}$. If $\varphi\colon A_2 \to B$ is a G-equivariant u.c.p. map such that $\varphi|_{A_1} = \iota|_{A_1}$, then $\varphi = \iota$.*

Proof. By G-rigidity, we have $\rho \circ \varphi = \mathrm{id}_{A_2}$. For $x \in A_2$, the Schwarz inequality yields

$$x^*x = \rho(\varphi(x))^*\rho(\varphi(x)) \leq \rho(\varphi(x)^*\varphi(x)) \leq \rho(\varphi(x^*x)) = x^*x,$$

so that $\rho(\varphi(x)^*\varphi(x)) = x^*x$. Hence $\varphi(A_2)$ and $\iota(A_2)$ are contained in the multiplicative domain of ρ, so for $x \in A_2$,

$$\rho((\iota(x) - \varphi(x))^*(\iota(x) - \varphi(x))) = \rho(\iota(x) - \varphi(x))^*\rho(\iota(x) - \varphi(x)) = 0.$$

Due to ρ being faithful, $\varphi(x) = x$, as wanted. □

Now we review the theory of reduced crossed products, with one specific motive: we want to be able to deduce G-injectivity of a C^*-algebra from injectivity of a crossed product and vice versa. It turns out that our usual reduced crossed product does not suffice for this objective, as the reduced crossed product may not be monotone complete. To remedy this, we instead use the model of the crossed product that one normally uses for von Neumann algebras.

If A is a unital G-C^*-algebra represented on some Hilbert space H, then for all $g \in G$ we may define maps $\iota_g\colon H \to H \otimes \ell^2(G)$ and $\pi_g\colon H \otimes \ell^2(G) \to H$ by

$$\iota_g(\xi) = \xi \otimes \delta_g, \quad \pi_g(\xi \otimes \delta_h) = \delta_{g,h}\xi, \quad \xi \in H,\ h \in G,$$

where $\delta_{g,h}$ denotes the Kronecker delta function. For an operator $x \in B(H \otimes \ell^2(G))$, we consider the matrix representation $[x_{g,h}]_{g,h\in G}$ of x given by $x_{g,h} = \pi_g x \iota_h \in B(H)$ for all $g, h \in G$. Notice that the maps $x \mapsto x_{g,h}$ are bounded. We now consider the norm-closed operator system

$$A \mathbin{\overline{\otimes}} B(\ell^2(G)) = \{x \in B(H \otimes \ell^2(G)) \mid x_{g,h} \in A \text{ for all } g, h \in G\}.$$

If $\mathscr{M} \subseteq B(H)$ is a von Neumann algebra, then $\mathscr{M} \mathbin{\overline{\otimes}} B(\ell^2(G))$ coincides with the von Neumann algebra tensor product of $\mathscr{M}$ and $B(\ell^2(G))$ in $B(H \otimes \ell^2(G))$. In fact, if A is a monotone complete C^*-algebra, then $A \mathbin{\overline{\otimes}} B(\ell^2(G))$ also has the structure of a monotone complete C^*-algebra [77, Theorem 3.12], but we will not need this fact. Define

$$M(A, G) = \{x \in A \mathbin{\overline{\otimes}} B(\ell^2(G)) \mid s^{-1}x_{g,h} = x_{gs,hs} \text{ for all } g, h, s \in G\}.$$

Letting $\pi\colon A \to B(H \otimes \ell^2(G))$ and $\lambda\colon G \to \mathcal{U}(H \otimes \ell^2(G))$ be a regular representation (see Section 3.2), then for $a \in A$ and $g \in G$,

$$(a\lambda_g)_{h,s} = \begin{cases} h^{-1}a & hs^{-1} = g \\ 0 & \text{otherwise} \end{cases} \tag{5.3.1}$$

for all $h, s \in G$, whenever we identify A with its image under π. Therefore $a\lambda_g \in M(A, G)$, so since $M(A, G)$ is norm-closed, it follows that $A \rtimes_r G \subseteq M(A, G)$. If $\mathscr{M}$ is a von Neumann algebra with a G-action by automorphisms, then $M(\mathscr{M}, G)$ coincides with the usual von Neumann algebra crossed product of $\mathscr{M}$ by G (cf., e.g., [134, V.7]). In the case of A being monotone complete, $M(A, G)$ can be given the structure of a monotone complete C^*-algebra as well [78, §3]. Both $A \mathbin{\overline{\otimes}} B(\ell^2(G))$ and $M(A, G)$ are independent of the choice of faithful representation $A \subseteq B(H)$, up to complete order isomorphism [77, Lemma 3.5].

We define a G-action on the operator system $A \mathbin{\overline{\otimes}} B(\ell^2(G))$ by

$$gx = \lambda_g x \lambda_g^*, \quad g \in G,\ x \in M(A, G).$$

It is easy to check that $M(A, G)$ is a G-invariant operator subsystem of $A \mathbin{\overline{\otimes}} B(\ell^2(G))$ and that the embedding $A \subseteq M(A, G)$ is G-equivariant.

Furthermore, we define a G-equivariant u.c.p. map $E\colon M(A, G) \to A$ given by $E(x) = x_{1,1}$. Then E satisfies $E|_A = \mathrm{id}_A$ and it is faithful: indeed, if $E(x) = 0$ for a positive element $x \in M(A, G)$, then $\pi_g x \iota_g = x_{g,g} = g^{-1}x_{1,1} = 0$ for all $g \in G$, so that $x^{1/2}\iota_g = 0$ for all $g \in G$. In fact, E extends the canonical faithful conditional expectation $A \rtimes_r G \to A$ to $M(A, G)$.

The following technical lemma comes from [79, Lemma 3.1].

Lemma 5.3.2. *Let A and B be unital G-C^*-algebras and let $\varphi\colon A \to B$ be a u.c.p. map. Then there exists a G-equivariant u.c.p. map $\tilde{\varphi}\colon A \mathbin{\overline{\otimes}} B(\ell^2(G)) \to B \mathbin{\overline{\otimes}} B(\ell^2(G))$ such that*

$$\tilde{\varphi}(x)_{g,h} = \varphi(x_{g,h}), \quad x \in A \mathbin{\overline{\otimes}} B(\ell^2(G)),\ g, h \in G.$$

Moreover, $\tilde{\varphi}$ (resp. $\tilde{\varphi}|_{A\rtimes_r G}$) is completely isometric if and only if φ is. If φ is G-equivariant, then $\tilde{\varphi}(M(A, G)) \subseteq M(B, G)$ and $\tilde{\varphi}|_{A\rtimes_r G}$ is the u.c.p. map satisfying

$$\tilde{\varphi}(a\lambda_g) = \varphi(a)\lambda_g$$

for all $a \in A$ and $g \in G$, so that $\tilde{\varphi}(A \rtimes_r G) \subseteq B \rtimes_r G$.

Proof. Assuming that $A \subseteq A^{**} \subseteq B(H)$ and $B \subseteq B^{**} \subseteq B(K)$ where A^{**} and B^{**} are the enveloping von Neumann algebras of A and B, respectively, we consider the bidual map $\varphi^{**}\colon A^{**} \to B^{**}$ which is a normal u.c.p. map. Now let $\tilde{\varphi}\colon A^{**} \mathbin{\overline{\otimes}} B(\ell^2(G)) \to B^{**} \mathbin{\overline{\otimes}} B(\ell^2(G))$ be the unique normal u.c.p. map satisfying $\tilde{\varphi}(x \otimes y) = \varphi^{**}(x) \otimes y$ for all $x \in A^{**}$ and $y \in B(\ell^2(G))$ [134, Proposition IV.5.13]. If $x \in A^{**} \mathbin{\overline{\otimes}} B(\ell^2(G))$, then consider the finite-rank operator $P_{g,h} \in B(\ell^2(G))$ given by $P_{g,h}\xi = \langle \xi, \delta_h\rangle \delta_g$ for $g, h \in G$. As $\iota_g = (1 \otimes P_{g,g})\iota_g$ and $(1 \otimes P_{g,g})x(1 \otimes P_{h,h}) = x_{g,h} \otimes P_{g,h}$ for all $g, h \in G$, then

$$\varphi^{**}(x_{g,h})\eta = \pi_g(\tilde{\varphi}(x_{g,h} \otimes P_{g,h}))\iota_h\eta = \pi_g((1 \otimes P_{g,g})\tilde{\varphi}(x)(1 \otimes P_{h,h}))\iota_h\eta = \tilde{\varphi}(x)_{g,h}\eta.$$

Hence $\tilde{\varphi}(x)_{g,h} = \varphi(x_{g,h})$ for all $g, h \in G$ and $x \in A \mathbin{\overline{\otimes}} B(\ell^2(G))$. For all $x \in A \mathbin{\overline{\otimes}} B(\ell^2(G))$ and $g, h, s \in G$,

$$\tilde{\varphi}(gx)_{h,s} = \varphi((gx)_{h,s}) = \varphi(x_{g^{-1}h,g^{-1}s}) = \tilde{\varphi}(x)_{g^{-1}h,g^{-1}s} = (g\tilde{\varphi}(x))_{h,s},$$

so that $\tilde{\varphi}$ is G-equivariant.

If φ is G-equivariant, one shows easily that $\tilde{\varphi}(M(A, G)) \subseteq M(B, G)$ and $\tilde{\varphi}(A \rtimes_r G) \subseteq B \rtimes_r G$. If φ is a *-homomorphism, then so are φ^{**}, $\tilde{\varphi}$ and the restriction of $\tilde{\varphi}$ to any C^*-subalgebra of $B(H \otimes \ell^2(G))$ contained in the operator system $A \mathbin{\overline{\otimes}} B(\ell^2(G))$.

If φ is a complete isometry, then due to [89, Proposition 2.6.13] any $x \in A \mathbin{\overline{\otimes}} B(\ell^2(G))$ satisfies

$$\begin{aligned}\|\tilde{\varphi}(x)\| &= \sup\{\|(\varphi(x_{gh}))_{g,h\in F}\| \mid F \subseteq G \text{ finite}\}\\ &= \sup\{\|(x_{gh})_{g,h\in F}\| \mid F \subseteq G \text{ finite}\}\\ &= \|x\|.\end{aligned}$$

For any $n \geq 1$, we have isomorphisms

$$M_n(A^{**} \mathbin{\overline{\otimes}} B(\ell^2(G))) \cong M_n(A^{**}) \mathbin{\overline{\otimes}} B(\ell^2(G)) \cong M_n(A)^{**} \mathbin{\overline{\otimes}} B(\ell^2(G))),$$

and similar ones for B. Therefore $\tilde{\varphi}^{(n)}$ can be identified with the u.c.p. map

$$M_n(A)^{**} \mathbin{\overline{\otimes}} B(\ell^2(G)) \to M_n(B)^{**} \mathbin{\overline{\otimes}} B(\ell^2(G))$$

induced by $\varphi^{(n)}$. The argument that $\tilde{\varphi}$ is an isometry therefore applies to $\tilde{\varphi}^{(n)}$, so $\tilde{\varphi}$ is a complete isometry. Since $\|\tilde{\varphi}(\pi(x))\| = \|\varphi(x)\|$ for all $x \in A$, where $\pi\colon A \to B(H \otimes \ell^2(G))$ is the regular representation, the converse is clear. ☐

We next give another characterization of G-injectivity for unital C^*-algebras, due to Hamana ([78, Lemma 3.1], [79, Lemma 3.2]). We give a self-contained proof that only requires the theorem of Choi and Effros (Corollary 5.2.12).

