Deformation Quantization of Endomorphism Bundles

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Ph.D. thesis

Approved July 2002

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Institute for Mathematical Sciences · University of Copenhagen · 2003

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ISBN 87-7834-542-1

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1. Introduction

A deformation quantization of a Poisson manifold $(M, \{\cdot,\cdot\})$ is an associative product $*$ on $C^\infty(M)[[\hbar]]$,

$$f * g = fg + \hbar P_1(f,g) + \hbar^2 P_2(f,g) + \dots ,$$

$*$ being $\hbar$ linear, P_i bidifferential and

$$[f,g] = i\hbar\{f,g\} + O(\hbar)$$

This notion was first introduced in [BFF$^+$78].

The question of existence and classification of deformation quantizations on general Poisson manifolds was solved in 1997 by Kontsevich in [Kon97].

The simpler case of existence of deformation quantizations of the canonical Poisson structure on symplectic manifolds was solved already in [WL83]. A simple geometric construction of deformation quantizations of symplectic manifolds were given by Fedosov in [Fed94]. The advantage of Fedosovs construction compared to the ones in [Kon97] and [WL83] is, that it is easy to handle and also suitable generalizable. The general most setting of the Fedosov construction is probably given in [NT01], where deformation quantizations of symplectic Lie algebroids is done. Also the classification of deformation quantizations becomes amenable in view of the Fedosov construction. In the case of symplectic manifolds this was done in [NT95a] and the classification of deformation quantizations on a symplectic manifold (M, ω) is given by the points (characteristic classes) θ in the space

$$\frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]]).$$

One of the main examples of deformation quantizations of symplectic manifolds are those coming from asymptotic calculus of pseudodifferential operators on manifolds, see for example [NT96]. If we consider the asymptotic calculus of pseudodifferential operators on a manifold M, we will get a deformation quantization of the cotangent bundle T^*M of M, where T^*M is equiped with the canonical symplectic structure.

This example gives the connection to index theory. On a deformation quantization of any symplectic $2n$ dimensional manifold there is a canonical trace, unique up to multiplication by a scalar, of the form

$$Tr(a) = \frac{1}{n!(i\hbar)^n}\left(\int_M a\omega^n + O(\hbar)\right). \tag{1.1}$$

By an appropriate choice of the representation of the quantisation, i.e. after applying a linear isomorphism of $C^\infty(M)[[\hbar]]$ of the form

$$f \to f + \hbar D_1(f) + \dots$$

one can assure that Tr has the form

$$Tr(a) = \frac{1}{n!(i\hbar)^n}\int_M a\omega^n,$$

which fixes it uniquely.

In most proofs of the Atiyah-Singer index theorem and related "local" index theorems, one of the main difficulty is to compute the trace of a certain operator on a Hilbert space, usually $L^2(M)$ as above. In order to compute this trace a scaling $\hbar \in \mathbb{R}_+$ of the operator is introduced and the asymptotic expansion of the trace as $\hbar \to 0$ becomes computable, or at least the constant term in the expansion. The computations coming out of this is computations like 1.1. This is why, computing the canonical trace on deformation quantizations is called algebraic index theory. Actually computing the trace on deformation quantizations, in a way that will be described now, implies the Atiyah-Singer index theorem, according to [NT96].

Many elements, not all, on which computing the trace is interesting, are first components in classes in cyclic periodic homology. The cyclic periodic homology or rather cohohmology was invented by Connes in [Con85]. It is the noncommutative analog of De Rahm cohomology and was already at the beginning intimately connected to index theory. A complex computing the cyclic periodic homology of a unital algebra A over a field k is given by

$$CC^{per}_{even}(A) = \prod_i A \otimes \bar{A}^{\otimes 2i}; \quad CC^{per}_{odd}(A) = \prod_i A \otimes \bar{A}^{\otimes 2i+1}$$

where $\bar{A} = A/k \cdot 1$ and the differential

$$CC^{per}_{even}(A) \xleftrightarrow{b+B} CC^{per}_{odd}(A)$$

is given by

$$b(a_0 \otimes \ldots \otimes a_n) = \sum_{k=0}^{n-1} (-1)^k a_0 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}$$

and

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{k=0}^{n} (-1)^k 1 \otimes a_k \otimes a_{k+1} \ldots \otimes a_n \otimes a_0 \ldots a_{k-1}$$

If for example $p \in M_n(A)$ is a projection, a class in $HC^{per}_{even}(A)$, the Chern character of p, is given by the formula

$$tr(p + \sum_{k \geq 1} \frac{(2k)!}{k!} (p - 1/2) \otimes p^{\otimes 2k}),$$

where

$$tr : M_n(A)^{\otimes k} \to A^{\otimes k}$$

is the map given by

$$(M_1 \otimes a_1) \otimes \ldots \otimes (M_k \otimes a_k) \mapsto Tr(M_1 \ldots M_k) a_1 \otimes \ldots \otimes a_k$$

Therefore $tr(p)$ can be regarded as the first component of a class in cyclic periodic homology.

Evaluating Tr on the first component gives a morphism of complexes

$$Tr : CC^{per}_*(A^\hbar_c) \to \mathbb{C}[[\hbar, \hbar^{-1}], \qquad (1.2)$$

where $A^\hbar_c$ is the algebra of compactly supported elements $(C^\infty_c(M)[[[\hbar]])$ in a deformation quantization $A^\hbar$, and $\mathbb{C}[[\hbar, \hbar^{-1}]$ is considered as a complex concentrated at degree zero with the trivial boundary map.

Computing trace on elements, that are first component of a class in cyclic periodic homology, is therefore the same as computing 1.2 at the level of homology.

In [NT95a] it is proven that

$$Tr(\cdot) \sim (-1)^n \int_M \hat{A}(TM) e^\theta \tilde{\mu}(\cdot)$$

where $\sim$ means, that the two side define the same morphism at the level of homology. Here θ is the characteristic class of the deformation and $\tilde{\mu}$ is the map $CC_*^{per}(A^\hbar) \to \Omega^*(M)$ given by

$$\tilde{\mu}(a_0 \otimes \ldots \otimes a_k) = \frac{1}{k!}\tilde{a}_0 d\tilde{a}_1 \cdots d\tilde{a}_k, \quad \tilde{a}_i = a_i \bmod \hbar$$

This settles the problem of computing the trace at the level of homology for deformation quantizations of symplectic manifold.

1.1. **Content of the thesis.** Below I propose a definition of deformation quantization of endomorphism bundles over a symplectic manifold. The motivation is clear: Deformation quantizations of the trivial line bundle is the algebraic analog of pseudodifferential operators in line bundles and therefore deformation quantization of an endomorphism bundle $End(E)$, E vector bundle over M, should be the algebraic analog of pseudodifferential operators in any vector bundle having $End(E)$ as endomorphism bundle.

The definition proposed consist in requiring a product $*$ on $\Gamma(End(E))[[\hbar]]$, so the algebra $(\Gamma(End(E))[[\hbar]], *)$ is locally isomorphic to $M_N(\mathcal{W}_n)$, $\mathcal{W}_n$ being the Weyl algebra, the canonical deformation quantization of the standard symplectic structure on $\mathbb{R}^{2n}$.

It turns out, that the Fedosov construction also works in this case, i.e. let $M_N(\mathbb{A}^\hbar)$ be the algebra of jets at zero of elements in $M_N(\mathcal{W}_n)$. There is associated to $(M, \omega, End(E))$ an algebra bundle $\mathbb{W}$ with fiber $M_N(\mathbb{A}^\hbar)$. Put $\mathfrak{g} = Der(M_N(\mathbb{A}^\hbar))$. There is a short exact sequence

$$0 \to \frac{1}{\hbar}\mathbb{C}[[\hbar]] \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of Lie algebras. The Fedosov construction then consist, for a given

$$\theta \in \frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]]),$$

in constructing a flat connection ∇ in $\mathbb{W}$ with values in $\mathfrak{g}$, such that $ker\nabla \simeq \Gamma(End(E))[[\hbar]]$ linearly and ∇ admits a lift $\tilde{\nabla}$ to a connection with values in $\tilde{\mathfrak{g}}$ and curvature θ. The product on $\Gamma(\mathbb{W})$ induces a product on $ker\nabla = \Gamma(End(E))[[\hbar]]$. This product gives a deformation quantization of $End(E)$ and θ will be an isomorphism invariant of the deformation quantization.

This construction is done in section 3. In this section it is also shown the following.

Theorem 1. *A deformation quantization of $End(E)$ is isomorphic to the flat sections of a Fedosov connection, and the isomorphism classes* *of deformation quantizations of $End(E)$ are classified by the points in*

$$\frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]]).$$

The principle, that a deformation quantization comes as flat sections in a certain infinite dimensional vector bundle, is not special to deformation quantizations. In section 2 it is shown, that sections of $\Gamma(End(E))$ are flat sections in an algebra bundle with fibre $M_N(\mathbb{C}[[\hat{x}_1, \dots, \hat{x}_n]])$.

The reason for redoing this construction for $End(E)$ is, that it is notationally simpler, and, hopefully, clarifies the construction. Therefore in section 3 only the differences in the construction for $End(E)$ and deformation quantizations of endomorpism bundles are spelled out.

Like in the scalar case there are canonical traces on deformation quantizations of endomorphism bundles. The rest of the thesis is devoted to index theory for these traces. The methods used for this, are the methods developed by Nest and Tsygan in [NT95a], [NT95b], [BNT99] and [NT01]. This method is based on the following.

(1) The action of $\overline{C}^{\lambda}_{*}(A)$, the reduced cyclic complex, on $CC^{per}_{*}(A)$:
$$\overline{C}^{\lambda}_{*}(A) \times CC^{per}_{*}(A) \xrightarrow{\cdot} CC^{per}_{*}(A)$$
(2) The construction of a special class, the fundamental class, in $\overline{C}^{\lambda}_{*}(A^{\hbar}_{E})$, $A^{\hbar}_{E}$ being the deformation quantization of $End(E)$; or rather the construction of a class in $\check{C}^{*}(M, \overline{C}^{\lambda}_{*}(A^{\hbar}_{E}))$, the Čech complex with values in the presheaf $V \to \overline{C}^{\lambda}_{*}(A^{\hbar}_{E|V})$.
(3) Computations in Lie algebra cohomology in order to identify the fundamental class at the level of cohomology.

The fundamental class U is lives in $\overline{C}^{\lambda}_{2n-1}(M_N(\mathcal{W}_n))$. Its role is, that it relates Tr to $\tilde{\mu}$ when evaluated at classes that are scalar mod $\hbar$. It has the effect that

$$Tr(U \cdot a) = (-1)^n \int \tilde{\mu(a)}, \quad a \in CC^{per}_{*}(\mathcal{W}_{n,c}).$$

Here, as before, the subscript "c" denotes the ideal of compactly supported elements in the deformation algebra in question.

The following plays the major role.

Theorem 2. *The class U has a, in cohomology, unique extension to a class, also denoted U, in $\check{C}^{*}(M, \overline{C}^{\lambda}_{*}(A^{\hbar}_{E}))$. On classes of the form $a_0 \otimes \ldots \otimes a_k \in CC^{per}_{*}(A^{\hbar}_{E,c})$, a_i scalar mod $\hbar$, the following holds.*

$$Tr(U \cdot a) = (-1)^n \int \tilde{\mu}(a) \ mod \ \hbar$$

It is not difficult to see, that this implies for general classes $a_0 \otimes \ldots \otimes a_k \in CC_*^{per}(A_{E,c}^{\hbar})$ that

$$Tr(U \cdot a_0 \otimes \ldots \otimes a_k) = (-1)^n \int ch(End(E))^{-1} ch(\nabla)(\tilde{a}_0 \otimes \ldots \otimes \tilde{a}_k) \bmod \hbar$$

where $ch(End(E))$ is the usual Chern character of $End(E)$ as a vector bundle, ∇ is a connection in $End(E)$ and

$$ch(\nabla)(\tilde{a}_0 \otimes \ldots \otimes \tilde{a}_k) = \int_{\Delta_k} tr(\tilde{a}_0 e^{-t_0 \nabla^2} \nabla(\tilde{a}_i) e^{-t_1 \nabla^2} \cdots \nabla(\tilde{a}_k) e^{-t_k \nabla^2}) dt_0 \cdots dt_{k-1},$$

the J.L.O. cocycle associated to ∇.

