CASPER GULDBERG

ON HOMOTOPY AUTOMORPHISMS OF KOSZUL SPACES

PHD THESIS

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Abstract

In this thesis we study the rational homotopy theory of the spaces of self-equivalences of Koszul spaces - that is, of simply connected spaces which are simultaneously formal and coformal in the language of rational homotopy theory. The primary tool to do so is the Homotopy Transfer Theorem for L_∞-algebras. We begin with a Lie model for the universal cover of $B\operatorname{aut} X$ where X is a Koszul space, and construct a well-behaved contraction to a smaller chain complex using relations between the cohomology algebra and homotopy Lie algebra of a Koszul space. Then we study the transferred structure which retains all information about the rational homotopy type, and derive several structural properties. We establish criteria for coformality of the universal cover of $B\operatorname{aut} X$, improving on existing results, and provide examples: highly connected manifolds and two-stage spaces, among others. Our main example is that of ordered configurations in $\mathbb{R}^n$, for which our model is small enough that we can compute several rational homotopy groups of the universal cover of $B\operatorname{aut} X$. Finally we study the group of components $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ for a Koszul space X, and establish a sufficient condition for it to be isomorphic to the group of algebra automorphisms of the cohomology algebra of X, or equivalently the Lie algebra automorphisms of the homotopy Lie algebra of X.

Resumé

I denne afhandling studerer vi den rationale homotopiteori for rummene af selvækvivalenser af Koszul rum - det vil sige, af enkeltsammenhængende rum der er både formelle og koformelle. Det primære værkstøj til at foretage studiet er "Homotopy Transfer Theorem" for L_∞-algebraer. Vi begynder med en Lie model for den universelle overlejring til $B\operatorname{aut} X$ hvor X er et Koszul rum, og konstruerer en pæn sammentrækning til et mindre kædekompleks ved hjælp af relationer mellem kohomologialgebraen og homotopi Lie algebraen for et Koszul rum. Så studerer vi den overførte struktur der husker al information om rational homotopi type, og udleder flere strukturelle egenskaber. Vi fastsætter kriterier for koformalitet af den universelle overlejring af $B\operatorname{aut} X$ der forbedrer eksisterende resultater og giver eksempler: højtsammenhængende mangfoldigheder og "two-stage" rum, blandt andre. Vores hovedeksempel er ordnede konfigurationer i $\mathbb{R}^n$, for hvilke vores model er tilstrækkelig lille til at vi kan udregne adskillige rationale homotopigrupper af den universelle overlejring for $B\operatorname{aut} X$. Endeligt studerer vi gruppen af komponenter $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ for et Koszul rum X, og finder tilstrækkelige betingelser for at denne er isomorf til gruppen af algebraautomorfier af kohomologialgebraen, eller ækvivalent af Lie algebraautomorfier af homotopi Lie algebraen for X.

Contents

1. Introduction

1.1. **Background.** This thesis is about the rational homotopy type of the space of self-homotopy equivalences, or homotopy automorphisms, of certain nice topological spaces, called *Koszul spaces*. These are simply connected spaces of finite $\mathbb{Q}$-type with the property that they are simultaneously formal and coformal in the language of rational homotopy theory. Returning to more explicit statements at a later point, we just note here that there are plenty of interesting examples of spaces which satisfies this condition: loop spaces, suspensions of connected spaces, various classes of manifolds, ordered configuration spaces for points in $\mathbb{R}^n$. We may also note that products and wedges of Koszul spaces are again Koszul spaces.

The terminology: Koszul space, is borrowed from Berglund [2], and justified by the fact that the rational cohomology algebra $H^*(X;\mathbb{Q})$ of such a space X is a graded commutative Koszul algebra and the rational homotopy Lie algebra $\pi_*(\Omega X)\otimes\mathbb{Q}$ is a graded Koszul Lie algebra, such that these are Koszul dual to one another under the duality between the operads governing respectively graded commutative algebras and graded Lie algebras.

It has long been known that for a simply connected space X of finite $\mathbb{Q}$-type with Quillen model $\mathscr{L}$, the space of homotopy automorphisms $\operatorname{aut} X$ is related to the derivations on $\mathscr{L}$, cf. Schlessinger-Stasheff [33]. Concretely the positive part of the homotopy quotient of the derivations by the inner derivations $\operatorname{Der}\mathscr{L}/\!/\mathscr{L}$, is a Lie model for the universal cover of $B\operatorname{aut} X$ (see Tanré [38]). See also [34] for a survey of the literature on homotopy theory of mapping spaces, and in particular spaces of self-equivalences.

Where as the model of Schlessinger-Stasheff and the work of Tanré does not address $\pi_0(\operatorname{aut} X)$, there are also results on this. Denote by $X_{\mathbb{Q}}$ the rationalisation of X. There are obvious maps

$$\pi_0(\operatorname{aut} X_{\mathbb{Q}}) \to \operatorname{aut} H^*(X;\mathbb{Q}),$$
$$\pi_0(\operatorname{aut} X_{\mathbb{Q}}) \to \operatorname{aut} \pi_*(\Omega X)\otimes\mathbb{Q},$$

given by sending a homotopy class to the induced map on respectively cohomology and homotopy. Sullivan [37] showed that the first of these maps is always surjective for a formal space, and Neisendorfer-Miller [30] showed that the second map is always surjective for a coformal space. Sullivan [37] and Wilkerson [40] also showed that $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ is a linear algebraic group if X is either a finite CW-complex, or has finite Postnikov tower.

There exist general models for mapping spaces expressed in terms of so called *Maurer-Cartan elements* of a simplicial dg Lie algebra constructed from models of source and target respectively. See Berglund [3] for details on this, or Buijs-Félix-Murillo [9] for a related approach. The first of these is particularly useful to investigate set of path components of $\operatorname{aut} X_{\mathbb{Q}}$, but for the main part our starting point is the Lie model of derivations given by Schlessinger-Stasheff, from which we proceed as follows.

1.2. **Overview.** For a formal space the Quillen model $\mathscr{L}$ is relatively small - we may take the Quillen construction on the cohomology $A = H^*(X;\mathbb{Q})$. For a space which is also coformal, the Quillen model is quasi-isomorphic to the homotopy Lie algebra L of the space. The explicit nature of the Koszul duality lets us do even better: there is always an explicit surjective quasi-isomorphism $f\colon \mathscr{L} \xrightarrow{\sim} L$. We extend this to a well-behaved contraction of $\mathscr{L}$ onto L, and by standard homological perturbation theory this induces a contraction of $\operatorname{Der}\mathscr{L}$ onto the f-derivations $\operatorname{Der}_f(\mathscr{L}, L)$. This in turn is isomorphic to the complex $sA\otimes L$ twisted by a Maurer-Cartan element κ corresponding to f. Thus the positive homology of this twisted complex $sA\otimes_\kappa L$ computes the positive rational homotopy groups $\pi_*(\operatorname{aut} X, 1_X)\otimes\mathbb{Q}$. This has been noticed by Berglund [2, 3]. However our approach here lets us obtain more information in two distinct ways.

Recall that the homotopy Lie algebra of a simply connected space X, is the graded abelian group $\pi_*(\Omega X)\otimes\mathbb{Q}$, equipped with the Samelson bracket. The first way we obtain more information is by the Homotopy Transfer Theorem for L_∞-algebras: the dg Lie structure on $\operatorname{Der}\mathscr{L}$ transfers along the contraction to $sA\otimes_\kappa L$, and further to the homology $H_*(sA\otimes_\kappa L)$. With this transferred structure, the homology computes $\pi_*(\operatorname{aut} X, 1_X)\otimes\mathbb{Q}$ not only as a graded abelian group, but as a graded Lie algebra - the homotopy Lie algebra of the 1-connected covering space $B\operatorname{aut} X\langle 1\rangle$. Even better: the L_∞-algebra $H_*(sA\otimes_\kappa L)$ completely determines the rational homotopy type of $B\operatorname{aut} X\langle 1\rangle$.

Secondly, the degree zero homology of the dg Lie algebra $\operatorname{Der}\mathscr{L}$ contains information about $\pi_0\operatorname{aut} X_{\mathbb{Q}}$. In addition to Sullivan and Wilkersons results mentioned above, Block-Lazarev [6] later identified the Lie algebra of $\pi_0\operatorname{aut} X_{\mathbb{Q}}$ in terms of the Harrison cohomology of the minimal Sullivan model for X, and we observe how the Lie algebra for $\pi_0\operatorname{aut} X_{\mathbb{Q}}$ is computed by $H_0(sA\otimes_\kappa L)$ in our case where X is a Koszul space.

To obtain more information about $\pi_0\operatorname{aut} X_{\mathbb{Q}}$ we also employ the model alluded to above. Berglund [3] shows that there is a bijection

$$[X_{\mathbb{Q}}, X_{\mathbb{Q}}]\simeq \pi_0(MC_\bullet(A\otimes L))$$

between the homotopy classes of self-maps of $X_{\mathbb{Q}}$ and the path components of the Kan complex $MC_\bullet(A\otimes L)$. This Kan complex is the simplicial set of Maurer-Cartan elements in the simplicial dg Lie algebra $\Omega_\bullet\otimes A\otimes L$, where $\Omega_\bullet$ is Sullivans simplicial de Rham algebra. We use this to identify a sufficient condition for when the group $\pi_0\operatorname{aut} X_{\mathbb{Q}}$ is isomorphic to the groups $\operatorname{aut} L$ and $\operatorname{aut} A$.

1.3. **Consequences of main results.** The following are some interesting consequences of the main technical results of the thesis discussed further below. First we express the rational homotopy of automorphisms of certain manifolds in terms of derivations of their homotopy Lie algebras.

Theorem 1.1 (cf. Example 4.14)**.** *For $n\geq 1$, let M be an n-connected manifold of dimension $d\leq 3n+1$, and let L denote the rational homotopy Lie algebra $\pi_*(\Omega M)\otimes\mathbb{Q}$.*

If $\operatorname{rank} H^*(M)>4$ then there are isomorphisms of graded Lie algebras

$$\pi_{>0}(\operatorname{aut}_* M, 1_M)\otimes\mathbb{Q}\simeq(\operatorname{Der} L)_{>0},$$
$$\pi_{>0}(\operatorname{aut} M, 1_M)\otimes\mathbb{Q}\simeq(\operatorname{Der} L/\operatorname{ad} L)_{>0}.$$

If $\operatorname{rank} H^*(M)=4$ then L may have a center, on which the derivations act, and there are isomorphisms of graded Lie algebras

$$\pi_{>0}(\operatorname{aut}_* M, 1_M)\otimes\mathbb{Q}\simeq(\operatorname{Der} L)_{>0}\ltimes sZ(L)_{>0},$$
$$\pi_{>0}(\operatorname{aut} M, 1_M)\otimes\mathbb{Q}\simeq(\operatorname{Der} L/\operatorname{ad} L)_{>0}\ltimes sZ(L)_{>0}.$$

If $\operatorname{rank} H^*(M) \geq 4$ *then the universal cover of the classifying space* $B \operatorname{aut}_* M\langle 1\rangle$ *is coformal, and* $B \operatorname{aut} M\langle 1\rangle$ *is coformal if the centre* $Z(L)$ *is zero.*

This generalises the result by Berglund-Madsen [5] who showed this for a $2d$-dimensional $(d-1)$-connected manifold M with $\operatorname{rank} H^*(M) > 4$. The strongest statement of this kind that we obtain here, is that the conclusion holds for any Poincaré duality space X which is Koszul and has cup length at most 2.

The following can be thought of as a Koszul dual statement to Theorem 1.1, which will be more clear from the context where it appears in the thesis.

Theorem 1.2 (cf. Theorem 4.19)**.** *Let* X *be a simply connected space with finitely generated cohomology* A *concentrated in even degrees, and let* q *be a homogeneous non-degenerate quadratic form in the generators of* A*, such that*

$$A \simeq \mathbb{Q}[x_1, \ldots, x_n]/(q).$$

Then there is an action of $(\operatorname{Der} \overline{A})_{>0}$ *on the centre* $Z(L)$ *which is* 1*-dimensional, and isomorphisms of graded Lie algebras*

$$\pi_{>0}(\operatorname{aut} X) \otimes \mathbb{Q} \simeq (\operatorname{Der} \overline{A})_{>0} \ltimes sZ(L),$$
$$\pi_{>0}(\operatorname{aut}_* X) \otimes \mathbb{Q} \simeq (\operatorname{Der} \overline{A})_{>0},$$

and $B \operatorname{aut}_* X\langle 1\rangle$ *is coformal.*

Secondly, for some Koszul spaces the group $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ is as small as it can possibly be. Recall that according to Sullivan [37] and Neisendorfer-Miller [30], $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ surjects onto $\operatorname{aut}(H^*(X;\mathbb{Q}))$ and respectively onto $\operatorname{aut}(\pi_*(\Omega X) \otimes \mathbb{Q})$ for a Koszul space, and compare to the following.

Theorem 1.3 (cf. Corollary 4.36)**.** *Let* X *be a Koszul space such that* $H^*(X;\mathbb{Q})$ *is generated as an algebra in a single cohomological degree* d*. Equivalently* $\pi_*(\Omega X)\otimes\mathbb{Q}$ *is generated as a Lie algebra in degree* $d-1$*. If*

(i) $H^i(X;\mathbb{Q}) = 0$ *for all* $i \geq d^2$*, or*
(ii) $\pi_i(\Omega X) \otimes \mathbb{Q} = 0$ *for all* $i \geq d(d-1)$*,*

then there are isomorphisms of groups

$$\operatorname{aut}(\pi_*(\Omega X) \otimes \mathbb{Q}) \simeq \pi_0(\operatorname{aut} X_{\mathbb{Q}}) \simeq \operatorname{aut}(H^*(X;\mathbb{Q}))$$

Two interesting classes of Koszul spaces with rational cohomology (or homotopy) generated in a single degree, arise as examples for Theorem 1.3: those

1) for which the cup length is less than the rational connectivity,
2) for which the Whitehead length is less than the rational connectivity.

Both cases are subsumed by the condition of having rational L.S.-category less than the rational connectivity.

A non-trivial example from the first class, is that of ordered configuration spaces. Consider the space $F(\mathbb{R}^n, k)$ of k ordered points in $\mathbb{R}^n$. Cohomology is generated in degree $n-1$ and vanishes above degree $(k-1)(n-1)$, which is less than $(n-1)^2$ provided that $k < n$. Thus we get

$$\operatorname{aut} \pi_*(\Omega F(\mathbb{R}^n, k)) \otimes \mathbb{Q} \simeq \pi_0(\operatorname{aut} F(\mathbb{R}^n, k)_{\mathbb{Q}}) \simeq \operatorname{aut} H^*(F(\mathbb{R}^n, k); \mathbb{Q}).$$

Staying with the example of configuration spaces, we produce several computational results about their rational homotopy groups. For $k = 3$ and even $n \geq 4$ we give closed formulae for the dimensions of all rational homotopy groups of $\operatorname{aut} F(\mathbb{R}^n, 3)$, and compute several rational homotopy groups in the cases $k = 4, 5, 6$. Further we identify the Lie algebra associated to the linear algebraic group $\pi_0(\operatorname{aut} F(\mathbb{R}^n, 3)_{\mathbb{Q}})$, and see that it is neither semi-simple, solvable or nilpotent.

Finally, we note that the simplicial techniques used to study $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ for a Koszul space X, immediately yields an explicit computation of $\pi_0(\operatorname{aut}(BG)_{\mathbb{Q}})$ and of $\pi_0(\operatorname{aut} G_{\mathbb{Q}})$ for a (simply) connected compact Lie group G - cf. Example 4.42. In both cases we get a product of general linear groups, and in the simply connected case, where both groups can be computed by our techniques, we see that they are isomorphic

$$\pi_0(\operatorname{aut} G_{\mathbb{Q}}) \simeq \prod_{j=1}^{m+1} GL(i_j, \mathbb{Q}) \simeq \pi_0(\operatorname{aut}(BG)_{\mathbb{Q}}),$$

where i_j is the number of generators of the rational cohomology algebras in a particular degree, and $m+1$ is the number of distinct degrees for generators. In the same way we easily compute $\pi_0(\operatorname{aut} V_{\mathbb{Q}})$ for a real, complex or quartenion Stiefel manifold V - cf. Example 4.43

$$\pi_0(\operatorname{aut} V_{\mathbb{Q}}) \simeq \prod_{j=1}^{n} \mathbb{Q}^{\times},$$

where n is the number of generators for the rational cohomology algebra.

1.4. **Technical results.** To state the main technical results of the thesis we need the following observation. Koszul algebras come equipped with a weight grading, that is

$$A = A(0) \oplus A(1) \oplus A(2) \oplus \cdots$$

and

$$L = L(1) \oplus L(2) \oplus \cdots$$

such that the multiplication and respectively bracket preserves the weight. The weight 1 parts are naturally identified with the indecomposables, and we may choose presentations for A and L such that all relations are quadratic, in particular we may identify the weight n parts with elements of word length and respectively bracket length n. The tensor product $A \otimes L$ is bigraded by weights, and we may define the *shifted weight grading* on the complex $sA \otimes_\kappa L$ by letting bidegree (p, q) be the elements in $sA(p+1) \otimes_\kappa L(q+1)$. Then the main technical result of the thesis is:

Theorem 1.4 (Corollary 4.2). *The L_∞-structure on $H_* s(A \otimes_\kappa L)$ transferred from the derivations* $\operatorname{Der} \mathscr{L} // \mathscr{L}$*, respects the shifted weight grading in the sense that for any $r \geq 1$ the operation ℓ_r has bidegree* $(2-r, 2-r)$*.*

From this we obtain several structural results about $H_*(sA \otimes_\kappa L)$, and thus the homotopy Lie algebra $\pi_*(\operatorname{aut} X, 1_X) \otimes \mathbb{Q}$. We identify part of the homology as $\operatorname{Der} L$ and part as $\operatorname{Der} A$ and show:

Theorem 1.5 (cf. Structure Theorem 4.8). *Let X be a Koszul space with homotopy Lie algebra L. The graded Lie algebra $\pi_*(\operatorname{aut}_* X, 1_X) \otimes \mathbb{Q}$ is a split extension of the positive derivations* $(\operatorname{Der} L)_{>0}$.

The kernel of the extension is known together with the action on it, but writing out the extension requires more detail than is appropriate here. We refer to the Structure Theorem 4.8 for details. There is a similar result for $\pi_*(\operatorname{aut} X, 1_X) \otimes \mathbb{Q}$, where we have to take the centre of L into account, and both versions have Koszul dual statements expressed in terms of the derivations on the rational cohomology algebra of X.

Theorem 1.6 (cf. Proposition 4.34)**.** *Let X be a Koszul space with cohomology algebra A and homotopy Lie algebra L. If all Maurer-Cartan elements of $A \otimes L$ are tensors of indecomposables in A and L, then the maps given by sending a homotopy class to the induced map*

$$\pi_0(\operatorname{aut} X_{\mathbb{Q}}) \to \operatorname{aut} A,$$
$$\pi_0(\operatorname{aut} X_{\mathbb{Q}}) \to \operatorname{aut} L,$$

are isomorphisms of groups.

Theorem 1.6 is the most general version of Theorem 1.3 that we obtain here. It gives a sufficient condition for the surjections by respectively Sullivan [37] and Neisendorfer-Miller [30] to be injective, under the assumption that X is simultaneously formal and coformal.

1.5. **Structure of the thesis.** The thesis consists of six sections including the current introduction. Second section is a collection of standard facts from the literature presented in a need to know fashion. That is, each subsection will cover only what the reader needs to know on a particular subject in order to read the thesis, and is not meant as a comprehensive introduction to the respective subjects. We also establish most notation in the second section.

Third section is a technical walk-through on how to obtain a nice contraction of the general Lie model for universal covers of classifying spaces of self-equivalences $B\operatorname{aut} X\langle 1\rangle$ presented by Tanré , in our case of a Koszul space X. Using this nice contraction we record the main technical results about the interplay between the L_∞-structure transferred along the contraction, and the various gradings present.

In the fourth section we study the homology of smaller complex produced by the contraction from above. The homology equipped with the transferred L_∞-structure retains all rational homotopy information about $B\operatorname{aut} X\langle 1\rangle$ - in particular the positive part is isomorphic to the rational homotopy Lie algebra of $B\operatorname{aut} X\langle 1\rangle$, when we only consider the binary operation of the L_∞-structure. From the results of Section 3 we obtain structural result about the homotopy Lie algebra, and identify sufficient conditions in for coformality of $B\operatorname{aut} X\langle 1\rangle$ in several interesting cases. The degree zero part of the homology is related to $\pi_0(\operatorname{aut} X)$, and Section 4 concludes by studying this, both from the point of the complex produced, and by simplicial methods using a very different model for the space $\operatorname{aut} X$.

The fifth section is dedicated to our main example for application of our theory: $F(\mathbb{R}^n, k)$ - spaces of configurations of ordered points in $\mathbb{R}^n$. We provide computations of several rational homotopy groups of $\operatorname{aut} F(\mathbb{R}^n, k)$ (all when $k = 3$ and n is even), and investigate the induced L_∞-structure on $\pi_*(\operatorname{aut} F(\mathbb{R}^n, 3)) \otimes \mathbb{Q}$ in an attempt to clarify if $B\operatorname{aut} F(\mathbb{R}^n, 3)$ is coformal.

Section six is the final section of the thesis and contains suggestions for further research based on the techniques presented the thesis.

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2. Preliminaries

In this section we will set up some facts for the rest of the thesis. Subsections are not intended to give exhaustive overviews justifying their titles, but just to give bare essentials within those areas, needed in the thesis.

We begin by fixing notation and conventions. First of all, we will always be working over the field of rational numbers $\mathbb{Q}$. We shall use the word *map* for a morphism of the relevant category, after we have described what these are. We will write composition of maps $g\colon A \to B$ and $f\colon B \to C$, either as $f \circ g$, or simply by juxta-position fg.

2.1. **Differential graded (co)algebras and Lie algebras.** All claims in this subsection are standard facts in the literature. See for example Loday-Vallette [27], Quillen [32] and Félix-Halperin-Thomas [13].

2.1.1. *Graded vector spaces.* A graded vector space V, is a vector space together with a decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_n$. An element $v \in V_n$ is said to have degree $|v| = n$. A morphism $(f, m) \in \mathrm{mor}(V, W)$ of graded vector spaces is a linear map $f\colon V \to W$ and an $m \in \mathbb{Z}$, such that $f(V_n) \subseteq W_{n+m}$. We will just denote this by f, and say that it has degree $|f| = m$. This is a symmetric monoidal category under the tensor product of graded vector spaces, where $(V \otimes W)_n = \bigoplus_{p+q=n} V_p \otimes W_q$. Notably, the symmetry isomorphism $\tau\colon V \otimes W \to W \otimes V$ is given by

$$\tau(v \otimes w) = (-1)^{pq} w \otimes v \tag{1}$$

for $v \in V_p$ and $w \in W_q$.

Graded vector spaces form a closed symmetric monoidal category by setting

$$\mathrm{Hom}(V, W)_n = \prod_{q-p=n} \mathrm{Hom}(V_p, W_q) = \{f \in \mathrm{mor}(V, W) \mid |f| = n\}$$

The rationals $\mathbb{Q}$ may be viewed as a graded vector space by setting

$$\mathbb{Q}_i = \begin{cases} \mathbb{Q} & \text{for } i = 0 \\ 0 & \text{else} \end{cases},$$

and the linear dual $V^\vee$ of a graded vector space V is the graded vector space $\mathrm{Hom}(V, \mathbb{Q})$. Note that $(V^\vee)_n \simeq (V_{-n})^*$, the n'th subspace of the linear dual, is isomorphic to the regular linear dual of $-n$'th subspace. We will avoid negative indexing and instead write V^n for V_{-n}, as is convention throughout the literature.

We say that a graded vector space V is bounded above (respectively below) if there exist a $m \in \mathbb{Z}$ such that $V_k = 0$ for all $k > m$ (respectively $k < m$). Thus, if V is bounded above, then $V^\vee$ is bounded below, and opposite.

We say that a graded vector space is of *finite type*, if it is finite dimensional in each degree.

2.1.2. *Chain complexes.* A chain complex (V, d_V) is a graded vector space V together with a differential: a linear map $d_V : V \to V$, of degree -1, such that $d_V^2 = 0$. For short we write V for this chain complex, and $d = d_V$. A chain map $f : V \to W$ is a map of graded vector spaces such that, such that $d_W \circ f = (-1)^{|f|} f \circ d_V$, and we denote the graded vector space of such by $\operatorname{Hom}_{dg}(V, W)$. Chain complexes also form a symmetric monoidal category by the tensor product of graded vector spaces, and defining the differential for $V \otimes W$ by

$$d(v \otimes w) = d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w).$$

It is a closed symmetric monoidal category with the Hom from graded vector spaces, equipped with the differential ∂ given by

$$\partial f = d_W \circ f - (-1)^{|f|} f \circ d_V, \qquad f \in \operatorname{Hom}(V, W).$$

For a chain complex (V, d), we define the *cycles* $Z \subseteq V$ to be the kernel of d, and the *boundaries* $B \subseteq V$ to be the image of d. Since the d squares to zero, we have $B \subseteq Z$ and we define the *homology* of V to be the quotient Z/B. We write $H_*(V, d)$ for the homology, with

$$H_n(V, d) := (H_*(V, d))_n = \ker(d\colon V_n \to V_{n-1}) / \operatorname{Im}(d\colon V_{n+1} \to V_n).$$

Accordingly the degree of an element in a chain complex is called the *homological degree*. By *cohomological degree* we shall mean the negative homological degree.

For a chain complex (V, d), we define the positive part V_+ as follows,

$$(V_+)_i = \left\{ \begin{array}{cc} V_i & i > 1 \\ \ker d\colon V_1 \to V_0 & i = 1 \\ 0 & i \leq 0 \end{array} \right. .$$

Denote by $s\mathbb{Q}$ the chain complex for which

$$s\mathbb{Q}_i = \left\{ \begin{array}{ll} \mathbb{Q} & \text{for } i = 1 \\ 0 & \text{else} \end{array} \right. .$$

I.e. $s\mathbb{Q}$ is generated by the single element s in degree 1. For a chain complex (V, d_V), we define the suspension (sV, d_{sV}) as $sV = s\mathbb{Q} \otimes V$, and there is a natural isomorphism $sV_i \simeq V_{i-1}$ for all i. The differential on sV is then given by $d_{sV} = -d_V$, with the sign enforced by (1). The suspension map is the isomorphism $V \to sV$ given by $v \mapsto s \otimes v$, the image denoted simply by sv.

The dual $(s\mathbb{Q})^\vee$ is generated by s^* in degree -1, and $s^*\mathbb{Q} \otimes V$ is called the desuspension of V, always written $s^{-1}V$. The isomorphism given by $v \mapsto s^* \otimes v$ is similarly called the desuspension isomorphism, with the image simply denoted by $s^{-1}v$.

2.1.3. *dg algebras.* By an algebra A, we shall mean a unital associative algebra: a vector space A equipped with a multiplication morphism $\mu\colon A \otimes A \to A$ and a unit morphism $\eta\colon \mathbb{Q} \to A$ satisfying the usual axioms to make it a monoid in the symmetric monoidal category of vector spaces. We will be explicit about other properties, i.e. whether A is commutative, or augmented. We say that A is augmented, if it comes equipped with an algebra homomorphism $\varepsilon\colon A \to \mathbb{Q}$. The kernel of ε is denoted by $\overline{A}$, and is called the augmentation ideal.

A dg (differential graded) algebra is a monoid in the symmetric monoidal category of chain complexes with only degree zero morphisms. Unfolding the definitions, it is a chain complex (A, d_A) with multiplication μ and unit η of degree zero: $\mu(A_p \otimes A_q) \subseteq A_{p+q}$ and $\eta(\mathbb{Q}) \subseteq A_0$, and the differential d_A is a derivation with respect to the multiplication: it satisfies the Leibniz rule $d_A(ab) = d_A(a)b + (-1)^{|a|} a d_A(b)$ for $a, b \in A$, which corresponds to the commutative diagram:

$$\begin{tikzcd}
A \otimes A \arrow[r, "\mu"] \arrow[d, "d_A\otimes 1 + 1 \otimes d_A"'] & A \arrow[d, "d_A"] \\
A \otimes A \arrow[r, "\mu"] & A.
\end{tikzcd}$$

Note that for a graded commutative algebra, the commutativity relation carries the sign from (1): $ab = (-1)^{|a||b|} ba$.

We say that a dg algebra A is *n-connected* if $A^0 = \mathbb{Q}$ and $A^k = 0$ for $n \geq k$. If A is 0-connected we just say that A is connected, or that A is a cochain algebra.

Let V be a graded vector space. Denote by ΛV the *free graded commutative algebra* on V. It is the tensor algebra

$$TV = \bigoplus_{n \geq 0} V^{\otimes n},$$

where multiplication is given by $v \cdot w = v \otimes w$, modulo the ideal generated by all elements of the form $v \otimes w - (-1)^{|v||w|} w \otimes v$ for $v, w \in TV$. Alternatively it can be defined using the coinvariants for the obvious symmetric action $\Lambda V = \bigoplus_{n\geq 0} \Lambda^n V$ where

$$\Lambda^n V = \left(V^{\otimes n}\right)_{\Sigma_n},$$

with the induced product from the tensor algebra. We write an element $v \in \Lambda^n V$ as $v = v_1 \wedge \ldots \wedge v_n$ where $v_i \in V$.

2.1.4. *dg coalgebras.* Completely dual to above, by a coalgebra C we mean a counital coassociative coalgebra, which may be cocommutative, or coaugmented. I.e. a vector space C with a comultiplication map $\Delta\colon C \to C \otimes C$ and a counit $\varepsilon\colon C \to \mathbb{Q}$ satisfying the usual axioms making it a comonoid in the symmetric monoidal category of vector spaces. We say that C is coaugmented, if it comes equipped with a coalgebra homomorphism $\eta\colon \mathbb{Q} \to C$. The cokernel of η is denoted by $\overline{C}$, and is called the coaugmentation coideal.

A dg coalgebra (C, d_C) is a comonoid in the symmetric monoidal category of chain complexes with only degree zero morphisms. It is a chain complex (C, d_C) with comultiplication and counit of degree zero as for an dg algebra, and d_C is a coderivation with respect to the comultiplication: the diagram

$$\begin{tikzcd} C\otimes C & C \arrow[l, "\Delta"'] \\ C\otimes C \arrow[u, "d_C\otimes 1+1\otimes d_C"] & C \arrow[l, "\Delta"] \arrow[u, "d_C"'] \end{tikzcd}$$

commutes.

We say that a coaugmented dg coalgebra C is *n-connected* if $\overline{C}_k = 0$ for $k \leq n$.

The linear dual of a dg coalgebra always has the structure of a dg algebra: there is a (degree zero) chain map $\varphi\colon C^\vee \otimes C^\vee \to (C\otimes C)^\vee$ and the composition

$$C^\vee \otimes C^\vee \xrightarrow{\varphi} (C\otimes C)^\vee \xrightarrow{\Delta^\vee} C^\vee$$

is indeed a multiplication on $C^\vee$. The linear dual of a dg algebra which is of finite type and bounded above or below, has the structure of a dg coalgebra: finite type and boundedness implies that φ is an isomorphism, and the composition

$$A^\vee \xrightarrow{\mu^\vee} (A\otimes A)^\vee \xrightarrow{\varphi^{-1}} A^\vee \otimes A^\vee$$

is indeed a comultiplication on $A^\vee$.

The free graded commutative algebra ΛV can be equipped with the *unshuffle coproduct*: the comultiplication defined by letting

$$\Delta(v_1 \wedge \ldots \wedge v_n) = \sum_{i=1}^{n-1} \sum_{\sigma} (-1)^\varepsilon (v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(i)}) \otimes (v_{\sigma(i+1)} \wedge \ldots \wedge v_{\sigma(n)}),$$

where σ runs over the set of $(i, n-i)$-unshuffles, i.e.

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(i) \quad \text{and} \quad \sigma^{-1}(i+1) < \cdots < \sigma^{-1}(n).$$

The sign is determined by

$$\varepsilon = \sum_{\substack{i<j \\ \sigma^{-1}(i)>\sigma^{-1}(j)}} |v_i||v_j|.$$

This comultiplication turns out to be cocommutative, and further makes ΛV the *cofree graded cocommutative (conilpotent) coalgebra* on V. When we think about it in this way we may sometimes denote it $\Lambda^c V$, not to be confused with the summands of the graded vector space structure.

In fact ΛV is even a Hopf algebra with the two defined operations above, but we will not be using this.

2.1.5. *dg Lie algebras.* Recall that a graded Lie algebra is a graded vector space L equipped with a Lie bracket: a graded anti-symmetric binary operation

$$[-,-]\colon L\otimes L \to L,$$

with any triple of elements satisfying the Jacobi relation:

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]].$$

A dg Lie algebra (L, d_L) is a graded Lie algebra L such that (L, d_L) is a chain complex, the Lie bracket has degree zero, and d_L is a derivation with respect to the

Lie bracket: the diagram

$$\begin{tikzcd} L \otimes L \arrow[r, "{[-,-]}"] \arrow[d, "d_L\otimes 1+1\otimes d_L"'] & L \arrow[d, "d_L"] \\ L \otimes L \arrow[r, "{[-,-]}"] & L \end{tikzcd}$$

commutes.

Again note that anti-commutativity, and Jacobi relations carry the signs enforced by (1), that is for all $x, y \in L$ we have

$$[x, y] = -(-1)^{|x||y|}[y, x].$$

We say that a dg Lie algebra L is *n-connected* if $L_k = 0$ for $k \leq n$.

For a dg Lie algebra L, the positive part L_+ is a Lie subalgebra.

Let V be graded vector space. Denote by $\mathbb{L}(V)$ the *free graded Lie algebra* on V. It is the smallest graded Lie subalgebra of TV such that $V \subset \mathbb{L}(V)$, where the tensor algebra on V is equipped with the bracket given by anti-symmetrising the usual multiplication:

$$[v, w] = v \otimes w - (-1)^{|v||w|} w \otimes v.$$

2.1.6. *Quasi-isomorphisms.* For any of the above types of dg (co)algebras, there is an induced structure of the same type on the homology of the underlying chain complex. A dg algebra homomorphism $A \to B$ is a *quasi-isomorphism* if the induced map $H_*(A) \to H_*(B)$ is an isomorphism. A quasi-isomorphism from A to B is often denoted $A \xrightarrow{\sim} B$.

We say that two dg algebras A_1 and A_n are quasi-isomorphic if there exist a zig-zag of quasi-isomorphisms

$$A_1 \xleftarrow{\sim} A_2 \xrightarrow{\sim} \cdots \xleftarrow{\sim} A_{n-1} \xrightarrow{\sim} A_n.$$

Similarly for dg Lie algebras and dg coalgebras.

2.1.7. *Contractions.* Let (W, d_W) and (V, d_V) be chain complexes. A *contraction* of W onto V is a diagram

$$\begin{tikzcd} W \arrow[loop left, "h"] \arrow[r, shift left, "p"] & V, \arrow[l, shift left, "i"] \end{tikzcd}$$

where p and i are chain maps such that $pi = 1_V$, $d_W h + h d_W = ip - 1_W$. The maps have degrees $|p| = |i| = 0$, and necessarily $|h| = 1$. Without loss of generality we may assume the *annihilation conditions*, that $h^2 = hi = ph = 0$, see [24]. Necessarily p and i are quasi-isomorphisms of chain complexes.