Lemma 5.3.3. *Let A be a unital G-C^*-algebra. Then A is G-injective if and only if $M(A, G)$ is injective.*

Proof. If $A \subseteq B(H)$ is G-injective, then A is injective by Lemma 5.1.6. Hence there exists a conditional expectation $B(H) \to A$ that extends to a u.c.p. projection $B(H \otimes \ell^2(G)) \to A \mathbin{\overline{\otimes}} B(\ell^2(G))$ due to Lemma 5.3.2. Therefore $\mathscr{A} = A \mathbin{\overline{\otimes}} B(\ell^2(G))$ is injective.

If $\kappa\colon A \to \mathscr{M} = \ell^\infty(G, B(H))$ is the canonical inclusion map, let $\omega\colon \mathscr{M} \to A$ be a G-equivariant u.c.p. map such that $\omega \circ \kappa = \mathrm{id}_A$. Let $\rho_g\colon G \to \mathcal{U}(H \otimes \ell^2(G))$ be the right regular representation, i.e., $\rho_g(\xi \otimes \delta_t) = \xi \otimes \delta_{tg^{-1}}$ for $\xi \in H$ and $g, t \in G$. We define a u.c.p. map

$$\tau'\colon \mathscr{A} \to \ell^\infty(G, B(H \otimes \ell^2(G))) \subseteq B((H \otimes \ell^2(G)) \otimes \ell^2(G))$$

by $\tau'(x)(g) = \rho_g x \rho_g^*$ for $g \in G$, and let $U \in \mathcal{U}(H \otimes \ell^2(G) \otimes \ell^2(G))$ be the unitary given by $U(\xi \otimes \delta_g \otimes \delta_h) = \xi \otimes \delta_h \otimes \delta_g$ for $\xi \in H$ and $g, h \in G$.

For all $g, h, s \in G$, one may show that $(U\tau'(x)U^*)_{g,h}(\xi \otimes \delta_s) = x_{gs,hs}\xi \otimes \delta_s$ for $\xi \in H$, meaning that $(U\tau(x)U^*)_{g,h} \in \ell^\infty(G, A)$ and $(U\tau'(x)U^*)_{g,h}(s) = x_{gs,hs}$ for all $s \in G$. We define $\tau(x) = U\tau'(x)U^* \in \ell^\infty(G, A) \mathbin{\overline{\otimes}} B(\ell^2(G))$ for all $x \in \mathscr{A}$.

Letting $\tilde{\omega}\colon \mathscr{M} \mathbin{\overline{\otimes}} B(\ell^2(G)) \to \mathscr{A}$ be the G-equivariant u.c.p. map induced by ω, then $\psi = \tilde{\omega} \circ \tau$ is a u.c.p. map. Let $g, h \in G$. Since

$$(s^{-1}\tau(x)_{g,h})(t) = \tau(x)_{g,h}(st) = x_{gst,hst} = \tau(x)_{gs,hs}(t)$$

for all $s, t \in G$, we have

$$s^{-1}\psi(x)_{g,h} = s^{-1}\omega(\tau(x)_{g,h}) = \omega(s^{-1}\tau(x)_{g,h}) = \omega(\tau(x)_{gs,hs}) = \psi(x)_{gs,hs}$$

for all $s \in G$. Hence $\psi(x) \in M(A,G)$. If $x \in M(A,G)$, then $\tau(x)_{g,h}(s) = x_{gs,hs} = s^{-1}x_{g,h} = \kappa(x_{g,h})(s)$ for all $s \in G$, so that

$$\psi(x)_{g,h} = \omega(\tau(x)_{g,h}) = \omega(\kappa(x_{g,h})) = x_{g,h}.$$

Since g and h were arbitrary, $\psi(x) = x$. This means that $M(A,G)$ is injective, since $\psi\colon \mathscr{A} \to M(A,G)$ is a u.c.p. projection.

If $M(A,G)$ is injective, the existence of a conditional expectation $E\colon M(A,G) \to A$ implies that A is injective. Hence $\mathscr{A} = A \overline{\otimes} B(\ell^2(G))$ can be endowed with the structure of a uniquely determined unital C^*-algebra. We then claim that $M(A,G)$ is in fact a C^*-subalgebra of $\mathscr{A}$. Indeed, for any $g \in G$, then by Lemma 5.3.2 the automorphism $g\colon A \to A$ extends to a complete order isomorphism $\tilde{g}\colon \mathscr{A} \to \mathscr{A}$, which is a *-automorphism of $\mathscr{A}$ when the latter is viewed as a C^*-algebra. Then one may check that $M(A,G)$ is the fixed point algebra of $\mathscr{A}$ with respect to the automorphisms $\tilde{g} \circ \mathrm{Ad}(\rho_g)$, $g \in G$.

Now let $\Psi\colon \mathscr{A} \to M(A,G)$ be a u.c.p. projection. Then Ψ is a conditional expectation, so that $\lambda_g \in M(A,G) \subseteq \mathrm{mult}(\Psi)$ with respect to the C^*-algebra structure on $\mathscr{A}$ for $g \in G$. In particular, Ψ is G-equivariant. Since $\ell^\infty(G,A)$ is a G-invariant G-operator subsystem of $\mathscr{A}$, we may define a G-equivariant u.c.p. map $\psi\colon \ell^\infty(G,A) \to A$ by $\psi = E \circ \Psi$. The canonical inclusion map $A \to \ell^\infty(G,A)$ coincides with the inclusion $A \subseteq M(A,G)$, and so $\psi \circ \kappa = \mathrm{id}_A$, implying that A is G-injective. $\square$

We finally give a proof of Hamana's characterization [79, Theorem 3.4] of G-injective envelopes via the injective envelope of a reduced crossed product.

Theorem 5.3.4. *Let A and B be unital G-C^*-algebras and suppose B contains A as a unital G-invariant C^*-subalgebra. Then the following are equivalent:*

(i) *There exists a unital complete isometry $B \rtimes_r G \to I(A \rtimes_r G)$ extending the canonical inclusion map $A \rtimes_r G \to I(A \rtimes_r G)$.*
(ii) *There exists a G-equivariant injective *-homomorphism $B \to I_G(A)$ extending the canonical inclusion $A \to I_G(A)$.*

Therefore $B \rtimes_r G$ is an essential extension of $A \rtimes_r G$ if and only if B is a G-essential extension of A.

Proof. (i) $\Rightarrow$ (ii): If $\iota\colon A \to B$ is the inclusion map, let $\tilde{\iota}\colon A \overline{\otimes} B(\ell^2(G)) \to B \overline{\otimes} B(\ell^2(G))$ be the induced G-equivariant complete isometry of Lemma 5.3.2. Lemma 5.2.8 then implies that $(B \rtimes_r G, \tilde{\iota})$ is an essential extension of $A \rtimes_r G$. Let $\mathscr{M} = \ell^\infty(G, B(H))$ for a Hilbert space H, and let $\varphi\colon B \to \mathscr{M}$ be a G-equivariant u.c.p. map such that $\varphi \circ \iota$ is completely isometric. If $\tilde{\varphi}\colon B \overline{\otimes} B(\ell^2(G)) \to \mathscr{M} \overline{\otimes} B(\ell^2(G))$ is the induced map of Lemma 5.3.2, then $\tilde{\varphi} \circ \tilde{\iota}|_{A \rtimes_r G}$ is completely isometric, so that $\tilde{\varphi}|_{B \rtimes_r G}$ is completely isometric. Thus φ is completely isometric, and (B, ι) is a G-essential extension of A. (Note that any G-operator system embeds equivariantly into $\mathscr{M}$ of the above form.)

(ii) $\Rightarrow$ (i): We assume that $A \subseteq I_G(A)$ as a G-invariant C^*-subalgebra. It suffices to show that $I_G(A) \rtimes_r G$ is an essential extension of $A \rtimes_r G$, as (ii) implies the existence of an injective *-homomorphism $B \rtimes_r G \to I_G(A) \rtimes_r G$ that extends the inclusion $A \rtimes_r G \subseteq I_G(A) \rtimes_r G$.

Let $\kappa\colon A \rtimes_r G \to I(A \rtimes_r G)$ be the canonical embedding. Since $M(I_G(A), G)$ is injective by Lemma 5.3.3 and contains $I_G(A) \rtimes_r G \supseteq A \rtimes_r G$, there is a complete isometry $\tau\colon I(A \rtimes_r G) \to M(I_G(A), G)$ such that $\tau \circ \kappa$ is the inclusion $A \rtimes_r G \subseteq M(I_G(A), G)$. Moreover, injectivity yields a u.c.p. map $\psi\colon I_G(A) \rtimes_r G \to I(A \rtimes_r G)$ such that $\psi|_{A \rtimes_r G} = \kappa$. Letting $E\colon M(I_G(A), G) \to I_G(A)$ be the canonical conditional expectation, we consider the diagram

$$\begin{tikzcd}
A \arrow[r] & I_G(A) \arrow[r, shift left] \arrow[r, bend left=40, "\tau\circ\psi"] & M(I_G(A), G). \arrow[l, shift left, "E"]
\end{tikzcd}$$

We note that $M(I_G(A), G)$ is injective and thus has a C^*-algebra structure, with respect to which the inclusion $I_G(A) \subseteq M(I_G(A), G)$ and E are still G-equivariant u.c.p. maps. Since $\tau \circ \psi|_A$ is the inclusion $A \subseteq I_G(A) \subseteq M(I_G(A), G)$, it follows from Lemma 5.3.1 that $\tau \circ \psi$ is the inclusion $I_G(A) \subseteq M(I_G(A), G)$. As $\tau(\psi(\lambda_g)) = \lambda_g$ for all $g \in G$ as well, $\tau \circ \psi$ is the identity map on all of $I_G(A) \rtimes_r G$ since $I_G(A) \cup C_r^*(G) \subseteq \operatorname{mult}(\tau \circ \psi)$. Therefore ψ is a complete isometry, so (i) holds. $\square$

5.4 Injective envelopes and the intersection property

In this section, we present our results from [26] on equivariant injective envelopes in view of wanting to uncover the ideal structure of a reduced crossed product. As noted in the beginning of Section 4.3, there is no immediate way of discerning this structure by means of information about the given C^*-dynamical system in the general setting, and most well-known ways of obtaining a clean picture of the ideals in the reduced crossed product are criteria ensuring simplicity or primeness (such as Proposition 3.2.3).

The results of Kalantar and Kennedy on C^*-simplicity were first realized by means of the G-injective envelope. In fact, the result that $I_G(\mathbb{C}) \rtimes_r G = C(\partial_F G) \rtimes_r G$ is simple whenever G is C^*-simple was simple was derivable from Hamana's original results: as $I_G(\mathbb{C}) \rtimes_r G \subseteq I(C_r^*(G))$ by Theorem 5.3.4 and any proper ideal J of $I_G(\mathbb{C}) \rtimes_r G$ satisfies $JC_r^*(G) \cup C_r^*(G)J \subseteq J$, simplicity of $C_r^*(G)$ will imply $J = \{0\}$ by Lemma 5.2.16. This argument is remarkably easy (once the theory is understood), and we will therefore discuss how injective envelopes may provide insight to the question of the ideal structure of a reduced crossed product. Many of our results are generalizations of recent ones by Kawabe [92] from the commutative to the non-commutative case.

In order to present our line of thinking as clearly as possible, we require the following notion, coined by Svensson and Tomiyama in [132].