To finish the index theory, the fundamental class has to be identified. This is done via Lie algebra cohomology. There is the Gelfand-Fuks morphism of complexes

$$C^*(\mathfrak{g}, \mathfrak{su}(N) + \mathfrak{u}(n); \overline{C}_*^{\lambda}(M_N(\mathbb{A}^{\hbar}))) \to \check{C}^*(M, \Omega^*(M, \overline{C}_*^{\lambda}(M_N(\mathbb{A}^{\hbar})))),$$

the latter complex being quasi isomorphic to $\check{C}^*(M, \overline{C}_*^{\lambda}(A_E^{\hbar}))$. As in the case of $M_N(\mathcal{W}_n)$, there is fundamental class $U \in \overline{C}_*^{\lambda}(M_N(\mathbb{A}^{\hbar}))$ extending uniquely in cohomology to a class in Lie algebra cohomology, also denoted U. It turns out, that $GF(U)$ is equivalent to $U \in \check{C}^*(M, \overline{C}_*^{\lambda}(A_E^{\hbar}))$ via the quasi isomorphism between $\check{C}^*(M, \overline{C}_*^{\lambda}(A_E^{\hbar}))$ and $\check{C}^*(M, \Omega^*(M, \overline{C}_*^{\lambda}(M_N(\mathbb{A}^{\hbar}))))$. Hence the question of computing or identifying U, is now a question of computations in

$$C^*(\mathfrak{g}, \mathfrak{su}(N) + \mathfrak{u}(n); \overline{C}_*^{\lambda}(M_N(\mathbb{A}^{\hbar}))).$$

It turns out to be useful to work with the differnential graded algebra $M_N(\mathbb{A}^{\hbar})[\eta]$, where η is a formal variable, $\eta^2 = 0$ and the differential is given by $\frac{\partial}{\partial \eta}$. The reason for doing this is to include the identity operation on $CC_*^{per}(M_N(\mathbb{A}^{\hbar}))$, i.e. the action of $\overline{C}_*^{\lambda}(M_N(\mathbb{A}^{\hbar}))$ on $CC_*^{per}(M_N(\mathbb{A}^{\hbar}))$ extends to an action of $\overline{C}_*^{\lambda}(M_N(\mathbb{A}^{\hbar})[\eta])$ also on $CC_*^{per}(M_N(\mathbb{A}^{\hbar}))$. With this action, the classes $\eta^{(k+1)} = k!\eta^{\otimes k+1}$ becomes the identity operations. The main technical theorem of this thesis, theorem 6.0.3, states the following

Theorem 3. *U is equivalent to*

$$\sum_{m \geq 0} (\hat{A} \cdot e^{\theta} \cdot ch)_{2m}^{-1} \cdot \eta^m$$

in $C^*(\mathfrak{g}, \mathfrak{su}(N) + \mathfrak{u}(n); \overline{C}_*^{\lambda}(M_N(\mathbb{A}^{\hbar})[\eta]))$, i.e. defines the same cohomology class.

Here $\hat{A}$ is the Lie algebraic $\hat{A}$ coming from $\mathfrak{u}(n)$, ch is the Lie algebraic Chern character coming from $\mathfrak{su}(N)$ and θ is the Lie algebraic class of deformation. Using the Gelfand-Fuks map, the index theorem follows.

Acknowledgements. I would like to thank Boris Tsygan for helpful conversations and Paulo Almeida and Nuno Martins for having hosted me for two month at the I.S.T. in Lisbon.

2. Sections of endomorphism bundles as flat sections in a profinite bundle

Let $\mathbb{O}_n = \mathbb{C}[[\hat{x}_1, \dots, \hat{x}_n]]$ and $M_N(\mathbb{O}_n) = M_N \otimes \mathbb{O}_n$, where M_N is the $N \times N$ complex matrices. We give $M_N(\mathbb{O}_n)$ a grading by

$$\deg(b \otimes \hat{x}_i) = 1, \quad b \in M_N$$

Furthermore we give $M_N(\mathbb{O}_n)$ the I-adic topology, where I is the ideal generated by elements of degree ≥ 1.

Definition 2.0.1. *Let G be the group of continuous automorphisms of $M_N(\mathbb{O}_n)$ with the following property:*

An automorphism belongs to G, if the induced automorphism on the centre $\mathbb{O}_n$ is an automorphism induced by an automorphism on $\mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n]]$.

Lemma 2.0.2. *An automorphism in G is the composition of an automorphism induced by an automorphism of $\mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n]]$ and an inner automorphism.*

Proof. Given an automorphism $\Phi \in G$ let φ be the induced automorphism on $\mathbb{O}_n$. Considering $\chi = \Phi \circ (\varphi \otimes id)^{-1}$ we have, that χ is an $\mathbb{O}_n$-module map.

For $A \in M_N(\mathbb{C})$ we have

$$\chi(A) = D_0(A) + \text{higher order terms }, \ D_0(A) \in M_N$$

Since D_0 is an automorphism of M_N, it is inner and hence extends to $M_N(\mathbb{O}_n)$. Let $\chi_1 = \chi \circ D_0^{-1}$.

$$\chi_1(A) = A + \sum_i \hat{x}_i D_i(A) + \text{ higher order terms}$$

Each D_i are derivations of M_N and hence given by commutators by elements $B_i \in M_N$. Therefore

$$\chi_1 \circ \exp(\sum_i ad(\hat{x}_i B_i))(A) = A + \text{ terms of order } \geq 2$$

Continuing by induction we get a sequence of elements $C_k \in M_N(\mathbb{O}_n)$, $\deg(C_k) = k$, with

$(\chi \circ \exp(ad(C_0)) \circ \ldots \circ \exp(ad(C_k))(A) = A +$ terms of order $\geq k+1$

Since the product

$$\exp(ad(C_0)) \circ \ldots \circ \exp(ad(C_k)) \circ \ldots$$

converge, we see, by the Hausdorff-Campbel formula, that χ is inner.

¿From lemma 2.0.2 follows

Proposition 2.0.3. *The Lie-group G has*

$$\mathbb{W}_n^0 \ltimes (\mathfrak{gl}_N(\mathbb{O}_n)/centre)$$

as Lie-algebra, where $\mathbb{W}_n^0$ is the Lie-algebra of formal vector fields vanishing in zero.

We note that $Der(M_N(\mathbb{O}_n))$ is larger than $\mathbb{W}_n^0 \ltimes (\mathfrak{gl}(\mathbb{O}_n)/centre)$, namely

$$\mathbb{W}_n \ltimes (\mathfrak{gl}_N(\mathbb{O}_n)/centre)$$

where $\mathbb{W}_n$ are all formal vector fields on $\mathbb{R}^n$.

Lemma 2.0.4. *Let X be a contractible open subset of $\mathbb{R}^k$, let $Aut(M_n)$ be the automorphism group of M_n and let $Der(M_n)$ be the derivations of M_n. Any smooth maps $\varphi_1 : X \to Aut(M_n)$ and $\varphi_2 : X \to Der(M_n)$ lifts to smooth maps $\tilde{\varphi}_1 : X \to Gl_n$ and $\tilde{\varphi}_2 : X \to M_n$.*

Proof. First the case of φ_1:

Let $\{e_{ij}\}$ be the standard matrix units. The families

$$x \to \varphi_1(x)(e_{ii})$$

of projections over X give rise to a family of line bundles $\{l_i\}$ over X by

$$l_i = (\varphi_1(x)(e_{ii}))(\mathbb{R}^n)$$

Since X is contractible, these line bundles are trivial. Let v_1 be a smooth nowhere vanishing section of l_1 and let

$$v_i(x) = (\varphi_1(x)(e_{i1}))(v_1(x))$$

Put

$$(\tilde{\varphi}_1(x))(e_i) = v_i(x)$$

where e_i is the vector $(a_1, \ldots, a_n) \in \mathbb{R}^n$ with $a_j = 0$, $j \neq i$ and $a_i = 1$.

We have $\tilde{\varphi}_1 : X \to Gl_n$ is smooth and

$$\tilde{\varphi}_1(x)A(\tilde{\varphi}_1(x))^{-1} = \varphi_1(x)(A), \quad A \in M_n$$

since

$$(\tilde{\varphi}_1(x)e_{ij}(\tilde{\varphi}_1(x))^{-1}v_k(x) = \tilde{\varphi}_1(x)e_{ij}e_k = \tilde{\varphi}_1(x)e_i\delta_{jk} = \delta_{jk}v_i(x)$$

and

$$\begin{aligned}\varphi_1(x)(e_{ij})v_k(x) &= (\varphi_1(x)(e_{ij}))(\varphi_1(x)(e_{k1}))v_1(x)\\ &= \delta_{kj}\varphi_1(x)(e_{i1})(v_1(x)) = \delta_{kj}v_i(x)\end{aligned}$$

In the case of φ_2, let $\varphi : X \times \mathbb{R} \to Aut(M_n)$ be defined by

$$\varphi(x,t) = \exp(t\varphi_2(x)).$$

This is clearly smooth and hence by the first part of the lemma we get a smooth lifting of φ to $\tilde{\varphi} : X \times \mathbb{R} \to Gl_n$.

Defining

$$\tilde{\varphi}_2(x) = \frac{\partial}{\partial t}(\tilde{\varphi})(x,0)$$

the lemma follows.

In view of lemma 2.0.4 the proof of lemma 2.0.2 gives some more

Lemma 2.0.5. *A smooth family of elements in G over a contractible open subset X of $\mathbb{R}^k$ lifts to a smooth family over X of elements in the group of invertible elements in $M_n(\mathbb{O}_n)$.*

2.1. **Jets.** Let A be an algebra bundle over a manifold M. In this case we let

$$I_m = \{a \in \Gamma(A) | a(m) = 0\}$$

Given this, we define

$$J_m A = \lim_{\underset{k}{\leftarrow}} \Gamma(A)/I_m^k$$

Denote by J_m the quotient map from $\Gamma(A)$ into $J_m A$. In the following we are only interested in the case where $A = End(E)$, E a vector bundle. Note that $J_m A \simeq M_N(\mathbb{O}_n)$ by choosing a trivialization; and that any other trivialization leads to an automorphism of $M_N(\mathbb{O}_n)$ in G.

If we are given a smooth path of automorphisms $\Phi_t \in G$, we can, according to lemma 2.0.5, write it as $\Phi_t = \varphi_t \circ \chi_t$, where φ_t is a smooth path the automorphisms induced by automorphisms of $\mathbb{R}[[\hat{x}_1, \ldots, \hat{x}_n]]$ and χ_t is a smooth path of inner automorphisms of $M_N(\mathbb{O}_n)$. It is well known, that φ_t lifts to a smooth path of local diffeomorphisms $\tilde{\Phi}_t$ of $\mathbb{R}^n$ preserving zero and since χ_t lifts to a smooth path of invertible elements in $M_N(\mathbb{O}_n)$, it lifts, by the Borel lemma, to a smooth path of invertible elements in $M_N(C^\infty(U))$, where U is an open subset of $\mathbb{R}^n$ containing zero. We thus get

Proposition 2.1.1. *Any smooth path of automorphism in G lifts to a smooth path of local bundle automorphism of $M_N(C^\infty(\mathbb{R}^n))$ preserving zero.*

2.2. **The frame bundle.** For a manifold M with a vector bundle E we define the following

Definition 2.2.1. *The frame bundle $\tilde{M}_E$ is given by*

$$\tilde{M}_E = \{(m, \Phi) | m \in M, \Phi : M_N(\mathbb{O}_n) \xrightarrow{\sim} J_m End(E)\}$$

We note, that $\tilde{M}_E$ is a profinite manifold and in fact a principal bundle over M with fibre G.

Proposition 2.2.2. *For all $(m, \Phi) \in \tilde{M}_E$ there is an isomorphism*

$$\omega : T_{(m,\Phi)} \tilde{M}_E \to Der(M_N(\mathbb{O}_n))$$

satisfying

$$\begin{aligned} \omega(A^*) &= A \qquad && A \in \mathbb{W}_n^0 \ltimes (\mathfrak{gl}_N(\mathbb{O}_n)/centre) \\ \varphi^* \omega &= ad\varphi^{-1}\omega && \text{for } \varphi \in G \\ d\omega + \tfrac{1}{2}[\omega, \omega] &= 0 \end{aligned}$$

where A^* is the fundamental vector field corresponding to A.