Contractions can be composed as follows:

$$\begin{tikzcd} W \arrow[loop left, "h"] \arrow[r, shift left, "p"] & V \arrow[l, shift left, "i"] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} V \arrow[loop left, "g"] \arrow[r, shift left, "q"] & U \arrow[l, shift left, "j"] \end{tikzcd}$$

compose to

$$\begin{tikzcd} W \arrow[loop left, "h+igp"] \arrow[r, shift left, "qp"] & U. \arrow[l, shift left, "ij"] \end{tikzcd}$$

This is easily checked by using the annihilation conditions.

Lemma 2.1. *For any chain complex V (over a field), we may choose a contraction*

$$h \circlearrowright V \underset{i}{\overset{p}{\rightleftarrows}} H_*(V),$$

where we consider the graded vector space $H_*(V)$ as a chain complex with zero differential.

Proof. Consider the short exact sequences

$$0 \longrightarrow Z_n \underset{j}{\overset{\tau}{\rightleftarrows}} V_n \underset{d}{\overset{\sigma}{\rightleftarrows}} B_{n-1} \longrightarrow 0$$

$$0 \longrightarrow B_n \overset{\rho}{\rightleftarrows} Z_n \underset{q}{\overset{\omega}{\rightleftarrows}} H_n(V) \longrightarrow 0.$$

Since we are working over a field, these are split exact, and we may choose splittings as already indicated. It is easy to check that the data

$$\sigma\rho\tau \circlearrowright V \underset{j\omega}{\overset{q\tau}{\rightleftarrows}} H_*(V),$$

is a contraction of V onto its homology. □

Finally for this section, note that the homotopy h of a contraction always satisfies the equation $dhd = d$.

2.2. **Twisting morphisms, bar and cobar constructions.** Given a dg coalgebra C and a dg Lie algebra L, the set of twisting morphisms is a certain subset of all linear maps $C \to L$, and in Appendix B of [32] Quillen shows that the bifunctor assigning this set to C and L is representable and corepresentable. We review parts of the theory below.

Definition 2.2. *Let (L, d) be a dg Lie algebra. A Maurer-Cartan element of L is a degree -1 element τ, satisfying the equation*

$$d\tau + \frac{1}{2}[\tau, \tau] = 0. \tag{2}$$

Any such Maurer-Cartan element τ gives rise to a twisted differential d^τ on L defined by

$$d^\tau = d + [\tau, -].$$

It is easy to check that this is a differential precisely when the equation (2) is satisfied. The set of Maurer-Cartan elements of a dg Lie algebra L is denoted $\mathrm{MC}(L)$, and clearly for a dg Lie map $f\colon L \to L'$ we have $f(\mathrm{MC}(L)) \subseteq \mathrm{MC}(L')$.

Let (C, d_C) be a coaugmented dg cocommutative coalgebra, and (L, d_L) a dg Lie algebra. The chain complex $\mathrm{Hom}(C, L)$ has the structure of a dg Lie algebra, called the *convolution dg Lie algebra*. The Lie bracket $[f, g]$ for $f, g \in \mathrm{Hom}(C, L)$ is given by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} L \otimes L \xrightarrow{[-,-]} L\ .$$

A Maurer-Cartan element of the convolution Lie algebra gives rise to a twisted differential ∂^τ on $\mathrm{Hom}(C, L)$, and we denote by $\mathrm{Hom}(C, L)^\tau$ the resulting dg Lie algebra.

Definition 2.3. *Let (C, d_C) be a coaugmented dg cocommutative coalgebra, and (L, d_L) a dg Lie algebra. A* twisting morphism *τ from C to L is a Maurer-Cartan element of the convolution Lie algebra $\tau \in \mathrm{MC}(\mathrm{Hom}(C, L))$, such that τ is zero on the coaugmentation of C. We write $\mathrm{Tw}(C, L)$ for the set of twisting morphism from C to L.*

The assignment of $\mathrm{Tw}(C, L)$ to the data C and L defines a bifunctor by precomposition of dg coalgebra maps, and composition of dg Lie maps. It turns out to be the representable and corepresentable bifunctor mentioned above, but before stating this as a proposition, we briefly review the corepresenting respectively representing objects.

Definition 2.4 (Cobar construction)**.** *Let C denote an coaugmented dg coalgebra. The cobar construction on C is denoted $\mathscr{L}(C)$, and it is the free Lie algebra $\mathbb{L}(s^{-1}\overline{C})$ on the desuspension of $\overline{C}$, equipped with a differential $d_{\mathscr{L}} = \delta_0 + \delta_1$. Here δ_0 and δ_1 are derivations given respectively by*

$$\delta_0(s^{-1}x) = -s^{-1}d_C(x),$$
$$\delta_1(s^{-1}x) = -\frac{1}{2}\sum_i (-1)^{|x_i'|}[s^{-1}x_i', s^{-1}x_i''],$$

for $x \in \overline{C}$, and the sum is given by the reduced comultiplication on C, which we write $\overline{\Delta}(x) = \sum_i x_i' \otimes x_i''$.

The cobar construction gives rise to corepresenting objects for $\mathrm{Tw}(-,-)$.

Definition 2.5 (Bar construction)**.** *Let L denote a dg Lie algebra. The bar construction on L is denoted $\mathscr{C}(L)$, and it is the cofree cocommutative coalgebra $\Lambda^c(sL)$ on the suspension of L, equipped with a differential $d_{\mathscr{C}} = d_0 + d_1$. The differential is given by the formulae*

$$d_0(sx_1 \wedge \ldots \wedge sx_n) = -\sum_{i=1}^{n}(-1)^{\epsilon_i} sx_1 \wedge \ldots \wedge sd_L(x_i) \wedge \ldots \wedge sx_n,$$
$$d_1(sx_1 \wedge \ldots \wedge sx_n) = \sum_{i<j}(-1)^{|sx_i|+\epsilon_{ij}} s[x_i, x_j] \wedge sx_1 \wedge \ldots \widehat{sx_i} \ldots \widehat{sx_j} \ldots \wedge sx_n,$$

for $x \in L$, with signs determined by

$$\epsilon_i = \sum_{i<j}|sx_j|,$$
$$\epsilon_{ij} = |sx_i|\sum_{r=1}^{i-1}|sx_r| + |sx_j|\sum_{r=1}^{j-1}|sx_r|.$$

The bar construction gives rise to representing objects for $\mathrm{Tw}(-,-)$.

Remark 2.6 Classically the bar and cobar construction takes dg algebras respectively coalgebras to dg Hopf algebras. The cobar construction defined above is exactly the dg Lie algebra of primitives in the classical setting. It also coincides with the Quillen construction on the dual $A := C^\vee$.

The bar construction as defined above coincides with the linear dual of the Chevalley-Eilenberg construction. It is not the same as taking the classical bar construction on the universal enveloping algebra UL, but the two constructions are quasi-isomorphic.

2.2.1. *Universal twisting morphisms.* By the definition of the cobar and bar constructions it follows immediately that projection to cogenerators and inclusion of generators give degree -1 morphisms

$$\pi\colon \mathscr{C}(L) \to L, \qquad \iota\colon C \to \mathscr{L}(C).$$

It is easy to check that these are twisting morphisms, and any twisting morphism $\tau\colon C \to L$ factors in two ways

$$\begin{tikzcd} & \mathscr{C}(L) \arrow[dr, "\pi"] & \\ C \arrow[ur, dashed, "\varphi_\tau"] \arrow[rr, "\tau"] \arrow[dr, "\iota"'] & & L \\ & \mathscr{L}(C) \arrow[ur, dashed, "\psi_\tau"'] & \end{tikzcd}$$

with φ_τ a dg coalgebra map and ψ_τ a dg Lie map, such that both are uniquely determined by τ. This is originally due to Quillen, see [32] App B, 6.1 and 6.2. We summarize as follows.

Proposition 2.7. *Let C be a coaugmented dg cocommutative coalgebra, and let L be a dg Lie algebra. The universal twisting morphisms induce bijections*

$$\mathrm{Hom}_{dgl}(\mathscr{L}(C), L) \xrightarrow[\iota^*]{\simeq} \mathrm{Tw}(C, L) \xleftarrow[\pi_*]{\simeq} \mathrm{Hom}_{dgc\,coalg}(C, \mathscr{C}(L)).$$

In particular the bar and cobar constructions form a pair of adjoint functors

$$\begin{tikzcd} DGCC \arrow[r, bend left, "\mathscr{L}"] & DGL \arrow[l, bend left, "\mathscr{C}"] \end{tikzcd}$$

between dg Lie algebras and coaugmented dg cocommutative coalgebras. In fact, when restricting to connected Lie algebras and 1-connected coalgebras this is even a Quillen equivalence [32].

2.3. **Koszul duality for Lie and commutative algebras.** Koszul algebras were first introduced by Priddy in [31], where the theory is developed for associative algebras. Ginzburg-Kapranov [16] developed Koszul duality for operads, and Koszul duality for algebras over certain Koszul operads. This in particular encompasses Koszul duality for Lie and commutative algebras. See also Milles [29] for this. An introduction to operads and Koszul duality for operads can be found in [27] by Loday-Vallette. Berglund [2] gives a short and concise review of parts of the theory, and we specialize some of that account here to the case of Lie and commutative algebras.

A weight grading on a coaugmented graded cocommutative coalgebra C, is a decomposition

$$\overline{C} = C(1) \oplus C(2) \oplus \cdots$$

such that the comultiplication of $\overline{C}$ is weight preserving, i.e. for $n \geq 2$

$$\overline{\Delta}(\overline{C}(n)) \subseteq \bigoplus_{p+q=n} \overline{C}(p) \otimes \overline{C}(q),$$

and $\overline{\Delta}(\overline{C}(1)) = 0$. Setting $C(0) \simeq \mathbb{Q}$ to be the image of the coaugmentation, we get a corresponding weight decomposition of C such that the comultiplication Δ on C is weight preserving.

Example 2.8 Let V be a graded vector space. The free graded cocommutative coalgebra $\Lambda^c V$ is coaugmented by the identification $\mathbb{Q} \simeq \Lambda^0 V$ and has natural weight grading with $\Lambda^c V(n) = \Lambda^n V$.

The cobar construction $(\mathscr{L}(C), \delta_0 + \delta_1)$ on a coaugmented graded cocommutative coalgebra C with a chosen weight grading is bigraded by *bracket length* ℓ_b and *total weight* ℓ_w. Bracket length is self-explanatory for a free graded Lie algebra, and total weight of an element is simply the sum of weights appearing in the brackets. E.g. suppose we have an element $x = [[x_1, x_2], x_3] \in \mathscr{L}(C)$ with $x_i \in \overline{C}(i)$, then $\ell_b(x) = 3$ and $\ell_w(x) = 1 + 2 + 3 = 6$.

Since any weight grading is positive, $\mathscr{L}(C)$ is concentrated in bigradings with $\ell_w \geq \ell_b$, and we let

$$D_{\mathscr{L}} := \{x \in \mathscr{L}(C) \mid \ell_w(x) = \ell_b(x)\} \subset \mathscr{L}(C)$$

denote the diagonal. Define the *Koszul dual graded Lie algebra to C* to be the quotient Lie algebra

$$C^{¡} := D_{\mathscr{L}} / D_{\mathscr{L}} \cap \mathrm{Im}(\delta_1).$$

The natural projection followed by the quotient $f\colon \mathscr{L}(C) \to C^{¡}$ gives rise to a twisting morphism $\kappa\colon C \to C^{¡}$ by restriction to generators (Proposition 2.7 and discussion above it).

Definition 2.9. *We say that C is a Koszul coalgebra if there exist a weight grading on C such that the natural projection followed by the quotient $\mathscr{L}(C) \to C^{¡}$ is a quasi-isomorphism of dg Lie algebras.*

A weight grading on a graded Lie algebra L, is a decomposition

$$L = L(1) \oplus L(2) \oplus \cdots$$

such that the bracket of L is weight preserving, i.e. $[L(p), L(q)] \subseteq L(p+q)$.

Example 2.10 Let V be graded vector space. The free graded Lie algebra on $\mathbb{L}(V)$ has a natural weight grading given by $\mathbb{L}(V)(n) = \mathbb{L}(V) \cap V^{\otimes n}$. This is just the bracket length in the free Lie algebra, and we also write it $\mathbb{L}^n(V)$.

The bar construction $(\mathscr{C}(L), d_0 + d_1)$ on a graded Lie algebra L with a chosen weight grading is bigraded by *wedge length* ℓ_b and total weight ℓ_w. Wedge length is the obvious grading for a cofree graded coalgebra, and total weight of an element is the sum of weights of letters appearing as letters in a word. E.g. suppose we have an element $y = [y_1, y_2] \wedge y_3 \wedge y_4 \in \mathscr{C}(L)$ with $y_i \in L(i)$. Then $\ell_b(y) = 3$ and $\ell_w(y) = (1 + 2) + 3 + 4 = 10$.

Since any weight grading is positive, $\mathscr{C}(L)$ is concentrated in bigradings with $\ell_w \geq \ell_b$, and we let

$$D_{\mathscr{C}} := \{x \in \mathscr{C}(L) \mid \ell_w(x) = \ell_b(x)\} \subset \mathscr{C}(L)$$

denote the diagonal. Define the *Koszul dual cocommutative coalgebra to L* to be the sub coalgebra

$$L^{¡} := D_{\mathscr{C}} \cap \ker(d_1) \subset \mathscr{C}(L).$$

The inclusion $g\colon L^{¡} \to \mathscr{C}(L)$ always gives rise to a twisting morphism $\kappa'\colon L^{¡} \to L$ by projection to cogenerators $\kappa' = \pi g$, cf. Proposition 2.7. Explicitly it is the composition

$$\kappa'\colon L^{¡} \rightarrowtail D_{\mathscr{C}} \twoheadrightarrow sL(1) \xrightarrow{s^{-1}} L(1) \rightarrowtail L.$$

Definition 2.11. *We say that L is a Koszul Lie algebra if there exist a weight grading on L such that the inclusion $L^{¡} \to \mathscr{C}(L)$ is a quasi-isomorphism of dg coalgebras.*

Notice that the quasi-isomorphisms of Definition 2.9 and 2.11 correspond to certain twisting morphism $L^{¡} \to L$, respectively $C \to C^{¡}$ cf. Proposition 2.7.

Definition 2.12. *An augmented graded commutative algebra A of finite type is a Koszul algebra if $A^\vee$ is a Koszul coalgebra.*

Definition 2.13. *Let L be a Koszul Lie algebra. The Koszul dual graded commutative algebra is $L^! := (L^{¡})^\vee$.*

Definition 2.14. *Let A be a Koszul algebra of finite type. The Koszul dual graded Lie algebra to A is $A^! := (A^\vee)^{¡}$.*

The following theorem is a special case of Berglund [2], Theorem 2.11.

Theorem 2.15. *Let L be a Koszul Lie algebra. Then L has a presentation of the form*

$$L = \mathbb{L}(V)/(R)$$

for some $R \subset \mathbb{L}(V)(2)$. The Koszul dual graded commutative algebra $L^!$ has a presentation

$$L^! = \Lambda((sV)^\vee)/(R^\perp),$$

where $R^\perp \subset \Lambda^2((sV)^\vee)$ is the annihilator of R with respect to the pairing of degree 2

$$\langle\,,\,\rangle\colon \Lambda^2((sV)^\vee) \otimes \mathbb{L}^2(V) \to \mathbb{Q},$$

induced from the standard pairing $V^\vee \otimes V \to \mathbb{Q}$.

Explicitly the induced pairing is given in terms of the standard pairing by the formula

$$\langle a \wedge b, [\alpha, \beta]\rangle = (-1)^{|b||\alpha|+|a|+|\alpha|}\langle a, \alpha\rangle\langle b, \beta\rangle - (-1)^{|\alpha||\beta|+|b||\beta|+|a|+|\beta|}\langle a, \beta\rangle\langle b, \alpha\rangle,$$

with $a, b \in sV^\vee$ and $\alpha, \beta \in V$.

2.4. **L_∞-algebras.** Our results rely on contracting the underlying complex of a dg Lie algebra, and studying what structure the contracted complex has. One might expect it to be a Lie algebra "up to homotopy" in some appropriate sense, and this is indeed the case. The correct notion is that of an L_∞-algebra or strongly homotopy Lie algebra, introduced by Lada and Stasheff [23], with reference to [33] by Schlessinger-Stasheff, unpublished at the time. Many of the ideas concerning L_∞-algebras are present in the work of Kontsevich, and a modern treatment is given in [27] by Loday-Vallette.

We follow the sign conventions from the latter.

Definition 2.16. *Let V be a graded vector space. An L_∞-*structure *on V is a family of linear maps*

$$\ell_n\colon V^{\otimes n} \to V, \quad n \geq 1,$$

of degree $n-2$, satisfying anti-symmetry

$$\ell_n(\ldots, x, y, \ldots) = -(-1)^{|x||y|}\ell_n(\ldots, y, x, \ldots),$$

and for all $n \geq 1$, the generalized Jacobi identities,

$$\sum_{p=1}^{n}\sum_{\sigma} \operatorname{sgn}(\sigma)(-1)^{\epsilon}\ell_{n+1-p}(\ell_p(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(p)}), x_{\sigma(p+1)}, \ldots, x_{\sigma(n)}) = 0,$$

where we sum over all $(p, n-p)$-unshuffles, i.e.

$$\sigma^{-1}(1) < \cdots < \sigma^{-1}(p) \quad \textit{and} \quad \sigma^{-1}(p+1) < \cdots < \sigma^{-1}(n).$$

The sign is given by

$$\epsilon = p(n-p) + \sum_{\substack{i<j \\ \sigma^{-1}(i) > \sigma^{-1}(j)}} |x_i||x_j|.$$

The generalized Jacobi identities for $n \leq 3$ are

$$\ell_1^2(x) = 0,$$

$$\ell_2(\ell_1(x), y) + (-1)^{|x|}\ell_2(x, \ell_1(y)) = \ell_1(\ell_2(x, y)),$$

$$\ell_2(\ell_2(x, y), z) + (-1)^{|y||z|+1}\ell_2(\ell_2(x, z), y) - \ell_2(x, \ell_2(y, z)) = -(\ell_1\ell_3 + \ell_3\ell_1)(x \otimes y \otimes z). \tag{3}$$

From this we see that ℓ_1 is a differential, and a derivation with respect to ℓ_2. For an L_∞-structure on a graded vector space V, the chain complex (V, ℓ_1) is called the underlying chain complex. We see from (3) that if either ℓ_1 or ℓ_3 are zero, then ℓ_2 is a Lie bracket, but in general (3) just states that the Jacobi relation holds up to chain homotopy in $V^{\otimes 3}$ given by $-\ell_3$ (note that we have abused notation slightly so the differential on $V^{\otimes 3}$ induced by ℓ_1, is also denoted by ℓ_1).

We shall say that ℓ_n for $n > 2$, is a *higher operation*. An L_∞-structure with trivial higher operations is just a dg Lie structure.

Definition 2.17. *An L_∞-morphism $g\colon (V, \ell) \to (W, l)$ is a family of graded alternating linear maps $\{g_n\colon V^{\otimes n} \to W\}_n$ of degree $n-1$, such that for every $n \geq 1$, g_n satisfies*

$$\sum_{p=1}^{n}\sum_{\sigma} \operatorname{sgn}(\sigma)(-1)^{\epsilon} g_{n+1-p}(\ell_p(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(p)}), x_{\sigma(p+1)}, \ldots, x_{\sigma(n)}) =$$

$$\sum_{\substack{k \geq 1 \\ i_1 + \cdots + i_k = n}} \sum_{\tau} \operatorname{sgn}(\tau)(-1)^{\eta} l_k(g_{i_1}(x_{\tau(1)}, \ldots, x_{\tau(i_1)}), \ldots, g_{i_k}(x_{\tau(i_{k-1}+1)}, \ldots, x_{\tau(i_k)})),$$

where σ is a $(p, n-p)$-unshuffles as above, and τ is an $(i_1, \ldots, i_k)$-unshuffles, i.e.

$$\tau^{-1}(i_j + 1) < \cdots < \tau^{-1}(i_{j+1}), \qquad \textit{for all } j \in \{0, \ldots, k-1\}, \quad i_0 := 0,$$

satisfying the extra condition that

$$\tau^{-1}(1) < \tau^{-1}(i_1 + 1) < \cdots < \tau^{-1}(i_1 + i_2 + \cdots + i_{k-1} + 1).$$

The signs are given by

$$\epsilon = p(n-p) + \sum_{\substack{i<j \\ \sigma^{-1}(i)>\sigma^{-1}(j)}} |x_i||x_j|,$$

$$\eta = \sum_{j=1}^{k}(k-j)(i_j-1) + \sum_{\substack{i<j \\ \tau^{-1}(i)>\tau^{-1}(j)}} |x_i||x_j| + \sum_{j=2}^{k}(i_j-1)\sum_{m=1}^{i_{j-1}} |x_{\tau(m)}|.$$

For $n=1$ the condition is simply that g_1 is a chain map. For $n=2$, it is

$$\begin{aligned}-g_2(\ell_1(x_1),x_2) - (-1)^{|x_1||x_2|+1}g_2(\ell_1(x_2),x_1) + g_1(\ell_2(x_1,x_2))\\ = l_2(g_1(x_1),g_1(x_2)) + (-1)^{|x_1||x_2|+1}l_2(g_1(x_2),g_1(x_1)) + l_1(g_2(x_1,x_2)),\end{aligned}$$

which we rearrange

$$\begin{aligned}g_2(\ell_1(x_1),x_2) + (-1)^{|x_1|}g_2(x_1,\ell_1(x_2)) + l_1(g_2(x_1,x_2))\\ = g_1(\ell_2(x_1,x_2)) - l_2(g_1(x_1),g_1(x_2))\end{aligned}$$

and see that the condition is precisely that g_2 is a chain homotopy between $g_1\ell_2$ and $l_2(g_1 \otimes g_1)$, so g_1 respects the binary operations up to homotopy. Similarly the higher maps $\{g_n\}_{n\geq 3}$ can be thought of as homotopies between homotopies, and so on.

Definition 2.18. *An L_∞* quasi-isomorphism *is an L_∞-morphism $\{g_n\}_n$, such that g_1 is a quasi-isomorphism of chain complexes.*

There is an equivalent definition of L_∞-algebras, which the following theorem expresses.

Theorem 2.19. *Let V be graded vector space. An L_∞-structure on V corresponds precisely to a square zero coderivation of degree -1 on the cofree cocommutative coalgebra $\Lambda(sV)$.*

An L_∞-morphisms is a dg coalgebra morphism under this correspondence.

A coderivation δ on a cofree coalgebra is completely determined by its corestriction $\pi\delta\colon \Lambda(sV) \to sV$. Write $\delta = \sum_{r\geq 0}\delta_r$, where δ_r lowers word length by r. I.e. for any $n \geq 0$ we have restrictions $\delta_r\colon \Lambda^n(sV) \to \Lambda^{n-r}(sV)$ and in particular $\delta_r\colon \Lambda(sV)^{r+1} \to sV$. If δ has degree -1, then the family of maps δ_r correspond to the operations ℓ_n for an L_∞-algebra, by setting

$$s\ell_r(v_1,\dots,v_r) = (-1)^{\sum_i i|x_{r-i}|+r}\delta_{r-1}(sv_1 \wedge \cdots \wedge sv_r).$$

The condition $\delta^2 = 0$ corresponds to the generalised Jacobi identities.

Taking the graded dual determines a dg algebra, and if V is of finite type and concentrated in positive degrees, then the opposite is true: A differential d on the free graded commutative algebra $\Lambda((sV)^\vee)$ determines an L_∞-structure on V. The differential is determined by restriction to $(sV)^\vee$, and similar to above we write $d_n\colon (sV)^\vee \to \Lambda^n((sV)^\vee)$ for the restriction, and the n-ary operation can be read of from this. Further, an L_∞-morphism is then just a dg algebra morphism.

This gives a convenient way of packaging the data of an L_∞-algebra with easy access to structural properties. For example, a minimal L_∞-structure on a positively graded vector space of finite type is given by a free graded commutative algebra equipped with a differential with no linear part.

Next, let (W, d_W) and (V, d_V) be chain complexes, and

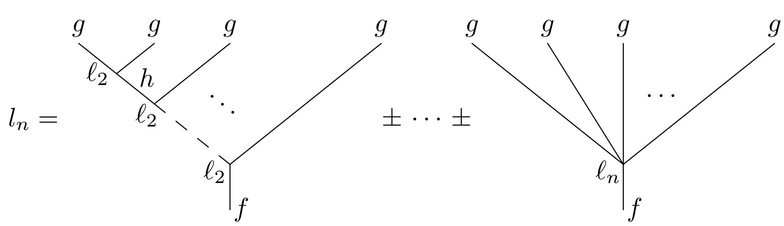
$$h \circlearrowright W \underset{g}{\overset{f}{\rightleftarrows}} V, \tag{4}$$

a contraction.

If W is an L_∞-algebra, then there is an induced L_∞-structure on V. This is the Homotopy Transfer Theorem for L_∞-algebras. It is mentioned in [23] without proof, and shown by Huebschmann [21]. A version with explicit formulae for the resulting structure appears in [27] and in [1] by Berglund. The latter also contains details on how to extend the maps occurring to L_∞-morphisms.

Before stating the theorem, suppose we are given a contraction (4) and an L_∞-structure $\{\ell_n\}$ on W, and consider rooted trees with each leaf labeled by g, each vertex by ℓ_n where $n+1$ is the valence of the vertex, each internal edge by h, and the root by f. Such a tree with n leaves may be taken as recipe for building a map $V^{\otimes n} \to V$, by composing from leaves to root, each leaf taking an input from one of the n copies of V in the source.

We can then form a signed sum over all such rooted trees with n leafs and labels as described, to get a map $l_n \colon V^{\otimes n} \to V$, which we may depict as



If we decorate the root of each tree by h instead of f, we get recipes for building maps $V^{\otimes n} \to W$, and forming the signed sum over all rooted trees with n leaves and this decoration, we get a map $g_n \colon V^{\otimes n} \to W$.

Theorem 2.20 (Homotopy Transfer Theorem). *Let $(L, \{\ell_n\})$ be an L_∞-algebra, and let (V, d_V) be a chain complex. Given a contraction*

$$h \circlearrowright L \underset{g}{\overset{f}{\rightleftarrows}} V,$$

the collection of maps $\{l_n\}$ as discussed above defines an L_∞-structure on V. The collection $\{g_n\}$ defines an extension of g to an L_∞ quasi-isomorphism $(V, \{l_n\}) \to (L, \{\ell_n\})$.

There is an extension of f to an L_∞ quasi-isomorphism, but the description is more complicated and we will not need it here, cf. [1].

Denote ℓ_2 by $[-,-]$ for now and let $x, y \in V$. For the binary and ternary transferred operations we get

$$l_2(x,y) = f([g(x), g(y)]),$$

and

$$\begin{aligned} l_3(x,y,z) = f \circ (&-[h[g(x),g(y)], g(z)] + (-1)^{|x|}[g(x), h[g(y),g(z)]] \\ &+(-1)^{|y||z|}[h[g(x),g(z)], g(y)] + \ell_3(g(x),g(y),g(z))) \end{aligned} \tag{5}$$

2.5. **Algebraic models for rational spaces.** There are two approaches by respectively Quillen [32] and Sullivan [37], to model spaces algebraically over $\mathbb{Q}$. With a few restrictions they succeed in providing complete algebraic invariants for the rational homotopy type of spaces. Standard references are [13] by Félix-Halperin-Thomas, and [7] by Bousfield-Gugenheim.

Definition 2.21. A rational space *is a simply connected space X such that the homotopy groups $\pi_n(X)$ are uniquely divisible for all k. Equivalently the homology groups $H_n(X)$ are uniquely divisible for $k > 0$.*

Recall that a uniquely divisible group is a group G where multiplication $G \xrightarrow{\cdot n} G$ is an isomorphism for all non-zero integers n. Equivalently it is a group which admits the structure of a vector space over $\mathbb{Q}$.

Definition 2.22. A rationalisation *of a simply connected space X is a rational space $X_{\mathbb{Q}}$ together with a map $r\colon X \to X_{\mathbb{Q}}$ inducing isomorphisms on rational homotopy groups*

$$\pi_n(X) \otimes \mathbb{Q} \xrightarrow{\sim} \pi_n(X_{\mathbb{Q}}), \quad n > 0.$$

Equivalently r induces isomorphisms on rational homology groups

$$H_n(X;\mathbb{Q}) \xrightarrow{\sim} H_n(X_{\mathbb{Q}}), \quad n > 0.$$

Definition 2.23. *A simplicial object in a category $\mathcal{C}$, is a functor $F\colon \Delta^{\mathrm{op}} \to \mathcal{C}$, where Δ is the category of finite non-empty linearly ordered sets: objects are finite non-empty linearly ordered sets, and morphisms are order preserving maps.*

Let $[n]$ denote the integers from 0 to n with the usual ordering, considered as an object of Δ. A common way to give the data of a simplicial object is then to specify the *n-simplicies* $F_n := F([n])$, together with *structure maps* $F(\varphi)\colon F_m \to F_n$ for $\varphi\colon [n] \to [m]$ in Δ.

A fundamental connection between topology and algebra is given by the Sullivan-de Rham algebra functor. We remind the reader that all algebras here are defined over $\mathbb{Q}$.

Definition 2.24. *The* simplicial de Rham algebra $\Omega^*_\bullet$ *is the simplicial commutative dg algebra with n-simplicies*

$$\Omega^*_n := \frac{\Lambda(t_i, dt_i \mid i = 0, \dots, n)}{(t_0 + \cdots + t_n - 1,\, dt_0 + \cdots + dt_n)}, \qquad |t_i| = 0,\, |dt_i| = 1,$$

differential as suggested by notation and structure maps $\varphi^*\colon \Omega^*_n \to \Omega^*_m$ given by

$$\varphi^*(t_i) = \sum_{j \in \varphi^{-1}(i)} t_j,$$

for $\varphi\colon [m] \to [n]$.

See [13] for more details.

Definition 2.25. *For a simplicial set X, the* Sullivan-de Rham algebra functor *is given by*

$$\Omega^*(X) := \mathrm{Hom}_{sSet}(X, \Omega^*_\bullet),$$

the set of simplicial maps from X to the simplicial de Rham algebra.

This is naturally a graded commutative cochain algebra, and integration defines a quasi-isomorphism to the singular cochains $\Omega^*(X) \to C^*(X;\mathbb{Q})$, cf. [7].

Definition 2.26. *For a graded commutative cochain algebra A, the* spatial realization functor *is given by*

$$\langle A\rangle := \mathrm{Hom}_{cdga}(A, \Omega^*_\bullet),$$

the simplicial set of algebra morphisms from A to the simplicial de Rham algebra.

The Sullivan-de Rham algebra and spatial realization functors form a contravariant adjunction, that is for a simplicial set X and a graded commutative cochain algebra A there is a bijection

$$\mathrm{Hom}_{cdga}(A, \Omega^*(X)) \simeq \mathrm{Hom}_{sSet}(X, \langle A\rangle),$$

and the two functors induce an equivalence of homotopy categories between simply connected rational Kan complexes of finite $\mathbb{Q}$-type, and *cofibrant* simply connected graded commutative cochain algebras of finite type. Together with the standard Quillen equivalence between simplicial sets and topological spaces, this justifies the study of rational homotopy theory by the algebraic models discussed below. We will not introduce too much model category language here, but simply note that all Sullivan algebras as defined in the following are cofibrant. See [13] for details.

In the following we will write $\Omega^*(X)$ for a topological space X, by which we mean $\Omega^*(S_\bullet X)$, where $S_\bullet X$ is the singular simplicial set of X.

2.5.1. *Sullivan models.* A *Sullivan algebra* is a cochain algebra A, which is free as a graded algebra $A = \Lambda(V)$ for some graded vector space $V = \bigoplus_{n\geq 1} V^n$, satisfying the following conditions. There is an increasing sequence of graded subspaces

$$V(0) \subseteq V(1) \subseteq \cdots \subseteq \bigcup_k V(k) = V$$

such that $d(V(0)) = 0$ and $d(V(n)) \subseteq \Lambda(V(n-1))$. A Sullivan algebra $A = \Lambda V$ is *minimal* if $d(V)$ is contained in the decomposables $\Lambda^{\geq 2}V$.

Remark 2.27 Recall from above that a dg commutative algebra for which the underlying algebra is free on a positively graded vector space V of finite type determines an L_∞-structure on sV. In particular a Sullivan algebra for a graded vector space V of finite type determines an L_∞-structure $\{\ell_n\}$ on sV, and minimality of the Sullivan algebra translates to the condition that $\ell_1 = 0$.

This is the defining property for a *minimal* L_∞-algebra in general.

Let X be a simply connected space. A cochain model for X is a cochain algebra A together with a quasi-isomorphism to the de Rham algebra $\Omega^*(X)$. A *Sullivan model* for X is a cochain model for which A is a Sullivan algebra. A minimal Sullivan model for X is a Sullivan model for which A is minimal.

One of the main points of Sullivan's theory is then the following.

Theorem 2.28 (See [13])**.** *There is always a minimal Sullivan model for a simply connected space. The minimal model is unique up to isomorphism.*

Thus we may speak of *the* minimal Sullivan model for a simply connected space.

The existence of minimal Sullivan models together with spatial realisation gives a convenient way of rationalising simply connected spaces.

Proposition 2.29. *Let X be a simply connected space of finite $\mathbb{Q}$-type with minimal Sullivan model $m_X \xrightarrow{\sim} \Omega^*(S_\bullet X)$. Then the realization of the adjoint map $S_\bullet X \to \langle m_X \rangle$ gives a rationalisation of X. In particular we may take $X_{\mathbb{Q}} = |\langle m_X \rangle|$.*

2.5.2. *Quillen models.* Let X be a simply connected space. A Lie model for X is a positively graded dg Lie algebra $L = L_{\geq 1}$ such that $\mathscr{C}(L)^\vee$ is a cochain model for X. A *Quillen model* for X is a Lie model for which the underlying Lie algebra L is free. A minimal Quillen model is a Quillen model for which the differential is decomposable.

Theorem 2.30 (See [13])**.** *There is always a minimal Quillen model for a simply connected space. The minimal model is unique up to isomorphism.*

Again, we may speak of *the* minimal Quillen model for a simply connected space.

2.5.3. *Formality and coformality.*

Definition 2.31. *(1) A commutative cochain algebra A is* formal *if A and $H^*(A)$ are quasi-isomorphic as commutative cochain algebras.*
(2) A simply connected space X is formal *if the commutative cochain algebra $\Omega^*(X)$ is formal.*

Equivalently X is formal if the minimal Sullivan model is quasi-isomorphic to $H^*(X)$.

Definition 2.32. *(1) A dg Lie algebra L is formal if L and $H_*(L)$ are quasi-isomorphic as dg Lie algebras.*
(2) A simply connected space X is coformal *if the minimal Quillen model is formal.*

The homology of the minimal Quillen model for X is the rational homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$.