Definition 5.4.1. Let G be a discrete group and let A be a G-C^*-algebra. We say that the action of G on A has the *intersection property* if every non-zero ideal of the reduced crossed product $A \rtimes_r G$ has non-zero intersection with A. The action of G on A is said to have the *residual intersection property* if it holds for all G-invariant ideals $I \subseteq A$ that the induced action of G on A/I has the intersection property.

One of the first necessary and sufficient criteria for the intersection property to hold was due to Kawamura and Tomiyama [93], in the case of an amenable group acting on a unital commutative C^*-algebra $C(X)$: the action on $C(X)$ has the intersection property if and only if the action on X is topologically free.

The best possible situation for uncovering the ideal structure problem is when a unital G-C^*-algebra A *separates the ideals* of the reduced crossed product $A \rtimes_r G$, i.e., when the map $I \mapsto I \rtimes_r G$ from the set of G-invariant ideals in A to the set of ideals in $A \rtimes_r G$ is a bijection. A theorem of Sierakowski [131, Theorem 1.10] gives a picture of exactly when this happens: A separates the ideals of $A \rtimes_r G$ if and only if the action of G on A is exact (see Section 4.3) and has the residual intersection property.

The class of exact groups is comparatively large – indeed it was not known until 2003 whether non-exact groups did exist, when Gromov gave the first example of a finitely generated, non-exact discrete group [66]. Due to Sierakowski's theorem and the abundance of exact groups (ensuring exactness of group actions), our aim will be to find criteria for the group action to satisfy the (residual) intersection property. As noted by Sierakowski, the action of G on A has the residual intersection property whenever $I \subseteq (I \cap A)\bar{\rtimes}_r G$ for any ideal $I \subseteq A \rtimes_r G$ (equality holds when the action is also exact). Recall from Section 4.3 that for any G-invariant ideal $Y \subseteq A$, $Y \bar{\rtimes}_r G \subseteq A \rtimes_r G$ is the kernel of the *-homomorphism $A \rtimes_r G \to (A/Y) \rtimes_r G$ induced by the quotient *-homomorphism $A \to A/Y$.

The *idée fixe* of our results was originally to consider what made the structure theorem in the previous chapter for maximal ideals (Theorem 4.3.4) possible. The strategy there, based on the proof of [23, Theorem 7.1], was to amplify a reduced crossed product by means of a natural extension with respect to the Furstenberg boundary, the centre of which a C^*-simple group could act freely on. Since $I_G(\mathbb{C}) = C(\partial_F G)$, one could ask whether $I_G(\mathbb{C})$ would always be contained in the centre of the G-injective envelope of any unital G-C^*-algebra A (in the same way it is contained in the natural extension $A \otimes C(\partial_F G)$). If one could then find a connection between the intersection property on A and $I_G(A)$, this might give another indication of why the aforementioned theorem is true, and perhaps even generalize the theorem.

As we shall now see, realizing parts of this idea is possible, but not all of it. The good news first, though: the following lemma is inspired by [23, Lemma 7.2].

Lemma 5.4.2. *Let A be a unital G-C^*-algebra and let B be a G-essential C^*-algebra extension of A. For any ideal $I \subseteq A \rtimes_r G$, let J be the ideal in $B \rtimes_r G$ generated by I. Then $I \cap A = \{0\}$ if and only if $J \cap B = \{0\}$.*

Proof. The "if" part is trivial. For the converse, we may assume that B is a unital G-invariant C^*-subalgebra of $I_G(A)$ containing A, by Remark 5.2.14. Let $\pi\colon A \rtimes_r G \to \mathscr{M}$ be a G-equivariant unital *-homomorphism with $\ker \pi = I$, where $\mathscr{M}$ is a G-injective G-C^*-algebra (for instance, we may take $\mathscr{M} = \ell^\infty(G, B(H))$, $(A \rtimes_r G)/I$ being represented faithfully on $B(H)$). By G-injectivity, we can extend π to a G-equivariant u.c.p. map $\tilde{\pi}\colon B \rtimes_r G \to \mathscr{M}$.

Let J be the ideal in $B \rtimes_r G$ generated by I and assume that $I \cap A = \{0\}$. As $I_G(A)$ is G-injective and π is completely isometric on A, there is a G-equivariant u.c.p. map $\varphi\colon \mathscr{M} \to I_G(A)$ such that $\varphi \circ \tilde{\pi}|_A = \varphi \circ \pi|_A$ is the inclusion of A into $I_G(A)$. Now let $\psi\colon I_G(A) \rtimes_r G \to \mathscr{M}$ be a G-equivariant u.c.p. map such that $\psi|_{B \rtimes_r G} = \tilde{\pi}$, so that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d] \arrow[dd, bend right] & B \arrow[r, hook] \arrow[d] & I_G(A) \arrow[d] \\
A \rtimes_r G \arrow[r] \arrow[dr, "\pi"] & B \rtimes_r G \arrow[r] \arrow[d, "\tilde{\pi}"] & I_G(A) \rtimes_r G \arrow[dl, dashed, "\psi"] \\
I_G(A) & \mathscr{M} \arrow[l, dashed, "\varphi"] &
\end{tikzcd}$$

Since $\varphi \circ \psi|_A$ is the inclusion map $A \to I_G(A)$, G-rigidity yields $\varphi \circ \psi|_{I_G(A)} = \mathrm{id}_{I_G(A)}$. In particular, $\varphi \circ \tilde{\pi}|_B$ is the identity map on B. It follows that $\tilde{\pi}(B) \subseteq \mathrm{mult}(\varphi)$, so that φ is a *-homomorphism on $C^*(\tilde{\pi}(B))$. Equipping $C^*(\tilde{\pi}(B))$ with the G-action given by conjugation by the unitaries $\pi(\lambda_g)$, $K = \ker \varphi \cap C^*(\tilde{\pi}(B))$ is a G-invariant ideal in $C^*(\tilde{\pi}(B))$. Now define

$$D = C^*(\tilde{\pi}(B \rtimes_r G)) = \overline{\mathrm{span}}(C^*(\tilde{\pi}(B)) \cdot \pi(C_r^*(G)))$$

and

$$L = \overline{\mathrm{span}}(K \cdot \pi(C_r^*(G))).$$

Both D and L are G-invariant C^*-subalgebras of $\mathscr{M}$. For any $g, h \in G$ and $x, y \in C^*(\tilde{\pi}(B))$ we see that

$$x\pi(\lambda_g)y\pi(\lambda_h) = x(\pi(\lambda_g)y\pi(\lambda_g)^*)\pi(\lambda_{gh}).$$

Hence if either x or y belongs to K, then $x\pi(\lambda_g)y\pi(\lambda_h) \in L$, so L is a G-invariant ideal of D.

Let $\phi\colon D \to D/L$ be the quotient map and let $(e_i)_{i\in I}$ be an approximate unit for K. Then $(e_i)_{i\in I}$ is an approximate unit for L as well, and any $d \in D$ belongs to L if and only if $e_i d \to d$. Therefore $L \cap C^*(\tilde{\pi}(B)) = K$. Now $\Phi = \phi \circ \tilde{\pi}\colon B \rtimes_r G \to D/L$ is multiplicative on $C_r^*(G)$, and since $\tilde{\pi}(x)^*\tilde{\pi}(x) - \tilde{\pi}(x^*x) \in \ker \varphi = K \subseteq L$ for all $x \in B$, it follows that Φ is a *-homomorphism. Note furthermore that $I \subseteq \ker \Phi$, so $J \subseteq \ker \Phi$ as well, and that Φ is G-equivariant. Finally, if $\Phi(x) = 0$ for $x \in B$ then $\tilde{\pi}(x) \in L \cap C^*(\tilde{\pi}(B)) = K$. Thus $x = \varphi(\tilde{\pi}(x)) = 0$ and $\ker \Phi \cap B = \{0\}$. This completes the proof. ☐

Theorem 5.4.3. *Let A be a G-C^*-algebra and let B be a G-essential C^*-algebra extension of A. Then the action of G on A has the intersection property if and only if the action of G on B has the intersection property.*

Proof. If $I \subseteq A \rtimes_r G$ is an ideal such that $I \cap A = \{0\}$, then let $J \subseteq B \rtimes_r G$ be the ideal generated by I. Since Lemma 5.4.2 yields that $J \cap B = \{0\}$, then if the action of G on B has the intersection property, it follows that $I \subseteq J = \{0\}$. Conversely, if $J \subseteq B \rtimes_r G$ is an ideal for which $J \cap B = \{0\}$, then $J \cap A = \{0\}$. Therefore, if the action of G on A has the intersection property, we have $J \cap (A \rtimes_r G) = \{0\}$. By Theorem 5.3.4 we may embed $B \rtimes_r G$ into $I(A \rtimes_r G)$ as a G-C^*-subalgebra containing $A \rtimes_r G$. Since $xJ + Jx \subseteq J$ for all $x \in A \rtimes_r G$, it follows from Lemma 5.2.16 that $J = \{0\}$. ☐

The above theorem generalizes part of a recent result due to Kawabe [92, Theorem 3.4]: for a compact G-space X, his result also equates the action of G on $C(X)$ having the intersection property to the action of G on the maximal ideal space Y of the commutative C^*-algebra $I_G(C(X))$ being free (see Remark 5.2.15 (ii)). This can be seen by first noting that the stabilizer subgroups in G of points in Y are amenable. Indeed, an invariant mean on $\ell^\infty(G_y)$ for any $y \in Y$ is obtained by the composition

$$\ell^\infty(G_y) \overset{\iota}{\hookrightarrow} \ell^\infty(G) \hookrightarrow \ell^\infty(G, C(Y)) \xrightarrow{\psi} C(Y) \xrightarrow{\delta_y} \mathbb{C}.$$

Here $\iota\colon \ell^\infty(G_y) \to \ell^\infty(G)$ denotes a G_y-equivariant inclusion (cf. p. 34), and the map $\psi\colon \ell^\infty(G, C(Y)) \to I_G(C(X)) = C(Y)$ is G-equivariant, u.c.p., and satisfies $\psi \circ \kappa = \mathrm{id}_{C(Y)}$ where $\kappa\colon C(Y) \to \ell^\infty(G, C(Y))$ is the canonical inclusion map (Lemma 5.2.2). Since Y is Stonean, the equivalence now follows from the theorems of Archbold and Spielberg (Propositions 3.2.3 and 3.2.4). We suspect that there is an appropriate notion of freeness of action that permits Kawabe's result to generalize to the non-commutative case as well, although we have not been able to realize which one.

We also note that Kawabe's result is a generalization of the Kalantar-Kennedy theorem (Theorem 3.2.6).

We next consider the centre of a G-injective C^*-algebra. Claim (i) in the following lemma is inspired by [79, Lemma 6.2], and claim (ii) is probably known, but we have been unable to find a reference for it. We first recall that the centre of a monotone closed C^*-algebra is monotone closed [128, Proposition 2.1.30], and that a commutative C^*-algebra is an AW^*-algebra if and only if it is injective (see, e.g., [70, Remark 2.5]).