In other words, ω is a flat connection $\tilde{M}_E$ with values in $Der(M_N(\mathbb{O}_n))$.

Proof. Let us suppose, we are given a path in $\tilde{M}_E$, so $\gamma(t) \in \tilde{M}_E$ with $\dot{\gamma}(0) = v$, $\gamma(0) = (m, \Phi)$ and $\gamma(t) = (m_t, \Phi_t)$. This lifts to a path of trivializations $\tilde{\gamma} = (m_t, \tilde{\Phi}_t)$, $\tilde{\Phi}_t : M_N(C^\infty(U)) \to \Gamma(End(E))$, that maps 0 to m_t.

Define $\omega(v)$ to be the derivation

$$J_0(a) \to J_0 \left(\frac{d}{dt} (\tilde{\Phi}^{-1} \circ \tilde{\Phi}_t(a)) \right), \qquad a \in M_N(\mathbb{C}^\infty(U))$$

This does not depend on the choice of γ or the lifting to $\tilde{\gamma}$ and will be an isomorphism.

The first identity follows, since ω is the canonical one form on the fibres of $\tilde{M}_E$.

For the second identity we have to compute $\omega(\varphi_*(v))$, $\varphi \in G$, but

$$\omega(\varphi_*(v)) = \left(J_0(a) \to J_0 \left(\frac{d}{dt} (\tilde{\varphi}^{-1} \circ \tilde{\Phi}^{-1} \circ \tilde{\Phi}_t \circ \tilde{\varphi}(a)) \right) \right) = ad(\varphi^{-1})\omega(v)$$

The third identity is equivalent to

$$\omega([\omega^{-1}(X), \omega^{-1}(Y)]) = [X, Y], \qquad X, Y \in Der(M_N(\mathbb{O}_n))$$

This will be a consequence of the first identity, because the statement is obvious for X, Y of the form $\frac{\partial}{\partial \hat{x}_i} \in Der(M_N(\mathbb{O}_n))$ and for $X, Y \in \mathbb{W}_n^0 \ltimes (\mathfrak{gl}_N(\mathbb{O}_n)/centre)$ it follows because ω is the canonical one form on the fibres of $\tilde{M}_E$. Hence it suffices to check the case when X is of the form $\frac{\partial}{\partial \hat{x}_i}$ and $Y \in \mathbb{W}_n^0 \ltimes (\mathfrak{gl}_N(\mathbb{O}_n)/centre)$. Therfore let φ_t be the one parameter group for Y on G. We have

$$\begin{aligned} \omega([\omega^{-1}(X), \omega^{-1}(Y)]) &= \omega(\lim_{t\to 0} \tfrac{1}{t}((\varphi_t)_* \omega^{-1}(X) - \omega^{-1}(X))) \\ &= \lim_{t\to 0} \tfrac{1}{t}(ad(\varphi_t^{-1})(X) - X) = [X, Y] \end{aligned}$$

The proposition follows from this.

Given $\tilde{M}_E$, we define the jet bundle of $End(E)$ by

$$JE = \tilde{M}_E \times_G M_N(\mathbb{O}_n)$$

The flat connection on $\tilde{M}_E$ gives a flat connection in JE in the following way: Choose a trivialization of $JE_{|U} \xrightarrow{\sim} U \times M_N(\mathbb{O}_n)$. This corresponds to a lift $\sigma : U \to \tilde{M}_{E|_U}$ of the projection $P : \tilde{M}_E \to M$. In this trivialization the connection ∇ is given by $\nabla = d - \sigma^*(\omega)$, where ω is the connection described in proposition 2.1.1.

Proposition 2.2.3. *The complex* $(\Omega^*(M, JE), \nabla)$ *is acyclic and the cohomology is isomorphic to* $\Gamma(End(E))$

Proof. There is an obvious map j from $\Gamma(End(E))$ into JE given by

$$j(\gamma) = ((m, \Phi), \Phi^{-1}(J_m\gamma)), \quad \gamma \in \Gamma(End(E)).$$

It is clearly injective and the image of j belongs to the kernel of ∇. To see this, choose a trivialization in the sense of a local bundle map $\Phi : M_N(C^\infty(U)) \to \Gamma(End(E))$, U open subset of $\mathbb{R}^n$. We will also denote the induced map from U to M by Φ. From this trivialization we get a special trivialization of $\tilde{M}_E$ by letting $\tilde{\Phi}_u$ denote the map $M_N(\mathbb{O}_n) = J_u(M_N(C^\infty(U))) \to J_{\Phi(u)}(\Gamma(End(E)))$ induced from Φ and then

$$U \times G \ni (u, g) \to (\Phi(u), g\tilde{\Phi}_u)$$

Using this trivialization, we get a local bundle isomorphism

$$C^\infty(U) \otimes M_N(\mathbb{O}_n) \to JE$$

and in this trivialization it is not difficult to see, that

$$\nabla = d - \sum_i dx_i \otimes \frac{\partial}{\partial \hat{x}_i} \tag{2.1}$$

If $\gamma \in M_N(C^\infty(U))$ we have that $j(\gamma)$ is just given by the Taylor expansion in each point, i.e.

$$j(\gamma)(u) = \sum_I \frac{\partial^{|I|}\gamma}{\partial x^I}\hat{x}^I$$

where I runs through all multi indices. Hence $j(\gamma) \in Ker(\nabla)$.

A computation in $(\Omega^*(U, M_N(\mathbb{O}_n)), \nabla)$, where ∇ is as in 2.1, gives, that $(\Omega^*(U, M_N(\mathbb{O}_n)), \nabla)$ is acyclic and the cohomology is $j(M_N(C^\infty(U))$.

We have thus seen, that $(\Omega^*(M, JE), \nabla)$ is locally acyclic and the cohomology is locally isomorphic to $\Gamma(End(E))$. Since we also have seen, that $\Gamma(End(E))$ is globally contained in $Ker\nabla$, i.e. the cohomology of $(\Omega^*(M, JE), \nabla)$ is a module over $\Gamma(End(E))$, the statement follows.

3. Deformation quantization of endomorphism bundles

We start with looking at $\mathbb{R}^{2n}$ with the standard symplectic structure. We will denote the coordinates by $x_1, \dots, x_n, \xi_1, \dots, \xi_n$. On $C^\infty(\mathbb{R}^{2n})[[\hbar]]$ we consider the Weyl quantization given by the product

$$(f * g)(x, \xi) = \exp\left(\frac{i\hbar}{2}\sum_{k=1}^{n}(\partial_{x_k}\partial_{\eta_k} - \partial_{\xi_k}\partial_{y_k})\right) f(x, \xi)g(y, \eta)|_{(x=y, \xi=\eta)}$$

We will denote the Weyl quantization by $\mathcal{W}_n$.

Since the definition of the product in the Weyl quantization of two functions f, g only uses derivatives of f, g, the Weyl quantization makes sense over any open subset U of $\mathbb{R}^{2n}$. We will in this case talk about the Weyl quantization over U.

Let (M, ω) be a symplectic manifold and let E be a vector bundle over M.

Definition 3.0.4. *A deformation quantization of $End(E)$ is a $\hbar$-linear associative product $*$ on $\Gamma(End(E))[[\hbar]]$, continuous in the $\hbar$-adic topology and*

$$f * g = fg + \hbar B_i(f, g) + \dots$$

where $f, g \in \Gamma(End(E))$ and the B_i are bidifferential expressions. Furthermore we require, that $(\Gamma(End(E))[[\hbar]], *)$ is locally isomorphic to $M_N(\mathcal{W}_n)$, where $\mathcal{W}_n$ is the Weyl algebra on some open subset of $\mathbb{R}^{2n}$.

Locally isomorphic in this case means that we are given a local bundle isomorphism $\Phi : M_N(C^\infty(U))[[\hbar]] \to \Gamma(End(E))[[\hbar]]|_U$ over a local symplectomorphism, such that the product $*'$, induced by $*$, on

$M_N(C^\infty(U))[[\hbar]]$ is isomorphic to $M_N(\mathcal{W}_n)$ in the sense, that there exist differential operators $\{D_i\}$, such that

$$\varphi : (M_N(C^\infty(U))[[\hbar]], *') \to M_N(\mathcal{W}_n)$$

given by

$$\varphi(a) = a + \hbar D_i(a) + \dots, \quad a \in M_N(C^\infty(U))$$

is an isomorhism of algebras.

We want to do the same construction for deformation quantizations as we did for endomorphism bundles. We therfore need the infinitesimal version of $M_N(\mathcal{W}_n)$. This is just given by considering $\mathbb{O}_{2n}[[\hbar]]$, $\mathbb{O}_{2n} = \mathbb{C}[[\hat{x}_1, \dots, \hat{x}_n, \hat{\xi}_1, \dots, \hat{\xi}_n]]$, with the same product as in the Weyl quantization. With this product we denote the algebra by $\mathbb{A}^\hbar$. The infinitesimal structure of $M_N(\mathcal{W}_n)$ will then be $M_N(\mathbb{A}^\hbar)$.

Definition 3.0.5. *A formal symplectomorphism of $\mathbb{O}_{2n}$ is a continuous automorphism of $\mathbb{O}_{2n}$ induced from an automorphism on*

$$\mathbb{R}[[\hat{x}_1, \dots, \hat{x}_n, \hat{\xi}_1, \dots, \hat{\xi}_n]]$$

that preserves the formal standard Poison bracket $\{\cdot, \cdot\}$ on $\mathbb{O}_{2n}$.

Let G be the subgroup of automorphisms of $M_N(\mathbb{A}^\hbar)$ given by:

$\Phi \in G$ if Φ is $\hbar$ linear, continuous, mod $\hbar$ Φ is an automorphism Φ_0 of $M_N(\mathbb{O}_{2n})$ and Φ_0 induces a formal symplectomorphism on $\mathbb{O}_{2n}$.

Let $\Phi \in G$ and let φ be the induced symplectomorphism on $\mathbb{O}_{2n}$. In this case we will say, that Φ is a automorphism over φ.

Lemma 3.0.6. *Any automorphism of $M_N(\mathbb{A}^\hbar)$ over the identity symplectomorphism is inner.*

Proof. Let Φ be such an automorphism. Since, mod $\hbar$, it is an automorphism over the identity, it is inner mod $\hbar$, and we can hence assume that Φ is the identity mod $\hbar$. In other words

$$\Phi(a) = a + \hbar D_1(a) + \dots, \quad a \in M_N(\mathbb{O}_{2n})$$

Since Φ is an automorphism, D_1 is a derivation $M_N(\mathbb{O}_{2n})$ and hence on the form $X + [A, \cdot]$, where X is a formal vector field and $A \in M_N(\mathbb{O}_{2n})$. If we assume that $a, b \in \mathbb{O}_{2n}$, we have, since Φ is an automorphism, that

$$\{a, D_1(b)\} + \{D_1(a), b\} = D_1(\{a, b\})$$

which means, that X is a formal hamiltonian vector field. Therefore there exist an element $x \in \mathbb{O}_{2n}$ such that $D_1(a) = \{x, a\}$. We hence have

$$\Phi \circ \exp(-ad(x + A)) = id \bmod \hbar^2$$

Continuing in this way, the result follows.

Since it is well known from [NT95a], that the Lie algebra of G, in the case where $N = 1$, is given by

$$\mathfrak{g}_0 = \left\{ \frac{ia}{\hbar} | a \in \mathbb{A}^\hbar, \quad a \text{ real mod } \hbar, \quad a \in (\hat{x}_1, \dots, \hat{\xi}_1, \dots)^2 \text{ mod } \hbar \right\} / \frac{\mathbb{C}[[\hbar]]}{\hbar}$$

and that any element of G is of the form $\exp(g)$, $g \in \mathfrak{g}_0$.

