Cohen-Macaulay Local Rings and Gorenstein Differential Graded Algebras

Anders Frankild

Ph.D. thesis

Approved May 2002

Thesis advisor: Hans-Bjørn Foxby, University of Copenhagen, Denmark

Evaluating committee: C. U. Jensen (chair), University of Copenhagen, Denmark
Luchezar L. Avramov, University of Nebraska, Lincoln, USA
John P. C. Greenlees, University of Sheffield, UK

Institute for Mathematical Sciences · University of Copenhagen · 2002

Anders Frankild
Matematisk Afdeling
Københavns Universitet
Universitetsparken 5
DK-2100 København Ø
Denmark
`frankild@math.ku.dk`

ISBN 87-7834-549-9

COHEN-MACAULAY LOCAL RINGS
AND
GORENSTEIN DIFFERENTIAL GRADED ALGEBRAS

ANDERS FRANKILD

This text constitute my Ph.D. thesis in mathematics from the University of Copenhagen submitted to the Faculty of Science. It consists of the following two bulleted segments:

- A elaborated synopsis of my work, titled *Cohen-Macaulay local ring and Gorenstein Differential Graded Algebras* (63 pages). It is divide into two parts:
  - The first part bears the title *Hyperhomological Algebra*, and consists of four sections. The first is a brief recap on notation. The following three sections describe [**1**], [**2**], and [**3**] (see below).
  - The second part bears the title *Differential Graded Algebras*, and consists of eight sections. Again, the first is a brief recap on notation. The following seven describe [**4**], [**5**], [**6**], [**7**], [**8**], [**9**], and [**10**] (see below).
- A collection of the 10 articles:

[**1**] *Quasi Cohen-Macaulay properties of local homomorphisms*, J. Algebra **235** (2001), 214–242.

[**2**] (with L. W. Christensen and H.-B. Foxby), *Restricted Homological Dimensions and Cohen-Macaulayness*, J. Algebra **251** (2002), 479–502.

[**3**] Vanishing of local homology, preprint, to appear in Math. Z..

[**4**] (with P. Jørgensen), *Foxby equivalence, complete modules, and torsion modules*, J. of Pure and Applied Algebra **174** (2002), 135–147.

[**5**] (with P. Jørgensen), Affine equivalence and Gorensteinness, preprint, to appear in Math. Scand.

[**6**] (with P. Jørgensen), Gorenstein Differential Graded Algebras, preprint, to appear in Israel J. of Math.

[**7**] (with P. Jørgensen), Dualizing DG-modules for Differential Graded Algebras, preprint (2001).

[**8**] (with S. Iyengar and P. Jørgensen), Dualizing DG modules and Gorenstein DG Algebras, preprint (2002), to appear in J. of the London Math. Soc.

[**9**] (with P. Jørgensen), Homological Identities for Differential Graded Algebras, to appear in J. Algebra.

[**10**] (with P. Jørgensen), Homological Identities for Differential Graded Algebras, II, preprint (2002).

Throughout this text, bold faced references will refer to my work only. Other references may be found on page 62.

Contents

Acknowledgments

The author expresses his deep gratitude to his thesis advisor, Hans-Bjørn Foxby, for his never ending support, encouragement, and tactful guidance.

Needless to say, that the author is deeply indebted to Peter Jørgensen, Lars Winther Christensen, and Srikanth Iyengar.

In particular, working with Peter Jørgensen *is* a great thrill.

PART I – HYPERHOMOLOGICAL ALGEBRA

A. Notation

(A.1) **Our universe.** Throughout the next three sections, we will work within the derived category of the category of modules over a noetherian commutative ring R.

(A.2) **The derived category.** An R-complex is a sequence of R-modules $\{X_n\}_{n\in\mathbb{Z}}$ equipped with an R-linear differential $\partial_n^X : X_n \longrightarrow X_{n-1}$, that is, $\partial_n^X \partial_{n+1}^X = 0$.

A morphism of complexes is a sequence of R-linear maps $\{\alpha_n\}_{n\in\mathbb{Z}}$ which commute with the involved differentials. A morphism $X \xrightarrow{\alpha} Y$ is called a quasi-isomorphism if the induced homomorphism in homology $\mathrm{H}(X) \xrightarrow{\mathrm{H}(\alpha)} \mathrm{H}(Y)$ is an isomorphism; this is denoted $X \xrightarrow{\simeq} Y$.

If we take the abelian category of R-complexes and formally invert all quasi-isomorphisms, we get the *derived category*, $\mathsf{D}(R)$, of the category of R-modules (see [40, chap. 10.]). A morphism $X \xrightarrow{\alpha} Y$ of complexes is an isomorphism in $\mathsf{D}(R)$ if only if it is quasi-isomorphism. We use the symbol $\cong$ to denote isomorphisms in $\mathsf{D}(R)$.

(A.3) **Subcategories.** For an R-complex X the *supremum*, $\sup X$ and the *infimum* $\inf X$ of $X \in \mathsf{D}(R)$ are the (possibly infinite) numbers $\sup\{\, i \mid \mathrm{H}_i(X) \neq 0\,\}$ and $\inf\{\, i \mid \mathrm{H}_i(X) \neq 0\,\}$, respectively. (Here we operate with the convention $\sup \varnothing = -\infty$ and $\inf \varnothing = \infty$.) The *amplitude* $\operatorname{amp} X$ is defined as $\operatorname{amp} X = \sup X - \inf X$.

The *full subcategories* $\mathsf{D}_-(R)$ and $\mathsf{D}_+(R)$ consist of complexes X for which, respectively, $\sup X < \infty$ and $\inf X > -\infty$, and we let $\mathsf{D}_\mathrm{b}(R) = \mathsf{D}_-(R) \cap \mathsf{D}_+(R)$. The full subcategory $\mathsf{D}_0(R)$ of $\mathsf{D}_\mathrm{b}(R)$ consists of X with $\mathrm{H}_i(X) = 0$ for $i \neq 0$. As each R-module M may be viewed (in a canonical way) as a complex concentrated in degree 0 (that is, $M \in \mathsf{D}_0(R)$), and since each $X \in \mathsf{D}_0(R)$ is isomorphic (in $\mathsf{D}(R)$) to the homology module $\mathrm{H}_0(X)$, we may, and will, identify $\mathsf{D}_0(R)$ with the category of R-modules. The full subcategory $\mathsf{D}^\mathrm{f}(R)$ of $\mathsf{D}(R)$ consists of complexes X for which all homology modules $\mathrm{H}_i(X)$ are finitely generated. The superscript f is also used in association with the full subcategories; for instance, $\mathsf{D}^\mathrm{f}_\mathrm{b}(R)$ consists of complexes X for which $\mathrm{H}(X)$ is bounded and finitely generated finite in every degree.

(A.4) **Homological dimensions.** As for modules, we may consider the *projective*, *injective*, and *flat dimensions* abbreviated pd, id, and fd, respectively, for any X in $\mathsf{D}(R)$. The full subcategories $\mathsf{P}(R)$, $\mathsf{I}(R)$, and $\mathsf{F}(R)$ of $\mathsf{D}_\mathrm{b}(R)$ consist of complexes of finite, respectively, projective, injective, and flat dimension (see [17, 1.4]). For instance, a complex sits inside $\mathsf{P}(R)$ precisely when it is isomorphic (in $\mathsf{D}(\mathrm{R})$) to a bounded complex of projectives. Again, we use the superscript f to denote finitely generated homology and the subscript 0 to denote modules; for instance, the full subcategory $\mathsf{F}^{\mathrm{f}}_0(R)$ denotes the category of finitely generated modules of finite flat dimension.

(A.5) **Derived functors.** On the abelian category of R-complexes the homomorphism functor, Hom, and the tensor product functor, $\otimes$, are defined in the usual way. On $\mathsf{D}(R)$ we may define the right-derived Hom, denoted RHom, and left-derived $\otimes$, denoted $\overset{\mathrm{L}}{\otimes}$; this is done via appropriate resolutions.

Let P, I, and F be R-complexes. We call P K-projective, I K-injective, and F K-flat, if $\mathrm{Hom}_R(P,-)$, $\mathrm{Hom}_R(-,I)$, and $F \otimes_R -$ send quasi-isomorphisms to quasi-isomorphisms.

We call P a K-projective resolution of the complex X, if $P \xrightarrow{\simeq} X$. In a similar way we define K-injective and K-flat resolutions. These types of resolution always exist (see [38]). Moreover, any complex P which is bounded to the right and consists of projectives is K-projective, as any complex I which is bounded to the left and consists of injectives is K-injective, as any complex F bounded to the right and consists of flats is K-flat.

We define $\mathrm{RHom}_R(X,Y)$ as $\mathrm{Hom}_R(P,Y)$ where P is a K-projective resolution of X, which is isomorphic to $\mathrm{Hom}_R(X,I)$ where I is a K-injective resolution of Y. We define $X \overset{\mathrm{L}}{\otimes}_R Y$ as $F \otimes_R Y$ where F is a K-flat resolution of X, which is isomorphic to $X \otimes_R G$ where G is a K-flat resolution of Y.

1. Quasi Cohen-Macaulay Properties of Local Homomorphisms

(1.1) **Infrastructure.** This paper is connected to the following papers:

Primary

- *None.*

Secondary

- **[2]** *Restricted Homological Dimensions and Cohen-Macaulayness.*
- **[4]** *Foxby equivalence, torsion modules, and complete modules.*

(1.2) **Setup.** Throughout this section, R and S will denote noetherian local commutative rings; $\mathfrak{m}$ will denote the maximal ideal in R, while $\mathfrak{n}$ will denote the maximal ideal in S. The $\mathfrak{m}$-adic completion of R is denoted $\widehat{R}$, while the $\mathfrak{n}$-adic completion of S is denoted $\widehat{S}$. By $\varphi : R \longrightarrow S$ we denote a local ring homomorphism; local meaning $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$. The completion of φ is the induced local ring homomorphisms $\widehat{\varphi} : \widehat{R} \longrightarrow \widehat{S}$. If $\mathfrak{p}$ is a prime ideal in R, then the field $R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ is denoted $k(\mathfrak{p})$. If $\mathfrak{q}$ is a prime ideal in S, then $\mathfrak{q} \cap R$ will denote the contraction of $\mathfrak{q}$ through φ.

We say that $\varphi : R \longrightarrow S$ is of finite flat dimension, when the R-module S is of finite flat dimension.

(1.3) **The Cohen-Macaulay defect of a ring.** Suppose R is local. The depth of R is defined as the unique maximal length of a regular sequence in R, and may be computed as,

$$\operatorname{depth} R = -\sup(\mathrm{RHom}_R(k, R)).$$

The (Krull) dimension of R is defined as the supremum of lengths taken over all strictly decreasing chains $\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$ of prime ideals of R, and may be computed as,

$$\dim R = \sup\{\, \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \mathsf{Spec}(R) \,\}.$$

The Cohen-Macaulay defect of R is defined as the non-negative integer

$$\operatorname{cmd} R = \dim R - \operatorname{depth} R.$$

For a noetherian commutative ring R the Cohen-Macaulay defect is defined as,

$$\operatorname{cmd} R = \sup\{\, \dim R_\mathfrak{p} - \operatorname{depth} R_\mathfrak{p} \mid \mathfrak{p} \in \mathsf{Spec}(R) \,\}.$$

(1.4) **Dualizing complexes.** A complex D is called a *dualizing complex* for R if:

- $D \in \mathsf{D}_\mathrm{b}^\mathrm{f}(R)$.
- The morphism $R \longrightarrow \mathrm{RHom}_R(D, D)$ is an isomorphism in $\mathsf{D}(R)$.
- $D \in \mathsf{I}(R)$.

Recall that a complete local ring admits a dualizing complex (see [9, thm. V.10.4]). When R admits a dualizing complex, D^R will denote a normalized dualizing complex, meaning $\inf D^R = \operatorname{depth} R$.

(1.5) **Auslander and Bass classes.** Let D be a dualizing complex for R, and consider the the pair of adjoint derived functors

$$(D \overset{\mathrm{L}}{\otimes}_R -, \operatorname{RHom}_R(D, -)).$$

Let η denote the unit and ϵ the counit of the adjoint functors.

The *Auslander* and *Bass classes* are full triangulated subcategories of the derived category $\mathsf{D}(R)$. The Auslander class is defined as:

$$\mathcal{A}_D(R) = \left\{ X \;\middle|\; \begin{array}{l} \eta_X : X \longrightarrow \operatorname{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R X) \\ \text{is an isomorphism, and } D \overset{\mathrm{L}}{\otimes}_R X \in \mathsf{D}_{\mathrm{b}}(R). \end{array} \right\},$$

and the *Bass class* is defined as:

$$\mathcal{B}_D(R) = \left\{ Y \;\middle|\; \begin{array}{l} \epsilon_Y : D \overset{\mathrm{L}}{\otimes}_R \operatorname{RHom}_R(D, Y) \longrightarrow Y \\ \text{is an isomorphism, and } \operatorname{RHom}_R(D, Y) \in \mathsf{D}_{\mathrm{b}}(R). \end{array} \right\}.$$

These classes were introduced in [9, sec. 3]. It is a central feature of the Auslander and Bass classes that we have the following full imbeddings of subcategories

$$\mathsf{F}(R) \subseteq \mathcal{A}_D(R) \qquad \text{and} \qquad \mathsf{I}(R) \subseteq \mathcal{B}_D(R).$$

In the next chapters we will study more abstract versions of Auslander and Bass classes associated to any pair of adjoint functors (see [**4**], [**5**], [**6**], and [**7**]). In particular, we will omit the above boundedness conditions imposed on $\mathcal{A}_D(R)$ and $\mathcal{B}_D(R)$.

(1.6) **Factorizations.** Consider a local homomorphism $\varphi : R \longrightarrow S$. Suppose one can find a commutative diagram consisting of local homomorphisms

$$\begin{tikzcd} & R' \arrow[dr, "\varphi'"] & \\ R \arrow[ur, "\dot{\varphi}"] \arrow[rr, "\varphi"'] & & S \end{tikzcd}$$

where $\dot{\varphi} : R \longrightarrow R'$ is flat and $\varphi' : R' \longrightarrow S$ is surjective. Such a diagram is called a *factorization* of φ.

We will call a factorization of φ *regular*, if $R'/\mathfrak{m}R'$ is regular; *Gorenstein* if $R'/\mathfrak{m}R'$ is Gorenstein.

A factorization is called a *Cohen factorization* if it is regular and R' is complete.

Given any local homomorphism $\varphi : R \longrightarrow S$, the composite $\dot{\varphi} : R \xrightarrow{\varphi} S \longrightarrow \widehat{S}$ admits a Cohen factorization (see [11, thm. (1.1)]).

(1.7) **Homomorphisms of finite G-dimension.** Suppose $\varphi : R \longrightarrow S$ is a local homomorphism. The homomorphism φ is of finite flat dimension ($S \in \mathsf{F}(R)$) if and only if $\widehat{S} \in \mathsf{F}(\widehat{R})$.

We say that φ is of finite G-dimension (and we write $\text{G-dim}\,\varphi < \infty$) if $\widehat{S} \in \mathcal{A}_{D^{\widehat{R}}}(\widehat{R})$ (see [9, sec. 4]).

If φ is of finite flat dimension, it is of finite G-dimension; the class of local homomorphisms of finite G-dimension encompasses that of local homomorphisms of finite flat dimension.

(1.8) **Definition.** Let $\varphi : R \longrightarrow S$ be a local homomorphism. A complex $C \in \mathsf{D}(S)$ is called *dualizing* for φ (see [9, sec. 5]), if

- $C \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(S)$.
- The morphism $S \longrightarrow \mathrm{RHom}_S(C, C)$ is an isomorphism in $\mathsf{D}(S)$.
- $D^{\widehat{R}} \overset{\mathrm{L}}{\otimes}_{\widehat{R}} (C \otimes_S \widehat{S}) \in \mathsf{I}(\widehat{S})$.

(1.9) **Definition ([1, def. (4.2)]).** Let $\varphi : R \longrightarrow S$ be a local homomorphism. If D^R and D^S exist, we use the symbol D^φ to denote the following S–complex

$$D^{\varphi} = \mathrm{RHom}_R(D^R, D^S).$$

By convention the symbol D^φ is only used when D^R and D^S exist.

If $\varphi : R \longrightarrow S$ is of finite G-dimension, and D^φ exist, then D^φ is dualizing for φ (see [9, thm. (5.3)]).

(1.10) **An analogy.** Consider a local ring R. Recall that the completed local ring $\widehat{R}$ admits a dualizing complex $D^{\widehat{R}}$. Moreover, R is Cohen-Macaulay if and only if $\widehat{R}$ is Cohen-Macaulay which is tantamount to $\operatorname{amp} D^{\widehat{R}} = 0$.

In essence: the Cohen-Macaulay property of R is completely encoded in the homological size (the amplitude) of the dualizing complex of its completion.

Next, consider a local homomorphism $\varphi : R \longrightarrow S$. A priori we do not know if φ admits a dualizing complex. However, the completion of φ do admit such a complex, namely

$$D^{\widehat{\varphi}} = \mathrm{RHom}_{\widehat{R}}(D^{\widehat{R}}, D^{\widehat{S}}).$$

It is therefore natural to make the following definition.

(1.11) **Definition ([1, def. (5.1)]).** Let φ be a local homomorphism of finite G-dimension. The *quasi dimension* of φ, denoted $\mathbf{q}\mathrm{dim}\ \varphi$, is defined as

$$\mathbf{q}\mathrm{dim}\ \varphi = \sup D^{\widehat{\varphi}},$$

and the *quasi Cohen–Macaulay defect* of φ, denoted $\mathbf{q}\mathrm{cmd}\ \varphi$, is defined as

$$\mathbf{q}\mathrm{cmd}\ \varphi = \operatorname{amp} D^{\widehat{\varphi}}.$$

The *depth* of φ was introduced by Avramov, Foxby, and B. Herzog by means of Cohen factorizations (see [11, (2.2)]), and it can also be detected by $D^{\widehat{\varphi}}$ since

$$\operatorname{depth}\varphi = \inf D^{\widehat{\varphi}} = \operatorname{depth} S - \operatorname{depth} R,$$

see [**1, def. (5.2) , (5.2), and lem. (4.5)**].

(1.12) **The numerical theory of D^{φ}.** Next, we must investigate the numerical (homological) invariants attached to D^{φ}. Some of these results are presented in the following.

(1.13) **Definition ([1, def. 4.8]).** Let R be local and let $\mathfrak{p} \in \mathsf{Spec}(R)$. We use the abbreviation

$$\mathrm{n}_R(\mathfrak{p}) = \dim(R/\mathfrak{p}),$$

and the symbol $\mathrm{m}_R(\mathfrak{p})$ is defined as

$$\begin{aligned}\mathrm{m}_R(\mathfrak{p}) &= \operatorname{depth} R_{\mathfrak{p}} + \dim(R/\mathfrak{p}) - \operatorname{depth} R\\ &= \operatorname{depth} R_{\mathfrak{p}} + \mathrm{n}_R(\mathfrak{p}) - \operatorname{depth} R.\end{aligned}$$

In particular, $\mathrm{m}_R(\mathfrak{p}) \geq 0$.

(1.14) **Theorem ([1, thm. 4.10]).** *Let φ be local and assume that* $\text{G-dim}\,\varphi$ *is finite. If D^{φ} exists, then for $\mathfrak{q} \in \mathsf{Spec}(S)$ one has the identity*

$$\mathrm{m}_S(\mathfrak{q}) - \mathrm{m}_R(\mathfrak{q} \cap R) = \inf\,(D^{\varphi})_{\mathfrak{q}} - \inf D^{\varphi}.$$

In particular, $\mathrm{m}_S(\mathfrak{q}) \geq \mathrm{m}_R(\mathfrak{q} \cap R)$.

(1.15) **Theorem ([1, thm. 4.12]).** *If* $\text{G-dim}\,\varphi$ *is finite and D^{φ} exists, then the following identity holds*

$$\operatorname{amp} D^{\varphi} = \sup\{\, \mathrm{m}_S(\mathfrak{q}) - \mathrm{m}_R(\mathfrak{q} \cap R) \mid \mathfrak{q} \in \mathsf{Spec}(S)\,\}.$$

(1.16) **The behavior of $\mathbf{q}\mathrm{cmd}$ on compositions.** Next, we study how the quasi Cohen-Macaulay defect behaves under compositions of local homomorphisms. With theorem (1.15) this turns out to be a very easy task.

Note, that the composition of two local homomorphisms of finite flat dimension again yield a local homomorphism of finite flat dimension.

This, however, is still an open problem when we consider local homomorphisms of finite G-dimension.

Returning to the study on how the quasi Cohen-Macaulay defect behaves under composition, we will ultimately see how it yields a theorem stating how the Cohen-Macaulay property ascents and descents along a local homomorphism of finite G-dimension (see theorem (1.21)).

(1.17) **Theorem ([1, thm. (5.13)]).** *Assume that ψ and φ are of finite G-dimension such that $\varphi\psi$ also is of finite G-dimension. Then the following hold*

$$\mathbf{q}\mathrm{dim}\ \varphi\psi \leq \mathbf{q}\mathrm{dim}\ \varphi + \mathbf{q}\mathrm{dim}\ \psi, \tag{1}$$

$$\mathbf{q}\mathrm{cmd}\ \varphi\psi \leq \mathbf{q}\mathrm{cmd}\ \varphi + \mathbf{q}\mathrm{cmd}\ \psi. \tag{2}$$

(1.18) **Theorem ([1, thm. (5.15)]).** *Assume that ψ and φ are of finite G-dimension such that $\varphi\psi$ is of finite G-dimension. Then the following hold*

$$\mathrm{depth}\,\psi + \mathbf{q}\mathrm{dim}\ \varphi \leq \mathbf{q}\mathrm{dim}\ \varphi\psi, \tag{1}$$

$$\mathbf{q}\mathrm{cmd}\ \varphi \leq \mathbf{q}\mathrm{cmd}\ \varphi\psi. \tag{2}$$

(1.19) **Theorem ([1, thm. (5.16)]).** *Assume that ψ and φ are of finite G-dimension such that $\varphi\psi$ also is finite G-dimension, and assume that* $\mathsf{Spec}(\widehat{S}) \longrightarrow \mathsf{Spec}(\widehat{R})$ *is surjective. Then the following hold*

$$\mathrm{depth}\,\varphi + \mathbf{q}\mathrm{dim}\ \psi \leq \mathbf{q}\mathrm{dim}\ \varphi\psi, \tag{1}$$

$$\mathbf{q}\mathrm{cmd}\ \psi \leq \mathbf{q}\mathrm{cmd}\ \varphi\psi. \tag{2}$$

(1.20) **Definition ([1, def. (6.2)]).** A local homomorphism φ is called *quasi Cohen–Macaulay* if G-dim$\,\varphi$ is finite and $\mathbf{q}\mathrm{cmd}\ \varphi = 0$.

(1.21) **Ascent–Descent Theorem ([1, (6.7)]).** *Let $\varphi : R \longrightarrow S$ be a local homomorphism. Then the following hold*

- (**A**) *If R is Cohen–Macaulay and φ is quasi Cohen–Macaulay, then S is Cohen–Macaulay.*
- (**D**) *If S is Cohen–Macaulay and* G-dim$\,\varphi$ *is finite, then φ is quasi Cohen–Macaulay.*

If, furthermore, the map of spectra $\mathsf{Spec}(\widehat{S}) \longrightarrow \mathsf{Spec}(\widehat{R})$ *is surjective one also has*

- (**D′**) *If S is Cohen–Macaulay and* G-dim$\,\varphi$ *is finite, then φ is quasi Cohen–Macaulay, and R is Cohen–Macaulay.*

(1.22) **The Cohen-Macaulay defect of a homomorphism.** By means of Cohen factorizations Avramov, Foxby and Herzog introduced the dimension (dim), depth (depth) and Cohen-Macaulay defect (cmd) of a local homomorphism (see [11] and [9, sec. 5]).

Let us briefly review the connection between the Cohen-Macaulay defect and the quasi Cohen-Macaulay defect of a local homomorphism.

If $\varphi : R \longrightarrow S$ is of finite G-dimension, then

$$\mathrm{cmd}\,\varphi \leq \mathbf{q}\mathrm{cmd}\,\varphi,$$

with equality if φ is of finite flat dimension or if R is Cohen-Macaulay (see [9, thm. (5.5)]).

(1.23) **Grothendieck's Localization problem.** In [28, (7.5.4)] Grothendieck posed the following localization problem for the Cohen-Macaulay property:

Let $\varphi : R \longrightarrow S$ be a flat homomorphism of local rings, and assume that for each $\mathfrak{p} \in \mathsf{Spec}(R)$ the formal fiber $k(\mathfrak{p}) \otimes_R \widehat{R}$ is Cohen–Macaulay. If the closed fiber $S/\mathfrak{m}S$ of φ at the maximal ideal $\mathfrak{m}$ of R is Cohen–Macaulay, then does each fiber $k(\mathfrak{p}) \otimes_R S$ of φ have the same property?

Recall that when $\varphi : R \longrightarrow S$ is flat, we have

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{cmd}\,\varphi = \mathrm{cmd}\,(S/\mathfrak{m}S),$$

by **[1, prop. (7.5)]**. In 1994 Avramov and Foxby solved the problem; the answer is positive. Later, in 1998, they came up with an extremely elegant solution to the problem. They showed that for $\varphi : R \longrightarrow S$ of finite flat dimension, $\mathfrak{q} \in \mathsf{Spec}(S)$ and $\mathfrak{p} = \mathfrak{q} \cap R \in \mathsf{Spec}(R)$, one has the beautiful inequality

$$\mathrm{cmd}\,\varphi_{\mathfrak{q}} + \mathrm{cmd}\,(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \mathrm{cmd}\,\varphi + \mathrm{cmd}\,(k(\mathfrak{p}) \otimes_R \widehat{R}),$$

see [7, thm. (5.3)]. Let us end this section by stating a theorem, which shows that this solution can be lifted to the realm of local homomorphisms locally of finite G-dimension, that is, local homomorphisms $\varphi : R \longrightarrow S$ for which all the localized (local) homomorphisms $\varphi_{\mathfrak{q}} : R_{\mathfrak{q} \cap R} \longrightarrow S_{\mathfrak{q}}$ are of finite G-dimension.

(1.24) **Theorem ([1, thm. (8.5)]).** *Let $\varphi : R \longrightarrow S$ be a local homomorphism locally of finite G-dimension. If $\mathfrak{q} \in \mathsf{Spec}(S)$ and $\mathfrak{p} = \mathfrak{q} \cap R \in \mathsf{Spec}(R)$, then there is an inequality*

$$\mathbf{q}\mathrm{cmd}\,\varphi_{\mathfrak{q}} + \mathrm{cmd}\,(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \mathbf{q}\mathrm{cmd}\,\varphi + \mathrm{cmd}\,(k(\mathfrak{p}) \otimes_R \widehat{R}).$$

2. Restricted Homological Dimensions and Cohen-Macaulayness

(2.1) **Infrastructure.** This paper is connected to the following papers:

Primary

- **[3]** *Vanishing of local Homology.*

Secondary

- **[1]** *Quasi Cohen-Macaulay Properties of Local homomorphisms.*

(2.2) **Setup.** Throughout this section, R will denote a noetherian commutative ring. When R, in addition, is local $\mathfrak{m}$ will denote its maximal ideal, and $k = R/\mathfrak{m}$ the residue class field.