Lemma 5.4.4. *Let A be a unital G-C^*-algebra. Then:*

(i) *There is a G-equivariant injective *-homomorphism $Z(A) \to Z(I_G(A))$.*
(ii) *If A is G-injective, then so is $Z(A)$.*

Proof. Let $\mathscr{M} = \ell^\infty(G, I(A))$, let $\kappa\colon A \to \mathscr{M}$ be a G-equivariant inclusion and let $\varphi\colon \mathscr{M} \to \mathscr{M}$ be a $\kappa(A)$-projection so that $I_G(A)$ can be taken to be the image $\varphi(\mathscr{M})$ with the Choi-Effros product. For $x \in Z(\mathscr{M})$ and $y \in \varphi(\mathscr{M})$,

$$\varphi(x) \circ y = \varphi(\varphi(x)y) = \varphi(xy) = \varphi(yx) = \varphi(y\varphi(x)) = y \circ \varphi(x)$$

by Lemma 5.2.9 and thus $\varphi(Z(\mathscr{M})) \subseteq Z(I_G(A))$. To see that (i) holds, note that $Z(A) \subseteq A' \cap I(A) = Z(I(A))$ by Remark 5.2.15 (i), so that $\kappa(Z(A)) \subseteq Z(\mathscr{M})$. Since $\kappa\colon A \to I_G(A)$ is a *-homomorphism and $\varphi \circ \kappa = \kappa$, (i) follows.

For (ii), note first that because A is monotone closed by Corollary 5.2.12, $Z = Z(A)$ is commutative and monotone closed and is therefore injective. Hence by Lemma 5.1.6 we only need to find a G-equivariant u.c.p. map $\varphi\colon \ell^\infty(G, Z) \to Z$ satisfying $\varphi \circ \kappa = \mathrm{id}_Z$ where κ is the canonical inclusion map.

Consider instead the canonical inclusion map $\kappa\colon A \to \mathscr{M} = \ell^\infty(G, A)$ for A and let $\varphi\colon \mathscr{M} \to A$ be a G-equivariant u.c.p. map such that $\varphi \circ \kappa = \mathrm{id}_A$. Since $\kappa(A) \subseteq \mathrm{mult}(\varphi)$, then for $z \in \ell^\infty(G, Z) = Z(\mathscr{M})$ we have

$$\varphi(z)x = \varphi(z)\varphi(\kappa(x)) = \varphi(z\kappa(x)) = \varphi(\kappa(x)z) = \varphi(\kappa(x))\varphi(z) = x\varphi(z)$$

whenever $x \in A$, so $\varphi(z) \in Z$. Thus φ maps $\ell^\infty(G, Z)$ into Z and $\varphi \circ \kappa|_Z = \mathrm{id}_Z$, so Z is indeed G-injective. ☐

For any G-operator system S, G-essentiality implies that there always exists a G-equivariant complete isometry $I_G(\mathbb{C}) \to S$ whenever S is G-injective. If we could always embed $I_G(\mathbb{C}) = C(\partial_F G)$ into any unital, commutative, G-injective G-C^*-algebra as a G-equivariant C^*-subalgebra, then any C^*-simple group G would act freely on the centre of any G-injective unital G-C^*-algebra by Theorem 3.2.6. Therefore, by Lemma 4.3.2 the action of G on all unital G-injective G-C^*-algebras would have the intersection property, and so the action of G on *all* unital G-C^*-algebras would have the intersection property due to Theorem 5.4.3. This sounds too good to be the case, and indeed, it is not – we have already seen an example.

Remark 5.4.5. Let A be the unital commutative G-C^*-algebra from Remark 4.3.7, that is, the unitization of $c_0(G/H)$ for some amenable subgroup $H \neq \{1\}$ of G. Then the action of G on A does not have the intersection property, as any proper ideal I of $c_0(G/H) \rtimes_r G$ satisfies $I \cap A = \{0\}$. Since $c_0(G/H) \rtimes_r G$ is Morita equivalent to $C^*(H)$ and $H \neq \{1\}$, it always contains at least one proper closed ideal, which is also a proper ideal of $A \rtimes_r G$.

If A is as given in Remark 5.4.5, $C(\partial_F G)$ is not a G-invariant C^*-subalgebra of the G-injective envelope $I_G(A)$ due to the above discussion. To clarify why this is, we observe that this G-injective envelope actually has a very concrete realization.

Proposition 5.4.6. *Let X be a discrete G-space. If A is the unitization of $c_0(X)$, then $I(A) = \ell^\infty(X)$. If $X = G/H$ for an amenable subgroup H of G, then $I_G(A) = \ell^\infty(G/H)$, where $\ell^\infty(G/H)$ is equipped with the G-action by left translation.*

Proof. We adapt an argument of Paulsen [116, Proposition 3.5] to prove the first claim. If $\phi\colon \ell^\infty(X) \to \ell^\infty(X)$ is a unital and positive map such that ϕ fixes $c_0(X)$, then any positive $f \in \ell^\infty(X)$ is the supremum of an increasing net of functions in $c_0(X)$. Thus $f \leq \phi(f)$. For $c > 0$ such that $r1 \geq f$, then $r1 - f \leq \phi(r1 - f) = r1 - \phi(f)$, meaning that $f \geq \phi(f)$. Hence $\phi(f) = f$, so that $A \subseteq \ell^\infty(X)$ is a rigid inclusion. Since $\ell^\infty(X)$ is injective, $I(A) = \ell^\infty(X)$.

Identifying $\ell^\infty(G/H)$ with a G-invariant C^*-subalgebra of $\ell^\infty(G)$, the second claim follows from the fact that there exists a G-equivariant conditional expectation $\ell^\infty(G) \to \ell^\infty(G/H)$ when H is amenable [29, Theorem 6 (iv)]. □

If an amenable subgroup H of G admits a unital G-equivariant *-homomorphism $C(\partial_F G) \to I_G(A) = \ell^\infty(G/H)$, where A is the unitization of $c_0(G/H)$. Any such map passes to a G-equivariant continuous surjection $\varphi\colon \beta(G/H) \to \partial_F G$. In particular, any $g \in H$ fixes $\varphi(H) \in \partial_F G$, so $H = \{1\}$ by freeness of the action. This explains why $C(\partial_F G)$ cannot be identified with a G-invariant C^*-subalgebra of $I_G(A)$.

Remark 5.4.7. Because $\ell^\infty(G/H)$ is G-injective for all amenable subgroups H of G, we observe that we can actually give a quick proof of one half of Kennedy's characterization of C^*-simplicity (Remark 3.2.8, C). Indeed, let H be an amenable subgroup of G and define $X = G/H$. By G-injectivity and G-essentiality, there exists a G-equivariant u.c.p. map $C(\partial_F G) \to \ell^\infty(X)$, which passes to a G-equivariant continuous map $\beta X \to \mathcal{P}(\partial_F G)$ by duality.

Suppose that H is recurrent, so that there exists a finite subset $F \subseteq G \setminus \{1\}$ such that $F \cap gHg^{-1} \neq \emptyset$ for all $g \in G$. Then $X \subseteq \bigcup_{s \in F} X^s$. Therefore $\beta X \subseteq \bigcup_{s \in F} \overline{X^s} =$

$\bigcup_{s\in F}(\beta X)^s$, because X is dense in βX. In particular, if $M \subseteq \beta X$ is a minimal subset, then $M \subseteq \bigcup_{s\in F} M^s$. By Proposition 2.4.1, the map $\beta X \to \mathcal{P}(\partial_F G)$ passes to a continuous G-equivariant surjection $M \to \partial_F G$, meaning that $\partial_F G \subseteq \bigcup_{s\in F}(\partial_F G)^s$. Since $F \subseteq G\setminus\{1\}$, the action of G on $\partial_F G$ is not free, so that G is not C^*-simple.

In light of the above discussion, we will now try to find a commutative G-injective envelope that the centre of a given G-injective envelope always does contain. We recall that if D is an AW^*-algebra, then just as in the von Neumann algebra case, any element $x \in D$ has a central support [17, 1.1.6], i.e., there exists a smallest central projection $C_x \in D$ such that $C_x x = x$. Further, if $z \in Z(D)$, then $zx = 0$ if and only if $zC_x = 0$.

Theorem 5.4.8. *Let A be a unital G-C^*-algebra. Then there is a G-essential C^*-algebra extension B of A such that $I_G(Z(I(A))$ embeds into $Z(B)$ as a unital G-invariant C^*-subalgebra. In particular, $I_G(Z(I(A)))$ embeds into $Z(I_G(A))$ as a unital G-invariant C^*-subalgebra.*

Proof. Since any G-essential extension of $I(A)$ is also a G-essential extension of A by Lemma 5.2.8 and Proposition 5.2.18, we may assume that A is injective. Let

$$Z = Z(A),\ \mathscr{M} = \ell^\infty(G, A),\ \mathcal{Z} = \ell^\infty(G, Z) = Z(\mathscr{M}),$$

and let $\kappa\colon A \to \mathscr{M}$ be the canonical inclusion map. Then $\kappa(Z) \subseteq \mathcal{Z}$. Let $\Psi\colon A\otimes\mathcal{Z} \to \mathscr{M}$ be the *-homomorphism given by

$$\Psi(x \otimes f) = \kappa(x)f, \quad x \in A,\ f \in \mathcal{Z}.$$

Let $A_{\mathcal{Z}}$ be the image of Ψ. Then $A_{\mathcal{Z}}$ is a unital G-invariant C^*-subalgebra of $\mathscr{M}$, and we have the following commutative diagram:

$$\begin{tikzcd}
Z \arrow[rr, "\kappa|_Z"] \arrow[dd, hook] & & \mathcal{Z} \arrow[dl, "1\otimes \mathrm{id}_{\mathcal{Z}}"'] \arrow[dd, hook] \\
& A\otimes\mathcal{Z} \arrow[dr, "\Psi"] & \\
A \arrow[ur, "\mathrm{id}_A\otimes 1"] \arrow[rr, "\kappa"] & & A_{\mathcal{Z}}
\end{tikzcd}$$

Let $\Phi\colon \mathscr{M} \to \mathscr{M}$ be a $\kappa(A)$-projection so that $\Phi(\mathscr{M}) = I_G(A)$. Since $\kappa(A) \subseteq \mathrm{mult}(\Phi)$ it is easy to see that $\Phi(\mathcal{Z}) \subseteq \kappa(A)' = \mathcal{Z}$, so by Lemma 5.2.6 and Lemma 5.4.4 we can let $\chi\colon \mathcal{Z} \to \mathcal{Z}$ be a $\kappa(Z)$-projection such that $\chi(\mathcal{Z}) = I_G(Z)$ and $\chi \prec \Phi|_{\mathcal{Z}}$. Let $B \subseteq \mathscr{M}$ be the image of the G-equivariant u.c.p. map

$$\tilde{\Psi} = \Psi \circ (\mathrm{id}_A \otimes \chi)\colon A \otimes \mathcal{Z} \to \mathscr{M}.$$

Note that $(\mathrm{id}_A \otimes 1)(A) \subseteq \mathrm{mult}(\tilde{\Psi})$, since $\tilde{\Psi} \circ (\mathrm{id}_A \otimes 1) = \kappa$. Further, κ maps A into B and

$$\Phi(\tilde{\Psi}(x \otimes f)) = \Phi(\kappa(x)\chi(f)) = \kappa(x)\chi(f) = \tilde{\Psi}(x \otimes f), \quad x \in A,\ f \in \mathcal{Z}.$$

This proves that $\Phi|_B = \mathrm{id}_B$, so $B \subseteq \Phi(\mathscr{M}) = I_G(A)$ and therefore (B, κ) is a G-essential extension of A by Lemma 5.2.8.

We now claim that B has the structure of a unital G-C^*-algebra. First endow the G-injective envelope $I_G(Z)$ with the Choi-Effros product of χ, i.e., define $x \circ y = \chi(xy)$ for $x, y \in I_G(Z)$, so that $I_G(Z)$ is a unital G-C^*-algebra with the involution and norm of $\mathcal{Z}$ and the product $\circ$.