We therefore see, that the Lie algebra $\mathfrak{g}_0$ of G for arbitrary N is given by

$$\begin{aligned} \mathfrak{g}_0 \quad = \quad & \left\{ \frac{ia}{\hbar} + b | b \in M_N(\mathbb{A}^\hbar), \quad a \in \mathbb{A}^\hbar, \quad a \text{ real mod } \hbar, \right. \\ & \left. a \in (\hat{x}_1, \dots, \hat{\xi}_1, \dots)^2 \text{ mod } \hbar \right\} / \frac{\mathbb{C}[[\hbar]]}{\hbar} \end{aligned}$$

To this Lie algebra we can add the derivations $\partial_{\hat{x}_1}, \dots, \partial_{\hat{\xi}_1}, \dots$. With these derivations added we call the Lie algebra $\mathfrak{g}$.

Let us suppose that we are given a deformation quantization $A^\hbar_E$ of $End(E)$. We define

$$I_m = \{a \in A^\hbar_E | a(m) = 0\}$$

and we let I^n_m denote the n-power of the ideal I_m in the undeformed product. The jet of $A^\hbar_E$ in m is defined by

$$J_m A^\hbar_E = \lim_{\overleftarrow{k}} A^\hbar_E / I^k_m$$

Since the value of the product in $A^\hbar_E$ in a point only depends on the derivatives in that point, the product descents to $J_m A^\hbar_E$.

If we choose a trivialization of $A^\hbar_E$ around m, we get an isomorphism $J_m A^\hbar_E \xrightarrow{\sim} M_N(\mathbb{A}^\hbar)$ and another trivialization will give an automorphism of $M_N(\mathbb{A}^\hbar)$ in G.

As in the case of $End(E)$ we do the following

Definition 3.0.7. *The frame bundle $\tilde{M}_{A^\hbar_E}$ is given by*

$$\tilde{M}_{A^\hbar_E} = \{(m, \Phi) | m \in M, \Phi : M_N(\mathbb{A}^\hbar) \xrightarrow{\sim} J_m \mathbb{A}^\hbar_E\}$$

As before, $\tilde{M}_{A^\hbar_E}$ is a principal bundle with fibre G.

Proposition 3.0.8. *For all $(m, \Phi) \in \tilde{M}_{A^\hbar_E}$ there exist an isomorphism*

$$\omega : T_{(m,\Phi)} \tilde{M}_{A^\hbar_E} \to \mathfrak{g}$$

satisfying

$$\begin{aligned} \omega(A^*) &= A \qquad A \in \mathfrak{g}_0 \\ \varphi^*\omega &= ad\varphi^{-1}\omega \quad \varphi \in G \\ d\omega + \tfrac{1}{2}[\omega, \omega] &= 0 \end{aligned}$$

where A is the fundamental vector field corresponding to A.

In other words ω is a flat connection with values in $\mathfrak{g}$.

Proof. The same as the case of endomorphism bundles.

As in the case of endomorphism bundles, we get a flat connection ∇ in the bundle

$$JA_E^\hbar = \tilde{M}_{A_E^\hbar} \times_G M_N(\mathbb{A}^\hbar)$$

Proposition 3.0.9. *The complex $(\Omega^*(M, JA_E^\hbar), \nabla)$ is acyclic and $Ker\nabla \xrightarrow{\sim} A_E^\hbar$.*

Proof. The same as the case of endomorphism bundles.

One sees, that a maximal compact subgroup of G is $H = U(n) \times SU(N)$. The $U(n)$-component comes from a maximal compact subgroup of $Sp(2n)$ and the $SU(N)$ comes from the maximal compact subgroup of the action of Gl_N on $M_N(\mathbb{C})$.

Since H is a maximal compact subgroup of G, we can reduce the bundle $\tilde{M}_{A^\hbar}$ to a H bundle P, which is easily seen to be a reduction of the principal bundle consisting of dual symplectic frames and frames of $End(E)$. We thus see, that $JA_E^\hbar$ is in fact isomorphic to $P \times_H \mathbb{A}^\hbar$. We will denote this bundle by $\mathbb{W}$.

We introduce a grading on $\mathbb{A}^\hbar$ in which $\hat{x}_1, \dots, \hat{x}_n, \hat{\xi}_1, \dots, \hat{\xi}_n$ has degree 1 and $\hbar$ has degree 2. This also gives a grading on $M_N(\mathbb{A}^\hbar)$. Furthermore we see, that the action of $U(n)$ on $\mathbb{A}^\hbar$ preserves the grading and we hence get a grading on $\mathbb{W}$.

We note, that we have an extension of Lie algebras

$$0 \to \frac{1}{\hbar} i\mathbb{R} + \mathbb{C}[[\hbar]] \to \tilde{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{3.1}$$

where

$$\tilde{\mathfrak{g}} = \left\{ \frac{ia}{\hbar} + b | a \in \mathbb{A}^\hbar, \quad a \text{ real mod } \hbar, \quad b \in M_N(\mathbb{A}^\hbar) \right\}$$

with bracket given by commutators. Since H acts semisimple on $\mathfrak{g}$, we get a H-equivariant lift of the quotient map $\tilde{\mathfrak{g}} \to \mathfrak{g}$. We can therefore lift the connection ∇ to a connection $\tilde{\nabla}$ taking values in $\tilde{\mathfrak{g}}$, meaning a collection of local $\tilde{\mathfrak{g}}$-oneforms $\{A_i\}$, i being labels of trivializations of $\mathbb{W}$ satisfying

$$A_i = g_{ij} dg_{ji} + g_{ij} A_j g_{ji}$$

where g_{ij} are the transition functions.

This connection however, is not flat, but because of the extension 3.1, the curvature $dA_i + \frac{1}{2}[A_i, A_i]$ is in $\Omega^2(M, \frac{1}{\hbar} i\mathbb{R} + \mathbb{C}[[\hbar]])$. Clearly the associated cohomology class is independent of the choice of lifting of ∇. By checking the definition of ∇, one sees, that the $\frac{1}{\hbar}$ component of $\tilde{\nabla}^2$ is $\frac{\omega}{i\hbar}$, where ω is the symplectic structure.

So we have for each deformation quantization of $End(E)$ a connection ∇ in $\mathbb{W}$, such that $Ker \nabla$ is isomorphic to the deformation quantization and the lifting of ∇ to a $\tilde{\mathfrak{g}}$-valued connection gives an element in

$$\frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]])$$

Proposition 3.0.10. *Let $A^{\hbar}_{1,E}$ and $A^{\hbar}_{2,E}$ be deformation quantizations with characteristic classes θ_1 and θ_2. Then $A^{\hbar}_{1,E} \simeq A^{\hbar}_{2,E}$ if and only if $\theta_1 = \theta_2$ in $\frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]])$.*

The proof of the above proposition relies on the following

Lemma 3.0.11. *Let $I_\omega : TM \to T^*M$ be the bundle isomorphism induced by ω. Since $T^*M \subset \mathbb{W}$, I_ω induces an element A in $\Omega^1(M, \mathbb{W})$. Put*

$$A_{-1} = \frac{A}{\hbar} \in \Omega(M, \frac{\mathbb{W}}{\hbar})$$

The complex

$$(\Omega^*(M, \mathbb{W}), AdA_{-1})$$

is acyclic and the cohomology is isomorphic to $\Gamma(End(E))[[\hbar]]$.

Proof. From lemma 3.12 in [NT01] we know, in the case of a trivial bundle E, that is $(\Omega^*(M, \mathbb{W}_t), Ad(A_{-1}))$ is acyclic and the cohomology is isomorphic to $C^\infty(M)[[\hbar]]$. Since

$$\begin{aligned}&(\Omega^*(M, \mathbb{W}), Ad(A_{-1})) = \\ &(\Omega^*(M, \mathbb{W}_t), Ad(A_{-1})) \otimes_{C^\infty(M)[[\hbar]]} \Gamma(End(E))[[\hbar]]\end{aligned}$$

and $\Gamma(End(E))$ is projective over $C^\infty(M)[[\hbar]]$, the result follows.

Proof(of proposition). Let $\tilde{\nabla}_1$ and $\tilde{\nabla}_2$ be the two connections with $\tilde{\nabla}_1^2 = \tilde{\nabla}_2^2 = \theta$. Note that we can actually assume, that $\tilde{\nabla}_1^2 = \tilde{\nabla}_2^2$ in $\frac{\omega}{i\hbar} + \Omega^2(M, \mathbb{C}[[\hbar]])$. We have

$$\tilde{\nabla}_1 - \tilde{\nabla}_2 = R_0 + R_1 + \dots, \quad R_i \in \Omega^1(M, \mathbb{W}^i)$$

From the equality of the curvatures we get $[A_{-1}, R_0] = 0$ and hence by lemma 3.0.11 we have an element $g_1 \in \Gamma(\mathbb{W}^1)$, such that $[g_1, A_{-1}] = R_0$. Therefore considering the connection $\nabla_{2,0} = ad(\exp g_1)\nabla_2$ we have

$$\tilde{\nabla}_1 - \tilde{\nabla}_{2,0} = R'_1 + \dots, \quad R'_i \in \Omega^1(M, \mathbb{W}^i)$$

Continuing by induction we get an element $g \in Inv(\Gamma(\mathbb{W}))$ conjugating ∇_1 into ∇_2, and hence the deformations will be isomorphic.

Let us now assume, that $A^\hbar_{1,E}$ and $A^\hbar_{2,E}$ are isomorphic. This induces an isomorphism between $\tilde{M}_{A^\hbar_{1,E}}$ and $\tilde{M}_{A^\hbar_{2,E}}$ compatible with the connections on $\tilde{M}_{A^\hbar_{1,E}}$ and $\tilde{M}_{A^\hbar_{2,E}}$. In particular we get an automorphism of $\mathbb{W}$ mapping ∇_1 to ∇_2, and we hence get, that $A^\hbar_{1,E}$ and $A^\hbar_{2,E}$ have the same characteristic class.

Theorem 3.0.12. *The deformation quantizations of an endomorphism bundle are classified by the affine space*

$$\frac{\omega}{i\hbar} + H^2(M, \mathbb{C}[[\hbar]])$$

Proof. We only need to prove, that for a class $\theta \in \frac{\omega}{i\hbar} + \Omega^2(M, \mathbb{C}[[\hbar]])$, we have a deformation quantization with characteristic class θ. To do this we start with a connection in P and thus get a H-connection ∇ in $\mathbb{W}$. We have that

$$[\nabla + A_{-1}, \nabla + A_{-1}] = \frac{\omega}{i\hbar} + 2[\nabla, A_{-1}] + [\nabla, \nabla]$$

One checks, that $[A_{-1}, [\nabla, A_{-1}]] = 0$ and according to lemma 3.0.11 we get an element $A_0 \in \Omega^1(M, \mathbb{W})$ such that $[A_{-1}, A_0] = [\nabla, A_{-1}]$. If we put $\nabla_0 = \nabla + A_{-1} + A_0$ we have

$$[\nabla_0, \nabla_0] - \theta = 0 \bmod \Omega^2(M, \mathbb{W})$$

Let us assume, that we have constructed ∇_n with $\nabla_n^2 - \theta = 0 \bmod \Omega^2(M, \mathbb{W}^{i \geq n})$. By the Bianchi identity we have $[\nabla_n, [\nabla_n, \nabla_n]] = 0$ and therfore have $[A_{-1}, ([\nabla_n, \nabla_n] - \theta)_n] = 0$, where $([\nabla_n, \nabla_n] - \theta)_n$ is the n-th component of $[\nabla_n, \nabla_n] - \theta$. As before, we get an element A_n with $[A_{-1}, A_n] = ([\nabla_n, \nabla_n] - \theta)_n$ and considering $\nabla_{n+1} = \nabla_n + A_n$ we have $[\nabla_{n+1}, \nabla_{n+1}] - \theta = 0 \bmod \Omega^2(M, \mathbb{W}^{i \geq n+1})$.

