Inversion Formulas for the Radon Transform

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Preface

Introduction

Let $n \in \mathbb{N}$ and $k \in \{1, \dots, n-1\}$ be given. Let $\mathbb{G}(k,n)$ be the space of all k-planes in $\mathbb{R}^n$. To a function f on $\mathbb{R}^n$ we associate the function $\hat{f}$ on $\mathbb{G}(k,n)$ defined by

$$\hat{f}(\xi) = \int_\xi f(x)\,dx, \qquad \xi \in \mathbb{G}(k,n).$$

The map $f \mapsto \hat{f}$ is called the k-dimensional Radon transform in $\mathbb{R}^n$, or the k-plane transform for short. In connection with the Radon transform we consider the dual transform $\varphi \mapsto \check{\varphi}$ which takes functions on $\mathbb{G}(k,n)$ to functions on $\mathbb{R}^n$ by

$$\check{\varphi}(x) = \int_{\{\xi \ni x\}} \varphi(\xi)\,d\xi, \qquad x \in \mathbb{R}^n,$$

i.e. the dual transform of a function at a point x is the average of that function over all k-planes passing through x.

It is possible to generalize the Radon transform to e.g. real hyperbolic space, $\mathbb{H}^n$. When

$$I_{n,1} = \left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & -1 \end{array}\right),$$

I_n being the identity matrix, this space can be realized as the set

$$M = \{x \in \mathbb{R}^{n+1} | x^t I_{n,1} x = -1 \wedge x_{n+1} > 0\}$$

equipped with the Riemannian structure

$$g_m(X,Y) = X^t I_{n,1} Y, \qquad m \in M.$$

Since k-planes are exactly the k-dimensional totally geodesic submanifolds of $\mathbb{R}^n$, the natural substitute for k-planes when passing to hyperbolic space is the set of totally geodesic submanifolds thereof, Ξ_k. From the Riemannian measure on $\mathbb{H}^n$ a measure is induced on any submanifold, and applying this the k-dimensional totally geodesic Radon transform of a function f on $\mathbb{H}^n$ is defined by

$$\hat{f}(\xi) = \int_\xi f(x)\,dx, \qquad \xi \in \Xi_k.$$

Again a dual transform is considered along side with the Radon transform:

$$\check{\varphi}(m) = \int_{\{\xi \ni m\}} \varphi(\xi)\,d\xi, \qquad x \in \mathbb{H}^n.$$

The areas in which the Radon transform plays a role are both multiple and varied. It is encountered both in pure and applied mathematics -

in the latter especially in connection with inversion problems as seen in e.g. CAT-scanners: is it possible to recover a function from it's Radon transform? This is an interesting problem in pure mathematics as well, and a number of inversion formulas are know. In the Euclidean case on such is the following:

$$f = (4\pi)^{-\frac{k}{2}} \frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n}{2})} I^{-k}(\hat{f})^{\vee}. \tag{1}$$

Here I^{-k} is a Riesz potential. These potentials arises when generalizing the notion of powers of the Laplacian from only positive integer powers to general complex powers. From the hyperbolic case we draw attention to the following inversion formula:

$$f = c \begin{cases} P_k(\Delta)(\hat{f})^{\vee} & , \quad k \text{ even} \\ ((2-n)-\Delta)K_-^1 P_k(\Delta)(\hat{f})^{\vee} & , \quad k \text{ odd} \end{cases}, \tag{2}$$

where

$$P_k(\Delta) = \prod_{i=0}^{[\frac{k}{2}]-1} ((k-2i-n)(k-2i-1)-\Delta),$$

$[\frac{k}{2}]$ denoting the integer part of $\frac{k}{2}$, and $c = (4\pi)^{-\frac{k}{2}} \frac{\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n}{2})}$, and K_-^1 is a certain convolution operator.

The Thesis

This thesis consists of two papers written between summer 2002 and the end of 2003. The first is to appear in *Math. Scand.* and deals with the inversion formula (1) for the Radon transform in the Euclidean case. The second, which has not yet been submitted, deals with the inversion formula (2) for the Radon transform in the hyperbolic case. Both papers prove the relevant inversion formula to hold not just for smooth functions of compact support, as is usually seen, but for functions of a certain regularity and meeting certain decay conditions.

When proving the inversion formula in the Euclidean case, Riesz potentials have a prominent place, so the main part of the first paper deals with the definition and manipulation of Riesz potentials.

The hyperbolic inversion formula is a close imitation of the Euclidean in the following sense: When deducing the Euclidean inversion formula a fundamental step is to realize that

$$(\hat{f})^{\vee}(x) = \frac{\Omega_{k-1}}{\Omega_{n-1}} \int_{\mathbb{R}^n} f(y)|x-y|^{k-n}\, dy = I^k f(x), \qquad x \in \mathbb{R}^n.$$

From this equality the inversion formula emerges by inverting I^k. Here the Riesz potentials play together in such a nice way that I^{-k} does the trick. The equivalent equality in the hyperbolic case is

$$(\hat{f})^{\check{}}(x) = \frac{\Omega_{k-1}}{\Omega_{n-1}} \int_M f(y) \sinh^{k-n}(d(y,m))\, dy = K^\alpha f(m), \qquad m \in M,$$

where in place of the Euclidean Riesz potential one defines a convolution operator K^α. This operator is not a generalized Riesz potential in the sense that it does not, as the Riesz potentials, generalize the notion of powers of the Laplacian. The inversion of K^α is somewhat less satisfying than the inversion of Riesz potential, but it is non the less doable by means of another closely related convolution operator, $K_-{}^1$. The first half of the second paper is dedicated to defining hyperbolic spaces, investigating the totally geodesic submanifolds thereof, and introducing the relevant integrals. The second half deals with the definition and behavior of the convolution operators K^α and K_-^α.

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PART I

The Euclidean Case

Sufficient Conditions for the Inversion Formula for the k-plane Radon Transform in $\mathbb{R}^n$.

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ABSTRACT: The inversion theorem (1) for the k-plane Radon transform in $\mathbb{R}^n$ is often stated for Schwartz functions, cf. [5, p.110], and lately for smooth functions on $\mathbb{R}^n$ fulfilling that $f(x) = O(|x|^{-N})$ for some $N > n$, cf. [6, Thm. I.6.2]. In this paper it will be shown, that it suffices to require that f is locally Hölder continuous and $f(x) = O(|x|^{-N})$ for some $N > k$ (N not necessarily an integer) in order for (1) to hold, and that the same decay on f but f only continuous implies an inversion formula only slightly weaker than (1).

INTRODUCTION

An important area in the theory of the k-plane Radon transform on $\mathbb{R}^n$ is the inversion theorems, which gives explicit formulas by which one can recover a function from its k-plane transform. Here we shall consider the formula

$$f = (4\pi)^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} I^{-k}(\hat{f})\check{}, \tag{1}$$

where " $\hat{}$ " denotes the k-plane transform and " $\check{}$ " the dual transform, while I^{-k} is a Riesz potential, cf. Section 4. It will be shown in this paper, that the formula holds for all functions in the space $C(k,n)$ (see Definition 1.3.), and that the formula with I^k replaced by $\lim_{\alpha \to -k_+} I^\alpha$ holds if $f \in C_a(\mathbb{R}^n)$ for some $a > k$.

Notice, that the decay requirement of $C(k,n)$ ($f(x) = O(|x|^{-N})$ for some $N > k$) on its member functions is, in some sense, the weakest possible in order for an inversion formula to hold: A sufficient condition for the integral in the k-plane transform of a continuous function f to be convergent is, that for every k-plane there exists an $\epsilon > 0$ such that $f(x) = O(|x|^{-k-\epsilon})$ *on this k-plane*. However this non-uniform decrease of f

is not enough to make the inversion formula valid. In [14], Zalcman shows the existence of a smooth function $f \neq 0$ on $\mathbb{R}^2$ satisfying $f(x) = O(|x|^{-2})$ *on every line*, which nonetheless has $\hat{f} = 0$. For further examples see e.g. [1] and [2].

The proof in this paper of the inversion formula is rooted in the basic definition of the Riesz potential, I^α ($\alpha \in \mathbb{C}$), which is

$$(I^\alpha f)(x) = \frac{1}{H_n(\alpha)} \int_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n}\,dy.$$

Here H_n is a certain meromorphic function. If f is continuous and $O(|x|^{-a})$ for some $a > 0$, the integral converges if $0 < \operatorname{Re}\alpha < a$. For values of α with $\operatorname{Re}\alpha \leq 0$, the Riesz potential can, depending on the regularity of f, be defined by analytic continuation (see e.g. [9, sec. 10.2, 10.7] for various ways of performing this extension). The key to the proof of the inversion formula is the identity $I^{-k}(I^k f) = f$, which will be established exactly for f in $C(k,n)$.

Inversion formulas for the Radon transform of L^p-functions also exists, but then the interpretation of the Riesz potentials is quite different. Examples can be found e.g. in [11] where Rubin verifies two inversion formulas for the case $k = n-1$. One of them is of the same nature as (1), and the other is of the type, where a suitably interpreted Riesz potential in applied *before* the dual transform instead of after. The last mentioned variant of inversion formula is in [10] proved for L^p-functions in the case of a general k under the assumption that $1 \leq p < \frac{n}{k}$. It is interesting to note, that given $f \in C(\mathbb{R}^n)$ such that it is $O(|x|^{-N})$, then $f \in L^p(\mathbb{R}^n)$ when $-Np < -n$, i.e. $p > \frac{n}{N}$. Thus Rubin's inversion formula can be used on this f when there exists a $p \geq 1$ with $\frac{n}{N} < p < \frac{n}{k}$, e.i. when $k < N$ which is precisely the decay condition in the inversion theorem of this paper.

The paper follows the lines of Helgason's exposition [6, Chap.V §5]: After the preliminaries, we study in Section 2 the analytic continuation of the map $\alpha \mapsto x_+^\alpha(f) = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(x)x^\alpha\,dx$. In Section 3 we use this to study the maps $\alpha \mapsto r^\alpha(f) = \frac{1}{\Gamma(\alpha+1)}\int_{\mathbb{R}^n} f(x)|x|^\alpha\,dx$, and in Section 4 we introduce Riesz potentials and establish the identity $I^{-k}(I^k f) = f$. Finally, in Section 5, we prove the two versions of the inversion formula.

The inversion formula in (1), expressed as it is in terms of Riesz potentials, holds for k both odd and even. If k is even it is well-known, that a similar inversion formula can be established using the Laplacian instead of Riesz potentials (see e.g. [6, p. 29]. Section 6 contains a brief discussion of the possible impact of the main result of the paper on the domain of this formula.

1. Preliminaries

For each $a > 0$ and $n \in \mathbb{N}$ we make the following definitions:

1.1. **Definition.** Define the function space $C_a(\mathbb{R}^n)$ by

$$C_a(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) \mid f(x) = O(|x|^{-a})\}.$$

1.2. **Definition.** For each $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $0 < \epsilon < 1$ and $x \in \mathbb{R}^n$ define the space $C^{l+\langle\epsilon\rangle,x}(\mathbb{R}^n)$ as the set of functions f on $\mathbb{R}^n$ such that f is C^l in some neighborhood $\mathcal{O}$ of x with each l'th order derivative of f Hölder continuous of index ϵ in that neighborhood, i.e.

$$\text{(2)}\quad \exists M > 0 \forall x_1, x_2 \in \mathcal{O} \forall \mathbf{l} \in \mathbb{N}_0^n, |\mathbf{l}| = l : \\ |(\partial^{\mathbf{l}} f)(x_1) - (\partial^{\mathbf{l}} f)(x_2)| \leq M|x_1 - x_2|^\epsilon.$$

Put

- $C^{l+\langle\epsilon\rangle}(\mathbb{R}^n) = \bigcap_{x\in\mathbb{R}^n} C^{l+\langle\epsilon\rangle,x}(\mathbb{R}^n) \subset C^l(\mathbb{R}^n)$,
- $C^{l+}(\mathbb{R}^n) = \bigcap_{x\in\mathbb{R}^n} \bigcup_{\epsilon>0} C^{l+\langle\epsilon\rangle,x} \subset C^l(\mathbb{R}^n)$

and

- $C_a^{l+\langle\epsilon\rangle,x}(\mathbb{R}^n) = C^{l+\langle\epsilon\rangle,x}(\mathbb{R}^n) \cap C_a(\mathbb{R}^n)$,
- $C_a^{l+\langle\epsilon\rangle}(\mathbb{R}^n) = C^{l+\langle\epsilon\rangle}(\mathbb{R}^n) \cap C_a(\mathbb{R}^n)$
- $C_a^{l+}(\mathbb{R}^n) = C^{l+}(\mathbb{R}^n) \cap C_a(\mathbb{R}^n)$
- $C_a^{l}(\mathbb{R}^n) = C^{l}(\mathbb{R}^n) \cap C_a(\mathbb{R}^n)$

1.3. **Definition.** Finally define for each $k \in \{1, \dots, n-1\}$ the space $C(k,n)$ as the set of functions f, such that $f \in C_{k+\delta}^{0+}(\mathbb{R}^n)$ for some $\delta > 0$. I.e. $f \in C(k,n)$ exactly when f is $O(|x|^{-k-\delta})$ for some $\delta > 0$, and there for each $x \in \mathbb{R}^n$ exists a neighborhood $\mathcal{O}$ and an ϵ, $0 < \epsilon < 1$, such that $|f(x_1) - f(x_2)|/|x_1 - x_2|^\epsilon$ is bounded for $x_1, x_2 \in \mathcal{O}$.

From now on, when the symbols a, n, l and ϵ are used, the assumption will be $a > 0$, $n \in \mathbb{N}$, $l \in \mathbb{N}_0$ and $0 < \epsilon < 1$, unless otherwise mentioned.

2. The Map $\alpha \mapsto x_+^\alpha(f)$

2.1. **Definition.** For each $\alpha \in \mathbb{C}$ with $-1 < \operatorname{Re}\alpha < a - 1$ define the map $x_+^\alpha : C_a(\mathbb{R}) \to \mathbb{C}$ by

$$x_+^\alpha(f) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x) x^\alpha dx. \tag{3}$$

2.2. *Remark.* The map x_+^α is well-defined since $-1 < \operatorname{Re}\alpha$ and $f \in C(\mathbb{R})$ makes the integrand integrable at 0, while $\operatorname{Re}\alpha < a - 1$ and $f(x) =$

$O(|x|^{-a})$ makes it integrable at ∞. Note, that the Γ-function is a non-vanishing meromorphic function with poles in $-\mathbb{N}_0$ and

$$\lim_{\alpha\to k}(\alpha-k)\Gamma(\alpha)=\frac{(-1)^{-k}}{(-k)!},\qquad k\in-\mathbb{N}_0. \tag{4}$$

2.3. **Proposition.** *Let $f\in C_a^{l+\langle\epsilon\rangle,0}(\mathbb{R})$. Then the map $\alpha\mapsto x_+^\alpha(f)$, defined on*

$$\{\alpha\in\mathbb{C}\mid -1<\operatorname{Re}\alpha<a-1\},$$

can be (uniquely) extended to a holomorphic map on

$$\{\alpha\in\mathbb{C}\mid -l-\epsilon-1<\operatorname{Re}\alpha<a-1\}.$$

This map will likewise be denoted $\alpha\mapsto x_+^\alpha(f)$. We have

$$x_+^\alpha(f)=(-1)^{(-\alpha-1)}f^{(-\alpha-1)}(0),\qquad \text{when } \alpha\in\{-l-1,\dots,-1\}. \tag{5}$$

Proof. The integral in (3) is not necessarily convergent in 0, when $\alpha\le -1$. But if we put

$$A(x)=f(x)-\sum_{k=0}^{l}\frac{f^{(k)}(0)}{k!}x^k\qquad\text{and}\qquad B(\alpha)=\sum_{k=0}^{l}\frac{f^{(k)}(0)\rho^{\alpha+k+1}}{k!(\alpha+k+1)}.$$

then, by calculating the integrals, one realizes that

$$x_+^\alpha(f)=\frac{1}{\Gamma(\alpha+1)}\Big(\int_0^\rho x^\alpha A(x)\,dx+\int_\rho^\infty x^\alpha f(x)\,dx+B(\alpha)\Big), \tag{6}$$

is an extension, cf. [4, p.57]. Here $0<\rho<1$ fulfills $\overline{B(0,\rho)}\subset\mathcal{O}$, where $\mathcal{O}$ is a neighborhood of 0 in which $f^{(l)}$ is Hölder continuous. This extension is well-defined on

$$S=\{\alpha\in\mathbb{C}\setminus-\mathbb{N}\mid -l-\epsilon-1<\operatorname{Re}\alpha<a-1\}.$$

To show this, only the first term needs thought. Since $f\in C^l(\mathcal{O})$, there exists, according to Taylors theorem, for any $x\in B(0,\rho)$ a y between 0 and x, such that

$$f(x)=\sum_{k=0}^{l}\frac{f^{(k)}(0)}{k!}x^k+\frac{f^{(l)}(y)-f^{(l)}(0)}{l!}x^l. \tag{7}$$

Because $f^{(l)}$ is Hölder continuous of index ϵ in $\mathcal{O}$ we therefore have

$$\int_0^\rho |x^\alpha A(x)|\,dx\le const\int_0^\rho x^{\operatorname{Re}\alpha+l+\epsilon}\,dx<\infty \tag{8}$$

since $\operatorname{Re}\alpha+l+\epsilon>-l-\epsilon-1+l+\epsilon=-1$.

Let $\alpha_0\in S$ be given. To show that $\alpha\mapsto x_+^\alpha(f)$ is holomorphic in α_0, choose $\delta>0$ such that

$$B(\alpha_0,\delta)\subset\{\alpha\in\mathbb{C}\setminus-\mathbb{N}\mid -l-\epsilon-1+\delta<\operatorname{Re}\alpha<a-1-\delta\}.$$

Clearly $\alpha \mapsto B(\alpha)$ is holomorphic in α_0. Thus we only need to show, that the two integrals in (6) are holomorphic in α_0. This will follow from the theorems of Cauchy and Morera, if it can be shown, that for any closed curve γ in $B(\alpha_0, \delta)$ the two integrals in each of the following expressions can be interchanged:

$$\int_\gamma \int_0^\rho x^\alpha A(x) \qquad \text{and} \qquad \int_\gamma \int_\rho^\infty x^\alpha f(x).$$

But for $x \in]0, \rho[$

$$\sup_{\alpha \in B(\alpha_0,\delta)} |x^\alpha A(x)| \leq |A(x)| x^{-l-\epsilon-1+\delta},$$

and this function is, as in (8), integrable over $]0, \rho[$. For $x \in]\rho, \infty[$ we have the existence of a constant c independent of x, such that

$$\sup_{\alpha \in B(\alpha_0,\delta)} |x^\alpha f(x)| \leq c\, x^{a-1-\delta-a} = c\, x^{-1-\delta}.$$

Now, let $m \in \{-l-1, \dots, -1\}$ be given. Choose $\delta' > 0$ such that

$$B(m, \delta') \setminus \{m\} \subset \{\alpha \in \mathbb{C} \setminus -\mathbb{N} \mid -l-\epsilon-1+\delta' < \operatorname{Re}\alpha < a-1-\delta'\}.$$

As before we have for $\alpha \in B(m, \delta')$, that

$$(9) \quad |\int_0^\rho x^\alpha A(x)\, dx| \leq C < \infty \qquad \text{and} \qquad |\int_\rho^\infty x^\alpha f(x)\, dx| \leq K < \infty,$$

where the constants C and K are independent of α. Thus for $\alpha \to m$ we have

$$(\alpha - m) \int_0^\rho x^\alpha A(x)\, dx \to 0 \qquad \text{and} \qquad (\alpha - m) \int_\rho^\infty x^\alpha f(x)\, dx \to 0.$$

Now (5) follows from (6) and (4). □

2.4. *Remark.* With the Hölder continuity condition on the derivatives of f replaced by ordinary continuity, the inequality in (8) changes to

$$\int_0^\rho |x^\alpha A(x)|\, dx \leq const \int_0^\rho x^{\operatorname{Re}\alpha + l}\, dx.$$

Thus when $f \in C_a^l(\mathbb{R})$, the extension of $\alpha \to x_+^\alpha(f)$ still exists but only on

$$\{\alpha \in C | -l-1 < \operatorname{Re}\alpha < a-1\}.$$

3. The Map $\alpha \mapsto r^\alpha(f)$

3.1. **Definition.** For each $\alpha \in \mathbb{C}$ with $-n < \operatorname{Re}\alpha < a - n$ define the map $r^\alpha : C_a(\mathbb{R}^n) \to \mathbb{C}$ by

$$r^\alpha(f) = \frac{1}{\Gamma(\alpha+n)} \int_{\mathbb{R}^n} |x|^\alpha f(x)\,dx. \tag{10}$$

3.2. *Remark.* As in Remark 2.2 it is seen, that r^α is well-defined.