Since A is injective, it is an AW^*-algebra. By [17, Proposition 1.1.10.1] $\mathscr{M}$ is an AW^*-algebra as well. Fix $x \in A$ and let $C_x \in Z$ be the central support of x. Then $\kappa(C_x) \in \mathcal{Z}$ is the central support of $\kappa(x)$ in $\mathscr{M}$. Indeed, if $f \in \mathcal{Z}$ is a projection such that $f\kappa(x) = \kappa(x)$, then $f(g)$ is a central projection in A and $f(g)g^{-1}x = g^{-1}x$ for all $g \in G$. Thus $g^{-1}C_x \leq f(g)$ for all $g \in G$, so that $\kappa(C_x) \leq f$. Now, suppose that $\kappa(x)f = 0$ for some $f \in \mathcal{Z}$. Then $\kappa(C_x)f = 0$. Since $\chi \circ \kappa|_Z = \kappa|_Z$, we have $\kappa(Z) \subseteq \operatorname{mult}(\chi)$ and

$$\kappa(C_x)\chi(f) = \chi(\kappa(C_x))\chi(f) = \chi(\kappa(C_x)f) = 0.$$

Hence $\kappa(x)\chi(f) = 0$. Now, for $n \geq 1$, let

$$I_n = \{y \in A \otimes \mathcal{Z} \otimes M_n(\mathbb{C}) \mid (\tilde{\Psi} \otimes \mathrm{id}_n)(y^*y) = (\tilde{\Psi} \otimes \mathrm{id}_n)(yy^*) = 0\}.$$

Then I_n is a closed ideal of $A \otimes \mathcal{Z} \otimes M_n(\mathbb{C})$. For instance, if $y \in I_n$, $x \in A$, $f \in \mathcal{Z}$ and $b \in M_n(\mathbb{C})$, then

$$\begin{aligned}
&(\tilde{\Psi} \otimes \mathrm{id}_n)((x \otimes f \otimes b)yy^*(x \otimes f \otimes b)^*) \\
&= (\kappa(x) \otimes b)(\tilde{\Psi} \otimes \mathrm{id}_n)(y(1 \otimes f^*f \otimes 1)y^*)(\kappa(x) \otimes b)^* \\
&\leq \|f\|^2(\kappa(x) \otimes b)(\tilde{\Psi} \otimes \mathrm{id}_n)(yy^*)(\kappa(x) \otimes b)^* \\
&= 0,
\end{aligned}$$

and

$$(\tilde{\Psi} \otimes \mathrm{id}_n)(y^*(x \otimes f \otimes b)^*(x \otimes f \otimes b)y) \leq \|x\|^2\|f\|^2\|b\|^2(\tilde{\Psi} \otimes \mathrm{id}_n)(y^*y) = 0,$$

meaning that $(x \otimes f \otimes b)y \in I_n$.

If $x \in A$ and $z \in \mathcal{Z} \otimes M_n(\mathbb{C})$ satisfy $(\Psi \otimes \mathrm{id}_n)(x \otimes z) = 0$, write $z^*z = \sum_{i,j} f_{ij} \otimes e_{ij}$ with respect to the canonical basis (e_{ij}) of matrix units in $M_n(\mathbb{C})$. Then $\sum_{i,j} \kappa(x^*x)f_{ij} \otimes e_{ij} = 0$, so that $\kappa(x^*x)f_{ij} = 0$ for all i, j. By what we have seen above, $\kappa(x^*x)\chi(f_{ij}) = 0$ for all i, j as well, so that

$$(\tilde{\Psi} \otimes \mathrm{id}_n)((x \otimes z)^*(x \otimes z))) = (\Psi \otimes \mathrm{id}_n)((\mathrm{id}_A \otimes \chi \otimes \mathrm{id}_n)((x \otimes z)^*(x \otimes z))) = 0.$$

Therefore it follows that $x \otimes z \in I_n$ whenever $x \otimes z \in \ker(\Psi \otimes \mathrm{id}_n)$ for $x \in A$ and $z \in \mathcal{Z} \otimes M_n(\mathbb{C})$. Since $\mathcal{Z} \otimes M_n(\mathbb{C})$ is exact, any closed ideal in $A \otimes \mathcal{Z} \otimes M_n(\mathbb{C})$ is generated by the elementary tensors it contains [19, Propositions 2.16–2.17], so $\ker(\Psi \otimes \mathrm{id}_n) \subseteq I_n$.

The above discussion shows that we may define a unital G-equivariant map $\varepsilon\colon A_{\mathcal{Z}} \to A_{\mathcal{Z}}$ with image B by

$$\varepsilon(\Psi(y)) = \tilde{\Psi}(y) = \Psi((\mathrm{id}_A \otimes \chi)(y)), \quad y \in \mathcal{A} \otimes \mathcal{Z},$$

so that the following diagram commutes:

$$\begin{tikzcd}
A \otimes \mathcal{Z} \arrow[r, "\Psi"] \arrow[d, "\mathrm{id}_A \otimes \chi"'] \arrow[dr, "\tilde{\Psi}"] & A_{\mathcal{Z}} \arrow[r, hook] \arrow[d, dashed, "\varepsilon"] & \mathscr{M} \arrow[d, "\Phi"] \\
A \otimes \mathcal{Z} \arrow[r, "\Psi"] & B \arrow[r, hook] & I_G(A)
\end{tikzcd}$$

For any two $x, y \in A \otimes \mathcal{Z} \otimes M_n(\mathbb{C})$ such that $(\Psi \otimes \mathrm{id}_n)(x) = (\Psi \otimes \mathrm{id}_n)(y)$ we have seen that $x - y \in I_n$, so that $(\tilde{\Psi} \otimes \mathrm{id}_n)(x) = (\tilde{\Psi} \otimes \mathrm{id}_n)(y)$. Since any positive element in $A_{\mathcal{Z}} \otimes M_n(\mathbb{C})$ lifts via the *-homomorphism $\Psi \otimes \mathrm{id}_n$ to a positive element in $A \otimes \mathcal{Z} \otimes M_n(\mathbb{C})$ for all n, ε is completely positive. Furthermore, if we let $\tilde{\chi} = \mathrm{id}_A \otimes \chi$, then

$$\varepsilon(\varepsilon(\Psi(y))) = \varepsilon(\Psi(\tilde{\chi}(y))) = \Psi(\tilde{\chi}(\tilde{\chi}(y))) = \Psi(\tilde{\chi}(y)) = \varepsilon(\Psi(y)),$$

so ε is idempotent. By Theorem 5.2.10 the image B of ε is a unital C^*-algebra when endowed with the Choi-Effros product, i.e.,

$$x * y = \varepsilon(xy), \quad x, y \in B.$$

This proves the claim. Furthermore, B is completely order isomorphic to this unital C^*-algebra. Since $\kappa\colon A \to B$ is a *-homomorphism with respect to the product on B, this means that B is a genuine G-essential C^*-algebra extension of A.

Define a map $\delta\colon I_G(Z) \to B$ given by $\delta(x) = \Psi(1 \otimes x)$. We claim that δ is in fact a unital G-equivariant injective *-homomorphism of $(I_G(Z), \circ)$ into the centre of $(B, *)$. In fact, δ is the inclusion map $I_G(Z) = \chi(\mathcal{Z}) \subseteq \tilde{\Psi}(A_{\mathcal{Z}}) = B$; we elect to use the above expression, as it makes calculations a bit neater. First, $\delta \circ \kappa|_Z = \kappa|_Z$ so that δ is a complete isometry, since $(I_G(Z), \kappa|_Z)$ is a G-essential extension of Z. Next, for $x, y \in \mathcal{Z}$ we observe that

$$\begin{aligned}
\delta(\chi(x) \circ \chi(y)) &= \Psi(1 \otimes \chi(\chi(x)\chi(y))) \\
&= (\Psi \circ \tilde{\chi})(1 \otimes \chi(x)\chi(y)) \\
&= \varepsilon(\Psi(1 \otimes \chi(x)\chi(y))) \\
&= \varepsilon(\Psi(1 \otimes \chi(x))\Psi(1 \otimes \chi(y))) \\
&= \Psi(1 \otimes \chi(x)) * \Psi(1 \otimes \chi(y)) \\
&= \delta(\chi(x)) * \delta(\chi(y)).
\end{aligned}$$

Therefore δ is a *-homomorphism of $(I_G(Z), \circ)$ into $(B, *)$. Finally, for $x \in \mathcal{Z}$ and $y \in A \otimes \mathcal{Z}$ we have $1 \otimes \chi(x) \in Z(A \otimes \mathcal{Z})$, and therefore

$$\begin{aligned}
\delta(\chi(x)) * \tilde{\Psi}(y) &= \varepsilon(\Psi(1 \otimes \chi(x))\tilde{\Psi}(y)) \\
&= \varepsilon(\Psi((1 \otimes \chi(x))\tilde{\chi}(y))) \\
&= \varepsilon(\Psi(\tilde{\chi}(y)(1 \otimes \chi(x)))) \\
&= \varepsilon(\Psi(\tilde{\chi}(y))\Psi(1 \otimes \chi(x))) \\
&= \tilde{\Psi}(y) * \delta(\chi(x)).
\end{aligned}$$

Thus δ is an injective G-equivariant *-homomorphism into the centre of B.

Finally, since $I_G(B) = I_G(A)$ by Lemma 5.2.8, we see that $Z(B)$ embeds into $Z(I_G(B)) = Z(I_G(A))$ as a unital G-invariant C^*-subalgebra due to Lemma 5.4.4. This proves the second claim. □

Remark 5.4.9. If a unital G-C^*-algebra A is prime, then the G-C^*-algebra B constructed in the proof of Theorem 5.4.8 is easily seen to be isomorphic to $A \otimes I_G(\mathbb{C})$, since $I(A)$ has trivial centre (Remark 5.2.17). Therefore $A \otimes I_G(\mathbb{C})$ is a G-essential extension of A.

Because of the above theorem, we obtain the following criterion for a group action on a C^*-algebra to have the intersection property:

Theorem 5.4.10. *Let A be a unital G-C^*-algebra and let $I(A)$ denote the injective envelope of A. Whenever the action of G on $Z(I(A))$ has the intersection property, then so does the action of G on A and $I(A)$.*

Proof. Due to [92, Theorem 3.4] (see also the discussion after Theorem 5.4.3) the action of G on the maximal ideal space of the G-injective envelope $I_G(Z(I(A)))$ is free, so by Theorem 5.4.8 the action of G on the maximal ideal space of $Z(I_G(A))$ is also free. Therefore the action of G on $I_G(A)$ has the residual intersection property by Lemma 4.3.2. The conclusion then follows from Theorem 5.4.3. □

We have not been able to recognize whether the action of G on $I_G(A)$ having the residual intersection property implies something similar for the action of G on A in the above proof, since it is not clear whether there is a correspondence between the closed G-invariant ideals of A and the closed G-invariant ideals of $I_G(A)$.

We finally present some applications for the case when the group G is C^*-simple.