For an ideal $\mathfrak{a}$ in R the set of prime ideals containing $\mathfrak{a}$ is denoted $\mathrm{V}(\mathfrak{a})$. If $\boldsymbol{a} = a_1, \dots, a_n$ is a sequence of elements in R, then $\mathrm{K}(\boldsymbol{a})$ denotes the Koszul complex on $\boldsymbol{a}$. It is a bounded complex of finitely generated free R-modules (see [14, chp. 5]); thus, the functors $-\otimes_R \mathrm{K}(\boldsymbol{a})$ and $- \overset{\mathrm{L}}{\otimes}_R \mathrm{K}(\boldsymbol{a})$ are naturally isomorphic, and we will henceforth not distinguish between them.

(2.3) **Restricted homological dimensions.** The *restricted flat dimension* and the *small restricted flat dimension* of $X \in \mathsf{D}_+(R)$ are defined respectively as:

$$\mathsf{Rfd}_R\, X = \sup\{\, \sup(T \overset{\mathrm{L}}{\otimes}_R X) \,|\, T \in \mathsf{F}_0(R)\,\},$$
$$\mathsf{rfd}_R\, X = \sup\{\, \sup(T \overset{\mathrm{L}}{\otimes}_R X) \,|\, T \in \mathsf{P}_0^{\mathrm{f}}(R)\,\},$$

see **[2, def. (2.1) and (2.9)]**.

The *restricted injective dimension* and the *small restricted injective dimension* of $Y \in \mathsf{D}_-(R)$ are defined respectively as:

$$\mathsf{Rid}_R\, Y = \sup\{\, -\inf(\mathrm{RHom}_R(T, Y) \,|\, T \in \mathsf{P}_0(R)\,\},$$
$$\mathsf{rid}_R\, Y = \sup\{\, -\inf(\mathrm{RHom}_R(T, Y) \,|\, T \in \mathsf{P}_0^{\mathrm{f}}(R)\,\},$$

see **[2, def. (5.10) and (5.1)]**.

The *restricted projective dimension* and the *small restricted projective dimension* of $X \in \mathsf{D}_+(R)$ are defined respectively as:

$$\mathsf{Rpd}_R\, X = \sup\{\, -\inf(\mathrm{RHom}_R(X, T) \,|\, T \in \mathsf{I}_0(R)\,\},$$
$$\mathsf{rpd}_R\, X = \sup\{\, \inf U - \inf(\mathrm{RHom}_R(X, U) \,|\, U \in \mathsf{I}^{\mathrm{f}}(R) \,\wedge\, \mathrm{H}(U) \neq 0\,\},$$

see **[2, def. (5.14) and (5.20)]**. Since a non-Cohen-Macaulay ring do not allow finitely generated modules of finite injective dimension, we use complexes from $\mathsf{I}^{\mathrm{f}}(R)$ (they always exist) as test objects. If R is of finite

(Krull) dimension the above restricted homological dimensions are always finite for bounded complexes (see **[2]**).

(2.4) **Depth.** Suppose R is local. The local depth of $Y \in \mathsf{D}_-(R)$ is defined as

$$\operatorname{depth}_R Y = -\sup(\operatorname{RHom}_R(k, Y).$$

When Y is just a finitely generated R-module $\operatorname{depth}_R Y$ is the maximal length of a Y-regular sequence in $\mathfrak{m}$. For an ideal $\mathfrak{a}$ in R the *non-local* depth of $Y \in \mathsf{D}_-(R)$ is defined as

$$\operatorname{depth}_R(\mathfrak{a}, Y) = -\sup(\operatorname{RHom}_R(\mathrm{K}(\boldsymbol{a}), Y)),$$

for any generating sequence $\boldsymbol{a}$ for $\mathfrak{a}$ (see [31, sec. 2]). Again, when Y is just a finitely generated R-module $\operatorname{depth}_R(\mathfrak{a}, Y)$ is the maximal length of a Y-regular sequence in $\mathfrak{a}$.

When R is local and $\mathfrak{a} = \mathfrak{m}$, then

$$\operatorname{depth}_R(\mathfrak{m}, Y) = \operatorname{depth}_R Y,$$

for any $Y \in \mathsf{D}_-(R)$ (see [31, sec. 2]).

(2.5) **Width.** Suppose R is local. The local width of $X \in \mathsf{D}_+(R)$ is defined as

$$\operatorname{width}_R X = \inf(k \overset{\mathrm{L}}{\otimes}_R X),$$

see [43, def. 2.1]. For an ideal $\mathfrak{a}$ in R the *non-local* width of $X \in \mathsf{D}_+(R)$ is defined as

$$\operatorname{width}_R(\mathfrak{a}, X) = \inf(\mathrm{K}(\boldsymbol{a}) \otimes_R X),$$

for any sequence of generators $\boldsymbol{a}$ of $\mathfrak{a}$ (see **[2, sec. 4]**).

When R is local and $\mathfrak{a} = \mathfrak{m}$, then

$$\operatorname{width}_R(\mathfrak{m}, X) = \operatorname{width}_R X,$$

for any $Y \in \mathsf{D}_+(R)$ (see **[2, cor. (4.11)]**).

(2.6) **Chouinard-like formulae.** In [15] it is shown that in case a module, M, over a ring, R, is of finite flat dimension, one can compute it as follows:

$$\operatorname{fd}_R M = \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \mathsf{Spec}(R)\}.$$

The restricted flat dimension displays the same feature: For any R-module:

$$\mathsf{Rfd}_R M = \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \mathsf{Spec}(R)\},$$

see **[2, thm. (2.4)]**. Dually, in [15] it is also shown that in case M is of finite injective dimension, one can compute it as follows:

$$\operatorname{id}_R M = \sup\{\operatorname{depth} R_\mathfrak{p} - \operatorname{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \mathsf{Spec}(R)\}.$$

The small restricted flat dimension and the small restricted injective dimension displays a similar behavior. But instead of taking a certain difference of *local* depths or depths and widths, we must take a certain difference of *non-local* depths or depths and widths. The results are listed below.

(2.7) **Theorem ([2, thm. (2.11)]).** *If* $X \in \mathsf{D}_\mathrm{b}(R)$, *then there are the next two equalities.*

$$\mathsf{rfd}_R\, X = \sup\{\, \sup(U \overset{\mathrm{L}}{\otimes}_R X) - \sup U \mid U \in \mathsf{P}^\mathrm{f}(R) \,\wedge\, \mathrm{H}(U) \neq 0 \,\},$$
$$\mathsf{rfd}_R\, X = \sup\{\, \mathrm{depth}_R(\mathfrak{p}, R) - \mathrm{depth}_R(\mathfrak{p}, X) \mid \mathfrak{p} \in \mathsf{Spec}(R) \,\}.$$

(2.8) **Theorem ([2, thm. (5.3)]).** *If* $Y \in \mathsf{D}_\mathrm{b}(R)$, *then there are the next two equalities.*

$$\mathsf{rid}_R\, X = \sup\{\, -\sup U - \inf(\mathrm{RHom}_R(U, Y) \mid U \in \mathsf{P}^\mathrm{f}(R) \,\wedge\, \mathrm{H}(U) \neq 0 \,\},$$
$$\mathsf{rid}_R\, X = \sup\{\, \mathrm{depth}_R(\mathfrak{p}, R) - \mathrm{width}_R(\mathfrak{p}, X) \mid \mathfrak{p} \in \mathsf{Spec}(R) \,\}.$$

(2.9) **Generalizing the Bass Formula.** Suppose R is a local ring. Recall that R is Cohen-Macaulay if and only if it admits a non-trivial finitely generated module of finite injective dimension.

How is the small restricted injective dimension connected to the injective dimension? Here is a result (see [**2, prop. (5.8)**]):

For every complex $Y \in \mathsf{D}_-(R)$ *there is an inequality*

$$\mathsf{rid}_R\, Y \leq \mathrm{id}_R\, Y,$$

and equality holds if $\mathrm{id}_R\, Y < \infty$ *and* $\mathrm{cmd}\ R \leq 1$.

Thus, the next corollary yields a generalization of the celebrated Bass Formula, which states, that over a local ring R, all non-trivial modules of finite injective dimension have *the same* injective dimension; namely $\mathrm{depth}\, R$.

(2.10) **Corollary ([2, cor. (5.5)]).** *If R is local,* $Y \in \mathsf{D}^\mathrm{f}_-(R)$, *and* $N \neq 0$ *is an R-module, then*

$$\mathsf{rid}_R\, Y = \mathrm{depth}\, R - \inf Y,$$
$$\mathsf{rid}_R\, N = \mathrm{depth}\, R.$$

(2.11) **Comment.** The next result shows that the local width behaves as expected when we consider $\mathrm{width}_R\, \mathrm{RHom}_R(X, Y)$ for $Y \in \mathsf{D}_+(R)$ and $X \in \mathsf{P}(R)$.

(2.12) **Theorem ([2, thm. (4.14)]).** *Let R be local and $Y \in \mathsf{D}_+(R)$. If $X \in \mathsf{P}(R)$, then*

$$\operatorname{width}_R(\operatorname{RHom}_R(X,Y)) = \operatorname{width}_R Y - \sup(X \overset{\mathrm{L}}{\otimes}_R k).$$

In particular: if $X \in \mathsf{P}^{\mathrm{f}}(R)$, then

$$\operatorname{width}_R(\operatorname{RHom}_R(X,Y)) = \operatorname{width}_R Y - \operatorname{pd}_R X.$$

(2.13) **Recognizing Cohen-Macaulay rings.** The restricted homological dimensions display a remarkable ability to detect rings which are very close to be Cohen-Macaulay, that is to say, rings for which the Cohen-Macaulay defect is at most one.

Therefore, one may think of these restricted dimensions as a *moral* Cohen-Macaulay dimension.

Recall, that A. Gerko in [26] defined the *Cohen-Macaulay-dimension*, $\text{CM-dim}_R M$, for every finitely generated module over a local ring. This homological dimension displays the following feature: the ring R is Cohen-Macaulay when and only when $\text{CM-dim}_R k$ is finite, which is tantamount to $\text{CM-dim}_R M$ being finite for all finite modules M.

The CM-dimension is connected to the restricted homological dimensions. Let us record the following result (see [**2, thm. (2.8)**]):

If R is local and M is a finitely generated R-module, then $\mathsf{Rfd}_R M \leq \text{CM-dim}_R M$ with equality if $\text{CM-dim}_R M$ is finite.

Two natural questions need to be answered:

- When does a restricted homological dimension and its small counterpart coincide?

  This property turns out to characterize almost Cohen-Macaulay rings.
- When does the (small) restricted flat and (small) projective dimension satisfy the Auslander-Buchsbaum formula (here R is local)?

  This property turns out to characterize Cohen-Macaulay rings.

We end this paragraph by listing the results dealing with the above questions.

(2.14) **Almost Cohen-Macaulay rings.** The next two results characterize almost Cohen-Macaulay rings.

(2.15) **Theorem ([2, thm. (3.2)]).** *If R is local, then the following are equivalent.*

(*i*) $\operatorname{cmd} R \leq 1$.
(*ii*) $\mathsf{rfd}_R X = \mathsf{Rfd}_R X$ *for all complexes* $X \in \mathsf{D}_+(R)$.
(*iii*) $\mathsf{rfd}_R M = \mathsf{Rfd}_R M$ *for all R-modules M.*

(2.16) **Theorem ([2, cor. (5.9)]).** *If R is local, then the following are equivalent.*

(*i*) $\operatorname{cmd} R \leq 1$.
(*ii*) $\mathsf{rid}_R Y = \operatorname{id}_R Y$ *for all complexes* $Y \in \mathsf{I}(R)$.
(*iii*) $\mathsf{rid}_R M = \operatorname{id}_R M$ *for all R-modules of finite injective dimension.*

(2.17) **Local Cohen-Macaulay rings.** The next two results characterize local Cohen-Macaulay rings.

(2.18) **Theorem ([2, thm. (3.4)]).** *If R is local, then the following are equivalent.*

(*i*) *R is Cohen-Macaulay.*
(*ii*) $\mathsf{Rfd}_R X = \operatorname{depth} R - \operatorname{depth}_R X$ *for all complexes* $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.
(*iii*) $\mathsf{rfd}_R M = \operatorname{depth} R - \operatorname{depth}_R M$ *for all finitely generated R-modules.*

(2.19) **Theorem ([2, thm. (5.22)]).** *If R is local, then the following are equivalent.*

(*i*) *R is Cohen-Macaulay.*
(*ii*) $\mathsf{Rpd}_R X = \operatorname{depth} R - \operatorname{depth}_R X$ *for all complexes* $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.
(*iii*) $\mathsf{rpd}_R M = \operatorname{depth} R - \operatorname{depth}_R M$ *for all finitely generated R-modules.*

3. Vanishing of Local Homology

(3.1) **Infrastructure.** This paper is connected to the following papers:

Primary

- **[2]** *Restricted Homological Dimensions and Cohen-Macaulayness.*

Secondary

- **[5]** *Affine equivalence and Gorensteinness.*

(3.2) **Setup.** Throughout this section, R will denote a noetherian commutative ring. When R, in addition, is local, $\mathfrak{m}$ will denote its maximal ideal.

For an ideal $\mathfrak{a}$ in R, the Čech complex, also known as the stable Koszul complex, is denoted $\mathrm{C}(\mathfrak{a})$ (see [14, chp. 5]).

(3.3) **Grothendieck's vanishing results.** Let us here review the important vanishing results for the famous local cohomology functors, introduced by Grothendieck.

Suppose $\mathfrak{a}$ is an ideal in R. We may consider the section functor with support in $\mathrm{V}(\mathfrak{a})$, which is defined on modules as

$$\Gamma_{\mathfrak{a}}(M) = \varinjlim \mathrm{Hom}_R(R/\mathfrak{a}^n, M).$$

This functor is left exact. Right deriving $\Gamma_{\mathfrak{a}}(-)$ we get the *local cohomology functors*. As usual we denote them $\mathrm{H}^i_{\mathfrak{a}}(-)$.

Next, suppose X is an object in $\mathsf{D}(R)$. By $\mathrm{R}\Gamma_{\mathfrak{a}}(-)$ we denoted the right derived local cohomology functor. This functor is obtained as follows: to any $X \in \mathsf{D}(R)$ one takes a K-injective resolution $X \xrightarrow{\simeq} I$ and define

$$\mathrm{R}\Gamma_{\mathfrak{a}}(X) = \Gamma_{\mathfrak{a}}(I).$$

When X is just an ordinary R-module we have $\mathrm{H}^i\,\mathrm{R}\Gamma_{\mathfrak{a}}(X) = \mathrm{H}^i_{\mathfrak{a}}(X)$. The local cohomology functors may be computed via the Čech complex on $\mathfrak{a}$, as one has the following isomorphism in $\mathsf{D}(R)$ (see [1, thm. 1.1(iv)]), namely

$$\mathrm{R}\Gamma_{\mathfrak{a}}(X) \cong \mathrm{C}(\mathfrak{a}) \otimes_R X \cong \mathrm{C}(\mathfrak{a}) \overset{\mathrm{L}}{\otimes}_R X,$$

where the second isomorphism follows since $\mathrm{C}(\mathfrak{a})$ is a bounded complex consisting of flat modules.

Next, one may like to study vanishing properties of $\mathrm{R}\Gamma_{\mathfrak{a}}(-)$. The theorems concerning vanishing properties of $\mathrm{R}\Gamma_{\mathfrak{a}}(-)$ are know as *Grothendieck's vanishing results.*

Assume $Y \in \mathsf{D}_-(R)$. The first vanishing result reads:

$$-\sup \mathrm{R}\Gamma_{\mathfrak{a}}(Y) = \mathrm{depth}_R(\mathfrak{a}, Y).$$

When $Y \in \mathsf{D}_{\mathrm{b}}(R)$ the second vanishing result reads:

$$-\inf \mathrm{R}\Gamma_{\mathfrak{a}}(Y) \leq \dim_R Y,$$

and equality is converted into an equality when R is local, $\mathfrak{a} = \mathfrak{m}$, and $Y \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(X)$ (see [22, prop. 7.10, thm. 7.8. and cor. 8.29], [31, thm. 6.1], and [14, chap. 6]) Here the Krull dimension of any Y is defined as,

$$\dim_R Y = \sup\{\, \dim(R/\mathfrak{p}) - \inf Y_{\mathfrak{p}} \mid \mathfrak{p} \in \mathsf{Spec}(R)\,\},$$

see [23, (16.3)].

(3.4) **Derived completion.** Suppose $\mathfrak{a}$ is an ideal in R. We may consider the completion functor with respect to $\mathfrak{a}$, which is defined on modules as

$$\Lambda^{\mathfrak{a}}(M) = \varprojlim (R/\mathfrak{a}^n \otimes_R M).$$

Left deriving $\Lambda^{\mathfrak{a}}(-)$ we get the so-called local homology functors. We denote them $\mathrm{H}^{\mathfrak{a}}_i(-)$. These was first studied by Matlis, when $\mathfrak{a}$ was generated by a regular sequence [34] and [35]. Then came the the work of Greenlees and May, settling the general case for modules [27], and finally Lipman, Lòpez, and Tarrío gave an exposition on local homology (and cohomology) on quasi compact separated schemes, in the context of derived categories [1].

Again, suppose X is an object in $\mathsf{D}(R)$. By $\mathrm{L}\Lambda^{\mathfrak{a}}(-)$ we denoted the left derived local homology functor. This functor is obtained as follows: to any $X \in \mathsf{D}(R)$ one takes a K-projective resolution $P \xrightarrow{\simeq} X$ and define

$$\mathrm{L}\Lambda^{\mathfrak{a}}(X) = \Lambda^{\mathfrak{a}}(P),$$

and we have $\mathrm{H}_i\, \mathrm{L}\Lambda^{\mathfrak{a}}(X) = \mathrm{H}^{\mathfrak{a}}_i(X)$. The local cohomology functors may also be computed via the Čech complex on $\mathfrak{a}$, as one has the following isomorphism in $\mathsf{D}(R)$, ([1, $(0.3)_{\mathrm{aff}}$, p. 4], and [18]) namely

$$\mathrm{L}\Lambda^{\mathfrak{a}}(X) \cong \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}), X).$$

Note, that in order to compute $\mathrm{L}\Lambda^{\mathfrak{a}}(-)$ via the Čech complex $\mathrm{C}(\mathfrak{a})$, it is imperative to work in $\mathsf{D}(R)$.

(3.5) **The pair** $(\mathrm{R}\Gamma_{\mathfrak{a}}(-), \mathrm{L}\Lambda^{\mathfrak{a}}(-))$**.** It follows from the above that

$$(\mathrm{R}\Gamma_{\mathfrak{a}}(-), \mathrm{L}\Lambda^{\mathfrak{a}}(-))$$

is an adjoint pair of functors.

Since there are results on the vanishing of local cohomology, it seems natural to study vanishing properties for local homology. The next vanishing result on local homology is completely analogous to the first of Grothendieck's vanishing result on local cohomology.

(3.6) **Theorem ([3, thm. (2.12)]).** *Let* $\mathfrak{a}$ *be an ideal in* R *and* $X \in \mathsf{D}_+(R)$. *Then there is an equality*

$$\inf \mathrm{L}\Lambda^{\mathfrak{a}}(X) = \mathrm{width}_R(\mathfrak{a}, X).$$

(3.7) **Bounds on** $\sup \mathrm{L}\Lambda^{\mathfrak{a}}(-)$ **([3, (2.13)]).** Suppose $X \in \mathsf{D}_{\mathrm{b}}(R)$, then one has the bound

$$\sup \mathrm{L}\Lambda^{\mathfrak{a}}(X) \leq \dim R - \mathrm{depth}_R(\mathfrak{a}, X).$$

(3.8) **Complete and derived complete objects.** Suppose $\mathfrak{a}$ is an ideal in R. An R-module M is called $\mathfrak{a}$-*complete* if the canonical homomorphism $M \longrightarrow \Lambda^{\mathfrak{a}}(M)$ is an isomorphism. We say that $X \in \mathsf{D}(R)$ is *derived* $\mathfrak{a}$-*complete* if the canonical morphism

$$X \longrightarrow \mathrm{L}\Lambda^{\mathfrak{a}}(X),$$

is an isomorphism. If M is an R-module which is $\mathfrak{a}$-complete, then M is derived $\mathfrak{a}$-complete when viewed as an object in $\mathsf{D}(R)$.

(3.9) **Torsion and derived torsion objects.** Suppose $\mathfrak{a}$ is an ideal in R. An R-module M is called $\mathfrak{a}$-*torsion* if the canonical homomorphism $\Gamma_{\mathfrak{a}}(M) \longrightarrow M$ is an isomorphism. We say that $X \in \mathsf{D}(R)$ is *derived* $\mathfrak{a}$-*torsion* if the canonical morphism

$$\mathrm{R}\Gamma_{\mathfrak{a}}(X) \longrightarrow X,$$

is an isomorphism. If M is an R-module which is $\mathfrak{a}$-torsion, then M is derived $\mathfrak{a}$-torsion when viewed as an object in $\mathsf{D}(R)$.

(3.10) **Vanishing results for Ext and Tor.** Suppose R is local, T a finitely generated R-module of finite projective dimension. By the Auslander-Buchsbaum Formula we know that the projective dimension of T equals the difference between depth of the ring and the depth of M; in symbols

$$\mathrm{pd}_R\, M = \mathrm{depth}\, R - \mathrm{depth}_R\, M.$$

On the other hand, we may also capture the projective dimension of M as the least integer i such that $\mathrm{Ext}^i_R(T, M) \neq 0$. To recapitulate; for T we have

$$\begin{aligned} \mathrm{Ext}^i_R(T, M) &= 0 \quad \text{for} \quad i > \mathrm{depth}\, R - \mathrm{depth}_R\, T, \quad \text{and} \\ \mathrm{Ext}^i_R(T, M) &\neq 0 \quad \text{for} \quad i = \mathrm{depth}\, R - \mathrm{depth}_R\, T. \end{aligned}$$

Suppose we drop the assumption of finite generation of T. We can still consider the difference $\mathrm{depth}\, R - \mathrm{depth}_R\, M$, but does this number contain any information on T?

Combining the vanishing results for local cohomology and homology with the restricted homological dimensions, we obtain results that indicate that this seems to be the case.

(3.11) **Theorem ([3, thm. (3.6)]).** *Let R be a local ring, and let $X \in \mathsf{D}_{\mathrm{b}}(R)$ be a non-trivial derived $\mathfrak{m}$-complete complex. Then*

$$\operatorname{depth} R - \inf X = \mathsf{rid}_R X = \mathsf{Rid}_R X.$$

(3.12) **Corollary ([3, cor. (3.7)]).** *Let R be a local ring. If M is a non-trivial R-module such that $\Lambda^{\mathfrak{m}}(M) \cong M$ and T is an R-module of finite projective dimension, then*

$$\begin{aligned}
&\operatorname{Ext}^i_R(T,M) = 0 \quad \textit{for} \quad i > \operatorname{depth} R - \operatorname{depth}_R T, \quad \textit{and}\\
&\operatorname{Ext}^i_R(T,M) \neq 0 \quad \textit{for} \quad i = \operatorname{depth} R - \operatorname{depth}_R T.
\end{aligned}$$

(3.13) **Corollary ([3, cor. (3.8)]).** *Let R be a complete local ring. If M is a non-trivial finitely generated R-module and T is an R-module of finite projective dimension, then*

$$\begin{aligned}
&\operatorname{Ext}^i_R(T,M) = 0 \quad \textit{for} \quad i > \operatorname{depth} R - \operatorname{depth}_R T, \quad \textit{and}\\
&\operatorname{Ext}^i_R(T,M) \neq 0 \quad \textit{for} \quad i = \operatorname{depth} R - \operatorname{depth}_R T.
\end{aligned}$$

(3.14) **Theorem ([3, thm. (3.9)]).** *Let R be a local ring, and let $X \in \mathsf{D}_{\mathrm{b}}(R)$ be a non-trivial derived $\mathfrak{m}$-torsion complex. Then*

$$\operatorname{depth} R + \sup X = \mathsf{rfd}_R X = \mathsf{Rfd}_R X.$$

(3.15) **Corollary ([3, cor. (3.10)]).** *Let R be a local ring. If M is a non-trivial R-module such that $\Gamma_{\mathfrak{m}}(M) \cong M$ and T is an R-module of finite flat dimension, then*

$$\begin{aligned}
&\operatorname{Tor}^R_i(T,M) = 0 \quad \textit{for} \quad i > \operatorname{depth} R - \operatorname{depth}_R T, \quad \textit{and}\\
&\operatorname{Tor}^R_i(T,M) \neq 0 \quad \textit{for} \quad i = \operatorname{depth} R - \operatorname{depth}_R T.
\end{aligned}$$

(3.16) **Theorem ([3, thm. (3.12)]).** *Let R be a local ring, and let $X \in \mathsf{D}_{\mathrm{b}}(R)$ be a non-trivial derived $\mathfrak{m}$-torsion complex. Then*

$$\operatorname{depth} R + \sup X = \mathsf{rpd}_R X = \mathsf{Rpd}_R X.$$

(3.17) **Corollary ([3, cor. (3.13)]).** *Let R be a local ring. If M is a non-trivial R-module such that $\Gamma_{\mathfrak{m}}(M) \cong M$ and T is an R-module of finite injective dimension, then*

$$\begin{aligned}
&\operatorname{Ext}^i_R(M,T) = 0 \quad \textit{for} \quad i > \operatorname{depth} R - \operatorname{width}_R T, \quad \textit{and}\\
&\operatorname{Ext}^i_R(M,T) \neq 0 \quad \textit{for} \quad i = \operatorname{depth} R - \operatorname{width}_R T.
\end{aligned}$$

PART II – DIFFERENTIAL GRADED ALGEBRAS

B. Notation

(B.1) **Our universe.** Throughout the next seven sections, we will work within the derived category of Differential Graded modules over a Differential Graded Algebra R.

(B.2) **Differential Graded Algebras.** Consider a graded algebra $R = \{R_n\}_{n\in\mathbb{Z}}$ over some commutative ring $\Bbbk$. Suppose R comes with a $\Bbbk$-linear differential $\partial_n^R : R_n \longrightarrow R_{n-1}$, that is, $\partial_n^R\partial_{n+1}^R = 0$, and that it satisfy the Leibnitz rule

$$\partial^R(rs) = \partial^R(r)s + (-1)^{|r|}r\partial^R(s),$$

where r is an element of degree $|r|$. In this case we call R a *Differential Graded Algebra*, henceforth abbreviated DGA. The *opposite* DGA of R is denoted R^{opp} and is simply R equipped with the multiplication

$$r \overset{\text{opp}}{\cdot} s = (-1)^{|r||s|}sr$$

for all elements r and s. A DGA R is called *commutative* if for all $r, s \in R$ we have $rs = (-1)^{|r||s|}sr$. Note, that we operate with Koszul's sign convention, that is, whenever two graded objects of degrees m and n are interchanged the sign $(-1)^{mn}$ will appear.