We will express r^α by x_+^α. To this end we introduce the mean value function:

3.3. **Definition.** For any $f \in C(\mathbb{R}^n)$ let $M_f : \mathbb{R} \to \mathbb{C}$ denote the mean value function of f around 0 defined by

$$M_f(t) = \frac{1}{\Omega_{n-1}} \int_{S^{n-1}} f(t\omega)\,d\omega. \tag{11}$$

3.4. *Remark.* Notice, that $t \mapsto M_f(t)$ is even, and that $M_f(0) = f(0)$.

3.5. **Lemma.** *When f is in $C_a^{l+\langle\epsilon\rangle,0}(\mathbb{R}^n)$ then M_f is in $C_a^{l+\langle\epsilon\rangle,0}(\mathbb{R})$.*

Proof. Standard arguments. □

3.6. *Remark.* Transition to polar coordinates in the defining expression (10) for r^α now gives $r^\alpha(f)$ in terms of x_+^α:

$$r^\alpha(f) = \Omega_{n-1} x_+^{\alpha+n-1}(M_f), \tag{12}$$

when $-1 < \operatorname{Re}\alpha + n - 1 < a - 1$, i.e. $-n < \operatorname{Re}\alpha < a - n$.

3.7. **Proposition.** *Let $f \in C_a^{l+\langle\epsilon\rangle,0}(\mathbb{R}^n)$. Then the map $\alpha \mapsto r^\alpha(f)$, defined on*

$$\{\alpha \in \mathbb{C} \mid -n < \operatorname{Re}\alpha < a - n\},$$

can be (uniquely) extended to a holomorphic map on

$$A = \{\alpha \in \mathbb{C} \mid -l - \epsilon - n < \operatorname{Re}\alpha < a - n\}.$$

This map will likewise be denoted $\alpha \to r^\alpha(f)$, and it satisfies (12). In specific

$$r^\alpha(f) = \Omega_{n-1}(-1)^{-\alpha-n} M_f^{(-\alpha-n)}(0), \qquad \textit{when } \alpha \in \{-l-n, \dots, -n\}. \tag{13}$$

Proof. Use (12) as definition and apply Proposition 2.3 using Lemma 3.5. □

4. Riesz Potentials

4.1. **Definition.** The meromorphic function H_n on $\mathbb{C}$ is defined by

$$H_n(\alpha) = 2^\alpha \pi^{\frac{n}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

4.2. *Remark.* Note that H_n has simple poles at each $\alpha \in -2\mathbb{N}_0$ and a zero in each $\alpha \in n + 2\mathbb{N}_0$.

4.3. **Definition.** We put $\mathbb{C}_n = \mathbb{C} \setminus (n + 2\mathbb{N}_0)$.

4.4. **Definition.** For each $x \in \mathbb{R}^n$, $f \in C_a(\mathbb{R}^n)$, and $\alpha \in \mathbb{C}_n$ with $0 < \operatorname{Re}\alpha < a$ the *αth Riesz potential, I^α, of f at x* is defined as

$$\begin{aligned}(I^\alpha f)(x) &= \frac{1}{H_n(\alpha)} \int_{\mathbb{R}^n} f(y)|x-y|^{\alpha-n}\, dy \\ &= \frac{1}{H_n(\alpha)} \int_{\mathbb{R}^n} f(x-y)|y|^{\alpha-n}\, dy. \end{aligned} \tag{14}$$

4.5. *Remark.* As in Remark 2.2 it is seen, that $I^\alpha f(x)$ is well-defined. Comparing with the defining expression (10) for r^α we see, that

$$(I^\alpha f)(x) = \frac{\Gamma(\alpha)}{H_n(\alpha)} r^{\alpha-n}(\tau_x f) \tag{15}$$

where $\tau_x f(y) = f(x-y)$.

4.6. **Proposition.** *Let $x \in \mathbb{R}^n$ be given. Assume that $f \in C_a^{l+\langle\epsilon\rangle,x}(\mathbb{R}^n)$. Then the map $\alpha \mapsto (I^\alpha f)(x)$, defined on the set*

$$\{\alpha \in \mathbb{C}_n \mid 0 < \operatorname{Re}\alpha < a\},$$

can be (uniquely) extended to a meromorphic map on

$$B = \{\alpha \in \mathbb{C} \mid -l - \epsilon < \operatorname{Re}\alpha < a\}.$$

This map will likewise be denoted $\alpha \to (I^\alpha f)(x)$. It satisfies (15) for $\alpha \in B \setminus ((-\mathbb{N}_0) \cup (n + 2\mathbb{N}_0))$. The poles, which are all simple, are in

$$(n + 2\mathbb{N}_0) \cup B.$$

Proof. Use (15) as definition and apply Proposition 3.7 to obtain a (unique) meromorphic extension to $\{\alpha \in \mathbb{C} \mid -l - \epsilon < \operatorname{Re}\alpha < a\}$. The possible poles are those of $\frac{\Gamma(\alpha)}{H_n(\alpha)} = \frac{1}{2}\pi^{-\frac{n+1}{2}}\Gamma(\frac{n-\alpha}{2})\Gamma(\frac{\alpha+1}{2})$. They are $\alpha \in 2\mathbb{N}_0 + n$ and $\alpha \in -2\mathbb{N}_0 - 1$, all simple. When $\alpha \in (-2\mathbb{N}_0 - 1) \cap B$ it follows from (13), that

$$r^{\alpha-n}(f) = \Omega_{n-1}(-1)^{-\alpha} M_f^{(-\alpha)}(0) = 0,$$

since M_f in an even function. Thus α is a removable singularity. $\square$

4.7. **Lemma.** *Let* $f \in C_a^{0+}$. *Then* $x \mapsto (I^0 f)(x)$ *is defined on all of* $\mathbb{R}^n$ *and*

$$I^0 f = f.$$

Proof. It follows from Proposition 4.6, that $x \mapsto (I^0 f)(x)$ is defined on all of $\mathbb{R}^n$. Since

$$\lim_{\alpha \to 0} \alpha H_n(\alpha) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \Omega_{n-1},$$

it follows from Proposition 4.6, (13), (4) and Remark 3.4, that

$$(I^0 f)(x) = \lim_{\alpha \to 0} \frac{\alpha \Gamma(\alpha)}{\alpha H_n(\alpha)} r^{\alpha - n}(\tau_x f) = M_{\tau_x f}(0) = f(x). \tag{16}$$

□

4.8. **Lemma.** *Let* $f \in C_a(\mathbb{R}^n)$. *Let* $\alpha \in \mathbb{C}$ *with* $0 < \operatorname{Re} \alpha < \min(a, n)$ *be given. Then*

$$I^\alpha f \in C_{b - \operatorname{Re} \alpha}(\mathbb{R}^n),$$

for any b *with* $\operatorname{Re} \alpha < b \leq \min(a, n)$ *if* $a \neq n$, *and for any* b *with* $\operatorname{Re} \alpha < b < n$ *if* $a = n$.

Proof. See [6, Prop. V.5.8.] with natural modifications to the proof in case $a = n$. □

4.9. **Proposition.** *Let* $f \in C_a(\mathbb{R}^n)$. *For any pair* $\alpha, \beta \in \mathbb{C}$ *satisfying*

$$\operatorname{Re} \alpha > 0 \quad \text{and} \quad \operatorname{Re} \beta > 0 \quad \text{and} \quad \operatorname{Re} (\alpha + \beta) < \min(a, n)$$

we have

$$I^\alpha I^\beta f = I^{\alpha + \beta} f. \tag{17}$$

4.10. *Remark.* Refer to e.g. [7, p. 43ff] or [8, Satz 9] in order to see how, when dealing with Riesz potentials as distributions, (17) can be expressed as a convolution of distributions. The distribution approach can prove Proposition 4.9 for a smaller class of functions.

Proof. That $0 < \operatorname{Re} \beta < \min(a, n)$ implies two things. First we get from Remark 4.5, that $I^\beta f$ is well-defined and given by

$$(I^\beta f)(z) = \frac{1}{H_n(\beta)} \int_{\mathbb{R}^n} f(y) |z - y|^{\beta - n} \, dy.$$

Secondly, we get the usage of Lemma 4.8 from which follows, that

$$I^\beta f \in C_{b - \operatorname{Re} \beta}(\mathbb{R}^n),$$

where b is chosen such that $\mathrm{Re}\,(\alpha+\beta) < b < \min(a,n)$. Thus, because $0 < \mathrm{Re}\,\alpha < b - \mathrm{Re}\,\beta$, $I^\alpha(I^\beta f)$ is well-defined and given by

$$\begin{aligned} & I^\alpha(I^\beta f)(x) \\ = \; & \frac{1}{H_n(\alpha)} \int_{\mathbb{R}^n} (I^\beta f)(z)|x-z|^{\alpha-n}\,dz \\ = \; & \frac{1}{H_n(\alpha)} \frac{1}{H_n(\beta)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(y)|z-y|^{\beta-n}\,dy|x-z|^{\alpha-n}\,dz. \end{aligned} \tag{18}$$

To show, that the order of integration can be interchanged, consider the expression

$$\int_{\mathbb{R}^n} |f(y)| \int_{\mathbb{R}^n} |z-y|^{\mathrm{Re}\,\beta-n}|x-z|^{\mathrm{Re}\,\alpha-n}\,dz\,dy. \tag{19}$$

By substituting $v = \frac{x-z}{|x-y|}$ in the inner integral and using the rotation invariance of the Lebesgue measure, this expression is rewritten as

$$\int_{\mathbb{R}^n} |f(y)||x-y|^{\mathrm{Re}\,\alpha+\mathrm{Re}\,\beta-n}\,dy \int_{\mathbb{R}^n} |e-v|^{\mathrm{Re}\,\beta-n}|v|^{\mathrm{Re}\,\alpha-n}\,dv,$$

where e is an arbitrary fixed unit vector. Now $0 < \mathrm{Re}\,(\alpha+\beta) < a$ makes the y-integral convergent. That the v-integral is convergent can be seen easily. Finally, it can be shown, e.g. using Fourier transform as in [13, p. 117-118], that

$$\int_{\mathbb{R}^n} |e-v|^{\beta-n}|v|^{\alpha-n}\,dv = \frac{H_n(\alpha)H_n(\beta)}{H_n(\alpha+\beta)}.$$

□

4.11. *Remark.* Let $x_0 \in \mathbb{R}^n$ be given. In what follows, we will often decompose a given function f on $\mathbb{R}^n$ as $f = f_1 + f_2$, where $f_1 = (1-\chi)f$ and $f_2 = \chi f$ for some compactly supported C^∞-function χ with $\chi(x) = 1$ in some neighborhood of x_0. Note that f_1 and f_2 have the same regularity as f, but f_1 is 0 in the neighborhood of x_0 and f_2 has compact support.

4.12. **Lemma.** *Let $f \in C^l_a(\mathbb{R}^n)$. Let $\alpha \in \mathbb{C}_n$ with $0 < \mathrm{Re}\,\alpha < a$ and $x_0 \in \mathbb{R}^n$ be given. Write $f = f_1 + f_2$ as in Remark 4.11. Then $I^\alpha f_1$ is smooth at x_0, and $I^\alpha f_2 \in C^l(\mathbb{R}^n)$ with $\partial^{\boldsymbol{p}}(I^\alpha f_2) = I^\alpha(\partial^{\boldsymbol{p}} f_2)$ for any $\boldsymbol{p} \in \mathbb{N}_0^n$ with $|\boldsymbol{p}| \le l$.*

Proof. Assume $\mathbf{p} \in \mathbb{N}_0^n$ to be given. Choose $\delta > 0$ such that $f_1 = 1$ in $B(x_0, 2\delta) \subset \mathcal{U}$. Then for any $x \in B(x_0,\delta)$

$$\begin{aligned} |f_1(y)\partial_x^{\mathbf{p}}|x-y|^{\alpha-n}| &\le c|f_1(y)||x-y|^{\mathrm{Re}\,\alpha-n-|\mathbf{p}|} \\ &\le c' 1_{\mathbb{R}^n\setminus B(x_0,2\delta)}(y)(|y|+1)^{-a}|y-x_0|^{\mathrm{Re}\,\alpha-n-|\mathbf{p}|}, \end{aligned}$$

since $\frac{1}{2}|x_0 - y| \le |x-y| \le 2|x_0 - y|$ for $y \notin B(x_0, 2\delta)$. Here c' does not depend on x. Since $-a + \operatorname{Re}\alpha - n - |\mathbf{p}| < -a + a - n = -n$, this gives us an integrable majorant of $\partial_x^{\mathbf{p}}(f_1(y)|x-y|^{\alpha-n})$ and it is independent of x.

To deal with $I^\alpha f_2$ assume $|\mathbf{p}| \le l$ and let $\mathcal{N}$ be any bounded subset of $\mathbb{R}^n$. Let $x \in \mathcal{N}$. Then

$$\begin{aligned} & |\partial_x^{\mathbf{p}}(f_2(x-y)|y|^{\alpha-n})| \\ = \;& |(\partial^{\mathbf{p}} f_2)(x-y)||y|^{\operatorname{Re}\alpha - n}| \\ \le \;& \sup|\partial^{\mathbf{p}} f_2|\, 1_{\mathcal{N}+(-\mathcal{K})}(y)|y|^{\operatorname{Re}\alpha-n}, \quad \forall y \in \mathbb{R}^n. \end{aligned} \tag{20}$$

Since $\operatorname{Re}\alpha > 0$ this is an integrable majorant of $\partial_x^{\mathbf{p}}(f_2(x-y)|y|^{\alpha-n})$ and it is independent of x. Thus $\partial^{\mathbf{p}}(I^\alpha f_2)$ exists in $\mathcal{N}$, $\mathcal{N}$ arbitrary, and thus in all of $\mathbb{R}^n$, and we see from (20) that $\partial^{\mathbf{p}}(I^\alpha f_2) = I^\alpha(\partial^{\mathbf{p}} f_2)$. □

4.13. **Lemma.** *Let $f \in C_a^l(\mathbb{R}^n)$. Let $\alpha \in \mathbb{C}_n$ with $0 < \operatorname{Re}\alpha < a$ be given. Then*

$$I^\alpha f \in C^l(\mathbb{R}^n) \tag{21}$$

and for any $x \in \mathbb{R}^n$ and $0 < \epsilon < 1$

$$f \in C^{l+\langle\epsilon\rangle,x}(\mathbb{R}^n) \Rightarrow I^\alpha f \in C^{l+\langle\epsilon\rangle,x}(\mathbb{R}^n). \tag{22}$$

Proof. Let $x_0 \in \mathbb{R}^n$ be given. Write $f = f_1 + f_2$ as in Remark 4.11. From the preceeding lemma $I^\alpha f_1$ is smooth at x_0 and $I^\alpha f_2 \in C^l(\mathbb{R}^n)$. Thus (21) holds.

Assume now, that $f \in C^{l+\langle\epsilon\rangle,x_0}(\mathbb{R}^n)$. Let $\mathbf{l} \in \mathbb{N}_0^n$ with $|\mathbf{l}| = l$ be given. To show the Hölder continuity of $\partial^{\mathbf{l}}(I^\alpha f_2)$ $(= I^\alpha(\partial^{\mathbf{l}} f_2)$ according to Lemma 4.12), let $\mathcal{K}$ be a compact neighborhood of x_0 in which $\partial^{\mathbf{l}} f$ is Hölder continuous of index ϵ and assume χ in the decomposition $f = f_1 + f_2 = (1-\chi)f + \chi f$ to have $\mathcal{K}$ as its support. Then $\partial^{\mathbf{l}} f_2$ is Hölder continuous of index ϵ in all of $\mathbb{R}^n$, so for any bounded neighborhood $\mathcal{N}$ of x_0 and any $x_1, x_2 \in \mathcal{N}$

$$\begin{aligned} & |\partial^{\mathbf{l}}(I^\alpha f_2)(x_1) - \partial^{\mathbf{l}}(I^\alpha f_2)(x_2)| \\ \le \;& \frac{1}{H_n(\alpha)} \int_{\mathcal{N}+(-\mathcal{K})} |\partial^{\mathbf{l}} f_2(x_1 - y) - \partial^{\mathbf{l}} f_2(x_2 - y)||y|^{\operatorname{Re}\alpha - n}\, dy \\ \le \;& M'|x_1 - x_2|^\epsilon \end{aligned}$$

for some $M' > 0$. □

4.14. **Lemma.** *Let $f \in C_a(\mathbb{R}^n)$. Let $\alpha \in \mathbb{C}_n$ with $\operatorname{Re}\alpha = 1$ be given. If $a > 1$ then*

$$f \in C^{l+\langle\epsilon\rangle,x}(\mathbb{R}^n) \Rightarrow \forall \epsilon', 0 < \epsilon' < \epsilon : I^\alpha f \in C^{(l+1)+\langle\epsilon'\rangle,x}(\mathbb{R}^n)$$

for any $x \in \mathbb{R}^n$ and $0 < \epsilon < 1$.

Proof. Let $x_0 \in \mathbb{R}^n$ be given. Write $f = f_1 + f_2$ as in Remark 4.11. Then from Lemma 4.12 $I^\alpha f_1$ is smooth at x_0, so only $I^\alpha f_2$ needs thought.

Pick $\mathbf{p} \in \mathbb{N}_0^n$ with $|\mathbf{p}| = l + 1$. Write $\mathbf{p} = \mathbf{l} + e_i$ for some $\mathbf{l} \in \mathbb{N}_0^n$ with $|\mathbf{l}| = l$, and some $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Let $\mathcal{K}$ be a compact neighborhood of x_0 in which $\partial^l f$ is Hölder continuous and assume χ in the decomposition $f = f_1 + f_2 = (1-\chi)f + \chi f$ to have $\mathcal{K}$ as its support. Put $g = \partial^{\mathbf{l}} f_2$. Then g is Hölder continuous of index ϵ in all of $\mathbb{R}^n$ and has support in $\mathcal{K}$. What needs to be shown is, that $\partial^{\mathbf{p}}(I^\alpha f_2) = \partial_i \partial^{\mathbf{l}}(I^\alpha f_2) = \partial_i(I^\alpha g)$ (Lemma 4.12) exists and is Hölder continuous in a neighborhood of x_0.