Corollary 5.4.11. *Let G be a C^*-simple group and let A be a unital G-C^*-algebra. If $Z(I(A))$ is G-simple, then the action of G on A has the intersection property.*

Proof. By [23, Theorem 7.1] (or Corollary 4.3.5), the crossed product $Z(I(A)) \rtimes_r G$ is simple, so the action of G on $Z(I(A))$ has the intersection property. Now apply Theorem 5.4.10. □

Corollary 5.4.12. *Let G be a C^*-simple group and let A be a unital G-C^*-algebra. If A is prime, then the action of G on A has the intersection property. In particular, $A \rtimes_r G$ is prime.*

Proof. Since A is prime, $I(A)$ has trivial centre (Remark 5.2.17), so by Corollary 5.4.11, the action of G on A has the intersection property.

If $J_1 \cap J_2 = \{0\}$ for ideals $J_1, J_2 \subseteq A \rtimes_r G$, then $(J_1 \cap A) \cap (J_2 \cap A) = \{0\}$ so $J_i \cap A = \{0\}$ and $J_i = \{0\}$ for some i. Therefore $A \rtimes_r G$ is prime. □

In light of the above result we recall from Remark 4.3.7 that C^*-simplicity in itself need not transform G-primeness of a C^*-algebra to primeness of the reduced crossed product. However, we are able to derive a weaker analogue of Theorem 4.3.4 for prime ideals:

Corollary 5.4.13. *Let G be a C^*-simple group and let A be a unital G-C^*-algebra. Then there is an injective map of the set of prime and G-invariant ideals to the set of prime ideals in $A \rtimes_r G$, given by $I \mapsto I \bar{\rtimes}_r G$.*

Proof. If $I \subseteq A$ is a prime, G-invariant ideal, then A/I is a prime C^*-algebra and $(A/I) \rtimes_r G$ is a prime C^*-algebra by Corollary 5.4.12. Thus $I \bar{\rtimes}_r G$ is a prime ideal of $A \rtimes_r G$, so the map $I \mapsto I \bar{\rtimes}_r G$ is well-defined, and it is injective since $(I \bar{\rtimes}_r G) \cap A = I$ for each G-invariant ideal $I \subseteq A$. □

Subjects for further research

In this final part of the thesis, we give an overview of some questions that have piqued our interest, some of which have been looked into in our research but that we have not been able to answer.

We have already discussed the first question in detail in Section 2.5:

Question 1. *Does there exist a non-trivial discrete group G, such that its action on the universal minimal compact G-space is proximal or strongly proximal?*

A simple unital C^*-algebra A, not isomorphic to the complex numbers, is said to be *purely infinite* if every non-zero hereditary C^*-subalgebra of A contains a projection that is Murray-von Neumann equivalent to the identity 1_A.

Question 2. *Let G be a discrete group and let X be a topologically free G-boundary (so that G is C^*-simple and $C(X) \rtimes_r G$ is simple by Theorem 3.2.6). Is $C(X) \rtimes_r G$ purely infinite?*

The above question has an answer in the affirmative for *some* discrete groups G and G-boundaries X. Indeed, some of the boundary actions considered in the present thesis satisfy a considerably stronger property, known as extreme proximality (a notion due to Glasner [61]). An action of G on a compact Hausdorff space X that contains more than two points is *extremely proximal* if for any proper, compact subset $K \subseteq X$ and non-empty open subset $U \subseteq X$ there exists $g \in G$ such that $gK \subseteq U$. Extremely proximal actions are boundary actions [61, Theorem 3.3], and they include

- the action of a non-elementary hyperbolic group G on its Gromov boundary ∂G [98, Example 2.1], and
- any action $G \mapsto \overline{\partial T}$ for a tree T satisfying the conditions of Proposition 2.2.14 [100, Proposition 4.26].

An example of a boundary action that is not extremely proximal is the action of $\mathrm{SL}(3, \mathbb{R})$ on real projective 2-space $\mathbb{P}^2(\mathbb{R})$ [61, pp. 332–333]. Finally, Laca and Spielberg gave a proof in 1996 that the reduced crossed product $C(X) \rtimes_r G$ is simple and purely infinite for any topologically free, extremely proximal compact G-space X [98, Theorem 5].

A 2000 result by Jolissaint and Robertson generalizes the above criterion of Laca and Spielberg. An action of a group G on a unital C^*-algebra A is *n-filling* if for any positive $x_1, \ldots, x_n \in A$ of norm 1 and $\varepsilon > 0$ there exist $g_1, \ldots, g_n \in G$ such that

$$\sum_{i=1}^{n} g_i x_i \geq 1 - \varepsilon.$$

If the action of G on a unital C^*-algebra A is properly outer and n-filling, and pAp is infinite-dimensional for all non-zero projections $p \in A$, the reduced crossed product $A \rtimes_r G$ is simple and purely infinite [88, Theorem 1.2].

For a compact G-space X, the action of G on $C(X)$ is n-filling if and only if for all non-empty open subsets $U_1, \ldots, U_n \subseteq X$ there exist $g_1, \ldots, g_n \in G$ such that $X = \bigcup_{i=1}^n g_i U_i$. If we say that an action of G on X with this latter property is also *n-filling*, the following holds:

- The action of $\mathrm{PSL}(n, \mathbb{Z})$ on $\mathbb{P}^{n-1}(\mathbb{R})$ for $n \geq 2$ is n-filling, but not $n-1$-filling [88, Example 2.1].
- Suppose that a discrete group G acts non-elementarily on a compact Hausdorff space X, such that $X = \overline{L_G}$ and G contains at least one hyperbolic homeomorphism of X. Then the action of G on X is n-filling for some $n \geq 2$ [88, Proposition 2.5].

Note also that the G-action on a compact G-space X is 2-filling if and only if it is extremely proximal.

The notion of a boundary action can be formulated so as to admit a non-commutative analogue, by the following observation, inspired by a theorem of Glasner and Furstenberg [61, Theorem 3.3], which might be of help in achieving a partial answer at the very least. If a discrete group G acts on a compact Hausdorff space X by homeomorphisms, then the following are equivalent:

(i) X is a G-boundary.

(ii) For all positive $f \in C(X)$ of norm 1 and $\varepsilon > 0$, there exist $g_1, \ldots, g_n \in G$ such that
$$\frac{1}{n} \sum_{i=1}^{n} g_i f \geq 1 - \varepsilon.$$

Proof. (i) $\Rightarrow$ (ii): Let $f \in C(X)$ be positive of norm 1, let $\varepsilon > 0$ and let $x \in X$ such that $f(x) = 1$. Now let (g_i) be a net in G such that $g_i \mu \to \delta_x$ in the weak* topology for all $\mu \in \mathcal{P}(X)$, by Corollary 2.1.17. It follows that $g_i f \to 1$ in the weak topology on $C(X)$. By the Hahn-Banach separation theorem, the identity 1 belongs to the norm-closed convex hull of $\{gf \mid g \in G\}$.

(ii) $\Rightarrow$ (i): Let $\mu \in \mathcal{P}(X)$, $x \in X$ and $\varepsilon > 0$. It suffices to find $g \in G$ such for any finite subset of functions $\{f_1, \ldots, f_k\} \subseteq C(X)$ where each f_j is positive, has norm 1 and equals 1 on a neighbourhood of x, then $|\mu(g^{-1} f_j) - f_j(x)| < \varepsilon$ for all $1 \leq j \leq k$. Indeed, the linear span of the set of such f_j is dense in $C(X)$ by the Stone-Weierstrass theorem.

Let U be an open neighbourhood of x such that $f_j|_U$ equals 1 for all j, and let $f \in C(X)$ be a positive norm-one function such that $\mathrm{supp}\,(f) \subseteq U$ and $f(x) = 1$, so that $f_j \geq f$ for all $1 \leq j \leq k$. Now there exist $g_1, \ldots, g_n \in G$ such that $\frac{1}{n} \sum_{i=1}^n g_i f \geq 1 - \varepsilon$. Hence there is $1 \leq i \leq n$ such that
$$\mu(g_i f_j) \geq \mu(g_i f) \geq 1 - \varepsilon = f_j(x) - \varepsilon$$
for all $1 \leq j \leq k$. $\square$

A result of Malyutin [102, Proposition 6] states that every minimal, proximal action of a discrete group G on the circle $\mathbb{T}$ is in fact automatically extremely proximal, so that the reduced crossed product $C(\mathbb{T}) \rtimes_r G$ is simple and purely infinite by the theorem of Laca and Spielberg. It would be interesting to know whether similar results held true for compact, connected manifolds of higher dimension.

In connection with our work on countable non-ascending HNN extensions in Sections 3.5 and 3.6, the following question is natural:

Question 3. *Let $\Gamma = \mathrm{HNN}(G, H, \theta)$ be a countable, strictly ascending HNN extension, i.e., one and only one of H and $\theta(H)$ equals G. Find criteria for Γ to be C^*-simple and/or have unique trace.*

This would be an especially interesting problem to attack because of the Thompson group F, which is isomorphic to a strictly ascending HNN extension of itself.

In Sections 4.3 and 4.4, we prove for any discrete group G and any multiplier $\sigma\colon G \times G \to \mathbb{T}$ that the twisted reduced group C^*-algebra $C^*_r(G, \sigma)$ is simple (resp. has a unique tracial state) whenever $C^*_r(G)$ is simple (resp. has a unique tracial state). Moreover, all C^*-simple groups have the unique trace property, as witnessed in Remark 3.2.7 and the result of Breuillard, Kalantar, Kennedy and Ozawa (Theorem 3.2.9). An answer to the following question would settle a “missing arrow”:

Question 4. *Let G be a discrete group and let $\sigma\colon G \times G \to \mathbb{T}$ be a multiplier. If $C^*_r(G, \sigma)$ is simple, does $C^*_r(G, \sigma)$ have unique trace?*

The converse has been shown to hold for any amenable group G, by Bédos and Omland [13], who have also proved that the properties are equivalent whenever G is FC-hypercentral. A vague idea for a solution would be to investigate if the behaviour of (G, σ) could be perhaps detected by the action of G on a Furstenberg boundary-like space, that would coincide with the original Furstenberg boundary for trivial σ.

Our final questions concern equivariant injective envelopes.

Question 5. *Does there exist an amenable discrete group G and an injective unital G-C^*-algebra A such that A is not G-injective?*

We gave an explanation at the end of Section 5.1 of why any injective von Neumann algebra A is G-injective whenever G is an amenable discrete group. Moreover, Argerami and Farenick have given a proof in [6, Theorem 2.2] that the injective envelope of a separable C^*-algebra A is a von Neumann algebra if and only if A contains a minimal essential ideal that is isomorphic to a direct sum of C^*-algebras of the form $K(H)$ (the compact operators on a Hilbert space H).

One idea for a proof of the general C^*-case could be to use the fact that any unital injective C^*-algebra is *perfect* [4, Theorem 2.1]. We briefly explain how to define the notion of perfection for C^*-algebras. The *atomic representation* of a unital C^*-algebra A is the direct sum $\pi_a = \bigoplus_{[\pi] \in \hat{A}} \pi$, where $\hat{A}$ denotes the set of equivalence classes of irreducible representations of A. Let $z \in A^{**}$ be the central support projection of the representation $\pi_a\colon A^{**} \to B(\bigoplus_{[\pi] \in \hat{A}} H_\pi)$. Now, A is said to be *perfect* if any $b \in zA^{**}$

such that b, b^*b and bb^* define weak*-continuous maps $P(A) \to \mathbb{C}$, actually belongs to zA. We refer to the memoir [1] of Akemann and Shultz for more information about perfect C^*-algebras.

Because of Theorem 5.4.10, another question is natural:

Question 6. *Let A be a unital G-C^*-algebra. Give sufficient criteria for G and A for the action of G on $Z(I(A))$ to have intersection property.*

The final two results of Section 5.4 focus on one of the simplest possible instances (when A is prime and G is C^*-simple). It would be interesting to find weaker conditions to ensure that Theorem 5.4.10 can be applied.