A morphism of DGAs over $\Bbbk$ is a morphism of graded algebras which is compatible with the involved differentials.

When R is a DGA the underlying graded algebra is denoted $R^\natural$.

(B.3) **Differential Graded modules.** Consider a graded left-module $M = \{M_n\}_{n\in\mathbb{Z}}$ over the graded algebra R. Suppose M comes with a $\Bbbk$-linear differential $\partial_n^M : M_n \longrightarrow M_{n-1}$, that is, $\partial_n^M\partial_{n+1}^M = 0$, and that it satisfy the Leibnitz rule

$$\partial^M(rm) = \partial^R(r)m + (-1)^{|r|}r\partial^M(m),$$

where r is a element of degree $|r|$. In this case we call M a *Differential Graded R-left-module*, henceforth abbreviated DG-R-left-module. A DG-R-right-module can be identified with a DG-R^{opp}-left-module

When M is a DG-module the underlying graded module is denoted $M^\natural$.

We may also consider DG-modules having more than one structure. Suppose R and S are DGAs and that M is a DG-R-left-S-right-module, in which case we indicate the structures as ${}_RM_S$. Always, when M has more than one structure we will assume they are compatible: for ${}_RM_S$ this amounts to $r(ms) = (rm)s$.

A morphism of DG-modules (for instance DG-R-left-S-modules) is a morphism of graded modules which is compatible with the involved differentials.

For a DG-R-left-module M we define the i'th suspension as, $(\Sigma^i M_j) = M_{j-i}$ and $\partial_j^{\Sigma^i M} = (-1)^i\partial_{j-i}^M$. The action of R on $\Sigma^i M$ is defined by

$r\Sigma^i(m) = (-1)^{|r|i}\Sigma^i(rm)$ for $r \in R$ and $m \in M$. If $M \xrightarrow{\alpha} N$ is a morphism of DG-R-left-module, say, then so is $\Sigma^i M \xrightarrow{\Sigma^i\alpha} \Sigma^i N$.

(B.4) **The center of a graded algebra.** A element c in a graded algebra R is said to be central if $rc = (-1)^{|r||c|}cr$ for all elements $r \in R$. The center of R is the set of all its central elements. Note, that R is commutative if all its elements are central.

(B.5) **Morphisms with central image.** Suppose that $Q \xrightarrow{\varphi} T$ is a morphism of DAGs, that Q is commutative, and that $\varphi(Q)$ is central in T. The morphism φ turns T into a DG-Q-left-Q-right-module and its Q-structures is compatible with its T-structure (as a DG-T-left-T-right-module). To recapitulate: Having φ we may, and will, turn T into a DG-module with structures as indicated ${}_{Q,T}T_{Q,T}$. Observe that the Q-structures on T are "balanced" in the sense $qt = (-1)^{|q||t|}tq$ for elements $q \in Q$ and $t \in T$.

(B.6) **Homology.** Suppose M is a DG-module. Since M comes with a $\Bbbk$-linear differential it has homology, which is denoted $\mathrm{H}(M)$. The product on R induce a product in $\mathrm{H}(R)$ making it a graded algebra; it also induce an action of $\mathrm{H}(R)$ on $\mathrm{H}(M)$ making it a graded $\mathrm{H}(R)$-module.

If a morphism of DG-modules $M \longrightarrow N$ induce a isomorphisms in homology $\mathrm{H}(M) \xrightarrow{\cong} \mathrm{H}(N)$ we call it a quasi-isomorphism; this is denoted $M \xrightarrow{\simeq} N$.

(B.7) **The derived category of DG-modules.** The category of DG-R-left-modules is an abelian category. As in (A.2) we can formally invert all quasi-isomorphisms of DG-R-left-modules and thus obtain the *derived category* of DG-R-left-modules, denoted $\mathsf{D}(R)$. Of course, we could do the same for the abelian category of DG-R-right-modules, thus obtaining $\mathsf{D}(R^{\mathrm{opp}})$.

Note, that the forgetful functor $(-)^\natural$ from $\mathsf{D}(R)$ to the category of graded $R^\natural$-left-modules, is additive, exact, faithful, and commutes with suspension.

(B.8) **Derived functors.** On $\mathsf{D}(R)$ we may also define the right-derived homomorphism functor, denoted RHom, and the left-derived tensor product functor, denoted $\overset{\mathrm{L}}{\otimes}$. As in paragraph (A.5) we do this via K-projective, K-injective, and K-flat resolutions of DG-modules; they exist see [33, secs. 3.1 and 3.2]. For instance, we say that a DG-module I is a K-injective resolution of a DG-module M if $M \xrightarrow{\simeq} I$ and $\mathrm{Hom}_R(-, I)$ sends quasi-isomorphisms to quasi-isomorphisms.

(B.9) **DG-modules over a ring.** Any ordinary ring, A, may be viewed as a DGA concentrated in degree zero, and when viewed as such, a DG-module module over A is simply an ordinary complex over A (viewed as a ring).

The various derived categories of A viewed as a DGA is the same as the various derived categories of A when viewed as a ring, and the have the same derived functors.

4. Foxby equivalence, complete modules, and torsion modules

(4.1) **Infrastructure.** This paper is connected to the following papers:

Primary

- [**5**] *Affine equivalence and Gorensteinness.*
- [**6**] *Gorenstein Differential Graded Algebras.*
- [**7**] *Dualizing DG-modules for Differential Graded Algebras.*
- [**8**] *Dualizing DG modules and Gorenstein DG Algebras.*

Secondary

- [**1**] *Quasi Cohen-Macaulay Properties of Local Homomorphisms.*

(4.2) **Setup.** Throughout this section, we investigate the following simple situation:

Consider two categories C, D and an adjoint pair of functors (F, G),

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "F"] & \mathsf{D}, \arrow[l, shift left, "G"] \end{tikzcd}$$

that is, there are natural transformations

$$\eta : 1_{\mathsf{C}} \longrightarrow GF \quad \text{and} \quad \epsilon : FG \longrightarrow 1_{\mathsf{D}}.$$

Here η is called the *unit* and ϵ the *counit* of the adjunction. Moreover, the compositions

$$1_{\mathsf{D}} : G \xrightarrow{\eta G} GFG \xrightarrow{G\epsilon} G,$$

and

$$1_{\mathsf{C}} : F \xrightarrow{F\eta} FGF \xrightarrow{\epsilon F} F,$$

are natural transformations of the identities, and for each $\mathsf{c} \in \mathsf{C}$ and $\mathsf{d} \in \mathsf{D}$ there is a bijection

$$\mathrm{Hom}_{\mathsf{D}}(F\mathsf{c}, \mathsf{d}) \xrightarrow{\varphi} \mathrm{Hom}_{\mathsf{C}}(\mathsf{c}, G\mathsf{d}),$$

where

$$\varphi f = Gf \circ \eta_{\mathsf{c}} \quad \text{and} \quad \varphi^{-1} g = \epsilon_{\mathsf{d}} \circ Fg.$$

Next, define full subcategories of C and D as,

$$\mathcal{A} = \{\mathsf{a} \in \mathsf{C} \mid \eta_{\mathsf{a}} \text{ is an isomorphism}\},$$
$$\mathcal{B} = \{\mathsf{b} \in \mathsf{D} \mid \epsilon_{\mathsf{b}} \text{ is an isomorphism}\}.$$

It follows that the functors F and G restrict to a pair of quasi-inverse equivalences of categories,

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "F"] & \mathcal{B}, \arrow[l, shift left, "G"] \end{tikzcd}$$

see **[4, thm (1.1)]**.

(4.3) **Blanket assumption.** In the rest of this section, R and S will denote DGAs, and ${}_{R,S}M$ will denote a DG-R-left-S-left-module.

(4.4) **A certain pair of adjoint functors.** Let R, S and ${}_{R,S}M$ be as in (4.3). Then

$$(- \overset{\mathrm{L}}{\otimes}_R M, \mathrm{RHom}_S(M, -))$$

is an adjoint pair of derived functors between derived categories

$$\mathsf{D}(R^{\mathrm{opp}}) \underset{\mathrm{RHom}_S(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathsf{D}(S).$$

Whenever, we, from now on, consider adjoint derived functors they will be of this form.

(4.5) **Generalized Foxby equivalence.** When R, S, and M are as in (4.3) we consider "our generic" pair of adjoint functors between derived categories of DG-modules,

$$\mathsf{D}(R^{\mathrm{opp}}) \underset{\mathrm{RHom}_R(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathsf{D}(S),$$

and by (4.2) we know that the above derived functors, $- \overset{\mathrm{L}}{\otimes}_R M$ and $\mathrm{RHom}_R(M, -)$, restrict to a pair of quasi-inverse equivalences of certain full subcategories, namely,

$$\mathcal{A}_M(R^{\mathrm{opp}}) \underset{\mathrm{RHom}_R(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathcal{B}_M(S).$$

We name these quasi-inverse equivalence *generalized Foxby equivalence*. As we will see, this simple functorial setup has some remarkable consequences.

(4.6) **The endomorphism DGA.** Suppose S is a ring, viewed a DGA concentrated in degree zero. Let M be a complex of S-left-modules. Define

$$R = \mathrm{Hom}_S(M, M).$$

A priori, R is just a complex of S-modules. Nevertheless, we may endow R with a multiplication. Suppose r_i is an element in R_i, that is, an S-linear map $M \xrightarrow{r_i} \Sigma^{-i}M$. If r_j is an element in R_j, then we define the product $r_i r_j$ as the composite $\Sigma^{-j}(r_i) \circ r_j$ which is an S-linear map $M \xrightarrow{r_i r_j} \Sigma^{-(i+j)}M$; an element in R_{i+j}. One may check that with this multiplication R becomes a DGA. The complex M acquires the structure of a DG-R-left-module with scalar multiplication $rm = r(m)$, and this structure is compatible with its S-structure. Thus, M becomes a DG-R-left-S-left-module ${}_{R,S}M$. Moreover, the identification map,

$$ {}_R R_R \xrightarrow{\cong} \operatorname{Hom}_S(M, M), $$

is an isomorphism of DG-R-left-R-right-modules. This observation is key in what follows.

(4.7) **Imposed conditions on M.** We will consider the following conditions imposed on M:

(1) We can resolve M by a DG-R-left-S-left-module which is K-projective over S, and the canonical morphism,
$$ R \xrightarrow{\rho} \operatorname{RHom}_S(M, M), $$
is an isomorphism.

(2) We can resolve M by a DG-R-left-S-left-module which is K-projective over R, and the canonical morphism,
$$ S \xrightarrow{\sigma} \operatorname{RHom}_R(M, M), $$
is an isomorphism.

(4.8) **Size of Auslander and Bass classes.** The next two corollaries investigate the size of the Auslander and Bass classes. But first a definition.

(4.9) **Definition.** If Q is a DGA, then we define two classes of DG-Q-left-modules by

$$ \mathcal{F}(Q) = \left\{ L \in \mathsf{D}(Q) \;\middle|\; \begin{array}{l} L \text{ is isomorphic in } \mathsf{D}(Q) \text{ to a} \\ K\text{-flat left-bounded DG-module} \end{array} \right\} $$

and

$$ \mathcal{I}(Q) = \left\{ N \in \mathsf{D}(Q) \;\middle|\; \begin{array}{l} N \text{ is isomorphic in } \mathsf{D}(Q) \text{ to a} \\ K\text{-injective right-bounded DG-module} \end{array} \right\}. $$

(4.10) **Corollary ([4, cor. (5.5)]).**

(1) Suppose that M satisfies condition (4.7)(1). Suppose moreover that when we forget the R-structure on M, we can resolve M by a K-projective DG-S-left-module, A, so that $({}_S A)^\natural$ is a direct summand in a finite coproduct of shifts of $S^\natural$.

Then the Auslander class $\mathcal{A}_M(R^{\mathrm{opp}})$ is all of $\mathsf{D}(R^{\mathrm{opp}})$.

(2) Suppose that M satisfies condition (4.7)(2). Suppose moreover that when we forget the S-structure on M, we can resolve M by a K-projective DG-R-left-module, B, so that $({}_R B)^\natural$ is a direct summand in a finite coproduct of shifts of $R^\natural$.

Then the Bass class $\mathcal{B}_M(S)$ is all of $\mathsf{D}(S)$.

(4.11) **Corollary ([4, cor. (2.7)]).**

(1) Suppose that M satisfies condition (4.7)(1). Suppose moreover the following:
- R and S are non-negatively graded.
- $\mathrm{H}_0(S)$ is left-noetherian, and each $\mathrm{H}_i(S)$ is finitely generated from the left over $\mathrm{H}_0(S)$.
- $\mathrm{H}(M)$ is bounded, and each $\mathrm{H}_i(M)$ is finitely generated over $\mathrm{H}_0(S)$.

Then
$$\mathcal{F}(R^{\mathrm{opp}}) \subseteq \mathcal{A}_M(R^{\mathrm{opp}}).$$

(2) Suppose that M satisfies condition (4.7)(2). Suppose moreover the following:
- R and S are non-negatively graded.
- $\mathrm{H}_0(R)$ is left-noetherian, and each $\mathrm{H}_i(R)$ is finitely generated from the left over $\mathrm{H}_0(R)$.
- $\mathrm{H}(M)$ is bounded, and each $\mathrm{H}_i(M)$ is finitely generated over $\mathrm{H}_0(R)$.

Then
$$\mathcal{I}(S) \subseteq \mathcal{B}_M(S).$$

(4.12) **Classical Foxby equivalence.** Suppose R is a noetherian commutative ring, viewed as a DGA concentrated in degree zero. Set $S = R$ and let M be a dualizing complex for R, that is, M is a bounded complex of R-modules, its homology modules are finitely generated, its injective dimension is finite, and the endomorphism DGA $\mathrm{RHom}_R(M,M)$ is particularly simple, in that the canonical morphism $R \xrightarrow{\rho} \mathrm{RHom}_R(M,M)$ is an isomorphism in $\mathsf{D}(R)$.

Since $R = R^{\mathrm{opp}}$ the adjoint pair pair of derived functors is,

$$\mathsf{D}(R) \underset{\mathrm{RHom}_R(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathsf{D}(R),$$

which is the functors known from classical Foxby equivalence. The Auslander and Bass classes $\mathcal{A}_M(R)$ and $\mathcal{B}_M(R)$ are simply the classes $\mathbf{A}(R)$ and $\mathbf{B}(R)$ of [9, def. (3.1)] (and paragraph (1.5)), except that we have avoided the (unnecessary) boundedness conditions imposed in [9]. The equivalence result in (4.2) essentially specializes to the equivalence theorem [9, thm. (3.2)].

Next, since R is commutative and R equals S, we may resolve M by a DG-R-left-S-module which is K-projective over S, namely, pick a K-projective resolution of M viewed as an R-complex. Since M is a dualizing complex for R we have $R \cong \mathrm{RHom}_S(M,M)$. Moreover, by assumptions on R, S, and M we see that the three conditions in corollary (4.11)(1) are met. Consequently, $\mathcal{A}_M(R)$ contains $\mathcal{F}(R)$. In particular, $\mathcal{A}_M(R)$ contains all complexes of finite flat dimension.

Symmetrically, by (4.11)(2) we see that $\mathcal{B}_M(R)$ contains $\mathcal{I}(R)$. In particular, $\mathcal{B}_M(R)$ contains all complexes of finite injective dimension.

(4.13) **Dwyer and Greenlees equivalence.** Suppose S is a ring, viewed as a DGA concentrated in degree zero. Let M be a bounded complex consisting of finitely generated projective S-left-modules, that is, M is a perfect complex.

Define R to be the endomorphism DGA of M, that is, $R = \mathrm{Hom}_S(M, M)$. By (4.6) M becomes a DG-R-left-S-left-module, and we get a pair of quasi-inverse equivalences between the Auslander and Bass classes,

$$\mathcal{A}_M(R^{\mathrm{opp}}) \underset{\mathrm{RHom}_S(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathcal{B}_M(S).$$

Next, since ${}_{R,S}M$ is perfect it is a K-projective resolution of itself when viewed as a complex over S. Moreover, by definition $R \xrightarrow{\rho} \mathrm{Hom}_S(M, M)$ is an isomorphism, and forgetting the R-structure on M we see that $({}_SM)^\natural$ is a direct summand in a finite coproduct of $S^\natural$.

Consequently, $\mathcal{A}_M(R^{\mathrm{opp}}) = \mathsf{D}(R^{\mathrm{opp}})$ by corollary (4.10)(1), and the above diagram takes on the form,

$$\mathsf{D}(R^{\mathrm{opp}}) \underset{\mathrm{RHom}_S(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathcal{B}_M(S).$$

which is identical to the right half of the diagram from Dwyer and Greenlees' Morita theorem [18, thm. 2.1]:

$$\mathbf{A}_{\mathrm{comp}} \underset{C}{\overset{E}{\rightleftarrows}} \mathsf{mod}\text{-}\mathcal{E} \underset{E}{\overset{T}{\rightleftarrows}} \mathbf{A}_{\mathrm{tors}}$$

(Note that Dwyer and Greenlees denote R by $\mathcal{E}$, and $D(R^{\mathrm{opp}})$ by $\mathsf{mod}\text{-}\mathcal{E}$). To see this, just check that:

- The functors $- \overset{\mathrm{L}}{\otimes}_R M$ and $\mathrm{RHom}_S(M, -)$ are identical with the functors T and E from [18].
- The Bass class $\mathcal{B}_M(S)$ equals $\mathbf{A}_{\mathrm{tors}}$ (see [18, thm. 2.1]).

Replacing M by $\mathrm{Hom}_S(M, S)$, generalized Foxby equivalence specialize to the other half of the diagram from [18, thm. 2.1].

(4.14) **Matlis equivalence.** Suppose R is a noetherian local commutative ring, with maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$. Let $\mathrm{E}(k)$ denote the injective hull of k.

One may ask what happens if we consider generalized Foxby equivalence with respect to the injective hull of k. As the functor $\operatorname{Hom}_R(-, \mathrm{E}(k))$, which provides a duality between artinian and noetherian R-modules over a complete ring, is called *Matlis duality*, we suggest that this instance of generalized Foxby equivalence should be named *Matlis equivalence*.

It turns out that this particular instance of generalized Foxby equivalence can tell if R is a Gorenstein ring or not. To be precise: R is Gorenstein, if and only if the Auslander class $\mathcal{A}_{\mathrm{E}(k)}(R)$ contains k, which is tantamount to the Bass class $\mathcal{B}_{\mathrm{E}(k)}(R)$ contains k.

Recall, if R admits a dualizing complex D, then classical Foxby equivalence displays the same feature, that is, R is Gorenstein, if and only if the Auslander class $\mathcal{A}_D(R)$ contains k, which is tantamount to the Bass class $\mathcal{B}_D(R)$ contains k (see [16, (3.1.12) and (3.2.10)]).

However, R does not always admit a dualizing complex.

(4.15) **Gorenstein sensitivity ([4, thm. (3.5)]).** Let R be as in (4.14). Then the following statements are equivalent:

(*i*) R is Gorenstein.
(*ii*) $k \in \mathcal{A}_{\mathrm{E}(k)}(R)$.
(*iii*) $k \in \mathcal{B}_{\mathrm{E}(k)}(R)$.

(4.16) **Comment.** In the next section we will encounter corollary (5.10) which is strongly connected to the Gorenstein sensitivity theorem.

5. Affine equivalence and Gorensteinness

(5.1) **Infrastructure.** This paper is connected to the following papers:

Primary

- None.

Secondary

- **[6]** *Gorenstein Differential Graded Algebras.*
- **[7]** *Dualizing DG-modules for Differential Graded Algebras.*
- **[3]** *Vanishing of Local Homology.*

(5.2) **Setup.** Throughout this section, R is a noetherian local commutative ring with maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$. By $\mathfrak{a}$ we denote an ideal in R. The $\mathfrak{a}$-adic completion of R is denoted $\widehat{R_{\mathfrak{a}}}$.

(5.3) **Essential images for derived section and completion.** The essential image of a functor is the closure of its range under isomorphisms. Let $\mathfrak{a}$ be an ideal in R. The essential image of the derived section functor, $\mathrm{R}\Gamma_{\mathfrak{a}}(-)$, is denoted $\mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}(R)$, while the essential image of the derived completion functor, $\mathrm{L}\Lambda^{\mathfrak{a}}(-)$, is denoted $\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}(R)$ (see [18], and paragraphs (3.8) and (3.9)).

(5.4) **Affine equivalence.** Suppose that R admits a dualizing complex D. Consider the adjoint pair of functors

$$\mathsf{D}(R) \underset{\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D),-)}{\overset{\mathrm{R}\Gamma_{\mathfrak{a}}(D)\overset{\mathrm{L}}{\otimes}_R -}{\rightleftarrows}} \mathsf{D}(R),$$

the corresponding Auslander and Bass class, that is, $\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R)$ and $\mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R)$, and the quasi-inverse equivalences of categories

$$\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R) \underset{\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D),-)}{\overset{\mathrm{R}\Gamma_{\mathfrak{a}}(D)\overset{\mathrm{L}}{\otimes}_R -}{\rightleftarrows}} \mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R).$$

We may also consider the adjoint pair of functors

$$\mathsf{D}(R) \underset{\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),-)}{\overset{\mathrm{C}(\mathfrak{a})\overset{\mathrm{L}}{\otimes}_R -}{\rightleftarrows}} \mathsf{D}(R),$$

see paragraph (3.5). From the beautiful work [18] it follows that the corresponding Auslander and Bass classes is $\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}(R)$ and $\mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}(R)$ respectively, and that we have quasi-inverse equivalences of categories

$$\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}(R) \underset{\mathrm{L}\Lambda^{\mathfrak{a}}(-)\simeq \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),-)}{\overset{\mathrm{R}\Gamma_{\mathfrak{a}}(-)\simeq \mathrm{C}(\mathfrak{a})\overset{\mathrm{L}}{\otimes}_R -}{\rightleftarrows}} \mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}(R).$$

In essence; in this paragraph we actually have two distinct instances of generalized Foxby equivalence, namely one which yields the Auslander and Bass classes $\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R)$ and $\mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R)$, while the other yields the Auslander and Bass classes $\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}(R)$ and $\mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}(R)$.

In [29] Hartshorne consider the functor $\mathrm{RHom}_R(-,\mathrm{R}\Gamma_{\mathfrak{a}}(R))$ over a *regular complete local ring* and produce a duality between *finite* objects (complexes in $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$), and what he calls *co-finite* objects (the essential image of $\mathrm{RHom}_R(-,\mathrm{R}\Gamma_{\mathfrak{a}}(R))$). He name this duality *affine duality.* Thus, is we suggest that the instance of generalized Foxby equivalence built on the adjoint functors $(\mathrm{R}\Gamma_{\mathfrak{a}}(D) \overset{\mathrm{L}}{\otimes}_R -, \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D),-))$ should be named *affine equivalence.*

(5.5) **Maximality.** It turns out that $\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}(R)$ and $\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R)$, and $\mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}(R)$ and $\mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R)$ are related in the sense that are inclusions

- $\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R) \subseteq \mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}(R)$
- $\mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R) \subseteq \mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}(R)$,

see [**5, prop. (1.6)**]. In other words: the maximal size of $\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R)$ is $\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}(R)$, while the maximal size of $\mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R)$ is $\mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}(R)$. The main theorem of [**5**] shows that Gorenstein rings are exactly the rings for which these maximal sizes are attained.

This feature, that a certain maximal size of the Auslander and Bass classes characterize the Gorenstein property, will appear later (see (7.17)).

(5.6) **The square root of completion.** Let us consider complexes X in $\mathsf{D}(R)$ for which the standard morphism

$$X \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a})) \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X,\mathrm{C}(\mathfrak{a})),\mathrm{C}(\mathfrak{a}))$$

is an isomorphism.

Now, the object $\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a}))$ in $\mathsf{D}(R)$ turns out to be very nice, in the sense that we have the following isomorphism in $\mathsf{D}(R)$,

$$\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a})) \cong \widehat{R_{\mathfrak{a}}},$$

which is a flat R-module, see [**5, lem. (1.9)**].

Consequently, we see that X has the property that the standard morphism

$$X \otimes_R \widehat{R_{\mathfrak{a}}} \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X,\mathrm{C}(\mathfrak{a})),\mathrm{C}(\mathfrak{a})),$$

is an isomorphism.

Recall, that if X is a complex in $\mathsf{D}(R)$ which is bounded to the right with finitely generated homology, then

$$\mathrm{L}\Lambda^{\mathfrak{a}}(X) \cong X \otimes_R \widehat{R_{\mathfrak{a}}},$$

see **[3, prop. (2.8)]**. Thus, the composed morphism

$$\mathrm{L}\Lambda^{\mathfrak{a}}(X) \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X, \mathrm{C}(\mathfrak{a})), \mathrm{C}(\mathfrak{a})),$$

is an isomorphism.

The main theorem of **[5]** shows that Gorenstein rings are exactly the rings for which the functor $\mathrm{RHom}_R(-, \mathrm{C}(\mathfrak{a}))$ is the "square root" of $\mathrm{L}\Lambda^{\mathfrak{a}}(-)$, when restricted to the full subcategory $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

Let us end this paragraph by stating the theorem.

(5.7) **The parameterized Gorenstein theorem ([5, thm. (2.2)]).** *Let R, k, $\mathfrak{a}$ and $\mathrm{C}(\mathfrak{a})$ be as above. Now the following conditions are equivalent:*

(*i*) *R is Gorenstein.*

(*ii*) *The standard morphism*

$$X \otimes_R \widehat{R_{\mathfrak{a}}} \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X, \mathrm{C}(\mathfrak{a})), \mathrm{C}(\mathfrak{a}))$$

is an isomorphism for $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

If R has a dualizing complex D, then the above conditions are also equivalent to the following

(*iii*) $k \in \mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R)$.

(*iv*) $\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R) = \mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}(R)$.

(*v*) $k \in \mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R)$.

(*vi*) $\mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}(R) = \mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}(R)$.

(5.8) **Two corollaries.** Specializing the parameterized Gorenstein theorem to its two extremal instances, namely $\mathfrak{a} = 0$ or $\mathfrak{a} = \mathfrak{m}$, we obtain the following two corollaries.

(5.9) **The parameterized Gorenstein theorem for $\mathfrak{a} = 0$ ([5, cor. (2.4)]).** *Let R and k be as above. Now the following conditions are equivalent:*

(*i*) *R is Gorenstein.*

(*ii*) *The standard morphism*

$$X \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X, R), R)$$

is an isomorphism for $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

If R has a dualizing complex D, then the above conditions are also equivalent to the following

(*iii*) $k \in \mathcal{A}_D(R)$.

(*iv*) $\mathcal{A}_D(R) = \mathsf{D}(R)$.