Let B be a symmetric, bounded neighborhood of 0 such that $\mathcal{K} \subset B + x = B_x$ for all x in some bounded, open neighborhood $\mathcal{O}$ of x_0. Let $\beta \in \mathbb{C}$ with $1 < \operatorname{Re}\beta < 2$ be given. Then for any $x \in \mathcal{O}$

$$\begin{aligned} \partial_i I^\beta g(x) &= c_n(\beta) \int_{B_x} g(y)(x_i - y_i)|x-y|^{\beta-n-2}\, dy \\ &= c_n(\beta) \int_B g(x-y) y_i |y|^{\beta-n-2}\, dy \end{aligned} \tag{23}$$

where $c_n(\beta) = \frac{\beta - n}{H_n(\beta)}$ and where the integral exists since $\operatorname{Re}\beta > 1$ and B is bounded. Furthermore, using the Hölder continuity of g

$$\int_B |(g(x-y) - g(x)) y_i |y|^{\alpha-n-2}|\, dy \le M \int_B |y|^{-n+\epsilon}\, dy < \infty,$$

i.e. the integral $\int_B (g(x-y) - g(x)) y_i |y|^{\alpha-n-2}\, dy$ exists. Using the symmetry of B we get

$$\begin{aligned} & |\frac{1}{c_n(\beta)} \partial_i I^\beta g(x) - \int_B (g(x-y) - g(x)) y_i |y|^{\alpha-n-2}\, dy| \\ \le \; & \int_B |(g(x-y) - g(x)) y_i (|y|^{\beta-n-2} - |y|^{\alpha-n-2})|\, dy \\ & \qquad + |g(x) \int_B y_i |y|^{\beta-n-2}\, dy| \\ \le \; & c' \int_B ||y|^{\beta-n-1+\epsilon} - |y|^{\alpha-n-1+\epsilon}|\, dy \end{aligned} \tag{24}$$

for some $c' > 0$. Now notice that when $n = 1$, then c_n has a removable singularity at $\beta = 1$, so that for any value of $n \in \mathbb{N}$, c_n is bounded and bounded away from 0 in a small enough neighborhood of α, i.e. $\lim_{\beta\to\alpha} \frac{1}{c_n(\beta)}$ exists and is not 0. Thus (24) shows that in the limit where $\operatorname{Re}\beta > 1$

$$\lim_{\beta\to\alpha} \partial_i I^\beta g(x) = c_n(\alpha) \int_B (g(x-y) - g(x)) y_i |y|^{\alpha-n-2}\, dy$$

uniformly on $\mathcal{O}$. So $\partial_i I^\alpha g$ does exist and

$$\partial_i I^\alpha g = c_n(\alpha)\int_B (g(x-y)-g(x))y_i|y|^{\alpha-n-2}\,dy$$

in all of $\mathcal{O}$. Given $0<\epsilon'<\epsilon$ put $s=\frac{\epsilon'}{\epsilon}$ and $t=1-s$. We then have for any $x_1,x_2\in\mathcal{O}$, that

$$\begin{aligned}
& |\partial_i I^\alpha g(x_1)-\partial_i I^\alpha g(x_2)| \\
= \;& c_n(\alpha)|\int_B (g(x_1-y)-g(x_1)-(g(x_2-y)-g(x_2)))y_i|y|^{\alpha-n-2}\,dy \\
\leq \;& c_n(\alpha)\int_B |(g(x_1-y)-g(x_1))-(g(x_2-y)-g(x_2))|^t \\
& \quad |(g(x_1-y)-g(x_2-y))-(g(x_1)-g(x_2))|^s|y|^{-n}\,dy \\
\leq \;& c_n(\alpha)\int_B (2M|y|^\epsilon)^t(2M|x_1-x_2|^\epsilon)^s|y|^{-n}\,dy \\
= \;& c_n(\alpha)|x_1-x_2|^{\epsilon s}2M\int_B |y|^{\epsilon t-n}\,dy = M'|x_1-x_2|^{\epsilon'}
\end{aligned}$$

for some constant $M'>0$. □

4.15. **Corollary.** *Let $f\in C_a(\mathbb{R}^n)$. Let $\alpha\in\mathbb{C}$ with $0<\operatorname{Re}\alpha<\min(a,n)$ be given. Then*

$$f\in C^{l+\langle\epsilon\rangle,x}(\mathbb{R}^n)\Rightarrow \forall\epsilon',0<\epsilon'<\epsilon : I^\alpha f\in C^{(l+[\operatorname{Re}\alpha])+\langle\epsilon'\rangle,x}(\mathbb{R}^n)$$

for any $x\in\mathbb{R}^n$ and $0<\epsilon<1$. Here $[\operatorname{Re}\alpha]$ denotes the integer part of $\operatorname{Re}\alpha$.

Proof. Write $\alpha=\beta+[\operatorname{Re}\alpha]$. Then $0\leq\operatorname{Re}\beta<1$. From Proposition 4.9 combined with Lemma 4.8

$$I^\alpha f = I^\beta(I^1(I^1(\ldots(I^1 f)\ldots))),$$

I^1 applied $[\operatorname{Re}\alpha]$ times. The claim now follows from Lemma 4.14 and Lemma 4.13. □

4.16. **Proposition.** *Let $k\in\{1,\ldots,n-1\}$ and $f\in C(k,n)$. Then*

$$I^{-k}(I^k f)=f.$$

Proof. Let $x\in\mathbb{R}^n$ be given and choose δ, $0<\delta<1$, such that $f\in C_{k+\delta}(\mathbb{R}^n)$. From Proposition 4.6 it follows, that there exists an δ', $0<\delta'<1$, such that the map

$$\alpha\mapsto (I^{\alpha+k}f)(x)$$

is holomorphic in $\{\alpha\in\mathbb{C}\mid -k-\delta'<\operatorname{Re}\alpha<\delta\}$. Since Lemma 4.8 and Corollary 4.15 with $a=b=k+\delta$ ensures, that

$$I^k f\in C^{k+}_\delta(\mathbb{R}^n),$$

we likewise get from Proposition 4.6, that there exists a δ'', $0 < \delta'' < 1$, such that the map

$$\alpha \mapsto (I^\alpha(I^k f))(x)$$

is well-defined and holomorphic in $\{\alpha \in \mathbb{C} \mid -k - \delta'' < \operatorname{Re}\alpha < \delta\}$. Proposition 4.9 gives us, that

$$I^\alpha I^k f(x) = I^{\alpha+k} f(x),$$

when $\alpha \in \{\alpha \in \mathbb{C} \mid 0 < \operatorname{Re}\alpha < \delta\}$. By analytic continuation this identity then holds on all of $\{\alpha \in \mathbb{C} \mid -k - \min(\delta', \delta'') < \operatorname{Re}\alpha < \delta\}$. In particular, using Lemma 4.7 with $a = k + \delta$

$$I^{-k} I^k f(x) = I^0 f(x) = f(x).$$

□

5. The Inversion Formula for the Radon Transform

Let $k \in \{1, \ldots, n-1\}$ be given. Let $f \in C_a(\mathbb{R}^n)$ for some $a > k$. For the k-plane transform one arrives, by calculating, at

$$(\hat{f})^\vee(x) = (4\pi)^{\frac{k}{2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)} (I^k f)(x), \tag{25}$$

cf. [3] or [6, p.29]. This will be used in what follows.

5.1. **Theorem.** *Let $k \in \{1, \ldots, n-1\}$. Assume, that $f \in C(k, n)$. Then f can be recovered from its k-plane transform by*

$$f = (4\pi)^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} I^{-k} (\hat{f})^\vee.$$

Proof. The claim follows from (25) by means of Proposition 4.16. □

5.2. *Remark.* Any differentiable function will also be locally Hölder continuous (but the inverse implication is not true). Thus the Hölder condition could in the entire paper have been replaced by demanding all functions to be one more time continuously differentiable. E.g. Theorem 5.1 is true for all $f \in C^1(\mathbb{R}^n)$ with $f(x) = O(|x|^{-k-\delta})$ for some $\delta > 0$.

An even lower regularity requirement on f can be bought at a small price:

5.3. **Theorem.** *Let $k \in \{1, \ldots, n-1\}$. Assume, that $f \in C_{k+\delta}$ for some $\delta > 0$. Then f can be recovered from its k-plane transform by*

$$f = (4\pi)^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \lim_{s \to -k_+} I^s (\hat{f})^\vee.$$

We will need the following lemma pointed out to me by Boris Rubin (cf. [12, Thm. I.2.6]):

5.4. **Lemma.** *Let $f \in C_a(\mathbb{R})$. Then*

$$\lim_{s \to -1_+} x_+^s(f) = f(0).$$

Proof. Let $\epsilon > 0$ be given and choose δ, $0 < \delta < 1$, such that $|f(x) - f(0)| \leq \epsilon$ when $|x| \leq \delta$. Write

$$x_+^s(f) = \frac{1}{\Gamma(s+1)} \left[\int_0^\delta (f(x) - f(0)) x^s \, dx + \int_\delta^\infty f(x) x^s \, dx + \int_0^\delta f(0) x^s \, dx \right].$$

Since (when $s > -1$),

$$\left| \frac{1}{\Gamma(s+1)} \int_0^\delta (f(x) - f(0)) x^s \, dx \right| \leq \frac{\epsilon}{\Gamma(s+2)} \delta^{s+1}$$

and (when $s - a < -1$)

$$\left| \frac{1}{\Gamma(s+1)} \int_\delta^\infty f(x) x^s \, dx \right| \leq \frac{c}{\Gamma(s+1)|s-a+1|} \delta^{s-a+1}$$

for some constant $c > 0$ and

$$\left| \frac{1}{\Gamma(s+1)} \int_0^\delta f(0) x^s \, dx - f(0) \right| \leq \left| f(0) \left(\frac{\delta^{s+1}}{\Gamma(s+2)} - 1 \right) \right|,$$

$|x_+^s(f) - f(0)|$ can be estimated by e.g. some multiple of ϵ when s is sufficiently close to -1. □

Also a parallel of Corollary 4.15 and thus of Lemma 4.14 for functions with the Hölder continuity of the derivatives replaced by ordinary continuity is needed:

5.5. **Lemma.** *Let $f \in C_a^l(\mathbb{R}^n)$. Let $\alpha \in \mathbb{C}_n$ with $Re\alpha = 1$ be given. If $a > 1$, then*

$$I^\alpha f \in C^{l + \langle \epsilon \rangle}(\mathbb{R}^n)$$

for any $0 < \epsilon < 1$.

Proof. Let $x \in \mathbb{R}^n$ and $0 < \epsilon < 1$ be given. Decompose $f = f_1 + f_2$ as in Remark 4.11. Then $I^\alpha f_1$ is smooth at x according to Lemma 4.12, so only $I^\alpha f_2$ needs thought.

From Lemma 4.12 $I^\alpha f_2$ is in $C^l(\mathbb{R}^n)$ with $\partial^{\mathbf{l}} I^\alpha f_2 = I^\alpha \partial^{\mathbf{l}} f_2$ for any $\mathbf{l} \in \mathbb{N}_0^n$ with $|\mathbf{l}| = l$. The claim is, that these derivatives are Hölder continuous of index ϵ at x. Therfore pick $\mathcal{O}$, a bounded neighborhood of x, and $x_1, x_2 \in \mathcal{O}$. Since f_2 has compact support $\mathcal{K}$, there exists $c > 0$ such that

$$|\partial^{\mathbf{l}} I^\alpha f_2(x_1) - \partial^{\mathbf{l}} I^\alpha f_2(x_2)| \leq c \int_{\mathcal{K}} \left| |x_1 - y|^{\alpha - n} - |x_2 - y|^{\alpha - n} \right| dy.$$

Thus it suffices to prove the existence of a constant $C > 0$ (independent of x_1 and x_2) such that

$$\int_{\mathcal{K}} \left||x_1 - y|^{\alpha-n} - |x_2 - y|^{\alpha-n}\right| dy \leq C|x_1 - x_2|^{\epsilon}. \tag{26}$$

Put

$$B_1 = B(x_1, \frac{2}{3}|x_1 - x_2|) \qquad B_2 = B(x_2, \frac{2}{3}|x_1 - x_2|) \qquad A = \mathcal{K} \setminus (B_1 \cup B_2).$$

Then $\mathcal{K} \subset B_1 \cup B_2 \cup A$, so (26) holds if it can be proved with $\mathcal{K}$ replaced by each of the three sets B_1, B_2 and A. But since $|x_2 - y| > \frac{1}{3}|x_1 - x_2|$ when $y \in B_1$

$$\begin{aligned} &\int_{B_1} \left||x_1 - y|^{\alpha-n} - |x_2 - y|^{\alpha-n}\right| dy \\ \leq\ &\int_{B_1} |x_1 - y|^{1-n}\, dy + \int_{B_1} |x_2 - y|^{1-n}\, dy \\ \leq\ &\int_{B(0,\frac{2}{3}|x_1-x_2|)} |y|^{1-n}\, dy + \int_{B(0,\frac{2}{3}|x_1-x_2|)} (\frac{1}{3}|x_1 - x_2|)^{1-n}\, dy \leq C_1|x_1 - x_2|. \end{aligned}$$

An equivalent calculation can be done for the integral on B_2. Thus we turn to the integral on A.

First let $y \in A$ with $|x_1 - y| \neq |x_2 - y|$ be given. Apply the mean value theorem to the function $t \mapsto \operatorname{Re} t^{\alpha-n}$ on the interval with endpoints $|x_1 - y|$ and $|x_2 - y|$ to obtain the existence of an $s_1 \in]0, 1[$ such that

$$\begin{aligned} &\left|\operatorname{Re}|x_1 - y|^{\alpha-n} - \operatorname{Re}|x_2 - y|^{\alpha-n}\right| \\ &\qquad \leq c'(s_1|x_1 - y| + (1 - s_1)|x_2 - y|)^{-n}\left||x_1 - y| - |x_2 - y|\right|. \end{aligned} \tag{27}$$

Then apply the mean value theorem to the function $t \mapsto \operatorname{Im} t^{\alpha-n}$ to obtain an s_2 and a similar evaluation of $|\operatorname{Im}|x_1-y|^{\alpha-n} - \operatorname{Im}|x_2-y|^{\alpha-n}|$. Conclude from this that for any $y \in A$

$$\left||x_1 - y|^{\alpha-n} - |x_2 - y|^{\alpha-n}\right| \leq c''(\min(|x_1 - y|, |x_2 - y|))^{-n}|x_1 - x_2|.$$

Choose $K > 0$ such that $B(y_1, K) \cap B(y_2, K) \supset A$ for all $y_1, y_2 \in \mathcal{O}$. Then K is independent of x_1 and x_2 and

$$\begin{aligned} &\int_A (\min(|x_1 - y|, |x_2 - y|))^{-n}\, dy \\ \leq\ &\int_{B(x_1,K)\setminus B_1} |x_1 - y|^{-n}\, dy + \int_{B(x_2,K)\setminus B_2} |x_2 - y|^{-n}\, dy \\ \leq\ &2\Omega_{n-1}(\log K - \log(\frac{2}{3}|x_1 - x_2|)) \leq C'(1 + |x_1 - x_2|^{\epsilon-1}), \end{aligned}$$

where C' is independent of x_1 and x_2. The last evaluation holds because $\epsilon - 1 < 0$. $\square$

5.6. **Corollary.** *Let $f \in C_a^l(\mathbb{R}^n)$. Let $\alpha \in \mathbb{C}$ with $1 \leq \operatorname{Re}\alpha < \min(a,n)$ be given. Then*

$$I^\alpha f \in C^{(l+[\operatorname{Re}\alpha]-1)+\langle\epsilon\rangle}(\mathbb{R}^n)$$

for any $0 < \epsilon < 1$.

Proof. Let $0 < \epsilon < 1$ be given. If $\operatorname{Re}\alpha = 1$ the claim is the previous lemma. If $\operatorname{Re}\alpha > 1$ write $\alpha = \beta + 1$. From Proposition 4.9

$$I^\alpha f = I^\beta(I^1 f).$$

According to the previous lemma $I^1 f \in C^{l+\langle\epsilon\rangle}(\mathbb{R}^n)$, so the claim follows from Corollary 4.15. □

Proof of Theorem 5.3. Use Remark 2.4 to modify the conclusions of Proposition 3.7 and 4.6 regarding the set of definition of the extension when the Hölder continuity on the derivatives of f is replaced by ordinary continuity. Use this in following the lines of the proof of Proposition 4.16: Let $x \in \mathbb{R}^n$ be given. The map

$$\alpha \mapsto (I^{\alpha+k} f)(x)$$

is holomorphic in $\{\alpha \in \mathbb{C} | -k < \operatorname{Re}\alpha < \delta\}$. Since Lemma 4.8 and Corollary 5.6 ensures, that

$$I^k f \in C_\delta^{(k-1)+\langle\epsilon\rangle}(\mathbb{R}^n)$$

for all $0 < \epsilon < 1$, the map

$$\alpha \mapsto (I^\alpha(I^k f))(x)$$

is holomorphic in

$$\bigcup_{0<\epsilon<1} \{\alpha \in \mathbb{C} | -(k-1+\epsilon) < \operatorname{Re}\alpha < \delta\} = \{\alpha \in \mathbb{C} | -k < \operatorname{Re}\alpha < \delta\}.$$

Thus by Proposition 4.9 and analytic extension

$$I^\alpha(I^k f)(x) = I^{\alpha+k} f(x), \tag{28}$$

when $-k < \operatorname{Re}\alpha < \delta$. The last step of the proof of Proposition 4.16 requires Lemma 4.7 the conclusion of which does not hold for an arbitrary $f \in C_a(\mathbb{R}^n)$ ($I^0(f)$ does not necessarily exist). But we can use Lemma 5.4 to replace Lemma 4.7 by (see (16))

$$\lim_{s\to 0_+}(I^s f)(x) = \lim_{s\to 0_+} \frac{\Gamma(s)}{H_n(s)} r^{s-n}(\tau_x f) = \lim_{s\to -1_+} x_+^s(M_{\tau_x f}) = M_{\tau_x f}(0) = f(x).$$

Thus, using (28), we have that

$$\lim_{s\to -k_+} I^s(I^k f(x)) = f.$$

This in connection with (25) proves the theorem. □

6. The Inversion Formula in Terms of the Laplacian

It is known, cf. [6], that if k is even, the inversion formula can be stated by means of the Laplacian, Δ, instead of the more complicated Riesz potentials. In fact

6.1. **Theorem.** *When k is even, and $f \in C^2(\mathbb{R}^n)$, and f and all its first and second order derivatives are $O(|x|^{-k-\epsilon})$ for some $\epsilon > 0$, then*

$$f = (4\pi)^{-\frac{k}{2}} \frac{\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} (-\Delta)^{\frac{k}{2}} (\hat{f})\check{}. \tag{29}$$

Proof. Follow the lines of [6, p.16-17]: First notice that it suffices for f to be continuous and $O(|x|^{-k-\epsilon})$ for some $\epsilon > 0$ in order to have formula [6, (34)] for the k-plane transform; that is,

$$(\hat{f})\check{}(x) = \Omega_{k-1} \int_0^\infty F(r,x) r^{k-1}\, dr, \tag{30}$$

for any $x \in \mathbb{R}^n$, where $F(r,x) = \frac{1}{\Omega_{n-1}} \int_{S^{n-1}} f(x + r\omega)\, d\omega$. Here $d\omega$ is the Haar measure on the unit sphere S^{n-1} in $\mathbb{R}^n$ with total mass $\Omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$. Then notice, that the demands on the decay of the derivatives of f allows us to apply the Laplacian (with respect to x) on (30) by interchanging it with the integration. By means of Darboux's equation, it can now be seen, as in [6, p.16-17], that

$$\Delta((\hat{f})\check{})(x) = \begin{cases} -\Omega_{k-1}(n-k) f(x) & k = 2 \\ -\Omega_{k-1}(n-k)(k-2) \int_0^\infty F(r,x) r^{k-3}\, dr & k \neq 2 \end{cases}.$$

When $k = 2$ this is (29). For $k \neq 2$ the expression is similar to (30) - the power of r in the integral has just been reduced and it is still larger than -1. Thus the Laplacian can be applied once more without inducing further demands on f or its derivatives. Continued iteration proves (29). $\square$

Can the Theorem 5.1 be used to enlarge the class of functions for which (29) holds? Not much, I think. Some relevant thoughts are the following: Let $\alpha_0 \in \mathbb{C}$ be given. Using Definition 4.4 and Green's formula it is not hard to see, that for $\varphi \in C^2(\mathbb{R}^n)$ with sufficient decay of φ and all it's first and second order derivatives ($O(|x|^{-2-\epsilon})$ for some $\epsilon > 0$ is enough),

$$I^\alpha \Delta \varphi = -I^{\alpha-2} \varphi \tag{31}$$

in some strip $\{\alpha \in \mathbb{C} | 2 < \operatorname{Re}\alpha < 2 + \delta\}$. If furthermore $\varphi \in C^{l+}(\mathbb{R}^n)$ for some integer $l \geq -\operatorname{Re}\alpha_0 + 2$, Proposition 4.6 can be used to extend both sides of (31) holomorphically to α_0 and thus prove (31) for $\alpha = \alpha_0$.