Bibliography

[1] C. A. Akemann and F. W. Shultz, *Perfect C^*-algebras*, Memoirs of the American Mathematical Society, 55 (1985), pp. xiii+117.

[2] R. Alperin and H. Bass, *Length functions of group actions on Λ-trees*, in Combinatorial group theory and topology (Alta, Utah, 1984), vol. 111 of Annals of Mathematics Studies, Princeton University Press, 1987, pp. 265–378.

[3] C. Anantharaman-Delaroche, *Action moyennable d'un groupe localement compact sur une algèbre de von Neumann*, Mathematica Scandinavica, 45 (1979), pp. 289–304.

[4] R. J. Archbold, *On perfect C^*-algebras*, Proceedings of the American Mathematical Society, 97 (1986), pp. 413–417.

[5] R. J. Archbold and J. S. Spielberg, *Topologically free actions and ideals in discrete C^*-dynamical systems*, Proceedings of the Edinburgh Mathematical Society, 37 (1992), pp. 119–124.

[6] M. Argerami and D. R. Farenick, *Local multiplier algebras, injective envelopes, and type I W^*-algebras*, Journal of Operator Theory, 59 (2008), pp. 237–245.

[7] W. B. Arveson, *Subalgebras of C^*-algebras*, Acta Mathematica, 123 (1969), pp. 141–224.

[8] E. Bédos, *Operator algebras associated with free products of groups with amalgamation*, Mathematische Annalen, 266 (1984), pp. 279–286.

[9] ——, *Discrete groups and simple C^*-algebras*, Mathematical Proceedings of the Cambridge Philosophical Society, 109 (1991), pp. 521–537.

[10] ——, *On the uniqueness of the trace of some simple C^*-algebras*, Journal of Operator Theory, 30 (1993), pp. 149–160.

[11] ——, *Simple C^*-crossed products with a unique trace*, Ergodic Theory and Dynamical Systems, 16 (1996), pp. 415–429.

[12] E. Bédos and R. Conti, *On maximal ideals in certain reduced twisted C^*-crossed products*, Mathematical Proceedings of the Cambridge Philosophical Society, 158 (2015), pp. 399–417.

[13] E. Bédos and T. Omland, *On twisted group C^*-algebras associated with FC-hypercentral groups and other related groups*, Ergodic Theory and Dynamical Systems, 36 (2016), pp. 1743–1756.

[14] B. Bekka, M. G. Cowling, and P. de la Harpe, *Simplicity of the reduced C^*-algebra of* $\mathrm{PSL}(n, \mathbb{Z})$, International Mathematics Research Notices, 7 (1994), pp. 285–291.

[15] ——, *Some groups whose reduced C^*-algebra is simple*, Publications Mathématiques de l'IHÉS, 80 (1994), pp. 117–134.

[16] M. B. Bekka and P. de la Harpe, *Groups with simple reduced C^*-algebras*, Expositiones Mathematicae, 18 (2000), pp. 215–230.

[17] S. K. Berberian, *Baer *-Rings*, no. 195 in Grundlehren der mathematischen Wissenschaften, Springer, 1972.

[18] B. Blackadar, *Operator Algebras: Theory of C^*-Algebras and von Neumann Algebras*, vol. 122 of Encyclopaedia of Mathematical Sciences, Springer, 2012.

[19] E. Blanchard and E. Kirchberg, *Non-simple purely infinite C^*-algebras: the Hausdorff case*, Journal of Functional Analysis, 207 (2004), pp. 461–513.

[20] F. Boca and V. Niţică, *Combinatorial properties of groups and simple C^*-algebras with a unique trace*, Journal of Operator Theory, 20 (1988), pp. 183–196.

[21] O. Bogopolski, *Introduction to Group Theory*, EMS Textbooks in Mathematics, European Mathematical Society, 1. ed., 2008.

[22] P. L. Bowers and K. Ruane, *Fixed points in boundaries of negatively curved groups*, Proceedings of the American Mathematical Society, 124 (1996), pp. 1311–1313.

[23] E. Breuillard, M. Kennedy, M. Kalantar, and N. Ozawa, *C^*-simplicity and the unique trace property for discrete groups*, 2016, `arXiv:1410.2518`.

[24] M. Bridson and A. Häfliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der mathematischen Wissenschaften, Springer Berlin Heidelberg, 2011.

[25] N. P. Brown and N. Ozawa, *C^*-algebras and Finite-Dimensional Approximations*, vol. 88 of Graduate Studies in Mathematics, American Mathematical Society, 2008.

[26] R. S. Bryder, *Injective envelopes and the intersection property*, 2017, `arXiv:1704.02723`.

[27] R. S. Bryder and M. Kennedy, *Reduced twisted crossed products over C^*-simple groups*, International Mathematics Research Notices, rnw296 (2016).

[28] R. C. Busby and H. A. Smith, *Representations of twisted group algebras*, Transactions of the American Mathematical Society, 149 (1970), pp. 503–537.

[29] P.-E. Caprace and N. Monod, *Relative amenability*, Groups, Geometry, and Dynamics, 8 (2014), pp. 747–774.

[30] I. Chiswell, *Introduction to Λ-trees*, World Scientific, 2001.

[31] M.-D. Choi, *A Schwarz inequality for positive linear maps on C^*-algebras*, Illinois Journal of Mathematics, 18 (1974), pp. 565–574.

[32] ——, *A simple C^*-algebra generated by two finite-order unitaries*, Canadian Journal of Mathematics, 31 (1979), pp. 867–880.

[33] M.-D. Choi and E. G. Effros, *Injectivity and operator spaces*, Journal of Functional Analysis, 24 (1977), pp. 156–209.

[34] H. B. Cohen, *Injective envelopes of Banach spaces*, Bulletin of the American Mathematical Society, 70 (1964), pp. 723–726.

[35] M. Coornaert, T. Delzant, and A. Papadopoulos, *Géométrie et théorie des groupes: Les groupes hyperboliques de Gromov*, Lecture Notes in Mathematics 1441, Springer-Verlag Berlin Heidelberg, 1 ed., 1990.

[36] M. M. Day, *Amenable semigroups*, Illinois Journal of Mathematics, 1 (1957), pp. 509–544.

[37] ——, *Fixed-point theorems for compact convex sets*, Illinois Journal of Mathematics, 5 (1961), pp. 585–590.

[38] Y. de Cornulier, *Infinite conjugacy classes in groups acting on trees*, Groups, Geometry, and Dynamics, 3 (2009), pp. 267–277.

[39] P. de la Harpe, *Reduced C^*-algebras of discrete groups which are simple with unique trace*, Springer Lecture Notes in Mathematics, 1132 (1985), pp. 230–253.

[40] ——, *Groupes hyperboliques, algèbres d'opérateurs et un théorème de Jolissaint*, Comptes Rendus des Séances de l'Académie des Sciences. Série I. Mathématique, 307 (1988), pp. 771–774.

[41] ——, *On simplicity of reduced C^*-algebras of groups*, Bulletin of the London Mathematical Society, 39 (2007), pp. 1–26.

[42] P. de la Harpe and M. Bridson, *Mapping class groups and outer automorphism groups of free groups are C^*-simple*, Journal of Functional Analysis, 212 (2004), pp. 195–205.

[43] P. de la Harpe and J.-P. Préaux, *C^*-simple groups: amalgamated free products, hnn extensions, and fundamental groups of 3-manifolds*, Journal of Topology and Analysis, 3 (2011), pp. 451–489.

[44] P. de la Harpe and G. Skandalis, *Powers' property and simple C^*-algebras*, Mathematische Annalen, 273 (1986), pp. 241–250.

[45] A. Derighetti, *Some remarks on $L^1(G)$*, Mathematische Zeitschrift, 164 (1978), pp. 189–194.

[46] J. Dixmier, *C^*-algebras*, North-Holland Publishing Company, 1977.

[47] S. Echterhoff, *Crossed products, the Mackey-Rieffel-Green machine and applications*, 2010, `arXiv:1006.4975`.

[48] R. Ellis, *Distal transformation groups*, Pacific Journal of Mathematics, 8 (1958), pp. 401–405.

[49] ——, *A semigroup associated with a transformation group*, Transactions of the American Mathematical Society, 94 (1960), pp. 272–281.

[50] ——, *Universal minimal sets*, Proceedings of the American Mathematical Society, 11 (1960), pp. 540–543.

[51] R. Engelking, *General topology*, vol. 6 of Sigma Series in Pure Mathematics, Heldermann Verlag, 1989.

[52] D. B. Epstein, J. W. Cannon, W. P. Thurston, M. S. Paterson, S. V. F. Levy, and D. F. Holt, *Word processing in groups*, Jones and Bartlett, 1992.

[53] R. Exel, *Exact groups and Fell bundles*, Mathematische Annalen, 323 (2002), pp. 259–266.

[54] J. Franks and M. Handel, *Global fixed points for centralizers and Morita's theorem*, Geometry & Topology, 13 (2009), pp. 87–98.

[55] Z. Frolík, *Maps of extremally disconnected spaces, theory of types, and applications*, in General Topology and Its Relations to Modern Analysis and Algebra, Academia Publishing House of the Czechoslovak Academy of Sciences, 1971, pp. 131–142.

[56] A. Furman, *On minimal strongly proximal actions of locally compact groups*, Isreal Journal of Mathematics, 136 (2003), pp. 173–187.

[57] H. Furstenberg, *A Poisson Formula for Semi-Simple Lie Groups*, Annals of Mathematics, 77 (1963), pp. 335–386.

[58] ——, *Boundary theory and stochastic processes on homogeneous spaces*, in Harmonic Analysis on Homogeneous Spaces, vol. 26, Proceedings of Symposia in Pure Mathematics, 1973.

[59] E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov*, vol. 83 of Progress in Mathematics, Birkhäuser Boston, 1990.

[60] T. Giordano and P. de la Harpe, *Groupes de tresses et moyennabilité intérieure*, Arkiv för Matematik, 29 (1991), pp. 63–72.

[61] S. Glasner, *Topological dynamics and group theory*, Transactions of the American Mathematical Society, 187 (1974), pp. 327–334.

[62] ——, *Compressibility properties in topological dynamics*, American Journal of Mathematics, 97 (1975), pp. 148–171.

[63] ——, *Proximal Flows*, vol. 517 of Lecture Notes in Mathematics, Springer-Verlag, 1976.

[64] F. P. Greenleaf, *Invariant Means on Topological Groups and Their Applications*, Van Nostrand-Reinhold Company, 1969.

[65] M. Gromov, *Hyperbolic Groups*, Springer New York, 1987, pp. 75–263.

[66] ——, *Random walk in random groups*, Geometric and Functional Analysis, 13 (2003), pp. 73–146.

[67] E. Guentner, N. Higson, and S. Weinberger, *The Novikov conjecture for linear groups*, Publications Mathématiques de l'IHÉS, 101 (2005), pp. 243–268.

[68] U. Haagerup, *A new look at C^*-simplicity and the unique trace property of a group*, 2015, `arXiv:arXiv:1509.5880`.

[69] U. Haagerup and L. Zsidó, *Sur la propriété de Dixmier pour les C^*-algebres*, Comptes Rendus de l'Acad'emie des Sciences, Série I, 298 (1984), pp. 173–176.