(*v*) $k \in \mathcal{B}_D(R)$.

(*vi*) $\mathcal{B}_D(R) = \mathsf{D}(R)$.

(5.10) **The parameterized Gorenstein theorem for $\mathfrak{a} = \mathfrak{m}$ ([5, cor. (2.6)]).** *Let R, k and $\mathrm{E}(k)$ be as in* (5.2). *Now the following conditions are equivalent:*

(*i*) *R is Gorenstein.*

(*ii*) *The standard morphism*

$$X \otimes_R \widehat{R} \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X, \mathrm{C}(\mathfrak{m})), \mathrm{C}(\mathfrak{m}))$$

is an isomorphism for $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

If R has a dualizing complex D, then the above conditions are also equivalent to the following

(*iii*) $k \in \mathcal{A}_{\mathrm{E}(k)}(R)$.

(*iv*) $\mathcal{A}_{\mathrm{E}(k)}(R) = \mathbf{A}^{\mathfrak{m}}_{\mathrm{comp}}(R)$.

(*v*) $k \in \mathcal{B}_{\mathrm{E}(k)}(R)$.

(*vi*) $\mathcal{B}_{\mathrm{E}(k)}(R) = \mathbf{A}^{\mathrm{tors}}_{\mathfrak{m}}(R)$.

6. Gorenstein Differential Graded Algebras

(6.1) **Infrastructure.** This paper is connected to the following papers:

Primary

- [**7**] *Dualizing DG-modules for Differential Graded Algebras.*
- [**8**] *Dualizing DG modules and Gorenstein DG Algebras.*

Secondary

- [**9**] *Homological Identities for Differential Graded Algebras.*
- [**10**] *Homological Identities for Differential Graded Algebras, II.*

(6.2) **Setup.** Throughout this section, R and S will denote DGAs for which $\mathrm{H}_0(R)$ and $\mathrm{H}_0(S)$ are noetherian rings.

(6.3) **How to detect a Gorenstein ring.** Suppose A is a noetherian local commutative ring. How do we detect when A is a Gorenstein ring?

One (functorial) way to characterize them is the following: A is a Gorenstein ring exactly when the contravariant functor $\mathrm{RHom}_A(-,A)$ gives a duality (that is, a pair of quasi-inverse contravariant equivalences of categories),

$$\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A) \underset{\mathrm{RHom}_A(-,A)}{\overset{\mathrm{RHom}_A(-,A)}{\rightleftarrows}} \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A),$$

where $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$ is the derived category of bounded complexes of with finitely generated homology (see [16, thm. (2.3.14)] and [**5, cor. (2.4)**]). This functorial characterization, however, is equivalent to the following two conditions:

- There is a natural isomorphism
$$M \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(M,A),A)$$
for M in $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$.
- $\mathrm{RHom}_A(-,A)$ sends $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$ to $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$.

This way of characterizing Gorenstein rings will be the cornerstone in our definition of a Gorenstein DGA.

But first we need a DG-analogue of $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

(6.4) **The category fin ([6, def. (0.8)]).** By $\mathsf{fin}(R)$ we denote the full subcategory of $\mathsf{D}(R)$ consisting of DG-modules M so that $\mathrm{H}(M)$ is bounded, and so that each $\mathrm{H}_i(M)$ is finitely generated over $\mathrm{H}_0(R)$.

(6.5) **Gorenstein DGAs ([6, def. (1.1)]).** We call R a *Gorenstein DGA* if it satisfies:

[G1] There is a quasi-isomorphism of DG-R-left-R-right-modules ${}_RR_R \xrightarrow{\simeq} {}_RI_R$ where ${}_RI$ and I_R are K-injective.

[G2] For $M \in \mathsf{fin}(R)$ and $N \in \mathsf{fin}(R^{\mathrm{opp}})$ the following standard morphisms are isomorphisms:

$$\mathrm{RHom}_{R^{\mathrm{opp}}}(R,R) \overset{\mathrm{L}}{\otimes}_R M \longrightarrow \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(M,R),R),$$
$$N \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(R,R) \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_{R^{\mathrm{opp}}}(N,R),R)$$

[G3] The functor $\mathrm{RHom}_R(-,R)$ maps objects from $\mathsf{fin}(R)$ to $\mathsf{fin}(R^{\mathrm{opp}})$, and the functor $\mathrm{RHom}_{R^{\mathrm{opp}}}(-,R)$ maps objects from $\mathsf{fin}(R^{\mathrm{opp}})$ to $\mathsf{fin}(R)$.

(6.6) **Gorenstein ring and conditions [G1]–[G3].** Let us briefly comment on the three conditions:

Condition [G1] is purely technical: The existence of I allows the formation of derived functors; the existence of I is (automatically) satisfied in all cases in which we are interested.

Condition [G2] singles out the duality property of Gorenstein rings: To us a Gorenstein DGA must be a sensible dualizing object. This is an integral feature of our definition. Since

$$\mathrm{RHom}_{R^{\mathrm{opp}}}(R,R) \cong R \cong \mathrm{RHom}_R(R,R),$$

as a DG-R-left-R-right module, the left-hand sides of the two morphisms in condition [G2] are isomorphic to M and N themselves. To phrase condition [G2] differently: If one takes an object in fin and dualize it twice with respect to R, the object will reappear up to natural isomorphism.

Condition [G3] supplements condition [G2]: It requires the operation of dualizing with respect to R to send fin to fin. So, when both conditions [G2] and [G3] are in force, dualization with respect to R is a duality (a contravariant equivalence of categories) between $\mathsf{fin}(R)$ and $\mathsf{fin}(R^{\mathrm{opp}})$, a feature displayed by an ordinary Gorenstein ring.

A moral comment: When both conditions [G2] and [G3] are in force one may think of this as a (moral) way of saying that R "is of finite injective dimension". This statement is *not well-defined!*

(6.7) **Gorenstein morphisms of DGAs.** Suppose that

$$A \xrightarrow{\varphi} B$$

is a local homomorphism between noetherian local commutative rings such that B viewed as a A module is finitely generated and of finite flat dimension. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals for A and B respectively. From [9, lem. (6.5), (7.7.1), and thm. (7.8)] we may conclude that φ is

"Gorenstein at $\mathfrak{n}$" in the sense of [9] if and only if

$$\Sigma^m B \cong \mathrm{RHom}_A(B, A),$$

for some $m \in \mathbb{Z}$. This purely *functorial characterization* of *Gorenstein homomorphisms* of (noetherian local commutative) rings suggests that this notion may be lifted to the realm of DGAs.

Suppose that

$$R \xrightarrow{\rho} S$$

is a "finite" morphism of DGAs. A priori, R and S may be non-commutative so the structures on

$$\mathrm{RHom}_R({}_R S_S, {}_R R_R) \quad \text{and} \quad \mathrm{RHom}_{R^{\mathrm{opp}}}({}_S S_R, {}_R R_R)$$

may be different. The first has the structure of a DG-S-left-R-right module while the second has the structure of a DG-S-right-R-left module. So as a first approximation to a definition of a "finite" Gorenstein morphism of DGAs one would (at least) have to consider the case

$$\Sigma^m({}_S S_R) \cong \mathrm{RHom}_R({}_R S_S, {}_R R_R) \quad \text{and} \quad \Sigma^m({}_R S_S) \cong \mathrm{RHom}_{R^{\mathrm{opp}}}({}_S S_R, {}_R R_R),$$

for some $m \in \mathbb{Z}$.

The above observation on Gorenstein homomorphisms will be key in our definition of what we call *finite Gorenstein morphisms of DGAs*. However, when dealing with non-commutative objects, one needs definitions which are "structure sensitive" (a fact already seen in [32]).

Let us review the definition of a finite Gorenstein morphism of DGAs.

(6.8) **Finite morphisms ([6. def. (2.1)]).** Suppose that

$$R \xrightarrow{\rho} S$$

is a morphism of DGAs. We call ρ a *finite morphism* if it satisfies:

- The functor ${}_S S_R \overset{\mathrm{L}}{\otimes}_R - : \mathsf{D}(R) \longrightarrow \mathsf{D}(S)$ sends $\mathsf{fin}(R)$ to $\mathsf{fin}(S)$.
- The functor $- \overset{\mathrm{L}}{\otimes}_R {}_R S_S : \mathsf{D}(R^{\mathrm{opp}}) \longrightarrow \mathsf{D}(S^{\mathrm{opp}})$ sends $\mathsf{fin}(R^{\mathrm{opp}})$ to $\mathsf{fin}(S^{\mathrm{opp}})$.
- The functor $\rho^* : \mathsf{D}(S) \longrightarrow \mathsf{D}(R)$, restricting scalars from S to R, satisfies
$$M \in \mathsf{fin}(S) \Leftrightarrow \rho^* M \in \mathsf{fin}(R).$$
- The functor $\rho^* : \mathsf{D}(S^{\mathrm{opp}}) \longrightarrow \mathsf{D}(R^{\mathrm{opp}})$, restricting scalars from S to R, satisfies
$$M \in \mathsf{fin}(S^{\mathrm{opp}}) \Leftrightarrow \rho^* M \in \mathsf{fin}(R^{\mathrm{opp}}).$$

(Note the slight abuse of notation in that ρ^* is used to denote the functor which restricts scalars from S to R both on DG-S-left-modules and on DG-S-right-modules.)

(6.9) **Induced morphisms.** Suppose that

$$R \xrightarrow{\rho} S$$

is a morphism of DGAs, and that R satisfies condition [G1].

Assume that we have a morphism

$$_SS_R \xrightarrow{\alpha} \mathrm{RHom}_R(_RS_S, \Sigma^n(_RR_R))$$

and a DG-S-left-module M. Then there is an induced morphism

$$\rho^* \mathrm{RHom}_S(M, {_SS_S}) \longrightarrow \mathrm{RHom}_R(\rho^* M, \Sigma^n(_RR_R)),$$

which is an isomorphism if α is an isomorphism.

Similarly, assume that we have have a morphism

$$_RS_S \xrightarrow{\beta} \mathrm{RHom}_{R^{\mathrm{opp}}}(_SS_R, \Sigma^n(_RR_R))$$

and a DG-S-right-module N. Then there is a morphism

$$\rho^* \mathrm{RHom}_{S^{\mathrm{opp}}}(N, {_SS_S}) \longrightarrow \mathrm{RHom}_{R^{\mathrm{opp}}}(\rho^* N, \Sigma^n(_RR_R)),$$

which is an isomorphism if β is an isomorphism.

(6.10) **Gorenstein morphisms ([6, def. (2.4)]).** Suppose that R satisfies condition [G1], and let $R \xrightarrow{\rho} S$ be a *finite* morphism of DGAs. We call ρ a *Gorenstein morphism* if it satisfies:

(1) There are isomorphisms
  (a) $\quad {_SS_R} \xrightarrow{\alpha} \mathrm{RHom}_R(_RS_S, \Sigma^n(_RR_R))$.
  (b) $\quad {_RS_S} \xrightarrow{\beta} \mathrm{RHom}_{R^{\mathrm{opp}}}(_SS_R, \Sigma^n(_RR_R))$.

(2) The isomorphisms α and β are compatible in the following sense:
  (a) For each DG-S-left-module M the following diagram is commutative,

$$\begin{tikzcd}
\rho^* M \arrow[r, "\rho^*\ell", "\cong"'] \arrow[d, "k", "\cong"'] & \rho^*(\mathrm{RHom}_{S^{\mathrm{opp}}}(_SS_S, {_SS_S}) \overset{\mathrm{L}}{\otimes}_S M) \arrow[d, "\rho^*(t)"] \\
\mathrm{RHom}_{R^{\mathrm{opp}}}(\Sigma^n(_RR_R), \Sigma^n(_RR_R)) \overset{\mathrm{L}}{\otimes}_R \rho^* M \arrow[d, "s"] & \rho^* \mathrm{RHom}_{S^{\mathrm{opp}}}(\mathrm{RHom}_S(M, {_SS_S}), {_SS_S}) \arrow[d, "\cong"', "b"] \\
\mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(\rho^* M, \Sigma^n(_RR_R)), \Sigma^n(_RR_R)) \arrow[r, "\cong", "a"'] & \mathrm{RHom}_{R^{\mathrm{opp}}}(\rho^* \mathrm{RHom}_S(M, {_SS_S}), \Sigma^n(_RR_R)),
\end{tikzcd}$$

where k and ℓ are the canonical identifications, s and t are standard morphisms like the ones in condition [G2], and a and b are induced by α and β as explained in (6.9).

  (b) For each DG-S-right-module N there is a commutative diagram constructed like the one above.

(6.11) **Gorenstein transfer.** Finite Gorenstein morphisms of DGAs ascent the Gorenstein property. We conjecture that they also descent the Gorenstein property, but are unable to prove this. Here is the ascent-result.

(6.12) **Theorem (Ascent ([6, thm. (2.6)])).** *Suppose that R and S satisfy condition* [G1], *and let $R \xrightarrow{\rho} S$ be a finite morphism of DGAs. Suppose that ρ is a Gorenstein morphism. Then*

$$R \text{ is Gorenstein } \Rightarrow S \text{ is Gorenstein.}$$

(6.13) **Gorenstein morphisms exits.** It turns out that interesting finite Gorenstein morphisms actually exist in nature. Let us review two important examples (which will play a central role in the rest of this thesis).

- Suppose A is a noetherian local commutative ring with maximal ideal $\mathfrak{m}$. Let $\boldsymbol{a} = (a_1, \dots, a_n)$ be a sequence of elements from $\mathfrak{m}$.

  The Koszul complex $\mathrm{K}(\boldsymbol{a})$ is a commutative DGA (see [40, exer. 4.5.1]), and $\mathrm{H}_0(\mathrm{K}(\boldsymbol{a}))$ is a noetherian (local) commutative ring since it is isomorphic to A modulo the ideal generated by $\boldsymbol{a}$.

  By definition, the zero component of $\mathrm{K}(\boldsymbol{a})$ is A. Thus, there is a canonical morphism of DGAs

  $$A \xrightarrow{\theta} \mathrm{K}(\boldsymbol{a}),$$

  where A is viewed as a DGA concentrated in degree zero. By [**6, lem. (3.3)**] it is a finite Gorenstein morphism.
- Suppose A is a noetherian local commutative ring. Let L be a bounded complex of finite generated projective A-modules.

  As noted in (4.6) L becomes a DG-$\mathcal{E}$-left-A-module, where $\mathcal{E} = \mathrm{Hom}_A(L, L)$ is the endomorphism DGA of L. Moreover, the next canonical morphism is an isomorphism

  $${}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}} \xrightarrow{\cong} \mathrm{Hom}_A({}_{A,\mathcal{E}}L, {}_{A,\mathcal{E}}L).$$

  Note that $\mathrm{H}_0(\mathcal{E})$ is a noetherian ring since it is a finitely generated A-module.

  The endomorphism DGA $\mathcal{E}$ and its homology $\mathrm{H}(\mathcal{E})$ are usually highly non-commutative. If, for instance, L is a projective resolution of a finitely generated A-module M of finite projective dimension, then $\mathrm{H}_0(\mathcal{E}) = \mathrm{End}_A(M)$. Moreover, $\mathcal{E}$ usually has non-zero homology in both positive and negative degrees.

  For any $a \in A$ we have a chain map $L \xrightarrow{a\cdot} L$ which is just multiplication by a, and as such it is an element in the zero component of $\mathcal{E}$, that is, an element in $\mathcal{E}_0 = \mathrm{Hom}_A(L, L)_0$. Thus, there is a canonical morphism of DGAs

  $$A \xrightarrow{\psi} \mathcal{E}, \qquad a \longmapsto (L \xrightarrow{a\cdot} L),$$

where A is viewed as a DGA concentrated in degree zero. By **[6, lem. (3.8)]** it is a finite Gorenstein morphism

(6.14) **Ascent-Descent theorems.** Now a natural question presents itself: How does the Gorenstein property for a noetherian local commutative ring A effect the Gorenstein property for the Koszul complex $\mathrm{K}(\boldsymbol{a})$ and the endomorphism DGA $\mathcal{E}$?

The answer is very simple and beautiful, and provided in the next two theorems.

(6.15) **Ascent-Descent: Koszul complex ([6, thm. (3.4)]).** *Let A be a noetherian commutative local ring, let $\boldsymbol{a} = (a_1, \dots, a_n)$ be a sequence of elements in A's maximal ideal, and let $\mathrm{K}(\boldsymbol{a})$ be the Koszul complex on $\boldsymbol{a}$. Then*

$$A \text{ is a Gorenstein ring } \Leftrightarrow \mathrm{K}(\boldsymbol{a}) \text{ is a Gorenstein DGA.}$$

(6.16) **Ascent-Descent: Endomorphism DGA ([6, thm. (3.9)]).** *Let A be a noetherian commutative local ring, let L be a bounded complex of finitely generated projective A-modules with $\mathrm{H}(L) \neq 0$, and let $\mathcal{E} = \mathrm{Hom}_A(L, L)$ be the endomorphism DGA of L. Then*

$$A \text{ is a Gorenstein ring } \Leftrightarrow \mathcal{E} \text{ is a Gorenstein DGA.}$$

7. Dualizing DG-modules for Differential Graded Algebras

(7.1) **Infrastructure.** This paper is connected to the following papers:

Primary

- [**6**] *Gorenstein Differential Graded Algebras.*
- [**8**] *Dualizing DG modules and Gorenstein DG Algebras.*

Secondary

- [**9**] *Homological Identities for Differential Graded Algebras.*
- [**10**] *Homological Identities for Differential Graded Algebras, II.*

(7.2) **Setup.** Throughout this section, R will denote a DGA for which $\mathrm{H}_0(R)$ is a noetherian ring.

(7.3) **When R is not a Gorenstein DGA.** Recall from section 6 that when R is Gorenstein, R itself is a good dualizing object meaning that there are quasi-inverse contravariant equivalences of categories,

$$\mathsf{fin}(R) \underset{\mathrm{RHom}_{R^{\mathrm{opp}}}(-,R)}{\overset{\mathrm{RHom}_R(-,R)}{\rightleftarrows}} \mathsf{fin}(R^{\mathrm{opp}}).$$

Now a natural question presents itself: what if R is *not* Gorenstein?

In this case we may ask for something weaker: a DG-R-left-R-right-module D with duality properties resembling the ones of a Gorenstein DGA.

We call such modules D for *dualizing DG-modules*; they are defined in definition (7.5).

(7.4) **Generalized Foxby equivalence.** A priori, a dualizing DG-module D must yield quasi-inverse *contravariant* equivalences of categories,

$$\mathsf{fin}(R) \underset{\mathrm{RHom}_{R^{\mathrm{opp}}}(-,D)}{\overset{\mathrm{RHom}_R(-,D)}{\rightleftarrows}} \mathsf{fin}(R^{\mathrm{opp}}).$$

But this is not all.

Recall the generalized Foxby equivalence from section 4; having D we also have quasi-inverse *covariant* equivalences of categories,

$$\mathcal{A}_D(R) \underset{\mathrm{RHom}_R(D,-)}{\overset{D\overset{\mathrm{L}}{\otimes}_R -}{\rightleftarrows}} \mathcal{B}_D(R).$$

In order to get a good definition of dualizing DG-modules, we will place further conditions on D ensuring that its corresponding Foxby equivalence is rigid (this statement will be made precise in what follows).

Let us end this paragraph and review the definition of a dualizing DG-module.

(7.5) **Dualizing DG-modules ([7, def. (1.1)]).** Let ${}_RD_R$ be a DG-R-left-R-right-module. We call ${}_RD_R$ a *weak dualizing DG-module* for R if it satisfies:

[D1] There are quasi-isomorphisms of DG-R-left-R-right-modules $P \xrightarrow{\simeq} D$ and $D \xrightarrow{\simeq} I$ such that ${}_RP$ and P_R are K-projective and ${}_RI$ and I_R are K-injective.

[D2] The following canonical morphisms in the derived category of DG-R-left-R-right-modules are isomorphisms,

$$R \xrightarrow{\rho} \mathrm{RHom}_R(D,D),$$
$$R \xrightarrow{\rho^{\mathrm{opp}}} \mathrm{RHom}_{R^{\mathrm{opp}}}(D,D).$$

[D3] For $M \in \mathsf{fin}(R)$ and $N \in \mathsf{fin}(R^{\mathrm{opp}})$ and ${}_RL_R$ equal to either ${}_RR_R$ or ${}_RD_R$, the following evaluation morphisms are isomorphisms:

$$\mathrm{RHom}_{R^{\mathrm{opp}}}(L,D) \overset{\mathrm{L}}{\otimes}_R M \longrightarrow \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(M,L),D),$$
$$N \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(L,D) \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_{R^{\mathrm{opp}}}(N,L),D).$$

We call D a *dualizing DG-module* for R if it also satisfies:

[D4] The functor $\mathrm{RHom}_R(-,D)$ maps $\mathsf{fin}(R)$ to $\mathsf{fin}(R^{\mathrm{opp}})$, and the functor $\mathrm{RHom}_{R^{\mathrm{opp}}}(-,D)$ maps $\mathsf{fin}(R^{\mathrm{opp}})$ to $\mathsf{fin}(R)$.

(7.6) **Finite injective dimension.** Again (as in (6.6)), conditions [D3] and [D4] is a (moral) way of saying that D "is of finite injective dimension over R". This statement is still *not well-defined!*

(7.7) **Dualizing DG-modules and Gorensteinness.** Note that R is a Gorenstein DGA when and only when ${}_RR_R$ is a dualizing DG-module for R; completely analogous to classical ring theory.

(7.8) **Weak dualizing complexes.** Suppose A is a noetherian local commutative ring admitting a dualizing complex D. If $\mathfrak{a}$ is an ideal in A, then $\mathrm{R}\Gamma_{\mathfrak{a}}(D)$ is a weak dualizing DG-module for A viewed as a DGA concentrated in degree zero. These objects was key in section 5.

(7.9) **Dualizing complexes and dualizing DG-modules.** Suppose A is a noetherian ring. Over such one may speak of *dualizing complexes*, see [45, def. 1.1].

If one should take our definition of dualizing DG-modules seriously, we must show that D is a dualizing DG-module for A when viewed a DGA concentrated in degree zero if and only if D is a dualizing complex for the ring A. Here are two such results.

(7.10) **Theorem ([7, thm. (1.7)]).** *Let A be a noetherian commutative ring of finite finitistic flat dimension (or equivalently finite finitistic injective dimension), and let $D \in \mathsf{D}(A)$. Then D is a dualizing DG-module for A, viewed as a DGA concentrated in degree zero, if and only if D is a dualizing complex for the ring A.*

(7.11) **Theorem ([7, thm. (1.9)]).** *Let A be a noetherian semi-local PI algebra over the field k, and let D be an object in $\mathsf{D}(A \otimes_k A^{\mathrm{opp}})$. Then D is a dualizing DG-module for A, viewed as a DGA concentrated in degree zero, if and only if D is a dualizing complex for the algebra A.*

(7.12) **A rigid Foxby equivalence.** Let us now return to the generalized Foxby equivalence one may consider when having a dualizing DG-module. It turns out that this instance of generalized Foxby equivalence displays additional rigidity. In order to qualify this statement we will need some notation.

(7.13) **A hierarchy of Auslander and Bass classes ([7, def. (2.3)]).** Suppose D is a weak dualizing DG-module for R, and consider the corresponding Auslander and Bass classes $\mathcal{A}_D(R)$ and $\mathcal{B}_D(R)$.

We will cut these full triangulated subcategories of $\mathsf{D}(R)$ appropriately down. Define the *finite* Auslander and Bass classes as

$$\mathcal{A}^{\mathrm{f}}_D(R) = \left\{ X \in \mathcal{A}(R) \;\middle|\; \begin{array}{l} X \in \mathsf{fin}(R) \text{ and} \\ D \overset{\mathrm{L}}{\otimes}_R X \in \mathsf{fin}(R) \end{array} \right\}$$

and

$$\mathcal{B}^{\mathrm{f}}_D(R) = \left\{ Y \in \mathcal{B}(R) \;\middle|\; \begin{array}{l} Y \in \mathsf{fin}(R) \text{ and} \\ \mathrm{RHom}_R(D, Y) \in \mathsf{fin}(R) \end{array} \right\}.$$

There are of course corresponding definitions for (finite) Auslander and Bass classes of DG-R-right-modules, denoted $\mathcal{A}_D(R^{\mathrm{opp}})$ and $\mathcal{A}^{\mathrm{f}}_D(R^{\mathrm{opp}})$, and $\mathcal{B}_D(R^{\mathrm{opp}})$ and $\mathcal{B}^{\mathrm{f}}_D(R^{\mathrm{opp}})$.

We can now formulate our first rigidity result.

(7.14) **(Finite) Foxby equivalence part I ([7, thm. (2.4)]).** *Let D be a weak dualizing DG-module for R. Then there are the following quasi-inverse equivalences,*

$$\begin{tikzcd} \mathcal{A}^{\mathrm{f}}_D(R) \arrow[r, shift left, "D \overset{\mathrm{L}}{\otimes}_R -"] & \mathcal{B}^{\mathrm{f}}_D(R). \arrow[l, shift left, "\mathrm{RHom}_R(D,-)"] \end{tikzcd}$$

There are of course corresponding quasi-inverse equivalences between $\mathcal{A}^{\mathrm{f}}_D(R^{\mathrm{opp}})$ and $\mathcal{B}^{\mathrm{f}}_D(R^{\mathrm{opp}})$.

(7.15) **The condition [Grade] ([7, def. (2.5)]).** Suppose that D is a dualizing DG-module for R, and that $M \xrightarrow{\mu} N$ is a morphism in $\mathsf{fin}(R)$. What condition imposed on R will ensure that

$$\left.\begin{array}{r} D \overset{\mathrm{L}}{\otimes}_R \mu \text{ is an isomorphism or} \\ \mathrm{RHom}_R(D,\mu) \text{ is an isomorphism} \end{array}\right\} \Rightarrow \quad \mu \text{ is an isomorphism?}$$

Of course we could ask the same question for a morphisms in $\mathsf{fin}(R^{\mathrm{opp}})$.

It turns out that the following condition imposed on R will do the trick.

We say that R satisfies [Grade] if the following hold:

- $M \in \mathsf{fin}(R)$ and $\mathrm{H}(M) \neq 0 \Longrightarrow \mathrm{H}(\mathrm{RHom}_R(M,R)) \neq 0$ and
- $N \in \mathsf{fin}(R^{\mathrm{opp}})$ and $\mathrm{H}(N) \neq 0 \Longrightarrow \mathrm{H}(\mathrm{RHom}_{R^{\mathrm{opp}}}(N,R)) \neq 0$.

The reason why we call this condition [Grade] comes from the following fact: if R is just a ring satisfying [Grade], then for $M \in \mathsf{fin}(R)$, the number

$$-\sup\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(M,R)) \neq 0 \,\},$$

also known as the $\mathrm{grade}_R(M)$, is not ∞, provided $\mathrm{H}(M) \neq 0$.

Next, let us list DGAs which satisfy [Grade].