We can thus for each class θ in $\frac{\omega}{i\hbar} + \Omega^2(M, \mathbb{C}[[\hbar]])$ construct a connection ∇_F with values in $\tilde{\mathfrak{g}}$ and curvature θ. We therefore only need to check, that the complex $(\Omega^*(M, \mathbb{W}), \nabla_F)$ is acyclic, the kernel is isomorphic to $\Gamma(End(E)[[\hbar]]$ and the product induced by the product on $\mathbb{W}$ gives a deformation quantization of $End(E)$. These results however, follows from the proof of the proposition 3.0.10, since we locally conjugate ∇_F to a connection on the form

$$d - \sum_{i=1}^{n} (\partial_{\tilde{x}_i} \otimes dx_i + \partial_{\tilde{\xi}_i} \otimes d\xi_i)$$

4. Lie algebra cohomology

Definition 4.0.13. *A differential graded Lie algebra $(\mathfrak{g}, d)$ over a commutative unital ring k is a ($\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$) graded k-module $\mathfrak{g}$ with a bracket operation $[\cdot,\cdot] : \mathfrak{g}^j \times \mathfrak{g}^i \to \mathfrak{g}^{i+j}$ and a differential $\partial : \mathfrak{g}^i \to \mathfrak{g}^{i-1}$ satisfying :*

$$\begin{gathered}
\partial[g_1, g_2] = [\partial g_1, g_2] + (-1)^{|g_1|}[g_1, \partial g_2] \\
[g_1, g_2] = -(-1)^{|g_1||g_2|}[g_2, g_1] \\
[g_1, [g_2, g_3]] + (-1)^{|g_3|(|g_1|+|g_2|)}[g_3, [g_1, g_2]] \\
+(-1)^{|g_1|(|g_2|+|g_3|)}[g_2, [g_3, g_1]] = 0
\end{gathered}$$

where $|\cdot|$ is the degree.

A $\mathfrak{g}$ module $\mathbb{L}^*$ is a complex $\mathbb{L}^*$ with an action of $\mathfrak{g}$, i.e. we have a map $\mathfrak{g}^i \times \mathbb{L}^j \to \mathbb{L}^{i+j}$ satisfying

$$g_1 g_2 l - (-1)^{|g_1||g_2|} g_2 g_1 l = [g_1, g_2] l$$

and

$$\partial_{\mathbb{L}^*}(gl) = (\partial_{\mathfrak{g}} g) l + (-1)^{|g|} g(\partial_{\mathbb{L}^*} l)$$

Given a differential graded Lie algebra $\mathfrak{g}$, we can define differential graded Lie algebra $\mathfrak{g}[\epsilon]$ as follows

$$\begin{aligned}
\mathfrak{g}[\epsilon] &= \mathfrak{g} + \epsilon\mathfrak{g} \qquad\qquad |\epsilon| = 1 \\
[g_1, \epsilon g_2] &= \epsilon[g_1, g_2] \\
[\epsilon g_1, \epsilon g_2] &= 0 \\
\partial(g_1 + \epsilon g_2) &= \partial_{\mathfrak{g}} g_1 + g_2 - \epsilon \partial_{\mathfrak{g}} g_2
\end{aligned}$$

Also one can construct the enveloping algebra by

$$U(\mathfrak{g}) = T(\mathfrak{g})/(g_1 \otimes g_2 - (-1)^{|g_1||g_2|} g_2 \otimes g_1 = [g_1, g_2])$$

where $T(\mathfrak{g})$ is the tensor algebra. Furthermore $U(\mathfrak{g})$ has a differential induced by the differential on $\mathfrak{g}$ by the graded Leibniz rule and a grading.

We note that $(U(\mathfrak{g}[\epsilon]), \partial)$ is a $\mathfrak{g}$ module. Let $\mathfrak{h}$ denote a Lie subalgebra of $\mathfrak{g}$. For a $\mathfrak{g}$ module $\mathbb{L}^*$ module we define

$$C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*) = \mathrm{Hom}_{U(\mathfrak{g})}(U(\mathfrak{g}[\epsilon]) \otimes_{U(\mathfrak{h}[\epsilon])} k, \mathbb{L}^*)$$

This will have a differential induced by the differential on $U(\mathfrak{g}[\epsilon])$ and the differential on $\mathbb{L}^*$. The homology of this complex will be denoted by $H^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*)$.

We are now going to give a construction of classes in $C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*)$ in special cases. First we assume, that $\mathbb{L}^*$ is homotopically constant in the following sense

Definition 4.0.14. *A $\mathfrak{g}$-module $\mathbb{L}^*$ is called homotopically constant if there exist operations*

$$\iota_g : \mathbb{L}^* \to \mathbb{L}^{*-1}$$

satisfying

$$[\partial, \iota_g] = L_g \quad [\partial, L_g] = 0$$
$$[L_{g_1}, \iota_{g_2}] = \iota_{[g_1,g_2]} \quad [\iota_{g_1}, \iota_{g_2}] = 0$$

where we have denoted the action of $\mathfrak{g}$ by L_g.

In other words, we have an action of the differential graded algebra $U(\mathfrak{g}[\epsilon])$ on $\mathbb{L}^*$.

If we furthermore assume, that there is a $\mathfrak{h}$-equivariant projection $\nabla : \mathfrak{g} \to \mathfrak{h}$ of the embedding $\mathfrak{h} \to \mathfrak{g}$, we get the usual Chern-Weil homomorphism, i.e. a map of complexes

$$CW : C^*(\mathfrak{h}[\epsilon], \mathfrak{h}; \mathbb{L}^*) \to C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*)$$

given in the following way:

For an element $g_1 \wedge g_2$ define $R(g_1 \wedge g_2) = [\nabla(g_1), \nabla(g_2)] - \nabla([g_1, g_2])$. Taking cup product gives $R^n : \wedge^{2n}\mathfrak{g} \to \wedge^n \epsilon\mathfrak{h}$. By composition this gives a map

$$\varphi : C^*(\mathfrak{h}[\epsilon], \mathfrak{h}; \mathbb{L}^*) \to C^*(\mathfrak{g}, \mathbb{L}^*)$$

There are operations λ^n on $\mathbb{L}^*$ given by

$$g_1 \wedge \ldots \wedge g_n \to \iota_{g_1 - \nabla g_1} \cdots \iota_{g_n - \nabla g_n} l$$

where $l \in \mathbb{L}^*$. Finally we set

$$CW(a) = \sum_n \lambda^n \cup \varphi(a)$$

where $\cup$ is the cup product. One checks that this gives a morphism of complexes.

Next we will give a construction of classes in $C^*(\mathfrak{h}[\epsilon], \mathfrak{h}; \mathbb{L}^*)$ for special cases of $\mathbb{L}^*$. To this end we need the following

Definition 4.0.15. *A $\mathfrak{g}$-module $\mathbb{L}^*$ is called very homotopically constant if $\mathbb{L}^*$ is homotopically constant and we have operations*

$$L_{\underline{g}} : \mathbb{L}^* \to \mathbb{L}^{*+1}$$
$$\iota_{\underline{g}} : \mathbb{L}^* \to \mathbb{L}^*$$

satisfying

$$[\partial, \iota_{\underline{g}}] = L_{\underline{g}} - \iota_g \quad [\partial, L_{\underline{g}}] = L_g \quad [\iota_{\underline{g}_1}, \iota_{\underline{g}_2}] = 0 \quad [\iota_{g_1}, \iota_{\underline{g}_2}] = 0$$
$$[\iota_{\underline{g}_1}, L_{g_2}] = \iota_{\underline{[g_1,g_2]}} \quad [\iota_{\underline{g}_1}, L_{\underline{g}_2}] = 0 \quad [L_{g_1}, L_{\underline{g}_2}] = L_{\underline{[g_1,g_2]}} \quad [L_{\underline{g}_2}, L_{\underline{g}_2}] = 0$$

in other words, we have an action of the differential graded algebra $U(\mathfrak{g}[\epsilon,\eta]) = U(\mathfrak{g}[\epsilon][\eta])$ on $\mathbb{L}^*$.

We now assume, that $\mathbb{L}^*$ is a very homotopically constant $\mathfrak{h}$-module. We denote by $\mathbb{L}^*_{\mathfrak{h}+\underline{\mathfrak{h}}}$ elements $l \in \mathbb{L}^*$ with $L_h l = 0$ and $L_{\underline{h}} l = 0$. We note that this is a complex. We get a morphism of complexes $\mathbb{L}^*_{\mathfrak{h}+\underline{\mathfrak{h}}} \to C^*(\mathfrak{h}[\epsilon], \mathfrak{h}; \mathbb{L}^*)$ by:

$$\mathbb{L}^*_{\mathfrak{h}+\underline{\mathfrak{h}}} \ni l \to (h_1 \cdots h_n \to \iota_{\underline{h}_1} \cdots \iota_{\underline{h}_n} l)$$

4.1. **Examples.** We are going to give some examples of relative classes in Lie algebra cohomology.

Example 1 Consider the extension 3.1 and choose a $\mathfrak{h}$-equivariant lift $\nabla : \mathfrak{g} \to \tilde{\mathfrak{g}}$ of the quotient map $\tilde{\mathfrak{g}} \to \mathfrak{g}$, $\mathfrak{h}$ being $\mathfrak{u}(n) + \mathfrak{su}(N)$. We then define the class θ in $C^*(\mathfrak{g}, \mathfrak{h}; \frac{1}{\hbar}\mathbb{C}[\hbar])$ by

$$\theta(g_1, g_2) = \nabla([g_1, g_2]) - [\nabla g_1, \nabla g_2] \tag{4.1}$$

We want to show, that θ actually comes from a sort of Chern-Weil map. Let $k : \tilde{\mathfrak{g}} \to \mathfrak{g}$ denote the quotient map and let $\tilde{\mathfrak{h}} = k^{-1}(\mathfrak{h})$, i.e.

$$\tilde{\mathfrak{h}} = \mathfrak{h} + \frac{i}{\hbar}\mathbb{R} + \mathbb{C}[[\hbar]]$$

Note that $C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*)$ and $C^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}; \mathbb{L}^*)$ are quasi isomorphic, when $\mathbb{L}^*$ is a $\mathfrak{g}$ module. We have a Chern-Weil homomorphism

$$CW : C^*(\tilde{\mathfrak{h}}[\epsilon], \tilde{\mathfrak{h}}; \mathbb{L}^*) \to C^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}; \mathbb{L}^*)$$

as before. A Choice of a $\tilde{\mathfrak{h}}$ equivariant split $\nabla' : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{h}}$ is given by $\nabla' = \nabla'' \circ k + id - \nabla \circ k$, where $\nabla'' : \mathfrak{g} \to \mathfrak{h}$ is a $\mathfrak{h}$-equivariant splitting of the embedding $\mathfrak{h} \to \mathfrak{g}$. Let θ be defined on $\tilde{\mathfrak{h}}$ to be the projection on $\frac{i}{\hbar}\mathbb{R} + \mathbb{C}[[\hbar]]$. It is now easy to see, that $CW(\theta)$ is, under the quasi isomorphism between $C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*)$ and $C^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}; \mathbb{L}^*)$, the same as the θ we defined in the start of this example.

Example 2 Some other classes in $C^*(\mathfrak{g}, \mathfrak{h})$, we need, are also coming from a Chern-Weil construction. We consider an $\mathfrak{h}$-equivariant splitting

∇' of the embedding $\mathfrak{h} \to \mathfrak{g}$. Composed with the the projection $\mathfrak{u}(n) + \mathfrak{su}(N) \to \mathfrak{su}(N)$ we get an $\mathfrak{h}$ equivariant map $\nabla : \mathfrak{g} \to \mathfrak{su}(N)$. Using this we therefore get a Chern-Weil homomorphism

$$CW : C^*(\mathfrak{su}(N)[\epsilon], \mathfrak{su}(N)) \to C^*(\mathfrak{g}, \mathfrak{su}(N))$$

It is clear, that this homomorphism in fact maps into $C^*(\mathfrak{g}, \mathfrak{h})$. We therefore get the usual classes, for example the usual chern character ch, which is

$$\exp(R) = \sum \frac{1}{n!} Tr(R^n)$$

where

$$R(g_1, g_2) = [\nabla g_1, \nabla g_2] - \nabla[g_1, g_2]$$

Here Tr denotes the usual non normalized trace on $\mathfrak{su}(N)$. This class is off course the Chern Weil map on the symmetric polynomium ch on $\mathfrak{su}(N)$ given by

$$ch(h_1, \dots, h_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{1}{n!} Tr(h_{\sigma(1)} \cdots h_{\sigma(n)})$$

Since $C^*(\mathfrak{su}(N)[\epsilon], \mathfrak{su}(N))$ embeds in $C^*(\tilde{\mathfrak{h}}[\epsilon], \tilde{\mathfrak{h}})$, the Chern Weil construction given in this example is just a particular case of the Chern Weil homomorphism

$$CW : C^*(\tilde{\mathfrak{h}}[\epsilon], \tilde{\mathfrak{h}}) \to C^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$$

Example 3 We can off course do the construction we did in example 2 for $\mathfrak{u}(n)$ instead of $\mathfrak{su}(N)$ and therfore get a Chern-Weil homomorphism

$$C^*(\mathfrak{u}(n)[\epsilon], \mathfrak{u}(n)) \to C^*(\mathfrak{g}, \mathfrak{h})$$

that again also can be viewed as the composition

$$C^*(\mathfrak{u}(n)[\epsilon], \mathfrak{u}(n)) \to C^*(\tilde{\mathfrak{h}}[\epsilon], \tilde{\mathfrak{h}}) \to C^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$$

We will in particular be interested in the $\hat{A}$ in this setting, that is the symmetric polynomium coming from

$$h \to \det\left(\frac{h/2}{\sinh(h/2)}\right)$$

5. Cyclic homology

We consider a differential graded unital algebra (A, δ) over a commutative ring k containing $\mathbb{Q}$, i.e. an algebra A, $A = \oplus_n A^n$, $A^n A^m \subset A^{n+m}$, $\delta : A^* \to A^{*-1}$ and $\delta(ab) = \delta(a)b + (-1)^{|a|} a\delta(b)$.