Proposition 2.33 (Cf. Loday-Vallette [27] Section 10.4)**.** *Let L and L' be dg Lie algebras, and consider them as L_∞-algebras with trivial higher operations. There exists an L_∞-quasi-isomorphism $L \xrightarrow{\sim} L'$ if and only if L and L' are quasi-isomorphic as dg Lie algebras.*

By this proposition, a dg Lie algebra L is formal if and only if there exist an L_∞ quasi-isomorphism $H_*(L) \xrightarrow{\sim} L$, where L and $H_*(L)$ are considered L_∞-algebras with trivial higher operations.

We may always choose a contraction

$$\begin{tikzcd} L \arrow[loop left, "k"] \arrow[r, shift left, "q"] & H_*(L) \arrow[l, shift left, "i"] \end{tikzcd} \tag{6}$$

onto the homology, cf. Lemma 2.1. The Homotopy Transfer Theorem for L_∞-algebras then produces a minimal L_∞-structure on $H_*(L)$ with l_2-operation the standard bracket induced on the homology, and an L_∞-quasi-isomorphism

$$H_*(L) \xrightarrow{\sim} L.$$

Corollary 2.34. *A dg Lie algebra L is formal if there exist a contraction* (6) *such that L_∞-structure $\{l_n\}$ on $H_*(L)$ produced by the Homotopy Transfer Theorem has $l_r = 0$ for all $r \geq 3$.*

The detection of a higher operation on the homology is not enough to conclude that a dg Lie algebra is not formal. There exist minimal L_∞-algebras with non-trivial higher operations which have isomorphic minimal structures where all higher operations vanish.

Example 2.35 Consider the minimal L_∞-algebra determined by

$$L := (\Lambda(s,t,u,v), du = v^2 + t^4 + s^6 + 2vt^2 + 2vs^3 + 2s^3t^2),$$

with $|s| = 2$, $|t| = 3$, $|v| = 6$, and $|u| = 11$. This has non-trivial operations up to arity 6. The map given by $v' \mapsto v + t^2 + s^3$ induces an isomorphism of dg algebras

$$(\Lambda(s,t,u,v'), du = v'^2) \to L$$

which determines a minimal L_∞-structure with no higher operations. Point in case is that we may have chosen a basis for the underlying graded vector space in which higher operations do not vanish, but they do by a "non-linear" change of basis.

However there is a way around this, as we shall see now.

2.5.4. *Massey brackets.* Completely analogous to Massey products, there are secondary operations on the homology of a dg Lie algebra L, called Massey brackets. For a formal dg Lie algebra all Massey brackets vanish. An L_∞-structure on the homology provides the choices needed for the construction of such, and detection of a higher operation for which the corresponding Massey bracket does not vanish is enough to conclude that a dg Lie algebra is not formal.

We will only introduce the triple Massey bracket here, as this is all we need to detect a ternary operation in our examples and conclude non-formality.

Definition 2.36. *Let (L,d) be a dg Lie algebra, and let $x,y,z \in L$ be cycles, representing homology classes $\overline{x}, \overline{y}$ and $\overline{z}$. The triple Massey bracket $M_3(\overline{x},\overline{y},\overline{z})$ is the set of homology classes*

$$\left\{\overline{(-1)^{|x|}[x,s] - [t,z] - (-1)^{|x||y|+|y|}[y,u]} \mid ds = [y,z],\, dt = [x,y],\, du = [x,z]\right\}.$$

Note that $M_3(\overline{x},\overline{y},\overline{z})$ is non-empty if and only if

$$\overline{[y,z]} = \overline{[x,y]} = \overline{[x,z]} = 0.$$

Now, let L be a dg Lie algebra with cycles x,y,z such that

$$[\overline{x},\overline{y}] = [\overline{x},\overline{z}] = [\overline{y},\overline{z}] = 0,$$

and choose $s,t,u \in L$ such that $ds = [y,z]$, $dt = [x,y]$ and $du = [x,z]$. Define the operation

$$\langle x,y,z\rangle_{s,t,u} := (-1)^{|x|}[x,s] - [t,z] - (-1)^{|x||y|+|y|}[y,u].$$

This defines a cycle in L by the Jacobi identity. Define the subgroup

$$A_{\overline{x},\overline{y},\overline{z}} := [\overline{x}, H_*(L)] + [\overline{y}, H_*(L)] + [\overline{z}, H_*(L)] \subseteq H_*(L).$$

We can show that

- The class $\overline{\langle x,y,z\rangle_{s,t,u}} + A_{\overline{x},\overline{y},\overline{z}} \in H_*(L)/A_{\overline{x},\overline{y},\overline{z}}$ does not depend on the representatives chosen for $\overline{x},\overline{y}$ and $\overline{z}$,
- The class $\overline{\langle x,y,z\rangle_{s,t,u}} + A_{\overline{x},\overline{y},\overline{z}}$ does not depend on the representatives chosen for s,t and u.

Thus $M_3(\overline{x},\overline{y},\overline{z})$, when non-empty, is a well-defined class in $H_*(L)/A_{\overline{x},\overline{y},\overline{z}}$.

Definition 2.37. *We say that M_3* vanishes *on L if $M_3(\alpha,\beta,\gamma) = 0 \in H_*(L)/A_{\alpha,\beta,\gamma}$ for all $\alpha,\beta,\gamma \in H_*(L)$ such that $M_3(\alpha,\beta,\gamma)$ is non-empty.*

The following lemmas are easy consequences of the bullets above.

Lemma 2.38. *Let L be a dg Lie algebra. If $d_L = 0$ then M_3 vanishes.*

Lemma 2.39. *Let $f\colon L \to L'$ be a map of dg Lie algebras. For all $\alpha,\beta,\gamma \in H_*(L)$ such that $M_3(\alpha,\beta,\gamma)$ is non-empty,*

(1) f induces a well-defined map $H_*(L)/A_{\alpha,\beta,\gamma} \to H_*(L')/A_{f_*(\alpha),f_*(\beta),f_*(\gamma)}$,
(2) $M_3(\alpha,\beta,\gamma)$ is mapped to $M_3(f_*(\alpha), f_*(\beta), f_*(\gamma))$ under this map.

Corollary 2.40. *Let $f\colon L \to L'$ be a quasi-isomorphism of dg Lie algebras. For all $\alpha,\beta,\gamma \in H_*(L)$ such that $M_3(\alpha,\beta,\gamma)$ is non-empty, $M_3(\alpha,\beta,\gamma) = 0$ if and only if $M_3(f_*(\alpha), f_*(\beta), f_*(\gamma)) = 0$.*

Proof. For a quasi-isomorphism f the induced map of Lemma 2.39 (1) is an isomorphism. □

Proposition 2.41. *Let L be a dg Lie algebra. If L is formal then M_3 vanishes on L.*

Proof. When L is formal, there is a zig-zag of quasi-isomorphisms $L \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} H_*(L)$. By Lemma 2.38 M_3 vanishes on $H_*(L)$, and by Corollary 2.40 we get that M_3 also vanishes on L. □

Proposition 2.42. *Let L be a dg Lie algebra, and denote by ℓ_3 the ternary operation on $H_*(L)$ transferred from L along some choice of contraction. If $M_3(\alpha,\beta,\gamma)$ is non-empty for $\alpha,\beta,\gamma \in H_*(L)$ then*

$$\ell_3(\alpha,\beta,\gamma) \in M_3(\alpha,\beta,\gamma).$$

Proof. Let $\alpha,\beta,\gamma \in H_*(L)$ be such that $M_3(\alpha,\beta,\gamma)$ is non-empty, and let

$$\begin{tikzcd} L \arrow[loop left, "k"] \arrow[r, shift left, "q"] & H_*(L) \arrow[l, shift left, "i"] \end{tikzcd}$$

be a contraction.

Recall the explicit formula (5) for ℓ_3 on $H_*(L)$, and note that all higher operations are zero on L. We insert

$$\begin{aligned} l_3(\alpha,\beta,\gamma) = q \circ (&-[k[i(\alpha), i(\beta)], i(\gamma)] + (-1)^{|\alpha|}[i(\alpha), k[i(\beta), i(\gamma)]] \\ &+(-1)^{|\beta||\gamma|}[k[i(\alpha), i(\gamma)], i(\beta)]), \end{aligned}$$

and make the choices

$$\begin{gathered} x = i(\alpha), \quad y = i(\beta), \quad z = i(\gamma) \\ s = k[i(\beta), i(\gamma)], \quad t = k[i(\alpha), i(\beta)] \quad \text{and} \quad u = k[i(\alpha), i(\gamma)]. \end{gathered}$$

Using the fact that $[i(\beta), i(\gamma)]$ is a cycle and thus $dk[i(\beta), i(\gamma)] = [i(\beta), i(\gamma)]$, we check that $ds = [y,z]$, and similarly that $dt = [x,y]$ and $du = [x,z]$. Further, it is easy to check that these choices exhibit $l_3(\alpha,\beta,\gamma)$ on the form

$$\overline{(-1)^{|x|}[x,s] - [t,z] - (-1)^{|x||y|+|y|}[y,u]}.$$

□

We conclude that the existence of a contraction such that the ternary transferred operation represents a non-zero class in some Massey bracket, is sufficient to conclude non-formality.

2.5.5. *Koszul Spaces.* The term *Koszul space* is coined by Berglund in [2]. The essentials for us are outlined below.

Theorem 2.43 (Berglund [2] Theorem 1.2)**.** *Let X be a simply connected space such that $H^*(X;\mathbb{Q})$ is of finite type. The following are equivalent:*

(1) X is both formal and coformal.
(2) X is formal and the cohomology algebra $H^*(X;\mathbb{Q})$ is a Koszul graded commutative algebra.
(3) X is coformal and the homotopy Lie algebra $\pi_*(\Omega X)\otimes\mathbb{Q}$ is a Koszul graded Lie algebra.

Definition 2.44. *A* Koszul space *is a simply connected space X such that $H^*(X;\mathbb{Q})$ is of finite type, satisfying the equivalent conditions of Theorem 2.43.*

Theorem 2.45 (Berglund [2] Theorem 1.3)**.** *Let X be a Koszul space with rational homotopy Lie algebra L and cohomology algebra A. The Koszul dual graded commutative algebra $L^!$ is isomorphic to A, and the Koszul dual graded Lie algebra $A^!$ is isomorphic to L.*

Corollary 2.46. *Let X be a Koszul space with homotopy Lie algebra L and cohomology algebra A. Then*

(1) $\mathscr{L}(A^\vee)$ is the minimal Quillen model for X.
(2) There is a surjective quasi-isomorphism

$$\mathscr{L}(A^\vee) \xrightarrow{\sim} L,$$

corresponding to a twisting morphism $\kappa\colon A^\vee \to L$.

Proof. Let m_X be a Sullivan model for X. Recall from section 2.2 that the bar and cobar constructions form a Quillen equivalence, so $\mathscr{L}(m_X^\vee)$ is a Quillen model for X. Since X is formal the minimal model m_X is quasi-isomorphic to the cohomology A, and the functor $\mathscr{L}$ preserves all weak equivalences so in particular $\mathscr{L}(A^\vee)$ is the minimal Quillen model for X.

That X is also coformal means that $\mathscr{L}(A^\vee)$ is quasi-isomorphic to L. We can do better: $A^\vee$ is a Koszul coalgebra so the natural projection followed by the quotient map is a surjective quasi-isomorphism of dg Lie algebras

$$\mathscr{L}(A^\vee) \xrightarrow{\sim} (A^\vee)^{¡},$$

which corresponds to the twisting morphism κ as discussed in section 2.3. By Theorem 2.45 we have $(A^\vee)^{¡} \simeq L$. $\square$

The existence of this explicit surjective quasi-isomorphism is the special feature of Koszul spaces upon which this thesis is build.

Remark 2.47 Recall from Theorem 2.15 that L is generated by some graded vector space V, and that A is generated by the shifted dual $(sV)^\vee$. The twisting morphism κ restricts to the canonical identification $((sV)^\vee)^\vee \simeq V$.

2.6. **Classification of fibrations.** The classification of fibrations with a given fibre is mainly due to Stasheff [36] and May [28].

Let X be a space homotopy equivalent to a finite CW-complex. Denote by $\operatorname{aut}(X)$ the topological monoid of homotopy equivalences $X \to X$, the *homotopy automorphisms* of X. Let $\operatorname{aut}_*(X)$ denote the submonoid of base point preserving maps.

The fibrations $E \to B$ with fibre homotopy equivalent to X are called X-fibrations in the following. They are classified by $B\operatorname{aut}(X)$ in the following sense. For any space B with the homotopy type of a CW-complex, the homotopy classes of maps $B \to B\operatorname{aut}(X)$ are in bijection with equivalence classes of X-fibrations $E \to B$, under the equivalence relation generated by fibre homotopy equivalences.

There exists a universal X-fibration $E_X \to B_X$, and the bijection is realised by pulling back along maps $B \to B_X$. The base B_X is homotopy equivalent to $B\operatorname{aut}(X)$.

In a similar way $B\operatorname{aut}_*(X)$ classifies fibrations with a section, and further, the inclusion of monoids $\operatorname{aut}_*(X) \to \operatorname{aut}(X)$ induces a map $B\operatorname{aut}_*(X) \to B\operatorname{aut}(X)$ which is equivalent to the universal X-fibration. See [28].

2.6.1. *Lie algebra derivations.* Having just discussed algebraic models for spaces, it is natural to ask how the universal fibrations are modeled. We here outline an answer by Schlessinger-Stasheff [33] in terms of Lie models. See also Tanré [38] for a detailed account, and Berglund-Madsen [5] for a recent discussion and expansion.

Let $f\colon L \to M$ be a map of dg Lie algebras. An f-derivation θ, is a graded linear map $\theta\colon L \to M$, such that

$$\theta([x,y]) = [\theta(x), f(y)] + (-1)^{|\theta||x|}[f(x), \theta(y)].$$

The graded vector space of f-derivations is denoted by $\operatorname{Der}_f(L,M)$, and it naturally is a subcomplex of $\operatorname{Hom}(L,M)$.

If $M = L$ and f is the identity on L, we write $\operatorname{Der} L$ and simply call elements derivations on L. The chain complex $\operatorname{Der} L$ has the structure of a dg Lie algebra, with the bracket given by the graded anti-symmetrized composition of maps. I.e. for $\theta, \theta' \in \operatorname{Der} L$,

$$[\theta, \theta'] = \theta \circ \theta' - (-1)^{|\theta||\theta'|}\theta' \circ \theta.$$

The Jacobi identity implies that the map $\operatorname{ad}_x\colon L \to L$ given by $\operatorname{ad}_x(y) = [x,y]$ is a derivation on L. The map $\operatorname{ad}\colon L \to \operatorname{Der} L$, mapping x to ad_x is a morphism of dg Lie algebras, and the image is a Lie ideal. The quotient $\operatorname{Der} L/\operatorname{ad} L$ is thus a dg Lie algebra, denoted $\operatorname{Out} L$ for outer derivations on L.

We may also consider the mapping cone (homotopy cofibre) of the map $\operatorname{ad}\colon L \to \operatorname{Der} L$, denoted by $\operatorname{Der} L/\!/\operatorname{ad} L$. This can also be equipped with a graded Lie bracket. As a graded vector space it is the direct sum $\operatorname{Der} L \oplus sL$. The bracket and differential are extensions of those on the derivations by

$$\begin{aligned} [\theta, sx] &= (-1)^{|\theta|} s\theta(x), \qquad [sx, sy] = 0 \\ d(sx) &= \operatorname{ad}_x - s d_L(x), \end{aligned}$$

for $\theta \in \operatorname{Der} L$ and $x, y \in L$.

For a dg Lie algebra L with a free underlying graded Lie algebra, the positive part $(\operatorname{Der} L/\!/\operatorname{ad} L)_+$ is the *Schlessinger-Stasheff classifying dg Lie algebra* for L. It classifies fibrations of Lie algebras with fibre L in way analogous to how $B\operatorname{aut}(X)$ classifies fibrations of spaces with fibre X. Even better:

Theorem 2.48 (Cf. Tanré [38] Corollaire VII.4(4))**.** *Let X be a simply connected space homotopy equivalent to a finite CW-complex. If $\mathscr{L}$ is a Quillen model for X, then the 1-connected cover of the map induced by inclusion of monoids*

$$B\operatorname{aut}_*(X)\langle 1\rangle \to B\operatorname{aut}(X)\langle 1\rangle$$

is modeled by the map of dg Lie algebras

$$(\operatorname{Der}\mathscr{L})_+ \to (\operatorname{Der}\mathscr{L} /\!/ \operatorname{ad}\mathscr{L})_+$$

given by the inclusion of the derivations.

3. Transferred L_∞-structure

The Lie model produced by Schlessinger-Stasheff and Tanré is fine for theoretical purposes, but it is very large. Quillen models are large in the first place, and taking the derivation Lie algebra on a Quillen model does not help this. In the case of homotopy automorphisms on a Koszul space we produce a much smaller L_∞-algebra which retains all the information of the larger model. This can be achieved due to the formality properties of Koszul spaces.

In this section we produce the smaller model, and study the L_∞-structure. The first part of this section sets up contractions and isomorphisms needed. The second part specializes to the case of interest: Koszul algebras. The third part is mostly technical. There are several gradings on the objects we study, and in the third part we record how these interact with the maps set up. This leads to the fourth part where we record some fairly immediate consequences as to how the L_∞-structure interacts with one of the gradings. This has interesting consequences for deciding formality of the model.

3.1. Induced contractions.

In this section we reduce the study of the derivations $\operatorname{Der}\mathscr{L}(C)$ to the study of a twisted version of the complex $C^\vee \otimes L$ by basic perturbation theory, and application of a standard isomorphism. There is little novelty in this, and some of the proofs are skipped.

Definition 3.1. *Let A be a dg commutative algebra, and L a dg Lie algebra. The chain complex $A\otimes L$ is equipped with a graded Lie bracket making it a dg Lie algebra. The bracket is given by*

$$[a\otimes x, b\otimes y] = (-1)^{|b||x|} ab \otimes [x,y],$$

for $a, b \in A$ and $x, y \in L$.

For any Maurer-Cartan element in this dg Lie algebra $\tau \in MC(A\otimes L)$, we have the twisted differential $d^\tau_{A\otimes L} = d_{A\otimes L} + \operatorname{ad}_\tau$, and write $A\otimes_\tau L$ for the resulting dg Lie algebra (in this way $A\otimes L$ equals $A\otimes_0 L$ - the dg Lie algebra twisted by the Maurer-Cartan element 0).

Let the following be a contraction of dg Lie algebras

$$\begin{tikzcd} M \arrow[loop left, "h"] \arrow[r, shift left, "f"] & L. \arrow[l, shift left, "g"] \end{tikzcd} \tag{7}$$

That is, L and M are dg Lie algebras, f is a quasi-isomorphism of dg Lie algebras, g is a chain map and h a chain homotopy, such that $fg = 1_L$ and $dh + hd = gf - 1_{\mathcal{L}}$.

We may assume that $hg = 0$, $h^2 = 0$ and $fh = 0$. Note that g and h are in general not Lie maps.

Lemma 3.2. *Given a graded commutative algebra A, a contraction (7) of dg Lie algebras and a Maurer-Cartan element $\tau \in MC(A \otimes M)$ such that A is nilpotent, there is an induced contraction of chain complexes*

$$\begin{tikzcd} A \otimes_\tau M \arrow[loop left, "h'"] \arrow[r, shift left, "1\otimes f"] & A \otimes_{(1\otimes f)(\tau)} L, \arrow[l, shift left, "g'"] \end{tikzcd}$$

where

(1) $1 \otimes f$ is a quasi-isomorphism of chain complexes,
(2) g' is given by the recursive formula $g' = 1 \otimes g + (1 \otimes h)\,\mathrm{ad}_\tau\, g'$,
(3) h' is given by the recursive formula $h' = 1 \otimes h + h'\,\mathrm{ad}_\tau (1 \otimes h)$.

These formulae converge because A is nilpotent.

For most of our applications A will be a finite dimensional Koszul algebra (see Remark 3.7 though), and thus nilpotent. See [1] for a discussion of weaker assumptions which may be adapted to our situation.

Proof. The contraction (7) induces a contraction

$$\begin{tikzcd} A \otimes M \arrow[loop left, "1\otimes h"] \arrow[r, shift left, "1\otimes f"] & A \otimes L. \arrow[l, shift left, "1\otimes g"] \end{tikzcd}$$

By the Basic Perturbation Lemma [8, 18] we obtain a new contraction

$$\begin{tikzcd} A \otimes_\tau M \arrow[loop left, "h'"] \arrow[r, shift left, "f'"] & (A \otimes L, 1 \otimes d_L + t'), \arrow[l, shift left, "g'"] \end{tikzcd}$$

where the maps are defined by the recursive formulae

$$\begin{aligned} f' &= 1 \otimes f + f'\,\mathrm{ad}_\tau (1 \otimes h) \\ g' &= 1 \otimes g + (1 \otimes h)\,\mathrm{ad}_\tau\, g' \\ h' &= 1 \otimes h + h'\,\mathrm{ad}_\tau (1 \otimes h) \\ t' &= f'\,\mathrm{ad}_\tau (1 \otimes g). \end{aligned}$$

Since f is a morphism of Lie algebras we have for any $a \otimes m \in A \otimes M$

$$(1 \otimes f)\,\mathrm{ad}_\tau (1 \otimes h)(a \otimes m) = \mathrm{ad}_{(1\otimes f)\tau}\, a \otimes fh(m),$$

and as $fh = 0$ we get $f' = 1 \otimes f$. Further for any $a \otimes l \in A \otimes L$ we get

$$t'(a \otimes l) = (1 \otimes f)\,\mathrm{ad}_\tau (1 \otimes g)(a \otimes l) = \mathrm{ad}_{(1\otimes f)(\tau)} (a \otimes l),$$

so that $t' = \mathrm{ad}_{(1\otimes f)(\tau)}$. □

Proposition 3.3. *Let C be a dg coalgebra, L a dg Lie algebra, and let* $\mathrm{Hom}(C, L)$ *denote the convolution dg Lie algebra. The map*

$$\varphi \colon C^\vee \otimes L \to \mathrm{Hom}(C, L),$$

given by

$$\varphi(f \otimes x)(c) = (-1)^{|c||x|} f(c)x,$$

is a map of dg Lie algebras with respect to the structure of Definition 3.1 on the left hand side, natural in C and L. If C and L are of finite type and either $C^\vee$ and

L are either both bounded above or both bounded below, then φ is an isomorphism (if either C or L is finite, the other need just be of finite type for φ to be an isomorphism).

For any Maurer-Cartan element $\tau \in \mathrm{MC}(C^\vee \otimes L)$, the same formula defines a map

$$\varphi\colon C^\vee \otimes_\tau L \to \mathrm{Hom}(C, L)^{\varphi(\tau)}$$

with the same properties.

The inverse to φ is given by sending a map $f\colon C \to L$ to the expression

$$\sum_i (-1)^{|c_i|(|f|+|c_i|)} c_i^* \otimes f(c_i),$$

where we have chosen a basis $\{c_i\}$ for C, and $\{c_i^*\}$ is the dual basis for $C^\vee$. Note that this is a finite sum if either C is finite dimensional or if $C^\vee$ and L both bounded above or both below.

Remark 3.4 We will need Proposition 3.3 to relate $sA \otimes L \simeq \mathrm{Hom}(s^{-1}C, L)$ in which case the signs work out as follows. For $sa \otimes x \in sA \otimes L$,

$$\varphi(sa \otimes x)(s^{-1}c) = (-1)^{|c||x|+|a|} a(c)x.$$

For $f \in \mathrm{Hom}(s^{-1}C, L)$,

$$\varphi^{-1}(f) = \sum_i (-1)^{|f||sc_i^*|+1} sc_i^* \otimes f(s^{-1}c_i).$$

Proposition 3.5. *Let $(L, [-,-]_L, d_L)$ be a dg Lie algebra, and (C, Δ_C, d_C) a coaugmented dg coalgebra. For any twisting morphism $\tau \in \mathrm{Tw}(C, L)$, restriction to generators gives a natural isomorphism of chain complexes*

$$\iota^*\colon\ \mathrm{Der}_f(\mathscr{L}(C), L) \xrightarrow{\simeq} \mathrm{Hom}(\overline{C}, L)^\tau$$

of degree -1, where f corresponds to τ under the bijection of Proposition 2.7.

Proof. Clearly restriction gives an isomorphism of graded vector spaces. We must show that ι^* is a chain map of degree -1, i.e. that

$$\iota^*(\partial(\theta)) = -\partial^\tau(\iota^*(\theta)) \tag{8}$$

for any f-derivation θ.

Observe first that ι is a twisting morphism. That is, it satisfies

$$0 = \partial(\iota) + \frac{1}{2}[\iota, \iota] = d_{\mathscr{L}(C)}\iota + \iota d_C + \frac{1}{2}[\iota, \iota]. \tag{9}$$

We expand the left hand side of (8) using (9)

$$\begin{aligned}
\iota^*(\partial(\theta)) &= \iota^*(d_L\theta - (-1)^{|\theta|}\theta d_{\mathscr{L}(C)}) \\
&= (-1)^{|\theta|-1} d_L\theta\iota + \theta d_{\mathscr{L}(C)}\iota \\
&= (-1)^{|\theta|-1} d_L\theta\iota + \theta(-\iota d_C - \frac{1}{2}[\iota, \iota]).
\end{aligned}$$

By definition of the bracket in the convolution Lie algebra, and the fact that θ is an f-derivation, we get

$$\begin{aligned}\iota^*(\partial(\theta)) &= (-1)^{|\theta|-1} d_L\theta\iota - \theta\iota d_C - \theta\frac{1}{2}[-,-]_L(\iota\otimes\iota)\Delta_C) \\ &= -d_L\iota^*(\theta) - (-1)^{|\theta|}\iota^*(\theta)d_C - \frac{1}{2}[-,-]_L(\theta\otimes f + f\otimes\theta)(\iota\otimes\iota)\Delta_C \\ &= -\partial(\iota^*(\theta)) - \frac{1}{2}((-1)^{|\theta|}[\iota^*(\theta),\tau] + [\tau,\iota^*(\theta)]) \\ &= -\partial(\iota^*(\theta)) - [\tau,\iota^*(\theta)].\end{aligned}$$

This is precisely $-\partial^\tau(\iota^*(\theta))$, and we are done. □

A map of dg Lie algebras $f\colon L\to M$ induces a chain map

$$f^*\colon \operatorname{Der} L \to \operatorname{Der}_f(M,L).$$

Composing with ad gives a natural chain map $L \to \operatorname{Der}_f(M,L)$, and we may consider the mapping cone

$$\operatorname{Der}_f(M,L)/\!/(f^*\circ \operatorname{ad})(L),$$

which we just write $\operatorname{Der}_f(M,L)/\!/L$ for short.

Corollary 3.6. *Let τ be a twisting morphism in $\operatorname{Tw}(C,L)$. Restriction to generators gives a natural isomorphism of chain complexes*

$$\operatorname{Der}_f(\mathscr{L}(C),L)/\!/L \xrightarrow{\simeq} s\operatorname{Hom}(C,L)^\tau.$$

Proof. The isomorphism of 3.5 extends to a natural isomorphism of graded vector spaces

$$\phi\colon \operatorname{Der}_f(\mathscr{L}(C),L)/\!/L \xrightarrow{\simeq} s\operatorname{Hom}(C,L)^\tau,$$

where $\phi(sx)$ for $x\in L$, is the (suspension of the) linear map which annihilates $\overline{C}$ and on the counit is given by $\phi(sx)(1) = x$. We check that this extension is still a chain map. On one hand

$$\begin{aligned}\partial^\tau(\phi(sx))(1) &= (\partial(\phi(sx)) + [\tau,\phi(sx)])(1) \\ &= d_L\phi(sx)(1) - (-1)^{|\phi(sx)|}\phi(sx)d_C(1) + [\tau(1),\phi(sx)(1)] \\ &= d_L(x),\end{aligned}$$

and

$$\begin{aligned}\partial^\tau(\phi(sx))(c) &= (\partial(\phi(sx)) + [\tau,\phi(sx)])(c) \\ &= -(-1)^{|\phi(sx)|}\phi(sx)d_C(c) + (-1)^{|c||x|}[\tau(c),\phi(sx)(1)] \\ &= (-1)^{|c||x|}[\tau(c),x],\end{aligned}$$

for $c\in\overline{C}$. On the other hand

$$\begin{aligned}\phi(\partial(sx))(1) &= \phi(\operatorname{ad}_x f - sd_L(x))(1) \\ &= -d_L(x),\end{aligned}$$

and

$$\begin{aligned}\phi(\partial(sx))(c) &= \phi(\mathrm{ad}_x f - sd_L(x))(c)\\ &= \phi(\mathrm{ad}_x f)(c)\\ &= (-1)^{|x|}[x, f(s^{-1}c)]\\ &= (-1)^{|x|+(|c|+1)|x|+1}[\tau(c), x]\\ &= (-1)^{|c||x|+1}[\tau(c), x]\end{aligned}$$

The formula we have given is for a map to $\mathrm{Hom}(C, L)^\tau$, and as such has degree -1. Thus the calculation shows that ϕ is a chain map. □

Let C be a coaugmented dg cocommutative coalgebra and let L be a dg Lie algebra, such that C or L is finite, or $C^\vee$ and L are both bounded above or both bounded below. Write $A := C^\vee$. Observe that

$$\mathrm{Der}\,\mathscr{L}(C) = \mathrm{Der}_{id}(\mathscr{L}(C), \mathscr{L}(C)),$$

and that the identity on $\mathscr{L}(C)$ corresponds to $\iota \in \mathrm{Tw}(C, \mathscr{L}(C))$ - the universal twisting morphism from Section 2.2. By Proposition 3.3 and 3.5 respectively 3.6, we get natural isomorphisms

$$\mathrm{Der}\,\mathscr{L}(C) \xrightarrow{\;r\;} s\,\mathrm{Hom}(\overline{C}, \mathscr{L}(C))^\iota \xrightarrow{\;\varphi^{-1}\;} s\overline{A} \otimes_{\varphi^{-1}(\iota)} \mathscr{L}(C)$$

$$\mathrm{Der}\,\mathscr{L}(C)/\!/\mathscr{L}(C) \xrightarrow{\;r\;} s\,\mathrm{Hom}(C, \mathscr{L}(C))^\iota \xrightarrow{\;\varphi^{-1}\;} sA \otimes_{\varphi^{-1}(\iota)} \mathscr{L}(C).$$

Combining these natural isomorphisms with the maps of Lemma 3.2, we get commutative diagrams of complexes

$$\begin{tikzcd} s\overline{A} \otimes_{\varphi^{-1}(\iota)} \mathscr{L}(C) \arrow[r, "1\otimes f"] \arrow[d, "\simeq"'] & s\overline{A} \otimes_{\varphi^{-1}(\tau)} L \arrow[d, "\simeq"] \\ \mathrm{Der}\,\mathscr{L}(C) \arrow[r, "f_*"'] & \mathrm{Der}_f(\mathscr{L}(C), L) \end{tikzcd}$$

respectively

$$\begin{tikzcd} sA \otimes_{\varphi^{-1}(\iota)} \mathscr{L}(C) \arrow[r, "1\otimes f"] \arrow[d, "\simeq"'] & sA \otimes_{\varphi^{-1}(\tau)} L \arrow[d, "\simeq"] \\ \mathrm{Der}\,\mathscr{L}(C)/\!/\mathscr{L}(C) \arrow[r, "f_*"'] & \mathrm{Der}_f(\mathscr{L}(C), L)/\!/L. \end{tikzcd}$$

Naturality ensures that $(1 \otimes f)(\varphi^{-1}(\iota)) = \varphi^{-1}(\tau)$. We will suppress the natural isomorphism φ in the notation from here on.

Under the correspondence expressed by these diagrams, the contraction produced in Lemma 3.2 is the same as in [5]. Notably, the positive parts of these diagrams provide contractions of the Schlessinger-Stasheff classifying dg Lie algebra for dg Lie algebra fibrations with kernels quasi-isomorphic to $\mathscr{L}(C)$.

There is no need to treat the case $\overline{A}$ (modeling $B\,\mathrm{aut}_*(X)\langle 1\rangle$, cf. Theorem 2.48) separately from the case with A (modeling $B\,\mathrm{aut}(X)\langle 1\rangle$), as we may think of $s\overline{A}\otimes_\kappa L$

as a subcomplex of $sA \otimes_\kappa L$, at least until we consider homology. Thus we proceed with only the one case.

Remark 3.7 Lemma 3.2 illustrates the need for A to be nilpotent, and this assumption on A is carried through the thesis until Section 4.3 where completely different techniques are employed. Our application of Lemma 3.2 will be for $M = \mathscr{L}(C)$ and f corresponding to the Koszul morphism κ of Section 2.3. Nilpotency of A may be replaced by nilpotency of L as follows.

Sullivan [37] showed that $\operatorname{Der} \mathscr{M}_X$ is a Lie model for $B\operatorname{aut} X\langle 1\rangle$ for a simply connected space X, when $\mathscr{M}_X$ is the minimal Sullivan model for X. See also Tanré [38]. For a Koszul space X the dual to the bar construction $\mathscr{C}(L)^\vee$ is the minimal Sullivan model for X. The injective quasi-isomorphism $L^{¡} \to \mathscr{C}(L)$ (cf. Section 2.3) gives rise to a contraction

$$h \circlearrowright \mathscr{C}(L)^\vee \underset{g}{\overset{f}{\rightleftarrows}} A.$$

If L is nilpotent, then for any Maurer-Cartan element $\tau \in \mathrm{MC}(\mathscr{C}(L)^\vee \otimes L)$ we get an induced contraction

$$h' \circlearrowright \mathscr{C}(L)^\vee \otimes_\tau L \underset{g'}{\overset{1\otimes f}{\rightleftarrows}} A \otimes_{(1\otimes f)(\tau)} L. \tag{10}$$

with formulae as in Lemma 3.2, now converging because L is nilpotent. Analogous to above and what follows in the next section for $\operatorname{Der} \mathscr{L}(C)$, we have an isomorphism of dg Lie algebras

$$\operatorname{Der} \mathscr{C}(L)^\vee \simeq \mathscr{C}(L)^\vee \otimes_\pi L.$$

The L_∞-structure transferred to $sA \otimes_{(1\otimes f)(\tau)} L$ along (10) will be L_∞-isomorphic to the one we produce below, because there are quasi-isomorphisms such that

$$\begin{tikzcd} \operatorname{Der} \mathscr{L}(C)/\!/\mathscr{L}(C) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \operatorname{Der} \mathscr{C}(L)^\vee \arrow[dl, "\sim"] \\ & sA \otimes_\kappa L & \end{tikzcd}$$

commutes (cf. proof of Theorem 4.22 for the top map). Thus everything goes through in the case where L and not A is nilpotent.

3.2. **Contractions for Koszul algebras.** In this section we specialize the results of the previous section to the setting of our primary interest, the case of Koszul algebras.

Suppose that C Koszul coalgebra of finite type with Koszul dual graded Lie algebra L, and that the twisting morphism τ from above is the morphism κ as described in Section 2.3. Recall that C and L have zero differentials.