Let A be a noetherian local commutative ring and let B be a non-commutative noetherian local PI algebra over a field (see [39]). The following DGAs satisfy [Grade]:

- The ring A viewed as a DGA concentrated in degree zero.
- The algebra B viewed as a DGA concentrated in degree zero.
- The Koszul complex $\mathrm{K}(\boldsymbol{a})$ on a sequence $\boldsymbol{a} = (a_1, \dots, a_n)$ of elements in the maximal ideal of A.
- The endomorphism DGA $\mathcal{E} = \mathrm{Hom}_A(L,L)$, where L is a bounded complex of finitely generated projective A-modules which is not exact.

The functors $D \overset{\mathrm{L}}{\otimes}_R -$ and $\mathrm{RHom}_R(D,-)$ ability to detect isomorphisms in fin is central in the proof of our second rigidity result.

(7.16) **(Finite) Foxby equivalence part II ([7, thm. (2.8)]).** *Let R satisfy* [Grade] *and let D be a dualizing DG-module for R. Then*

- $$\left.\begin{array}{r} X \in \mathsf{fin}(R) \textit{ and} \\ D \overset{\mathrm{L}}{\otimes}_R X \in \mathcal{B}^{\mathrm{f}}(R) \end{array}\right\} \Rightarrow \quad X \in \mathcal{A}^{\mathrm{f}}(R).$$
- $$\left.\begin{array}{r} Y \in \mathsf{fin}(R) \textit{ and} \\ \mathrm{RHom}_R(D,Y) \in \mathcal{A}^{\mathrm{f}}(R) \end{array}\right\} \Rightarrow \quad Y \in \mathcal{B}^{\mathrm{f}}(R).$$

There are of course corresponding results for DG-R-right-modules.

(7.17) **Maximality.** Suppose A is a noetherian local commutative ring admitting a dualizing complex.

In [16], (see also The parameterized Gorenstein theorem (5.7)) one can find theorems characterizing Gorenstein rings in terms of the maximal size for the Auslander and Bass classes.

In the presence of a dualizing DG-module, generalized Foxby equivalence displays the same feature; it can detect Gorenstein DGAs in terms of the maximality of $\mathcal{A}_D(R)$ and $\mathcal{B}_D(R)$.

Note, from (7.13) that we have full embeddings of categories

- $$\begin{cases} \mathcal{A}_D^{\mathrm{f}}(R) \subseteq \mathsf{fin}(R) \quad \text{and} \\ \mathcal{A}_D^{\mathrm{f}}(R^{\mathrm{opp}}) \subseteq \mathsf{fin}(R^{\mathrm{opp}}). \end{cases}$$
- $$\begin{cases} \mathcal{B}_D^{\mathrm{f}}(R) \subseteq \mathsf{fin}(R) \quad \text{and} \\ \mathcal{B}_D^{\mathrm{f}}(R^{\mathrm{opp}}) \subseteq \mathsf{fin}(R^{\mathrm{opp}}), \end{cases}$$

showing that the maximal possible size of either of the classes $\mathcal{A}_D^{\mathrm{f}}$, and $\mathcal{B}_D^{\mathrm{f}}$ is fin. We end this section by stating a theorem which characterizes the DGAs for which this maximal size is attained.

(7.18) **Gorenstein Theorem ([7, thm. (2.9)]).** *Let R be a DGA satisfying* [Grade] *and* [G1]. *Moreover, let D be a dualizing DG-module for R. The following conditions are equivalent:*

(*i*) *R is a Gorenstein DGA (i.e.,* [G2] *and* [G3] *hold).*

(*ii*) $$\begin{cases} \mathcal{A}_D^{\mathrm{f}}(R) = \mathsf{fin}(R) \quad \textit{and} \\ \mathcal{A}_D^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}}). \end{cases}$$

(*iii*) $$\begin{cases} \mathcal{B}_D^{\mathrm{f}}(R) = \mathsf{fin}(R) \quad \textit{and} \\ \mathcal{B}_D^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}}). \end{cases}$$

8. Dualizing DG modules and Gorenstein DG Algebras

(8.1) **Infrastructure.** This paper is connected to the following papers:

Primary

- **[6]** *Gorenstein Differential Graded Algebras.*
- **[7]** *Dualizing DG-modules for Differential Graded Algebras.*

Secondary

- **[9]** *Homological Identities for Differential Graded Algebras.*
- **[10]** *Homological Identities for Differential Graded Algebras, II.*

(8.2) **Setup.** Throughout this section, A denotes a noetherian commutative ring, and R a DGA.

(8.3) **Existence of dualizing DG-modules.** In section 7 we defined dualizing DG-modules (see definition (7.5)). However, we did not address the following obvious question: When do dualizing DG-modules exist?

In this section we will make up for this. We will provide examples of DGAs which admit dualizing DG-modules. In search of these examples the following trick is paramount.

(8.4) **Coinduction.** Suppose we have the following data:

- $C \in \mathsf{D}(A)$ is a dualizing complex for A
- $A \xrightarrow{\varphi} R$ is a morphism of DGAs which has image inside the center of R (see paragraph (B.5)).
- As an A-complex R is bounded with finitely generated homology (that is, $R \in \mathsf{fin}(A)$).

Define D to be the *coinduced object* of C over R, that is, let

$$ {}_R D_R = \mathrm{Hom}_A({}_{A,R}R_R, {}_A C) \cong \mathrm{RHom}_A({}_{A,R}R_R, {}_A C). $$

(8.5) **Proposition ([8, prop. 2.5]).** *Let A, R and C be as above. Suppose the following conditions hold for the coinduced object*

$$ D = \mathrm{RHom}_A(R, C). $$

(1) *The following canonical morphism is an isomorphism in the derived category of DG-R-left-R-right-modules,*

$$ \mathrm{RHom}_A(C, C) \overset{\mathrm{L}}{\otimes}_A R \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(R, C), C). $$

(2) *For $M \in \mathsf{fin}(R)$ and $N \in \mathsf{fin}(R^{\mathrm{opp}})$, the following evaluation morphisms are isomorphisms in $\mathsf{D}(R)$ and $\mathsf{D}(R^{\mathrm{opp}})$,*

$$\mathrm{RHom}_A(C,C) \overset{\mathrm{L}}{\otimes}_A (D \overset{\mathrm{L}}{\otimes}_R M) \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(D \overset{\mathrm{L}}{\otimes}_R M, C), C),$$
$$(N \overset{\mathrm{L}}{\otimes}_R D) \overset{\mathrm{L}}{\otimes}_A \mathrm{RHom}_A(C,C) \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(N \overset{\mathrm{L}}{\otimes}_R D, C), C).$$

(3) *There is a quasi-isomorphism of DG-R-left-R-right modules $P \xrightarrow{\simeq} D$ where ${}_R P$ and P_R are K-projective.*

Then D is a dualizing DG-module for R.

(8.6) **DGAs admitting dualizing modules.** With this result we can prove the following collection of theorems and one proposition.

(8.7) **Theorem.** *Let A be a noetherian commutative ring admitting a dualizing complex. The the following statements hold:*

(1) *The Koszul complex $\mathrm{K}(\boldsymbol{a})$ on each finite set of elements $\boldsymbol{a}$ in A admits a dualizing DG-module (*[**8, thm. 2.1**]*).*

(2) *For each bounded complex of finitely generated projectives, P, with $\mathrm{H}(P) \neq 0$, the endomorphism DGA $\mathcal{E} = \mathrm{Hom}_A(P,P)$ admits a dualizing DG-module (*[**8, thm. 2.3**]*).*

(3) *Suppose that A is local and that $\varphi : A' \longrightarrow A$ is a local homomorphism of finite flat dimension. The DG-fibre of φ (see* [12, (3.7)]*) admits a dualizing DG-module (*[**8, thm. 2.2**]*).*

(4) *Let G be a topological monoid. Suppose that the singular homology of G with coefficients in A is finitely generated over A. Then the chain DGA $\mathrm{C}_*(G;A)$ admits a dualizing DG-module (*[**8, thm. 2.4**]*).*

(5) *Let X be simply connected topological space. Suppose the cohomology of X with coefficients in a field k is finitely generated over k. Then the cochain DGA $\mathrm{C}^*(X;k)$ admits a dualizing DG-module (*[**8, prop. 5.2**]*).*

(8.8) **Local DGAs.** Suppose R is a *commutative* DGA (see paragraph (B.5)). We call it *local* if it also satisfy the following conditions:

- $R_i = 0$ for $i < 0$ (that is, R is a chain DGA).
- R_0 is noetherian.
- $\mathrm{H}_0(R)$ is local with residue class field k and each $\mathrm{H}_i(R)$ is finitely generated over $\mathrm{H}_0(R)$.

Note that k may be viewed as an DG-R-left-R-right-module.

A DG-R-left-R-right-module M is called balanced if $rm = (-1)^{|r||m|}mr$ for elements $r \in R$ and $m \in M$, that is, its left and right structures determine each other.

We can now formulate an existence result which is a DG-analogue of [30, V.3.4].

(8.9) **Theorem ([8, thm. 3.1]).** *Let $R \in \mathsf{fin}(R)$ be a commutative local DGA, let D be a balanced DG-R-module, and let $n \in \mathbb{Z}$. The following conditions are equivalent.*

(i) *D is a dualizing DG-module for R.*

(ii) *$D \in \mathsf{fin}(R)$ and $\mathrm{RHom}_R(k, D) \cong \Sigma^n k$.*

(8.10) **Uniqueness.** When A is a noetherian local commutative ring any pair of balanced dualizing complexes are isomorphic up to suspension (see [30, chap. 5]). Here is a DG-analogue of this result (which contains the previous theorem, if R admits a balanced dualizing DG-module).

(8.11) **Theorem ([8, thm. 3.2]).** *Let $R \in \mathsf{fin}(R)$ be a commutative local DGA, let D and E be balanced dualizing DG-R-modules, and let $n, r \in \mathbb{Z}$. The following conditions are equivalent.*

(i) *$E \cong \Sigma^n D^r$.*

(ii) *$\mathrm{RHom}_R(k, E) \cong \Sigma^n k^r$.*

In particular, if E is a balanced dualizing DG-module, then it is isomorphic to an appropriate suspension of D.

(8.12) **Gorenstein DGAs.** Again, when R is a commutative local DGA (not necessarily with the property $R \in \mathsf{fin}(R)$) we have

$$\mathrm{RHom}_R(k, R) \cong \Sigma^n k$$

for some $n \in \mathbb{Z}$ if it is Gorenstein (see **[8, prop. 3.3.(a)]**).

For a commutative local DGA with $R \in \mathsf{fin}(R)$, Avramov and Foxby defined it to be *Gorenstein* if

$$\mathrm{RHom}_R(k, R) \cong \Sigma^n k$$

for some $n \in \mathbb{Z}$ (see [6]).

Thus, when restricting to commutative local DGAs for which $R \in \mathsf{fin}(R)$, the class of Gorenstein DGAs (in the sense of definition (6.5)) could be smaller than the class of *Gorenstein* DGAs (in the sense of Avramov and Foxby). However, the two a priori different Gorenstein notions turn out to be equivalent on the above class. Here is the result.

(8.13) **Theorem ([8, thm. 4.3]).** *Let R a commutative local DGA for which $R \in \mathsf{fin}(R)$, and let $n \in \mathbb{Z}$. The following conditions are equivalent.*

(i) *R is Gorenstein.*

(ii) *$\mathrm{RHom}_R(k, R) \cong \Sigma^n k$.*

(8.14) **Cochain DGAs.** Suppose R is a DGA and let k be a field. We call R a cochain DGA if it satisfy the following conditions:

- $R^i = 0$ for $i < 0$ (that is, R is a cochain DGA).
- $R^0 = k$ and $R^1 = 0$.
- $\dim_k \mathrm{H}(R) < \infty$.

Note, that k can be viewed in a canonical way as a DG-R-left-R-right-module concentrated in degree zero. Next, we presents two results investigating the Gorenstein property of such DGAs.

(8.15) **Dualizing DG-module.** It turns out that our cochain DGAs always admits a dualizing DG-module.

(8.16) **Proposition ([8, prop. 5.2]).** *Let R be a cochain DGA. Then*

$$D = \mathrm{Hom}_k(R, k) = \mathrm{RHom}_k(R, k)$$

is a dualizing DG-module for R.

(8.17) **Theorem ([8, thm. 5.3]).** *Let R be a cochain DGA such that $\mathrm{H}(R)$ is commutative. If*

$$\mathrm{RHom}_R(k, R) \cong \Sigma^n k$$

for some $n \in \mathbb{Z}$, then R is Gorenstein.

(8.18) **Theorem ([8, thm. 5.4]).** *Let R be a commutative cochain DGA. If R is Gorenstein, then*

$$\mathrm{RHom}_R(k, R) \cong \Sigma^n k$$

for some $n \in \mathbb{Z}$.

(8.19) **Gorenstein topological spaces.** Suppose X is a topological space and let k be a field. By $\mathrm{C}^*(X, k)$ we denote the complex of cochains on X with coefficients in k. This complex is equipped with a multiplication which is defined by the so-called Alexander-Whitney map making it into a cochain DGA (see [12]). Félix, Halperin, and Thomas defines the topological space X to be *Gorenstein at k* if

$$\mathrm{RHom}_{\mathrm{C}^*(X;k)}(k, \mathrm{C}^*(X; k)) \cong \Sigma^n k,$$

for some $n \in \mathbb{Z}$ (see [19]). Applying theorems (8.17) and (8.18) we get a result which ties the notion of Gorenstein topological spaces together with Gorenstein DGAs (in the sense of (6.5)).

(8.20) **Theorem ([8, thm. 5.6]).** *Suppose X is a simply connected topological space such that $\mathrm{H}^*(X; k)$ is finitely generated over k. If X is Gorenstein at k, then the DGA $\mathrm{C}^*(X; k)$ is Gorenstein. The converse holds if the characteristic of k is zero.*

9. Homological Identities for Differential Graded Algebras

(9.1) **Infrastructure.** This paper is connected to the following papers:

Primary

- [**10**] *Homological Identities for Differential Graded Algebras, II.*
- [**7**] *Dualizing DG-modules for Differential Graded Algebras.*
- [**8**] *Dualizing DG modules and Gorenstein DG Algebras.*

Secondary

- [**2**] *Restricted Homological Dimensions and Cohen-Macaulayness.*

(9.2) **Setup.** Throughout this section, R will denote a DGA satisfying:
- $R_i = 0$ for $i < 0$ (that is, R is a chain DGA).
- $\mathrm{H}_0(R)$ is a noetherian ring which is local in the sense that it has a unique maximal two sided ideal J such that $\mathrm{H}_0(R)/J$ is a skew field.
- ${}_RR \in \mathsf{fin}(R)$ and $R_R \in \mathsf{fin}(R^{\mathrm{opp}})$.

We denote the skew field $\mathrm{H}_0(R)/J$ by k.

Note that k can be viewed in a canonical way as a DG-R-left-R-right-module concentrated in degree zero.

(9.3) **Imposed conditions on the dualizing DG-module.** In section 7 we encountered dualizing DG-modules.

Throughout, we will assume that R admits a dualizing DG-module D with the property

$$\mathrm{RHom}_R({}_Rk, {}_RD_R) \cong k_R \quad \text{and} \quad \mathrm{RHom}_{R^{\mathrm{opp}}}(k_R, {}_RD_R) \cong {}_Rk.$$

When R is just a noetherian local commutative ring admitting a dualizing complex D, then by [**7, thm. (1.7)**] D is a dualizing DG-module for R viewed as a DGA concentrated in degree zero, and as such it (or an appropriate suspension of D) meets the above requirements (see [30, prop. V.3.4]).

But there are other natural DGAs which satisfy the requirements in question. Let us end this paragraph by listing the ones we know:

- The DG-fibre $F(\alpha')$, where $A' \xrightarrow{\alpha'} A$ is a local ring homomorphism of finite flat dimension between noetherian local commutative rings A' and A, and where A admits a dualizing complex.
- The Koszul complex $\mathrm{K}(\boldsymbol{a})$, where $\boldsymbol{a} = (a_1, \ldots, a_n)$ is a sequence of elements in the maximal ideal of the noetherian local commutative ring A, and where A again admits dualizing complex.

- The chain DGA $\mathrm{C}_*(G;k)$ where k is a field and G is a path connected topological monoid with $\dim_k \mathrm{H}_*(G;k) < \infty$ (see [20, chap. 8]).

(9.4) **Dualizing complexes and homological identities.** It is well-documented that the theory of dualizing complexes in ring theory provides very slick proofs for (at least) the following important results from homological algebra:

- The *Auslander-Buchsbaum Formula.*
- The *Bass Formula.*
- The *No Holes theorem.*

See [4, thm. 3.7], [13, lem. (3.3)], [21, thm. (1.1)], [25], [37, thm. 2]), [41, thm. 0.3], and [42, thm. 1.1].

It is therefore natural to suspect that dualizing DG-modules ultimately will provide means which will enable us to generalize the above results to the realm of DGAs. That this is indeed the case will be demonstrated in this and the next section.

(9.5) **Dagger Duality ([9, (0.3)]).** Let us here introduce some new notation. Suppose D is a dualizing DG-module for R.

For any DG-R-left-module M and any DG-R-right-module N we may define the *dagger duals* (with respect to D) as,

$$M^\dagger = \mathrm{RHom}_R(M,D) \qquad \text{and} \qquad N^\dagger = \mathrm{RHom}_{R^{\mathrm{opp}}}(N,D).$$

Since we only consider one particular D, we will henceforth suppress it from our notation.

Dagger duality will denote the pair of quasi-inverse contravariant equivalences of categories between $\mathsf{fin}(R)$ and $\mathsf{fin}(R^{\mathrm{opp}})$,

$$\mathsf{fin}(R) \underset{(-)^\dagger}{\overset{(-)^\dagger}{\rightleftarrows}} \mathsf{fin}(R^{\mathrm{opp}}),$$

see also paragraph (7.4). Note the slight abuse of notation in that $(-)^\dagger$ denotes two different functors.

The duality displays the following feature: for $M, N \in \mathsf{fin}(R)$ we have

$$\mathrm{RHom}_{R^{\mathrm{opp}}}(N^\dagger, M^\dagger) = \mathrm{RHom}_R(M,N)$$

The name "dagger duality" is due to Foxby; it plays a central role in the proofs of the results presented here.

(9.6) **Semi-free resolutions.** The semi-free resolutions also act as a central ingredient in [**9**].

Let us here review some important facts:

- Suppose M is a DG-R-left-module with $\mathrm{H}(M)$ bounded to the right and each $\mathrm{H}_i(M)$ finitely generated as an $\mathrm{H}_0(R)$-module. Set $v = \inf\{i \mid \mathrm{H}_i(M) \neq 0\}$.
- We can construct a minimal semi-free resolution $F \xrightarrow{\simeq} M$ which has a semi-free filtration with quotients as indicated,

$$
\begin{array}{ccccccccccccccc}
 & & & \Sigma^{v+1}R^{\gamma_0} & & & & \Sigma^{v+2}R^{\gamma_1} & & & \cdots & & \\
 & & / & & \backslash & & / & & \backslash & & & & \\
0 & \subseteq & F(0) & \subseteq & L(0) & \subseteq & F(1) & \subseteq & L(1) & \subseteq & \cdots & \subseteq & F \\
 & \backslash & / & & & \backslash & / & & & & & & \\
 & \Sigma^{v}R^{\delta_0} & & & & \Sigma^{v+1}R^{\delta_1} & & & & & \cdots, & &
\end{array}
$$

  where all γ_j and δ_j are finite. We choose to present the above diagram in order to emphasize the analogy to semi-free resolutions considered in the next section (see paragraph (10.6)).
- We can write $F^\natural$ as

$$F^\natural \cong \coprod_{v \leq j} \Sigma^j (R^\natural)^{\beta_j}.$$

  where each β_j is finite.

Minimality of F means that the differential ∂^F maps into $\mathfrak{m}F$, where $\mathfrak{m}$ is the DG-ideal

$$\cdots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow J \longrightarrow 0 \longrightarrow \cdots.$$

As consequence, $\mathrm{Hom}_R(F, k)$ and $k \otimes_R F$ have vanishing differentials. See [3, prop. 2], [10], and [19, lem. (A.3)(iii)].

(9.7) **Definition ([9, def. (1.1)]).** For a DG-R-left-module M, we define the *k-projective dimension*, the *k-injective dimension*, and the *depth* as

$$
\begin{aligned}
k.\mathrm{pd}_R\, M &= -\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(M,k)) \neq 0 \,\}, \\
k.\mathrm{id}_R\, M &= -\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(k,M)) \neq 0 \,\}, \\
\mathrm{depth}_R\, M &= -\sup\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(k,M)) \neq 0 \,\}.
\end{aligned}
$$

(9.8) **Observation ([9, (1.2)]).** When $M \in \mathsf{fin}(R)$ and employing minimal resolutions we see that

$$k.\mathrm{pd}_R\, M = \sup\{\, i \mid \mathrm{H}_i(k \overset{\mathrm{L}}{\otimes}_R M) \neq 0 \,\}.$$

(9.9) **Definition.** For a DG-R-left-module M, we define the j'th *Bass number* as

$$\mu_R^j(M) = \dim_k \mathrm{H}_{-j}(\mathrm{RHom}_R(k, M)).$$

(Note that $\mu_R^j(M)$ may well equal $+\infty$.)

(9.10) **A pivot result.** In order to prove the three main result of this section we need the following central result.

(9.11) **Proposition ([9, prop. (1.8)]).** *Let M and N be DG-R-left-modules with $\mathrm{H}(M)$ and $\mathrm{H}(N)$ bounded to the right, and each $\mathrm{H}_i(M)$ and each $\mathrm{H}_i(N)$ finitely generated as an $\mathrm{H}_0(R)$-module. Suppose that $k.\mathrm{pd}_R\, M$ is finite. Then*

$$\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(M,N)) \neq 0 \,\} = -k.\mathrm{pd}_R\, M + \inf\{\, i \mid \mathrm{H}_i(N) \neq 0 \,\}.$$

(9.12) **Homological identities for chain DGAs.** Let us now list our three main results. As stated in paragraph (9.10) their proofs hinge on proposition (9.11).

It should be noted that when specializing the DGA R to an ordinary noetherian local commutative ring, the classical Auslander-Buchsbaum Formula, the classical Bass Formula, and the classical No-Holes Theorem reemerge (see [**9, (3.3)**]).

(9.13) **The Auslander-Buchsbaum Formula ([9, thm. (2.3)]).** *Let M be in $\mathsf{fin}(R)$ and suppose that $k.\mathrm{pd}_R\, M$ is finite. Then*

$$k.\mathrm{pd}_R\, M + \mathrm{depth}_R\, M = \mathrm{depth}_R\, R.$$

(9.14) **The Bass Formula ([9, thm. (2.4)]).** *Let N be in $\mathsf{fin}(R)$ and suppose that $k.\mathrm{id}_R\, N$ is finite. Then*

$$k.\mathrm{id}_R\, N + \inf\{\, i \mid \mathrm{H}_i(N) \neq 0 \,\} = \mathrm{depth}_R\, R.$$

(9.15) **Chain Gorenstein DGAs.** Suppose $M \in \mathsf{fin}(R)$ have finite k-projective and k-injective dimension. It is reasonable to conjecture that R in this case must be a Gorenstein DGA.

(9.16) **Gap Theorem ([9, thm. (2.5)]).** *Let M be in $\mathsf{fin}(R)$ and let $g \in \mathbb{Z}$ satisfy $g > \mathrm{amp}\, R$. Assume that the sequence of Bass numbers of M has a gap of length g, in the sense that there exists $\ell \in \mathbb{Z}$ such that*

- $\mu_R^{\ell}(M) \neq 0$.
- $\mu_R^{\ell+1}(M) = \cdots = \mu_R^{\ell+g}(M) = 0$.
- $\mu_R^{\ell+g+1}(M) \neq 0$.

Then we have

$$\mathrm{amp}\, M \geq g + 1.$$

(9.17) **Gaps in Bass series of DGAs.** As indicated in the above theorem we say that the sequence of Bass numbers of a DG-R-left-module M has a *gap of length g* if there exists an ℓ with

- $\mu_R^{\ell}(M) \neq 0$,
- $\mu_R^{\ell+1}(M) = \cdots = \mu_R^{\ell+g}(M) = 0$,

- $\mu_R^{\ell+g+1}(M) \neq 0$.

Avramov and Foxby defines the *Bass series* of M as

$$I_M(t) = \sum_n \mu_R^n(M)t^n,$$

and defines the *gap of* $I_M(t)$ by

$$\operatorname{gap} I_M(t) = \sup\left\{ g \;\middle|\; \begin{array}{l}\text{the sequence of Bass numbers}\\ \text{of } M \text{ has a gap of length } g\end{array} \right\}.$$

(see [6, sec. 3]). Avramov and Foxby now pose the following question (see [6, question (3.10)]): is the gap in the Bass series for R always less or equal the amplitude of R?

We see that the above theorem answers this in the affirmative (see [**9, (3.1)**]).

(9.18) **A topological application.** Since the chain DGA $\mathrm{C}_*(G;k)$ of a path connected topological monoid G taking coefficients in a field k is an example of the DGAs we consider in this section, it would not be a complete surprise if our general homological results on these DGAs could provide information on topological setups. Here is one such result.

(9.19) G**-Serre-fibrations ([9, (3.2)]).** Let G and k be as above, and let

$$G \longrightarrow P \xrightarrow{p} X$$

be a G-Serre-fibration (see [20, chap. 2]). Suppose that the homology of G, P, and X (with coefficients in k) are finitely dimensional over k.

It follows that $\mathrm{C}_*(P;k)$ sits inside $\mathsf{fin}(\mathrm{C}_*(G;k))$, and it turns out that the k-projective dimension of $\mathrm{C}_*(P;k)$ equals $\sup\{\, i \mid \mathrm{H}_i(X;k) \neq 0 \,\}$ which is finite by assumption.

Evoking the Auslander-Buchsbaum Formula (9.13) we get

$$\sup\{\, i \mid \mathrm{H}_i(P;k) \neq 0 \,\} = \sup\{\, i \mid \mathrm{H}_i(G;k) \neq 0 \,\} + \sup\{\, i \mid \mathrm{H}_i(X;k) \neq 0 \,\},$$

yielding that the homological dimension is additive on G-Serre-fibrations; this results is usually obtained using the Serre spectral sequence associated to the fibration in question.

10. Homological Identities for Differential Graded Algebras, II

(10.1) **Infrastructure.** This paper is connected to the following papers:

Primary

- **[9]** *Homological Identities for Differential Graded Algebras.*
- **[7]** *Dualizing DG-modules for Differential Graded Algebras.*
- **[8]** *Dualizing DG modules and Gorenstein DG Algebras.*

Secondary

- **[2]** *Restricted Homological Dimensions and Cohen-Macaulayness.*

(10.2) **Setup.** Throughout this section, k will denote a field and R a DGA over k satisfying:
- $R^i = 0$ for $i < 0$ (that is, R is a cochain DGA).
- $R^0 = k$ and $R^1 = 0$.
- $\dim_k \mathrm{H}(R) < \infty$.