[70] D. Hadwin and V. I. Paulsen, *Injectivity and projectivity in analysis and topology*, Science China Mathematics, 54 (2011), pp. 2347–2359.

[71] J. Hakeda and J. Tomiyama, *On some extension properties of von Neumann algebras*, Tohoku Mathematical Journal, 19 (1967), pp. 315–323.

[72] M. Hamana, *Injective envelopes of Banach modules*, Tohoku Mathematical Journal, Second Series, 30 (1978), pp. 439–453.

[73] ——, *Injective envelopes of C^*-algebras*, Journal of the Mathematical Society of Japan, 31 (1979), pp. 181–197.

[74] ——, *Injective envelopes of operator systems*, Publications of the Research Institute for Mathematical Sciences, 15 (1979), pp. 773–785.

[75] ——, *Regular embeddings of C^*-algebras in monotone complete C^*-algebras*, Journal of the Mathematical Society of Japan, 33 (1981), pp. 159–183.

[76] ——, *The centre of the regular monotone completion of a C^*-algebra*, Journal of the London Mathematical Society (2), 26 (1982), pp. 522–530.

[77] ——, *Tensor products for monotone complete C^*-algebras. I*, Japanese Journal of Mathematics, New Series, 8 (1982), pp. 259–283.

[78] ——, *Tensor products for monotone complete C^*-algebras. II*, Japanese Journal of Mathematics, New Series, 8 (1982), pp. 285–295.

[79] ——, *Injective envelopes of C^*-dynamical systems*, Tohoku Mathematical Journal, 15 (1985), pp. 463–487.

[80] Y. Hartman, K. Juschenko, O. Tamuz, and P. Vahidi Ferdowsi, *Thompson's group F is not strongly amenable*, 2016, `arXiv:1607.04915`. Accepted to Ergodic Theory and Dynamical Systems.

[81] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. Volume I*, Springer-Verlag, 1963.

[82] G. Higman, B. H. Neumann, and H. Neumann, *Embedding theorems for groups*, Journal of the London Mathematical Society, Second Series, 24 (1949), pp. 247–254.

[83] N. Hindman and D. Strauss, *Algebra in the Stone-Čech Compactification: Theory and Applications*, vol. 27 of De Gruyter Studies in Mathematics, Walter de Gruyter, 2012.

[84] A. Hulanicki, *Means and Følner condition on locally compact groups*, Studia Mathematica, 27 (1966), pp. 87–104.

[85] J. R. Isbell, *Uniform spaces*, no. 12 in Mathematical Surveys, American Mathematical Society, 1964.

[86] N. A. Ivanov, *On the structure of some reduced amalgamated free product C^*-algebras*, International Journal of Mathematics, 22 (2011), pp. 281–306.

[87] N. A. Ivanov and T. Omland, *C^*-simplicity of free products with amalgamation and radical classes of groups*, Journal of Functional Analysis, 272 (2017), pp. 3712–3741.

[88] P. Jolissaint and G. Robertson, *Simple purely infinite C^*-algebras and n-filling actions*, Journal of Functional Analysis, 175 (2000), pp. 197–213.

[89] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Volume I*, Pure and Applied Mathematics, Academic Press, Inc., 1983.

[90] M. Kalantar and M. Kennedy, *Boundaries of reduced C^*-algebras of discrete groups*, Journal für die Reine und Angewandte Mathematik, 727 (2017), pp. 247–267.

[91] I. Kaplansky, *Functional analysis*, in Some aspects of analysis and probability, vol. 4 of Surveys in Applied Mathematics, John Wiley & Sons, Inc., New York, 1958, pp. 1–34.

[92] T. Kawabe, *Uniformly recurrent subgroups and the ideal structure of reduced crossed products*, 2017, `arXiv:1701.03413`.

[93] S. Kawamura and J. Tomiyama, *Properties of Topological Dynamical Systems and Corresponding C^*-Algebras*, Tokyo Journal of Mathematics, 13 (1990), pp. 251–257.

[94] M. Kennedy, *Characterizations of C^*-simplicity*, 2015, `arXiv:1509.1870`.

[95] M. Kennedy and S. Raum, *Traces on reduced group C*-algebras*, 2017, arXiv:1706.05903.

[96] E. Kirchberg and S. Wassermann, *Exact groups and continuous bundles of C^*-algebras*, Mathematische Annalen, 315 (1999), pp. 169–203.

[97] A. Kishimoto, *Outer automorphisms and reduced crossed products of simple C^*-algebras*, Communications in Mathematical Physics, 81 (1981), pp. 429–435.

[98] M. Laca and J. Spielberg, *Purely infinite C^*-algebras from boundary actions of discrete groups*, Journal für die Reine und Angewandte Mathematik, 480 (1996), pp. 125–139.

[99] A. Le Boudec, *C^*-simplicity and the amenable radical*, Inventiones Mathematicae, 209 (2017), pp. 159–174.

[100] A. Le Boudec and N. Matte Bon, *Subgroup dynamics and C^*-simplicity of groups of homeomorphisms*, 2016, arXiv:1605.01651.

[101] R. I. Loebl, *Injective von Neumann algebras*, Proceedings of the American Mathematical Society, 44 (1974), pp. 46–48.

[102] A. V. Malyutin, *Classification of group actions on the line and the circle*, Rossiĭskaya Akademiya Nauk. Algebra i Analiz, 19 (2007), pp. 156–182.

[103] J. Melleray, L. Nguyen Van Thé, and T. Tsankov, *Polish groups with metrizable universal minimal flows*, 2014, arXiv:1404.6167.

[104] N. Monod and Y. Shalom, *Cocycle superrigidity and bounded cohomology for negatively curved spaces*, Journal of Differential Geometry, 67 (2004), pp. 395–455.

[105] C. C. Moore, *Compactifications of symmetric spaces*, American Journal of Mathematics, 86 (1964), pp. 201–218.

[106] G. D. Mostow, *Strong Rigidity of Locally Symmetric Spaces*, vol. 78 of Annals of Mathematics Studies, Princeton University Press, 1973.

[107] F. J. Murray and J. von Neumann, *On Rings of Operators. IV*, Annals of Mathematics, 44 (1943), pp. 716–808.

[108] A. Y. Olshanskii and D. V. Osin, *C^*-simple groups without free subgroups*, Groups, Geometry, and Dynamics, 8 (2014), pp. 933–983.

[109] A. J. Ol'šanskiĭ, *On the question of the existence of an invariant mean on a group*, Akademiya Nauk SSSR i Moskovskoe Matematicheskoe Obshchestvo. Uspekhi Matematicheskikh Nauk, 35 (1980), pp. 199–200.

[110] T. Omland, *Dynamical systems and operator algebras associated to Artin's representation of braid groups*, 2016, arXiv:1609.04737.

[111] N. Ozawa, *Lecture on the Furstenberg boundary and C^*-simplicity.* http://www.kurims.kyoto-u.ac.jp/~narutaka/notes/yokou2014.pdf, 2014.

[112] J. A. Packer and I. Raeburn, *Twisted crossed products of C^*-algebras*, Mathematical Proceedings of the Cambridge Philosophical Society, 106 (1989), pp. 293–311.

[113] A. Papadopoulos, *Metric Spaces, Convexity and Nonpositive Curvature*, IRMA lectures in mathematics and theoretical physics, European Mathematical Society, 2014.

[114] W. L. Paschke and N. Salinas, *C^*-algebras associated with free products of groups*, Pacific Journal of Mathematics, 82 (1979), pp. 211–221.

[115] V. Paulsen, *Completely Bounded Maps and Operator Algebras*, vol. 78 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2002.

[116] V. I. Paulsen, *Weak expectations and the injective envelope*, Transactions of the American Mathematical Society, 363 (2011), pp. 4735–4755.

[117] R. R. Phelps, *Lectures on Choquet's Theorem*, vol. 1757 of Lecture Notes in Mathematics, Springer-Verlag, 2 ed., 2001.

[118] D. R. Pitts, *Structure for regular inclusions*, 2012, `arXiv:1202.6413`.

[119] S. Popa, *On the relative Dixmier property for inclusions of C^*-algebras*, Journal of Functional Analysis, 171 (2000), pp. 139–154.

[120] R. T. Powers, *Simplicity of the C^*-algebra associated with the free group on two generators*, Duke Mathematical Journal, 42 (1975), pp. 151–156.

[121] T. Poznansky, *Characterizations of linear groups whose reduced C^*-algebras are simple*, 2008, `arXiv:0812.2486`.

[122] S. D. Promislow, *A class of groups producing simple, unique trace C^*-algebras*, Mathematical Proceedings of the Cambridge Philosophical Society, 114 (1993), pp. 223–233.

[123] J. C. Quigg, *Duality for reduced twisted crossed products of C^*-algebras*, Indiana University Mathematics Journal, 35 (1986), pp. 549–571.

[124] I. Raeburn, A. Sims, and D. P. Williams, *Twisted actions and obstructions in group cohomology*, in C^*-algebras (Münster, 1999), Springer, Berlin, 2000, pp. 161–181.

[125] I. Raeburn and D. Williams, *Morita Equivalence and Continuous-Trace C^*-Algebras*, vol. 60 of Mathematical Surveys and Monographs, American Mathematical Society, 1998.

[126] S. Raum, *C^*-simplicity of locally compact Powers groups*, 2015, `arXiv:1505.0779`. To appear in Journal für die reine und angewandte Mathematik.

[127] N. W. Rickert, *Amenable groups and groups with the fixed point property*, Transactions of the American Mathematical Society, 127 (1967), pp. 221–232.

[128] K. Saitô and J. D. M. Wright, *Monotone Complete C^*-algebras and Generic Dynamics*, Springer-Verlag, 2015.

[129] J.-P. Serre, *Trees*, Springer-Verlag, Berlin-New York, 1980.

[130] A. Sierakowski, *Discrete crossed product C^*-algebras*, PhD thesis, University of Copenhagen, 2009.

[131] A. Sierakowski, *The ideal structure of reduced crossed products*, Münster Journal of Mathematics, 3 (2010), pp. 237–261.

[132] C. Svensson and J. Tomiyama, *On the commutant of $C(X)$ in C^*-crossed products by $\mathbb{Z}$ and their representations*, Journal of Functional Analysis, 256 (2009), pp. 2367–2386.

[133] M. Takesaki, *On the cross-norm of the direct product of C^*-algebras*, Tohoku Mathematical Journal, 16 (1964), pp. 111–122.

[134] ——, *Theory of Operator Algebras I*, Studies in Advanced Mathematics, Springer-Verlag New York Inc., 1979.

[135] J. Tomiyama, *On the projection of norm one in W^*-algebras*, Proceedings of the Japan Academy, 33 (1957), pp. 608–612.

[136] ——, *On the product projection of norm one in the direct product of operator algebras*, The Tohoku Mathematical Journal, 11 (1959), pp. 305–313.

[137] R. D. Tucker-Drob, *Shift-minimal groups, fixed price* 1*, and the unique trace property*, 2012, `arXiv:1211.6395`.

[138] V. Uspenskij, *On universal minimal compact G-spaces*, in Topology Proceedings, vol. 25, Auburn University, 2000, pp. 301–308.

[139] W. A. Veech, *Topological dynamics*, Bulletin of the American Mathematical Society, 83 (1977), pp. 775–830.

[140] G. Zeller-Meier, *Produits croisés d'une C^*-algèbre par un groupe d'automorphismes*, Journal de Mathématiques Pures et Appliquées. Neuvième Série, 47 (1968), pp. 101–239.