Define an operator τ on $A^{\otimes(n+1)}$ by

$$\tau(a_0 \otimes \ldots \otimes a_n) = (-1)^{(|a_n|+1)\sum_{i=0}^{n-1}(|a_i|+1)} a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}$$

and consider the complex

$$\ldots \xleftarrow{b+\delta} C^n(A)/Im(1-\tau) \xleftarrow{b+\delta} C^{n+1}(A)/Im(1-\tau) \xleftarrow{b+\delta} \ldots$$

where $C^{n+1}(A)$ is elements on the form $a_0 \otimes \ldots \otimes a_k$ with $k + \sum |a_i| = n$,

$$\begin{aligned} b(a_0 \otimes \ldots \otimes a_n) = & \sum_{k=0}^{n-1} (-1)^{k+\sum_{i=0}^{k} |a_i|} a_0 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_n \\ & + (-1)^{(|a_n|+1)\sum_{i<n}(|a_i|+1)+|a_n|} a_n a_0 \otimes \ldots \otimes a_{n-1} \end{aligned}$$

and

$$\delta(a_0 \otimes \ldots \otimes a_n) = \sum_{k=0}^{n} (-1)^{\sum_{i=1}^{k-1}(|a_i|+1)} a_0 \otimes \ldots \otimes \delta(a_k) \otimes \ldots \otimes a_n$$

The complex is denoted $C_*^\lambda(A)$, the homology is the cyclic homology of A and is denoted by $HC_*(A)$.

The reduced cyclic homology is given by the homology of the complex

$$\ldots \xleftarrow{b+\delta} \overline{C}^n(A)/Im(1-\tau) \xleftarrow{b+\delta} \overline{C}^{n+1}(A)/Im(1-\tau) \xleftarrow{b+\delta} \ldots$$

where $\overline{C}^*(A)$ comes from considering $\bar{A}^{\otimes *}$ instead of $A^{\otimes *}$, where $\bar{A} = A/k \cdot 1$.

The reduced cyclic homology of A is denoted by $\overline{HC}_*(A)$.

It is well known, see [Lod98] and [BNT99], that there is an exact sequence

$$\ldots HC_n(k) \to HC_n(A) \to \overline{HC}_n(A) \to HC_{n-1}(k) \to \ldots \tag{5.1}$$

We will briefly give a construction, due to Brodzki, of the connecting morphism $\overline{HC}_*(A) \to HC_{*-1}(k)$ at the level of complexes, see [Bro93] and [BNT99]; i.e. a morphism of complexes

$$Br : \overline{C}_*^\lambda(A) \to C_{*-1}^\lambda(k)$$

given the connecting homomorphism at the level of homology. Let $l : A \to k$ be a k-linear map with $l(1) = 1$. Put

$$\begin{aligned} \rho(a) &= l(\delta(a)) \quad a \in A \\ \rho(a_1 \otimes a_2) &= l(a_1)l(a_2) - l(a_1 a_2) \quad a_1 \otimes a_2 \in A^{\otimes 2} \\ \rho &= 0 \text{ on } A^{\otimes m},\ m \geq 3 \end{aligned}$$

and define $Br : \overline{C}_*^\lambda(A) \to C_{*-1}^\lambda(k)$ by:

On $\overline{C}_{2n+1}^\lambda$,

$$\begin{aligned} Br(a_0 \otimes \ldots \otimes a_m) = & \textstyle\sum_{i=0}^{m} (-1)^{\sum_{k<i}(|a_k|+1)\sum_{k\geq i}(|a_k|+1)} (\rho \otimes \ldots \otimes \rho) \\ & (a_i \otimes \ldots \otimes a_0 \otimes a_m \otimes \ldots \otimes a_{i-1})(n+1)! \cdot 1^{\otimes 2n+1} \end{aligned}$$

On $\overline{C}^{\lambda}_{2n}$ Br is zero.

We now consider the differential graded algebra $k[\eta]$, where η has degree one, $\eta^2 = 0$ and differential given by ∂_η. For a differential graded algebra A we define $A[\eta]$ to be $A \otimes k[\eta]$, where $\otimes$ is the tensor product of differential graded algebras. It is not difficult to see, that $HC_*(A[\eta]) = 0$ and we therfore have

Proposition 5.0.1. *The morphism*

$$Br : \overline{C}^{\lambda}_{*}(A[\eta]) \to C^{\lambda}_{*-1}(k)$$

is a quasi isomorphism.

Since it is not standard, we mention, that the reduced cyclic homology is Morita invariant, at least in the case of algebras and matrices over these algebras. To see this, let A be an algebra, and let $l : A \to k$ be a map, that is needed in the construction of Br. Let tr denote the normalized trace $tr : M_n(A) \to A$. We now have a commutative diagram

$$\begin{array}{ccccccc}
\to & HC^*(M_N(A)) & \to & \overline{HC}^*(M_N(A)) & \xrightarrow{Br} & HC^{*-1}(k) & \to \\
 & \downarrow tr & & \downarrow tr & & \| & \\
\to & HC^*(A) & \to & \overline{HC}^*(A) & \xrightarrow{Br} & HC^{*-1}(k) & \to
\end{array}$$

where $Br : \overline{HC}^*(M_N(A)) \to HC^{*-1}(k)$ is induced by $l \circ tr$. According to [Lod98], $tr : HC^*(M_n(A)) \to HC^*(A)$ is an isomorphism. The result therefore follows from 5.1.

5.1. Operations on the periodic complex.

For the periodic cyclic complex we consider $\prod_n A \otimes \bar{A}^{\otimes n}$. We give this a $\mathbb{Z}/2\mathbb{Z}$ grading by

$$|a_0 \otimes \ldots \otimes a_n| = n + \sum_i |a_i| \bmod 2$$

On this we consider the differential $b + B + \delta$ where b and δ are given as before and

$$B(a_0 \otimes \ldots \otimes a_n)$$
$$= \sum_{i=0}^{n} (-1)^{\sum_{j \le i}(|a_j|+1) \sum_{j \ge i+1}(|a_j|+1)} 1 \otimes a_i \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}$$

We will denote this complex by $CC^{per}_*(A)$.

The main feature about cyclic periodic homology, that we are going to need, is the following; see [NT98] for complete formulas

Theorem 5.1.1. *There is a morphism of complexes*

$$\overline{C}^{\lambda}_{*}(A[\eta]) \otimes CC^{per}_*(A) \to CC^{per}_*(A)$$

satisfying the following

- *$n!\eta^{\otimes n+1} \cdot a = a$ for $a \in CC^{per}_*(A)$*
- *The component in A of $(b_1 \otimes \ldots \otimes b_n) \cdot (a_0 \otimes \ldots \otimes a_m)$, where $b_1 \otimes \ldots \otimes b_n \in \overline{C}^{\lambda}_*(A)$, is zero when $m \neq n$ and equal to*

$$\sum_i \frac{1}{n!}(-1)^{i(n-1)} a_0[b_{i+1}, a_1] \cdots [b_n, a_{n-i}][b_1, a_{n-i+1}] \cdots [b_i, a_n]$$

when $m = n$.

The framework underlying theorem 5.1.1 also gives other operations on $CC^{per}_*(A)$, see [NT98] for details. Let $(C^*(A,A), b)$ be the Hochshild cohomological complex, i.e. $C^*(A,A) = Hom_k(\bar{A}^{\otimes *}, A)$ and

$$b\varphi(a_1, \otimes, a_{n+1}) = (-1)^n a_1 \varphi(a_2, \ldots, a_{n+1})$$
$$+ \sum_{j=1}^n (-1)^{n+j} \varphi(a_1, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) - \varphi(a_1, \ldots, a_n) a_{n+1}.$$

Given two elements $\varphi \in C^n(A,A)$, $\psi \in C^m(A,A)$, define

$$\varphi \circ \psi(a_1, \ldots, a_{n+m-1})$$
$$= \sum_{j \geq 0} (-1)^{(n-1)j} \varphi(a_1, \ldots, a_j, \psi(a_{j+1}, \ldots, a_{j+m}), \ldots).$$

Set

$$[\varphi, \psi] = \varphi \circ \psi - (-1)^{(n+1)(m+1)} \psi \circ \varphi.$$

With this bracket and with a suitable defined grading, $C^*(A,A)$ actually becomes a differential graded Lie algebra.

For $\varphi \in C^n(A,A)$ one can construct operations

$$L_\varphi : CC^{per}_*(A) \to CC^{per}_{*-n+1}(A)$$
$$I_\varphi : CC^{per}_*(A) \to CC^{per}_{*-n}(A)$$

such that

$$[L_\varphi, L_\psi] = L_{[\varphi,\psi]} \quad [I_\varphi, L_\psi] = I_{[\varphi,\psi]} \quad [B + b, I_\varphi] = I_{b\varphi} + L_\varphi$$

6. The fundamental Class

We consider the reduced cyclic homology complex of $M_N(\mathbb{A}^\hbar)[\hbar^{-1}]$. According to lemma 5.1.1. in [BNT99] and Morita invariance of reduced cyclic homology, the homology is given in the following way

$$\overline{HC}_i(M_N(\mathbb{A}^\hbar)[\hbar^{-1}]) = \mathbb{C}[[\hbar, \hbar^{-1}], \quad i = 1, 3, \ldots, 2n-1$$
$$\overline{HC}_i(M_N(\mathbb{A}^\hbar)[\hbar^{-1}]) = 0 \quad \text{otherwise}$$

A concrete generator for the homology in dimension $2n-1$ is given by

$$U_0 = \frac{1}{2n(i\hbar)^n} \sum_{\sigma \in S_{2n}} (v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(2n)})$$

where $(v_1, \ldots, v_{2n}) = (\hat{x}_1, \hat{\xi}_1, \ldots, \hat{x}_n, \hat{\xi}_n)$.

Let $\tilde{\mathfrak{h}}$ denote the inverse image of $\mathfrak{h}$ under the map $\tilde{\mathfrak{g}} \to \mathfrak{g}$. Note that U_0 is invariant under the action of $\tilde{\mathfrak{h}}$ and therefore, by the result on $\overline{HC}_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}]))$, extends uniquely in homology to a class

$$U \in C^*(\tilde{\mathfrak{h}}[\epsilon], \tilde{\mathfrak{h}}; \overline{C}_*^\lambda(M_N(\mathbb{A}^\hbar[\hbar^{-1}])))$$

We want to work with the differential graded algebra $M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]$ instead of $M_N(\mathbb{A}^\hbar[\hbar^{-1}])$. In the complex

$$\overline{C}_*^\lambda(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])$$

we define $\eta^{(k)} = k!\eta^{\otimes k+1}$. We note, that the complex $\overline{C}_*^\lambda(M_N(\mathbb{A}^\hbar[\hbar^{-1}]))$ is homotopically constant as a module over the commutator Lie algebra of $M_N(\mathbb{A}^\hbar[\hbar^{-1}])$ by putting

$$\begin{aligned} &\iota_g(a_0 \otimes \ldots \otimes a_p) \\ &= \sum_{i=0}^{p} (-1)^{\sum_{p \geq i}(|a|_p+1)(|g|+1)} a_0 \otimes \ldots \otimes a_i \otimes g \otimes \ldots \otimes a_p \end{aligned}$$

Using these operations, we define classes

$$\eta^{[k]} \in C^*(\tilde{\mathfrak{h}}[\epsilon], \tilde{\mathfrak{h}}; \overline{C}_*^\lambda(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$$

by

$$(h_1\epsilon, \ldots, h_p\epsilon) \to \iota_{h_1\eta} \cdots \iota_{h_p\eta}\eta^{(k)}$$

since $\eta^{(k)}$ is a class in $\overline{C}_*^\lambda(M_N(\mathbb{A}^\hbar[\hbar^{-1}])_{\mathfrak{h}+\underline{\mathfrak{h}}}$.