Note that k can be viewed in a canonical way as a DG-R-left-R-right-module concentrated in degree zero.

(10.3) **Comment.** This section is a direct counterpart to the previous one (see also paragraph (10.9)) .

(10.4) **A particular simple dualizing DG-module.** From section 8 we know that

$$D = \mathrm{RHom}_k(R, k) \cong \mathrm{Hom}_k(R, k)$$

is a dualizing DG-module for R. Thus, reading paragraph (9.5) we see, using adjointness, that dagger duality simply implodes into dualization with respect to the field k.

(10.5) **Duality.** For any DG-R-left-module M and any DG-R-right-module N we may define the k-duals

$$M' = \mathrm{RHom}_k({}_{R,k}M_k, k) \qquad \text{and} \qquad N' = \mathrm{RHom}_k({}_kN_{R,k}, k).$$

There is now a pair of quasi-inverse contravariant equivalences of categories between $\mathsf{G}(R)$ and $\mathsf{G}(R^{\mathrm{opp}})$, where $\mathsf{G}(R)$ denotes the category of DG-R-left-modules for which $\mathrm{H}(M)$ is finite dimensional over k,

$$\begin{tikzcd} \mathsf{G}(R) \arrow[r, shift left, "(-)'"] & \mathsf{G}(R^{\mathrm{opp}}). \arrow[l, shift left, "(-)'"] \end{tikzcd}$$

Note that when M and N are DG-R-left-modules with $\mathrm{H}(M)$ and $\mathrm{H}(N)$ finite dimensional over k we have

$$\mathrm{RHom}_{R^{\mathrm{opp}}}(N', M') \cong \mathrm{RHom}_R(M, N),$$

see also paragraph (9.5). Moreover, the following feature is also handy,

$$\inf\{\, i \mid \mathrm{H}^i(M') \neq 0 \,\} = -\sup\{\, i \mid \mathrm{H}^i(M) \neq 0 \,\}.$$

(10.6) **Semi-free resolutions ([10, thm. (A.2)]).** Again, semi-free resolutions act as a central ingredient in [**10**].

Let us here review some important facts:

- Suppose M is a DG-R-left-module with $\mathrm{H}(M)$ non-zero and bounded to the left, and each $\mathrm{H}^i(M)$ finite dimensional over k. Set $u = \inf\{\, i \mid \mathrm{H}^i(M) \neq 0 \,\}$.
- We can construct a minimal semi-free resolution $F \xrightarrow{\simeq} M$ which has a semi-free filtration with quotients as indicated,

$$\begin{tikzcd}[column sep=small]
 & & & \Sigma^{-u-1}R^{\delta_1} \arrow[dl, dash] \arrow[dr, dash] & & & & \Sigma^{-u-2}R^{\delta_2} \arrow[dl, dash] \arrow[dr, dash] & & & \cdots & & \\
0 & \subseteq & F(0) & \subseteq & L(0) & \subseteq & F(1) & \subseteq & L(1) & \subseteq & \cdots & \subseteq & F \\
 & \Sigma^{-u}R^{\gamma_0} \arrow[ul, dash] \arrow[ur, dash] & & & & \Sigma^{-u}R^{\gamma_1} \arrow[ul, dash] \arrow[ur, dash] & & & & & \cdots, & &
\end{tikzcd}$$

  where γ_j and δ_j are finite.
- We can write $F^\natural$ as

$$F^\natural \cong \coprod_{j \leq -u} \Sigma^j (R^\natural)^{\beta_j},$$

  where each β_j is finite .
- If the filtration terminates, then there exists a semi-split (that is, the sequence is split after applying $(-)^\natural$) exact sequence of DG-R-left-modules

$$0 \to P \longrightarrow F \longrightarrow \Sigma^w R^\alpha \to 0$$

  with $\alpha \neq 0$, with P being K-projective, and with

$$P^\natural \cong \coprod_{w \leq j} \Sigma^j (R^\natural)^{\epsilon_j}.$$

(10.7) **Definition ([10, def. (1.1)]).** For a DG-R-left-module M, we define *width* and *depth* by

$$\begin{aligned}
\operatorname{width}_R M &= -\sup\{\, i \mid \mathrm{H}^i(k \overset{\mathrm{L}}{\otimes}_R M) \neq 0 \,\},\\
\operatorname{depth}_R M &= \inf\{\, i \mid \mathrm{H}^i(\mathrm{RHom}_R(k, M)) \neq 0 \,\}.
\end{aligned}$$

(10.8) **Definition.** For a DG-R-left-module M, we define the j'th *Bass number* as

$$\mu^j(M) = \dim_k \mathrm{H}^j(\mathrm{RHom}_R(k, M)).$$

(Note that $\mu^j(M)$ may well equal $+\infty$.)

(10.9) **A dictionary between invariants.** The so-called *looking glass principle* formulated by Avramov and Halperin (see [12]) tells us that there exists a deep symmetry between chain and cochain DGAs. Morally speaking, it tells us that a result in the world of chain DGAs should have a mirror version in the world of cochain DGAs. Indeed, the results presented here on cochain DGAs are a direct consequence of the looking glass principle; they all are mirror versions of the results from the previous section. However, the symmetry is *not* automatic, that is, symmetric statements need *not* have symmetric proofs.

This "defect" is very distinct in the two non-symmetric proofs of the two symmetric propositions (9.11) and (10.11).

As a chain DGA, S, may be visualized as

$$\cdots \longrightarrow S_n \longrightarrow \cdots \longrightarrow S_1 \longrightarrow S_0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \cdots$$

and a cochain DGA, R, as

$$\cdots \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow k \longrightarrow 0 \longrightarrow R^2 \longrightarrow R^3 \longrightarrow \cdots$$

we can formulate the following dictionary between invariants for chain and cochain DGAs.

Homological invariants	Cohomological invariants
$\inf\{\, i \mid \mathrm{H}_i(M) \neq 0 \,\}$	$\inf\{\, i \mid \mathrm{H}^i(N) \neq 0 \,\}$
$\operatorname{amp} M$	$\operatorname{amp} N$
$k.\mathrm{pd}_S\, M$	$-\operatorname{width}_R N$
$k.\mathrm{id}_S\, M$	$-\operatorname{depth}_R N$
$\operatorname{depth}_S M$	$-\sup\{\, i \mid \mathrm{H}^i(N) \neq 0 \,\}$

(10.10) **A pivot result.** As in (9.10), we need the following proposition in order to prove the three main result of this section.

(10.11) **Proposition ([10, prop. (1.6)]).** *Let M and N be DG-R-left-modules with $\mathrm{H}(M)$ and $\mathrm{H}(N)$ bounded to the left. Suppose that each $\mathrm{H}^i(M)$ is finite dimensional over k, and that $\operatorname{width}_R M$ is finite. Then*

$$\inf\{\, i \mid \mathrm{H}^i(\mathrm{RHom}_R(M,N)) \neq 0 \,\} = \operatorname{width}_R M + \inf\{\, i \mid \mathrm{H}^i(N) \neq 0 \,\}.$$

(10.12) **Homological identities for cochain DGAs.** Let us now list our three main results.

(10.13) **Cochain Auslander-Buchsbaum Formula ([10, thm. (2.1)]).** *Let M be a DG-R-left-module with $\mathrm{H}(M)$ finite dimensional over k, and suppose that $\mathrm{width}_R M$ is finite. Then*

$$\mathrm{width}_R M + \sup\{\, i \mid \mathrm{H}^i(M) \neq 0 \,\} = \sup\{\, i \mid \mathrm{H}^i(R) \neq 0 \,\}.$$

(10.14) **Cochain Bass Formula ([10, thm. (2.2)]).** *Let N be a DG-R-left-module with $\mathrm{H}(N)$ finite dimensional over k, and suppose that $\mathrm{depth}_R N$ is finite. Then*

$$\mathrm{depth}_R N - \inf\{\, i \mid \mathrm{H}^i(N) \neq 0 \,\} = \sup\{\, i \mid \mathrm{H}^i(R) \neq 0 \,\}.$$

(10.15) **Cochain Gorenstein DGAs.** Suppose M is a DG-R-left-module with $\mathrm{H}(M)$ finitely generated over k with finite depth and width. It is reasonable to conjecture that R in this case must be a Gorenstein DGA (see paragraph (9.15)).

(10.16) **Cochain Gap Theorem ([10, thm. (2.3)]).** *Let M be a DG-R-left-module with $\mathrm{H}(M)$ finite dimensional over k, and let g be an integer satisfying $g > \sup\{\, i \mid \mathrm{H}^i(R) \neq 0 \,\}$. Assume that the sequence of Bass numbers of M has a gap of length g, in the sense that there exists an integer ℓ such that*

- $\mu^{\ell}(M) \neq 0$.
- $\mu^{\ell+1}(M) = \cdots = \mu^{\ell+g}(M) = 0$.
- $\mu^{\ell+g+1}(M) \neq 0$.

Then we have

$$\mathrm{amp}\, M \geq g + 1.$$

(10.17) **A topological application.** Here is a result which is parallel to the application in paragraph (9.19). We consider the cochain DGA $\mathrm{C}^*(Y;k)$ of a simply connected topological space Y taking coefficients in a field k.

(10.18) **Fibrations of topological spaces ([10, (3.2)]).** Let Y and k be as above, and let

$$F \longrightarrow X \xrightarrow{p} Y$$

be a fibration (see [20, chap. 2]). Suppose that the cohomology of F, X, and Y (with coefficients in k) are finite dimensional over k.

As Y is simply connected and its cohomology is finite dimensional over k, we may, and will, interchange $\mathrm{C}^*(Y;k)$ with an equivalent cochain DGA, R, which satisfies the condition from setup (10.2).

Note that $\mathrm{H}(\mathrm{C}^*(X;k)) = \mathrm{H}^*(X;k)$ which is finite dimensional by assumption, and it turns out that the width of the DG-$\mathrm{C}^*(Y;k)$-left-module $C^*(X,k)$ equals $-\sup\{\, i \mid \mathrm{H}^i(F;k) \neq 0\,\}$ which is finite (again by assumption).

Evoking the Cochain Auslander-Buchsbaum Formula (10.13) we get

$$\sup\{\, i \mid \mathrm{H}^i(X;k) \neq 0\,\} = \sup\{\, i \mid \mathrm{H}^i(F;k) \neq 0\,\} + \sup\{\, i \mid \mathrm{H}^i(Y;k) \neq 0\,\},$$

yielding that the cohomological dimension is additive on fibrations; this results is usually obtained using the Eilenberg-Moore spectral sequence associated to the fibration in question.

Bold faced references. Recall, that references **[1]**,...,**[10]** correspond to the articles on page 2.

References

[1] L. Alonso Tarrío, A. Jeremías López, and J. Lipman, *Local homology and cohomology on schemes*, Ann. Sci. École Norm. Sup. (4) **30** (1997), 1–39.

[2] D. Apassov, *Dualizing modules over differential graded rings*, 41–47 in "Complexes and Differential Graded Modules", ph.d. thesis, Lund University, 1999.

[3] D. Apassov, *Homological dimensions over differential graded rings*, 25–39 in "Complexes and Differential Graded Modules", ph.d. thesis, Lund University, 1999.

[4] M. Auslander and D. Buchsbaum, *Homological dimension in Noetherian rings*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 36–38.

[5] M. Auslander and D. Buchsbaum, *Homological dimensions in Noetherian rings. II*, Trans. Amer. Math. Soc. **88** (1958), 194–206.

[6] L. L. Avramov and H.-B. Foxby, *Locally Gorenstein homomorphisms*, Amer. J. Math. **114** (1992), 1007–1047.

[7] ―― and ――, *Cohen–Macaulay Properties of Ring Homomorphisms*, Adv. in Math. **133** (1998), 54–95.

[8] ―― and ――, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), 129–155.

[9] ―― and ――, *Ring homomorphisms and finite Gorenstein dimension,* Proc. London Math. Soc. (3) **75** (1997), 241–270.

[10] ――, ――, and S. Halperin, Differential graded homological algebra, in preparation.

[11] ――, ―― and B. Herzog, *Structure of Local Homomorphisms*, Journal of Algebra. **164** (1994), 124–145.

[12] L. L. Avramov and S. Halperin, *Through the looking glass: a dictionary between rational homotopy theory and local algebra*, 1–27 in "Algebra, algebraic topology and their interactions" (proceedings of the conference in Stockholm, 1983), Lecture Notes in Math., Vol. 1183, Springer, Berlin, 1986.

[13] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.

[14] M. P. Brodmann and R. Y. Sharp, "Local cohomology: An algebraic introduction with geometric applications", Cambridge Stud. Adv. Math., Vol. 60, Cambridge University Press, Cambridge, 1998.

[15] L. G. Chournard, II, *On finite weak and injective dimensions*, Proc. Amer. Math. Soc. **60** (1976), 57–60.

[16] L. W. Christensen, "Gorenstein Dimensions", Lecture Notes in Math., Vol. 1747, Springer, Berlin, 2000.

[17] ――, *Semi-dualizing complexes and their Auslander categories,* to appear in Trans. Amer. Math. Soc.

[18] W. G. Dwyer and J. P. C. Greenlees, Complete modules and torsion modules, preprint (2000).

[19] Y. Félix, S. Halperin, and J.-C. Thomas, *Gorenstein spaces*, Adv. Math. **71** (1988), 92–112.

[20] ――, ――, and ――, "Rational Homotopy Theory", Grad. Texts in Math., Vol 205, Springer, Berlin, 2000.

[21] R. Fossum, H.-B. Foxby, P. Griffith, and I. Reiten, *Minimal injective resolutions with applications to dualizing modules and Gorenstein modules*, Inst. Hautes Études Sci. Publ. Math. **45** (1975), 193–215.

[22] H.-B. Foxby, A homological theory for complexes of mosules, preprint (1981), 19a&b, Købehavns Universitet.

[23] H.-B. Foxby, Hyperhomological Algebra & Commutative Rings, manuscript (1998).

[24] ———, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), 5–19.

[25] ———, *On the μ^i in a minimal injective resolution*, Math. Scand. **29** (1971), 175–186.

[26] A. A. Gerko, *On homological dimensions*, Sbornik Math. **192** (2001) 1165–1179.

[27] J. P. C. Greenlees and J. P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), no. 2, 438–453.

[28] A. Grothendieck, "Éléments de géométrie algébrique" IV. Étude local des schémas et des morphismes de schémas, *Inst. Hautes Études Sci. Publ. Math.* **20, 24, 28, 32** (1964–1967).

[29] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9** (1970), 145–164.

[30] ———, *Residues and duality*, Lecture Notes in Math., Vol. 20, Springer, Berlin, 1966, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne.

[31] S. Iyengar, *Depth for complexes, and intersection theorems*, Math. Z. **230** (1999), no. 3, 545–567.

[32] P. Jørgensen, *Gorenstein homomorphisms of noncommutative rings*, J. Algebra **211** (1999), 240–267.

[33] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 63–102.

[34] E. Matlis, *The Koszul complex and duality*, Comm. Algebra **1** (1974), 87–144.

[35] ———, The higher properties of R-sequences, J. Algebra **50** (1978), no. 1, 77–112.

[36] J. Rickard, *Morita theory for derived categories,* J. London Math. Soc. (2) **39** (1989), 436–456.

[37] P. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) **9** (1976), 103–106.

[38] N. Spaltenstein, *Resolutions of unbounded complexes*, Comp. Math. **65** (1988), 121–154.

[39] R. Vámos, *Semi-local noetherian PI rings*, Bull. London Math. Soc. **9** (1977) 251–256.

[40] C. A. Weibel, "An introduction to homological algebra", Cambridge Stud. Adv. Math., Vol. 38, Cambridge University Press, Cambridge, 1997, paperback edition.

[41] Q. S. Wu and J. J. Zhang, *Dualizing Complexes over Noncommutative Local Rings*, J. Algebra **239** (2001), 513–548.

[42] ——— and ———, *Some homological invariants of local PI algebras*, J. Algebra **225** (2000), 904–935.

[43] S. Yassemi, *Width of complexes of modules*, Acta. Math. Vietnam. **23** (1998), no. 1, 161–169.

[44] A. Yekutieli, *Dualizing complexes over noncommutative graded algebras*, J. Algebra **153** (1992), 41–84.

[45] ——— and J. J. Zhang, *Rings with Auslander Dualizing Complexes*, J. Algebra **213** (1999), 1–51.

Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, DK–2100 København Ø, Denmark

E-mail address: frankild@math.ku.dk

Journal of Algebra
vol. **235,** 214–242 (2001)

QUASI COHEN–MACAULAY PROPERTIES OF LOCAL HOMOMORPHISMS

ANDERS FRANKILD

Abstract. For a large class of local homomorphisms $\varphi : R \to S$, including those of finite G–dimension studied by Avramov and Foxby in [8], we assign a new numerical invariant called the *quasi Cohen–Macaulay defect* of φ, and a local homomorphism is called *quasi Cohen–Macaulay* if it is of finite G–dimension and has trivial quasi Cohen–Macaulay defect. We show among other things the following

Ascent–Descent Theorem. Let $\varphi : R \to S$ be a local homomorphism.

(**A**) If R is Cohen–Macaulay and φ is quasi Cohen–Macaulay, then S is Cohen–Macaulay.

(**D**) If S is Cohen–Macaulay and G–dim φ is finite, then φ is quasi Cohen–Macaulay.

If furthermore the map of spectra Spec $\widehat{S} \to$ Spec $\widehat{R}$ is surjective one also has

(**D′**) If S is Cohen–Macaulay and G–dim φ is finite, then φ is quasi Cohen–Macaulay, and R is Cohen–Macaulay.

1. Introduction

One aspect of Grothendieck's approach to algebraic geometry and commutative algebra is an extensive study of morphisms instead of just objects. In EGA IV, [19], Grothendieck develops in great detail the theory of local properties of locally Noetherian schemes and their morphisms. Using Grothendieck's terminology, a morphism is said to have a given property (e.g. Cohen–Macaulay), if the morphism is *flat* and its non–trivial fibers have the geometric form of the corresponding property.

Over the last years L. L. Avramov and H.–B. Foxby have carried out extensive studies on the ability of a local homomorphism $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ of finite flat dimension, to ascent a given (homological) property of the source to that of the target, and vice versa. A crucial aspect in their investigation is the extensive use of numerical invariants attached to a given local homomorphism, cf. [4, 5, 8]. These invariants (e.g. depth and dimension) measure the "size" of a local homomorphism

1991 *Mathematics Subject Classification.* Primary 13H10, 13D25, 14E40; Secondary 13C15.

exactly as their ring theoretical counterparts measure the “size” of a local ring.

In 1992 Avramov, Foxby, and B. Herzog assigned to every local homomorphism $\varphi : R \to S$ an integer called the *Cohen–Macaulay defect* of the homomorphism, denoted $\operatorname{cmd}\varphi$, and a local homomorphism is thus called Cohen–Macaulay if it has trivial Cohen–Macaulay defect and is of finite flat dimension, cf. [11].

In [8] Avramov and Foxby expand their study of local homomorphisms by weakening the homological assumption on the maps by introducing the class of local homomorphisms of finite *Gorenstein dimension* or G–dimension (if φ is of finite G–dimensions we use the abbreviation $\text{G–dim}\,\varphi < \infty$), which is based on M. Auslander's theory of G–dimension of finite modules. A key ingredient in their study is the notion of a *dualizing complex* for a local homomorphism. With this new object, they give a beautiful description of the so–called *quasi Gorenstein homomorphisms.* The theory of Gorenstein homomorphisms has been developed to a non–commutative setting by P. Jørgensen in [21]; and it should be possible to do the same with the results presented in this text.

Inspired by the work in [8] we assign to every local homomorphism $\varphi : R \to S$ of finite G–dimension a new numerical invariant called the *quasi Cohen–Macaulay* defect of φ, denoted $\mathbf{q}\mathrm{cmd}\,\varphi$, and a local homomorphism is said to be *quasi Cohen–Macaulay* if it is of finite G–dimension and has trivial quasi Cohen–Macaulay defect. With this new invariant we show the *Ascent Theorem:* If R is Cohen–Macaulay and φ is quasi Cohen–Macaulay, then S is Cohen–Macaulay. Also we show the *Descent Theorem:* If S is Cohen–Macaulay and φ is of finite G–dimension, then φ is quasi Cohen–Macaulay; if a certain extra condition is met, then R is Cohen–Macaulay too.

This paper is organized as follows: **Section** 3 serves as a brief introduction to hyperhomological algebra. In **section** 4 we define the so–called *Auslander* and *Bass* categories and some extra terminology is introduced. **Section** 5 investigates how a dualizing complex for a local homomorphism of finite G–dimension behaves under localization. The section ends with a key technical result which is a generalized version of a theorem due to Avramov and Foxby characterizing the (homological) amplitude of the dualizing complex for a local homomorphism of finite flat dimension (between rings admitting dualizing complexes). Under the above assumptions we show that if C is the dualizing complex for a homomorphism of finite G–dimension, one has the following

$$\operatorname{amp} C = \sup\{\, \mathrm{m}_S(\mathfrak{q}) - \mathrm{m}_R(\mathfrak{q} \cap R) \mid \mathfrak{q} \in \operatorname{Spec} S \,\},$$

where $\mathrm{m}_S(\mathfrak{q}) = \operatorname{depth} S_{\mathfrak{q}} + \dim_R(S/\mathfrak{q}) - \operatorname{depth} S$ for $\mathfrak{q} \in \operatorname{Spec} S$.

With this result we study the new invariant $\mathbf{q}\mathrm{cmd}\,\varphi$ in **section** 6. We show that for the local structure homomorphism $\eta : \mathbb{Z}_{(p)} \to S$, where $p = \mathrm{char}(S/\mathfrak{n})$, one has $\mathbf{q}\mathrm{cmd}\,\eta = \mathrm{cmd}\,S$, and that $\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{cmd}(S/\mathfrak{m}S)$ when φ is flat. Employing a result due to Avramov and Foxby, we show that the new invariant is identical to $\mathrm{cmd}\,\varphi$ when $\mathrm{fd}\,\varphi$ is finite or R is Cohen–Macaulay. In general when $\mathrm{G{-}dim}\,\varphi$ is finite it is shown that $\mathbf{q}\mathrm{cmd}\,\varphi \geq \mathrm{cmd}\,\varphi$.

Furthermore, the behavior of $\mathbf{q}\mathrm{cmd}$ under composition of local homomorphisms is studied: If $\mathrm{G{-}dim}\,\psi$, $\mathrm{G{-}dim}\,\varphi$ and $\mathrm{G{-}dim}\,\varphi\psi$[1] are finite, then

$$\mathbf{q}\mathrm{cmd}\,\varphi \leq \mathbf{q}\mathrm{cmd}\,\varphi\psi \leq \mathbf{q}\mathrm{cmd}\,\varphi + \mathbf{q}\mathrm{cmd}\,\psi\,.$$

In particular

$$\mathbf{q}\mathrm{cmd}\,\varphi \leq \mathrm{cmd}\,S \leq \mathbf{q}\mathrm{cmd}\,\varphi + \mathrm{cmd}\,R,$$

and the Ascent and Descent Theorems (**A**) and (**D**) follows. It is known, cf. [4], that R is Cohen–Macaulay, when S is Cohen–Macaulay and φ is of finite flat dimension; ($\mathbf{D}'$) is a partial result in this direction.

In **section** 7 we introduce and study the *quasi Cohen–Macaulay homomorphisms*, and show that this class of homomorphisms is remarkable rigid under composition and decomposition. **Section** 8 contains some results on how $\mathbf{q}\mathrm{cmd}$ behaves under composition when $\psi : Q \to R$ is of finite G–dimension and $\varphi : R \to S$ is flat. These results are important for the closing section. **Section** 9 is devoted to Grothendieck's *Localization Problem* for the Cohen–Macaulay property, [19, (7.5.4)]:

Let $\varphi : R \to S$ be a flat homomorphism of local rings, and assume that for each $\mathfrak{p} \in \mathrm{Spec}\,R$ the formal fiber $k(\mathfrak{p}) \otimes_R \widehat{R}$ is Cohen–Macaulay. If the closed fiber $S/\mathfrak{m}S$ of φ at the maximal ideal $\mathfrak{m}$ of R is Cohen–Macaulay, then does each fiber $k(\mathfrak{p}) \otimes_R S$ of φ have the same property?

A positive answer was provided in 1994 by Avramov and Foxby, cf. [7], and in 1998 they showed, cf. [4], that for a local homomorphism φ of finite flat dimension, $\mathfrak{q} \in \mathrm{Spec}\,S$ and $\mathfrak{p} = \mathfrak{q} \cap R$, one has the inequality

$$\mathrm{cmd}\,\varphi_{\mathfrak{q}} + \mathrm{cmd}(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \mathrm{cmd}\,\varphi + \mathrm{cmd}(k(\mathfrak{p}) \otimes_R \widehat{R}),$$

where $\varphi_{\mathfrak{q}} : R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is the induced local homomorphism, thereby giving an elegant solution to the Localization Problem. We generalize this result to homomorphisms locally of finite G–dimension, that is, $\varphi_{\mathfrak{q}}$ is of finite G–dimension for all $\mathfrak{q} \in \mathrm{Spec}\,S$ by proving

$$\mathbf{q}\mathrm{cmd}\,\varphi_{\mathfrak{q}} + \mathrm{cmd}(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \mathbf{q}\mathrm{cmd}\,\varphi + \mathrm{cmd}(k(\mathfrak{p}) \otimes_R \widehat{R}),$$

for $\mathfrak{q}$ and $\mathfrak{p}$ as above.

[1]It is still not known whether or not the G–dimension is transitive, cf. [8, Remrk. (4.8)]

2. Acknowledgments

This paper will be included in the author's Ph.D. thesis. The author would like to express his deep gratitude to his advisor, Prof. Hans–Bjørn Foxby, for his tactful guidance. It is a pleasure to thank Peter Jørgensen and Lars Winther Christensen for many fruitful discussions.

3. Hyperhomological Algebra

(3.1) ***Comment***. All results in this text are formulated and proved within the derived category of the category of R–modules. This section serves as a brief re cap on the vocabulary and some basic but important results concerning this category.

(3.2) ***Conventions***. In this text all rings are commutative, Noetherian and non–trivial. The symbol $(R, \mathfrak{m}, k)$ denotes a local ring, that is, $\mathfrak{m}$ is the unique maximal ideal of R and k denotes the residue class field $R/\mathfrak{m}$. The completion of R with respect to the $\mathfrak{m}$–adic topology is denoted $\widehat{R}$. A ring homomorphism $\varphi : R \to S$ is said to be local if R and S are local and $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, where $\mathfrak{n}$ denotes the maximal ideal of S. When we consider the completion of φ, that is, $\widehat{\varphi} : \widehat{R} \to \widehat{S}$, we always think of S completed in the $\mathfrak{n}$–adic topology.