Iterating (31) and then using Lemma 4.7 proves that when k is even and positive, and $h \in C^{k+}(\mathbb{R}^n)$, and h and all it's derivatives of order less than or equal to k have a certain decay ($O(|x|^{-2-\epsilon})$ for some $\epsilon > 0$ is enough), then

$$(-\Delta)^{\frac{k}{2}} h = I^{-k} h.$$

Thus we see from Theorem 5.1 that (29) holds also for $f \in C(k,n)$ when, in stead of decay demands on derivatives of f, we demand a certain decay of $(\hat{f})^{\check{}}$ and all it's derivatives of order less than or equal to k ($O(|x|^{-2-\epsilon})$ is enough). Notice, that since $(\hat{f})^{\check{}}$ is proportional to $I^k f$ the derivatives of $(\hat{f})^{\check{}}$ do exist according to Corollary 4.15.

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PART II

The Hyperbolic Case

Sufficient Conditions for an Inversion Formula for the k-dimensional Totally Geodesic Radon Transform on Real Hyperbolic Space.

Sine R. Jensen

ABSTRACT: The inversion formula (26) for the k-dimensional totally geodesic Radon transform on real hyperbolic space is indicated by B. Rubin in [26, p.219-221] for smooth, compactly supported functions. In this paper it will be shown that it suffices for the function to be twice continuously differentiable with all derivatives of order up to and including 2 meeting a certain decay requirement.

INTRODUCTION

Geometric integral transforms and inversion formulas in connection with these are many and varied. Consider e.g. the k-plane Radon transform in n-dimensional Euclidean space, $k \in \{1, \ldots, n-1\}$. One of the more esthetically pleasing inversion formulas for this transform is Helgason's formula:

$$f = c\, I^{-k}(\hat{f})^{\vee} \tag{1}$$

(cf. [12, Thm. I.6.2]). Here (and in the following) $\hat{f}$ denotes the Radon transform under consideration applied to f, while $\check{\varphi}$ is the dual Radon transform of the function φ. By I^{-k} is meant the $(-k)$'th Riesz potential, and c is some constant. For k even I^{-k} is nothing but the Laplacian to the power of $\frac{k}{2}$ (cf. [7]). In the hyperplane case ($k = n-1$) the formula goes back to Radon [22] for $n = 3$, and to John [16] for arbitrary n. Originally, (1) was proved for $f \in C_c^\infty(\mathbb{R}^n)$, the space of smooth compactly supported functions on $\mathbb{R}^n$. This result has been stepwise expanded (cf. [8], [12] and others). In [15] the author defines a, in some sense, 'natural maximal' function space on which to consider the k-plane transform and shows (1) to hold on this space using the setup in [12].

In the present paper we will move on from the k-plane transform to it's hyperbolic analog. This is the k-dimensional totally geodesic Radon transform on real hyperbolic space, $\mathbb{H}^n$, as introduced by Helgason in the fundamental paper [7]. Note that $\mathbb{H}^n$ is the classical n-dimensional Riemannian space of constant curvature -1 realized e.g. as the upper sheet of the two-sheeted hyperboloid in $\mathbb{R}^{n+1}$. Helgason proves in that same paper [7] the following hyperbolic version of (1): For k even and f in $C_c^\infty(\mathbb{H}^n)$

$$f = P_k(\Delta)(\hat{f})\check{}, \tag{2}$$

where $P_k(\Delta)$ is some polynomial of degree $\frac{k}{2}$ in the Laplace-Beltrami operator. A second formula of Helgason [9, Thm. 3.1] covers all $1 \leq k \leq n-1$. It has the form

$$f(x) = c\left[\left(\frac{d}{d(u^2)}\right)^k \int_0^u ((\hat{f})^\vee_{\cosh^{-1}(v^{-1})})(x)(u^2-v^2)^{\frac{k}{2}-1}\,dv\right]_{u=1},$$

for $f \in C_c^\infty(\mathbb{H}^n)$, where $((\hat{f})^\vee_{\cosh^{-1}(v^{-1})})(x)$ is the average of $\hat{f}$ over all totally geodesic submanifolds of distance $\cosh^{-1}(v^{-1})$ from x, and c is some constant. See also [10] for a variation. Meanwhile, it is in fact possible to generalize (2) to the case k odd. This was done by Berenstein and Tarabusi [1] and [2] using Fourier transform and distributions: For k odd and f in $C_c^\infty(\mathbb{H}^n)$

$$f = P(\Delta)S(\hat{f})\check{},$$

where $P(\Delta)$ is again some polynomial in the Laplace-Beltrami operator, and where S is a convolution operator with kernel $r \mapsto \sinh^{k-n}(r)\cosh(r)$. An alternative proof was indicated by Rubin [26, p.219-221] and his approach is the one adopted here.

The goal of this paper is firstly to define (as in the Euclidean case) some 'natural maximal' function space on which to consider the k-dimensional totally geodesic Radon transform on $\mathbb{H}^n$. This space will be denoted $C_k(\mathbb{H}^n)$ (cf. Definitions 4.1 and 4.4). It consists, as in the Euclidean case, of continuous functions of a certain decay at infinity (cf. the Introduction in [15]), but the decay requirement must be modified according to the measure on hyperbolic space. Secondly, we will prove (2) for k even and a generalized version of (2) for k odd, both on a certain subspace of $C_k(\mathbb{H}^n)$ (cf. Theorem 8.2). We would like to imitate the proof of (1) in [15] by generalizing Riesz potentials starting out from the hyperbolic formula

$$(\hat{f})\check{}(x) = c\int_0^\infty (M^r f)(x)(\sinh r)^{k-1}\,dr, \qquad f \in C_c^\infty(\mathbb{H}^n) \tag{3}$$

(cf. [12, (28) p.92]) and the Euclidean formula $(\hat{f})\check{} = c\,I^k$. Unfortunately, these generalized Riesz potentials, defined by the right side of (3), turns out not to abide by the power rule $I^\alpha I^\beta = I^{\alpha+\beta}$, which is crucial to the proof of (1). Instead, we have used the approach of Berenstein-Tarabusi and Rubin which involves not only the kernel $r \mapsto \sinh^{\alpha-n}$ but also the kernel $r \mapsto \sinh^{\alpha-n} r \cosh r$ [26]. The different nature of this approach compared to the one in [15] is reflected in the way the constraints on f in the hyperbolic inversion formula (Theorem 8.2) deviate from those on f in the Euclidean inversion formula [15, Thm. 5.1]. In Theorem 8.2 they involve decay requirements not only on f as in [15] but also on the derivatives of f up to the second order. The precise statement involves the universal enveloping algebra of the Lie algebra $\mathfrak{so}(n,1)$.

We note in passing the viewpoint applied by Kurusa [18]. For $k = n-1$ he considers any $f \in C_c^\infty(\mathbb{H}^n)$ as expressed in terms of a power expansion in spherical harmonics: $f(p,\omega) \sim \sum_{l,m}^\infty f_{l,m}(p) Y_{l,m}(\omega)$. Kurusa inverts the Radon transform by determining the coefficients $\{f_{l,m}\}_{l,m}$. In that same paper he establishes an inversion formula a kind to the Euclidean formula

$$f = c(\Lambda \hat{f})\check{}, \qquad f \in \mathcal{S}(\mathbb{R}^n)$$

[12, Thm. I.3.6], where Λ involves differentiations and, when n is even, a Hilbert transform. In this connection see also [20].

The problem of establishing inversion formulas for larger classes of functions than smooth compactly supported functions has been addressed in particular by Rubin and Berenstein. In [4] and [5] they consider the k-dimensional totally geodesic Radon transform on $\mathbb{H}^n$ and show that given $f \in L^p(\mathbb{R}^n)$ then $\hat{f}(\xi)$ is defined for a.a. totally geodesic submanifolds ξ of $\mathbb{H}^n$ if $\frac{k-1}{n-1} < \frac{1}{p} \leq 1$. For $\frac{1}{p} \leq \frac{k-1}{n-1}$ they provide a counter example. Instead of generalizing (2) they prove an inversion formula that relies on the choice of a suitable 'wavelet' function. The inversion formula is expressed as a limit in L^p-norm (cf. [5, Thms. 5.3 and 5.5]) and is thus not a pointwise formula.

This paper is organized as follows: Section 1 through 3 presents some well known background material as in [12] but with more details. In Section 1 the construction of general pseudo-Riemannian hyperbolic space is carried through. Then, after restricting to the Riemannian case of negative curvature, $\mathbb{H}^n$, Sections 2 and 3 deals with characterizing the totally geodesic submanifolds and introducing integration thereon. The aforementioned space $C_k(\mathbb{H}^n)$ is defined in section 4, whereafter the Radon transform is presented. Section 5 presents the dual transform and proves (3) on $C_k(\mathbb{H}^n)$. In Section 6 the convolution operators of [26], K^α and K_-^α, which takes the place of Riesz potentials, are carefully defined. This includes establishing certain mapping properties. Note, that Rubin and Berenstein in [5, Lemma 4.1] also remarks upon the problem in the first part of Lemma 6.4 which is, given $f \in L^p(\mathbb{H}^n)$, to find conditions on p and

α under which $K^\alpha f$ is well-defined a.e. on $\mathbb{H}^n$. Lemma 6.4 and [5, Lemma 4.1] express the exact same conditions on p and α execpt that Berenstein and Rubin points out that $K^\alpha f$ is in fact well-defined *everywhere* on $\mathbb{H}^n$ when $\frac{\mathrm{Re}\,\alpha-1}{n-1} < \frac{1}{p} < \frac{\mathrm{Re}\,\alpha}{n}$. In Section 7 the operators K^α and K^α_- are investigated. The entire section prepares for Proposition 7.6 and Corollary 7.7, which are the versions of [26, Thm. 4.4] and [26, Cor. 4.7] needed to deal with functions not necessarily of compact support. In Section 8 the inversion formula in finally stated and proved using a result from [26, p.220-221] concerning a certain convolution.

1. Hyperbolic Spaces

Let $p, q \in \mathbb{N}_0$ and $n \in \mathbb{N}$ with $p+q = n$ be given, and make the following

1.1. **Definition.** Put

$$I_{p,q} = \left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right) \in GL(\mathbb{R}^{p+q}),$$

where I_m, $m \in \mathbb{N}$, denotes the $m \times m$ identity matrix. For $\epsilon = \pm 1$ define the quadratic form $B_\epsilon(x) : \mathbb{R}^{n+1} \to \mathbb{R}$ by

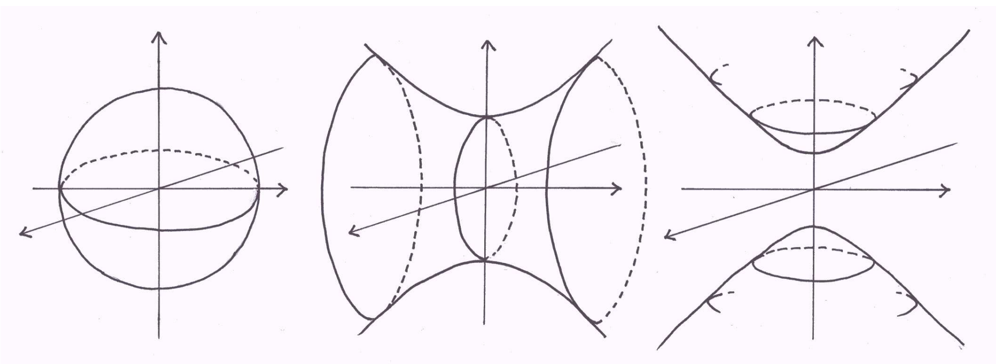
$$\begin{aligned} B_\epsilon(x) = B_\epsilon^{p,q}(x) &= {x_1}^2 + \ldots + {x_p}^2 + \epsilon {x_{p+1}}^2 - {x_{p+2}}^2 - \ldots - {x_{p+q+1}}^2 \\ &= \begin{cases} x^t I_{p+1,q} x & , \quad \epsilon = +1 \\ x^t I_{p,q+1} x & , \quad \epsilon = -1 \end{cases}. \end{aligned}$$

Then B_ϵ generates the following subset of $\mathbb{R}^{n+1}$

$$Q_\epsilon = Q_\epsilon^{p,q} = \{x \in \mathbb{R}^{n+1} | B_\epsilon x = \epsilon\}.$$

1.2. *Example.* For $n = 2$ the following subsets arises

$Q^{2,0}_{+1}$ and $Q^{0,2}_{-1}$ are the unit sphere.
$Q^{1,1}_{+1}$ and $Q^{1,1}_{-1}$ are one-sheeted hyperboloids.
$Q^{0,2}_{+1}$ and $Q^{2,0}_{-1}$ are two-sheeted hyperboloids.



The set Q_ϵ is furnished with the subspace topology.

1.3. **Lemma.** *If* $p > 0$ *then* $Q^{p,q}_{+1}$ *is connected and if* $p = 0$ *then* $Q^{p,q}_{+1}$ *has two components.*
If $q > 0$ *then* $Q^{p,q}_{-1}$ *is connected and if* $q = 0$ *then* $Q^{p,q}_{-1}$ *has two components.*

1.4. *Remark.* Since $Q^{p,q}_{+1} \cong Q^{q,p}_{-1}$, the two above claims are equivalent.

Proof of Lemma 1.3. Consider the case $\epsilon = -1$.
If $p = 0$ then $q > 0$ and Q_{-1} is the unit sphere in $\mathbb{R}^{n+1}$ which is connected.
If $p \geq 1$ let $x \in Q_{-1}$ be given. Put

$$a = \operatorname{sign}(x_1)\sqrt{\sum_{i=1}^{p} {x_i}^2} \qquad \text{and} \qquad b = \operatorname{sign}(x_{p+q+1})\sqrt{\sum_{i=p+1}^{p+q+1} {x_i}^2},$$

where the definition of sign(0) as either 1 or -1 is not essential. Note that $|b| \geq 1$, and define $\psi_+ : [1, b] \to Q_{-1}$ for $b \geq 1$ and $\psi_- : [b, -1] \to Q_{-1}$ for $b \leq -1$ by

$$\psi_\pm(t) = (\operatorname{sign}(x_1)\sqrt{t^2 - 1}, 0, \ldots, 0, t).$$

Then ψ_+ is a continuous curve in $Q^+_{-1} = \{y \in Q_{-1} | y_{p+q+1} \geq 0\}$ connecting o with $(a, 0, \ldots, 0, b)$, and ψ_- is a continuous curve in $Q^-_{-1} = \{y \in Q_{-1} | y_{p+q+1} \leq 0\}$ connecting $-o$ with $(a, 0, \ldots, 0, b)$. If $p = 1$ then $a = x_1$. If $p \geq 2$ then $(a, 0, \ldots, 0)$ can be connected to $(x_1, \ldots, x_p)$ by a continuous curve within the sphere with center 0 and radius $|a|$ in $\mathbb{R}^p$, $S^p_{|a|}$, since this is a connected set exactly when $p \geq 2$. If $q = 0$ then $b = x_{p+q+1}$. If $q \geq 1$ then $(0, \ldots, 0, b)$ can be connected to $(x_{p+1}, \ldots, x_{p+q+1})$ within $S^{q+1}_{|b|}$ as before. From this follows, that Q^+_{-1} and Q^-_{-1} are connected subsets of Q_{-1}. For $q \geq 1$, $Q^+_{-1} \cap Q^-_{-1} \neq \emptyset$ so Q_{-1} is connected. For $q = 0$, Q^+_{-1} and Q^-_{-1} are separated by the plane $\{y \in \mathbb{R}^{p+q+1} | y_{p+q+1} = 0\}$. $\square$

Consider the Lie group

$$O(p, q) = \{g \in GL(p + q) | g^t I_{p,q} g = I_{p,q}\}.$$

Then $O(p + 1, q)$ respectively $O(p, q + 1)$ is the group of all linear maps on $\mathbb{R}^{n+1}$ that preserves $B^{p,q}_{+1}$ respectively $B^{p,q}_{-1}$. Put

$$o_\epsilon = \begin{cases} (1, 0, \ldots, 0) \in Q_{+1} & , \quad \epsilon = +1 \\ (0, \ldots, 0, 1) \in Q_{-1} & , \quad \epsilon = -1 \end{cases}.$$

1.5. **Lemma.**
The group $O_e(p + 1, q)$ *acts transitively on the* o_{+1}*-component of* $Q^{p,q}_{+1}$*.*
The group $O_e(p, q + 1)$ *acts transitively on the* o_{-1}*-component of* $Q^{p,q}_{-1}$*.*
Here the subfix e denotes identity component, and the action is that of matrix multiplication on the left.

In connection with this and the following lemma, it will be useful to note that (see [12, Lemma IV.1.4]):

1.6. **Lemma.** *When $p, q \geq 1$ the four components of $O(p,q)$ are determined by*

$$\begin{array}{lll} \det g_p \geq 1 & \text{and} & \det g_q \geq 1, \\ \det g_p \leq 1 & \text{and} & \det g_q \leq -1, \\ \det g_p \geq -1 & \text{and} & \det g_q \geq 1, \\ \det g_p \leq -1 & \text{and} & \det g_q \leq -1, \end{array}$$

where g_p is the sub-matrix $(g_{ij})_{1\leq i,j\leq p}$ and g_q is the sub-matrix $(g_{ij})_{p+1\leq i,j\leq p+q}$ for any $g \in O(p,q)$.

Proof of Lemma 1.5. Consider the case $\epsilon = -1$. Let $x \in Q_{-1}$ be given. Put $a = \sqrt{\sum_{1=1}^{p} {x_i}^2}$ and $b = \sqrt{\sum_{i=p+1}^{p+q+1} {x_i}^2}$. Use the transitivity of $SO(m) = \{g \in O(m) | \det g = 1\}$ on any sphere in $\mathbb{R}^m$ with center 0 for $m = p$ and $m = q+1$ to construct $g \in SO(p, q+1) = \{g \in O(p, q+1) | \det g = 1\}$ such that $g \cdot (0, \ldots, 0, a, b, 0, \ldots, 0)^t = x$. Then construct $g' \in SO(p, q+1)$ such that $g' \cdot o = (0, \ldots, 0, a, b, 0, \ldots, 0)^t$. This shows that the separable Lie group $SO(p, q+1)$ acts transitively on the connected manifold which is the o_{-1}-component of Q_{-1}. Thus so does the identity component of that group ([11, Thm. II.3.2 and Prop. II.4.3]). ☐

1.7. **Lemma.**
The isotropy subgroup of $O_e(p+1,q)$ at o_{+1} is isomorphic to $O_e(p,q)$.
The isotropy subgroup of $O_e(p,q+1)$ at o_{-1} is isomorphic to $O_e(p,q)$.
The isomorphisms are given by

$$\begin{aligned} O_e(p,q) \ni g &\mapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & g \end{array}\right) \in O_e(p+1,q), \\ O_e(p,q) \ni g &\mapsto \left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right) \in O_e(p,q+1). \end{aligned} \tag{4}$$

Proof. Consider the case $\epsilon = -1$. Then 4 is easily seen to be an isomorphism between $O(p,q)$ and the isotropy subgroup of $O(p,q+1)$ at o_{-1}. From Lemma 1.6 $O_e(p,q+1) \cap O(p,q) = O_e(p,q)$. ☐

In order to simplify notation we assume $\epsilon \in \{\pm 1\}$ to be given and make the following

1.8. **Definition.** Put

$$\begin{array}{rcl} o & = & o_\epsilon \\ M & = & \text{the } o\text{-component of } Q_\epsilon^{p,q} \\ G & = & \left\{ \begin{array}{lcl} O_e(p+1,q) & , & \epsilon = +1 \\ O_e(p,q+1) & , & \epsilon = -1 \end{array} \right. \\ H & = & O_e(p,q) \end{array}$$

Furthermore, for each $g \in G$ denote the action of g on $m \in M$ by $g \cdot m$, and let $l_g : M \to M$ denote the diffeomorphism $l_g(m) = g \cdot m$.