We have

Lemma 6.0.2. *One has*

$$U = \sum_{m \geq 0} (\hat{A} \cdot e^\theta \cdot ch)_{2m}^{-1} \cdot \eta^{[m]}$$

in $H^*(\tilde{\mathfrak{h}}[\epsilon], \tilde{\mathfrak{h}}; \overline{C}_*^\lambda(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$.

Proof. It is well known from [BNT99], that there is a splitting principle, i.e. the inclusion morphism

$$\begin{gathered} H^*((\tilde{\mathfrak{h}})[\epsilon], (\tilde{\mathfrak{h}}); \overline{C}_*^\lambda(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])) \to \\ H^*((\tilde{\mathfrak{d}}_n + \mathfrak{su}(N))[\epsilon], (\tilde{\mathfrak{d}}_n + \mathfrak{su}(N)); \overline{C}_*^\lambda(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])) \end{gathered}$$

where $\tilde{\mathfrak{d}}_n = \mathfrak{d}_n + \hbar^{-1}\mathbb{C}[[\hbar]]$ and $\mathfrak{d}_n$ is $n \times n$ diagonal matrices, is injective.

Therefore, we only have to identify the two classes in

$$H^*((\tilde{\mathfrak{d}}_n + \mathfrak{su}(N))[\epsilon], (\tilde{\mathfrak{d}}_n + \mathfrak{su}(N)); \overline{C}_*^\lambda(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])).$$

We next note, that we can factor the classes at stake in the following way: Write

$$M_N(\mathbb{A}^\hbar[\hbar^{-1}]) = \mathbb{A}_1^\hbar[\hbar^{-1}] \otimes \ldots \otimes \mathbb{A}_1^\hbar[\hbar^{-1}] \otimes M_N(\mathbb{A}_1^\hbar)[\hbar^{-1}]$$

where $\mathbb{A}_1^\hbar$ is the formal Weyl algebra in one variable. We can write $U = U_1 \times \ldots \times U_1 \times U_1'$, where U_1 is the extension of the fundamental class in $C^*(\mathfrak{d}_1[\epsilon], \mathfrak{d}_1; \overline{C}_*^\lambda(\mathbb{A}^\hbar[\hbar^{-1}, \eta]))$ and U_1' is the extension of the fundamental class in

$$C^*((\tilde{\mathfrak{d}}_1 + \mathfrak{su}(N))[\epsilon], \tilde{\mathfrak{d}}_1 + \mathfrak{su}(N); \overline{C}_*^\lambda(M_N(\mathbb{A}_1^\hbar[\hbar^{-1}])[\eta]))$$

We thus need to identify U_1 and U_1'.

In the first case we can represent U_1 by

$$U_1 = \sum_{m=1}^{\infty} \frac{1}{m}((i\hbar)^{-1}\hat{\xi} \otimes \hat{x})^{\otimes m} c_1^{m-1} \tag{6.1}$$

where c_1 is the first Chern class.

Recall from the definition of $\mathbb{A}_1^\hbar[\hbar^{-1}]$, that this is $\mathbb{C}[[\hat{x}, \hat{\xi}]][[\hbar, \hbar^{-1}]$ with a product $*$. Given an element $f \in \mathbb{A}_1^\hbar[\hbar^{-1}]$, we can regard f as a function in the variables $\hat{x}, \hat{\xi}$ with values in $\mathbb{C}[[\hbar, \hbar^{-1}]$. Hence given $f \in \mathbb{A}_1^\hbar[\hbar^{-1}]$, we can define $l(f) = f(0,0)$. With this l we get, according to section 5, a quasi isomorphism of complexes

$$Br : \overline{C}_*^\lambda(\mathbb{A}^\hbar[\hbar^{-1}, \eta]) \to C_*^\lambda(\mathbb{C}[[\hbar, \hbar^{-1}])$$

One checks, that this gives a morphism of complexes

$$C^*(\mathfrak{d}_1[\epsilon], \mathfrak{d}_1; \overline{C}_*^\lambda(\mathbb{A}_1^\hbar[\hbar^{-1}, \eta])) \xrightarrow{Br} C^*(\mathfrak{d}_1[\epsilon], \mathfrak{d}_1; C_*^\lambda(\mathbb{C}[[\hbar, \hbar^{-1}]))$$

and therefore a quasi isomorphism of complexes.

A computation now gives

$$Br(U_1) = \sum_{m=0}^{\infty} 1^{(m)} \hat{A}_{2m}^{-1}$$

where $1^{(m)} = m!(m+1)!1^{\otimes(2m+1)}$ and $\hat{A}$ is given in example 3 in section 4. On the other hand we have $Br(\eta^{[m]}) = 1^{(m-1)}$, where $\eta^{[m]}$ is the class

$$\epsilon d_1, \ldots, \epsilon d_k \to \iota_{d_1} \cdots \iota_{d_k} \eta^{(m)}, \quad d_i \in \mathfrak{d}_1$$

Since Br is a quasi isomorphism, we have

$$U_1 = \sum_{m=0}^{\infty} \eta^{[m]} \hat{A}_{2m}^{-1}$$

in $H^*(\mathfrak{d}_1[\epsilon], \mathfrak{d}_1; \hat{C}_*^\lambda(\mathbb{A}_1^\hbar[\hbar^{-1}, \eta]))$.

We note, that we have a morphism of complexes

$$C^*((\tilde{\mathfrak{d}}_1+\mathfrak{su}(N))[\epsilon],\tilde{\mathfrak{d}}_1+\mathfrak{su}(N);\overline{C}^{\lambda}_*(M_N(\mathbb{A}^{\hbar}[\hbar^{-1}])))$$
$$Tr\downarrow$$
$$C^*((\tilde{\mathfrak{d}}_1+\mathfrak{su}(N))[\epsilon],\tilde{\mathfrak{d}}_1+\mathfrak{su}(N);\overline{C}^{\lambda}_*(\mathbb{A}^{\hbar}[\hbar^{-1}]))$$

where in the bottom row $\mathfrak{su}(N)$ acts trivially and Tr denotes the morphism of complexes

$$\overline{C}^{\lambda}_*(M_N(\mathbb{A}^{\hbar}_1)[\hbar^{-1}])\to\overline{C}^{\lambda}_*(\mathbb{A}^{\hbar}_1[\hbar^{-1}])$$

induced by the normalized trace Tr.

In the top row we have the class U_1', that in homology is the unique extension of the fundamental class. In the bottom row we have a class U', given by the same formula as in 6.1. Also this is, in homology, a unique extension of the fundamental class. Therefore $Tr(U)=U'$ in

$$H^*((\tilde{\mathfrak{d}}_1+\mathfrak{su}(N)[\epsilon],\tilde{\mathfrak{d}}_1+\mathfrak{su}(N);\overline{C}^{\lambda}_*(\mathbb{A}^{\hbar}[\hbar^{-1}]))$$

We further note, that there are morphisms of complexes

$$C^*((\tilde{\mathfrak{d}}_1+\mathfrak{su}(N))[\epsilon],\tilde{\mathfrak{d}}_1+\mathfrak{su}(N);\overline{C}^{\lambda}_*(M_N(\mathbb{A}^{\hbar}[\hbar^{-1}])[\eta]))$$
$$Tr\downarrow$$
$$C^*((\tilde{\mathfrak{d}}_1+\mathfrak{su}(N))[\epsilon],\tilde{\mathfrak{d}}_1+\mathfrak{su}(N);\overline{C}^{\lambda}_*(\mathbb{A}^{\hbar}[\hbar^{-1}][\eta]))$$
$$Br\downarrow$$
$$C^*((\tilde{\mathfrak{d}}_1+\mathfrak{su}(N))[\epsilon],\tilde{\mathfrak{d}}_1+\mathfrak{su}(N);C^{\lambda}_*(\mathbb{C}[[\hbar,\hbar^{-1}]))$$

where Br is the Brodzki map as before. We thus have

$$Br(Tr(U_1'))=\sum_{m\leq 0}\hat{A}^{-1}_{2m}1^{(m)}$$

Note that

$$Br(Tr(\eta^{[m]}))=\sum_{l=0}^{\infty}1^{(m+l)}(e^{\theta}ch)_{2l}$$

where θ is given in example 1 in section 4 and ch is given in example 2 in section 4.

Since $Br\circ Tr$ is a quasi isomophism we get

$$U_1'=\sum_{m=0}^{\infty}(\hat{A}\cdot e^{\theta}\cdot ch)^{-1}_{2m}\eta^{[m]}$$

in $H^*((\tilde{\mathfrak{d}}_1+\mathfrak{su}(N))[\epsilon],\tilde{\mathfrak{d}}_1+\mathfrak{su}(N);\overline{C}^{\lambda}_*(M_N(\mathbb{A}^{\hbar}_1[\hbar^{-1}])[\eta]))$.

The lemma now follows, since $\hat{A}$ is multiplicative.

It is well known, see [BNT99], that the restriction homomorphism

$$C^*(\tilde{\mathfrak{g}}[\epsilon], \tilde{\mathfrak{h}}; \overline{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])) \to C^*(\tilde{\mathfrak{h}}[\epsilon], \tilde{\mathfrak{h}}; \overline{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$$

is a quasi isomorphism. But in $C^*(\tilde{\mathfrak{g}}[\epsilon], \tilde{\mathfrak{h}}; \overline{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$ there are two classes, that under the restriction homomorphism maps to $\eta^{[m]}$, namely $CW(\eta^{[m]})$ and the class

$$(g_1\epsilon, \dots, g_p\epsilon) \to \iota_{g_1\eta} \cdots \iota_{g_p\eta}\eta^{(m)} \tag{6.2}$$

Therefore $CW(\eta^{[m]})$ and 6.2 are equivalent in

$$C^*(\tilde{\mathfrak{g}}[\epsilon], \tilde{\mathfrak{h}}; \overline{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$$

Considering the restriction homomorphism

$$C^*(\tilde{\mathfrak{g}}[\epsilon], \tilde{\mathfrak{h}}; \overline{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta])) \to C^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}; \overline{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$$

we get, that in the righthand side, $CW(\eta^{[m]})$ is equivalent to $\eta^{(m)}$. For $\mathfrak{g}$-modules $\mathbb{L}^*$ we have, that $C^*(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}; \mathbb{L}^*)$ is quasi isomorphic to $C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*)$. We hence get

Theorem 6.0.3. *Let U be an extension of U_0 to a class in*

$$C^*(\mathfrak{g}, \mathfrak{h}; \overline{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$$

Then we have the following equality

$$U = \sum_{m\geq 0} (\hat{A} \cdot e^\theta \cdot ch)^{-1}_{2m} \cdot \eta^{(m)}$$

in $H^*(\mathfrak{g}, \mathfrak{h}; \overline{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])[\eta]))$

7. The Gelfand-Fuks construction

We now consider a $\mathfrak{g}$-module $\mathbb{L}^*$, where $\mathfrak{g}$ is as in section 3. Given a deformation quantization $A^\hbar_E$ we can consider the bundle

$$\tilde{M}_{A^\hbar_E} \times_G \mathbb{L}^*$$

and also consider the differential forms with values in this bundle. We will denote this by $\Omega^*(M, \mathbb{L}^*)$. Furthermore we get a flat connection ∇ induced from the connection on $\tilde{M}_{A^\hbar_E}$. Using this connection and the differential on $\mathbb{L}^*$, we get a complex $\Omega(M, \mathbb{L}^*)$. The Gelfand-Fuks construction gives a morphism of complexes

$$GF : C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*) \to \Omega^*(M, \mathbb{L}^*)$$

defined in the following way:

Choose $U(N) \times SU(N)$ trivializations of $\tilde{M}_{A_E^\hbar} \times_G \mathbb{L}^*$. In a given trivialization we write $\nabla = d + A$, where A is the connection one form. Given vector fields $X_1, \dots, X_p$ and $l \in C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*)$ we define

$$GF(l)(X_1, \dots, X_p) = l(X_1, \dots X_p)$$

If we look at the examples of classes in $C^*(\mathfrak{g}, \mathfrak{h}; \mathbb{L}^*)$ we constructed in section 4 we see, that for $\theta \in C^*(\mathfrak{g}, \mathfrak{h}; \frac{1}{\hbar}\mathbb{C}[[\hbar]])$ we get, that $GF(\theta)$ is the characteristic class of the deformation quantization. In the case of $\hat{A}$ we see, that $GF(\hat{A})$ is the $\hat{A}$ class of TM and the case of ch this is just the Chern character of $End(E)$.