Lemma 3.8. *Let C be a Koszul coalgebra of finite type with Koszul dual graded Lie algebra L. The surjective quasi-isomorphisms $f\colon \mathscr{L}(C) \to L$ associated to the twisting morphism κ, gives rise to a contraction*

$$h \circlearrowright \mathscr{L}(C) \underset{g}{\overset{f}{\rightleftarrows}} L,$$

where we may choose h and g to have the following properties:

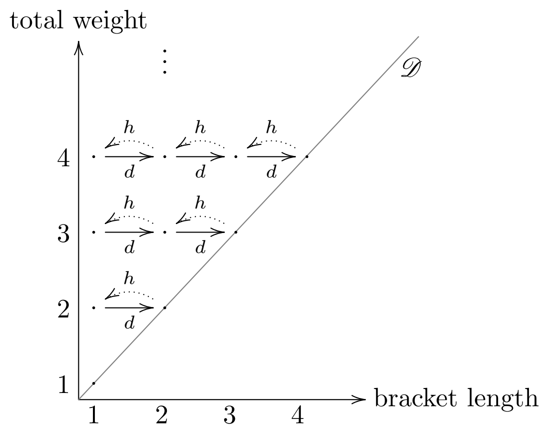
(i) *the image of g is contained in the diagonal*

$$\mathscr{D}_{\mathscr{L}} = \{x \in \mathscr{L}(C) \mid \ell_w(x) = \ell_b(x)\},$$

(ii) *h preserves the total weight,*

(iii) *the contraction satisfies the annihilation conditions: $fh = 0$, $hg = 0$ and $h^2 = 0$.*

We may illustrate the structure of $\mathscr{L}(C)$ and the above claims with the following picture.

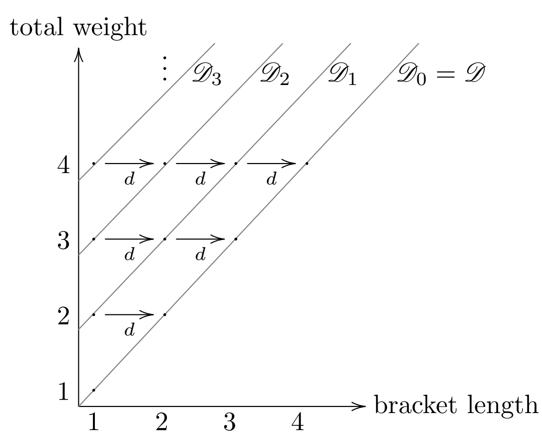


In the picture f is the quotient map to the cokernel of the right most differential in each row and it is zero outside the diagonal. Since L is Koszul dual to C, it is isomorphic to this cokernel. The diagonal is the free Lie algebra on $s^{-1}C(1)$, and by Koszul duality each generator corresponds to a dual generator of L. The quotient map f assigns to each generator of $C(1)$ the dual generator of L.

Definition 3.9. *For $i \geq 0$, define the i'th offset diagonal*

$$\mathscr{D}_i = \left\langle x \in \mathscr{L}(C) \mid \begin{array}{c} x \text{ is homogeneous for } \ell_w \text{ and } \ell_b, \\ \ell_w(x) - \ell_b(x) = i \end{array} \right\rangle.$$

We may illustrate these subspaces as follows



Observe that the differential lowers offset index by 1, i.e. $d(\mathscr{D}_{i+1}) \subseteq \mathscr{D}_i$ for all i.

Proof of Lemma 3.8. For point (i) recall that f is given by projection to $\mathscr{D}$ followed by the quotient map to $\operatorname{coker} d$. A section g of the quotient map $\mathscr{D} \to \operatorname{coker} d$ is trivially a chain map, it is a section of f, and it has the desired property.

For point (ii) consider the bounded below chain complex

$$\cdots \xrightarrow{d} \mathscr{D}_2 \xrightarrow{d} \mathscr{D}_1 \xrightarrow{d} \mathscr{D}_0 \xrightarrow{d} 0.$$

The maps f and g then gives rise to chain maps

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \mathscr{D}_2 \arrow[r, "d"] \arrow[d, shift left, "f_2"] & \mathscr{D}_1 \arrow[r, "d"] \arrow[d, shift left, "f_1"] & \mathscr{D}_0 \arrow[r, "d"] \arrow[d, shift left, "f_0"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u, shift left, "g_2"] & 0 \arrow[r] \arrow[u, shift left, "g_1"] & L \arrow[r] \arrow[u, shift left, "g_0"] & 0
\end{tikzcd}$$

which we also denote f and g. We now have a chain complex of vector spaces (projective modules) $\mathscr{D}_*$, and a diagram

$$\begin{tikzcd}
 & \mathscr{D}_* \arrow[d, "f", "\sim"'] \\
\mathscr{D}_* \arrow[ur, bend left, "gf"] \arrow[ur, "1"'] \arrow[r, "f"'] & L
\end{tikzcd}$$

where both gf and 1 are lifts of f along the surjective quasi-isomorphism f. The standard proof of the fact that gf and 1 are then homotopic, proceeds by constructing a homotopy.

First note that $gf - 1$ factors through $\ker f$, so we may construct a homotopy $h\colon \mathscr{D}_* \to \ker f \cap \mathscr{D}_*$. That is, a family of maps $h_i\colon \mathscr{D}_i \to \ker f \cap \mathscr{D}_{i+1}$ such that

$$dh_i + h_{i-1}d = g_i f_i - 1_i. \tag{11}$$

We will do this such that and h_i preserves the total weight and the annihilation conditions are satisfied.

First set $h_{-2} = h_{-1} = 0$, since $\mathscr{D}_i = 0$ for $i < 0$. Clearly these maps preserve the total weight and satisfy the annihilation conditions. Now for $n \geq 0$ suppose that we have constructed h_i with the desired property for all $i < n$.

Consider the map $g_n f_n - 1_n - h_{n-1}d\colon \mathscr{D}_n \to \ker f \cap \mathscr{D}_n$. If we apply the differential and use the fact that $gf - 1$ is a chain map together with the equation (11), we get

$$\begin{aligned}
d(g_n f_n - 1_n - h_{n-1}d) &= (g_n f_n - 1_n)d - dh_{n-1}d \\
&= (g_n f_n - 1_n)d - (g_n f_n - 1 - h_{n-2}d)d \\
&= 0.
\end{aligned}$$

Thus $g_n f_n - 1_n - h_{n-1}d$ factors through the cycles $Z(\ker f \cap \mathscr{D}_n)$ which is exactly the boundaries $B(\ker f \cap \mathscr{D}_n)$ since f is a quasi-isomorphism. Then we get a diagram

$$\begin{tikzcd}
 & (\ker f \cap \mathscr{D}_{n+1}) \arrow[d, "d"] \\
\mathscr{D}_n \arrow[ur, dashed, "h_n"] \arrow[r, "g_n f_n - 1_n - h_{n-1}d"'] & B(\ker f \cap \mathscr{D}_n)
\end{tikzcd}$$

with a lift as indicated because $\mathscr{D}_n$ is a vector space (thus projective) and the differential is surjective from $\ker f$ onto the boundaries. Such a lift is just a choice of pre-images $d^{-1}((g_n f_n - 1_n - h_{n-1}d)(x_j))$ for a linear basis $\{x_j\}$ of $\mathscr{D}_n$. The map $g_n f_n - 1_n - h_{n-1}d$ preserves the total weight by part (i) and the assumption on h_{n-1}, and since d preserves the total weight, we can always choose pre-images such that h_n preserves total weight.

Clearly we have $f_{n+1}h_n = 0$, and clearly $h_n g_n$ is zero for $n > 0$. Now $h_0 g_0$ is a lift of

$$(g_0 f_0 - 1_0 - h_{-1}d)g_0 = (g_0 f_0 - 1_0)g_0 = 0,$$

along the differential. We may choose h_0 to vanish on $\operatorname{Im} g_0$ without violating the condition that h_0 preserves total weight. Similarly $h_{n+1}h_n$ is a lift of

$$\begin{aligned}(g_{n+1}f_{n+1} - 1_{n+1} - h_n d)h_n &= -h_n - h_n d h_n \\ &= -h_n - h_n(g_n f_n - 1_n - h_{n-1}d) \\ &= h_n h_{n-1} d,\end{aligned}$$

along the differential. Inductively this is zero, and again we may choose h_{n+1} to be zero on $\operatorname{Im} h_n$ without violating the condition that h_{n+1} preserves total weight.

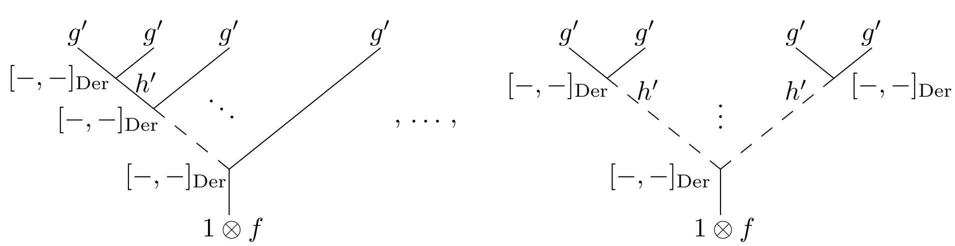
Finally h_n then satisfies (11) for $i = n$ and by construction the resulting homotopy h has the properties (ii) and (iii). $\square$

Now the dg Lie structure on $\operatorname{Der}\mathscr{L}(C)/\!/\mathscr{L}(C)$ induces one on $sA \otimes_\iota \mathscr{L}(C)$ by the natural isomorphisms in Proposition 3.3 and Corollary 3.6. This is not the same as the one from Definition 3.1. We denote the induced bracket by $[-,-]_{\mathrm{Der}}$ to remind our selfs that it comes from the derivations.

Suppose now that A is nilpotent. By Lemma 3.2 there are contractions along which we may transfer the structure using the Homotopy Transfer Theorem for L_∞-algebras to get L_∞-structures $\{\ell_n\}$ and $\{l_n\}$ as below

$$\begin{tikzcd} sA \otimes_\iota \mathscr{L}(C) \arrow[r, shift left] \arrow[d, dotted, no head] & sA \otimes_\kappa L \arrow[l, shift left] \arrow[r, shift left] \arrow[d, dotted, no head] & H_*(sA \otimes_\kappa L). \arrow[l, shift left] \arrow[d, dotted, no head] \\ {[-,-]_{\mathrm{Der}}} & \{\ell_n\} & \{l_n\} \end{tikzcd}$$

Recall that the structure $\{\ell_n\}$ on $sA \otimes_\kappa L$ produced by the Homotopy Transfer Theorem is obtained by composing maps according to decorations on rooted trees



Since we transfer a dg Lie structure (no higher operations), we only need to consider binary rooted trees as depicted above.

It is easy to check that if we transfer the structure further along the contraction to the homology

$$\begin{tikzcd} sA \otimes_\kappa L \arrow[loop left, "k"] \arrow[r, shift left, "q"] & H_*(sA \otimes_\kappa L), \arrow[l, shift left, "i"] \end{tikzcd}$$

we get the same result as we do by transferring the original one along the composed contraction, and the resulting structure can be obtained by composing maps according to the same trees where the decorations are changed so that g' is replaced by $g'i$, the homotopy h' is replaced by $h' + g'k(1 \otimes f)$, and finally $(1 \otimes f)$ at the root of a tree is replaced by $q(1 \otimes f)$.

This prompts us to investigate what can be said about all of these maps in general.

3.3. **Contractions and gradings.** The bracket length in L is well-defined since there is a presentation with purely quadratic relations, cf. Theorem 2.15. Fix this as weight grading for L in the following.

As mentioned in Example 2.10, there is a natural weight grading for a free graded Lie algebra given by bracket length. The underlying graded Lie algebra of $\mathscr{L}(C)$ is free, so we may choose the bracket length as a weight grading for this. This is reflected in the notation: we refer to elements of bracket length q by $\mathscr{L}(q)$, and elements in the subspace $\mathscr{D}_i$ of bracket length q by $\mathscr{D}_i(q)$.

The Koszul coalgebra C has a weight grading, and the dual algebra $A = C^\vee$ has an induced weight grading $A(p) = C(p)^\vee$. This is clearly preserved by the induced multiplication on A. We still assume that C or L is finite or that $A = C^\vee$ and L are both bounded in the same direction.

Lemma 3.10 (Weight Lemma)**.** *The given maps f and ι, and the maps g and h chosen as in Lemma 3.8, interact with the weight gradings of A, L and $\mathscr{L}$ as follows. For $p \geq 0$, $q \geq 1$ and $i \geq 0$,*

$$\begin{aligned} f &\colon \mathscr{L}(q) \to L(q) \\ g &\colon L(q) \to \mathscr{L}(q) \\ h &\colon \mathscr{D}_i(q) \to \mathscr{D}_{i+1}(q-1) \\ \iota &\colon sA(p) \otimes \mathscr{D}_i(q) \to \bigoplus_{m \geq 1} sA(p+m) \otimes \mathscr{D}_{i+m-1}(q+1). \end{aligned}$$

Recall that we write ι as short hand for the map ad_ι using the structure from Definition 3.1.

Proof of Lemma 3.10. By construction of $C^{¡} \simeq L$ the bracket length in L corresponds to that of $\mathscr{L}(C(1)) = \mathscr{D}$. Both f and g factor through the diagonal $\mathscr{D}$. Since h preserves total weight and raises the offset index for the diagonals, it lowers bracket length by 1.

The map ι splits as a sum $\iota = \sum_{m \geq 1} \iota_m$, with one summand for each weight m in a linear basis for A. The term ι_m raises weight by m in A, bracket length by 1 in $\mathscr{L}(C)$, and raises total weight by m in $\mathscr{L}(C)$. $\square$

In other words, f and g both preserve homological degree and weights (for both $\mathscr{L}$ and C), and h increase homological degree by 1, decrease weight by 1 and preserve the total weight. As observed in the proof above the map ι splits as a sum

$\iota = \sum_{m\geq 1} \iota_m$. Each term ι_m decrease homological degree by 1, increase weight in A by m and in $\mathscr{L}$ by 1, and increase total weight m.

A consequence of the Weight Lemma is the following proposition, which we mention for completeness even though the lemma itself is the key element in what is to come after the proposition.

Proposition 3.11. *The maps g', h' and $1\otimes f$ interact with the weight gradings of A, L and $\mathscr{L}$ as follows. For $p \geq 0$ and $q \geq 1$,*

$$\begin{aligned} 1\otimes f\colon sA(p)\otimes_\kappa \mathscr{L}(q) &\to sA(p)\otimes_\kappa L(q) \\ g'\colon sA(p)\otimes_\kappa L(q) &\to sA(\geq p)\otimes_\kappa \mathscr{L}(q), \\ h'\colon sA(p)\otimes_\kappa \mathscr{L}(q) &\to sA(\geq p)\otimes_\kappa \mathscr{L}(q-1). \end{aligned}$$

Proof. By Proposition 3.2 we can identify g' and h' from g, h and ι.

$$g' = \sum_{i\geq 0}((1\otimes h)\iota)^i(1\otimes g) \qquad \text{and} \qquad h' = \sum_{i\geq 0}((1\otimes h)\iota)^i(1\otimes h). \tag{12}$$

The proposition now follows by combining these formulae with the Weight Lemma (Lemma 3.10). $\square$

Similar properties can be deduced for the maps in the contraction to the homology.

Proposition 3.12. *There is a contraction*

$$\begin{tikzcd} sA\otimes_\kappa L \arrow[loop left, "k"] \arrow[r, shift left, "q"] & H_*(sA\otimes_\kappa L) \arrow[l, shift left, "i"] \end{tikzcd},$$

such that q and i preserve the weight grading in L and the homotopy k decreases the weight by 1 in both A and L.

Proof. It is easy to check that the splittings as in Lemma 2.1, can be chosen such that the contraction produced in that lemma has the desired properties. $\square$

3.4. **Transferred operations and gradings.** Combining the findings of the previous section, we can work out how the transferred L_∞-structure interacts with the weight gradings. Theorem 3.18 of this section is the main technical theorem of the thesis.

From here we may write $\mathscr{L}$ for short of $\mathscr{L}(C)$ for notational convenience. We begin by giving a formula for the Lie bracket $[-,-]_{\mathrm{Der}}$ on $sA\otimes_\iota \mathscr{L}$.

Definition 3.13. *For $a\in \overline{A}$ and $x\in\mathscr{L}$, denote by $x\frac{\partial}{\partial a}$ the unique derivation on $\mathscr{L}(C)$ extending the linear map $\varphi(sa\otimes x)\colon s^{-1}C\to\mathscr{L}(C)$, given by*

$$\varphi(sa\otimes x)(s^{-1}c) = (-1)^{|c||x|+|a|}a(c)x$$

(cf. Proposition 3.3).

Lemma 3.14. *The Lie bracket on $sA\otimes_\kappa\mathscr{L}$ induced by the isomorphism*

$$\mathrm{Der}\,\mathscr{L}//\mathscr{L} \simeq sA\otimes_\iota \mathscr{L}$$

is given by

(13)

$$[sa\otimes x, sb\otimes y]_{\mathrm{Der}} = \begin{cases} (-1)^\alpha sb\otimes x\frac{\partial}{\partial a}y - (-1)^\beta sa\otimes y\frac{\partial}{\partial b}x & a,b\in\overline{A} \\ (-1)^{|a|+|x|+1}s1\otimes x\frac{\partial}{\partial a}y & a\in\overline{A},\ b\in A(0) \\ 0 & a,b\in A(0) \end{cases}$$

for $x, y \in \mathscr{L}(C)$. The signs are given by

$$\alpha = (|x| + |a|)(|b| + 1) + 1,$$
$$\beta = |x|(|y| + |b| + 1) + |a|.$$

Proof. Let $\{c_i\}_i$ be a basis for $\overline{C}$. The bracket $[-,-]_{\mathrm{Der}}$ is by definition the composition

$$\varphi^{-1} \circ [-,-] \circ (\varphi \otimes \varphi).$$

For $x, y \in \mathscr{L}$ and basis elements $a, b \in \overline{A}$ we get

$$[sa \otimes x, sb \otimes y]_{\mathrm{Der}} = \sum_i (-1)^{\epsilon} sc_i^* \otimes \left[x\frac{\partial}{\partial a}, y\frac{\partial}{\partial b} \right] (s^{-1}c_i) \tag{14}$$

where the sign is given by

$$\begin{aligned} \epsilon &= \left(\left| x\frac{\partial}{\partial a} \right| + \left| y\frac{\partial}{\partial b} \right| \right) |sc_i^*| + 1 \\ &= (|x| + |a| + |y| + |b|)|sc_i^*| + 1, \end{aligned}$$

according to Remark 3.4. We evaluate:

$$\left[x\frac{\partial}{\partial a}, y\frac{\partial}{\partial b} \right] (s^{-1}c_i) = \left(x\frac{\partial}{\partial a} \circ y\frac{\partial}{\partial b} - (-1)^{(|x|+|a|+1)(|y|+|b|+1)} y\frac{\partial}{\partial b} \circ x\frac{\partial}{\partial a} \right) (s^{-1}c_i)$$

and see that first term is non-zero only if $c_i^* = b$, and second term is non-zero only if $c_i^* = a$. Thus the sum (14) reduces to

$$\begin{aligned} &(-1)^{(|b|+1)(|x|+|y|+|a|)+1} sb \otimes x\frac{\partial}{\partial a} \circ y\frac{\partial}{\partial b}(s^{-1}b^*) \\ &\quad - (-1)^{(|x|+|a|+1)(|y|+|b|+1)+(|a|+1)(|x|+|y|+|b|)+1} sa \otimes y\frac{\partial}{\partial b} \circ x\frac{\partial}{\partial a}(s^{-1}a^*) = \\ &(-1)^{(|x|+|a|)(|b|+1)+1} sb \otimes x\frac{\partial}{\partial a}y \\ &\quad - (-1)^{|x|(|y|+|b|+1)+|a|} sa \otimes y\frac{\partial}{\partial b}x. \end{aligned}$$

The result follows by extending linearly.

Let x, y, a be as above and now $b = 1 \in A(0)$. Recall that $\varphi(s1 \otimes y)$ is the linear map $s^{-1}C \to \mathscr{L}$ which is non-zero only on $C(0) \simeq \mathbb{Q}$, and $\varphi(s1 \otimes y)(1) = sy$. Thus

$$[-,-] \circ (\varphi \otimes \varphi)(sa \otimes x \otimes s1 \otimes y) = \left[x\frac{\partial}{\partial a}, sy \right] = (-1)^{|a|+|x|+1} sx\frac{\partial}{\partial a}y,$$

by the definition of the bracket restricted to $\operatorname{Der} \mathscr{L} \otimes s\mathscr{L}$. The inverse φ^{-1} on $s\mathscr{L}$ is given by $sx \mapsto s1 \otimes x$, so we get

$$[sa \otimes x, s1 \otimes y]_{\mathrm{Der}} = (-1)^{|a|+|x|+1} s1 \otimes x\frac{\partial}{\partial a}y.$$

Finally for $a = b = 1 \in A(0)$ we have $[sx, sy] = 0$, and thus

$$[sa \otimes x, sb \otimes y]_{\mathrm{Der}} = 0.$$

$\square$

Lemma 3.15. *The bracket* $[-,-]_{\mathrm{Der}}$ *interacts with the weight gradings of* A *and* $\mathscr{L}$ *as follows. For* $p_1, p_2 \geq 0$ *and* $q_1, q_2 \geq 1$

$$\begin{array}{c} sA(p_1) \otimes_\iota \mathscr{L}(q_1) \otimes sA(p_2) \otimes_\iota \mathscr{L}(q_2) \\ \Big\downarrow {\scriptstyle [-,-]_{\mathrm{Der}}} \\ sA(p_1) \otimes_\iota \mathscr{L}(q_1+q_2-1) \\ \oplus sA(p_2) \otimes_\iota \mathscr{L}(q_1+q_2-1). \end{array}$$

Proof. Without loss of generality we may assume that a, b, x, y in the formula (13) each is presented by a single term which is a (bracketed) word in the generators of A respectively $\mathscr{L}$. Thus they are concentrated in a single weight each. The composition $y\frac{\partial}{\partial b} \circ x\frac{\partial}{\partial a}$ is given by,

$$y\frac{\partial}{\partial b} \circ x\frac{\partial}{\partial a}(s^{-1}c) = (-1)^{|c||x|+|a|} a(c) y\frac{\partial}{\partial b}(x)$$

and the recursive formula

(15)

$$y\frac{\partial}{\partial b}(x) = \begin{cases} (-1)^{|x||y|+|b|} b(x) y & x \in \mathscr{L}(1) = s^{-1}C \\ ([y\frac{\partial}{\partial b}(x_1), x_2] + (-1)^{|y\frac{\partial}{\partial b}||x_1|}[x_1, y\frac{\partial}{\partial b}(x_2)]) & x = [x_1, x_2] \end{cases}.$$

From this we see that the weight of $y\frac{\partial}{\partial b} \circ x\frac{\partial}{\partial a}(c)$ is the sum of weights of x and y minus 1. Further, all terms of (13) for which $a(c_i) = 0$, vanish. Therefore all the sc_i^* appearing in the resulting sum will have the same weight as sa. The same holds mutatis mutandis, for the other composition $x\frac{\partial}{\partial a} \circ y\frac{\partial}{\partial b}$. □

Example 3.16 Let $\{a_i\}$ be a basis for A, and let $\{c_i\}$ be the dual basis for C. We can then calculate the first term of the bracket $[sa_1 \otimes [c_1, c_2], sa_2 \otimes [c_1, [c_2, c_3]]]_{\mathrm{Der}}$ (but leave out the signs):

$$\begin{aligned} & sa_1 \otimes [c_1, c_2]\frac{\partial}{\partial sa_2}[c_1, [c_2, c_3]] \\ &\qquad = sa_1 \otimes \left([[c_1, c_2]\frac{\partial}{\partial sa_2}(c_1), [c_2, c_3]] + [c_1, [c_1, c_2]\frac{\partial}{\partial sa_2}[c_2, c_3]]\right) \\ &\qquad = sa_1 \otimes \left([c_1, [[c_1, c_2]\frac{\partial}{\partial sa_2}(c_2), c_3]] + [c_1, [c_2, [c_1, c_2]\frac{\partial}{\partial sa_2}(c_3)]]\right) \\ &\qquad = sa_1 \otimes [c_1, [[c_1, c_2], c_3]]. \end{aligned}$$

Effectively we have scanned the word $[c_1, [c_2, c_3]]$ for occurrences of the letter $a_2^* = c_2$, and replaced it with the word $[c_1, c_2]$.

Notice how sa_1 is preserved in the first term, and that the bracket lengths in $\mathscr{L}(C)$ goes from $2 + 3 = 5$ in the input, to $4 = 5 - 1$ in the output. The second term is computed in the same way.

Definition 3.17. *The complex* $sA \otimes_\kappa L$ *is bigraded by weight in* A *and* L*. The* shifted weight grading *is the bigrading which in degree* (p, q) *is* $sA(p+1) \otimes_\kappa L(q+1)$*, for* $p, q \geq 0$*.*

The differential κ has bidegree $(1, 1)$ in the standard weight grading, and so also in the shifted weight grading. This is a special case of the following theorem on the entire L_∞-structure.

Theorem 3.18. *Let C be a Koszul graded cocommutative coalgebra with Koszul dual graded Lie algebra L, such that the linear dual $A = C^\vee$ is nilpotent. The L_∞-structure on $sA \otimes_\kappa L$ transferred from the derivations* $\operatorname{Der} \mathscr{L}(C)//\mathscr{L}(C)$ *through the chosen contraction, respects the shifted weight grading in the sense that for any $r \geq 1$ the operation ℓ_r has bidegree $(2-r, 2-r)$.*

Recall that the r-ary operation of an L_∞-algebra has homological degree $r-2$, so it is reasonable to say that the L_∞-structure stated in the theorem respects the grading. However there is no a priori connection to the homological grading in what is discussed.

Proof of Theorem 3.18. We introduce yet another grading for $sA \otimes_\iota \mathscr{L}(C)$: the *mass* $m(sa \otimes x)$ of an element $sa \otimes x$ is the total weight $\ell_w(x)$ of x in $\mathscr{L}(C)$ minus the weight $w(a)$ of a in A. We verify that h, ι and $[-,-]_{\mathrm{Der}}$ preserve the mass grading.

It straightforward to see that h and ι preserve the mass by Lemma 3.10. From the formulae (13) and (15) we see that also $[-,-]_{\mathrm{Der}}$ preserves the mass: in the general case we get

$$\begin{aligned} m([sa \otimes x, sb \otimes y]_{\mathrm{Der}}) &= (\ell_w(x) - w(a)) + (\ell_w(y) - w(b)) \\ &= (\ell_w(x) - w(b) + \ell_w(y)) - w(a) \\ &= (\ell_w(y) - w(a) + \ell_w(x)) - w(b), \end{aligned}$$

where second and third line is the mass of respectively first and second term of the right hand side expression of (13) in the first case. The other cases are similar.

For f and g the total weight in $\mathscr{L}(C)$ agrees with the weight in L since f vanish outside $\mathscr{D}_0$ and $\operatorname{Im} g$ is contained in this. Thus there is a natural way to speak of the interaction of f and g with the mass grading, and in this sense they both preserve it.

In particular, the maps $1 \otimes f$, g' and h' all preserve the mass grading. Now consider the operation

$$\begin{array}{c} \bigotimes_{k=1}^{r} sA(p_k) \otimes_\kappa L(q_k) \\ \Big\downarrow{\scriptstyle \ell_r} \\ sA \otimes_\kappa L. \end{array}$$

It is composed of maps which all preserve mass, as we have just verified. Then since the weight grading of L coincides with the total weight grading on $\mathscr{D}_0$, we have for any element x in the source that

$$\begin{aligned} A\text{-weight of } \ell_r(x) &= L\text{-weight of } \ell_r(x) - \text{mass of } \ell_r(x) \\ &= L\text{-weight of } \ell_r(x) - \text{mass of } x \end{aligned}$$

The mass of x is $\sum_{k=1}^r q_k - p_k$, and by Lemma 3.20 below, the image $\ell_r(x)$ has weight $\sum_{k=1}^r q_k - 2r + 3$ in L. Thus the image $\ell_r(x)$ has weight

$$\sum_{k=1}^{r} q_k - 2r + 3 - \sum_{k=1}^{r} (q_k - p_k) = \sum_{k=1}^{r} p_k - 2r + 3$$

in A, and ℓ_r has bidegree $(2-r, 2-r)$ in the shifted weight grading. $\square$

Remark 3.19 It is straight forward to check that none of the maps defining the transferred operations decrease the weight in A. Thus the condition $\sum_{k=1}^{r} p_k \geq 2r-3$ gives a lower bound on the weight in A for where ℓ_r is non-zero. E.g. ℓ_3 restricted to $A(1)\otimes L$ is zero: in the shifted weight grading it is an operation from three copies of weight $(0,*)$ to weight $(-1,*)$, but then one of the maps defining ℓ_3 would have lowered the weight in A.

Lemma 3.20. *For $r \geq 1$ the operation ℓ_r on $sA \otimes_\kappa L$, interacts with the weight grading of L as follows*

$$\begin{tikzcd} \bigotimes_{k=1}^{r} sA \otimes_\kappa L(q_k) \arrow[d, "\ell_r"] \\ sA \otimes_\kappa L(\sum_{k=1}^{r} q_k - 2r + 3). \end{tikzcd}$$

Proof. By the Homotopy Transfer Theorem ℓ_r is given by composing along binary rooted trees with r leaves, decorated by maps as established earlier. For each vertex we apply the bracket, and for each internal edge we apply the homotopy. Both decrease bracket length in $\mathscr{L}$ by 1 and there are $(r-1)+(r-2) = 2r-3$ vertices and internal edges. The other maps do not change the bracket length in $\mathscr{L}$ or L. □

If we consider only the transferred binary operation we get the following.

Corollary 3.21. *Consider the graded anti-commutative (non-associative) algebra $(sA \otimes L, \ell_2)$. Then:*

1. *$sA \otimes_\kappa L(1)$ is a subalgebra of $sA \otimes_\kappa L$,*
2. *$sA \otimes_\kappa L(j)$ is a module over $sA \otimes_\kappa L(1)$ for $j \geq 0$,*
3. *$\bigoplus_{j \geq m} sA \otimes_\kappa L(j)$ is a subalgebra of $sA \otimes_\kappa L$ for all $m \geq 0$.*
4. *$sA(1) \otimes_\kappa L$ is a subalgebra of $sA \otimes_\kappa L$,*
5. *$sA(i) \otimes_\kappa L$ is a module over $sA(1) \otimes_\kappa L$ for $i \geq 0$,*
6. *$\bigoplus_{i \geq m} sA(i) \otimes_\kappa L$ is a subalgebra of $sA \otimes_\kappa L$ for all $m \geq 0$.*

4. On homology

Having produced a smaller L_∞-model for the cover of the classifying space of the homotopy automorphisms, with some knowledge of the structure, we now proceed to investigate what can be said in general about the L_∞-structure on the homology of the model.

We begin the section by noticing that Theorem 3.18 and Corollary 3.21 carries over to homology. Then we give a "Recognition Proposition", identifying a small part of the homology in terms of derivations, not on the Quillen model of the underlying Koszul space, but on the homotopy Lie algebra.

Having identified the derivations, we use the homology version of Corollary 3.21 to write the positive part of the homology $H_{>0}(A \otimes_\kappa L)$ as an extension of the derivations, and we establish sufficient conditions for coformality of $B\operatorname{aut} X\langle 1\rangle$, which can be verified immediately, and provide examples.

Finally we investigate the homology in degree zero, which is a separate story entirely. We have two ways of approaching this, and have dedicated one subsection to each below. The first relies on work by Block-Lazarev [6], and lets us state similar results for a Lie algebra associated to $\pi_0 \operatorname{aut} X$ as we have for the higher homotopy groups. The second approach is by studying a Kan complex associated to the Maurer-Cartan elements of $A \otimes L$. Hinich [19] and Getzler [15] have both studied this Kan complex, and Berglund [3] and Buijs-Félix-Murillo [9] have used it as a model for mapping spaces.

Notation 4.1 Denote the differential on $sA \otimes_\kappa L$ by κ. It increases weight in both factors by 1, and so it is convenient to restrict κ to certain weight components. We will denote by κ^i the restriction of κ to $sA(i) \otimes_\kappa L$, and set

$$H^i(sA \otimes_\kappa L)_* := \ker \kappa^i / \operatorname{Im} \kappa^{i-1}.$$

Similarly we define κ_j to be the restriction of κ to $sA \otimes_\kappa L(j)$, and set

$$H_j(sA \otimes_\kappa L)_* := \ker \kappa_j / \operatorname{Im} \kappa_{j-1}.$$

Both $H^i(sA \otimes_\kappa L)_*$ and $H_j(sA \otimes_\kappa L)_*$ are graded vector spaces, and $*$ is a placeholder for the homological grading, not to be confused with gradings induced from the weight gradings of A and L. We omit i respectively j from the notation if no restriction is made, so that $H(sA \otimes L)_* := H^*_*(sA \otimes L)_*$.

Corollary 4.2. *The L_∞-structure on $H(sA \otimes_\kappa L)_*$ transferred from the derivations $\operatorname{Der} \mathscr{L} /\!/ \mathscr{L}$, respects the shifted weight grading in the sense that for any $r \geq 1$ the operation ℓ_r has bidegree $(2-r, 2-r)$.*

Proof. The maps to and from homology i, q preserve weights and so also the shifted weight. The contracting homotopy k decreases weights by 1, and the result follows from counting homotopies appearing in the tree formulae for operations transferred from $sA \otimes_\kappa L$ to $H(sA \otimes_\kappa L)_*$. □

Thus Corollary 3.21 is also valid once we pass to homology.

Corollary 4.3. *Consider the graded Lie algebra $(H(sA \otimes_\kappa L)_*, l_2)$. Then:*

(1) $H_1(sA \otimes_\kappa L)_*$ *is a Lie subalgebra of* $H(sA \otimes_\kappa L)_*$,
(2) $H_j(sA \otimes_\kappa L)_*$ *is a Lie module over* $H_1(sA \otimes_\kappa L)_*$ *for* $j \geq 0$,
(3) $\bigoplus_{j \geq m} H_j(sA \otimes_\kappa L)_*$ *is a Lie subalgebra of* $H(sA \otimes_\kappa L)_*$ *for all* $m \geq 0$.
(4) $H^1(sA \otimes_\kappa L)_*$ *is a Lie subalgebra of* $H(sA \otimes_\kappa L)_*$,
(5) $H^i(sA \otimes_\kappa L)_*$ *is a Lie module over* $H^1(sA \otimes_\kappa L)_*$ *for* $i \geq 0$,
(6) $\bigoplus_{i \geq m} H^i(sA \otimes_\kappa L)_*$ *is a Lie subalgebra of* $H(sA \otimes_\kappa L)_*$ *for all* $m \geq 0$.

The L_∞-structure on the homology ultimately comes from derivations on $\mathscr{L}(C)$, so it should perhaps not surprise that we are able to recognize derivations on L as part of the homology.

Proposition 4.4 (Recognition Proposition)**.** *Consider the dg anti-commutative algebra $(sA \otimes_\kappa L, \ell_2)$. The map $F\colon \operatorname{Der} L \to sA \otimes L$ given by*

$$F(\theta) = \sum_i (-1)^{|sa_i||\theta|+1} sa_i \otimes \theta(\alpha_i)$$

where $\{a_i\}$ is a linear basis for $A(1)$, and $\{\alpha_i\}$ the dual basis for $L(1)$, induces isomorphisms of graded Lie algebras

(1) $\operatorname{Der} L \simeq \ker \kappa^1$, *and*
(2) $\operatorname{ad} L \simeq \operatorname{Im} \kappa^0$.