Give M the obvious differential structure (note that $dim(M) = n$). Then G is a transitive Lie transformation group of M. Thus ([11, Thm. II.3.2] and [11, Thm. II.4.3]) H is closed in G and the map $gH \mapsto g \cdot o$ of G/H onto M is a diffeomorphism, when G/H is given the unique differential structure with the property that G is a Lie transformation group of G/H ([11, Tmh. II.4.3]);

$$G/H \simeq M \qquad \text{(diffeomorphism)}.$$

1.9. *Remark.* A tangent space $T_m(M)$ of M will always be thought of as a subspace of $\mathbb{R}^{n+1}$. As an example let $m \in M$ and $g \in G$ be given. Then the differential $(dl_g)_m : T_m(M) \to T_{g \cdot m}(M)$ is left matrix multiplication of g on $X \in T_m(M) \subset \mathbb{R}^{n+1}$. That is $(dl_g)_m(X) = gX \in T_{g \cdot m}(M) \subset \mathbb{R}^{n+1}$.

1.10. **Definition.** Let $\sigma : G \to G$ be the involutive isomorphism

$$g \overset{\sigma}{\longmapsto} \begin{cases} I_{1,p+q} g I_{1,p+q} & , \quad \epsilon = +1 \\ I_{p+q,1} g I_{p+q,1} & , \quad \epsilon = -1 \end{cases} .$$

1.11. *Remark.* The effect of σ on a matrix is the following change of sign of the entries of that matrix:

$$\left(\begin{array}{c|c} + & - \\ \hline - & + \end{array}\right) \text{ when } \epsilon = +1, \quad \left(\begin{array}{c|c} + & - \\ \hline - & + \end{array}\right) \text{ when } \epsilon = -1.$$

1.12. **Lemma.** *The triple (G, H, σ) is a symmetric space.*

Proof. Use Remark 1.11 to see, that the fixed point group for σ in G consists of those elements of G which has the form

$$\left(\begin{array}{c|ccc} \pm 1 & 0 & \dots & 0 \\ \hline 0 & * & \dots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \dots & * \end{array}\right) \text{ when } \epsilon = +1 \quad \left(\begin{array}{ccc|c} * & \dots & * & 0 \\ \vdots & & \vdots & \vdots \\ * & \dots & * & 0 \\ \hline 0 & \dots & 0 & \pm 1 \end{array}\right) \text{ when } \epsilon = -1.$$

From this and Lemma 1.6 it follows, that the identity component of the fixed point subgroup for σ is H. $\square$

1.13. *Remark.* Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ that of H. Let $\varphi : G \to M$ be defined by $\varphi(g) = g \cdot o$. Since (G, H, σ) is a symmetric space, the Lie algebra of G can be written as a direct sum, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the (-1)-eigenspace of $(d\sigma)_e$, and where $(d\varphi)_e|_{\mathfrak{m}} : \mathfrak{m} \to T_o(M)$ is an isomorphism ([17, Thm.s XI.3.3 and XI.3.2]). This identification will be used often in what follows.

1.14. **Lemma.** *A basis for $\mathfrak{m}$ is*

$$Y_i = \begin{cases} -E_{1,i} + E_{i,1} & , \quad 2 \le i \le p+1 \\ E_{1,i} + E_{i,1} & , \quad p+2 \le i \le p+q+1 \end{cases}$$

when $\epsilon = +1$ *and*

$$Y_i = \begin{cases} E_{i,p+q+1} + E_{p+q+1,i} & , \quad 1 \leq i \leq p \\ E_{i,p+q+1} - E_{p+q+1,i} & , \quad p+1 \leq i \leq p+q \end{cases}$$

when $\epsilon = -1$*. Here* $E_{i,j}$ *denotes the matrix with the ij'th entry equal to 1 and 0 elsewhere.*

Proof. Note that for $X \in \mathfrak{g}$

$$X \in \mathfrak{m} \Leftrightarrow (d\sigma)_e(X) = -X \Leftrightarrow \sigma(X) = -X.$$

Thus from Remark 1.11 it follows that $X \in \mathfrak{m}$ exactly when $X \in \mathfrak{g}$ and has the form

$$\left(\begin{array}{c|ccc} 0 & * & \dots & * \\ \hline * & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ * & 0 & \dots & 0 \end{array}\right) \text{ when } \epsilon = +1 \qquad \left(\begin{array}{ccc|c} 0 & \dots & 0 & * \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & * \\ \hline * & \dots & * & 0 \end{array}\right) \text{ when } \epsilon = -1.$$

The lemma follows, since $\mathfrak{g} = \mathfrak{so}(p+1, q)$ if $\epsilon = +1$, and $\mathfrak{g} = \mathfrak{so}(p, q+1)$ if $\epsilon = -1$. □

Let $m \in M$ be given. Restrict the quadratic form B_ϵ to $T_m(M)$. This restriction induces a bilinear, symmetric map, g_m, on $T_m(M)$:

$$g_m(X, Y) = \begin{cases} X^t I_{p+1,q} Y & , \quad \epsilon = +1 \\ X^t I_{p,q+1} Y & , \quad \epsilon = -1 \end{cases}.$$

1.15**. Lemma.** *The 2-form, which restricted to each tangent space* $T_m(M)$ *of* M *is* g_m*, defines a* G*-invariant pseudo-Riemannian structure on* M*.*

Proof. For notational convenience assume $\epsilon = -1$. Let $g \in G$ be given and put $m = g \cdot o$. For each $X, Y \in T_o(M)$ (cf. Remark 1.9)

$$\begin{aligned} g_m((dl_g)_o(X), (dl_g)_o(Y)) &= g_m(gX, gY) \\ &= (gX)^t I_{p,q+1}(gY) = X^t I_{p,q+1} Y = g_o(X, Y), \end{aligned}$$

which shows the G-invariance. To see that g_m is non-degenerate for each $m \in M$, note that this is true for $m = o$ and use the G-invariance. □

1.16. *Remark.* Note, that the proof in fact proved any element of $SO(p+1, q)$ respectively $SO(p, q+1)$ to be an isometry of M.

1.17. *Remark.* For each $X \in \mathfrak{g}$

$$\begin{aligned} (d\varphi)_e(X) = (d\varphi)_e(\tfrac{d}{dt}|_{t=0} \exp(tX)) &= \tfrac{d}{dt}|_{t=0} \varphi(\exp(tX)) \\ &= \tfrac{d}{dt}|_{t=0} \exp(tX) \cdot o = Xo, \end{aligned}$$

exp being matrix exponentiation. Thus $(d\varphi)_e(Y_i) = e_i$ for all i ($2 \le i \le p+q+1$ for $\epsilon = +1$, $1 \le i \le p+q$ for $\epsilon = -1$), so the bases from Lemma 1.14 are orthonormal in the sense that

$$g_o(d\varphi_e(Y_i), d\varphi_e(Y_j)) = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j, \quad i \le p \\ -1 & \text{for } i = j, \quad i > p \end{cases} .$$

When a pseudo-Riemannian structure on a differentiable manifold is represented on some tangent space by a matrix, this matrix will be diagonalizable because of the symmetry of the pseudo-Riemannian structure. It can, in fact, be uniquely diagonalized (up to rearrangement of the diagonal entries) to a diagonal matrix with only ± 1 on the diagonal (the non-degeneracy prevents zeros from appearing on the diagonal). Let p' be the number of occurrences of 1 on the diagonal and q' that of -1. Then (p', q') is the same for all points of a connected manifold and is called the *signature* of the pseudo-Riemannian manifold. Note that the manifold is actually Riemannian if and only if $q' = 0$ (note $p' = 0$: negative definite). By considering the pseudo-Riemannian structure of M at o, it is seen that the signature of M is (p, q).

Introduce on M the (pseudo-)Riemannian connection and the curvature tensor derived thereof, R.

1.18. **Proposition.** *The manifold M has constant sectional curvature ϵ.*

Proof. The sectional curvature of any two-dimensional subspace, ν, of $T_o(M)$ is

$$(5) \qquad K(\nu) = K(X, Y) = -\frac{g_o(R(X,Y)X, Y)}{g_o(X,X)g_o(Y,Y) - g_o(X,Y)^2}$$

independently of $X, Y \in \nu$, $\{X, Y\}$ linearly independent (cf. [11, p.65]). Thus if

$$(6) \qquad g_o(R(X,Y)X, Y) = -\epsilon(g_o(X,X)g_o(Y,Y) - g_o(X,Y)^2)$$

for all $X, Y \in T_o(M)$, then $K(\nu) = \epsilon$ for all 2-dimensional subspaces ν of $T_o(m)$. In fact it is enough to show (6) for a basis for $T_o(M)$, everything being linear. But since (G, H, σ) is a symmetric space, the curvature tensor at $o \in M$ is given by

$$R(X,Y)Z = -[[X,Y], Z]$$

for all $X, Y, Z \in T_o(M)$ when $T_o(M)$ is identified with $\mathfrak{m}$ by means of $d\varphi_e$ as in Remark 1.13 (cf. [11, p.215]). Direct calculation now proves (6) for all X, Y in the basis $\{d\varphi_e(Y_i)\}_i$ (see Lemma 1.14 and Remark 1.17), so all sectional curvatures of M at o is ϵ.

Since G/H is symmetric the curvature tensor is invariant under parallel translation ([17, Thm.s XI.3.3 and XI.3.2]) and then so is the sectional curvature as defined by (5). Thus all sectional curvatures at o equal to ϵ implies all sectional curvatures at any point equal to ϵ. $\square$

The spaces $Q^{p,q}_{\pm 1}$ with the structures introduced so fare are called *hyperbolic spaces* and are denoted by $H^{p,q}_{\pm 1}$, and the special case $q = 0$ and $\epsilon = -1$ is also denoted $\mathbb{H}^p$. They are (up to a constant factor on the pseudo-Riemannian structure and local isometries) the only pseudo-Riemannian manifolds of constant curvature and signature (p, q) (cf. [12, Thm. IV.1.3]).

2. Totally Geodesic Submanifolds of $\mathbb{H}^n$

In what follows let

$$n \in \mathbb{N} \setminus \{1\} \qquad \text{and} \qquad k \in \{1, \ldots, n-1\}$$

be given. The intention is to define a Radon transform on a hyperbolic space of dimension n a kind to the k-plane Radon transform on $\mathbb{R}^n$. For this recall that a totally geodesic submanifold ξ of a manifold M is a submanifold of M such that any M-geodesic tangential to ξ at some point is in fact a curve in ξ (see e.g. [11, p. 79]). Thus the k-planes of $\mathbb{R}^n$ are exactly the k-dimensional totally geodesic submanifolds of $\mathbb{R}^n$, so it is an obvious choice to let the set of k-dimensional totally geodesic submanifolds of M take the place of the set of k-planes.

2.1**. Definition.** The space of k-dimensional totally geodesic submanifolds of M will be denoted Ξ (or Ξ_k when the specification seems necessary).

In order to work with as simple a setup as possible consider from now on only the *Riemannian* hyperbolic space of *negative* curvature of dimension n. That is

$$M = H^{n,0}_{-1} = \mathbb{H}^n, \qquad G = O_e(n, 1), \qquad H = O_e(n).$$

Then the restriction of the Riemannian structure on M to any $\xi \in \Xi$ gives a Riemannian structure and thus a Riemannian measure on ξ. Note that the restriction of a pseudo-Riemannian structure on a manifold to some submanifold does not necessarily give a pseudo-Riemannian structure on that submanifold. To be sure of this, the structure must be either positive definite, which happens for $H^{p,q}_{-1}$ exactly when $q = 0$, or negative definite, which happens for $H^{p,q}_{-1}$ exactly when $p = 0$. But $H^{0,q}_{-1}$ is the unit sphere and thus compact, which makes what follows later mostly irrelevant if $M = H^{0,q}_{-1} = S^q$. For a discussion of that case we refere to e.g. [25], [24] and [9, Thm. 3.2].

2.2. *Remark.* Under the new assumption - that is $M = H^{n,0}_{-1}$ - the (pseudo-) Riemannian structure on $T_m(M)$ is seen to correspond to the Riemannian structure induced from $\mathbb{R}^{n+1}$ exactly when $m = o$.

2.3. **Lemma.** *The geodesics of M are exactly the non-empty intersections of M with 2-dimensional subspaces of $\mathbb{R}^{n+1}$.*

Proof. As isometries of M, G preserves the space of geodesics on M. As linear maps on $\mathbb{R}^{n+1}$, G preserves the space of 2-dimensional subspaces of $\mathbb{R}^{n+1}$. The subgroup $H \simeq O_e(n)$ of G (see Lemma 1.7) acts transitively on $S^{n-1} \subset T_o(M) \subset \mathbb{R}^{n+1}$ (see Remark 1.9). Thus H acts transitively on both the set of geodesics on M through o and the set of 2-dimensional subspaces of $\mathbb{R}^{n+1}$ containing o. From the transitivity of G it now follows that G acts transitively on both the set of geodesics on M and the set of 2-dimensional subspaces of $\mathbb{R}^{n+1}$. Thus it is enough to show $\xi = M \cap U$ for just one geodesic ξ through o and one 2-dimensional subspace U of $\mathbb{R}^{n+1}$ containing o.

Let γ be the geodesic such that $\gamma(0) = o$ and $\gamma'(0) = e = (1, 0, \ldots, 0)$, and put $U = \operatorname{span}\{o, e\}$. Note that γ is pointwise fixed by the isometry $x \mapsto Ax$ where

$$A = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix} \in O(n,1)$$

(see Remark 1.16). Thus $\gamma \subset M \cap U = \{(t, 0, \ldots, 0, \sqrt{t^2+1}) \in \mathbb{R}^{n+1} | t \in \mathbb{R}\}$. As a closed subspace of $\mathbb{R}^n$, M is complete as a topological space. Hence it is geodesically complete (cf. [6, Thm. VII.2.8]), so γ has infinite length on both sides of o. Thus equality must hold: $\gamma = M \cap U$. $\square$

2.4. *Remark.* This lemma cannot hold when M is a general hyperbolic space, because then the intersection of M and some 2-dimensional subspace of $\mathbb{R}^{n+1}$ can have more than one component (consider e.g. $H^{1,1}_{\pm 1}$). The proof fails in the general case because $H \simeq O_e(n)$ was invoked.

2.5. **Definition.** For each $m \in M$ put $G_m = \{g \in G | g \cdot m = m\}$.

2.6. **Lemma.** *The group G acts transitively on the set*

$$\{\operatorname{Exp}_m(V) | m \in M, V \text{ is a } k\text{-dimensional subspace of } T_m(M)\}.$$

Furthermore, for each $m \in M$, the group G_m acts transitively on the set

$$\{\operatorname{Exp}_m(V) | V \text{ is a } k\text{-dimensional subspace of } T_m(M)\}.$$

Proof. Let $m_1, m_2 \in M$ and V_1, V_2 k-dimensional subspaces of $T_{m_1}(M)$ respectively $T_{m_2}(M)$ be given. Pick $g_1, g_2 \in G$ such that $g_i \cdot o = m_i$ for $i = 1, 2$. Then

$$g_i^{-1} \cdot \operatorname{Exp}_{m_i}(V_i) = \operatorname{Exp}_o(dl_{g_i^{-1}}(V_i)), \qquad i = 1, 2.$$

The subgroup $H = O_e(n)$ of G (cf. Lemma 1.7) acts transitively on the set of k-dimensional subspaces of $T_o(M)$, so it is possible to pick $h \in H$ such that

$$dl_h(dl_{g_1^{-1}}(V_1)) = dl_{g_2^{-1}}(V_2).$$

Now $g_2 h g_1^{-1} \mathrm{Exp}_{m_1}(V_1) = \mathrm{Exp}_{m_2}(V_2)$.

If $m_1 = m_2$ let (with notation as before) $g_1 = g = g_2$. Then $g_2 h g_1^{-1} = ghg^{-1} \in gHg^{-1} = G_m$. $\square$

2.7. **Lemma.** *Let $m \in M$ and V a k-dimensional subspace of $T_m(M)$ be given. Put $\xi = \mathrm{Exp}_m(V)$. Then there exists an isomorphism between $G' = O_e(k,1)$ and a subgroup of G such that*

$$\xi \simeq \mathbb{H}^k \qquad (G'\text{-invariant isometry})$$

when the submanifold ξ of M is given the Riemannian structure induced from M, and when G' is considered as a subgroup of G through this isomorphism. In particular G' acts transitively on ξ.

Proof. Because G is a group of isometries of M which acts transitively on $\{\mathrm{Exp}_m(V) | m \in M, V \text{ is a } k\text{-dimensional subspace of } T_m(M)\}$ according to Lemma 2.6, it can be assumed that $m = o$ and $V = \{x \in \mathbb{R}^{n+1} | x_1 = \ldots = x_{n-k} = x_{n+1} = 0\}$. But according to Lemma 2.3, the geodesics of M through o whose tangent vector at o lies in V are exactly the intersections of M with subspaces of the form span$\{o, v\}$, $v \in V$. Thus

$$\begin{aligned} \xi &= M \cap (\cup_{v \in V} \mathrm{span}\,\{o, v\}) \\ &= M \cap \{x \in \mathbb{R}^{n+1} | x_1 = \ldots = x_{n-k} = 0\} \\ &= \{x \in \mathbb{R}^{n+1} | x_1 = \ldots = x_{n-k} = 0 \\ &\qquad\qquad \text{and } {x_{n+1-k}}^2 + \ldots + {x_n}^2 - {x_{n+1}}^2 = -1\} \end{aligned}$$

which is diffeomorphic to $\mathbb{H}^k$ by the projection

$$\xi \ni (x_1, \ldots, x_{n+1}) \overset{\Pi}{\mapsto} (x_{n+1-k}, \ldots, x_{n+1}) \in \mathbb{H}^k.$$

Consider $G' = O_e(k,1)$ as a subgroup of G by

$$G' \ni g' \overset{\iota}{\mapsto} \left(\begin{array}{c|c} I_{n-k} & 0 \\ \hline 0 & g' \end{array} \right) \in G.$$

That Π is G'-linear is seen through explicit calculations verifying $g' \cdot \Pi(m) = \Pi(\iota(g') \cdot m)$ for all $g' \in G'$ and $m \in \xi$. Further calculations show that the induced Riemannian structure on ξ equals

$$(X, Y) \mapsto (d\Pi(X))^t I_{k,1} (d\Pi(Y))$$

on $T_o(\xi)$. This corresponds to the Riemannian structure of $\mathbb{H}^k$ on $T_o(\mathbb{H}^k)$ which is $(X', Y') \mapsto (X')^t I_{k,1} Y$. But since Π is a G'-linear bijection, G' acts transitively not only on $\mathbb{H}^k$ but also on ξ, in both cases as an isometry. Thus it follows that Π is an isometry. $\square$

2.8. **Lemma.** *The space of k-dimensional totally geodesic submanifolds of M is given by*

$$\Xi_k = \{\mathrm{Exp}_m(V) | m \in M, V \text{ is a } k\text{-dimensional subspace of } T_m(M)\}.$$

Proof. Let $\xi \in \Xi_k$ be given. Pick $m \in \xi$, and put $V = T_m(\xi) \subset T_m(M)$. Then $\xi \supset \mathrm{Exp}_m(V)$, since ξ is a totally geodesic submanifold of M. That ξ is a totally geodesic submanifold of M also implies that any curve in ξ which is an M-geodesic, is a ξ-geodesic, and that any ξ-geodesic is an M-geodesic (cf. [11, Lemmas I.14.2 and I.14.3]). Thus $\mathrm{Exp}_m(V)$ consists of all ξ-geodesics through m. Now, ξ is a complete Riemannian manifold as a totally geodesic submanifold of the complete Riemannian manifold M (cf. [11, Lemma I.14.3]). Since any two points in a complete Riemannian manifold can be joined by a geodesic (cf. [6, Thm. VII.2.8]), it follows that $\xi \subset \mathrm{Exp}_m(V)$. We have proved $\xi = \mathrm{Exp}_m(V)$.