The main example we are going to look at, is the case where $\mathbb{L}^*$ is $\overline{C}_*^\lambda(M_N(\mathbb{A}^\hbar)[\hbar^{-1}])$. We note that

Lemma 7.0.4. *Let $A_E^\hbar$ be a deformation quantization over $\mathbb{R}^{2n}$. The complex $(\Omega^*(\mathbb{R}^{2n}, \overline{C}_*^\lambda(M_N(\mathbb{A}^\hbar))), \nabla)$ is acyclic and the cohomology is jets on the diagonal of elements in $\overline{C}_*^\lambda(M_N(\mathcal{W}_n))$.*

Proof. We can assume, that ∇ is on the form $d - \sum_i(\partial_{\hat{x}_i} \otimes dx_i + \partial_{\hat{\xi}_i} \otimes d\xi_i)$. We have a short exact sequence of complexes

$$0 \to (\Omega^*(\mathbb{R}^{2kn}/\Delta, M_N(\mathbb{A}^\hbar)^{\otimes k}), \nabla) \to (\Omega^*(\mathbb{R}^{2kn}, M_N(\mathbb{A}^\hbar)^{\otimes k}), \nabla) \xrightarrow{\varphi^*} (\Omega^*(\mathbb{R}^{2k}, M_N(\mathbb{A}^\hbar)^{\otimes k}), \nabla) \to 0$$

where $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2kn}$ is the map onto the diagonal δ. From the associated long exact sequence we see, that $(\Omega^*(\mathbb{R}^{2k}, M_N(\mathbb{A}^\hbar)^{\otimes k}), \nabla)$ is acyclic and that the cohomology is jets on the diagonal of elements of $M_N(\mathcal{W}_n)^{\otimes k}$. By considering the following short exact sequence of complexes

$$\begin{aligned} 0 \to (\Omega^*(\mathbb{R}^{2k}, 1 \otimes M_N(\mathbb{A}^\hbar) \otimes \dots \otimes M_N(\mathbb{A}^\hbar) + \dots \\ + M_N(\mathbb{A}^\hbar) \otimes \dots \otimes M_N(\mathbb{A}^\hbar) \otimes 1), \nabla) \\ \to (\Omega^*(\mathbb{R}^{2k}, M_N(\mathbb{A}^\hbar)^{\otimes k}), \nabla) \to (\Omega^*(\mathbb{R}^{2k}, \overline{M_N(\mathbb{A}^\hbar)}^{\otimes k}), \nabla) \to 0 \end{aligned}$$

we see, that $(\Omega^*(\mathbb{R}^{2k}, \overline{M_N(\mathbb{A}^\hbar)}^{\otimes k}), \nabla)$ is acyclic and that the cohomology is jets on the diagonal of elements in $\overline{M_N(\mathcal{W}_n)}^{\otimes k}$.

Let us suppose, we have an element a in $\Omega^*(M, \overline{M_N(\mathbb{A}^\hbar)}^{\otimes k}/Im(1-\tau))$ with $\nabla(a) = 0$. We can then lift a to an element $\tilde{a} \in \Omega^*(\mathbb{R}^{2k}, \overline{M_N(\mathbb{A}^\hbar)}^{\otimes k})$, where $\nabla(\tilde{a}) \in \Omega^*(\mathbb{R}^{2k}, Im(1-\tau))$. However $b = \frac{1}{k}\sum_{i=0}^{k-1} \tau^i(\tilde{a})$ is also a lift of a and $\nabla(b) = \sum_{i=0}^{k-1} \tau^i \nabla(\tilde{a}) = 0$. The lemma follows from this.

8. Traces on Deformation quantizations and Index Theory

We consider a deformation quantization $A^\hbar_E$ of an endomorphism bundle $End(E)$ over a symplectic manifold M of dimension n. Let $A^\hbar_{E,c}$ be the algebra of elements in $A^\hbar_E$ with compact support. This algebra has a canonical $\mathbb{C}[[\hbar,\hbar^{-1}]$ valued trace defined in the following way:

Let (V_i,Φ_i) be a cover of M and $\Phi_i^{-1} : M_N(\mathcal{W}_n) \to A^\hbar_E$ local isomorphisms over V_i. Let ρ_{V_i} be a partition of unity with respect to the covering. For an element $a \in A^\hbar_{E,c}$ we define

$$Tr(a) = \sum_i \int \frac{1}{n!(i\hbar)^n} tr(\Phi_i(\rho_i * a))\omega^n_{st}$$

where ω_{st} is the standard symplectic form on $\mathbb{R}^{2n}$. That this is independent of the choices made and a trace, hinges on the following two propositions

Proposition 8.0.5. *Let E be the trivial line bundle. The Tr is a trace and independent of the choices made.*

Proof. See [Fed96].

Proposition 8.0.6. *Let $\mathcal{W}_n$ be the Weyl algebra over some contractible open subset U of $\mathbb{R}^{2n}$. Any automorphism over the identity map of $M_N(\mathcal{W}_n)$ is inner.*

Proof. More or less the same as lemma 3.0.6, see also lemma 2.0.4.

We consider the trace as a functional on $CC^{per}_*(A^\hbar_{E,c})$ and we want to compute the trace at the level of homology. To this end we consider the following:

Given an element b in $\overline{C}^\lambda_*(A^\hbar_E[\hbar^{-1},\eta])$, we define $\chi_{Tr}(b)$ in $CC^*_{per}(A^\hbar_{E,c})$ in the following way

$$\chi_{Tr}(b)(a) = Tr(b \cdot a)$$

where $\cdot$ means the action of $\overline{C}^\lambda_*(A^\hbar_E[\hbar^{-1},\eta])$ on $CC^{per}_*(A^\hbar_{E,c}[\hbar^{-1}])$, see theorem 5.1.1. We will now extend this to elements in $\check{C}^*(M,\overline{C}^\lambda_*(A^\hbar_E[\hbar^{-1},\eta])$, the Čech complex with values in the presheaf $V \to \overline{C}^\lambda_*(A^\hbar_{E|V}[\hbar^{-1},\eta])$. This is done in the following

Proposition 8.0.7. *Let* $\{b_{V_0\ldots V_p}\} \in \check{C}^*(M, \overline{C}^\lambda_*(A^\hbar_E[\hbar^{-1},\eta]))$ *and* $a \in CC^{per}_*(A^\hbar_{E,c}[\hbar^{-1}])$*. Define*

$$\chi_{Tr}(\{b_{V_0\ldots V_p}\})(a) = \sum_{V_0,\ldots,V_p} Tr(I_{\rho_{V_0}}[B+b, I_{\rho_{V_1}}]\ldots[B+b, I_{\rho_{V_p}}]a)$$

This gives a morphism of complexes

$$\check{C}^*(M, \overline{C}^\lambda_*(A^\hbar_E[\hbar^{-1},\eta]) \otimes CC^{per}_*(A^\hbar_{E,c}[\hbar^{-1}]) \to \mathbb{C}[[\hbar,\hbar^{-1}]$$

Proof. See [NT95a].

8.1. **The fundamental class in the Čech complex.** Recall that we have the canonical coordinates $x_1,\ldots,x_n,\xi_1,\ldots,\xi_n$ on $\mathbb{R}^{2n}$. We will also use this notation for the associated coordinate functions and consider these coordinate functions as elements in $\mathcal{W}_n$.

We can consider the fundamental class U_0 in $\overline{C}^\lambda_*(M_N(\mathcal{W}_n)[\hbar^{-1}])$ given by

$$U_0 = \frac{1}{2n(i\hbar)^n}\sum_{\sigma\in S_{2n}}(v_{\sigma_1}\otimes\ldots\otimes v_{\sigma_{2n}})$$

where $(v_1,\ldots,v_{2n}) = (x_1,\ldots,x_n,\xi_1,\ldots,\xi_n)$.

By the same argument as in the section on the fundamental class in Lie-algebra cohomology, this class extends uniquely in cohomology to a class U in $\check{C}^*(M, \overline{C}^\lambda_*(A^\hbar_E[\hbar^{-1}]))$. In order to connect this class to the fundamental class defined in Lie algebra cohomology we introduce the complex

$$\check{C}^*(M, \Omega^*(M, \overline{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])))$$

According to lemma 7.0.4 this is quasi isomorphic to $\check{C}^*(M, \overline{C}^\lambda_*(A^\hbar_E[\hbar^{-1}]))$. Furthermore we have the Gelfand-Fuks morphism

$$GF : C^*(\mathfrak{g},\mathfrak{h}; \overline{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])) \to \check{C}^*(M, \Omega^*(M, \overline{C}^\lambda_*(M_N(\mathbb{A}^\hbar[\hbar^{-1}])))$$

Because of uniquenes we get, that in cohomology we have $GF(U) = U$. Because of theorem 6.0.3 we therefore get

Theorem 8.1.1. *In the complex* $\check{C}^*(M, \overline{C}^\lambda_*(A^\hbar_E[\hbar^{-1},\eta]))$ *the two classes* U *and*

$$\sum_{m\le 0}(\hat{A}\cdot e^\theta\cdot ch)^{-1}_{2m}\cdot\eta^{(m)}$$

are equivalent.

With this we are now in position to prove

Theorem 8.1.2. *We have that* $\chi_{Tr}(U)(a_0 \otimes \ldots \otimes a_k)$ *has no singularities in* $\hbar$ *and*

$$\chi_{Tr}(U)(a_0 \otimes \ldots \otimes a_k) = \int ch^{-1}(End(E))ch(\nabla)(\tilde{a}_0 \otimes \ldots \otimes \tilde{a}_k) \ mod\ \hbar$$

Here $\tilde{a}_i$ *is* a_i *mod* $\hbar$ *and* $ch(\nabla)$ *is the J.L.O. cocycle associated to* ∇ *(see also* [Gor99]*), i.e.*

$$ch(\nabla)(\tilde{a}_0 \otimes \ldots \otimes \tilde{a}_k) =$$
$$\int_{\Delta_k} tr(\tilde{a}_0 e^{-t_0\nabla^2}\nabla(\tilde{a}_i)e^{-t_1\nabla^2}\cdots\nabla(\tilde{a}_k)e^{-t_k\nabla^2})dt_0\cdots dt_{k-1}$$

where tr *is the normalized trace on* $End(E)$.

We therefore have, according to theorem 8.1.1, that

$$Tr(a_0 \otimes \ldots \otimes a_k \cdot e^{\theta}) = \int \hat{A}\cdot ch(\nabla)(\tilde{a}_0 \otimes \ldots \otimes \tilde{a}_k) \ mod\ \hbar$$

Proof. Because of Morita equivalence it is enough to look at the case where $\tilde{a}_i$ is scalar for all i. We have that

$$\chi_{Tr}(U)(a_0 \otimes \ldots \otimes a_k) = \sum_{V_0} \chi_{tr}(U_0)(I_{\rho V_0}(a_0 \otimes \ldots \otimes a_k)) + \ldots$$

and it is not difficult to see, that $\ldots$ is zero modulo $\hbar$. The explicit formula for $\chi_{Tr}(U_0)$ gives

$$\sum_{V_0} \chi_{Tr}(U_0)(I_{\rho V_0}(a_0 \otimes \ldots \otimes a_{2n})) = \frac{1}{2n!}\int \tilde{a}_0 d\tilde{a}_1 \cdots d\tilde{a}_{2n} \text{ mod } \hbar$$

The result follows from this.

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