Further, $\ker \kappa^0 \simeq sZ(L)$ *the suspension of the centre of L.*

It follows directly from the proposition that

$$H^0(sA \otimes_\kappa L)_* \simeq sZ(L), \quad \text{and} \quad H^1(sA \otimes_\kappa L)_* \simeq \operatorname{Der} L/L = \operatorname{Out} L.$$

Before we prove this we will need some notation and a lemma.

Recall from Theorem 2.15 that L has a presentation $L = \mathbb{L}(V)/(R)$, and that the Koszul dual commutative algebra has a presentation $A = \Lambda((sV)^\vee)/(R^\perp)$. In particular we may take $A(1) = (sV)^\vee$. Let $f\colon \mathbb{L}(V) \to L$ denote the quotient map. Then $f^*\colon \operatorname{Der} L \to \operatorname{Der}_f(\mathbb{L}(V), L)$ is injective, and the formula for F defines an isomorphism of graded vector spaces

$$F'\colon \operatorname{Der}_f(\mathbb{L}(V), L) \to s(sV)^\vee \otimes L.$$

Since $\kappa = (1 \otimes f)(\iota)$ we get the formula

$$\kappa^1(sa \otimes x) = \sum_i (-1)^{|\alpha_i||a|+1} sa_i a \otimes [\alpha_i, x], \qquad a \in A(1),\, x \in L$$

where $\{a_i\}$ is a basis for $A(1)$ and $\{\alpha_i\}$ the dual basis for $L(1)$. The same formula also defines a map

$$(\kappa^1)'\colon s(sV)^\vee \otimes L \to s\Lambda^2(sV^\vee) \otimes L.$$

Lemma 4.5. *An f-derivation θ' is in the image of f^* if and only if $(\kappa^1)'F'(\theta')$ is in $sR^\perp \otimes L$.*

Proof. The pairing

$$\langle\, ,\, \rangle\colon \Lambda^2(sV^\vee) \otimes \mathbb{L}^2(V) \to \mathbb{Q}$$

of Theorem 2.15 induces a map

$$\begin{tikzcd} s\Lambda^2(sV^\vee) \otimes L \otimes \mathbb{L}^2(V) \arrow[r, "p"] \arrow[d, "symm.\otimes 1"'] & sL \\ sL \otimes \Lambda^2(sV^\vee) \otimes \mathbb{L}^2(V). \arrow[ur, "1\otimes\langle\,,\,\rangle"'] & \end{tikzcd}$$

The expression

$$(\kappa^1)'F'(\theta') = \sum_{i,j} (-1)^{|\alpha_j||a_i|+|sa_i||\theta|} sa_j a_i \otimes [\alpha_j, \theta'(\alpha_i)] \tag{16}$$

is in $sR^\perp \otimes L$ if and only if

$$p((\kappa^1)'F'(\theta') \otimes \lambda) = 0,$$

for all $\lambda = \sum_{k,l} \lambda_{k,l}[\alpha_k, \alpha_l]$ in $R \subseteq \mathbb{L}^2(V)$. Using the following properties:

- the pairing is given by the formula

$$\langle ab, [\alpha,\beta]\rangle = (-1)^{|b||\alpha|+|a|+|\alpha|}\langle a,\alpha\rangle\langle b,\beta\rangle - (-1)^{|\alpha||\beta|+|b||\beta|+|a|+|\beta|}\langle a,\beta\rangle\langle b,\alpha\rangle,$$

- without loss of generality we assume that $\lambda_{k,l} = (-1)^{|\alpha_k||\alpha_l|+1}\lambda_{l,k}$,
- θ' is an f-derivation,

we compute

$$p((\kappa^1)'F'(\theta') \otimes \lambda) = 2\sum_{i,j} (-1)^{|\alpha_i||\theta|+(|\alpha_i|+|\theta'|+|\alpha_j|)} \lambda_{i,j} s[\alpha_i, \theta'(\alpha_j)] = s\theta'(\lambda)$$

to see that (16) is in $sR^\perp \otimes \mathbb{L}(V)$ precisely when θ' vanishes on R and thus induces a map $L \to L$. It is easy to check that this map is a derivation. $\square$

Proof of the Recognition Proposition 4.4. Consider the commutative diagram

$$\begin{tikzcd}
 & & 0 \\
\operatorname{Der} L \arrow[r, hook, "F"] \arrow[d, hook, "f^*"] & sA(1) \otimes L \arrow[r, "\kappa"] \arrow[d, equal] & sA(2) \otimes_\kappa L \arrow[u] \\
\operatorname{Der}_f(\mathbb{L}, L) \arrow[r, "\simeq", "F'"'] & sA(1) \otimes L \arrow[r, "\kappa'"] \arrow[dr, dashed] & s\Lambda^2 sV^\vee \otimes L \arrow[u] \\
 & & sR^\perp \otimes L \arrow[u] \\
 & & 0 \arrow[u]
\end{tikzcd}$$

where the right column is exact. By Lemma 4.5 an f-derivation θ' maps to $sR^\perp \otimes L$ if and only if it comes from a derivation θ on L. By exactness $F(\theta)$ is in the kernel of κ. On the other hand, suppose x is in the kernel of κ. Then by exactness the f-derivation $F'^{-1}(x)$ maps to $sR^\perp \otimes L$, and by Lemma 4.5 there is derivation ψ such that $F(\psi) = F'F'^{-1}(x) = x$.

Therefore we get an isomorphism of graded vector spaces $\overline{F}\colon \operatorname{Der} L \xrightarrow{\simeq} \ker \kappa^1$. Even better, in [5] it is shown that there is an L_∞-morphism ψ, such that the left triangle of the following diagram commutes,

$$\begin{tikzcd}
\operatorname{Der} \mathscr{L} \arrow[r, "f_*"] & \operatorname{Der}_f(\mathscr{L}, L) \arrow[r, "\simeq"] & s\overline{A} \otimes_\kappa L \\
 & \operatorname{Der} L \arrow[ul, "\psi"] \arrow[u, "f^*"] \arrow[ur, "F"'] &
\end{tikzcd}.$$

In particular f^* extends to an L_∞-morphism when $\operatorname{Der}_f(\mathscr{L}, L)$ is given the transferred L_∞-structure, and so is a map of graded Lie algebras on homology. The right triangle also commutes. Indeed by Remark 3.4 we get

$$\varphi^{-1} r f^*(\theta) = \varphi^{-1}(\theta f|_{s^{-1}\overline{C}}) = \sum_i (-1)^{|sa_i||\theta|+1} sa_i \otimes \theta(\alpha_i) = F(\theta),$$

where φ is the isomorphism of Proposition 3.3, and r denotes restriction to generators. Thus $\overline{F}\colon \operatorname{Der} L \overset{\simeq}{\to} \ker \kappa^1$ is even an isomorphism of graded Lie algebras. Note here that even though we are only considering $sA \otimes L$ as a graded anti-commutative algebra, the elements in $\ker \kappa^1$ do satisfy the Jacobi identity.

For point (2), we identify of image and kernel of κ^0, again using that $\kappa = (1 \otimes f)(\iota)$. For $x \in L$,

$$F(\operatorname{ad}_x) = \sum_i (-1)^{|sa_i||x|+1} sa_i \otimes [x, \alpha_i] = \sum_i sa_i \otimes [\alpha_i, x] = \kappa^0(s1 \otimes x)$$

Therefore the image of κ^0 is $F(\operatorname{ad}_L)$, isomorphic to the inner derivations on L, and the kernel is obviously the centre of L suspended. □

A direct consequence of Corollary 4.3 is the following proposition.

Proposition 4.6. *The exact sequence of graded vector spaces*

$$0 \longrightarrow H^{\geq 2}(sA \otimes_\kappa L)_* \longrightarrow H(sA \otimes_\kappa L)_* \longrightarrow H^{\leq 1}(sA \otimes_\kappa L)_* \longrightarrow 0, \tag{17}$$

is a split extension of graded Lie algebras with brackets given by the transferred operation l_2.

By the Recognition Proposition 4.4 we know the right term as a graded vector space:

$$H^{\leq 1}(sA \otimes_\kappa L)_* \simeq \operatorname{Der} L/\operatorname{ad} L \oplus sZ(L),$$

and from Corollary 4.3 we know that $sZ(L)$ is a Lie module over $\operatorname{Der} L/\operatorname{ad} L$.

There is a similar split extension for $H(s\overline{A} \otimes_\kappa L)_*$, and it follows from the Recognition Proposition 4.4 that

$$H^{\leq 1}(s\overline{A} \otimes_\kappa L)_* \simeq \operatorname{Der} L.$$

Remark 4.7 There is a Koszul dual story to the above. The proof of Proposition 4.4 can be dualized to yield $H_1(s\overline{A} \otimes_\kappa L)_* \simeq \operatorname{Der} \overline{A}$. For the non-augmented case we note that as a graded vector space $H_1(sA \otimes_\kappa L)_*$ is a direct sum of $H_1(s\overline{A} \otimes_\kappa L)_*$ and the kernel of $\kappa_1 \colon sA(0) \otimes_\kappa L(1) \to sA(1) \otimes_\kappa L(2)$. This last summand we may identify as $sZ(L)(1) := sZ(L) \cap sL(1)$. It is an abelian Lie algebra, but in general $\operatorname{Der} \overline{A}$ may act non-trivially on it.

There is a split exact sequence of graded Lie algebras (with transferred bracket), similar to (17)

$$0 \longrightarrow H_{\geq 2}(sA \otimes_\kappa L)_* \longrightarrow H(sA \otimes_\kappa L)_* \longrightarrow H_{\leq 1}(sA \otimes_\kappa L)_* \longrightarrow 0.$$

In the next sections we investigate these extensions when A and L arise from a Koszul space. We begin with positive homological degrees, and then proceed with degree zero.

4.1. **Positive homological part.** Let X be a Koszul space with nilpotent cohomology algebra A and homotopy Lie algebra L. The Schlessinger-Stasheff classifying Lie algebra models the homotopy automorphisms of X, and so far we have produced a contraction of this model, whilst keeping track of the L_∞-structure to some extent. The content of this is that the homology $H(sA \otimes_\kappa L)_{>0}$ computes $\pi_{>0}(\operatorname{aut} X) \otimes \mathbb{Q}$ as a graded vector space, and the minimal L_∞-structure on the homology is compatible with the graded Lie structure on rational homotopy.

Theorem 4.8 (Structure Theorem)**.** *Let X be a Koszul space with nilpotent cohomology algebra A and homotopy Lie algebra L. There are split extensions of graded Lie algebras*

$$0 \longrightarrow H^{\geq 2}(s\overline{A} \otimes_\kappa L)_{>0} \longrightarrow \pi_{>0}(\operatorname{aut}_* X) \otimes \mathbb{Q} \longrightarrow (\operatorname{Der} L)_{>0} \longrightarrow 0$$

$$0 \longrightarrow H^{\geq 2}(sA \otimes_\kappa L)_{>0} \longrightarrow \pi_{>0}(\operatorname{aut} X) \otimes \mathbb{Q} \longrightarrow (\operatorname{Der} L/\operatorname{ad} L)_{>0} \ltimes sZ(L) \longrightarrow 0,$$

and according to Remark 4.7,

$$0 \longrightarrow H_{\geq 2}(s\overline{A} \otimes_\kappa L)_{>0} \longrightarrow \pi_{>0}(\operatorname{aut}_* X) \otimes \mathbb{Q} \longrightarrow (\operatorname{Der} \overline{A})_{>0} \longrightarrow 0,$$

$$0 \longrightarrow H_{\geq 2}(sA \otimes_\kappa L)_{>0} \longrightarrow \pi_{>0}(\operatorname{aut} X) \otimes \mathbb{Q} \longrightarrow (\operatorname{Der} \overline{A})_{>0} \ltimes sZ(L)(1) \longrightarrow 0.$$

Proof. Combining Proposition 4.6 with the above observations regarding the Recognition Proposition 4.4, and with Corollary 4.2 we get the first extension. For the second extension, recall that $H^0(sA \otimes L)_*$ is identified with the centre of L which is concentrated in positive homological degrees. By Corollary 4.2 the centre is a Lie module for $\operatorname{Der} L/\operatorname{ad} L$.

The last two extensions are similarly produced by using Remark 4.7. $\square$

Corollary 4.9. *Let X be a Koszul space with nilpotent cohomology algebra A and homotopy Lie algebra L, for which $H^{\geq 2}(sA \otimes_\kappa L)_{>0} = 0$. Then*

(1) $\pi_{>0}(\operatorname{aut}_* X) \otimes \mathbb{Q} \simeq (\operatorname{Der} L)_{>0}$, and $B \operatorname{aut}_* X\langle 1\rangle$ is coformal,
(2) $\pi_{>0}(\operatorname{aut} X) \otimes \mathbb{Q} \simeq (\operatorname{Der} L/\operatorname{ad} L)_{>0} \ltimes sZ(L)$, and $B \operatorname{aut} X\langle 1\rangle$ is coformal if L has no centre.

Proof. The isomorphisms of graded Lie algebras are direct consequences of Theorem 4.8. All higher operations of the L_∞-structure on $\pi_{>0}(\operatorname{aut}_* X) \otimes \mathbb{Q}$ vanish because they respect the shifted weight grading and $H^*(s\overline{A} \otimes L)_{>0}$ is concentrated in weight $* = 1$.

In the second case we recall that $H^1(sA \otimes_\kappa L)_*$ corresponds to $\operatorname{Der} L/\operatorname{ad} L$ and $H^0(sA \otimes_\kappa L)_*$ corresponds to $sZ(L)$. There is a priori the option for the operation l_3 to go from three copies of bidegree $(0, *)$ to bidegree $(-1, *)$ in the shifted weight grading, but if the centre is zero this operation is then also zero. It is easy to see that even higher operations also vanish by the same reasoning. $\square$

Corollary 4.10. *Let X be a Koszul space with nilpotent cohomology algebra A and homotopy Lie algebra L, for which $H_{\geq 2}(sA \otimes_\kappa L)_{>0} = 0$. Then*

(1) $\pi_{>0}(\operatorname{aut}_* X) \otimes \mathbb{Q} \simeq (\operatorname{Der} \overline{A})_{>0}$, and $B \operatorname{aut}_* X\langle 1\rangle$ is coformal,
(2) $\pi_{>0}(\operatorname{aut} X) \otimes \mathbb{Q} \simeq (\operatorname{Der} \overline{A})_{>0} \ltimes sZ(L)(1)$, and $B \operatorname{aut} X\langle 1\rangle$ is coformal if $sZ(L) \cap L(1) = 0$ (in particular if L has no centre).

The proof is analogous to that of Corollary 4.9, but easier as $L(0) = 0$. Several interesting examples arise from these corollaries. In particular when considering Poincaré duality spaces. We first give an example which do not rely on Poincaré duality.

Example 4.11 Let X be a connected space of finite $\mathbb{Q}$-type. Then the suspension ΣX is rationally equivalent to a wedge of, say n, spheres, and thus a Koszul space with trivial cohomology algebra generated by the reduced cohomology. The homotopy Lie algebra is free on the n dual generators, and we write it $\mathbb{L} = \mathbb{L}(x_1, \ldots, x_n)$. In particular $H^{\geq 2}(sA \otimes_\kappa L)_* = 0$, and Corollary 4.9 states that

$$\pi_{>0}(\operatorname{aut}_* \Sigma X) \otimes \mathbb{Q} \simeq (\operatorname{Der} \mathbb{L})_{>0},$$

and $B \operatorname{aut}_*(\Sigma X)\langle 1\rangle$ is coformal. If $n > 1$ then there is no centre for the free Lie algebra, so

$$\pi_{>0}(\operatorname{aut} \Sigma X) \otimes \mathbb{Q} \simeq (\operatorname{Der} \mathbb{L}/\operatorname{ad} \mathbb{L})_{>0},$$

and $B \operatorname{aut}(\Sigma X)\langle 1\rangle$ is coformal.

Note that the model by Sclessinger-Stasheff and Tanré is only slightly larger in this case. It is obtained as follows. For each odd generator x_i, add to to the free Lie algebra a generator y_i of degree $2|x_i| - 1$, and define a differential $d_{\mathbb{L}}$ by mapping $[x_i, x_i] \mapsto y_i$. Then the Schlessinger-Stasheff model is $(\operatorname{Der} \mathbb{L}(x_1, \ldots, x_n, y_i, d_{\mathbb{L}}))_+$. In particular the two models are equal if there are no odd generators, that is ΣX is equivalent to a wedge of odd spheres.

Lemma 4.12. *Let X be a Koszul space which satisfies Poincaré duality. Let n be the largest number such that such that $A(m) = 0$ for all $m > n$. Then $A(n)$ is generated by a single class ν, and with $t = |\nu|$ we have*

$$H^n(sA \otimes_\kappa L)_* \simeq (s^t L(1))_*.$$

Further, $L(1)$ is concentrated in homological degrees less than $|t|$, so we get

$$H^n(sA \otimes_\kappa L)_{\geq 0} = 0.$$

Proof. By Poincaré duality there is a top cohomological degree d for A, and A^d is generated by a single element ω. Suppose a non-zero element $\nu' \in A(n)$ has degree $c < d$. Then there is a non-zero element $\rho \in A^{d-c}$ such that $\nu' \cdot \rho = \omega$. Since the multiplication respects the weight grading in A, we have $\rho \in A(0)$ and $\omega \in A(n)$. For a Koszul algebra $A(0) \simeq \mathbb{Q}$, and the multiplication $A(0) \otimes A(n) \to A(n)$ is just scalar multiplication, so $\nu' = \rho^{-1}\omega$ for $\rho \in \mathbb{Q}$ and thus $A(n)$ is the 1-dimensional graded vector space generated by ω.

It follows that the pairings $A(p) \otimes A(n-p) \to A(n) \simeq \mathbb{Q}$ for all $0 \leq p \leq n$ are non-degenerate.

The complex $A \otimes_\kappa L$ has the form

$$\cdots \longrightarrow A(n-1) \otimes_\kappa L \xrightarrow{\kappa_{n-1}} A(n) \otimes_\kappa L \xrightarrow{\kappa_n} 0 \tag{18}$$

Set $d = \dim L(1)$. Let $\{\alpha_i\}$ be a basis for $L(1)$, and let $\{a_i\}$ be the dual basis for $A(1)$. Now choose a basis $\{b_i\}$ for $A(n-1)$ such that $a_i b_j = \delta_{ij}\omega$, and denote the homological degrees $|b_i| = h_i$ for $1 \leq i \leq d$. The complex (18) is then isomorphic to

$$\cdots \longrightarrow s^{h_1} L \otimes \cdots \otimes s^{h_d} L \xrightarrow{\partial} s^t L \longrightarrow 0$$

where $\partial(\zeta_1, \ldots, \zeta_d) = \sum_i (-1)^{|b_i||\alpha_i|}[\alpha_i, \zeta_i]$ for $\zeta_j \in s^{h_j} L$. The image of ∂ is then $s^t[L, L]$. We may identify $L(1) \simeq L/[L, L]$ since L is a Koszul Lie algebra, and the first claim follows.

From the non-degenerate pairing $A(1) \otimes A(n-1) \to A(n)$ it also follows that the homological degrees of the generators of $A(1)$ are $t - h_i$ for $0 \leq i \leq d$. Since L is Koszul dual to A the generators of $L(1)$ have homological degrees $h_i - t - 1$, and so $s^t L(1)$ is concentrated in degrees $h_i - 1 < 0$. □

Corollary 4.13. *Let X be a Poincaré duality space which is also a Koszul space. If the cup length of X is less than 2 then*

(1) $\pi_{>0}(\operatorname{aut}_* X) \otimes \mathbb{Q} \simeq (\operatorname{Der} L)_{>0}$, *and* $B\operatorname{aut}_* X\langle 1\rangle$ *is coformal,*
(2) $\pi_{>0}(\operatorname{aut} X) \otimes \mathbb{Q} \simeq (\operatorname{Der} L/\operatorname{ad} L)_{>0} \ltimes sZ(L)$, *and* $B\operatorname{aut} X\langle 1\rangle$ *is coformal if* L *has no centre .*

Proof. Since the cup length for X is less than 2, we have $A(n) = 0$ and thus $H^n(sA \otimes_\kappa L)_* = 0$, for $n > 2$. By Lemma 4.12 also $H^2(sA \otimes_\kappa L)_* = 0$. The result now follows from Corollary 4.9. □

With Corollary 4.13 we have recovered the result by Berglund-Madsen [5] that highly connected manifolds (of sufficiently high rank and even dimension) have coformal homotopy automorphism spaces. However there are no assumptions on connectivity or parity of dimension in the present, only that it is Koszul and the condition that all triple cup products vanish.

Example 4.14 For $n \geq 1$, let M be an n-connected manifold of dimension $d \leq 3n+1$. Then M is formal by [30], and the cohomology algebra has a natural weight grading

$$A = A(0) \oplus A(1) \oplus A(2),$$

respected by the cup product. In particular the cup length is less than 2 (and it is equal to 2 by Poincaré duality). If $\dim A(1) \geq 2$, then M is also coformal [30], and Corollary 4.13 applies:

$$\pi_{>0}(\operatorname{aut}_* M) \otimes \mathbb{Q} \simeq (\operatorname{Der} L)_{>0},$$

and $B \operatorname{aut}_* X\langle 1\rangle$ is coformal, and

$$\pi_{>0}(\operatorname{aut} M) \otimes \mathbb{Q} \simeq (\operatorname{Der} L/\operatorname{ad} L)_{>0} \ltimes sZ(L),$$

and $B \operatorname{aut} X\langle 1\rangle$ is coformal if L has no centre.

Further if $\dim A(1) \geq 3$, then there is no centre for the homotopy Lie algebra L, as shown in [4]: A non-trivial centre for L implies that the Euler characteristic

$$\chi(L) := \sum (-1)^i \dim \operatorname{Ext}^i_{UL}(\mathbb{Q}, \mathbb{Q})$$

is zero. Koszul duality implies that $\operatorname{Ext}^i_{UL}(\mathbb{Q}, \mathbb{Q}) \simeq A(i)$, so that

$$\chi(L) = 2 - \dim A(1).$$

Examples of Koszul spaces which do not satisfy the conditions of Corollary 4.9 are H-spaces. Instead they satisfy the conditions of Corollary 4.10.

Example 4.15 Let X be a simply connected H-space. Then X is rationally equivalent to a product of Eilenberg-MacLane spaces, and thus Koszul with abelian homotopy Lie algebra $L = L/[L, L] = L(1)$, and free cohomology algebra A. Now A is in general not nilpotent, but L is (very much so) and by Remark 3.7 this is good enough.

The differential κ is zero, so $H(sA \otimes_\kappa L)_* = sA \otimes L$, and in particular

$$H^k(sA \otimes_\kappa L)_* = sA(k) \otimes L \neq 0$$

for all $k \geq 0$.

However, as observed $L = L(1)$ in this case, so $H_{\geq 2}(sA \otimes_\kappa L)_* = 0$ and by Corollary 4.10 we get that

$$\pi_{>0}(\operatorname{aut}_* X) \otimes \mathbb{Q} \simeq (\operatorname{Der} \overline{A})_{>0},$$

and $B \operatorname{aut}_* X\langle 1\rangle$ is coformal, and

$$\pi_{>0}(\operatorname{aut} X) \otimes \mathbb{Q} \simeq \big(\operatorname{Der} \overline{A})\big)_{>0} \ltimes sZ(L)(1).$$

We can not conclude coformality of $B \operatorname{aut} X\langle 1\rangle$ from Corollary 4.10 in this case, but since the higher operations respect the shifted weight grading and $L = L(1)$ these must again all be zero, and $B \operatorname{aut} X\langle 1\rangle$ is indeed coformal.

Note again that $Z(L)(1) = L(1) = L$, and by Koszul duality $L(1) \simeq (sA(1))^\vee$, so we have $sZ(L)(1) \simeq A(1)^\vee$. Any derivation is uniquely determined by its value on $A(1)$, and since A is free any map $A(1) \to A$ determines a derivation on A, in particular $A(1)^\vee$ corresponds bijectively to derivations with values in $A(0) \simeq Q$. Noting that any derivation on A is identically zero on $A(0)$ we thus get

$$\big(\operatorname{Der} \overline{A})\big)_{>0} \ltimes A(1)^\vee \simeq (\operatorname{Der} A))_{>0}$$

as a graded vector space, and the action of $\big(\operatorname{Der}\overline{A})\big)_{>0}$ on $A(1)^\vee$ corresponds precisely to the bracket on $(\operatorname{Der} A))_{>0}$, as can be checked simply by chasing through the construction.

In this case where A is free, it is also the minimal Sullivan model for X so we just recover the models for $B\operatorname{aut}_* X\langle 1\rangle$ and $B\operatorname{aut} X\langle 1\rangle$ in terms of the minimal Sullivan model for X, which are implicit in [33].

Note that if A is generated in a single degree then $(\operatorname{Der}\overline{A})_{>0} = 0$, since a derivation is determined by its value on generators and A is concentrated in negative degrees. So for a simply connected H-space X with rational cohomology generated in a single degree, the positive rational homotopy groups $\pi_{>0}(\operatorname{aut}_* X)\otimes\mathbb{Q}$ are all zero.

Example 4.16 Consider the simply connected H-space S^3. The rational homotopy Lie algebra is abelian with a single generator α in degree 2, and the cohomology algebra is free on a generator x in degree 3. The the complex $sA\otimes_\kappa L$ is then a graded vector space generated by

$$s1\otimes\alpha, \quad \text{and} \quad sx\otimes\alpha,$$

in degrees 3 and 0 respectively, and the differential is trivial. The augmented version $s\overline{A}\otimes_\kappa L$ is the 1-dimensional graded vector space generated by $sx\otimes\alpha$ in degree zero. The positive part of $s\overline{A}\otimes_\kappa L$ is thus zero, corresponding to the fact that $(\operatorname{Der}\overline{A})_{>0} = 0$. We get

$$\pi_{>0}(\operatorname{aut}_* S^3)\otimes\mathbb{Q} = 0,$$

as we should expect, since $\pi_i(\operatorname{aut}_* S^3)\otimes\mathbb{Q}\simeq\pi_{i+3}S^3\otimes\mathbb{Q} = 0$ for $i>0$.

The positive part of $sA\otimes_\kappa L$ is 1-dimensional generated by $s1\otimes\alpha$, corresponding to the class $s\alpha\in sZ(L) = sL$. We get

$$\pi_{>0}(\operatorname{aut} S^3)\otimes\mathbb{Q}\simeq sL,$$

a 1-dimensional graded vector space in degree 3. This is also what we should expect from the long exact sequence in rational homotopy associated to the fibration given by the evaluation map $\operatorname{aut} S^3\to S^3$:

$$0 = \pi_3\operatorname{aut}_* S^3\otimes\mathbb{Q}\to\pi_3\operatorname{aut} S^3\otimes\mathbb{Q}\to\pi_3 S^3\otimes\mathbb{Q}\to\pi_2\operatorname{aut}_* S^3\otimes\mathbb{Q} = 0.$$

Remark 4.17 Suppose that X is a simply connected formal space such that L is finite dimensional. Félix-Halperin [12] show that such an X is a so called “two-stage space”: a simply connected space X such that the rationalisation $X_\mathbb{Q}$ is the total space of a principal fibration

$$K_1 \longrightarrow X_\mathbb{Q} \longrightarrow K_0,$$

for a pair of generalised Eilenberg-MacLane spaces K_0 and K_1, such that the indecomposables of the homotopy Lie algebra L is identified with K_0. One consequence is that L has brackets of length at most 2, and there is a weight grading on L such that $L = L(1)\oplus L(2)$ by separating the indecomposables and the decomposables. If X is Koszul, then this weight grading realises L as a Koszul Lie algebra with Koszul dual $A = H^*(X;\mathbb{Q})$.

Our study of Koszul spaces such that L is finite dimensional thus reduces to a study of formal and coformal two-stage spaces. The homotopy automorphisms of formal two-stage spaces have previously been studied by Smith [35] by different methods, and without the condition of coformality. On the other hand, some formal two-stage spaces are automatically Koszul as we explain below.

A two-stage space X is the total space of a principal fibration

$$\prod_n K(W_n, n) \longrightarrow X_{\mathbb{Q}} \longrightarrow \prod_n K(V_n, n),$$

for some finite dimensional graded vector spaces V and W. The Sullivan minimal model of X is given by $\mathscr{M}_X = (\Lambda(V^\vee) \otimes \Lambda(W^\vee), d_X)$ where the differential satisfies $d_X(V^\vee) = 0$ and $d_X(W^\vee) \subseteq \Lambda(V^\vee)$, and we may assume that the restriction $d_X \colon W^\vee \to \Lambda(V^\vee)$ is injective. Choose bases $x_1, \ldots, x_m$ for $V^\vee$ and $y_1, \ldots, y_n$ for $W^\vee$. Then $d_X(x_i) = 0$ and $d_X(y_j) = P_j(x_1, \ldots x_m)$, a polynomial which by minimality has no linear terms.

The space X is formal if and only if the sequence $P_1, \ldots, P_n$ is a regular sequence in $\Lambda(V)$ - that is P_j is not a zero divisor in $\Lambda(V)/(P_1, \ldots, P_{j-1})$ for $1 \leq j \leq n$. Then X is Koszul if and only if P_j is quadratic for all j.

Remark 4.18 Let X be a Koszul space with finite dimensional homotopy Lie algebra. Then X is a two-stage space as noted above. The complex $sA \otimes_\kappa L$ then takes the form

$$0 \longrightarrow sA \otimes L(1) \xrightarrow{\kappa} sA \otimes L(2) \longrightarrow 0,$$

which is isomorphic to the complex produced by Smith [35]. As a special case of Corollary 3.21 we reproduce his result that $H_2(sA \otimes_\kappa L)_{>0}$ is an abelian ideal in $H(sA \otimes_\kappa L)_{>0} \simeq \pi_{>0}(\operatorname{aut} X) \otimes \mathbb{Q}$, in the case where X is Koszul. In addition we get that $H_1(sA \otimes_\kappa L)_*$, and in particular $H_1(sA \otimes_\kappa L)_{>0}$, is a Lie subalgebra, and that this Lie subalgebra supports no higher operations.

If the differential κ happens to be surjective in positive degrees, then $B \operatorname{aut}_* X\langle 1 \rangle$ is coformal by Corollary 4.10. If $Z(L)(1) = 0$ in addition, then $B \operatorname{aut} X\langle 1 \rangle$ is coformal, also by Corollary 4.10.

Theorem 4.19. *Let X be a simply connected space with finitely generated cohomology A concentrated in even degrees, and let q be a homogeneous non-degenerate quadratic form in the generators of A, such that*

$$A \simeq \mathbb{Q}[x_1, \ldots, x_n]/(q).$$

Then there is an abelian extension of graded Lie algebras

$$H_2(sA \otimes_\kappa L)_{>0} \longrightarrow \pi_{>0}(\operatorname{aut} X) \otimes \mathbb{Q} \longrightarrow (\operatorname{Der} \overline{A})_{>0},$$

where $H_2(sA \otimes_\kappa L)_{>0}$ is 1-dimensional and identifies with the suspension of the centre $sZ(L)$, and an isomorphism of graded Lie algebras

$$\pi_{>0}(\operatorname{aut}_* X) \otimes \mathbb{Q} \simeq (\operatorname{Der} \overline{A})_{>0},$$

and $B \operatorname{aut}_* X\langle 1 \rangle$ is coformal.

Note that q is homogeneous with respect to homological degrees when A as here is the cohomology of a space, as opposed to just being a quadratic form.

Proof. A Sullivan model for X is given by $(\Lambda(x_1, \ldots, x_n, y), dy = q)$, so X is clearly formal, and the homotopy Lie algebra is finite dimensional, in particular X is also two-stage and it follows that X is coformal since q is quadratic, and from Theorem 2.15 that $L(2)$ is 1-dimensional.

The only non-zero brackets of L are given by the pairings of generators from $q=\sum c_{ij}x_ix_j$. I.e. if we denote the generators dual to x_i by α_i, then the non-zero brackets are $[\alpha_i,\alpha_j]$ for i,j such that $c_{ij}\neq 0$ (and they are all linearly dependent). The complex $sA\otimes_\kappa L$ splits as a sum with summands

$$0 \longrightarrow sA(p)\otimes L(1) \xrightarrow{\kappa} sA(p+1)\otimes L(2) \longrightarrow 0\ .$$

for $p\geq -1$. For $p\geq 0$ the differential is surjective because q is non-degenerate. Choose a basis $\{\tilde{x}_i\}_i$ for $A(1)$ such that $q=\sum_i d_i\tilde{x}_i^2$, and let $\{\tilde{\alpha}_i\}_i$ be the dual basis for $L(1)$. Notably the coefficients d_i need not be rational, so we tensor the complex with $\mathbb{R}$ and show that κ is surjective as a linear map between real vector spaces. Since κ is defined as a map between rational vector spaces, that is the case if and only if κ is surjective as a linear map between rational vector spaces.

In the new basis κ is given by the adjoint action of $\sum_i\tilde{x}_i\otimes\tilde{\alpha}_i$. An element of $sA(p+1)\otimes L(2)$ is a linear combination of elements of the form

$$\prod_{j=1}^{p+1}\tilde{x}_{i_j}\otimes[\tilde{\alpha}_{i_1},\tilde{\alpha}_{i_1}]=\pm\kappa\left(\prod_{j=2}^{p+1}\tilde{x}_{i_j}\otimes\tilde{\alpha}_{i_1}\right)$$

if $p\geq 1$, and

$$\tilde{x}_{i_1}\otimes[\tilde{\alpha}_{i_1},\tilde{\alpha}_{i_1}]=\pm\kappa\left(1\otimes\tilde{\alpha}_{i_1}\right)$$

if $p=0$. The same is true when restricting to the positive part of the complex.

For $p=-1$ the complex is just $A(0)\otimes L(2)$ with no differentials, and the (positive) element $s1\otimes\alpha$ spans the entire homology $H_2(sA\otimes_\kappa L)$, where α is some choice of basis for $L(2)$.

The results now follow from the Structure Theorem 4.8 (note that $\alpha\in L(2)$ spans the entire centre of L), and we conclude coformality using again that the transferred structure respects the shifted weight grading.

□

Example 4.20 Let B be a space with rational cohomology free on finitely many even generators

$$H^*(B;\mathbb{Q})\simeq\mathbb{Q}[x_1,\dots x_n],$$

for example $B=BG$, the classifying space of a compact connected Lie group G. Consider a sphere bundle

$$S^{2m-1}\to X\to B,$$

with an odd dimensional sphere. Then the cohomology of X is of the form

$$\mathbb{Q}[x_1,\dots x_n]/(f)$$

where f is the Euler class of the bundle (cf. [14]), and X is Koszul if and only if f is quadratic. If f is non-degenerate then Proposition 4.19 applies (f is homogeneous of degree $2m$).

4.2. **Degree zero.** The connection between algebra and topology for us, has so far been the Schlessinger-Stasheff classifying Lie algebra $(\operatorname{Der}\mathscr{L}/\!/\mathscr{L})_+$, but our study has been of the non-truncated Lie algebra $\operatorname{Der}\mathscr{L}/\!/\mathscr{L}$.