(3.3) ***Complexes***. An *R–complex* is a sequence of R–modules X_i and R–linear maps, called differentials, $\partial_i^X : X_i \to X_{i-1}$ for $i \in \mathbb{Z}$ such that $\partial_{i-1}\partial_i = 0$. The module X_i is the module in degree i. If $X_i = 0$ for $i \neq 0$ we identify X with the module in degree 0 and a *module* M is thought of as the complex $0 \to M \to 0$ concentrated in degree 0.

If m is an integer, the symbol $\Sigma^m X$ denotes the complex X *shifted* m degrees (to the left). It is given by $(\Sigma^m X)_i = X_{i-m}$ and $\partial_i^{\Sigma^m X} = (-1)^m \partial_{i-m}^X$.

To capture the position of a given complex R–complex X we introduce the numbers *supremum*, *infimum* and *amplitude*. These are defined by

$$\begin{aligned} \sup X &= \sup \{ \, i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0 \, \}, \\ \inf X &= \inf \{ \, i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0 \, \}, \\ \operatorname{amp} X &= \sup X - \inf X. \end{aligned}$$

By convention $\sup X = -\infty$ and $\inf X = \infty$ if X is *homological trivial*, that is, if $\mathrm{H}(X) = 0$.

A *morphism* $\alpha : X \to Y$ of R–complexes is a sequence of R–linear maps $(\alpha_i : X_i \to Y_i)_{i \in \mathbb{Z}}$ such that $\partial_i^Y \alpha_i - \alpha_{i-1}\partial_i^X = 0$ for $i \in \mathbb{Z}$. A morphism is called a *quasi–isomorphism* if it induces an isomorphism in homology, that is, $\mathrm{H}(\alpha_i) : \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is an isomorphism for all

$i \in \mathbb{Z}$. We use the symbol $\simeq$ to indicate quasi–isomorphisms, while $\cong$ is used to indicate isomorphisms of complexes (and hence modules).

(3.4) ***Derived Category***. The *derived category* of R–modules is the category of R–complexes localized with respect to the class of all quasi–isomorphisms and is denoted $\mathbf{D}(R)$, cf. [27, Chap. 10]. We use the symbol $\simeq$ to denote isomorphisms in $\mathbf{D}(R)$ and $\sim$ is used to denote isomorphisms up to shift. The first notation corresponds to the fact that a morphism in the category of R–complexes is a quasi–isomorphism if and only if it represents an isomorphism in the derived category.

The full subcategories $\mathbf{D}_+(R), \mathbf{D}_-(R), \mathbf{D}_\mathrm{b}(R)$ and $\mathbf{D}_0(R)$ consists of complexes X with $\mathrm{H}_i(X) = 0$ for respectively $i \ll 0, i \gg 0, |i| \gg 0$ and $i \neq 0$. The symbol $\mathbf{D}^\mathrm{f}(R)$ denotes the full subcategory of $\mathbf{D}(R)$ consisting of complexes with $\mathrm{H}_i(X)$ finitely generated for every $i \in \mathbb{Z}$. In general we define $\mathcal{S}^\mathrm{f}(R) = \mathcal{S}(R) \cap \mathbf{D}^\mathrm{f}(R)$ and $\mathcal{S}_0(R) = \mathcal{S}(R) \cap \mathbf{D}_0(R)$ if $\mathcal{S}(R)$ is a subcategory of $\mathbf{D}(R)$.

(3.5) ***Functors***. The left derived functor of the tensor product functor is denoted $- \otimes^{\mathbf{L}}_R -$, and the right derived functor of the homomorphism functor is denoted $\mathbf{R}\mathrm{Hom}_R(-,-)$. By [26] and [3] no boundedness condition are imposed on the arguments, and $X \otimes^{\mathbf{L}}_R Y$ and $\mathbf{R}\mathrm{Hom}_R(X,Y)$ for $X, Y \in \mathbf{D}(R)$ are uniquely determined up to isomorphism in $\mathbf{D}(R)$ and enjoy the usual functorial properties.

(3.6) ***Notation***. Let $X, Y \in \mathbf{D}(R)$. For $i \in \mathbb{Z}$ we define

$$\mathrm{Tor}^R_i(X,Y) = \mathrm{H}_i(X \otimes^{\mathbf{L}}_R Y)$$

and

$$\mathrm{Ext}^i_R(X,Y) = \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(X,Y)).$$

These symbols are called the *hyper* Tor module, respectively, the *hyper* Ext module of the complexes X and Y. *Caution:* When M is a module and Y is a complex $\mathrm{Tor}^R_i(M,Y)$ does *not* denote the additive functor $\mathrm{Tor}^R_i(M,-)$ applied to the complex Y; the latter is a complex, not necessarily a module. But for modules X and Y these definitions coincide with the ones from classical homological algebra.

(3.7) ***Localization***. Let $\mathfrak{p}$ be a prime ideal of R. When $X, Y \in \mathbf{D}(R)$ there is an isomorphism of $R_\mathfrak{p}$–complexes

$$(X \otimes^{\mathbf{L}}_R Y)_\mathfrak{p} \simeq X_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} Y_\mathfrak{p}.$$

If $Y \in \mathbf{D}_-(R)$ and $Z \in \mathbf{D}^\mathrm{f}_+(R)$ then one has the isomorphism of $R_\mathfrak{p}$–complexes

$$\mathbf{R}\mathrm{Hom}_R(Z,Y)_\mathfrak{p} \simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(Z_\mathfrak{p}, Y_\mathfrak{p}).$$

See [3, Lem. 5.2].

(3.8) ***Bounds***. Let $R \to S$ be a ring homomorphism. Then the following hold

(a) If $X \in \mathbf{D}^{\mathrm{f}}_{+}(R)$ and $X' \in \mathbf{D}^{\mathrm{f}}_{+}(S)$, then $X \otimes^{\mathbf{L}}_{R} X' \in \mathbf{D}^{\mathrm{f}}_{+}(S)$.

Moreover, for R–complexes $X \in \mathbf{D}_{+}(R)$ and $Y \in \mathbf{D}_{+}(R)$ one has

(b) $\inf (X \otimes^{\mathbf{L}}_{R} Y) \geq \inf X + \inf Y$.

If $X, Y \in \mathbf{D}_{+}(R)$ then the last inequality is actually an equality if and only if $\mathrm{H}_i(X) \otimes_R \mathrm{H}_j(Y) \neq 0$ where $i = \inf X$ and $j = \inf Y$. In particular, if R is local and $X, Y \in \mathbf{D}^{\mathrm{f}}_{+}(R)$, equation holds and is known as *Nakayama's Lemma for complexes*, cf. [16, Lem. 2.1(2)].

(3.9) ***Homological Dimensions***. For $X \in \mathbf{D}(R)$ the *projective, injective* and *flat* dimension of X is defined by, respectively,

$$\begin{aligned}
\mathrm{pd}_R X &= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(X, N)) \mid N \in \mathbf{D}_0(R)\},\\
\mathrm{id}_R X &= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(N, X)) \mid N \in \mathbf{D}_0(R)\},\\
\mathrm{fd}_R X &= \sup\{\sup(N \otimes^{\mathbf{L}}_{R} X) \mid N \in \mathbf{D}_0(R)\}.
\end{aligned}$$

These numerical invariants can also be defined by using suitable bounded projective, injective and flat resolutions of X [3, Sec. 2.P, 2.I and 2.F]. The full subcategories consisting of complexes of finite projective, injective and flat dimension are denoted $\mathbf{P}(R)$, $\mathbf{I}(R)$ and $\mathbf{F}(R)$ respectively.

(3.10) ***Homological Dimensions and Bounds***. Let $(R, \mathfrak{m}, k)$ be a local ring. Then $\mathbf{F}^{\mathrm{f}}(R) = \mathbf{P}^{\mathrm{f}}(R)$ and the following holds for $Z \in \mathbf{D}^{\mathrm{f}}_{-}(R)$

$$\mathrm{fd}_R Z = \mathrm{pd}_R Z = \sup (Z \otimes^{\mathbf{L}}_{R} k).$$

See [3, Cor. 2.10.F].

(3.11) ***Canonical Morphisms***. Let $R \to S$ be a ring homomorphism. When $Y, Z \in \mathbf{D}(R)$ and $X', Y', Z' \in \mathbf{D}(S)$, the following canonical isomorphisms exist in $\mathbf{D}(S)$. First consider the associativity and adjointness isomorphisms

(a) $$(Z \otimes^{\mathbf{L}}_{R} Y') \otimes^{\mathbf{L}}_{S} X' \simeq Z \otimes^{\mathbf{L}}_{R} (Y' \otimes^{\mathbf{L}}_{S} X').$$

(b) $$\mathbf{R}\mathrm{Hom}_S(Z \otimes^{\mathbf{L}}_{R} X', Y') \simeq \mathbf{R}\mathrm{Hom}_R(Z, \mathbf{R}\mathrm{Hom}_S(X', Y')).$$

(c) $$\mathbf{R}\mathrm{Hom}_R(Z' \otimes^{\mathbf{L}}_{S} X', Y) \simeq \mathbf{R}\mathrm{Hom}_S(Z', \mathbf{R}\mathrm{Hom}_R(X', Y)).$$

Furthermore the derived tensor product is commutative, that is

(d) $$X \otimes^{\mathbf{L}}_{R} Y \simeq Y \otimes^{\mathbf{L}}_{R} X.$$

We also consider the evaluation morphisms

(e) $\omega_{ZY'X'} : \mathbf{R}\mathrm{Hom}_R(Z, Y') \otimes^{\mathbf{L}}_{S} X' \to \mathbf{R}\mathrm{Hom}_R(Z, Y' \otimes^{\mathbf{L}}_{S} X')$.

(f) $\theta_{ZX'Y'} : Z \otimes^{\mathbf{L}}_{R} \mathbf{R}\mathrm{Hom}_S(X', Y') \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(Z, X'), Y')$.

In general the morphisms (e) and (f) are not isomorphisms. But under certain extra conditions they turn out to be. The following hold for $Z \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$

$\omega_{ZY'X'}$ is an isomorphism when $Y' \in \mathbf{D}_-(S)$, and $X' \in \mathbf{F}(S)$ or $Z \in \mathbf{P}(R)$.

$\theta_{ZX'Y'}$ is an isomorphism when $X' \in \mathbf{D}_{\mathrm{b}}(S)$, and $Y' \in \mathbf{I}(S)$ or $Z \in \mathbf{P}(R)$.

See [3, Lem. 4.4] and [15, Chap. 9].

(3.12) ***Poincaré Series.*** Let $(R, \mathfrak{m}, k)$ be a local ring. For $X \in \mathbf{D}^{\mathrm{f}}_+(R)$ we define

$$\beta_i^R(X) = \operatorname{rank}_k(\operatorname{Tor}_i^R(X,k)).$$

These numbers are called the *Betti numbers* of X. Note that $\beta_i^R(X)$ is finite and vanish for $i \ll 0$. The formal Laurent series defined by

$$\mathrm{P}_X^R(t) = \sum_{i\in\mathbb{Z}} \beta_i^R(X) t^i$$

is the *Poincaré series* of X; it has non–negative coefficients and by Nakayama's Lemma and (3.10) we get the following equations for the order and the degree of $\mathrm{P}_X^R(t)$

$$\operatorname{ord} \mathrm{P}_X^R(t) = \inf X,$$

and

$$\deg \mathrm{P}_X^R(t) = \operatorname{pd}_R X,$$

the latter might be infinite, cf. [15, (11.17)].

(3.13) ***Support.*** Let X be a R–complex. The usual (or large) support, $\operatorname{Supp}_R X$, of the complex X consists of all $\mathfrak{p} \in \operatorname{Spec} R$ such that $X_{\mathfrak{p}}$ is not homological trivial. Thus

$$\begin{aligned} \operatorname{Supp}_R X &= \{\mathfrak{p} \in \operatorname{Spec} R |\ \mathrm{H}(X_{\mathfrak{p}}) \neq 0\} \\ &= \bigcup_{i\in\mathbb{Z}} \operatorname{Supp}_R \mathrm{H}_i(X). \end{aligned}$$

If X is a module, then $\operatorname{Supp}_R X$ is precisely the classical support of X.

(3.14) ***Depth.*** Let R be local and let k be the residue class field. The *depth* of a R–complex X is defined by

$$\operatorname{depth}_R X = -\sup(\mathbf{R}\mathrm{Hom}_R(k,X)).$$

If $M \in \mathbf{D}_0(R)$ we have

$$\operatorname{depth}_R M = \inf\{\, i \in \mathbb{Z} \mid \operatorname{Ext}^i_R(k,M) \neq 0 \,\},$$

and when M is finite this agrees with the classical definition of the depth of a module (the maximal length of a regular sequence). If $Y \in \mathbf{D}_-(R)$, then

$$-\sup Y \leq \operatorname{depth}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}},$$

where $\mathfrak{p} \in \operatorname{Spec} R$ (here R is not necessarily local). If Y is not homological trivial the above inequality turns out to be an equality precisely when $\mathfrak{p} \in \operatorname{Ass}_R \mathrm{H}_s(Y)$ where $s = \sup Y$. Moreover when $Y \in \mathbf{D}^{\mathrm{f}}_{-}(R)$ the next inequality holds

$$\operatorname{depth}_R Y \leq \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} + \dim_R(R/\mathfrak{p}),$$

for all $\mathfrak{p} \in \operatorname{Spec} R$. By [12, Lem. (3.1)] the inequality follows for finite modules; for a complex version of this result consult [15, (13.35)].

(3.15) ***Krull Dimension***. The *dimension* (or *Krull dimension*) of $X \in \mathbf{D}(R)$ is defined by

$$\dim_R X = \sup\{\, \dim(R/\mathfrak{p}) - \inf X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R X \,\}.$$

When $M \in \mathbf{D}_0(R)$ we have $\inf M_\mathfrak{p} = 0$ for every $\mathfrak{p} \in \operatorname{Supp}_R M$, thus

$$\dim_R M = \sup\{\, \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R M \,\},$$

and when M is finite this again agrees with the classical definition of the dimension of a module. Note that $\dim_R X = -\infty$ precisely when X is homological trivial, and that $\dim_R X = \infty$ if $\inf X = -\infty$. Moreover if $\dim_R R < \infty$, $X \in \mathbf{D}_+(R)$, and X is not homological trivial, then the next inequalities hold

$$-\infty < -\inf X \leq \dim_R X \leq \dim R - \inf X < \infty,$$

and if $\mathfrak{p} \in \operatorname{Spec} R$ also

$$\dim_R X \geq \dim_{R_\mathfrak{p}} X_\mathfrak{p} + \dim(R/\mathfrak{p}).$$

(3.16) ***Cohen–Macaulay Defect***. Let R be local and let $X \in \mathbf{D}_\mathrm{b}(R)$ and assume $\operatorname{depth}_R X < \infty$, then

$$\dim_R X \geq \operatorname{depth}_R X,$$

by [15, (16.31)]. If $X \in \mathbf{D}_\mathrm{b}(R)$ we define the *Cohen–Macaulay defect* of X, denoted $\operatorname{cmd}_R X$, as

$$\operatorname{cmd}_R X = \dim_R X - \operatorname{depth}_R X.$$

Note that if $\mathfrak{p} \in \operatorname{Supp}_R X$ and $X \in \mathbf{D}^{\mathrm{f}}_\mathrm{b}(R)$, then the next inequality holds

$$\operatorname{cmd}_{R_\mathfrak{p}} X_\mathfrak{p} \leq \operatorname{cmd}_R X.$$

If $X \in \mathbf{D}_\mathrm{b}(R)$ with $\operatorname{depth}_R X < \infty$, then $\operatorname{cmd}_R X$ is non–negative. In particular, $\operatorname{cmd}_R X$ is non–negative, when $X \in \mathbf{D}^{\mathrm{f}}_\mathrm{b}(R)$ and X is not homological trivial. An R–complex is called a *Cohen–Macaulay complex* if $\operatorname{cmd}_R X = 0$.

(3.17) ***Formal Identity***. Let $R \to S$ be a local ring homomorphism. If $X \in \mathbf{D}^{\mathrm{f}}_+(R)$ and $X' \in \mathbf{D}^{\mathrm{f}}_+(S)$, then we have the next formula

$$\mathrm{P}^S_{X \otimes^{\mathbf{L}}_R X'}(t) = \mathrm{P}^R_X(t)\, \mathrm{P}^S_{X'}(t).$$

See [8, Lem. (1.5.3)(a)].

(3.18) ***Flat Extensions***. Let $R \to S$ be a flat local homomorphism. If $X \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ the next useful equations hold

(a) $$\dim_S(X \otimes_R S) = \dim_R X + \dim_R(S/\mathfrak{m}S).$$

(b) $$\operatorname{depth}_S(X \otimes_R S) = \operatorname{depth}_R X + \operatorname{depth}(S/\mathfrak{m}S).$$

(c) $$\operatorname{cmd}_S(X \otimes_R S) = \operatorname{cmd}_R X + \operatorname{cmd}(S/\mathfrak{m}S).$$

(d) $$\sup\,(X \otimes_R S) = \sup X.$$

(e) $$\inf\,(X \otimes_R S) = \inf X.$$

(f) $$\operatorname{amp}(X \otimes_R S) = \operatorname{amp} X.$$

For the statements (a), (b) and (c), cf. e.g., [4, Prop. p.60]. Using that S is faithful flat over R (d), (e) and (f) are immediate.

4. Auslander and Bass Categories

(4.1) ***Comment***. In this section we give a brief introduction to the so–called *Auslander* and *Bass* categories. These categories furnishes a natural environment for studying local homomorphisms of finite G–dimension, cf. [8, Sec. 3] and [14, Sec. 4].

(4.2) ***Convention***. Throughout the rest of this paper all rings will be local.

(4.3) **Definition.** Let $C \in \mathbf{D}(R)$. We say that C is a *semi–dualizing complex* (for R), cf. [14, Def. (2.1)] if

(a) $C \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$;
(b) The homothety–morphism $\chi^C_R : R \longrightarrow \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism.

(4.4) ***Comment***. Note that the ring R is a semi–dualizing complex for R.

(4.5) **Definition.** An R–complex D is called *dualizing* if D is semi–dualizing and $D \in \mathbf{I}(R)$, cf. [20, Chap. V] or [15, Chap. 15].

(4.6) ***Existence***. Any homomorphic image of a Gorenstein ring admits a dualizing complex. In particular, any complete ring admits a dualizing complex, cf. [20, Chap. V].

(4.7) ***Uniqueness***. Let $D, D' \in \mathbf{D}(R)$ be dualizing complexes for R. Then D and D' are isomorphic up to shift in $\mathbf{D}(R)$. In particular the number $\operatorname{amp} D$ is uniquely determined, and one has $\operatorname{amp} D = \operatorname{cmd} R$.

A dualizing complex D for which $\sup D = \dim R$ is called a *normalized* dualizing complex and is denoted D^R. Note that $\inf D^R = \operatorname{depth} R$. See [20, Thm. V.3.1 and Prop. V.2.1], [17, Prop. 3.14] and [15, (16.20)].

(4.8) ***Localization***. If D is dualizing for R, then $D_\mathfrak{p}$ is dualizing for $R_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} R$. Moreover the $R_\mathfrak{p}$–complex $\Sigma^{-\dim(R/\mathfrak{p})}(D^R)_\mathfrak{p}$ is normalized, cf. [20, Chap. V] or [15, (15.17)].

(4.9) ***Biduality (Dagger Duality)***. Consider a semi–dualizing complex C. If $M \in \mathbf{D}_\mathrm{b}^\mathrm{f}(R)$ we define the *dagger dual* of M with respect to C, denoted $M^{\dagger_C}$, by letting

$$M^{\dagger_C} = \mathbf{R}\mathrm{Hom}_R(M, C),$$

cf. [14, Def. (2.7)]. If $C = D^R$ is a normalized dualizing complex, the symbol $M_R^\dagger$ is often used to denote the dagger dual of M with respect to D^R. We will adopt this notation. By [20, Prop. V.2.1] see also [15, (15.10)] we have the following useful canonical isomorphism: If $M \in \mathbf{D}_+^\mathrm{f}(R)$, then $M \simeq M_{R\,R}^{\dagger\,\dagger}$.

(4.10) ***Auslander Categories***. Assume that R admits a dualizing complex D. The canonical morphisms

$$\gamma_X^D : X \to \mathbf{R}\mathrm{Hom}_R(D, D \otimes_R^\mathbf{L} X)$$

and

$$\iota_Y^D : D \otimes_R^\mathbf{L} \mathbf{R}\mathrm{Hom}_R(D, Y) \to Y$$

where $X, Y \in \mathbf{D}_\mathrm{b}(R)$, are defined by requiring commutativity of the following diagrams. For γ_X^D it is the diagram

$$\begin{tikzcd} X \arrow[r, "\gamma_X^D"] \arrow[d, "\simeq"] & \mathbf{R}\mathrm{Hom}_R(D, D \otimes_R^\mathbf{L} X) \\ R \otimes_R^\mathbf{L} X \arrow[r, "\eta", "\simeq"'] & \mathbf{R}\mathrm{Hom}_R(D, D) \otimes_R^\mathbf{L} X, \arrow[u, "\omega_{DDX}"'] \end{tikzcd}$$

where $\eta = \chi_R^D \otimes_R^\mathbf{L} X$. For ι_Y^D it is the diagram

$$\begin{tikzcd} D \otimes_R^\mathbf{L} \mathbf{R}\mathrm{Hom}_R(D, Y) \arrow[r, "\iota_Y^D"] \arrow[d, "\theta_{DDY}"] & Y \\ \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(D, D), Y) \arrow[r, "\nu", "\simeq"'] & \mathbf{R}\mathrm{Hom}_R(R, D), \arrow[u, "\simeq"'] \end{tikzcd}$$

where $\nu = \mathbf{R}\mathrm{Hom}_R(\chi_R^D, Y)$. Next we introduce the full subcategories of $\mathbf{D}_\mathrm{b}(R)$ called the *Auslander* and the *Bass* categories. The objects in the Auslander category, denoted $\mathbf{A}(R)$, are specified by

(a) X belongs to $\mathbf{A}(R)$ precisely when $D \otimes^{\mathbf{L}}_R X \in \mathbf{D}_{\mathrm{b}}(R)$ and the canonical morphism $\gamma^D_X : X \to \mathbf{R}\mathrm{Hom}_R(D, D \otimes^{\mathbf{L}}_R X)$ is an isomorphism.

The objects in the Bass category, denoted $\mathbf{B}(R)$, are specified by

(b) Y belongs to $\mathbf{B}(R)$ precisely when $\mathbf{R}\mathrm{Hom}_R(D, Y) \in \mathbf{D}_{\mathrm{b}}(R)$ and the canonical morphism $\iota^D_M : D \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, Y) \to Y$ is an isomorphism.

By (3.11)(e) and (f), we have the following natural embeddings, namely

$$\mathbf{F}(R) \subseteq \mathbf{A}(R) \qquad \text{and} \qquad \mathbf{I}(R) \subseteq \mathbf{B}(R).$$

(4.11) ***Factorizations***. Let $\varphi : R \to S$ be a local homomorphism. A factorization of φ is a commutative diagram consisting of local homomorphisms

$$\begin{tikzcd} & R' \arrow[dr, "\varphi'"] & \\ R \arrow[ur, "\dot{\varphi}"] \arrow[rr, "\varphi"'] & & S \end{tikzcd}$$

where $\dot{\varphi} : R \to R'$ is flat, while $\varphi' : R' \to S$ is surjective. A given factorization is called *regular* or *Gorenstein* if the closed fiber $R'/\mathfrak{m}R'$ has the corresponding property. A *Cohen* factorization is a regular factorization in which R' is a complete local ring. For any local homomorphism $\varphi : R \to S$, the composite $\grave{\varphi} : R \xrightarrow{\varphi} S \to \widehat{S}$ admits a Cohen factorization, cf. [11, Thm. (1.1)]. Note that if $R \to R' \to S$ is a Gorenstein factorization of φ, then $R \to \widehat{R'} \to \widehat{S}$ is a Cohen factorization of $\grave{\varphi}$ and $\widehat{R} \to \widehat{R'} \to \widehat{S}$ is one for $\widehat{\varphi}$.

(4.12) **Definition.** Let $\varphi : R \to S$ be a local homomorphism. We say, that φ is of *finite flat dimension*, $\mathrm{fd}\,\varphi < \infty$, if $S \in \mathbf{F}(R)$, and this is equivalent to $\widehat{S} \in \mathbf{F}(\widehat{R})$, cf. [11, Lem. (3.2)]. The homomorphism φ is of *finite G–dimension*, $\mathrm{G\text{–}dim}\,\varphi < \infty$, if $\widehat{S} \in \mathbf{A}(\widehat{R})$. The following holds: If φ is of finite flat dimension, then φ is of finite G–dimension, cf. [8, Sec. 4.] for details.

5. Dualizing Complexes for Local Homomorphisms

(5.1) ***Comment***. In this section we review the so–called *dualizing complex for a local homomorphism.* This object was first introduced in [8], and was used to give a very beautiful description of *quasi Gorenstein homomorphisms.* In this paper we use the dualizing complex for a local homomorphism to define the *quasi Cohen–Macaulay defect* of it (see section 6).

(5.2) **Definition.** Let $\varphi : R \to S$ be a local homomorphism. A complex $C \in \mathbf{D}(S)$ is called *dualizing* for φ, cf. [8, Sec. 5], if

(a) $C \in \mathbf{D}^{\mathrm{f}}_{\mathrm{b}}(S)$;
(b) The homothety–morphism $\chi^C_S : S \to \mathbf{R}\mathrm{Hom}_S(C, C)$ is an isomorphism;
(c) $D^{\hat{R}} \otimes^{\mathbf{L}}_{\hat{R}} (C \otimes_S \widehat{S}) \in \mathbf{I}(\widehat{S})$.

(5.3) **Definition.** Let $\varphi : R \to S$ be a local homomorphism. If D^R and D^S exist, we use the symbol D^φ to denote the following S–complex

$$D^\varphi = \mathbf{R}\mathrm{Hom}_R(D^R, D^S).$$

By convention the symbol D^φ is only used when D^R and D^S exist.