Let $m \in M$ and V a k-dimensional subspace of $T_m(M)$ be given. Put $\xi = \mathrm{Exp}_m(V)$. Then $V = T_m(\xi)$. It is to be proved that ξ is totally geodesic. Let γ be a geodesic of M tangential to ξ at $m' \in \xi$. According to Lemma 2.7 there exists a subgroup G' of G that acts transitively on ξ. Pick $g \in G'$ such that $g \cdot m = m'$. Then $\xi = g \cdot \xi$, since G' acts on ξ, so $\xi = g \cdot \mathrm{Exp}_m(V) = \mathrm{Exp}_{m'}(dl_g V) \supset \gamma$. $\square$

2.9. **Definition.** For each $m \in M$ put $\Xi(m) = \{\xi \in \Xi | m \in \xi\}$.

2.10. **Corollary.** *The group G acts transitively on Ξ. Furthermore, for each $m \in M$, the group G_m acts transitively on $\Xi(m)$.*

Proof. Combine the previous lemma with Lemma 2.6. $\square$

3. Integration on Totally Geodesic Submanifolds of $\mathbb{H}^n$

From e.g. [11, p.215] it is known that

3.1. **Proposition.** *The differential of* $\mathrm{Exp}_o : T_o(M) \to M$ *at* $X \in T_o(M)$ *is*

$$(d\mathrm{Exp}_o)_X = (dl_{\exp X})_o \circ \sum_{n=0}^{\infty} \frac{\mathcal{T}_X^n}{(2n+1)!}.$$

Here the identifications $T_X(T_o(M)) \simeq T_o(M) \simeq \mathfrak{m}$ (Remark 1.13) is used to define $\exp X$*, and to define $\mathcal{T}_X : T_X(T_o(M)) \to T_o(M)$ by*

$$\mathcal{T}_X(Y) = [X, [X, Y]].$$

3.2. *Remark.* It is worthwhile to note that

$$\mathrm{Exp}_o((d\varphi)_e(X)) = \exp(X) \cdot o$$

for all $X \in \mathfrak{m}$ (see e.g. [11, p.208]), where as previously $\varphi : G \to M$ is given by $\varphi(g) = g \cdot o$.

3.3. *Remark.* Let $h \in H$, $X \in \mathfrak{m}$ and $f \in C^\infty(M)$ be given. Then

$$\begin{aligned}(dl_h)_o((d\varphi)_e X)(f) &= \frac{d}{dt}\Big|_0 f(h\,\exp(tX)\cdot o)\\ &= \frac{d}{dt}\Big|_0 f(\exp(tAd(h)X)\cdot o) = (d\varphi)_e(Ad(h)X)(f),\end{aligned}$$

i.e. $h((d\varphi)_e X) = (d\varphi)_e(Ad(h)X)$ (see Remark 1.9).

3.4. **Lemma.** *Let $m \in M$ and $X \in T_m(M)$ be given. Then*

$$\det(d\mathrm{Exp}_m)_X = \left(\frac{\sinh|X|}{|X|}\right)^{n-1}, \qquad |X| = \sqrt{g_m(X,X)}.$$

Proof. It will be enough to consider the case $m = o$, since $d\mathrm{Exp}_{g\cdot o} = dl_g\, d\mathrm{Exp}_o\, d(dl_{g^{-1}})$, where $\det d(dl_{g^{-1}}) = \det dl_{g^{-1}} = 1$ and $\det dl_g = 1$, because $l_{g^{-1}}$ and l_g are isometries.

Define the linear map $\mathcal{A}_X$ on $T_o(M)$ by

$$\mathcal{A}_X = \sum_{n=0}^{\infty} \frac{\mathcal{T}_X^n}{(2n+1)!},$$

where $\mathcal{T}_X(Y) = [X,[X,Y]]$ using the identification $T_o(M) \simeq \mathfrak{m}$ (Remark 1.13). Then $\det(d\mathrm{Exp}_o)_X = \det(dl_{\exp(X)})_o \det \mathcal{A}_X = \det \mathcal{A}_X$ according to Lemma 3.1 (again since $l_{\exp(X)}$ is an isometry). Put $c = \sqrt{g_o(X,X)}$. For each $h \in H$ (cf. Remark 3.3)

$$\begin{aligned}\mathcal{T}_{hX}(Y) &= [Ad(h)X,[Ad(h)X,Y]]\\ &= Ad(h)[X,[X,Ad(h^{-1})Y]] = Ad(h)\mathcal{T}_X Ad(h^{-1})(Y).\end{aligned}$$

Thus $\det \mathcal{A}_X = \det \mathcal{A}_{hX}$ for each $h \in H$, and therefore, since $H \simeq O_e(n)$ acts transitively on $S_c^{n-1} = \{x \in \mathbb{R}^n \,|\, |x| = c\} \subset T_o(M)$, it suffices to consider $det\mathcal{A}_{cY_1}$, where Y_1 stems from the basis $\{Y_i\}_i$ from Lemma 1.14. But calculations show that

$$\mathcal{T}_{cY_1}(Y_i) = \begin{cases} 0 & i = 1\\ c^2 Y_i & i \neq 1\end{cases},$$

so the linear map $\mathcal{T}_{cY_1}$ on $T_o(M)$ has the eigenvalues 0 of multiplicity 1 and c^2 of multiplicity $n-1$. Thus $\det \mathcal{A}_{cY_1}$ is equal to the determinant of a diagonal matrix with $\sum_{n=0}^{\infty} \frac{0^n}{(2n+1)!} = 1$ as one diagonal entry and $\sum_{n=0}^{\infty} \frac{c^{2n}}{(2n+1)!} = \frac{\sinh c}{c}$ as the other $n-1$ diagonal entries. $\square$

3.5. *Remark.* Let $m \in M$ be given. Since M is a complete Riemannian manifold any two points can be joined by a geodesic (cf. [6, Thm. VII.2.8]), so $\mathrm{Exp}_m : T_m(M) \to M$ is surjective. According to Lemma 2.3 it is also injective. Thus from Lemma 3.4 and the inverse function theorem, Exp_m is a diffeomorphism.

3.6. *Remark.* Since Exp_m is a diffeomorphism according to Remark 3.5, it is seen by Lemma 3.4 and transition to polar coordinates that integration of $f \in L^1(M)$ with respect to the Riemannian measure on M can be expressed by

$$\begin{aligned}\int_M f\,dm &= \int_{\mathbb{R}^n} f\circ \operatorname{Exp}_m(x)\left(\frac{\sinh|x|}{|x|}\right)^{n-1} dx \\ &= \int_0^\infty \int_{S^{n-1}} f\circ \operatorname{Exp}_m(r\omega)\,d\omega\,\sinh(r)^{n-1}\,dr\end{aligned}$$

for any $m \in M$.

3.7. **Lemma.** *Integration with respect to the Riemannian measure dm on $\xi \in \Xi_k$ is given by*

$$\int_\xi f\,dm = \int_{\mathbb{R}^k} f\circ \operatorname{Exp}_m(x)\left(\frac{\sinh|x|}{|x|}\right)^{k-1} dx, \quad f \in L^1(\xi), \tag{7}$$

for any $m \in \xi$.

Proof. Integration with respect to the Riemannian measure on $\mathbb{H}^k$ can be expressed by the right hand side of (7) according to Remark 3.6. But $\xi \simeq \mathbb{H}^k$ (isometry) according to Lemmas 2.7 and 2.8, so the lemma follows from the transformation theorem for integration on Riemannian spaces (cf. [13, Thm. I.1.3]). □

3.8. *Remark.* A Riemannian measure is invariant under isometries, so for any $\xi \in \Xi_k$ the map $L^1(\xi) \ni f \mapsto \int_\xi f\,dm$ is invariant under the subgroup of G which preserves ξ. In particular $L^1(M) \ni f \mapsto \int_M f\,dm$ is G-invariant.

4. A Radon Transform on $\mathbb{H}^n$

Let d denote the geodesic distance on M. Then d is continuous on $M \times M$ because the topology induced by d is the same as the original (cf. [6, Thm.VII.2.8]). Furthermore

$$d(m, \operatorname{Exp}_m(X)) = g_m(X,X)^{\frac{1}{2}} \tag{8}$$

for all $m \in M$ and $X \in T_m(M)$, because M is a normal neighborhood of each of its points (Remark 3.5), so that all geodesics are globally minimizing. Note that d is G-invariant in the sense that

$$d(g\cdot m_1, g\cdot m_2) = d(m_1, m_2),$$

for all $g \in G$ and $m_1, m_2 \in M$, because any $g \in G$ is an isometry. Define

$$|m| := d(m, o), \qquad m \in M,$$

and let $f(m) = \mathcal{O}(g(m))$ signify the existence of constants $c, c' > 0$ such that

$$|f(m)| \leq c' \, |g(m)| \qquad \text{for} \qquad |m| > c,$$

when f and g are functions on M.

4.1. **Definition.** For each $a, b \in \mathbb{R}$ and each $m \in M$ put

$$C_{a,b}(M) = \{f \in C(M) | f(m) = \mathcal{O}(\sinh^a(d(m,o))(d(m,o))^b)\}.$$

It will be useful to note the following

4.2. **Lemma.** *Let $a, b \in \mathbb{R}$ be given. For each compact subset K of G and each $m_0 \in M$ there exists constants $c, c' > 0$ such that*

$$\sinh^a(d(k \cdot m, m_0))(d(k \cdot m, m_0))^b \leq c' \sinh^a(d(m, m_0))(d(m, m_0))^b$$

for all $k \in K$ and $m \in M$ with $|m| > c$.

Proof. It suffices to prove the existence of constants c_1, c_2 and C_1, C_2 such that

$$c_1 \, d(k \cdot m, m_0) \leq d(m, m_0) \leq c_2 \, d(k \cdot m, m_0)$$

and

$$C_1 \sinh(d(k \cdot m, m_0)) \leq \sinh(d(m, m_0)) \leq C_2 \sinh(d(k \cdot m, m_0))$$

(and such that $d(k \cdot m, m_0)$ is bounded away from 0) for all $k \in K$ and $m \in M$ with $|m|$ big enough, because then, no matter whether a and b are positive or negative, there exists $c, c' > 0$ depending on a and b such that

$$(d(k \cdot m, m_0))^b < c' \, (d(m, m_0))^b$$

and

$$\sinh^a(d(k \cdot m, m_0)) < c' \sinh^a(d(m, m_0))$$

for all $k \in K$ and $m \in M$ with $|m| > c$.

Put $s = \sup\{d(k \cdot m_0, m_0) | k \in K\} < \infty$. Use the triangle inequality and the G-invariance of d to see that

$$\begin{aligned} d(m, m_0) - s &\leq d(m, m_0) - d(k \cdot m_0, m_0) \\ &\leq d(k \cdot m, m_0) \\ &\leq d(m, m_0) + d(k \cdot m_0, m_0) \leq d(m, m_0) + s \end{aligned}$$

for all $k \in K$ and $m \in M$. Since

$$d(m, m_0) \geq d(m, o) - d(m_0, o)$$

on M, it is possible to pick $C > 0$ such that

$$d(m, m_0) - s > 0 \qquad \text{when} \qquad |m| > C.$$

It follows that

$$\frac{d(m, m_0)}{d(m, m_0) + s} \leq \frac{d(m, m_0)}{d(k \cdot m, m_0)} \leq \frac{d(m, m_0)}{d(m, m_0) - s} \tag{9}$$

and

$$\frac{\sinh(d(m,m_0))}{\sinh(d(m,m_0)+s)} \le \frac{\sinh(d(m,m_0))}{\sinh(d(k\cdot m,m_0))} \le \frac{\sinh(d(m,m_0))}{\sinh(d(m,m_0)-s)}, \tag{10}$$

for all $k \in K$ and all $m \in M$ with $|m| > C$. The left and right hand sides of (9) has limit 1 for $|m| \to \infty$. The left respectively right hand side of (10) has limit e^{-s} respectively e^s for $|m| \to \infty$. □

4.3**. Corollary.** *Given $a, b \in \mathbb{R}$ and $g \in G$ the space $C_{a,b}(M)$ is invariant under l_g, i.e. $f \in C_{a,b}(M) \Rightarrow f \circ l_g \in C_{a,b}(M)$.*

Proof. Let $f \in C_{a,b}(M)$ be given. Then

$$f \circ l_g(m) = f(g \cdot m) = \mathcal{O}(\sinh^a(d(g\cdot m, o))(d(g \cdot m, o))^b),$$

so it follows from the lemma with $K = \{g\}$ and $m_0 = o$ that

$$f \circ l_g(m) = \mathcal{O}(\sinh^a(d(m,o))(d(m,o))^b),$$

i.e. $f \circ l_g \in C_{a,b}(M)$. □

4.4**. Definition.** For each $a \in \mathbb{R}$ put

$$C_a(M) = \bigcup_{\varepsilon>0} C_{1-a,-1-\varepsilon}(M).$$

4.5*. Remark.* Given $a \in \mathbb{R}$ and $g \in G$ the space $C_a(M)$ is invariant under l_g according to Corollary 4.3.

4.6**. Definition.** Let $f \in C_k(M)$. Then the integral on the right hand side of (7) exists and is finite for each $\xi \in \Xi_k$, so $f\big|_\xi \in L^1(\xi)$ for each $\xi \in \Xi_k$. The k-dimensional totally geodesic Radon transform of $f \in C_k(M)$ is now defined to be the function $\hat{f}$ on Ξ_k given by

$$\hat{f}(\xi) = \int_\xi f\,dm.$$

5. The Dual Transform on $\mathbb{H}^n$

The group G acts transitively on Ξ (cf. Corollary 2.10). Thus, as was the case with $M \simeq G/H$, Ξ can be identified with G/L when L is the isotropy subgroup of G at some chosen $\xi_0 \in \Xi$ (let us assume that $\xi_0 \ni o$). Give Ξ the final topology induced by $gL \mapsto g \cdot \xi_0$.

5.1**. Lemma.** *Let $m \in M$ be given. Then $\Xi(m)$ is compact in Ξ, and $\Xi(m)$ is a homogeneous space under the compact group G_m (cf. Definitions 2.5 and 2.9).*

Proof. Pick $g \in G$ such that $g \cdot o = m$. Then $G_m = gHg^{-1}$, so G_m is compact. Since H acts transitively on $\Xi(o)$ and G acts transitively on Ξ (cf. Corollary 2.10), we have $\Xi(g \cdot o) = g \cdot \Xi(o) = \{gh \cdot \xi_0 | h \in H\}$. But the map $H \ni h \mapsto gh \cdot \xi_0 \in \Xi$ is continuous. Thus $\Xi(m)$ is compact.

To see that $\Xi(m)$ is a homogeneous space under G_m only the requirement that $G_m \ni g \mapsto g \cdot \xi \in \Xi(m)$ is open for each $\xi \in \Xi(m)$ needs consideration. Given $\xi \in \Xi(m)$ put $G_\xi = \{g \in G | g \cdot \xi = \xi\}$. Use the compactness of G_m to see that $G_m/G_\xi \ni gG_\xi \mapsto g \cdot \xi \in \Xi(m)$ takes closed sets to closed sets. As a bijective map it is then open. And the projection $G_m \ni g \mapsto gG_\xi \in G_m/G_\xi$ is open as well. $\square$

Similar to the Euclidean case, a transform dual to the Radon transform is introduced:

5.2. **Definition.** Let $\varphi \in C(\Xi)$. The dual transform of φ is defined by

$$\check{\varphi}(m) = \int_{\Xi(m)} \varphi(\xi)\, d\xi, \qquad m \in M,$$

where $\int_{\Xi(m)} d\xi$ is the normalized Haar integral on $\Xi(m)$ under G_m (cf. Lemma 5.1).

5.3. *Remark.* Let $\int_H dh$ be the normalized Haar integral on H. Then the dual transform can be expressed

$$\check{\varphi}(m) = \check{\varphi}(g \cdot o) = \int_H \varphi(gh \cdot \xi_0)\, dh,$$

since this indeed is a normalized G_m-invariant positive integral on $\Xi(m)$.

5.4. **Definition.** Let $f \in C(M)$. For each $m \in M$ let $\mathbb{R} \ni r \mapsto M^r f(m) \in \mathbb{C}$ denote the meanvalue function of f around m defined by

$$M^r f(m) = \frac{1}{\Omega_{n-1}} \int_{S^{n-1}} f \circ \operatorname{Exp}_m(r\omega)\, d\omega,$$

where $\int_{S^{n-1}} d\omega$ is the unique Haar integral on S^{n-1} under $O(n)$ with total mass $\Omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$.

5.5. *Remark.* Let $\int_H dh$ be the normalized Haar integral on H. For each $r \in \mathbb{R}$ let y_r be any element of M such that $d(y, o) = |r|$. Then the meanvalue function can be expressed

$$M^r f(m) = M^r f(g \cdot o) = \int_H f(gh \cdot y_r)\, dh,$$

since this indeed is a normalized $O(n)$-invariant positive integral on S^{n-1}.