It should not be surprising that there is information about $\pi_0 \operatorname{aut} X$ contained in the non-truncated version. Here we compare it to the Harrison cohomology of a Sullivan model for X, which Block-Lazarev show is related to $\pi_0 \operatorname{aut} X$. Block-Lazarev use the term André-Quillen cohomology, which agrees with Harrison cohomology in characteristic zero, but they define it using the Harrison complex.

4.2.1. *Harrison cohomology.* We introduce Harrison cohomology for dg algebras following the definition of Block-Lazarev [6], and with sign conventions from Loday [26]. For an associative graded algebra A we may consider the truncated Hochschild complex

$$\cdots \longrightarrow A^{\otimes 3} \longrightarrow A^{\otimes 2} \longrightarrow 0,$$

which we denote $\overline{C}(A, A)$, where the differential is defined as follows. For all $n \geq 3$, there are maps $d_i \colon A^{\otimes n+1} \to A^{\otimes n}$ given by

$$\begin{aligned} d_i(a_0 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \qquad 0 \leq i < n \\ d_n(a_0 \otimes \cdots \otimes a_n) &= (-1)^{\epsilon_n} a_n a_0 \otimes \cdots \otimes a_n, \end{aligned}$$

with $\epsilon_n = |a_{n+1}| \sum_{j=0}^n |a_j|$. The differential of $\overline{C}(A, A)$ is given by $\sum_{i=1}^n (-1)^i d_i$.

For a dga A, the complex $\overline{C}(A, A)$ is a double complex. For all $n \geq 2$, the differential $d_{A^{\otimes n}}$ on $A^{\otimes n}$ is defined as $\sum_{i=1}^n (-1)^i \delta_i$, with $\delta_i \colon A^{\otimes n} \to A^{\otimes n}$ given by

$$\delta_i(a_0 \otimes \cdots \otimes a_n) = (-1)^{|a_0| + \cdots + |a_{i-1}|} a_0 \otimes \cdots \otimes d_A(a_i) \otimes \cdots \otimes a_n, \qquad 0 \leq i \leq n$$

and the total differential on $\overline{C}(A, A)$ is then $\sum_{i=0}^n (-1)^i (d_i - d_{A^{\otimes n}})$.

The *shuffle product* $\mu \colon \overline{C}(A, A) \otimes \overline{C}(A, A) \to \overline{C}(A, A)$ is given by:

$$\begin{aligned} &\mu((a \otimes a_1 \cdots \otimes a_n) \otimes (b \otimes a_{n+1} \otimes \cdots \otimes a_{n+m})) \\ &\qquad = \sum_\sigma (-1)^\epsilon ab \otimes a_{\sigma(1)} \otimes \cdots a_{\sigma(n+m)}, \end{aligned}$$

where σ runs through all (n, m)-shuffles, and the sign is given by

$$\epsilon = |b| \sum_{i=1}^n |a_i| + \sum_{\substack{i<j \\ \sigma^{-1}(i) > \sigma^{-1}(j)}} |a_i||a_j|.$$

If A is graded commutative then $\overline{C}(A, A)$ is a non-unital dg algebra over A with the shuffle product, and the indecomposables

$$C_*^{\mathrm{Har}}(A, A) := \overline{C}(A, A)/\overline{C}(A, A)^2$$

is the *Harrison complex.* This is what Block-Lazarev call the André-Quillen complex.

Definition 4.21. *For a dg commutative algebra A we define the complex*

$$C^*_{Har}(A, A) := \operatorname{Hom}_A(C_*^{Har}(A, A), A),$$

and the cohomology of this is Harrison cohomology, denoted $H^*_{Har}(A, A)$.

The Harrison complex of a graded commutative algebra A comes equipped with a Lie bracket. Block-Lazarev argues that one way to see this is to identify $C^*_{\mathrm{Har}}(A,A)$ with $\operatorname{Coder} CA$ the space of coderivations on the cofree Lie coalgebra on A, which is naturally a dg Lie algebra.

The following theorem is also a direct consequence of Block-Lazarev [6], based on work of Schlessinger-Stasheff [33].

Theorem 4.22. *Let $\mathscr{L}$ be a Quillen model for a simply connected space X of finite type. The dg Lie algebras $\operatorname{Der}\mathscr{L}/\!/\mathscr{L}$ and $C^*_{Har}(\mathscr{C}(\mathscr{L})^\vee, \mathscr{C}(\mathscr{L})^\vee)$ are quasi-isomorphic.*

Proof. Schlessinger and Stasheff [33] (Theorem 3.17), show that for a dg Lie algebra $\mathscr{L}$ such that the underlying graded Lie algebra is free there is a quasi-isomorphism

$$\operatorname{Der}\mathscr{L}/\!/\mathscr{L} \xrightarrow{\simeq} \operatorname{Coder}\mathscr{C}(\mathscr{L}).$$

If $\mathscr{L}$ is finitely generated in each degree, then dualizing gives an isomorphism

$$\operatorname{Coder}\mathscr{C}(\mathscr{L}) \simeq \operatorname{Der}\mathscr{C}(\mathscr{L})^\vee.$$

Now $C^*_{\mathrm{Har}}(A,A)$ is isomorphic to $\operatorname{Coder} CA$, and by Theorem 2.8(3) [6] the dg Lie algebras $\operatorname{Der} A$ and $\operatorname{Coder} CA$ are quasi-isomorphic if A is cofibrant. Now apply this for $A = \mathscr{C}(\mathscr{L})^\vee$. If $\mathscr{L}$ is a Quillen model for a simply connected space X of finite type, then $\mathscr{C}(\mathscr{L})^\vee$ is a Sullivan model and in particular cofibrant, and the underlying graded Lie algebra of $\mathscr{L}$ is free, so the result follows. $\square$

Note that $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ is a group. Sullivan [37] and Wilkerson [40] showed that $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ is linear algebraic group over $\mathbb{Q}$ if X is a finite CW-complex or has a finite Postnikov tower. See Block-Lazarev [6] for a modern treatment. To avoid introducing a lot of terminology from algebraic geometry, we refer the reader to Hochschild [20] and Waterhouse [39] for details on algebraic groups and related constructions. For us a linear algebraic group over $\mathbb{Q}$ is just a subgroup of $GL_n(\mathbb{Q})$ for some $n > 0$, defined in terms of polynomial equations. E.g.

$$SL(n,\mathbb{Q}) = \{M \in GL_n(\mathbb{Q}) \mid \det M = 1\},$$

and the determinant of a matrix is a polynomial in the entries.

To any linear algebraic group G we may associate a Lie algebra $\operatorname{Lie}(G)$, which retains a lot of information about G, but not all, as we here give a brief introduction to. Again, see [20, 39] for definitions and properties of $\operatorname{Lie}(G)$ and what else follows. In good cases the exponential power series converges and $\exp(\operatorname{Lie}(G))$ carries a group structure given by the *Baker-Campbell-Hausdorff formula*:

$$\begin{aligned} x \cdot y &= \log(e^x e^y) \\ &= x + y + \frac{1}{2}[x,y] + \frac{1}{12}([x,[x,y]] + [y,[y,x]]) - \frac{1}{24}[y,[x,[x,y]]] - \cdots \end{aligned}$$

for $x, y \in L$. See also [22] for details on this. There is a surjection from $\exp(\operatorname{Lie}(G))$ to the connected component of the identity element e of G in the Zariski topology, and the kernel can be identified with $\pi_1(G,e)$. When working over $\mathbb{Q}$, it is necessary that G is nilpotent to qualify as a "good case".

Theorem 4.23 (cf. [6] Theorem 3.4)**.** *Let X be a nilpotent CW complex which is either finite or has a finite Postnikov tower, and let A denote a Sullivan model. Then $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ is a linear algebraic group, and the Lie algebra $\operatorname{Lie}(\pi_0(\operatorname{aut} X_{\mathbb{Q}}))$ is isomorphic to $H^0_{\mathrm{Har}}(A,A)$.*

For Koszul spaces Theorem 4.22 and 4.23 combine to the following.

Proposition 4.24. *Let X be a Koszul space with homotopy Lie algebra L and cohomology algebra A, such that A or L is finite. Then the Lie algebra* $\operatorname{Lie}(\pi_0(\operatorname{aut} X_{\mathbb{Q}}))$ *is isomorphic to $H(sA \otimes_\kappa L)_0$ equipped with the transferred bracket.*

The analogue to Theorem 4.8 then reads:

Theorem 4.25. *Let X be a Koszul space with homotopy Lie algebra L and cohomology algebra A, such that A or L is finite. Then there are a split extension of Lie algebras*

$$0 \longrightarrow H^{\geq 2}(sA \otimes_\kappa L)_0 \longrightarrow \operatorname{Lie}(\pi_0(\operatorname{aut} X_{\mathbb{Q}})) \longrightarrow (\operatorname{Der} L)_0 \longrightarrow 0,$$
$$0 \longrightarrow H_{\geq 2}(sA \otimes_\kappa L)_0 \longrightarrow \operatorname{Lie}(\pi_0(\operatorname{aut} X_{\mathbb{Q}})) \longrightarrow (\operatorname{Der} A)_0 \longrightarrow 0.$$

Remark 4.26 For a simply connected space X we have $\pi_0(\operatorname{aut}_* X) \simeq \pi_0(\operatorname{aut} X)$, and thus the same split extensions for $\operatorname{aut}_* X$.

Example 4.27 We return to the examples of suspensions and H-spaces. Let X be a connected space such that ΣX is simply connected and rationally equivalent to wedge of spheres. The cohomology A of ΣX is a trivial graded algebra generated by the reduced cohomology and $H^{\geq 2}(sA \otimes_\kappa L)_* = 0$. The homotopy Lie algebra of ΣX is free, so we denote it $\mathbb{L}$ and Theorem 4.25 gives us

$$\operatorname{Lie}(\pi_0(\operatorname{aut} \Sigma X_{\mathbb{Q}})) \simeq (\operatorname{Der} \mathbb{L})_0.$$

If $\mathbb{L}$ is generated on the graded vector space V then $(\operatorname{Der} \mathbb{L})_0$ is in bijection with linear maps $V \to \mathbb{L}$ of degree zero.

Dually, if X is a simply connected H-space, then X is equivalent to a product of Eilenberg-MacLane spaces. The cohomology A is free, and the homotopy Lie algebra is abelian. In particular $H_{\geq 2}(sA \otimes_\kappa L)_0 = 0$, and Theorem 4.25 gives us

$$\operatorname{Lie}(\pi_0(\operatorname{aut} X_{\mathbb{Q}})) \simeq (\operatorname{Der} A)_0.$$

If A is generated on V then $(\operatorname{Der} A)_0$ is in bijection with linear maps $V \to A$ of degree zero.

In general $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ is not nilpotent for a Koszul space X, so extracting information about it from $\operatorname{Lie}(\pi_0(\operatorname{aut} X_{\mathbb{Q}}))$ is not straight forward. We do know the following.

Proposition 4.28. *The Lie algebras $H^{\geq 2}(sA \otimes_\kappa L)_0$ and $H_{\geq 2}(sA \otimes_\kappa L)_0$ are nilpotent if A or L is finite.*

Proof. Suppose that A is finite. The transferred bracket on $H(sA \otimes_\kappa L)_*$ respects the shifted weight grading, so $H^{\geq 2}(sA \otimes_\kappa L)_0$ is nilpotent because there is a maximal weight m for A such $A(m) \neq 0$. For $H_{\geq 2}(sA \otimes_\kappa L)_0$, we note first that L is concentrated in positive homological degrees, and since the transferred bracket respects the shifted weight grading it also increases homological degree strictly in the L factor. As A is finite and concentrated in non-positive homological degrees it follows that $H_{\geq 2}(sA \otimes_\kappa L)_0$ is nilpotent.

The argument if L is finite is completely analogous. □

This proposition implies that the kernels of the extensions in Theorem 4.25 $H^{\geq 2}(sA \otimes L)_0$ and $H_{\geq 2}(sA \otimes L)_0$, correspond to normal subgroups of $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ by the Campbell-Baker-Hausdorff formula. It is also possible to identify a Lie subalgebra of the derivations which is nilpotent in both cases. This Lie subalgebra is also a Lie ideal and the complement is the derivations which preserve the weight gradings in L and respectively A. In both cases these are identified with degree zero of the kernel of the restriction of κ:

$$\ker(sA(1) \otimes_\kappa L(1) \xrightarrow{\kappa} sA(2) \otimes_\kappa L(2))_0.$$

This narrows down the problem of non-nilpotency, but as we shall see in Example 5.11, we cannot expect to circumvent it entirely.

Instead we take a different approach to gain information about degree zero in the next section.

4.3. **Degree zero by simplicial methods.** Consider $A \otimes L$ as a dg Lie algebra with the structure given by Definition 3.1, and zero differential. Recall that a Maurer-Cartan element of $A \otimes L$ is a element $\tau \in (A \otimes L)_{-1}$ satisfying $\frac{1}{2}[\tau, \tau] = 0$. We may form the simplicial dg Lie algebra $\Omega_\bullet \otimes A \otimes L$, and consider the simplicial set of Maurer-Cartan elements

$$MC_\bullet(A \otimes L) := MC(\Omega_\bullet \otimes A \otimes L).$$

Proposition 4.29 ([3]). *The simplical set $MC_\bullet(A \otimes L)$ is a Kan complex* [15] *and it is homotopy equivalent to the space of maps* $\mathrm{Map}(X_\mathbb{Q}, X_\mathbb{Q})$*. In particular there is a bijection*

$$[X_\mathbb{Q}, X_\mathbb{Q}] \simeq \pi_0 MC_\bullet(A \otimes L).$$

The path components are given by equivalence classes of Maurer-Cartan elements in $A \otimes L$, with the usual notion of equivalence in a Kan complex: τ and τ' are equivalent if there is a 1-simplex γ such that $d_0\gamma = \tau$ and $d_1\gamma = \tau'$. A 1-simplex γ is a Maurer-Cartan element in $\Omega_1 \otimes A \otimes L$, and we recall the standard identification

$$\Omega_1 := \frac{\Lambda(t_0, t_1, dt_0, dt_1)}{(t_0 + t_1 - 1, dt_0 + dt_1)} \simeq \Lambda(t, dt),$$

where $|t_i| = |t| = 0$ and $|dt_i| = |dt| = 1$ with the differential mapping t_i to dt_i and t to dt as suggested by notation. So γ can be identified with a polynomial $\gamma(t, dt)$ in the commuting variables t and dt where $dt^2 = 0$ because $|dt|$ is odd, and with coefficients in $A \otimes L$. Denote by α_i the coefficient for t^i, and by β_j the coefficient for $t^j dt$. Since $|\gamma| = -1$ we get that $|\alpha_i| = -1$ and $|\beta_j| = 0$ for all i, j. So we can write

$$\gamma(t, dt) = \alpha(t) + \beta(t)dt := \sum_i \alpha_i t^i + \sum_j \beta_j t^j dt.$$

Recall that $d_{A\otimes L} = 0$. The Maurer-Cartan equation for γ expands to

$$\begin{aligned} 0 &= \frac{1}{2}[\gamma(t, dt)\gamma(t, dt)] + d\gamma(t, dt) \\ &= \frac{1}{2}[\alpha(t) + \beta(t)dt, \alpha(t) + \beta(t)dt] + d(\alpha(t)) + d(\beta(t)dt) \\ &= \frac{1}{2}[\alpha(t), \alpha(t)] + [\alpha(t), \beta(t)dt] + \frac{d}{dt}\alpha(t). \end{aligned}$$

The simplicial structure maps are then given by evaluating $d_0(\gamma) = \gamma(0, 0)$, and $d_1(\gamma) = \gamma(1, 0)$, so by unfolding the definitions we get $\tau \sim \tau'$ if and only if there exist polynomials $\alpha(t) \in (A \otimes L)_{-1}[t]$ and $\beta(t) \in (A \otimes L)_0[t]$ such that

$$[\alpha(t), \alpha(t)] = 0, \qquad [\alpha(t), \beta(t)] = -\frac{d}{dt}\alpha(t), \qquad \alpha(0) = \tau, \quad \alpha(1) = \tau'.$$

Definition 4.30. *The* linear part *of a Maurer-Cartan element $\tau \in MC(A \otimes L)$ is the image of τ under the projection*

$$\pi \colon A \otimes L \twoheadrightarrow A(1) \otimes L(1).$$

We say that τ is linear if $\pi(\tau) = \tau$.

Proposition 4.31. *Two Maurer-Cartan elements $\tau, \tau' \in MC(A \otimes L)$ are in the same path component of $MC_\bullet(A \otimes L)$ only if their linear parts agree, that is $\pi(\tau) = \pi(\tau')$.*

Proof. Suppose that $\tau \sim \tau'$. Then there exist polynomials $\alpha(t)$ and $\beta(t)$ as above. Write $\alpha(t) = \sum \alpha_i t^i$ and $\beta(t) = \sum_j \beta_j t^j$. The equation $[\alpha(t), \beta(t)] = -\frac{d}{dt}\alpha(t)$ implies that

$$-(n+1)\alpha_{n+1} = \sum_{i+j=n} [\alpha_i, \beta_j], \qquad n \geq 0 \tag{19}$$

from which we see that α_{n+1} is decomposable for $n \geq 0$, and so has no linear part. In particular the linear part of $\alpha_0 = \tau$ equals the linear part of $\sum_{i \geq 0} \alpha_i = \tau'$. □

Choose a basis $\{a_i\}_I$ for $A(1)$, and let $\{\alpha_i\}_I$ denote the dual basis for $L(1)$. Then a linear Maurer-Cartan element has a presentation $\tau = \sum_{i,j} \lambda_{ij} a_i \otimes \alpha_j$ for some $\lambda_{ij} \in \mathbb{Q}$. In this way an $I \times I$ matrix with $\mathbb{Q}$-coefficients (λ_{ij}) may represent a Maurer-Cartan element. Such a matrix also represents a linear map $A(1) \to A(1)$, and a linear map $L(1) \to L(1)$ since $A(1)$ and $L(1)$ are dual (up to a shift).

Proposition 4.32. *The following are equivalent: an $I \times I$ matrix with $\mathbb{Q}$-coefficients represents*

1. *a linear Maurer-Cartan element of $A \otimes L$,*
2. *an endomorphism of A,*
3. *an endomorphism of L.*

The proof of the proposition uses the same techniques as the proof of the Recognition Proposition 4.4. It is technical and with no separate interest, so we skip it.

Proposition 4.33 (cf. Sullivan [37], and Neisendorfer-Miller [30])**.** *For any space X there are maps*

$$\pi_0(\operatorname{aut} X_{\mathbb{Q}}) \to \operatorname{aut} H^*(X; \mathbb{Q}) \tag{20}$$

$$\pi_0(\operatorname{aut} X_{\mathbb{Q}}) \to \operatorname{aut}(\pi_*(\Omega X) \otimes \mathbb{Q}) \tag{21}$$

by functoriality. For a simply connected space X the map (20) *is surjective if and only if X is formal, and the map* (21) *is surjective if and only if X is coformal.*

Proposition 4.34. *Let X be a Koszul space with cohomology algebra A and homotopy Lie algebra L. If all Maurer-Cartan elements of $A \otimes L$ are linear, then the maps* (20) *and* (21) *are isomorphisms of groups.*

Proof. Since all Maurer-Cartan elements are linear we get by Proposition 4.31 that

$$\pi_0 MC_\bullet(A \otimes L) \simeq MC(A \otimes L),$$

since two elements are in the same component if and only if they are equal. The bijections

$$\mathrm{Hom}(L, L) \simeq MC(A \otimes L) \simeq \mathrm{Hom}(A, A)$$

resulting from Proposition 4.32 then give us

$$\mathrm{Hom}(L, L) \simeq [X_{\mathbb{Q}}, X_{\mathbb{Q}}] \simeq \mathrm{Hom}(A, A)$$

by Proposition 4.29, and the image of a self-map $X_{\mathbb{Q}} \to X_{\mathbb{Q}}$ can be identified with the induced map on respectively homotopy and cohomology. Such a self-map is a homotopy equivalence if and only if it induces isomorphisms in homotopy and cohomology ($X_{\mathbb{Q}}$ is simply connected and $\mathbb{Q}$-local), and so by restricting we get bijections

$$\operatorname{aut} L \simeq \pi_0 \operatorname{aut} X_{\mathbb{Q}} \simeq \operatorname{aut} A$$

□

From the proof we see that the hypotheses of Proposition 4.34 can be weakened slightly. It is enough that all the Maurer-Cartan elements *which correspond to homotopy equivalences* are linear.

Corollary 4.35. *If all Maurer-Cartan elements are linear and A or L is finite, then*

$$(\mathrm{Der}\, L)_0 \simeq \mathrm{Lie}(\pi_0 \operatorname{aut} X_{\mathbb{Q}}) \simeq (\mathrm{Der}\, A)_0.$$

Proof. Since A or L is finite, $\pi_0 \operatorname{aut} X_{\mathbb{Q}}$ is a linear algebraic group by Theorem 4.23, and we may apply $\mathrm{Lie}(-)$ to the isomorphisms of Proposition 4.34 to get

$$\mathrm{Lie}(\pi_0 \operatorname{aut} X_{\mathbb{Q}}) \simeq \mathrm{Lie}(\operatorname{aut} L) \simeq (\mathrm{Der}\, L)_0$$

and

$$\mathrm{Lie}(\pi_0 \operatorname{aut} X_{\mathbb{Q}}) \simeq \mathrm{Lie}(\operatorname{aut} A) \simeq (\mathrm{Der}\, A)_0.$$

□

Here we have used two more facts from algebraic geometry:

(1) the association of a Lie algebra to a linear algebraic group given by $\mathrm{Lie}(-)$ is functorial, and
(2) for a finite dimensional graded algebra A (not necessarily associative) the group of automorphisms is a linear algebraic group, and when working in characteristic zero $\mathrm{Lie}(\operatorname{aut} A)$ identifies with $(\mathrm{Der}\, A)_0$, the derivations of degree zero.

Corollary 4.36. *Let X be a Koszul space such that $H^*(X;\mathbb{Q})$ is generated in a single cohomological degree d. Equivalently $\pi_*(\Omega X) \otimes\ Q$ is generated in degree $d-1$. If*

(i) $H^i(X;\mathbb{Q}) = 0$ for all $i \geq d^2$, or
(ii) $\pi_i(\Omega X) \otimes \mathbb{Q} = 0$ for all $i \geq d(d-1)$,

then

$$\pi_0(\operatorname{aut} X_{\mathbb{Q}}) \simeq \operatorname{aut}(H^*(X;\mathbb{Q})) \simeq \operatorname{aut}(\pi_*(\Omega X) \otimes \mathbb{Q}).$$

Proof. Since A is generated in homological degree $-d$, and the homotopy Lie algebra L is generated in degree $d-1$, we get

$$(A \otimes L)_{-1} = \bigoplus_{i \geq 0} A(1+i(d-1)) \otimes L(1+id)$$

The assumption on A respectively L, implies that $(A \otimes L)_{-1} = A(1) \otimes L(1)$ since all summands with $i \geq 1$ vanish. In particular all Maurer-Cartan elements must be linear and the corollary follows by Proposition 4.34. $\square$

Corollary 4.36 is formulated to highlight the analogous roles played by cohomology and homotopy, but the apparent (almost) symmetric conditions may be a bit misleading. Recall from Remark 4.17 that a formal space X has finite dimensional homotopy only if it is a two-stage space. Thus if $\pi_*(\Omega X) \otimes \mathbb{Q}$ is generated in degree $d-1$, then $\pi_i(\Omega X) \otimes \mathbb{Q} = 0$ for all $i > 2(d-1)$ and necessarily $d \geq 3$, so condition (ii) is automatically satisfied if the homotopy is finite. In that case the assumption that L is generated in a single degree can be weakened.

Example 4.37 Let X be a Koszul space with finite dimensional rational homotopy Lie algebra L, and suppose L is generated in degrees $5, 7$ and 9. Then the rational cohomology algebra is generated in cohomological degrees $6, 8$ and 10. Now only the linear part of $A \otimes L$ contributes to degree -1, as we will show.

We know that X is two-stage, so

$$(A \otimes L)_{-1} = (A \otimes L(1))_{-1} \oplus (A \otimes L(2))_{-1}.$$

The Lie algebra L contributes positively to degrees of elements in the tensor product, and A contributes negatively. Thus we just have to check that no linear combination of $6, 8$ and 10 equal $n+m+1$ for $n, m \in \{5, 7, 9\}$, so that

$$(A \otimes L(2))_{-1} = 0,$$

and that no linear combination of two or more of $6, 8$ and 10 equal $n+1$ for $n \in \{5, 7, 9\}$, so that

$$\bigoplus_{i \geq 2} (A(i) \otimes L(1))_{-1} = 0.$$

The first condition is satisfied because linear combinations of even numbers are even, and $n+m+1$ is odd, when n and m are odd. The second is satisfied by inspection: $2 \cdot 6 > 9 + 1$.

In conclusion all Maurer-Cartan elements are linear, and Proposition 4.34 applies.

In general if X is a Koszul space with finite dimensional homotopy L generated in degrees $\{n_i\}_{i=1}^m$, the condition is that

$$\sum_{i=1}^{l} (n_{i_l} + 1) = n_j + n_k + 1$$

has no solutions for $1 \leq j, k, l \leq m$ and $1 \leq i_l \leq m$, and

$$\sum_{i=1}^{l} (n_{i_l} + 1) = n_j + 1$$

has no solutions for $1 \leq j, l \leq m$ and $1 \leq i_l \leq m$.

Example 4.37 illustrates a special case where this can easily be checked, and we may formulate that as

Corollary 4.38. *Let X be a Koszul space such that $L = \pi_*(\Omega X) \otimes \mathbb{Q}$ is finite dimensional, and generated in odd degrees $n_1 \leq n_2 \leq \cdots \leq n_m$ with $2n_1 > n_m$. Then*

$$\pi_0(\operatorname{aut} X_{\mathbb{Q}}) \simeq \operatorname{aut}(H^*(X;\mathbb{Q})) \simeq \operatorname{aut}(\pi_*(\Omega X) \otimes \mathbb{Q})$$

The proof is completely analogous to the reasoning in Example 4.37.

Example 4.39 Consider the classifying space $BSU(3)$ of the classical Lie group $SU(3)$. The rational cohomology algebra is free on two generators in degrees 4 and 6, and the rational homotopy Lie algebra is abelian on two generators in degrees 3 and 5. By Corollary 4.38 we conclude that

$$\pi_0(\operatorname{aut} BSU(3)_{\mathbb{Q}}) \simeq \operatorname{aut}(H^*(BSU(3);\mathbb{Q})) \simeq \mathbb{Q}^\times \times \mathbb{Q}^\times.$$

Example 4.40 Let K be rationally equivalent to a finite product of rational Eilenberg-MacLane spaces concentrated in even degrees as above

$$K \simeq_{\mathbb{Q}} \prod_{i=1}^{n} K(\mathbb{Q}, 2k_i)$$

Consider an odd sphere bundle

$$S^{2m-1} \to X \to K$$

with quadratic non-degenerate Euler class defined over $\mathbb{Q}$ cf. Example 4.20. Then the total space X satisfies the conditions of Corollary 4.38.

Rationally we have $BSU(3) \simeq_{\mathbb{Q}} K(\mathbb{Q},4) \times K(\mathbb{Q},6)$. For $K = BSU(3)$ in the above, the only non-degenerate quadratic forms are multiples of xy where x and y are the generators of $H^*(BSU(3);\mathbb{Q})$. Then

$$\pi_0(\operatorname{aut} X_{\mathbb{Q}}) \simeq \operatorname{aut}\left(\mathbb{Q}[x,y]/(xy)\right) \simeq \mathbb{Q}^\times \times \mathbb{Q}^\times.$$

Example 4.39 also (and perhaps better) serves for the following

Proposition 4.41. *Let X be a Koszul space such that $A = H^*(X;\mathbb{Q})$ is free on generators $x_1, \ldots, x_n$ such that*

$$|x_1| = \cdots = |x_{i_1}| < |x_{i_1+1}| = \cdots = |x_{i_2}| < \cdots < |x_{i_m+1}| = \cdots = |x_n|$$

(equivalently $L = \pi_*(\Omega X) \otimes \mathbb{Q}$ is abelian on generators satisfying the above). Then

$$\pi_0(\operatorname{aut} X_{\mathbb{Q}}) \simeq \prod_{j=1}^{m+1} GL(i_j, \mathbb{Q}),$$

that is, a product with a one factor $GL(i,\mathbb{Q})$ for each degree of generator, such that i is the number of generators of that particular degree, and $m+1$ is the number of distinct degrees for the generators.

Proof. Since L is abelian, also $\Omega_\bullet \otimes A \otimes L$ is abelian. Then two Maurer-Cartan elements $\tau, \tau' \in \mathrm{MC}(A \otimes L)$ are in the same path component of $\mathrm{MC}_\bullet(A \otimes L)$ if and only if there exists a polynomial $\alpha(t) \in (A \otimes L)_{-1}[t]$ such that $\frac{d}{dt}\alpha(t) = 0$, with $\alpha(0) = \tau$ and $\alpha(1) = \tau'$. But then $\alpha(t)$ is constant and $\tau = \tau'$, so

$$\pi_0(\mathrm{MC}_\bullet(A \otimes L)) = \mathrm{MC}(A \otimes L).$$

As in the proof of Proposition 4.34 we conclude that

$$\pi_0(\operatorname{aut} X_{\mathbb{Q}}) \simeq \operatorname{aut} H^*(X;\mathbb{Q}),$$

which we identify with the product we wanted. $\square$

For the following examples we refer to [14] for computations of cohomology algebras.

Example 4.42 Let G be a compact connected Lie group. Then BG is a Koszul space with free rational cohomology, and by Proposition 4.41 we get

$$\pi_0(\operatorname{aut} BG_{\mathbb{Q}}) \simeq \prod_{j=1}^{m+1} GL(i_j,\mathbb{Q}),$$

with notation as in the proposition.

Let G be a compact simply connected Lie group. Then G is also a Koszul space with free rational cohomology (on odd generators). The generators for the cohomology are the same as for BG, only shifted once in degree, and in particular we get

$$\pi_0(\operatorname{aut} G_{\mathbb{Q}}) \simeq \prod_{j=1}^{m+1} GL(i_j,\mathbb{Q}) \simeq \pi_0(\operatorname{aut} BG_{\mathbb{Q}}).$$

Surely $\pi_0(\operatorname{aut} G_{\mathbb{Q}})$ and $\pi_0(\operatorname{aut} BG_{\mathbb{Q}})$ have been studied, but we are not aware of any references at this time.

Example 4.42 also illustrates that having only linear Maurer-Cartan elements is only a sufficient condition for the maps (20) and (21) to be isomorphisms. Consider the compact simply connected Lie group $SU(7)$, with

$$A = H^*(SU(7);\mathbb{Q}) \simeq \Lambda(x_3,\ldots,x_{15})$$

with cohomological degrees $|x_i| = i$, and abelian homotopy Lie algebra L on dual generators $\alpha_3,\ldots,\alpha_{15}$ with homological degrees $|\alpha_i| = i-1$. Every element in $(A\otimes L)_{-1}$ is a Maurer-Cartan element, in particular the non-linear

$$x_3x_5x_7 \otimes \alpha_{15}.$$

This does not correspond to a homotopy equivalence however, and we have not decided if having only linear Maurer-Cartan elements *which correspond to homotopy equivalences* is necessary for the maps (20) and (21) to be isomorphisms in general.

Example 4.43 Let V be a real, complex or quarternion Stiefel manifold. Then the rational cohomolgy $H^*(V;\mathbb{Q})$ is free with generators all in distinct degrees, and by Proposition 4.41 we get

$$\pi_0(\operatorname{aut} V_{\mathbb{Q}}) \simeq \prod_{j=1}^{n} \mathbb{Q}^\times$$

where n is the number of generators.

Example 4.44 Denote by $F(\mathbb{R}^n,k)$ the space of ordered configurations of k points in $\mathbb{R}^n$. By Theorem 5.1 in the next section $A = H^*(F(\mathbb{R}^n,k);\mathbb{Q})$ is generated in a single cohomological degree $n-1$, and $A(i) = 0$ for all $i \geq k$. Thus if $k \leq n-1$ then Corollary 4.36 is satisfied, and we get

$$\pi_0(\operatorname{aut} F(\mathbb{R}^n,k)_{\mathbb{Q}}) \simeq \operatorname{aut} H^*(F(\mathbb{R}^n,k);\mathbb{Q}) \simeq \operatorname{aut}(\pi_*(\Omega F(\mathbb{R}^n,k))\otimes\mathbb{Q}).$$

Remark 4.45 Compare Corollary 4.35 to Theorem 4.25. If $H^{\geq 2}(sA \otimes L)_0$ is non-zero, the map from Theorem 4.25

$$\mathrm{Lie}(\pi_0(\mathrm{aut}\, X_{\mathbb{Q}})) \longrightarrow (\mathrm{Der}\, L)_0,$$

has a non-trivial kernel. By Example 4.44 above the map (20) is an isomorphism for $X = F(\mathbb{R}^n, k)$ when $k \leq n-1$, but from the calculations in Section 5 we see that $H^{\geq 2}(sA \otimes L)_0 \neq 0$ for e.g. $k = 3$ and even $n \geq 4$.

In conclusion, the map from Theorem 4.25 is not in general the image of the map (20) under the functor Lie.

5. Configuration spaces

Another virtue of using the complex $sA \otimes L$ as a model for the cover of the classifying space of homotopy automorphisms, is the possibility of computing the ranks of the rational homotopy in particular cases. It was done in [5] for highly connected manifolds, and we give a different example here.

Denote by $F(\mathbb{R}^n, k)$ the space of (ordered) configurations of k points in $\mathbb{R}^n$. These spaces are Koszul for all n and k (see [2] and [25]). Recall the structure of the cohomology algebra and the homotopy Lie algebra of $F(\mathbb{R}^{n+1}, k)$.

Theorem 5.1 (cf. [11])**.** *The cohomology algebra $H^*(F(\mathbb{R}^{n+1}, k); \mathbb{Q})$ is a free graded algebra generated by elements a_{pq} of cohomological degree n where $1 \leq p < q \leq k$, subject to the Arnold relations*

$$a_{pq}a_{qr} + a_{qr}a_{rp} + a_{rp}a_{pq} = 0, \quad p, q, r \,\text{distinct},$$
$$a_{pq}^2 = 0$$

with the convention that $a_{pq} = (-1)^{n+1}a_{qp}$, for $p > q$. This algebra has a linear basis consisting of all monomials $a_{i_1 j_1} \ldots a_{i_r j_r}$ such that $i_1 < \cdots < i_r$ and $i_p < j_p$ for $1 \leq p \leq r$.

Theorem 5.2 (cf. [10])**.** *The rational homotopy Lie algebra $\pi_*(\Omega F(\mathbb{R}^{n+1}, k)) \otimes \mathbb{Q}$ is generated by elements α_{pq} of homological degree $n-1$, with $1 \leq p < q \leq k$, subject to the (orthogonal to the above) relations*

$$[\alpha_{pq}, \alpha_{rs}] = 0, \quad \{p, q\} \cap \{r, s\} = \emptyset$$
$$[\alpha_{pq}, \alpha_{pr} + \alpha_{qr}] = 0, \quad p, q, r \,\text{distinct}$$

with the convention that $\alpha_{pq} = (-1)^{n+1}\alpha_{qp}$ for $p > q$.

Lemma 5.3. *The dimensions of the graded components of the cohomology algebra $A := H^*(F(\mathbb{R}^n, k); \mathbb{Q})$ are given by the Stirling numbers of first kind. Explicitly, in weight j we have*

$$\dim A(j) = \begin{bmatrix} k \\ k-j \end{bmatrix}.$$

Proof. The homogeneous polynomial

$$\prod_{m=0}^{k-1} (y + mx)$$

generates the Stirling numbers of first kind when $x = 1$ (backwards), and the Poincaré polynomial for $H^*(F(\mathbb{R}^n, k); \mathbb{Q})$ when $y = 1$ (with the substitution $x = z^{n-1}$). □

Corollary 5.4. *The dimensions of the graded components of the homotopy Lie algebra $L := \pi_*(\Omega F(\mathbb{R}^n,k))\otimes\mathbb{Q}$ are given by the following formula*

$$\dim L(r) = \frac{1}{r}\sum_{d|r}(-1)^{n(d+r)}\mu\Big(\frac{r}{d}\Big)\sum_{j=1}^{k-1} j^d\,.$$

Proof. The Poincaré polynomial for $A := H^*(F(\mathbb{R}^n,k);\mathbb{Q})$ is

$$\prod_{m=0}^{k-1}(1+mx)$$

giving the following equation for the generating series of the dimension of the universal enveloping algebra

$$\prod_{m=0}^{k-1}(1-mx)^{-1} = \begin{cases} \prod_{m=0}^{\infty}(1-x^m)^{\epsilon_m} & n \text{ even} \\ \prod_{m=0}^{\infty}(1-(-x)^m)^{(-1)^{(m+1)}\epsilon_m} & n \text{ odd}\,. \end{cases}$$

Taking logarithms and comparing coefficients in the resulting Taylor series we get

$$-\frac{1}{r}\sum_{m=1}^{k-1} m^r = \begin{cases} \sum_{d|r}\epsilon_d\frac{d}{r} & n \text{ even} \\ \sum_{d|r}(-1)^{d+r}\epsilon_d\frac{d}{r} & n \text{ odd} \end{cases}$$

and the corollary follows by the Möbius inversion formula. ☐

It follows from the given presentation of homotopy Lie algebra that is has the structure of an iterated semi-direct product of free Lie algebras:

$$\pi_*(\Omega F(\mathbb{R}^{n+1},k))\otimes\mathbb{Q} \simeq \mathbb{L}_{k-1}\ltimes(\mathbb{L}_{k-2}\ltimes(\cdots\ltimes\mathbb{L}_1)\cdots), \tag{22}$$

where $\mathbb{L}_i$ is the free Lie algebra on the generators α_{ij} of which there are $k-i$. Indeed there are no relations among generators with the same first index, and the relations

$$\begin{aligned} [\alpha_{pq},\alpha_{rs}] &= 0, \quad \{p,q\}\cap\{r,s\}=\emptyset \\ [\alpha_{pq},\alpha_{pr}+\alpha_{qr}] &= 0, \quad p,q,r \text{ distinct} \end{aligned}$$

describe the action of the free Lie algebras of greater first index on those with lesser.

The isomorphism (22) gives a linear basis. As a graded vector space the Lie algebra is isomorphic to the direct sum of the free Lie algebras, and the natural bases for those provide one which we will call the *unmixed basis.* This also gives an alternative way of arriving at the dimension formula. Simply add up the dimensions of the free Lie algebras in question. These dimension are given by the well known Witt formula

$$\dim\mathbb{L}_i(r) = \frac{1}{r}\sum_{d|r}(-1)^{d+r}\mu\big(\tfrac{r}{d}\big)\,(k-i)^d.$$

5.1. **Rational homotopy groups.**

Example 5.5 Consider $F(\mathbb{R}^n,3)$ for $n\geq 6$ and even, with cohomology A and homotopy Lie algebra L. In non-negative degrees the complex $(sA\otimes_\kappa L)_+$ splits as a direct sum with summands

$$0 \longrightarrow sA(0)\otimes_\kappa L(t) \xrightarrow{\kappa^0} sA(1)\otimes_\kappa L(t+1) \xrightarrow{\kappa^1} sA(2)\otimes_\kappa L(t+2) \longrightarrow 0,$$

for $t\geq 1$ (with the obvious restrictions of κ^0 and κ^1).

By Proposition 4.4 κ^0 can be identified with the map $\operatorname{ad}\colon L \to \operatorname{Der} L$, and so the kernel is the center of L. When n is even the center is 1-dimensional, spanned by the element $\alpha_{12}+\alpha_{13}+\alpha_{23} \in L(1)$. That is, for $t \geq 2$ the map κ^0 is injective.

Recall that (a_{12}, a_{13}, a_{23}) is a basis for $A(1)$ and $(a_{12}a_{23}, a_{13}a_{23})$ is a basis for $A(2)$. Using the presentation of A given in Theorem 5.1, we find that the matrices representing the maps given by multiplication by a generator, are

$$a_{12} : \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad a_{13} : \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad a_{23} : \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Now by slight abuse of notation, let α_{ij} denote the matrix representing $ad_{\alpha_{ij}}$. Then κ^1 is represented by the block matrix

$$\begin{pmatrix} -\alpha_{13}-\alpha_{23} & \alpha_{12} & \alpha_{12} \\ \alpha_{13} & -\alpha_{12}-\alpha_{23} & \alpha_{13} \end{pmatrix}.$$

As noted above, the element $\sum_{ij} \alpha_{ij}$ is central in L, and so the matrices denoted the same way also satisfy $\sum_{ij} \alpha_{ij} = 0$. Apply this and reduce the resulting block matrix:

$$\begin{aligned} \begin{pmatrix} -\alpha_{13}-\alpha_{23} & \alpha_{12} & \alpha_{12} \\ \alpha_{13} & -\alpha_{12}-\alpha_{23} & \alpha_{13} \end{pmatrix} &= \begin{pmatrix} \alpha_{12} & \alpha_{12} & \alpha_{12} \\ \alpha_{13} & \alpha_{13} & \alpha_{13} \end{pmatrix} \\ &\rightsquigarrow \begin{pmatrix} 0 & 0 & \alpha_{12} \\ 0 & 0 & \alpha_{13} \end{pmatrix}. \end{aligned}$$

The Lie algebra L is isomorphic to

$$\mathbb{L}(\alpha_{23}) \ltimes \mathbb{L}(\alpha_{12}, \alpha_{13}),$$

so the maps given by α_{12} and α_{13} respectively are injective on $L(\geq 2)$, and therefore κ^1 has rank $\dim L(t+1)$ when restricted to the t-summand.

We can now compute the homology of each summand. For $t=1$ we found that κ^0 has rank 2 and κ^1 has rank 1. The dimensions of $A(i)$ and $L(j)$ are found in Corollary 5.3 and Corollary 5.4 and if we write the $t=1$ summand with just the dimension of each vector space, and the rank of the maps decorating the arrows:

$$0 \longrightarrow 3 \xrightarrow{2} 3 \xrightarrow{1} 4 \longrightarrow 0,$$

we find that the homology is of dimension 1, 0 and 3 respectively. Since A is generated in homological degree $1-n$ and L is generated in homological degree $n-2$, these graded vector spaces are concentrated in degrees $n-2$, $n-3$ and $n-4$ respectively.

For the summands $t \geq 2$ the map κ^0 is injective, and we get homology of dimension

$$\frac{2}{t+1} \sum_{d|t+1} \mu\!\left(\tfrac{t+1}{d}\right)(1+2^d) - \frac{1}{t} \sum_{d|t} \mu\!\left(\tfrac{t}{d}\right)(1+2^d) \quad \text{and}$$
$$\frac{2}{t+2} \sum_{d|(t+2)} \mu\!\left(\tfrac{t+2}{d}\right)(1+2^d) - \frac{1}{t+1} \sum_{d|t+1} \mu\!\left(\tfrac{t+1}{d}\right)(1+2^d),$$

concentrated in degrees $(t+1)(n-2)-n+1$ and $(t+1)(n-2)-n$ respectively. Clearly this is a recurring pattern.

When $n \geq 6$, only a single summand contribute to each degree of the homology of the entire complex $sA \otimes L$, and so we have computed the dimensions of

$$H(sA \otimes_\kappa L)_i \simeq \pi_i(\operatorname{aut} F(\mathbb{R}^n, 3), 1_{F(\mathbb{R}^n,3)}) \otimes \mathbb{Q}, \qquad i \geq 1.$$

Take e.g. $n = 6$: The first non-zero rational homotopy groups is obtained for $t = 1$:

$$\dim \pi_2 = 3, \quad \dim \pi_3 = 0, \quad \dim \pi_4 = 1.$$

The next ones appear for $t = 2$, and from the formulae above we get:

$$\dim \pi_6 = 3, \quad \dim \pi_7 = 4,$$

and so they keep coming in pairs for every $t \geq 2$. We list a few more for $3 \leq t \leq 6$:

$$\begin{aligned} \dim \pi_{10} &= 9, \quad \dim \pi_{11} = 4 \\ \dim \pi_{14} &= 12, \quad \dim \pi_{15} = 9 \\ \dim \pi_{18} &= 30, \quad \dim \pi_{19} = 12. \end{aligned}$$

In general we observe that there are infinitely many homotopy groups of the rationalised space $(\operatorname{aut} F(\mathbb{R}^n, 3))_\mathbb{Q}$ and that they grow exponentially: the expression for the dimension in degree $(t+1)(n-2)-n$ is dominated by the term $\frac{2}{t+2}\mu(1)(1+2^{t+2})$.

Another observation is that each connected component of $(\operatorname{aut} F(\mathbb{R}^n, 3))_\mathbb{Q}$ is $(n-5)$-connected.

Remark 5.6 The same approach works for $F(\mathbb{R}^4, 3)$ also, but in that case $sA(2)\otimes_\kappa L(3)$ is concentrated in degree zero, and removed when we pass to $(sA\otimes_\kappa L)_+$. Other than that everything in the above example goes through.

The trick used to compute the rational homotopy groups for $F(\mathbb{R}^n, 3)$ by reducing the block matrix representing the differential κ does not work in general. In the next example we present the results of computer aided calculations for configurations with more than three points.

Example 5.7 In general the complex $(sA \otimes_\kappa L)_+$ for $F(\mathbb{R}^n, k)$ splits as a direct sum with summands

$$0 \longrightarrow A(0) \otimes L(t) \longrightarrow sA(1) \otimes L(t+1) \longrightarrow \cdots \longrightarrow sA(k-1) \otimes L(t+k-1),$$

for $t \geq 1$. We have computed the homology of some of these summands when n is even, and $k = 4, 5, 6$. The computations are implemented in Magma, and the code is available upon request. Our results are summarised in the following tables. To give some perspective we first record the dimensions of the algebras in consideration.

The dimensions for $A(i)$ and $3 \leq k \leq 6$ are:

i k	3	4	5	6
0	1	1	1	1
1	3	6	10	15
2	2	11	35	85
3	0	6	50	225
4	0	0	24	274
5	0	0	0	120
6	0	0	0	0

The dimensions for $L(i)$ and $3 \le k \le 6$ are:

i $\quad k$	3	4	5	6
1	3	6	10	15
2	1	4	10	20
3	2	10	30	70
4	3	21	81	231
5	6	54	258	882
6	9	125	795	3375
7	18	330	2670	13830

The computations in Magma are actually implemented for the complex $s\overline{A} \otimes_\kappa L$. Since we have identified the first differential $A(0) \otimes L \to A(1) \otimes L$ with $\mathrm{ad}\colon L \to \operatorname{Der} L$, and the centre of L has dimension 1 (generated by $\sum_{ij} \alpha_{ij}$), it is easy to correct for this. The case $k = 3$ has already been exhaustively covered. When $k = 4$ the Betti numbers of the summands in the complex are:

summand				
1	1	0	45	80
2	0	0	81	230
3	0	1	230	501
4	0	0	502	1410
5	0	1	1410	3515
6	0	0	3516	9571

For $k = 5$ we get:

summand					
1	1	0	0	1254	4355
2	0	0	10	4355	13070
3	0	1	0	13079	45660
4	0	0	?	?	?

Finally for $k = 6$ we have computed:

summand						
1	1	0	0	?	?	?
2	0	0	15	?	?	?
3	0	1	?	?	?	?

There are several observations to be made from this, but we postpone this to Section 6.

5.2. L_∞**-structure.** In the following we will study the induced L_∞-structure on $H(sA \otimes_\kappa L)_*$ for the configuration spaces. First we make some of the general formulae produced in previous sections explicit in this case. With notation from previous sections the structure is transferred along the contraction

$$h'+g'kf \circlearrowright sA \otimes_\iota \mathscr{L} \underset{g'i}{\overset{qf}{\rightleftarrows}} H(sA \otimes_\kappa L)_*,$$

where we recall that

$$g' = \sum_{j \ge 0} ((1 \otimes h)\iota)^j (1 \otimes g) \qquad \text{and} \qquad h' = \sum_{j \ge 0} ((1 \otimes h)\iota)^j (1 \otimes h).$$

The map ι splits as a sum $\iota = \sum_{m\geq 1} \iota_m$ where ι_m increases the total weight in $\mathscr{L}$ by m and bracket length by 1. The homotopy h preserves total weight and decreases bracket length by 1, so for any $j \geq 1$ the map $((1\otimes h)\iota)^j$ preserves bracket length and strictly increases total weight. In our case $A(p) = 0$ for $p \geq 3$, so only the diagonals $\mathscr{D}_0$ and $\mathscr{D}_1$ are non-zero, and in particular $(1\otimes h)\iota(1\otimes h) = 0$. In conclusion we have

$$g' = 1 \otimes g + (1\otimes h)\iota(1\otimes g) \qquad \text{and} \qquad h' = 1\otimes h.$$

The map i is given by a choice of cycle representatives for the homology, and the map g as produced by Lemma 3.8 is given by a choice of basis for L. Cycle representatives will be chosen later, while the unmixed basis for L gives a choice of g. Finally h is inductively constructed according to the proof of Lemma 3.8.

In the following we assume that n is even, but similar calculation can be made for odd n. To construct h we first need formulae for the differential on $\mathscr{L}$. It is given by the reduced comultiplication on $A^\vee$. It is dual to the multiplication on $\overline{A}$, and with the bases (a_{12}, a_{13}, a_{23}) for $A(1)$ and $(a_{12}a_{23}, a_{13}a_{23})$ for $A(2)$ the multiplication $A(1)\otimes A(1) \to A(2)$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \end{pmatrix}.$$

From which we get

$$\begin{aligned} \overline{\Delta}(a_{12}a_{23}^*) &= a_{12}^* \otimes a_{13}^* + a_{12}^* \otimes a_{23}^* - a_{13}^* \otimes a_{12}^* - a_{23}^* \otimes a_{12}^* \\ \overline{\Delta}(a_{13}a_{23}^*) &= -a_{12}^* \otimes a_{13}^* + a_{13}^* \otimes a_{12}^* + a_{13}^* \otimes a_{23}^* - a_{23}^* \otimes a_{13}^*. \end{aligned}$$

The differential on $\mathscr{L}$ is then given by

$$\begin{aligned} d(x_{12}x_{23}) &= [x_{12}, x_{13}] + [x_{12}, x_{23}] \\ d(x_{13}x_{23}) &= [x_{13}, x_{23}] - [x_{12}, x_{13}], \end{aligned}$$

where we write x_{ij} for $s^{-1}a_{ij}^*$ and $x_{ij}x_{kl}$ for $s^{-1}a_{ij}a_{kl}^*$ when denoting elements of $\mathscr{L} = \mathbb{L}(s^{-1}\overline{A}^\vee, d)$.

Now we can begin to construct h. We know that we may choose $hg = 0$, and that g is given by choosing the unmixed basis. Explicitly that means that g behaves as a Lie map on any bracketed word in the generators α_{12} and α_{13} and sends α_{12} to x_{12} and α_{13} to x_{13}, e.g.

$$g([\alpha_{12}, [\alpha_{12}, \alpha_{13}]]) = [x_{12}, [x_{12}, x_{13}]].$$

Further $g(\alpha_{23}) = x_{23}$, and if α_{23} appears in an expression as part of a bracket, then we can think of g as rewriting the expression using the relations of L, such that it no longer contains α_{23} and map it as before, e.g.

$$g([\alpha_{12}, \alpha_{23}]) = g([\alpha_{13}, \alpha_{12}]) = [x_{13}, x_{12}].$$

We immediately get

$$\begin{aligned} h(x_{ij}) &= 0, \quad \text{for all } ij \\ h([x_{12}, x_{13}]) &= 0. \end{aligned}$$

Now we will have to make choices for the values of h. To find the valid choices we apply the differential, and the use inductive definition of h.

$$\begin{aligned} dh([x_{12},x_{23}]) &= gf([x_{12},x_{23}]) - [x_{12},x_{23}] - hd([x_{12},x_{23}]) \\ &= g([\alpha_{12},\alpha_{23}]) - [x_{12},x_{23}] \\ &= -[x_{12},x_{13}] - [x_{12},x_{23}]. \end{aligned}$$

We see that this is $-d(x_{12}x_{23})$ and choose $h([x_{12},x_{23}]) = -x_{12}x_{23}$. Completely similarly we find $h([x_{13},x_{23}]) = -x_{13}x_{23}$ as a valid choice.

As above it follows immediately that

$$h([x_{12},[x_{12},x_{13}]]) = h([x_{13},[x_{12},x_{13}]]) = 0,$$

and using that the differential is a derivation we compute (and choose)

$$\begin{aligned} h([x_{12},[x_{12},x_{23}]]) &= -[x_{12},x_{12}x_{23}], \\ h([x_{12},[x_{13},x_{23}]]) &= -[x_{12},x_{13}x_{23}], \\ h([x_{13},[x_{12},x_{23}]]) &= -[x_{13},x_{12}x_{23}], \\ h([x_{13},[x_{13},x_{23}]]) &= -[x_{13},x_{13}x_{23}]. \end{aligned}$$

By the Jacobi identity we further find

$$\begin{aligned} h([x_{23},[x_{12},x_{13}]]) &= h[[x_{23},x_{12}],x_{13}] + h[x_{12},[x_{23},x_{13}]] \\ &= [x_{12},x_{13}x_{23}] - [x_{13},x_{12}x_{23}]. \end{aligned}$$

All that remain in weight 3 is $[x_{23},[x_{12},x_{23}]]$ and $[x_{23},[x_{13},x_{23}]]$. By the same reasoning as above we find

$$\begin{aligned} h([x_{23},[x_{12},x_{23}]]) &= [x_{13}-x_{23},x_{12}x_{23}] - [x_{12},x_{13}x_{23}], \\ h([x_{23},[x_{13},x_{23}]]) &= [x_{12}-x_{23},x_{13}x_{23}] - [x_{13},x_{12}x_{23}]. \end{aligned}$$

From this we can build a table for the values of g' up to weight 2 in L expressed in the chosen bases. Recall again that $g' = 1 \otimes g + (1 \otimes h)\iota(1 \otimes g)$. We get:

$$\begin{aligned}
g'(1 \otimes \alpha_{12}) &= 1 \otimes x_{12} + a_{23} \otimes x_{12}x_{23},\\
g'(1 \otimes \alpha_{13}) &= 1 \otimes x_{13} + a_{23} \otimes x_{13}x_{23},\\
g'(1 \otimes \alpha_{23}) &= 1 \otimes x_{23} - a_{12} \otimes x_{12}x_{23} - a_{13} \otimes x_{13}x_{23},\\
g'(1 \otimes [\alpha_{12}, \alpha_{13}]) &= 1 \otimes [x_{12}, x_{13}] + a_{23} \otimes ([x_{12}, x_{13}x_{23}] - [x_{13}, x_{12}x_{23}]),\\
g'(a_{12} \otimes \alpha_{12}) &= a_{12} \otimes x_{12} - a_{12}a_{23} \otimes x_{12}x_{23},\\
g'(a_{12} \otimes \alpha_{13}) &= a_{12} \otimes x_{13} - a_{12}a_{23} \otimes x_{13}x_{23},\\
g'(a_{12} \otimes \alpha_{23}) &= a_{12} \otimes x_{23} + (a_{12}a_{23} - a_{13}a_{23}) \otimes x_{13}x_{23},\\
g'(a_{13} \otimes \alpha_{12}) &= a_{13} \otimes x_{12} - a_{13}a_{23} \otimes x_{12}x_{23},\\
g'(a_{13} \otimes \alpha_{13}) &= a_{13} \otimes x_{13} - a_{13}a_{23} \otimes x_{13}x_{23},\\
g'(a_{13} \otimes \alpha_{23}) &= a_{13} \otimes x_{23} + (a_{13}a_{23} - a_{12}a_{23}) \otimes x_{12}x_{23},\\
g'(a_{23} \otimes \alpha_{12}) &= a_{23} \otimes x_{12},\\
g'(a_{23} \otimes \alpha_{13}) &= a_{23} \otimes x_{13}\\
g'(a_{23} \otimes \alpha_{23}) &= a_{23} \otimes x_{23} - a_{12}a_{23} \otimes x_{12}x_{23} - a_{13}a_{23} \otimes x_{13}x_{23},\\
g'(a_{12} \otimes [\alpha_{12}, \alpha_{13}]) &= a_{12} \otimes [x_{12}, x_{13}] - a_{12}a_{23} \otimes ([x_{12}, x_{13}x_{23}] - [x_{13}, x_{12}x_{23}]),\\
g'(a_{13} \otimes [\alpha_{12}, \alpha_{13}]) &= a_{13} \otimes [x_{12}, x_{13}] - a_{13}a_{23} \otimes ([x_{12}, x_{13}x_{23}] - [x_{13}, x_{12}x_{23}]),\\
g'(a_{23} \otimes [\alpha_{12}, \alpha_{13}]) &= a_{23} \otimes [x_{12}, x_{13}].
\end{aligned}$$

Because ι restricted to $A(2) \otimes L$ is zero, we also get

$$g'(a_{12}a_{23} \otimes \alpha) = a_{12}a_{23} \otimes g(\alpha), \qquad g'(a_{13}a_{23} \otimes \alpha) = a_{13}a_{23} \otimes g(\alpha),$$

for any $\alpha \in L$.

The following proposition shows that there are no higher operations on the truncated complex $(sA \otimes_\kappa L)_+$ associated to the configuration spaces $F(\mathbb{R}^n, 3)$. This may be an indicator that there are no higher operations on the positive homology of the complex either, which we will discuss further in Section 6. The condition that we only look at the positive part of the complex is important, as illustrated by Example 5.10 which follows after the proposition.

Proposition 5.8. *For the spaces $F(\mathbb{R}^n, 3)$, there are no higher operations on $(sA \otimes_\kappa L)_+$.*

Proof. Consider the composition

$$\nu := h[-,-]_{\mathrm{Der}}(g' \otimes g').$$

Any higher operation on $sA \otimes L$ will factor through ν, cf. Section 3.2. The claim is now that on the positive part $(sA \otimes_\kappa L)_+$, we have $\operatorname{Im} \nu \subseteq sA \otimes \operatorname{Im} g \oplus sA \otimes \mathscr{D}_1$, and so the annihilation condition, together with the fact that $h(\mathscr{D}_1) \subseteq \mathscr{D}_2 = 0$, forces any higher operation to be zero.

Since g is given by rewritings to the unmixed basis, the claim reduces to *that $\operatorname{Im} \nu \cap \mathscr{D}$ consists of sums of elements of the form $a \otimes x$ for $a \in A$ and where x is a bracketed word in the letters x_{12} and x_{13} or $x = x_{23}$.*

The image of g' is contained in

$$sA \otimes \operatorname{Im} g \oplus sA(2) \oplus \operatorname{Im} h,$$

where we recall that $\operatorname{Im} g \subseteq \mathscr{D}$ and $\operatorname{Im} h \subseteq \mathscr{D}_1$. As explained in Example 3.16, the bracket $[sa\otimes x, sb\otimes y]_{\mathrm{Der}}$ effectively scans the word x for occurrences of the letter b and replaces it with the word y (and the other way around). The bracket $[-,-]_{\mathrm{Der}}$ is bilinear, so without loss of generality we may assume that a and b are weight homogeneous, and that x and y each is contained in either $\operatorname{Im} g$ or $\operatorname{Im} h$ when both $sa\otimes x$ and $sb\otimes y$ are in $\operatorname{Im} g'$. The possible scenarios are then:

- x and y are in $\operatorname{Im} g$. In this case x and y are either both bracketed words in the letters x_{12} and x_{13}, and so $[sa\otimes x, sb\otimes y]_{\mathrm{Der}}$ is again in $sA\otimes \operatorname{Im} g$; or one or both of x and y equals x_{23}. Suppose $x = x_{23}$. Then we must have $a\in A(0)$ or else $sa\otimes x$ would have non-positive homological degree. The only possibly non-zero term of the bracket $[sa\otimes x, sb\otimes y]_{\mathrm{Der}}$ is then a multiple of $1\otimes y\in sA\otimes \operatorname{Im} g$, which happens if b is in the subspace spanned by a_{23}. The same holds mutatis mutandis for $y = x_{23}$.
- x is in $\operatorname{Im} g$ and y is in $\operatorname{Im} h$ (again we may switch x and y with the appropriate changes). Scanning x for b and replacing by y yields a term in $sA\otimes\mathscr{D}_1$. Scanning y for a and replacing by x yields a term in $sA\otimes\mathscr{D}_1$ if $a\in A(1)$. None of these cases are of interest.

  Scanning y for a and replacing by x yields a term in $sA\otimes\operatorname{Im} g$ if $a\in A(2)$ as we now explain: y is contained in $\mathscr{D}_1$, so precisely one letter x_0 (in each term) is of weight 2 in $A^\vee$, and we get a non-zero term only if a^* is in the subspace spanned by x_0. In this case $x\in\operatorname{Im} g$ replaces x_0, and it remains to argue that the rest of the letters in y are either x_{12} or x_{13}.

  We use that $sb\otimes y$ is not only in $\operatorname{Im}(1\otimes h)$ but in $\operatorname{Im}(1\otimes h)\iota(1\otimes g)$. The first part $(1\otimes g)$ gives a bracketed word w in the letters x_{12} and x_{13}. Then ι produces a term $[x,w]$ for each linear generator x of $A^\vee$, that is $x_{12}, x_{13}, x_{23}, x_{12}x_{23}$ and $x_{13}x_{23}$ are set as prefixes to w. Only the term for x_{23} is not in the kernel of $1\otimes h$, as the first two are in $\operatorname{Im} g$ and the last two are in $\mathscr{D}_1$. Use the Jacobi relation to write

  $$[x_{23}, w] = [w', [x_{12}, x_{23}]] + [w'', [x_{13}, x_{23}]],$$

  where w', w'' are sums of words in the letters x_{12} and x_{13}. Now

  $$\begin{aligned} dh[x_{23}, w] &= gf[x_{23}, w] - [x_{23}, w] \\ &= gf[w', [x_{12}, x_{23}]] + [w'', [x_{13}, x_{23}]] - [w', [x_{12}, x_{23}]] - [w'', [x_{13}, x_{23}]] \\ &= -[w', [x_{12}, x_{13}]] + [w'', [x_{12}, x_{13}]] - [w', [x_{12}, x_{23}]] - [w'', [x_{13}, x_{23}]] \\ &= -d([w', x_{12}x_{23}] + [w'', x_{13}x_{23}]), \end{aligned}$$

  so $h[x_{23}, w] = -([w', x_{12}x_{23}] + [w'', x_{13}x_{23}])$ is a valid choice, and with this we see that all letters except the one (in each term) of weight 2, are x_{12} or x_{13}.
- both x and y are in $\operatorname{Im} h$. Scanning x for b and replacing by y lands us in $sA\otimes\mathscr{D}_1$, and similar the other way around.

□

Remark 5.9 In the following we do computations using the formula for the bracket $[-,-]_{\mathrm{Der}}$ from Lemma 3.14, for the configuration spaces $F(\mathbb{R}^n, 3)$. If n is even then A is generated in odd degrees, and L is generated in even degrees. In that case α and β are both odd. If n is odd then A is generated in even degrees and L is generated in odd degrees. In that case α and β are both even.

Example 5.10 We stay with the example of $F(\mathbb{R}^n, 3)$ with n even for now. Here we shall see that in contrast to Proposition 5.8 there are indeed higher operations on the non-truncated complex $sA \otimes_\kappa L$.

Clearly we must involve a non-positive element, as we see from the example above. However, a single non-positive element is enough: consider the expression

$$\ell_3(a_{12} \otimes \alpha_{23}, a_{12} \otimes [\alpha_{12}, \alpha_{13}], a_{13}a_{23} \otimes [\alpha_{12}, [\alpha_{12}, \alpha_{13}]]),$$

where we suppress the suspensions in the notation. Recall from the general formula (5), that

$$\begin{aligned}\ell_3(x,y,z) = f \circ (-[h'[g'(x), g'(y)], g'(z)] + (-1)^{|x|}[g'(x), h'[g'(y), g'(z)]] \\ +(-1)^{|y||z|}[h'[g'(x), g'(z)], g'(y)]),\end{aligned} \tag{23}$$

and note that, in our case $|x| = 0$, $|y| = n-2$ and $|z| = n-3$. If $n \geq 4$, there is only the degree zero element $x = a_{12} \otimes \alpha_{23}$.

We worked out formulae for g' and h' above for n even, and with these we find

$$\begin{aligned}&h'[g'(a_{12} \otimes \alpha_{23}), g'(a_{12} \otimes [\alpha_{12}, \alpha_{13}])]_{\mathrm{Der}} \\ &= h'(a_{13}a_{23} \otimes ([x_{13}, x_{12}x_{23}] + [x_{12}, x_{13}x_{23}]) - a_{12} \otimes [x_{23}, x_{13}] \\ &\qquad + a_{12}a_{23} \otimes [x_{23}, x_{13}x_{23}]) \\ &= -a_{12} \otimes x_{13}x_{23}\end{aligned}$$

Next we take the bracket of this and $g'(a_{13}a_{23} \otimes [\alpha_{12}, [\alpha_{12}, \alpha_{13}]])$ and apply f to get the first term of (23).

$$\begin{aligned}&[-a_{12} \otimes x_{13}x_{23}, g'(a_{13}a_{23} \otimes [\alpha_{12}, [\alpha_{12}, \alpha_{13}]])]_{\mathrm{Der}} \\ &= -a_{12} \otimes [x_{12}, [x_{12}, x_{13}]] + a_{13}a_{23} \otimes ([x_{13}x_{23}, [x_{12}, x_{13}]] + [x_{12}, [x_{13}x_{23}, x_{13}]])\end{aligned}$$

which maps to $-a_{12} \otimes [\alpha_{12}, [\alpha_{12}, \alpha_{13}]]$ under f, so first term of (23) is $a_{12} \otimes [\alpha_{12}, [\alpha_{12}, \alpha_{13}]]$. We compute the last term:

$$\begin{aligned}&h'[g'(a_{12} \otimes \alpha_{23}), g'(a_{13}a_{23} \otimes [\alpha_{12}, [\alpha_{12}, \alpha_{13}]])]_{\mathrm{Der}} \\ &= h'(a_{12}a_{23} \otimes [x_{12}, [x_{12}, x_{13}]] - a_{13}a_{23} \otimes ([x_{23}, [x_{12}, x_{13}]] + [x_{12}, [x_{23}, x_{13}]])) \\ &= -a_{13}a_{23} \otimes (2[x_{12}, x_{13}x_{23}] - [x_{13}, x_{12}x_{23}]),\end{aligned}$$

and this we bracket with $g'(a_{12} \otimes [\alpha_{12}, \alpha_{13}])$

$$\begin{aligned}&[-a_{13}a_{23} \otimes (2[x_{12}, x_{13}x_{23}] - [x_{13}, x_{12}x_{23}]), g'(a_{12} \otimes [\alpha_{12}, \alpha_{13}])]_{\mathrm{Der}} \\ &= -a_{13}a_{23} \otimes (2[[x_{12}, x_{13}], x_{13}x_{23}] + [x_{13}, ([x_{13}, x_{12}x_{23}] + [x_{12}, x_{13}x_{23}])] \\ &\qquad + a_{12}a_{23} \otimes [x_{12}, (2[x_{12}, x_{13}x_{23}] - [x_{13}, x_{12}x_{23}]).\end{aligned}$$

This expression is contained in $sA \otimes \mathscr{D}_1$ so f maps it to zero. It is an example to illustrate the following.

Let x be an element in $sA \otimes \mathscr{L}$, and $y \in sA \otimes L$, and write

$$\begin{aligned}g'(y) &= s1 \otimes v_0 + sa_1 \otimes v_1 + sa_2 \otimes v_2, \qquad a_i \in A(i), v_i \in \mathscr{L}, \\ h'(x) &= sb_1 \otimes w_1 + sb_2 \otimes w_2, \qquad b_i \in A(i), w_i \in \mathscr{L}.\end{aligned}$$

Necessarily $w_i \in \mathscr{D}_1$, so for $[h'(x), g'(y)]_{\mathrm{Der}}$ to be non-zero, a_2^* must be in the span of some letter of the word w_1 or w_2, or b_i^* is in the span of some letter of the word v_j for some i, j. In the latter case, the bracket takes values in $sA \otimes \mathscr{D}_1$ again. In the first case v_2 replaces the letter which matches a_2, and if y is in $sA(1) \otimes L$ then

v_2 is in the image of h, and in particular in $\mathscr{D}_1$. In both cases f vanishes on the bracket.

From this we see that also the middle term of (23) is zero, and in conclusion

$$\ell_3(a_{12}\otimes\alpha_{23}, a_{12}\otimes[\alpha_{12},\alpha_{13}], a_{13}a_{23}\otimes[\alpha_{12},[\alpha_{12},\alpha_{13}]]) = a_{12}\otimes[\alpha_{12},[\alpha_{12},\alpha_{13}]],$$

which is then a non-zero higher operation on the non-truncated complex $sA\otimes_\kappa L$.