In the Euclidean case an inversion formula for the Radon transform is based on the fact, that calculation of $(\hat{f})^\vee$ gives a handy result (in particular $(\hat{f})^\vee$ is well-defined). On the hyperbolic space M we have:

5.6. **Lemma.** *For each $f \in C_k(M)$, $\hat{f}$ is continuous on Ξ.*

Proof. It is enough to show that $g \mapsto \hat{f}(g \cdot \xi_0)$ is continuous on G. Let $g \in G$ be given. Since $\mathrm{Exp}_{g\cdot o}(y) = g \cdot \mathrm{Exp}_o(dl_{g^{-1}}y)$ for all $y \in T_{g\cdot o}$ where $\det(dl_{g^{-1}}) = 1$ since g^{-1} is an isometry, Lemma 3.7 combined with the transformation theorem for integration on Riemannian spaces (cf. [13, Thm. I.1.3]) shows that

$$\hat{f}(g \cdot \xi_0) = \int_{\mathbb{R}^k} f(g \cdot \mathrm{Exp}_o(y)) \left(\frac{\sinh|y|}{|y|}\right)^{k-1} dy. \tag{11}$$

Since $f \in C_k(M)$ there exists $\epsilon > 0$ and $c > 0$ such that

$$|f(g \cdot \mathrm{Exp}_o(y))| \leq c \sinh^{1-k}(d(g \cdot \mathrm{Exp}_o(y), o))(d(g \cdot \mathrm{Exp}_o(y), o))^{-1-\epsilon}$$

for all $y \in \mathbb{R}^k$ and $g \in G$ with $d(g \cdot \mathrm{Exp}_o(y), o) > c$. Thus when K is a compact subset of G there exists, according to Lemma 4.2, a $c' > 0$ independent of g in K such that

$$\begin{aligned} |f(g \cdot \mathrm{Exp}_o(y))| &\leq c' \sinh^{1-k}(d(\mathrm{Exp}_o(y), o))(d(\mathrm{Exp}_o(y), o))^{-1-\epsilon} \\ &= c' \sinh^{1-k}|y||y|^{-1-\epsilon} \end{aligned}$$

for all $y \in \mathbb{R}^k$ with $|y| = d(\mathrm{Exp}_o(y), o) > c'$ and all $g \in K$. From this and (11) in combination with Lebesgue's Dominated Convergence Theorem, the continuity of $g \mapsto \hat{f}(g \cdot \xi_0)$ follows. □

5.7. **Lemma.** *Let $f \in C_k(M)$ be given. Then*

$$(\hat{f})^{\vee}(m) = \frac{\Omega_{k-1}}{\Omega_{n-1}} \int_M f(y) \sinh^{k-n}(d(y, m))\, dy, \qquad m \in M.$$

Proof. According to the previous lemma $\hat{f} \in C(\Xi)$, so $(\hat{f})^{\vee}$ is well-defined on M. Now

$$\begin{aligned}
(\hat{f})^{\vee}(m) = (\hat{f})^{\vee}(g\cdot o) &= \int_H \hat{f}(gh\cdot\xi_0)\,dh \\
&= \int_H \int_{\xi_0} f(gh\cdot x)\,dx\,dh \\
&= \int_{\xi_0} \int_H f(gh\cdot x)\,dh\,dx \\
&= \int_{\xi_0} M^{d(x,o)} f(m)\,dx \\
&= \int_0^\infty \int_{S^{k-1}} M^{d(\mathrm{Exp}_o(r\omega),o)} f(m)\sinh^{k-1}(r)\,d\omega\,dr \\
&= \int_0^\infty \int_{S^{k-1}} M^r f(m)\sinh^{k-1}(r)\,d\omega\,dr \\
&= \Omega_{k-1}\int_0^\infty M^r f(m)\sinh^{k-1}(r)\,dr \\
&= \frac{\Omega_{k-1}}{\Omega_{n-1}}\int_0^\infty \int_{S^{n-1}} f\circ \mathrm{Exp}_m(r\omega)\,d\omega\,\sinh^{k-1}(r)\,dr \\
&= \frac{\Omega_{k-1}}{\Omega_{n-1}}\int_M f(y)\sinh^{k-n}(d(y,m))\,dy.
\end{aligned}$$

It was possible to interchange $\int_H$ and $\int_{\xi_0}$ by applying Lemma 4.2 in the same way as in the proof of Lemma 5.6. □

6. Convolution Operators

Let $\int_{G/H} dg_H$ be the Haar measure on G/H corresponding to the Riemannian measure on M:

$$\int_{G/H} f(g\cdot o)\,dg_H = \int_M f(m)\,dm, \quad f\in C_c(M).$$

Since H is compact, there exists a unique Haar measure on G, $\int_G dg$, such that

$$\int_G \eta(g)\,dg = \int_{G/H}\int_H \eta(gh)\,dh\,dg, \quad \eta\in C_c(G), \tag{12}$$

where $\int_H dh$ is the normalized Haar measure on H. The Haar measure on G will be both left and right invariant, since G is unimodular as a semisimple Lie group. Note that for $f \in C_c(M)$

$$
\begin{aligned}
\int_M f(m)\,dm &= \int_{G/H} f(g\cdot o)\,dg_H \\
&= \int_{G/H}\int_H f(gh\cdot o)\,dh\,dg_H = \int_G f(g\cdot o)\,dg. \qquad (13)
\end{aligned}
$$

Convolution of two functions η_1 and η_2 on G is defined using the measure from (12):

$$\eta_1 * \eta_2(g_0) = \int_G \eta_1(g)\eta_2(g^{-1}g_0)\,dg.$$

To see when this has meaning, the following well-known result will later be used (cf. [14, Cor. (20.14), Thm. (20.16), Thm. (20.18) and Thm. (20.2)]):

6.1. **Proposition.** *Let G be a locally compact, unimodular group. Assume that $\eta_1 \in L^p(G)$ and that $\eta_2 \in L^q(G)$ for some $1 \le p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} \ge 1$. Then the integral $\eta_1 * \eta_2(g)$ exists and is finite for a.a. $g \in G$, and $\eta_1 * \eta_2 \in L^r(G)$ where $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$ (if $\frac{1}{p} + \frac{1}{q} = 1$ then $r = \infty$).*

Convolution of two functions f_1 and f_2 on M is defined using the convolution on G:

$$f_1 * f_2(g_0\cdot o) = (f_1\circ\varphi) * (f_2\circ\varphi)(g_0) = \int_G f_1(g\cdot o) f_2(g^{-1}g_0\cdot o)\,dg, \qquad (14)$$

where as previously $\varphi : G \to M$ is given by $\varphi(g) = g\cdot o$.

6.2. **Definition.** For each $\alpha \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < n$ put

$$k_+^\alpha(m) = \frac{1}{H_n(\alpha)} \sinh^{\alpha-n}(d(o,m))$$

and

$$k_-^\alpha(m) = \frac{1}{H_n(\alpha)} \sinh^{\alpha-n}(d(o,m))\cosh(d(o,m)),$$

where

$$H_n(\alpha) = 2^\alpha \pi^{\frac{n}{2}} \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

6.3. *Remark.* The convolution of a function f on M with $k_\pm^\alpha$ is expressible as an integral on M using the G-invariance of d:

$$
\begin{aligned}
f * k_+^\alpha(m) &= \frac{1}{H_n(\alpha)} \int_G f(g\cdot o)\sinh^{\alpha-n}(d(g\cdot o, m))\,dg \\
&= \frac{1}{H_n(\alpha)} \int_M f(y)\sinh^{\alpha-n}(d(y,m))\,dy,
\end{aligned}
$$

(cf. (13)). Similarly

$$f * k_-^\alpha(m) = \frac{1}{H_n(\alpha)} \int_M f(y) \sinh^{\alpha-n}(d(y,m)) \cosh(d(y,m))\, dy.$$

6.4. **Lemma.** *Let $\alpha \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < n$ be given.*
If $f \in L^p$ for some $p \in \mathbb{R}_+$ with $\frac{\operatorname{Re}\alpha - 1}{n-1} < \frac{1}{p} \le 1$, then the integral $f * k_+^\alpha(m)$ exists and is finite for a.a. $m \in M$. In fact $f * k_+^\alpha \in L^r(M)$ for all $r \in \mathbb{R}_+$ with $\frac{1}{p} - \frac{\operatorname{Re}\alpha}{n} < \frac{1}{r} < \frac{1}{p} - \frac{\operatorname{Re}\alpha - 1}{n-1}$ and $\frac{1}{r} \le \frac{1}{p}$.
If $f \in L^p$ for some $p \in \mathbb{R}_+$ with $\frac{\operatorname{Re}\alpha}{n-1} < \frac{1}{p} \le 1$, then the integral $f * k_-^\alpha(m)$ exists and is finite for a.a. $m \in M$. In fact $f * k_-^\alpha = a^\alpha + b^\alpha$, where $a^\alpha \in L^r(M)$ for all $r \in \mathbb{R}_+$ with $\frac{1}{p} - \frac{\operatorname{Re}\alpha}{n} < \frac{1}{r} \le \frac{1}{p}$, and $b^\alpha \in L^s(M)$ for all $s \in \mathbb{R}_+$ with $\frac{1}{s} < \frac{1}{p} - \frac{\operatorname{Re}\alpha}{n-1}$.
If $0 < \operatorname{Re}\alpha < \frac{n}{2}$, then the integral $k_+^\alpha * k_-^\alpha(m)$ exists and is finite for a.a. $m \in M$.

Proof. Write

$$k_\pm^\alpha = \varphi_\pm^\alpha + \psi_\pm^\alpha,$$

where

$$\varphi_\pm^\alpha(m) = 1_{[0,1]}(d(o,m)) k_\pm^\alpha(m).$$

By Remark 3.6 combined with (8), a counting of powers of sinh and cosh shows that

$$\text{(15)} \qquad \varphi_\pm^\alpha \in L^q(M) \Leftrightarrow q(\operatorname{Re}\alpha - n) + (n-1) > -1 \Leftrightarrow \frac{1}{q} > \frac{n - \operatorname{Re}\alpha}{n},$$

and

$$\text{(16)} \qquad \psi_+^\alpha \in L^q(M) \Leftrightarrow q(\operatorname{Re}\alpha - n) + (n-1) < 0 \Leftrightarrow \frac{1}{q} < \frac{n - \operatorname{Re}\alpha}{n-1},$$

and

$$\text{(17)} \quad \psi_-^\alpha \in L^q(M) \Leftrightarrow q(\operatorname{Re}\alpha + 1 - n) + (n-1) < 0 \Leftrightarrow \frac{1}{q} < \frac{n - 1 - \operatorname{Re}\alpha}{n-1}.$$

Assume that $f \in L^p(M)$ for some $p \in \mathbb{R}_+$. Apply Proposition 6.1 to obtain the following:

Because of (15) the integral $f * \varphi_\pm^\alpha(m)$ exists and is finite for a.a. $m \in M$ if $p \ge 1$ and $\frac{1}{p} + \frac{1}{q} \ge 1$ for some $q \in \mathbb{R}_+$ such that $\frac{n - \operatorname{Re}\alpha}{n} < \frac{1}{q} \le 1$; that is if

$$\frac{1}{p} \le 1.$$

Then $f * \varphi_\pm^\alpha \in L^r(M)$ for all $r \in \mathbb{R}_+$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$ for some $q \in \mathbb{R}_+$ such that $\frac{n - \operatorname{Re}\alpha}{n} < \frac{1}{q} \le 1$; that is for r such that

$$\frac{1}{p} - \frac{\operatorname{Re}\alpha}{n} < \frac{1}{r} \le \frac{1}{p}.$$

Because of (16) the integral $f*\psi_+^\alpha(m)$ exists and is finite for a.a. $m \in M$ if $p \geq 1$ and $\frac{1}{p}+\frac{1}{q} \geq 1$ for some $q \in \mathbb{R}_+$ such that $\frac{1}{q} \leq 1$ and $\frac{1}{q} < \frac{n-\operatorname{Re}\alpha}{n-1}$; that is if

$$\frac{\operatorname{Re}\alpha - 1}{n-1} < \frac{1}{p} \leq 1.$$

Then $f*\psi_+^\alpha \in L^r(M)$ for all $r \in \mathbb{R}_+$ such that $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}-1$ for some $q \in \mathbb{R}_+$ such that $\frac{1}{q} \leq 1$ and $\frac{1}{q} < \frac{n-\operatorname{Re}\alpha}{n-1}$; that is for r such that

$$\frac{1}{r} \leq \frac{1}{p} \qquad \text{and} \qquad \frac{1}{r} < \frac{1}{p} - \frac{\operatorname{Re}\alpha - 1}{n-1}.$$

Because of (17) the integral $f*\psi_-^\alpha(m)$ exists and is finite for a.a. $m \in M$ if $p \geq 1$ and $\frac{1}{p}+\frac{1}{q} \geq 1$ for some $q \in \mathbb{R}_+$ such that $\frac{1}{q} < \frac{n-1-\operatorname{Re}\alpha}{n-1}$; that is if

$$\frac{\operatorname{Re}\alpha}{n-1} < \frac{1}{p} \leq 1.$$

Then $f*\psi_-^\alpha \in L^r(M)$ for all $r \in \mathbb{R}_+$ such that $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}-1$ for some $q \in \mathbb{R}_+$ such that $\frac{1}{q} < \frac{n-1-\operatorname{Re}\alpha}{n-1}$; that is for r such that

$$\frac{1}{r} < \frac{1}{p} - \frac{\operatorname{Re}\alpha}{n-1}.$$

For the last part of the lemma note that according to (15), φ_+^α and φ_-^α are L^1-functions on M when $\operatorname{Re}\alpha < n$, so that from Proposition 6.1

$$\varphi_+^\alpha * \varphi_-^\alpha \text{ exists and is finit for a.a. } m \in M \text{ when } \operatorname{Re}\alpha < n,$$

and so that, when combining Proposition 6.1 with (16) and (17),

$$\varphi_+^\alpha * \psi_-^\alpha \text{ exists and is finit for a.a. } m \in M \text{ when } \operatorname{Re}\alpha < n-1,$$

and

$$\psi_+^\alpha * \varphi_-^\alpha \text{ exists and is finit for a.a. } m \in M \text{ when } \operatorname{Re}\alpha < n.$$

Finally

$$\psi_+^\alpha * \psi_-^\alpha \text{ exists and is finit for a.a. } m \in M \text{ when } \operatorname{Re}\alpha < \frac{n}{2},$$

which again follows from Proposition 6.1, (16) and (17) since

$$\frac{n-\operatorname{Re}\alpha}{n-1} + \frac{n-1-\operatorname{Re}\alpha}{n-1} > 1 \Leftrightarrow \operatorname{Re}\alpha < \frac{n}{2}.$$

□

6.5. *Remark.* Let $f \in C_a(M)$ for some $a \in \mathbb{R}$ be given. If $\alpha \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < n$, the integral $f*k_+^\alpha(m)$ exists and is finite for all $m \in M$ if $\operatorname{Re}\alpha \leq a$, because then

$$y \mapsto f(y)\sinh^{\alpha-n}(d(y,m))$$

is integrable on M for all $m \in M$ (cf. Remarks 6.3 and 3.6). In fact, counting powers of sinh we see that f will be in $L^p(M)$ if $p(1-a)+(n-1) \leq 0$ that is if $\frac{1}{p} \leq \frac{a-1}{n-1}$. Thus the previous lemma gives information on the integrability of $f * k_+^\alpha$ when $\operatorname{Re}\alpha < a$.

Similarly, the integral $f * k_-^\alpha(m)$ exists and is finite for all $m \in M$ if $\operatorname{Re}\alpha \leq a-1$, and the previous lemma gives information on the integrability of $f * k_-^\alpha$ when $\operatorname{Re}\alpha < a-1$.

Using Lemma 6.4 and Remark 6.5 the definition of the following convolution operators (cf. [26]) now makes sense (see Remark 6.3):

6.6. **Definition.** Let $\alpha \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < n$ be given.
For each $f \in L^p(M)$, where $p \in \mathbb{R}_+$ with $\frac{\operatorname{Re}\alpha-1}{n-1} < \frac{1}{p} \leq 1$, or $f \in C_a(M)$ for some $a \geq \operatorname{Re}\alpha$, define

$$(K^\alpha f)(m) = f * k_+^\alpha(m) = \frac{1}{H_n(\alpha)} \int_M f(y) \sinh^{\alpha-n}(d(y,m))\, dy,$$

the integral existing and being finite for a.a. $m \in M$ if $f \in L^p$ and everywhere if $f \in C_a$.
For each $f \in L^p(M)$, where $p \in \mathbb{R}_+$ with $\frac{\operatorname{Re}\alpha}{n-1} < \frac{1}{p} \leq 1$, or $f \in C_a(M)$ for some $a-1 \geq \operatorname{Re}\alpha$, define

$$(K_-^\alpha f)(m) = f * k_-^\alpha(m) = \frac{1}{H_n(\alpha)} \int_M f(y) \sinh^{\alpha-n}(d(y,m)) \cosh(d(y,m))\, dy,$$

the integral existing and being finite for a.a. $m \in M$ if $f \in L^p$ and everywhere if $f \in C_a$.

7. Some Results Involving K^α and K_-^α

We will need a series of lemmas concerning the convolution operators of Definition 6.6.

7.1. **Lemma.** *Assume that $f \in C_a(M)$ for some $0 < a < n$. Let $m \in M$ be given. Then the map*

$$\alpha \mapsto K^\alpha f(m)$$

is well-defined and continuous on $\{\alpha \in \mathbb{C} | 0 < \operatorname{Re}\alpha < a\}$.

Proof. Note that $\alpha \mapsto K^\alpha f(m)$ is indeed well-defined on $\{\alpha \in \mathbb{C} | 0 < \operatorname{Re}\alpha < a\}$ according to Remark 6.5. From Remark 3.6 and Definition 5.4

$$(K^s f)(m) = \frac{\Gamma(s)}{H_n(s)} \Omega_{n-1} \frac{1}{\Gamma(s)} \int_0^\infty F_m(r,s) r^{s-1}\, dr, \tag{18}$$

where

$$F_m(r,s) = (M^r f)(m) \left(\frac{\sinh(r)}{r}\right)^{s-1}.$$

A standard application of Lebesgue's Dominated Convergence Theorem proves the continuity. □

7.2. **Lemma.** *Assume that $f \in C_a(M)$ for some $a > 0$. Then*
$$\lim_{s \to 0_+} (K^s f) = f$$
pointwise on M.

Proof. Let $m \in M$ and $s > 0$ be given. Rewrite $(K^s f)(m)$ as in (18), and note that
$$\frac{\Gamma(s)}{H_n(s)} = \frac{(s-0)\Gamma(s)\Gamma(\frac{n-s}{2})}{2(\frac{s}{2}-0)2^s \pi^{\frac{n}{2}}\Gamma(\frac{s}{2})} \to \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}} = \frac{1}{\Omega_{n-1}} \text{ for } s \to 0.$$
Let $\varepsilon > 0$ so small that $f \in C_{1-a,-1-\varepsilon}(M)$ as defined in 4.1 be given. Since $(r,s) \mapsto F_m(r,s)$ is continuous on $\mathbb{R}^2$ it is possible to pick $\delta \in]0,1[$ such that
$$|F_m(r,s) - F_m(0,0)| < \varepsilon$$
when $|r|, |s| < \delta$. Now for $0 < s$ sufficiently small (at least so small that $s < \min\{\delta, a\}$),
$$|\frac{1}{\Gamma(s)} \int_0^\delta (F_m(r,s) - F_m(0,0)) r^{s-1}\, dr| \le \frac{\varepsilon}{\Gamma(s+1)}\delta^s < 2\varepsilon,$$
and
$$|\frac{1}{\Gamma(s)} \int_\delta^\infty F_m(r,s) r^{s-1}\, dr| \le \frac{c}{\Gamma(s)} |\int_\delta^\infty r^{-1-\varepsilon}\, dr| = \frac{c}{\Gamma(s)\varepsilon\delta^\varepsilon} < \varepsilon,$$
where c is a constant independent of δ, ε and s. It was used here that $f \in C_{1-a,-1-\varepsilon}(M)$ so that
$$F_m(r,s) = \mathcal{O}(\sinh^{s-a}(r) r^{-s-\varepsilon}).$$
Finally
$$|(\frac{1}{\Gamma(s)} \int_0^\delta r^{s-1}\, dr - 1) F_m(0,0)| = |\frac{\delta^s}{\Gamma(s+1)} - 1||F_m(0,0)| < \varepsilon$$
for s sufficiently small. Thus
$$|\frac{1}{\Gamma(s)} \int_0^\infty F_m(r,s) r^{s-1}\, dr - F_m(0,0)| < 4\varepsilon$$
for s sufficiently small. □

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ (over $\mathbb{R}$). Put $\mathcal{U}_l(\mathfrak{g}) = \{u \in \mathcal{U}(\mathfrak{g}) | \text{order of } u \le l\}$. Define the representation $X \mapsto L_X$ of $\mathfrak{g}$ on $C^1(M)$ by
$$(L_X f)(m) = \frac{d}{dt}\Big|_0 f(\exp(-tX) \cdot m).$$
This representation generates representations of $\mathcal{U}_l(\mathfrak{g})$ on $C^l(M)$ for each $l \in \mathbb{N}$, $\mathcal{U}_l(\mathfrak{g}) \ni u \mapsto L_u$, such that when $\Omega \in \mathcal{U}(\mathfrak{g})$ is the Casimir element

then L_Ω is the Laplacian on M (times a non-zero constant) (cf. [13, Chap. II]).