Example 5.11 Consider again $F(\mathbb{R}^n,3)$ for $n\geq 6$ and even, with cohomology A and homotopy Lie algebra L. In this example we describe the Lie algebra $H(sA\otimes_\kappa L)_0$, which by Proposition 4.24 is the Lie algebra of the algebraic group $\pi_0(\operatorname{aut} F(\mathbb{R}^n,3)_{\mathbb{Q}})$. Note that by Example 4.44, we also have

$$\operatorname{aut} L \simeq \pi_0(\operatorname{aut} F(\mathbb{R}^n,3)_{\mathbb{Q}}) \simeq \operatorname{aut} A.$$

Since $n\geq 6$ the only contribution to homology in degree zero comes from the linear part of the complex, i.e. the kernel of κ^1

$$0 \longrightarrow sA(1)\otimes_\kappa L(1) \xrightarrow{\kappa^1} sA(2)\otimes_\kappa L(2) \longrightarrow 0.$$

Again (a_{12},a_{13},a_{23}) is a basis for $A(1)$ and $(a_{12}a_{23},a_{13}a_{23})$ is a basis for $A(2)$. The unmixed basis for L gives a basis $(\alpha_{12},\alpha_{13},\alpha_{23})$ for $L(1)$ and $([\alpha_{12},\alpha_{13}])$ for $L(2)$. With the standard choice of bases for the tensor products, the map κ^1 restricted to $A(1)\otimes L(1)$ is then represented by the matrix

$$\begin{pmatrix} 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 \end{pmatrix},$$

and in particular it is surjective. Thus $H(sA\otimes L)_0$ is a 7-dimensional Lie algebra, and with the bases chosen we can find generating cycles for the homology. One set of choices is

$$\begin{aligned} a &:= (a_{12}-a_{13})\otimes\alpha_{12}, \quad & b &:= (a_{12}-a_{13})\otimes\alpha_{13}, \quad & c &:= (a_{12}-a_{13})\otimes\alpha_{23},\\ d &:= (a_{12}-a_{23})\otimes\alpha_{12}, \quad & e &:= (a_{12}-a_{23})\otimes\alpha_{13}, \quad & f &:= (a_{12}-a_{23})\otimes\alpha_{23},\\ & & & & g &:= \kappa = a_{12}\otimes\alpha_{12} + a_{13}\otimes\alpha_{13} + a_{23}\otimes\alpha_{23}, \end{aligned}$$

where we have suppressed the suspension from notation. The bracket is given by the transferred operation l_2 which on the chosen representatives equals the composition $f[g'(-),g'(-)]_{\mathrm{Der}}$ followed by the map to homology (cf. end of Section 3.2). Notice that the map to homology on $A\otimes L(1)$ is just the projection to cycles, as there are no boundaries here.

Recall that $g' = g+htg+(ht)^2g+\cdots$, and that since $hth=0$ actually $g'=g+htg$. So we get

$$g'\colon A(1)\otimes L(1)\to A(1)\otimes\mathscr{D}_0\oplus A(2)\otimes\mathscr{D}_1,$$

and thus $[g'(-),g'(-)]_{\mathrm{Der}}$ also maps to

$$A(1)\otimes\mathscr{D}_0\oplus A(2)\otimes\mathscr{D}_1,$$

where the projection to $A(1)\otimes\mathscr{D}_0$ comes solely from $[g(-),g(-)]_{\mathrm{Der}}$. The summand $A(2)\otimes\mathscr{D}_1$ is in the kernel of f, so the restriction of l_2 to $A(1)\otimes L(1)$ is given by $f[g(-),g(-)]_{\mathrm{Der}}$ followed by the map to homology. Now g on $L(1)$ just maps α_{ij} to x_{ij}, and f on $\mathscr{D}(1)$ is the inverse mapping x_{ij} to α_{ij}.

We compute a few values of $f[g(-),g(-)]_{\mathrm{Der}}$ to illustrate the method:

$$\begin{aligned}
f[g(a),g(b)]_{\mathrm{Der}} &= (a_{12}-a_{13})\otimes(x_{13}\frac{\partial}{\partial a_{12}}x_{12}-x_{13}\frac{\partial}{\partial a_{13}}x_{12})\\
&\quad-(a_{12}-a_{13})\otimes(x_{12}\frac{\partial}{\partial a_{12}}x_{13}-x_{12}\frac{\partial}{\partial a_{13}}x_{13})\\
&=(a_{12}-a_{13})\otimes x_{13}+(a_{12}-a_{13})\otimes x_{12}\\
&=b+a,\\
f[g(a),g(c)]_{\mathrm{Der}} &= (a_{12}-a_{13})\otimes(x_{23}\frac{\partial}{\partial a_{12}}x_{12}-x_{23}\frac{\partial}{\partial a_{13}}x_{12})\\
&\quad-(a_{12}-a_{13})\otimes(x_{12}\frac{\partial}{\partial a_{12}}x_{23}-x_{12}\frac{\partial}{\partial a_{13}}x_{23})\\
&=(a_{12}-a_{13})\otimes x_{23}\\
&=c.
\end{aligned}$$

The images are cycles, as they are in general, and so we can explicitly compute l_2 on all of the generating cycles. The entire structure of the Lie algebra is encoded by the (anti-symmetric) multiplication matrix for l_2:

$$\begin{pmatrix}
0 & a+b & c & a-d & b+d & c & 0\\
 & 0 & -c & -e & e & 0 & 0\\
 & & 0 & -a-f & -b+f & -c & 0\\
 & & & 0 & e & f+d & 0\\
 & & & & 0 & e & 0\\
 & & & & & 0 & 0\\
 & & & & & & 0
\end{pmatrix}.$$

We see that $g=\kappa$ spans a central ideal. Another observation is that this Lie algebra is not nilpotent: e.g.

$$[a,[a,b]]=[a,a+b]=[a,b].$$

With slightly more effort we also find that $H(sA\otimes_\kappa L)_0$ is not semi-simple. If we quotient by $\langle g\rangle$ we find that the Lie ideal generated by $[a,b]=a+b$ is a proper ideal of $H(sA\otimes_\kappa L)_0/\langle g\rangle$. It has a linear basis given by

$$a+b,\ c,\ a-d,\ a+f,\ \text{and } e,$$

It is easy to see now that $H(sA\otimes_\kappa L)_0$ is not semi-simple.

The ideal $\langle a+b\rangle$ coincides with $[H(sA\otimes_\kappa L)_0, H(sA\otimes_\kappa L)_0]$, and the derived series stabilises at this ideal. Thus $H(sA\otimes_\kappa L)_0$ is not solvable either. Using the computer algebra program Magma, we find that $H(sA\otimes_\kappa L)_0$ has a Levi decomposition

$$H(sA\otimes_\kappa L)_0\simeq A_1\ltimes R$$

where R denotes the radical and the semisimple factor is isomorphic to the classical simple Lie algebra A_1. Thus R is of dimension 4, and 4-dimensional solvable Lie algebras over any field are completely classified [17]. Again using Magma, we find that R is isomorphic to the Lie algebra generated by elements x_1,x_2,x_3 and x_4 with non-zero brackets

$$[x_4,x_1]=x_1,[x_4,x_2]=x_3,[x_4,x_3]=x_3.$$

We have not had time to find a presentation in the basis given for $H(sA\otimes_\kappa L)_0$.

6. Suggestions for further research

In this section we finish the thesis with a discussion of what further research might be undertaken directly from the open ends we leave. We have dedicated a subsection to a few suggestions that are better developed than others. Before we explore those, we give a short list of more vague questions

- What is the meaning of the negative part of the complex $sA \otimes_\kappa L$?
- Are there other interesting examples of Koszul spaces to apply the theory to?
- How does the information obtained about $\operatorname{aut} F(\mathbb{R}^n, k)$ give information about automorphisms of E_n-operads?

6.1. **The role of non-linear Maurer-Cartan elements.** Recall Proposition 4.31 and Proposition 4.34: Two Maurer-Cartan elements $\tau, \tau' \in \mathrm{MC}(A \otimes L)$ are in the same component of $\mathrm{MC}_\bullet(A \otimes L)$ only if their linear parts agree, that is $\pi(\tau) = \pi(\tau')$. If all Maurer-Cartan elements are linear then $\pi_0(\operatorname{aut} X_{\mathbb{Q}})$ is isomorphic to $\operatorname{aut} A$ and $\operatorname{aut} L$.

Using this technique, we can say almost nothing if there a non-linear Maurer-Cartan elements of $A \otimes L$. In this section we will discuss the role of such, and connect the two rather different approaches used in the thesis. One being the complex $sA \otimes_\kappa L$ cooked up from contractions of a Lie model, and the other being the analysis of the components of the Kan complex $\mathrm{MC}_\bullet(A \otimes L)$.

Let $\tau \in MC(A \otimes L)$ be a Maurer-Cartan element. Denote by $\tilde{\tau}$ the part of τ in $\ker \pi$. We call this the non-linear part of τ. We begin with a lemma.

Lemma 6.1. *For $\tau \in MC(A \otimes L)$ both the linear and non-linear part of τ are again Maurer-Cartan elements, and $\tilde{\tau}$ is a $\pi(\tau)$-cycle (and vice versa).*

Proof. The Lie algebra $A \otimes L$ is bigraded by weight in respectively A and L. Write $x_{(i,j)}$ for the part of an element $x \in A \otimes L$ contained in $A(i) \otimes L(j)$. Then $\tau = \sum_{i,j} \tau_{(i,j)}$, (that is $\pi(\tau) = \tau_{(1,1)}$) and expanding we get

$$0 = [\tau, \tau] = \sum_{\substack{i,k \geq 0 \\ j,l \geq 1}} [\tau_{(i,j)}, \tau_{(k,l)}],$$

where the term $[\tau_{(i,j)}, \tau_{(k,l)}]$ is contained in $A(i+k) \otimes L(j+l)$. For degree reasons $\tau_{(0,1)} = 0$, so the only contribution to the part in $A(2) \otimes L(2)$ comes from the linear part $[\tau_{(1,1)}, \tau_{(1,1)}]$. In particular both the linear and non-linear parts of τ are Maurer-Cartan elements, and $\tilde{\tau}$ is a $\pi(\tau)$-cycle (and vice versa). □

It is natural to ask how the Maurer-Cartan elements with non-linear parts affect the picture. As noted after Proposition 4.34, we need only worry about those corresponding to homotopy equivalences. A first thing to observe is that such non-linear Maurer-Cartan elements can not in general lie in the same path components of $\mathrm{MC}_\bullet(A \otimes L)$ as linear elements. Otherwise Proposition 4.34 would hold without the assumption of linearity. But that is not very much information. Instead we can try to connect them to the other approach, namely the complex $sA \otimes_\kappa L$, where they occur as elements in degree zero. We will show that *a non-linear Maurer-Cartan element gives rise to a non-linear κ-cycle.*

Should it happen that a Maurer-Cartan element $\tau \in \mathrm{MC}(A \otimes L)$ is in the same path component as κ in $MC_\bullet(A \otimes L)$, then $\pi(\tau) = \kappa$ by Proposition 4.31, and $\tilde{\tau}$

is a non-linear κ-cycle, as established above. This is equivalent to τ corresponding to a homotopy equivalence which induces the identity on both cohomology and homotopy. As observed this is not always the case though.

Recall that there is a one-to-one correspondence between linear Maurer-Cartan elements and endomorphisms of A respectively L. We can show that

Lemma 6.2. *Composition of endomorphisms correspond to matrix products under the correspondence expressed by Proposition 4.32.*

Suppose now that τ is not necessarily in the path component of κ, but just corresponds to some homotopy equivalence $X_{\mathbb{Q}} \to X_{\mathbb{Q}}$. We can show that

Lemma 6.3. *The $I \times I$-matrix (cf. Proposition 4.32) associated to $\pi(\tau)$, and the corresponding endomorphisms φ and ψ of respectively A and L are all invertible.*

The inverse matrix represents a Maurer-Cartan element which we will denote $\pi(\tau)^{-1}$, and it represents the inverse automorphisms φ^{-1} and ψ^{-1}.

Extend the linear bases $\{a_i\}_{i\in I}$ and $\{\alpha_j\}_{j\in I}$ for respectively $A(1)$ and $L(1)$ to linear bases $\{a_i\}_{i\in I'}$ and $\{\alpha_j\}_{j\in I''}$ for A and L, and write

$$\tau = \sum_{i\in I', j\in I''} \lambda_{ij} a_i \otimes \alpha_j.$$

Now the linear part of

$$\varphi^{-1}.\tau := \sum_{i\in I', j\in I''} \lambda_{ij} \varphi^{-1}(a_i) \otimes \alpha_j$$

equals κ because the matrix representing φ^{-1} is inverse to λ_{ij} for $i, j \in I$. At the same time $\varphi^{-1}.\tau$ is again a Maurer-Cartan element because

$$\begin{aligned}
[\varphi^{-1}.\tau, \varphi^{-1}.\tau] &= \sum_{i,k\in I', j,l\in I''} (-1)^{|a_k||\alpha_j|} \lambda_{ij}\lambda_{kl} \varphi^{-1}(a_i)\varphi^{-1}(a_k) \otimes [\alpha_j, \alpha_l] \\
&= \sum_{j,l\in I''} \varphi^{-1}\left(\sum_{i,k\in I'} (-1)^{|a_k||\alpha_j|} \lambda_{ij}\lambda_{kl} a_i a_k \right) \otimes [\alpha_j, \alpha_l]
\end{aligned}$$

is zero if and only if

$$\sum_{j,l\in I''} \sum_{i,k\in I'} (-1)^{|a_k||\alpha_j|} \lambda_{ij}\lambda_{kl} a_i a_k \otimes [\alpha_j, \alpha_l] = [\tau, \tau]$$

is zero, and τ is a Maurer-Cartan element. Then by Lemma 6.1 the non-linear part of $\varphi^{-1}.\tau$ is a degree zero κ-cycle in $sA \otimes_\kappa L$.

In this way there is a map from non-linear Maurer-Cartan elements which correspond to homotopy equivalences, to non-linear κ-cycles of homological degree zero given by mapping τ to $\varphi^{-1}.\tau$. This completes the task of connecting the two approaches, but for all practical purposes we are interested in the positive part of $sA \otimes_\kappa L$, so we may hope to do better than just finding a κ-cycle of degree zero.

We may use the induced L_∞-structure to produce a positive cycle.

Lemma 6.4. *If there is a κ-cycle $\gamma \in (sA \otimes_\kappa L)_{>0}$ such that the induced bracket $\ell_2(\varphi^{-1}.\tau, \gamma)$ is non-zero, then this bracket is a non-linear element of positive degree representing a non-zero class in the kernel of (at least one of) the extensions in Theorem 4.8.*

Proof. The induced operation ℓ_2 respects the shifted weight grading, so $\ell_2(\varphi^{-1}.\tau, \gamma)$ is again non-linear. The induced operation also has homological degree zero, so $\ell_2(\varphi^{-1}.\tau, \gamma)$ has positive degree, and finally ℓ_2 maps cycles to cycles. $\square$

It would be interesting to find conditions for when such an element γ exists. If it can always be found, then we have shown that existence of non-linear Maurer-Cartan elements corresponding to homotopy equivalences, implies a non-trivial kernel for the extensions in Theorem 4.8. This again allows for the potential of higher operations on $\pi_{>0}(\operatorname{aut} X) \otimes \mathbb{Q}$, but does not guarantee it.

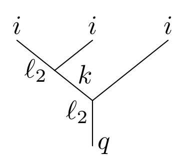
We expect that the phenomenon is connected to the action of $\pi_0 \operatorname{aut} X_{\mathbb{Q}} \simeq \pi_1 B \operatorname{aut} X_{\mathbb{Q}}$ on the higher homotopy groups. Indeed there is an induced action on homology given by $\operatorname{ad}_{[\varphi^{-1}.\tau]}$ the adjoint action of the class of $\varphi^{-1}.\tau$, which may or may not agree with the usual action of π_1.

6.2. **Coformality of $B \operatorname{aut} F(\mathbb{R}^n, 3)$.** Even though we have studied the induced structure ℓ_n on $sA \otimes_\kappa L$ for $\operatorname{aut} F(\mathbb{R}^n, 3)$ extensively, we have not succeeded in deciding if there are higher operations on the homology $H_*(sA \otimes_\kappa L)$ giving rise to Massey brackets, and thus preventing coformality of $B \operatorname{aut} F(\mathbb{R}^n, 3)$. We take the opportunity while suggesting further research, to present our progress and challenges to this task. The rest of the section proceeds as an analysis of how to potentially obtain a Massey bracket, or alternatively show that all such vanish.

From Proposition 5.8 we know that there are no higher operations on the positive part of $sA \otimes_\kappa L$. Thus a higher operation on $H(sA \otimes_\kappa L)_*$ comes from the transfer of the induced bracket ℓ_2 to homology along the contraction

$$\begin{tikzcd} sA \otimes_\kappa L \arrow[loop left, "k"] \arrow[r, shift left, "q"] & H(sA \otimes_\kappa L)_*, \arrow[l, shift left, "i"] \end{tikzcd}$$

cf. Proposition 3.12. It is natural to begin the search for a ternary operation, which is necessarily a sum of compositions along trees of the following form



From the interplay between the weight gradings and the maps involved we can limit the search significantly. First write

$$\begin{tikzcd} H^{p_1}_{q_1}(sA \otimes_\kappa L) \otimes H^{p_2}_{q_2}(sA \otimes_\kappa L) \otimes H^{p_3}_{q_3}(sA \otimes_\kappa L) \arrow[d, "l_3"] \\ H(sA \otimes_\kappa L). \end{tikzcd}$$

For degree reasons we must have $\sum_i p_i \geq 3$ for l_3 to be non-zero, and we must have $q_i > p_i$ to stay in the positive part of the complex. Now if $p_1 = p_2 = p_3 = 1$ then the image is in $H^0(sA \otimes_\kappa L)$ which we have identified with the centre of L. But $\sum q_i \geq \sum_i (p_i + 1) \geq 6$, and l_3 respects the shifted weight grading, so the image is in bidegree $(0, \sum_i q_i - 3)$. That is, in weight greater than 3 in L, while the centre of L is concentrated in weight 1. Thus at least one p_i is greater than 1.

If $p_1 = p_2 = p_3 = 2$ then $l_3 = 0$ for degree reasons, so it is enough to analyse the situation where

$$(p_1, p_2, p_3) \in \{(0,2,2), (1,1,2), (1,2,2)\}.$$

Let us note first that a basis for the cycles of $sA(1) \otimes_\kappa L$ is given by

$$\{(a_{12} - a_{23}) \otimes \alpha, (a_{13} - a_{23}) \otimes \beta \mid \alpha, \beta \text{ in the unmixed basis for } L\},$$

and a basis for the cycles of $sA(2) \otimes_\kappa L$ is given by

$$\{(a_{12}a_{23}) \otimes \alpha, (a_{13}a_{23}) \otimes \beta \mid \alpha, \beta \text{ in the unmixed basis for } L\},$$

where we recall that the unmixed basis for L is just iterated brackets of α_{12} and α_{13}. That is, we may view α, β as elements of the free Lie algebra $\mathbb{L}(\alpha_{12}, \alpha_{13})$. The cycles of $sA(0) \otimes_\kappa L$ where identified with the centre of L, so they are spanned by the single element $1 \otimes \alpha_{12} + \alpha_{13} + \alpha_{23}$.

We now analyse the composition

$$\begin{tikzcd} H^{p_1}_{q_1}(sA \otimes_\kappa L) \otimes H^{p_2}_{q_2}(sA \otimes_\kappa L) \arrow[d, "k\ell_2(i \otimes i)"] \\ sA \otimes_\kappa L \end{tikzcd} ,$$

corresponding to the first part of the trees above which any operation l_3 must factor through. If $p_1 = p_2 = 2$ this is zero for degree reasons. If $p_1 = p_2 = 1$ then the image is in $sA(0) \otimes_\kappa L$ also for degree reasons, and it must be of the form $1 \otimes g(\gamma)$ for some $\gamma \in L$ with weight greater than 1. Similarly, if $p_1 = 0$ and $p_2 = 2$ then we end up in $sA(0) \otimes_\kappa L$.

In both cases the next part of the composition making up the l_3 operation is given by

$$\begin{tikzcd} sA(0) \otimes_\kappa L \otimes sA(2) \otimes_\kappa L \arrow[d, "\ell_2"] \\ sA \otimes_\kappa L \end{tikzcd} ,$$

and in both cases the part in $sA(0) \otimes L$ is of the form $1 \otimes \alpha$ where α is a bracketed word in the letters α_{12} and α_{13}. Note that

$$g'(1 \otimes \alpha) = 1 \otimes g(\alpha) + a_{23} \otimes h[x_{23}, g(\alpha)]$$

because h vanish on the image of g. Denote the other input to ℓ_2 by $a \otimes \beta \in sA(2) \otimes L$, and expand

$$\ell_2(1 \otimes \alpha, a \otimes \beta) = f([1 \otimes g(\alpha) + a_{23} \otimes h[x_{23}, g(\alpha)], a \otimes g(\beta)]_{\mathrm{Der}}).$$

Since $a \in A(2)$ the only possible non-zero term from this is

$$a_{23} \otimes f\left(g(\beta)\frac{\partial}{\partial a} h[x_{23}, g(\alpha)]\right),$$

but this is not a cycle in $sA \otimes_\kappa L$, so such terms of l_3 all contribute with zero. Thus a non-zero contribution to l_3 must come from the case $p_1 = 1$ and $p_2 = 2$.

Recall from Section 2.5.4 that to decide formality we must produce a non-zero Massey bracket, or show that they all vanish. That is, we should look elements $x, y, z \in H(sA \otimes_\kappa L)_*$ such that l_2 on any pair is zero, but $l_3(x, y, z) \neq 0$, or show that this can not be the case. In combination with the above analysis we see that we should search for elements $x \in H^1(sA \otimes L)$ and $y \in H^2(sA \otimes L)$ such that

$$l_2(x, y) = q\ell_2(i(x), i(y)) = 0,$$

while

$$k\ell_2(i(x), i(y)) \neq 0.$$

Recall that q and k both factor through the projection to cycles, and q then quotient by boundaries, while k is defined by projection to boundaries and a choice of lift along the differential. Thus x and y should satisfy that the cycle part of $\ell_2(i(x), i(y))$ is a (non-zero) boundary. For degree reasons $\ell_2(i(x), i(y))$ is contained in $sA(2) \otimes_\kappa L$, and all boundaries there are of the form

$$a_{12}a_{23} \otimes [\alpha_{12}, \alpha] + a_{13}a_{23} \otimes [\alpha_{13}, \alpha],$$

for some $\alpha \in L$. For $i(x)$ and $i(y)$ in the chosen bases for the cycles, we get that $\ell_2(i(x), i(y))$ is a linear combination of

$$\begin{gathered}
a_{12}a_{23} \otimes g(\beta)\frac{\partial}{\partial a_{12}a_{23}}h[x_{23}, g(\alpha)] + a_{12}a_{23} \otimes g(\alpha)\frac{\partial}{\partial a_{12}}g(\beta) \\
a_{13}a_{23} \otimes g(\beta)\frac{\partial}{\partial a_{12}a_{23}}h[x_{23}, g(\alpha)] + a_{12}a_{23} \otimes g(\alpha)\frac{\partial}{\partial a_{13}}g(\beta) \\
a_{12}a_{23} \otimes g(\beta)\frac{\partial}{\partial a_{13}a_{23}}h[x_{23}, g(\alpha)] + a_{13}a_{23} \otimes g(\alpha)\frac{\partial}{\partial a_{12}}g(\beta) \\
a_{13}a_{23} \otimes g(\beta)\frac{\partial}{\partial a_{13}a_{23}}h[x_{23}, g(\alpha)] + a_{13}a_{23} \otimes g(\alpha)\frac{\partial}{\partial a_{13}}g(\beta).
\end{gathered}$$

We have not been able to decide if it is possible to obtain a non-zero boundary from this. The analysis for even higher operations is similar but longer, and we are stuck at the same point.

6.3. **On** $\operatorname{aut} F(\mathbb{R}^n, k)$ **for** $k \geq 4$**.** As advertised in Example 5.7 there are several observations to be made regarding the tables of Betti numbers. Recall that we identified the first differential

$$\kappa \colon sA(0) \otimes L \longrightarrow sA(1) \otimes L$$

with the map $\operatorname{ad} \colon L \to \operatorname{Der} L$. Thus the first column of each table of Betti numbers is dimensions of the centre of L in each weight. As noted, the centre is concentrated in $L(1)$, and is 1-dimensional spanned by the element $\sum_{ij} \alpha_{ij}$. Second column in each table is then the dimensions of the space of outer derivations $\operatorname{Out} L$, which raise weight by respectively $1, 2, 3, \ldots$. There is one such outer derivation raising weight by 3 (and 5 for $k = 4$), and we might guess that there is one raising degree with $2m + 1$ for all $m \geq 1$ in each case $k \geq 4$. We give a short discussion of how we might approach this.

Consider the Hochschild-Serre spectral sequence for computing the Lie algebra cohomology $H^1(L, L) \simeq \operatorname{Out} L$ using the split extensions

$$\mathbb{L} \longrightarrow L_k \longrightarrow L_{k-1},$$

and the fact that L_2 is a free Lie algebra on one generator. This gives an inductive tool to approach higher values of k. Notice also that there is a Σ_k action on $F(\mathbb{R}^n, k)$ permuting the k ordered points, and this induces actions on both A and L. We can show that the differential κ is equivariant for this action, and it seems likely that there is information to be gained by studying the vector spaces as Σ_k-representations.

From the very sparse data, we might also hope that $H^{\leq k-3}(sA \otimes_\kappa L)_*$ is relatively small in general. This suggest that it can be possible to check if higher operations on $H_*(sA \otimes_\kappa L)$ vanish or not, as we know that they respect the shifted weight grading and so, if they should not vanish for degree reasons, must mainly have input from small values of i_m, when writing

$$\ell_r \colon \bigotimes_{m=1}^{r} sA(i_m) \otimes L(j_m) \to sA(\sum i_m - 2r + 3) \otimes L(\sum j_m - 2r + 3).$$

Another pattern appears which we are in fact able to explain, and which may help calculations.

Proposition 6.5. *For any value of k, there are exact sequences of graded vector spaces*

$$0 \longrightarrow H_j^0(sA \otimes_\kappa L) \longrightarrow H_j^1(sA \otimes_\kappa L) \longrightarrow \cdots \longrightarrow H_j^{k-1}(sA \otimes_\kappa L) \longrightarrow 0$$

for $j \geq 1$, with maps induced by multiplication by any fixed algebra generator $a \in A$.

Recall the table of Betti numbers for $k = 4$:

summand				
1	1	0	45	80
2	0	0	81	230
3	0	1	230	501
4	0	0	502	1410
5	0	1	1410	3515
6	0	0	3516	9571

The exact sequences of Proposition 6.5 can be read on the diagonals from bottom left to top right. The table is only concerned with positive homological degree, so the first occurrences of such sequences are

$$0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}^{81} \longrightarrow \mathbb{Q}^{80} \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow \mathbb{Q}^{230} \longrightarrow \mathbb{Q}^{230} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}^{502} \longrightarrow \mathbb{Q}^{501} \longrightarrow 0$$

and so on.

Proof of Proposition 6.5. It is easy to check that the maps $m_a \colon A \to A$ given by multiplication by a fixed generator a gives rise to exact sequences

$$0 \longrightarrow A(0) \longrightarrow A(1) \longrightarrow \cdots \longrightarrow A(k-1) \longrightarrow 0,$$

and that the squares

$$\begin{tikzcd}
A(i) \otimes L(j) \arrow[r, "\kappa"] \arrow[d, "m_a"'] & A(i+1) \otimes L(j+1) \arrow[d, "m_a"] \\
A(i+1) \otimes L(j) \arrow[r, "\kappa"] & A(i+2) \otimes L(j+1)
\end{tikzcd}$$

commute in the graded sense for all i, j. Thus $sA \otimes_\kappa L$ is a double complex which is exact in one direction, and bounded above and below in either direction for any fixed degree. We get two spectral sequences, both with $E_0^{i,j} = sA(i-j) \otimes L(j)$ and both converging to the homology of the total complex. Since one direction is exact the total complex has zero homology, and so the sequence with

$$\begin{aligned} E_1^{*,*} &= H_*(sA \otimes L, \kappa), \quad \text{and} \\ E_2^{*,*} &= H_*(H_*(sA \otimes L, \kappa), (m_a)_*) \end{aligned}$$

converges to zero. In fact it collapses at the E_2-page, as we will show below in Lemma 6.6, and so the maps induced from multiplication by a generator $(m_a)_*$ again give exact sequences

$$0 \longrightarrow H_*^0(sA \otimes_\kappa L) \longrightarrow H_*^1(sA \otimes_\kappa L) \longrightarrow \cdots \longrightarrow H_*^{k-1}(sA \otimes_\kappa L) \longrightarrow 0$$

and the maps $(m_a)_*$ preserve the weight in L. □

These exact sequences give a way of computing dimensions of $H_j^i(sA \otimes_\kappa L)$ for greater values of i, using what we might know about the space for lesser values of i. In combination with the suggested approach of studying the spaces as Σ_k representations, and gaining knowledge about $H_j^1(sA \otimes_\kappa L)$ using the Hochschild-Serre spectral sequence, we might be able to push the calculations further, and perhaps even give formulae for the Betti numbers in general.

Lemma 6.6. *The spectral sequence for the double complex $sA \otimes_\kappa L$, with*

$$\begin{aligned} E_1^{*,*} &= H_*(sA \otimes L, \kappa), \quad \textit{and} \\ E_2^{*,*} &= H_*(H_*(sA \otimes L, \kappa), (m_a)_*), \end{aligned}$$

collapses at the E_2-page.

Proof. The higher differentials of the spectral sequence for a double complex are all constructed in similar fashion, and we only show that all differentials on the E_2-page are zero. The rest follows by similar arguments. Let $[x] \in E_2$ represent the class of an element $x \in E_1$, in particular $(m_a)_*(x) = 0$. The element x is itself the class of $\tilde{x} \in E_0$, so in particular $[\kappa, \tilde{x}] = 0$. The condition that $(m_a)_*(x) = 0$ is equivalent to $a \cdot \tilde{x} = [\kappa, y]$ for some $y \in E_0$. Now

$$[\kappa, a \cdot y] = \pm a \cdot [\kappa, y] = \pm a^2 \cdot \tilde{x} = 0,$$

so $a \cdot y$ is a κ-cycle. Then $z := [a \cdot y] \in E_1$ is a cycle too, as

$$(m_a)_*(z) = [a^2 \cdot y] = 0.$$

The differential on E_2 is then defined by $d_2([x]) := [z] \in E_2$. For any such $[x]$ we may write $z = a \cdot y + \operatorname{Im} \kappa$ (notice that z has weight greater than 1 in A), and we have

$$a \cdot [\kappa, v] = \pm[\kappa, a \cdot v] \in \operatorname{Im} \kappa,$$

because the bracket on $sA \otimes L$ is essentially just that of L induced by extension of scalars by A. Thus $[z] = 0$ as claimed. □

References

[1] A. Berglund, *Homological perturbation theory for algebras over operads*, Algebr. Geom. Topol. 14 (2014), no.5, 2511–2548.

[2] A. Berglund, *Koszul spaces*, Trans. Amer. Math. Soc. 366 (2014), no. 9, 4551–4569.

[3] A. Berglund, *Rational homotopy theory of mapping spaces via Lie theory for L-infinity algebras*, arXiv:1110.6145 [math.AT].

[4] A. Berglund, I. Madsen, *Homological stability of diffeomorphism groups*, Pure Appl. Math. Q. **9** (2013), no. 1, 1–48.

[5] A. Berglund, I. Madsen, *Rational homotopy theory of automorphisms of highly connected manifolds*, arXiv:1401.4096 [math.AT].

[6] J. Block, A. Lazarev, *André-Quillen cohomology and rational homotopy of function spaces*, Adv. Math. **193** (2005), no. 1, 18–39.

[7] A. Bousfield, V.K.A.M. Gugenheim, *On PL DeRham theory and rational homotopy type*, Mem. Amer. Math. Soc., vol. 8, No. 179, Amer. Math. Soc., Providence, R.I., 1976.

[8] R. Brown, *The twisted Eilenberg-Zilber theorem*, 1965 Simposio di Topologia (Messina, 1964) 33–37 Edizioni Oderisi, Gubbio.

[9] U. Buijs, Y. Félix, A. Murillo, L_∞ *rational homotopy of mapping spaces*, Rev. Mat. Complut. **26** (2013), no. 2, 573–588.

[10] F. Cohen, S. Gitler, *On loop spaces of configuration spaces*, Trans. Amer. Math. Soc. **354** (2002), no. 5, 1705–1748.

[11] F. Cohen, T. Lada, P. May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics, Vol. 533, Springer 1976.

[12] Y. Felix, S. Halperin, *Formal spaces with finite dimensional rational homotopy*, Trans. Amer. Math. Soc. **270** (1982) 575–588.

[13] Y. Félix, S. Halperin, J-C. Thomas, *Rational Homotopy Theory*, Graduate texts in Mathematics 205, Springer, 2001.

[14] Y. Félix, J. Oprea, D. Tanré, *Algebraic models in geometry*, Oxford Graduate Texts in Mathematics 17, Oxford University Press, 2008.

[15] E. Getzler, *Lie theory for nilpotent* L_∞*-algebras*, Ann. of Math. (2) **170** (2009), no. 1, 271–301.

[16] V. Ginzburg, M. Kapranov, *Koszul duality for Operads*, Duke Math. J. **76** (1994), no. 1, 203–272.

[17] W. de Graaf, *Classification of solvable Lie algebras*, Experiment. Math. **14** (2005), no. 1, 15–25.

[18] V.K.A.M. Gugenheim, *On the chain complex of a fibration*, Illinois J. Math. **16** (1972) 398–414.

[19] V. Hinich, *Descent of Deligne groupoids*, Internat. Math. Res. Notices (1997), no. 5, 223–239.

[20] G. P. Hochschild, *Basic theory of algebraic groups and Lie algebras*, Graduate Texts in Mathematics 75, Springer, 1981.

[21] J. Huebschmann, *The sh-Lie algebra perturbation lemma*, Forum Math. **23** (2011), no. 4, 669–691.

[22] N. Jacobson, *Lie algebras*, Interscience Tracts in Pure and Applied Mathematics, No. 10, Interscience Publishers (a division of John Wiley & Sons), 1962.

[23] T. Lada, J. Stasheff, *Introduction to SH Lie algebras for physicists*, Internat. J. Theoret. Phys. **32** (1993), no. 7, 1087–1103.

[24] L. Lambe, J. Stasheff, *Applications of perturbation theory to iterated fibrations*, Manuscripta Math. **58** (1987), no. 3, 363–376.

[25] P. Lambrechts, I. Volić, *Formality of the little* N*-disks operad*, Mem. Amer. Math. Soc. **230** (2014), no. 1079.

[26] J-L. Loday, *Cyclic homology*, Grundlehren der Mathematischen Wissenschaften 301, Springer, 1998.

[27] J-L. Loday, B. Vallette, *Algebraic operads*, Grundlehren der Mathematischen Wissenschaften 346, Springer 2012.

[28] J. P. May, *Classifying spaces and fibrations*, Mem. Amer. Math. Soc. **1** (1975), no. 155.

[29] J. Millès, *The Koszul complex is the cotangent complex*, International Mathematics Research Notices, rnr034, (2011) 44p.

[30] J. Neisendorfer, T. Miller, *Formal and coformal spaces*, Illinois J. Math. **22** no. 4 (1978), 565–580.

[31] S.B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970) 39–60.

[32] D. Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969) 205–295.

[33] M. Schlessinger, J. Stasheff, *Deformation theory and rational homotopy type*, arXiv:1211.1647 [math.QA].

[34] S. Smith, *The homotopy theory of function spaces: a survey*, Contemp. Math. **519** (2010) 3–39.

[35] S. Smith, *The rational homotopy Lie algebra of classifying spaces for formal two-stage spaces*, J. Pure Appl. Algebra **160** no. 2-3 (2001), 333–343.

[36] J. Stasheff, *A classification theorem for fibre spaces*, Topology **2** (1963), 239–246.

[37] D. Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. No. 47 (1977), 269–331 (1978).

[38] D. Tanré, *Homotopie rationnelle: modèles de Chen, Quillen, Sullivan*, Lecture Notes in Mathematics 1025, Springer 1983.

[39] W. C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics 66, Springer, 1979.

[40] C. W. Wilkerson *Applications of minimal simplicial groups*, Topology **15** (1976), no. 2, 111–130.