7.3. **Lemma.** *Let $l \in \mathbb{N}$ be given. Let $\alpha \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < n$ and $f \in C^l(M)$ be given such that*

$$\forall u \in \mathcal{U}_l(\mathfrak{g}) : L_u f \in C_{\operatorname{Re}\alpha}(M). \tag{19}$$

Then $K^\alpha f$ is in $C^l(M)$ and

$$\forall u \in \mathcal{U}_l(\mathfrak{g}) : L_u(K^\alpha f) = K^\alpha(L_u f).$$

Proof. Note that $u = 1$ in (19) makes $K^\alpha f$ well-defined. Furthermore, by induction, it is enough to prove the lemma for $l = 1$. Let $u \in \mathcal{U}(\mathfrak{g})$ of order 1 be given. Then $u = X \in \mathfrak{g}$ and

$$(L_u f)(m) = \frac{d}{dt}\Big|_0 f(\exp(-tX)\cdot m), \quad m \in M.$$

We will prove that for each $m \in M$ integration and differentiation in the following expression (20) can be interchanged, because then it follows from the G-invariance of d and of the integral on M that $K^\alpha f$ is in $C^1(M)$ with

$$\begin{aligned} &(L_u(K^\alpha f))(m) \\ = \;& \frac{d}{dt}\Big|_0 \int_M f(\exp(-tX)\cdot y)\sinh^{\alpha-n}(d(y,m))\,dy \\ = \;& \int_M \frac{d}{dt}\Big|_0 f(\exp(-tX)\cdot y)\sinh^{\alpha-n}(d(y,m))\,dy \\ = \;& (K^\alpha(L_u f))(m) \end{aligned} \tag{20}$$

Let $m \in M$ be given. For any $t_0 \in \mathbb{R}$

$$\frac{d}{dt}\Big|_{t_0} f(\exp(-tX)\cdot m) = \frac{d}{dt}\Big|_0 f(\exp(-(t+t_0)X)\cdot m) = (L_u f)(\exp(-t_0X)\cdot m).$$

Therefore, with $g_t = \exp(-tX)$, it follows from (19) that

$$\frac{d}{dt} f(\exp(-tX)\cdot m) = \mathcal{O}(\sinh^{1-\operatorname{Re}\alpha}(d(g_t\cdot m, o))(d(g_t\cdot m, o)^{-1-\varepsilon})$$

for some $\varepsilon > 0$. Thus, according to Lemma 4.2, there exists $c > 0$ such that

$$|\frac{d}{dt} f(\exp(-tX)\cdot m)| \le c\,\sinh^{1-\operatorname{Re}\alpha}(d(m,o))(d(m,o))^{-1-\varepsilon}$$

for all t in some compact neighborhood of 0. Hence

$$m \mapsto \frac{d}{dt} f(\exp(-tX)\cdot m)\sinh^{\alpha-n}(d(y,m))$$

is dominated by an integrable function on M, which is independent of t in some neighborhood of 0. $\square$

7.4. **Lemma.** *Let $l \in \mathbb{N}$ and $f \in C^l(M)$ be given. Then*

$$\forall u \in \mathcal{U}_l(M) : M^r(L_u f) = L_u(M^r f).$$

Proof. By induction it is enough to prove the lemma for $l = 1$. Let $u \in \mathcal{U}_1(\mathfrak{g})$ be given. Then $u = X \in \mathfrak{g}$ so for each $m \in M$

$$\begin{aligned}
M^r(L_u f)(m) &= \frac{1}{\Omega_{n-1}} \int_{S^{n-1}} (L_u f) \circ \mathrm{Exp}_m(r\omega)\, d\omega \\
&= \frac{1}{\Omega_{n-1}} \int_{S^{n-1}} \frac{d}{dt}\Big|_0 f(\exp(-tX) \cdot \mathrm{Exp}_m(r\omega))\, d\omega \\
&= \frac{d}{dt}\Big|_0 \frac{1}{\Omega_{n-1}} \int_{S^{n-1}} f(\mathrm{Exp}_{\exp(-tX)\cdot m}(dl_{\exp(-tX)}(r\omega))\, d\omega \\
&\overset{(\star)}{=} \frac{d}{dt}\Big|_0 \frac{1}{\Omega_{n-1}} \int_{S^{n-1}} f(\mathrm{Exp}_{\exp(-tX)\cdot m}(r\omega))\, d\omega \\
&= \frac{d}{dt}\Big|_0 (M^r f)(\exp(-tX) \cdot m) \\
&= (L_u(M^r f))(x).
\end{aligned}$$

In the transformation $(\star)$ it was used, that $dl_{\exp(-tX)}$ is an isometry of the unit sphere of $T_m(M)$ on the unit sphere of $T_{\exp(-tX)\cdot m}(M)$, because $l_{\exp(-tX)}$ is an isometry of M. □

7.5. **Lemma.** *Let $l \in \mathbb{N}$ and $a \in \mathbb{R}$ be given. Assume of $f \in C^l(M)$ that*

$$\forall u \in \mathcal{U}_l(\mathfrak{g}) : L_u f \in C_a(M). \tag{21}$$

Given $m \in M$ we will then have that

$$\frac{d^l}{dr^l} M^r f(m) = \mathcal{O}(\sinh^{1-a}(r) r^{-1-\varepsilon})$$

for some $\varepsilon > 0$.

Proof. Pick $g \in G$ such that $g \cdot o = m$. Then

$$\begin{aligned}
M^r f(m) &= \frac{1}{\Omega_{n-1}} \int_{S^{n-1}} f \circ \mathrm{Exp}_m(r\omega)\, d\omega \\
&= \frac{1}{\Omega_{n-1}} \int_{S^{n-1}} f(g\mathrm{Exp}_o(r dl_{g^{-1}}\omega))\, d\omega = M^r(f \circ l_g(o)),
\end{aligned}$$

since $dl_{g^{-1}}$ is an isometry of the unit sphere in $T_m(M)$ on the unit sphere in $T_o(M)$, because $l_{g^{-1}}$ is an isometry of M. Furthermore, for $u \in \mathcal{U}_l(\mathfrak{g})$

$$L_u(f \circ l_g) = (L_{Ad(g)u} f) \circ l_g,$$

since when e.g. $u \in \mathcal{U}_1(\mathfrak{g})$, i.e. $u = X \in \mathfrak{g}$, then

$$\begin{aligned}
(L_u(f \circ l_g))(m) &= \frac{d}{dt}\Big|_0 f(g \exp(-tX) \cdot m) \\
&= \frac{d}{dt}\Big|_0 f(g \exp(-tAd(g)X) g \cdot m) = ((L_{Ad(g)u} f) \circ l_g)(m)
\end{aligned}$$

for each $m \in M$. Since $C_a(M)$ is invariant under l_g (cf. Corollary 4.3), we may thus assume that $m = o$.

Note that

$$\frac{d^l}{dr^l} M^r f(o) = \frac{1}{\Omega_{n-1}} \int_{S^{n-1}} \frac{d^l}{dr^l} f \circ \mathrm{Exp}_o(r\omega)\, d\omega.$$

Let $\omega \in S^{n-1} \subset T_o(M)$ be given. Using Remark 3.2 and, for notational convenience, not distinguishing between ω and $(d\varphi_e)^{-1}\omega$, we see that for any $r_0 \in \mathbb{R}$

$$\begin{aligned} \frac{d^l}{dr^l}\Big|_{r_0} f \circ \mathrm{Exp}_o(r\omega) &= \frac{d^l}{dr^l}\Big|_0 f(\exp((r + r_0)\omega) \cdot o) \\ &= \frac{d^l}{dr^l}\Big|_0 f(\exp(-r(-\omega)) \exp(r_0\omega) \cdot o) \\ &= (L_Y f)(\exp(r_0\omega) \cdot o), \end{aligned}$$

where $Y = (-1)^l \omega \cdot \ldots \cdot \omega$ (ω appearing l times). Therefore it follows from (21) that for some $\varepsilon > 0$

$$\begin{aligned} \frac{d^l}{dr^l} f \circ \mathrm{Exp}_o(r\omega) &= \mathcal{O}(\sinh^{1-a}(d(\exp(r\omega) \cdot o, o))(d(\exp(r\omega) \cdot o, o))^{-1-\varepsilon}) \\ &= \mathcal{O}(\sinh^{1-a}(|r|)|r|^{-1-\varepsilon}), \end{aligned}$$

since (cf. Remark 3.2 and (8))

$$d(\exp(r\omega) \cdot o, o) = d(\mathrm{Exp}_o(r\omega), o) = |r\omega| = |r|.$$

□

7.6. **Proposition.** *Let $\alpha \in \mathbb{C}$ with $2 < \mathrm{Re}\,\alpha$ and $f \in C^2(M)$ be given such that*

$$\forall u \in \mathcal{U}_2(\mathfrak{g}) : L_u f \in C_{\mathrm{Re}\,\alpha}(M). \tag{22}$$

Then

$$((\alpha - n)(\alpha - 1) - \Delta) K^\alpha f = K^{\alpha-2} f.$$

Proof. Note, that (22) for $u = 1$ makes $K^\alpha f$ well-defined.

According to Lemma 7.3, Remark 3.6 and Lemma 7.4

$$\begin{aligned} H_n(\alpha) \Delta_m K^\alpha f(m) &= \int_M (\Delta f)(y) \sinh^{\alpha-n}(d(y, m))\, dy \\ &= \Omega_{n-1} \int_0^\infty M^r(\Delta f)(m) \sinh^{\alpha-1}(r)\, dr. \end{aligned}$$

Thus using the Darboux equation (see e.g. [12, p. 90-91])

$$H_n(\alpha) \Delta_m K^\alpha f(m) = \int_0^\infty (\Delta_{rad} M^r f)(m) \sinh^{\alpha-1}(r)\, dr,$$

where Δ_{rad} is the radial part of the Laplacian on M:

$$\Delta_{rad} = \frac{\partial^2}{\partial r^2} + (n-1)\coth(r)\frac{\partial}{\partial r}.$$

For each $N > 1$ we have by way of partial integration, that

$$\begin{aligned}
(23)\quad & \int_{\frac{1}{N}}^{N} (\frac{d^2}{dr^2} M^r f)(m)\ \sinh^{\alpha-1}(r)\, dr \\
&= \left[(\frac{d}{dr}M^r f)(m)\sinh^{\alpha-1}(r)\right]_{\frac{1}{N}}^{N} - \left[(\alpha-1)(M^r f)(m)\sinh^{\alpha-2}(r)\cosh(r)\right]_{\frac{1}{N}}^{N} \\
&\qquad + (\alpha-1)\int_{\frac{1}{N}}^{N} (M^r f)(m)\frac{d}{dr}(\sinh^{\alpha-2}(r)\cosh(r))\, dr
\end{aligned}$$

and

$$\begin{aligned}
(24)\quad & \int_{\frac{1}{N}}^{N} \coth(r)(\frac{d}{dr}M^r f)(m)\sinh^{\alpha-1}(r)\, dr \\
&= \left[(M^r f)(m)\sinh^{\alpha-2}(r)\cosh(r)\right]_{\frac{1}{N}}^{N} \\
&\qquad - \int_{\frac{1}{N}}^{N} (M^r f)(m)\frac{d}{dr}(\sinh^{\alpha-2}(r)\cosh(r))\, dr.
\end{aligned}$$

Note that

$$|M^r f(m)| \le \frac{1}{\Omega_{n-1}} \int_{S^{n-1}} |f\circ \mathrm{Exp}_m(r\omega)|\, d\omega = |f\circ \mathrm{Exp}_m(r\omega)|,$$

so that $f \in C_a$ implies

$$(25)\qquad \forall m \in M \exists \varepsilon > 0 : M^r f(m) = \mathcal{O}(\sinh^{1-a}(r) r^{-1-\varepsilon}).$$

Since

$$\frac{d}{dr}(\sinh^{\alpha-2}(r)\cosh(r)) = (\alpha-2)\sinh^{\alpha-3}(r) + (\alpha-1)\sinh^{\alpha-1}(r),$$

it follows from (25) that $r \mapsto (M^r f)(m)\frac{d}{dr}(\sinh^{\alpha-2}(r)\cosh(r))$ is dominated on $[0,\infty[$ by an integrable function. Thus letting N go to ∞ in (23) and (24) while applying Lemma 7.5 and (25), we obtain

$$\begin{aligned}
& \int_0^\infty (\Delta_{rad} M^r f)(m)\sinh^{\alpha-1}(r)\, dr \\
= \quad & (\alpha-n)\int_0^\infty (M^r f)(m)((\alpha-2)\sinh^{\alpha-3}(r) + (\alpha-1)\sinh^{\alpha-1}(r))\, dr.
\end{aligned}$$

Hence

$$\begin{aligned}
& ((\alpha-n)(\alpha-1)-\Delta)K^\alpha f \\
= \; & (\alpha-n)(\alpha-1)K^\alpha f \\
& \qquad -\frac{\alpha-n}{H_n(\alpha)}((\alpha-2)H_n(\alpha-2)K^{\alpha-2}f+(\alpha-1)H_n(\alpha)K^\alpha f) \\
= \; & \frac{(\alpha-n)(\alpha-2)}{H_n(\alpha)}H_n(\alpha-2)K^{\alpha-2}f = K^{\alpha-2}f.
\end{aligned}$$

□

7.7. **Corollary.** *Assume that $n > 2$. Let $f \in C^2(M)$ be given such that*

$$\forall u \in \mathcal{U}_2(\mathfrak{g}) : L_u f \in C_a(M)$$

for some $a > 2$. Then

$$((2-n)-\Delta)K^2 f = f.$$

Proof. Both $\alpha \mapsto K^\alpha f(m)$ and $\alpha \mapsto \Delta K^\alpha f(m) = K^\alpha(\Delta f(m))$ are continuous on $\{\alpha \in \mathbb{C} | 0 < \operatorname{Re}\alpha < 2\}$ (cf. Lemma 7.1 and Lemma 7.3), so according to Lemma 7.2 and Proposition 7.6

$$\begin{aligned}
((2-n)(2-1)-\Delta)K^2 f(m) &= \lim_{s\to 2_+}((s-n)(s-1)-\Delta)K^s f(m) \\
&= \lim_{s\to 2_+} K^{s-2}f(m) = f(m).
\end{aligned}$$

□

8. The Inversion Theorem

8.1. **Lemma.** *Let $\alpha \in \mathbb{C}$ with $0 < \operatorname{Re}\alpha < \min\{n-1, \frac{n}{2}\}$ be given. Then*

$$(f * k_+^\alpha) * k_-^\alpha = f * (k_+^\alpha * k_-^\alpha)$$

a.e. on M, whenever $f \in L^p(M)$ for some $p \in \mathbb{R}_+$ such that

$$\max\{\frac{\operatorname{Re}\alpha}{n-1}, \frac{2\operatorname{Re}\alpha - 1}{n-1}\} < \frac{1}{p} \leq 1.$$

Proof. Let $f \in L^p(M)$ for some $p \in \mathbb{R}_+$. Then, according to Lemma 6.4, the integral $f * k_+^\alpha(m)$ exists and is finite for a.a. $m \in M$ if

$$\frac{\operatorname{Re}\alpha - 1}{n-1} < \frac{1}{p} \leq 1,$$

and then $f * k_+^\alpha \in L^r(M)$ for all $r \in \mathbb{R}_+$ such that

$$\frac{1}{p} - \frac{\operatorname{Re}\alpha}{n} < \frac{1}{r} < \frac{1}{p} - \frac{\operatorname{Re}\alpha - 1}{n-1} \text{ and } \frac{1}{r} \leq \frac{1}{p}.$$

Again according to Lemma 6.4 the integral $(f * k_+^\alpha) * k_-^\alpha$ exists and is finite for a.a. $m \in M$ if $f * k_+^\alpha \in L^r(M)$ for some $r \in \mathbb{R}_+$ such that

$$\frac{\operatorname{Re}\alpha}{n-1} < \frac{1}{r} \leq 1.$$

This is especially true when f is replaced by $|f|$, so since $k_\pm^\alpha$ are positive functions, the function (cf. (14))

$$(g, h) \mapsto f(h \cdot o) k_+^\alpha(h^{-1} g \cdot o) k_-^\alpha(g^{-1} g_0 \cdot o)$$

is in $L^1(G \times G)$ for a.a. $m = g_0 \cdot o$ in M, when

$$\frac{\operatorname{Re}\alpha - 1}{n-1} < \frac{1}{p} \leq 1 \quad \text{and} \quad \frac{\operatorname{Re}\alpha}{n-1} < \min\{\frac{1}{p}, \frac{1}{p} - \frac{\operatorname{Re}\alpha - 1}{n-1}\},$$

that is when

$$\max\{\frac{\operatorname{Re}\alpha}{n-1}, \frac{2\operatorname{Re}\alpha - 1}{n-1}\} < \frac{1}{p} \leq 1.$$

From Fubini's Theorem we get

$$\begin{aligned}
((f * k_+^\alpha) * k_-^\alpha)(g_0 \cdot o) &= \int_G (f * k_+^\alpha)(g \cdot o) k_-^\alpha(g^{-1} g_0 \cdot o)\, dg \\
&= \int_G \int_G f(h \cdot o) k_+^\alpha(h^{-1} g \cdot o)\, dh\, k_-^\alpha(g^{-1} g_0 \cdot o)\, dg \\
&= \int_G f(h \cdot o) \int_G k_+^\alpha(g \cdot o) k_-^\alpha(g^{-1} h^{-1} g_0 \cdot o)\, dg\, dh \\
&= \int_G f(h \cdot o)(k_+^\alpha * k_-^\alpha)(h^{-1} g_0 \cdot o)\, dh \\
&= (f * (k_+^\alpha * k_-^\alpha))(g_0 \cdot o),
\end{aligned}$$

where $k_+^\alpha * k_-^\alpha$ is well-defined according to the last part of Lemma 6.4, since $\operatorname{Re}\alpha < \frac{n}{2}$. $\square$

8.2. **Theorem.** *Assume that $n > 2$, and let $k \in \{1, \ldots, n-1\}$ be given. Let $f \in C^2(M)$ be given such that*

$$\forall u \in \mathcal{U}_2(\mathfrak{g}) : L_u f \in C_a(M)$$

for some a with $a > 2$ and $a \geq k$. Then f can be recovered from its k-dimensional Radon transform by

$$f = c \begin{cases} P_k(\Delta)(\hat{f})^\vee & , \quad k \text{ even} \\ ((2-n) - \Delta) K_-^1 P_k(\Delta)(\hat{f})^\vee & , \quad k \text{ odd} \end{cases}, \tag{26}$$

where

$$P_k(\Delta) = \prod_{i=0}^{[\frac{k}{2}]-1} ((k - 2i - n)(k - 2i - 1) - \Delta),$$

$[\frac{k}{2}]$ denoting the integer part of $\frac{k}{2}$, and $c = (4\pi)^{-\frac{k}{2}} \frac{\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n}{2})}$.

Proof. According to Lemma 5.7 and Definition 6.6

$$(\hat{f})^{\check{}}(m) = \frac{\Omega_{k-1}}{\Omega_{n-1}} H_n(k) K^k f(m) = c^{-1} K^k f(m)$$

for all $m \in M$ since $f \in C_k(M)$. Using Proposition 7.6 repeatedly, and in the case of k even applying Corollary 7.7, we see that

$$cP_k(\Delta)(\hat{f})^{\check{}} = P_k(\Delta) K^k f = \begin{cases} f & , \quad k \text{ even} \\ K^1 f & , \quad k \text{ odd} \end{cases} .$$

We wish to apply Lemma 8.1 with $\alpha = 1$ on f. But $n > 2$, so $1 < \min\{n-1, \frac{n}{2}\}$. And according to Remark 6.5, $f \in L^p(M)$ for all $p \in \mathbb{R}_+$ with $\frac{1}{p} \leq \frac{a-1}{n-1}$, where $\frac{1}{n-1} < \frac{a-1}{n-1}$ since $a > 2$. Thus the prerequisites of Lemma 8.1 are met, so

$$K_-^1 K^1 f = (f * k_+^1) * k_-^1 = f * (k_+^1 * k_-^1).$$

According to [26, p.220-221], $k_+^1 * k_-^1 = k_+^2$. Hence

$$K_-^1 K^1 f = K^2 f.$$

From Corollary 7.7 the result follows. □

The case $n = 2$ is discussed in e.g. [26, p.221], [12, Chap. III] and [20].

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