(5.4) **Lemma.** *Let φ be local and assume that D^φ exists. Then the following hold*

$$D^\varphi \simeq (D^R \otimes^{\mathbf{L}}_R S)^\dagger_S \quad \textit{and} \quad (D^\varphi)^\dagger_S \simeq D^R \otimes^{\mathbf{L}}_R S.$$

Proof. The first isomorphism is just the canonical isomorphism

$$\mathbf{R}\mathrm{Hom}_R(D^R, D^S) \simeq \mathbf{R}\mathrm{Hom}_S(D^R \otimes^{\mathbf{L}}_R S, D^S).$$

Notice next that since $D^R \otimes^{\mathbf{L}}_R S \in \mathbf{D}^{\mathrm{f}}_{+}(S)$, dagger duality yields

$$(D^\varphi)^\dagger_S \simeq (D^R \otimes^{\mathbf{L}}_R S)^{\dagger\,\dagger}_{SS} \simeq D^R \otimes^{\mathbf{L}}_R S.$$

Hence the proof is complete. $\square$

(5.5) **Lemma.** *If* G–dim φ *is finite and D^φ exists, then*

$$\operatorname{depth}_S D^\varphi = \operatorname{depth} R.$$

Proof. Since $\varphi : R \to S$ is local we conclude, using (3.8), that

$$\inf (D^R \otimes^{\mathbf{L}}_R S) = \inf D^R.$$

Thus, now applying (5.4), we construct the following chain

$$\begin{aligned} \operatorname{depth}_S D^\varphi &= \operatorname{depth}_S (D^R \otimes^{\mathbf{L}}_R S)^\dagger_S \\ &\overset{(a)}{=} \inf (D^R \otimes^{\mathbf{L}}_R S) \\ &= \inf D^R \\ &= \operatorname{depth} R. \end{aligned}$$

Here (a) is due to [15, (16.20)]. This concludes the proof. $\square$

(5.6) **Lemma.** *If* G–dim φ *is finite and D^φ exists, then the following holds*

$$\inf D^\varphi = \operatorname{depth} S - \operatorname{depth} R.$$

Proof. As G–dim φ is finite [8, Lem. (6.2)(b)] informs us that $D^S \in \mathbf{B}(R)$. Thus we have the isomorphism

$$D^R \otimes^{\mathbf{L}}_R D^\varphi = D^R \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D^R, D^S) \simeq D^S.$$

Consequently $D^R \otimes^{\mathbf{L}}_R D^\varphi$ is dualizing for S. Applying (3.17) we get

$$\mathrm{P}^S_{D^S}(t) = \mathrm{P}^S_{D^R \otimes^{\mathbf{L}}_R D^\varphi}(t) = \mathrm{P}^R_{D^R}(t)\, \mathrm{P}^S_{D^\varphi}(t),$$

yielding the equation $\operatorname{ord} \mathrm{P}^S_{D^S}(t) = \operatorname{ord} \mathrm{P}^R_{D^R}(t) + \operatorname{ord} \mathrm{P}^S_{D^\varphi}(t)$. As D^R and D^S are normalized, we conclude from Nakayama's Lemma that

$$\operatorname{depth} S = \operatorname{depth} R + \inf D^\varphi,$$

proving the assertion. □

(5.7) ***Comment***. Consider a local ring R and a finitely generated R–module M. It is well known (see (3.14)) that for $\mathfrak{p} \in \operatorname{Supp}_R M$ one has the inequality

$$\operatorname{depth}_R M \leq \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim(R/\mathfrak{p}).$$

Hence the integer

$$\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim(R/\mathfrak{p}) - \operatorname{depth}_R M$$

is non–negative.

(5.8) ***Comment***. To capture the behavior of a dualizing complex for a local homomorphism under localization, we introduce the following definition.

(5.9) **Definition.** Let R be local and let $\mathfrak{p} \in \operatorname{Spec} R$. We use the abbreviation

$$\mathrm{n}_R(\mathfrak{p}) = \dim(R/\mathfrak{p}),$$

and the symbol $\mathrm{m}_R(\mathfrak{p})$ is defined as

$$\begin{aligned} \mathrm{m}_R(\mathfrak{p}) &= \operatorname{depth} R_\mathfrak{p} + \dim(R/\mathfrak{p}) - \operatorname{depth} R \\ &= \operatorname{depth} R_\mathfrak{p} + \mathrm{n}_R(\mathfrak{p}) - \operatorname{depth} R. \end{aligned}$$

In particular, $\mathrm{m}_R(\mathfrak{p}) \geq 0$.

(5.10) **Lemma.** *Assume that D^φ exists. Then $D^{\varphi_\mathfrak{q}}$ exists for every $\mathfrak{q} \in \operatorname{Spec} S$, and one has the following isomorphism over $S_\mathfrak{q}$*

$$\Sigma^{\mathrm{n}_S(\mathfrak{q})} D^{\varphi_\mathfrak{q}} \simeq \Sigma^{\mathrm{n}_R(\mathfrak{q} \cap R)} (D^\varphi)_\mathfrak{q}.$$

Proof. The assertion follows directly from the fact that

$$\mathbf{R}\mathrm{Hom}_R(D^R, D^S)_\mathfrak{q} \simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}((D^R)_\mathfrak{p}, (D^S)_\mathfrak{q}),$$

where $\mathfrak{p} = \mathfrak{q} \cap R$, and (4.8). □

(5.11) **Theorem.** *Let φ be local and assume that* G–dim φ *is finite. If* D^φ *exists, then for* $\mathfrak{q} \in \operatorname{Spec} S$ *and* $\mathfrak{p} = \mathfrak{q} \cap R$ *one has the identity*

$$\mathrm{m}_S(\mathfrak{q}) - \mathrm{m}_R(\mathfrak{p}) = \inf (D^\varphi)_\mathfrak{q} - \inf D^\varphi.$$

In particular $\mathrm{m}_R(\mathfrak{p}) \leq \mathrm{m}_S(\mathfrak{q})$.

Proof. By (5.10) we have the following

$$\inf (D^\varphi)_\mathfrak{q} = \inf (\Sigma^{-\mathrm{n}_R(\mathfrak{p})+\mathrm{n}_S(\mathfrak{q})} D^{\varphi_\mathfrak{q}}) = \inf D^{\varphi_\mathfrak{q}} - \mathrm{n}_R(\mathfrak{p}) + \mathrm{n}_S(\mathfrak{q}),$$

where $\mathfrak{q} \in \operatorname{Spec} S$ and $\mathfrak{p} = \mathfrak{q} \cap R$. Since D^φ exists, D^R exists (by convention), and R has Gorenstein formal fibers, cf. [15, (22.26)]. Consequently G–dim $\varphi_\mathfrak{q}$ is finite by [8, Prop. (4.5)]. Hence $D^{\varphi_\mathfrak{q}}$ is dualizing for $\varphi_\mathfrak{q}$. Applying (5.6) we get the equations

$$\begin{aligned} \inf D^\varphi &= \operatorname{depth} S - \operatorname{depth} R, \\ \inf D^{\varphi_\mathfrak{q}} &= \operatorname{depth} S_\mathfrak{q} - \operatorname{depth} R_\mathfrak{p}. \end{aligned}$$

Using this information we can construct the following chain

$$\begin{aligned} \inf (D^\varphi)_\mathfrak{q} - \inf D^\varphi &= \inf D^{\varphi_\mathfrak{q}} - \mathrm{n}_R(\mathfrak{p}) + \mathrm{n}_S(\mathfrak{q}) - \inf D^\varphi \\ &= \operatorname{depth} S_\mathfrak{q} - \operatorname{depth} R_\mathfrak{p} - \mathrm{n}_R(\mathfrak{p}) + \mathrm{n}_S(\mathfrak{q}) - \operatorname{depth} S + \operatorname{depth} R \\ &= \mathrm{m}_S(\mathfrak{q}) - \mathrm{m}_R(\mathfrak{p}), \end{aligned}$$

and we have shown the desired result. □

(5.12) **Lemma.** *Let* G–dim φ *be finite and assume that* D^φ *exists. If* $\mathfrak{n} \in \operatorname{Ass}_S \mathrm{H}_s(D^\varphi)$ *for* $s = \sup D^\varphi$, *then* $\operatorname{amp} D^\varphi = 0$.

Proof. If $\mathfrak{n} \in \operatorname{Ass}_S \mathrm{H}_s(D^\varphi)$, then it follows from [15, (12.6)] that

$$-\operatorname{depth}_S D^\varphi = \sup D^\varphi.$$

From this we deduce the following chain

$$\begin{aligned} \operatorname{amp} D^\varphi &= \sup D^\varphi - \inf D^\varphi \\ &= -\operatorname{depth}_S D^\varphi - \inf D^\varphi \\ &\overset{(a)}{=} -\operatorname{depth} R - \operatorname{depth} S + \operatorname{depth} R \\ &= -\operatorname{depth} S \leq 0. \end{aligned}$$

Here (a) follows from (5.5) and (5.6). This completes the proof. □

(5.13) **Theorem.** *If* G–dim φ *is finite and* D^φ *exists, then the following identity holds*

$$\operatorname{amp} D^\varphi = \sup \{\, \mathrm{m}_S(\mathfrak{q}) - \mathrm{m}_R(\mathfrak{q} \cap R) \mid \mathfrak{q} \in \operatorname{Spec} S \,\}.$$

Proof. We start by showing the inequality

$$\operatorname{amp} D^\varphi \leq \sup \{\, \mathrm{m}_S(\mathfrak{q}) - \mathrm{m}_R(\mathfrak{q} \cap R) \mid \mathfrak{q} \in \operatorname{Spec} S \,\}.$$

Let $s = \sup D^{\varphi}$ and choose $\mathfrak{q} \in \operatorname{Ass}_S \mathrm{H}_s(D^{\varphi})$. Applying (5.10) and (5.12) produces the following equalities

$$\operatorname{amp}(D^{\varphi})_{\mathfrak{q}} = \operatorname{amp} D^{\varphi_{\mathfrak{q}}} = 0,$$

and hence $\inf (D^{\varphi})_{\mathfrak{q}} = \sup (D^{\varphi})_{\mathfrak{q}}$ which equals $\sup D^{\varphi}$ by the choice of $\mathfrak{q}$. Consequently the next chain is obtained using (5.11)

$$\begin{aligned} \mathrm{m}_S(\mathfrak{q}) - \mathrm{m}_R(\mathfrak{q} \cap R) &= \inf (D^{\varphi})_{\mathfrak{q}} - \inf D^{\varphi} \\ &= \sup D^{\varphi} - \inf D^{\varphi} \\ &= \operatorname{amp} D^{\varphi}, \end{aligned}$$

which establishes our assertion.

Conversely, let $\mathfrak{q} \in \operatorname{Spec} S$ be arbitrary. By (5.11) we have

$$\mathrm{m}_S(\mathfrak{q}) - \mathrm{m}_R(\mathfrak{q} \cap R) = \inf (D^{\varphi})_{\mathfrak{q}} - \inf D^{\varphi},$$

which gives rise to the following chain

$$\begin{aligned} \mathrm{m}_S(\mathfrak{q}) - \mathrm{m}_R(\mathfrak{q} \cap R) &= \inf (D^{\varphi})_{\mathfrak{q}} - \inf D^{\varphi} \\ &\leq \sup (D^{\varphi})_{\mathfrak{q}} - \inf D^{\varphi} \\ &\leq \sup D^{\varphi} - \inf D^{\varphi} \\ &= \operatorname{amp} D^{\varphi}. \end{aligned}$$

Hence we have shown

$$\operatorname{amp} D^{\varphi} \geq \sup \{ \mathrm{m}_S(\mathfrak{q}) - \mathrm{m}_R(\mathfrak{q} \cap R) \mid \mathfrak{q} \in \operatorname{Spec} S \},$$

and the proof is complete. □

6. Quasi Cohen–Macaulay Defect of a Local Homomorphism

(6.1) ***Comment***. In this section we define the *quasi dimesion* and the *quasi Cohen–Macaulay defect* of a local homomorphism of finite G–dimension. We investigate how the the quasi Cohen–Macaulay defect behaves under composition of local homomorphisms, and the results obtained here will play a central role in the next section.

(6.2) **Definition.** Let φ be a local homomorphism of finite G–dimension . The *quasi dimension* of φ, denoted $\mathbf{q}\dim \varphi$, is defined as

$$\mathbf{q}\dim \varphi = \sup D^{\widehat{\varphi}},$$

and the *quasi Cohen–Macaulay defect* of φ, denoted $\mathbf{q}\mathrm{cmd}\, \varphi$, is defined as

$$\mathbf{q}\mathrm{cmd}\, \varphi = \operatorname{amp} D^{\widehat{\varphi}}.$$

(6.3) **Depth of φ.** The depth of a local homomorphism $\varphi : R \to S$ was introduced in [11, Depth (2.2)] as $\operatorname{depth} \varphi = \operatorname{depth} S - \operatorname{depth} R$. Applying (5.6) we get, assuming G–dim φ is finite

$$\inf D^{\widehat{\varphi}} = \operatorname{depth} \widehat{S} - \operatorname{depth} \widehat{R} = \operatorname{depth} S - \operatorname{depth} R = \operatorname{depth} \varphi.$$

Thus when G–dim φ is finite we have the following useful formula

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathbf{q}\mathrm{dim}\,\varphi - \mathrm{depth}\,\varphi.$$

This is the reason for not introducing the quasi depth of a local homomorphism.

(6.4) **Observation.** Let S be a local ring and consider the structure homomorphism $\eta : \mathbb{Z}_{(p)} \to S$. Since $\mathbb{Z}_{(p)}$ is Gorenstein it follows immediately that $D^{\widehat{S}}$ is dualizing for $\widehat{\eta}$ and thus $\mathbf{q}\mathrm{cmd}\,\eta = \mathrm{cmd}\,S$. Moreover: Assume that $\varphi : R \to S$ is a local homomorphism of finite G–dimension. By [8, Size (5.5)] we obtain the next inequality

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{amp}\,D^{\widehat{\varphi}} \geq \mathrm{cmd}\,\varphi.$$

Equality holds when $\mathrm{fd}\,\varphi$ is finite or if R is Cohen–Macaulay. Finally it is worth mentioning that in section 6 (see (8.5)) we establish the following result: If $\varphi : R \to S$ is flat, then $\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{cmd}(S/\mathfrak{m}S)$.

Informally speaking, the rest of this text investigates how much of the theory developed in [4] one can adopt for the quasi Cohen–Macaulay defect. As (hopefully) will become apparent, the quasi Cohen–Macaulay defect enjoys most of the properties of that of the Cohen–Macaulay defect of a local homomorphism, cf. [11, 8, 4].

(6.5) **Quasi Cohen–Macaulay defect and φ'.** Let φ be a local homomorphism of finite G–dimension and let $R \to R' \xrightarrow{\varphi'} \widehat{S}$ be a Cohen factorization of $\grave{\varphi}$. As $\varphi' : R' \to \widehat{S}$ is surjective the $\widehat{S}$–complex

$$C = \mathbf{R}\mathrm{Hom}_{R'}(\widehat{S}, R'),$$

is dualizing for φ' by [8, Lem. (6.5)]. Now $D^{\varphi'} = D^{\widehat{\varphi'}}$ since both R' and $\widehat{S}$ are complete, so $D^{\varphi'} \sim C$ by [8, Uniq. (5.4)], and it follows that

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{amp}\,C = \mathrm{amp}\,D^{\varphi'} = \mathbf{q}\mathrm{cmd}\,\varphi'\,.$$

(6.6) **Coding $\mathbf{q}$cmd .** Let φ be as above. Assume that C is dualizing for φ. Then the complex $C \otimes_S \widehat{S}$ is dualizing for $\widehat{\varphi}$ hence $D^{\widehat{\varphi}} \sim C \otimes_S \widehat{S}$. Thus we have, cf. (3.18)

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{amp}(C \otimes_S \widehat{S}) = \mathrm{amp}\,C.$$

In particular, when D^{φ} exists we get

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{amp}\,D^{\varphi}.$$

(6.7) **Definition.** Let M be a R–module. The *quasi–imperfection* of M, denoted $\mathbf{q}\mathrm{imp}_R\,M$, is defined by letting

$$\mathbf{q}\mathrm{imp}_R\,M = \text{G–dim}_R\,M - \mathrm{grade}_R\,M.$$

A proper ideal $\mathfrak{a}$ of R is called *quasi–perfect* if $R/\mathfrak{a}$ is a quasi–perfect module i.e. $\mathbf{q}\mathrm{imp}_R\,(R/\mathfrak{a}) = 0$, cf. [18].

(6.8) **Lemma.** *Let $\varphi : R \to S$ be of finite G–dimension . If φ is finite, then*

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathbf{q}\mathrm{imp}_R\, S.$$

Proof. Assume that $\varphi : R \to S$ is finite and let $C = \mathbf{R}\mathrm{Hom}_R(S, R)$. By [8, Lem. (6.5)] this S–complex is dualizing for φ and by (6.6) we get $\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{amp}\, C$. By [8, Gor. dim. (4.1.3)] we are informed that $-\inf C = \mathrm{G\text{–}dim}_R\, S$, and using the homological characterization of grade, we moreover obtain $\sup C = -\,\mathrm{grade}_R\, S$. Whence

$$\mathbf{q}\mathrm{cmd}\,\varphi = \sup C - \inf C = -\,\mathrm{grade}_R\, S + \mathrm{G\text{–}dim}_R\, S = \mathbf{q}\mathrm{imp}_R\, S,$$

thereby concluding the proof. □

(6.9) **Proposition.** *Let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ be Cohen factorization of $\dot{\varphi}$ and assume that* G–dim φ *is finite. Then the following holds*

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathbf{q}\mathrm{imp}_{R'}\, \widehat{S}.$$

Proof. As $\varphi' : R' \to \widehat{S}$ is surjective and of finite G–dimension, the assertion is immediate by (6.8) and (6.6). □

(6.10) **Invariant of a Cohen factorization.** Let $\varphi : R \to S$ be of finite G–dimension. If $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ is a Cohen factorization of $\dot{\varphi}$, then the $\widehat{S}$–complex

$$C = \mathbf{R}\mathrm{Hom}_{R'}(\widehat{S}, R')$$

is dualizing for $\dot{\varphi}$ and $\widehat{\varphi}$. This is the conclusion of [8, Lem. (6.5) and Lem. (6.7)]. Moreover since $\dot{\varphi}$ is flat and $R'/\mathfrak{m}R'$ is regular, hence Cohen–Macaulay, the following hold

$$\begin{aligned}\dim R' - \dim R &= \dim(R'/\mathfrak{m}R')\\ &= \mathrm{depth}(R'/\mathfrak{m}R')\\ &= \mathrm{depth}\, R' - \mathrm{depth}\, R,\end{aligned}$$

and this number is denoted d.

(6.11) **Lemma.** *If $\varphi : R \to S$ is of finite G–dimension and $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ is a Cohen factorization of $\dot{\varphi}$, then*

$$\Sigma^d\, C \simeq D^{\widehat{\varphi}},$$

where C and d are as above.

Proof. As G–dim φ is finite, we are informed by [8, Thm. (4.3)] that $\mathrm{G\text{–}dim}_{R'}\, \widehat{S}$ is finite. Using [8, Gor. dim. (4.1.2)] and the fact that $\widehat{S}$ is finite over R', we may compute as follows

$$\begin{aligned}\text{G–dim}_{R'} \widehat{S} &= \operatorname{depth} R' - \operatorname{depth}_{R'} \widehat{S} \\ &= \operatorname{depth} R' - \operatorname{depth} \widehat{S} \\ &= \operatorname{depth} R' - \operatorname{depth} S.\end{aligned}$$

Then apply [8, Gor. dim. (4.1.3)] to establish

$$\text{G–dim}_{R'} \widehat{S} = -\inf\left(\mathbf{R}\mathrm{Hom}_{R'}(\widehat{S}, R')\right) = -\inf C.$$

Thus we have seen

$$\inf C = \operatorname{depth} S - \operatorname{depth} R'.$$

As (5.6) yields the equation

$$\inf D^{\hat{\varphi}} = \operatorname{depth} S - \operatorname{depth} R,$$

we get

$$\inf D^{\hat{\varphi}} - \inf C = \operatorname{depth} R' - \operatorname{depth} R = d.$$

Therefore we conclude by [8, Uniq. (5.4)] that $\Sigma^d C \simeq D^{\hat{\varphi}}$ and the proof is complete. □

(6.12) **Theorem.** *If* $\varphi : R \to S$ *is local and* G–dim φ *is finite, then*

$$\mathbf{q}\mathrm{dim}\, \varphi \leq \dim(S/\mathfrak{m}S).$$

Proof. Let $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ be a Cohen factorization of $\dot{\varphi}$. From (6.11) we get the isomorphism $\Sigma^d C \simeq D^{\hat{\varphi}}$. Using the homological characterization of grade we obtain

$$\begin{aligned}\sup D^{\hat{\varphi}} &= \sup \Sigma^d C \\ &= \sup C + d \\ &= -\operatorname{grade}_{R'} \widehat{S} + \dim(R'/\mathfrak{m}R')\end{aligned}$$

Let $\mathfrak{a}$ denote the kernel of φ' and let $\overline{\mathfrak{a}}$ denote the kernel of the induced surjective homomorphism $R'/\mathfrak{m}R' \to \widehat{S}/\mathfrak{m}\widehat{S}$. Since $R'/\mathfrak{m}R'$ is regular and hence Cohen–Macaulay, we may choose a $(R'/\mathfrak{m}R')$–regular sequence in $\overline{\mathfrak{a}}$ of the form $b_1 + \mathfrak{m}R', \ldots, b_h + \mathfrak{m}R'$, where $h = \operatorname{ht} \overline{\mathfrak{a}}$, cf. [22, Thm. 17.4]. As $\overline{\mathfrak{a}} = \mathfrak{a}(R'/\mathfrak{m}R')$ we may assume that $b_i \in \mathfrak{a}$ for $i = 1, \ldots, h$. Since $\dot{\varphi} : R \to R'$ is flat, we conclude, using [22, Cor. 22.5] (see also [11, Lem. (1.3)]), that $b_1, \ldots, b_h$ is actually R'–regular. As $\operatorname{grade}_{R'}(R'/\mathfrak{a})$ is the maximal length of a R'–regular sequence from $\mathfrak{a}$, we have shown

$$\operatorname{grade}_{R'} \widehat{S} = \operatorname{grade}_{R'}(R'/\mathfrak{a}) \geq \operatorname{ht} \overline{\mathfrak{a}}.$$

Again using the Cohen–Macaulayness of $R'/\mathfrak{m}R'$ we are informed by [22, Thm. 17.4] that

$$\dim(\widehat{S}/\mathfrak{m}\widehat{S}) - \dim(R'/\mathfrak{m}R') = -\operatorname{ht} \overline{\mathfrak{a}},$$

and bearing in mind the isomorphism $\widehat{S}/\mathfrak{m}\widehat{S} \cong \widehat{S/\mathfrak{m}S}$ we finally obtain the next chain

$$\begin{aligned}\dim(S/\mathfrak{m}S) - \mathbf{q}\dim\varphi &= \dim(\widehat{S/\mathfrak{m}S}) - \sup D^{\widehat{\varphi}} \\ &= \dim(\widehat{S}/\mathfrak{m}\widehat{S}) - \dim(R'/\mathfrak{m}R') + \operatorname{grade}_{R'}\widehat{S} \\ &= -\operatorname{ht}\overline{\mathfrak{a}} + \operatorname{grade}_{R'}\widehat{S} \\ &\geq 0.\end{aligned}$$

This completes the proof. □

(6.13) ***Comment***. Let $\psi : Q \to R$ and $\varphi : R \to S$ be local homomorphisms of finite flat dimension. Then it is easy to see that the composite $\varphi\psi : Q \to S$ is of finite flat dimension. Since a local homomorphism of finite flat dimension has finite G–dimension, it is natural to ask the following question: Is the G–dimension transitive? This is still an open problem, cf. [8, Remrk. (4.8) and Cor. (7.10)]. One could argue that this problem corresponds to the fact that we do not have a functor which measures G–dimension *a priori*. At this moment, it is only possible to measure G–dimension with a functor, namely $\mathbf{R}\mathrm{Hom}_R(-, R)$, *when* $\text{G–dim}_R M$ *is finite*, cf. [8, Gor. dim. (4.1.3)]. Here M denotes a finitely generated R–module.

(6.14) **Theorem.** *Assume that ψ and φ are of finite G–dimension such that $\varphi\psi$ also is of finite G–dimension . Then the following hold*

(1) $$\mathbf{q}\dim\varphi\psi \leq \mathbf{q}\dim\varphi + \mathbf{q}\dim\psi,$$

(2) $$\mathbf{q}\mathrm{cmd}\,\varphi\psi \leq \mathbf{q}\mathrm{cmd}\,\varphi + \mathbf{q}\mathrm{cmd}\,\psi\,.$$

Proof. By (5.13) there exists a $\mathfrak{q}^* \in \operatorname{Spec}\widehat{S}$ such that

$$\mathbf{q}\mathrm{cmd}\,\varphi\psi = \operatorname{amp} D^{\widehat{\varphi\psi}} = \mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) - \mathrm{m}_{\widehat{Q}}(\mathfrak{q}^* \cap \widehat{Q}).$$

Rewriting this integer as

$$\mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) - \mathrm{m}_{\widehat{Q}}(\mathfrak{q}^*\cap\widehat{Q}) = (\mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) - \mathrm{m}_{\widehat{R}}(\mathfrak{q}^*\cap\widehat{R})) + (\mathrm{m}_{\widehat{R}}(\mathfrak{q}^*\cap\widehat{R}) - \mathrm{m}_{\widehat{Q}}(\mathfrak{q}^*\cap\widehat{Q})),$$

and using (5.13) we obtain the following inequality

$$\mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) - \mathrm{m}_{\widehat{Q}}(\mathfrak{q}^* \cap \widehat{Q}) \leq \mathbf{q}\mathrm{cmd}\,\varphi + \mathbf{q}\mathrm{cmd}\,\psi,$$

establishing (2). Using (6.3) the statement under (1) is now immediate. □

(6.15) ***Comment***. It is still not known if there exists local homomorphisms ψ and φ *both of finite flat dimension*, for which the strict inequalities of (6.14) holds, cf. [4, Sec. 4]. As a partial answer to this question, we prove the following two theorems.

(6.16) **Theorem.** *Assume that ψ and φ are of finite G–dimension such that $\varphi\psi$ is of finite G–dimension . Then the following hold*

$$\text{(1)} \qquad \operatorname{depth}\psi + \mathbf{q}\!\dim\varphi \leq \mathbf{q}\!\dim\varphi\psi,$$

$$\text{(2)} \qquad \mathbf{q}\mathrm{cmd}\,\varphi \leq \mathbf{q}\mathrm{cmd}\,\varphi\psi\,.$$

Proof. Again (5.13) guarantees the existence of a $\mathfrak{q}^* \in \operatorname{Spec}\widehat{S}$ realizing the equation

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) - \mathrm{m}_{\widehat{R}}(\mathfrak{q}^* \cap \widehat{R}).$$

From (5.11) we have the inequality

$$\mathrm{m}_{\widehat{R}}(\mathfrak{q}^* \cap \widehat{R}) - \mathrm{m}_{\widehat{Q}}(\mathfrak{q}^* \cap \widehat{Q}) \geq 0.$$

Consequently we have shown

$$\mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) - \mathrm{m}_{\widehat{R}}(\mathfrak{q}^*\cap\widehat{R}) \leq (\mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) - \mathrm{m}_{\widehat{R}}(\mathfrak{q}^*\cap\widehat{R})) + (\mathrm{m}_{\widehat{R}}(\mathfrak{q}^*\cap\widehat{R}) - \mathrm{m}_{\widehat{Q}}(\mathfrak{q}^*\cap\widehat{Q})),$$

and therefore we obtain the following chain

$$\begin{aligned}\mathbf{q}\mathrm{cmd}\,\varphi &\leq (\mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) - \mathrm{m}_{\widehat{R}}(\mathfrak{q}^* \cap \widehat{R})) + (\mathrm{m}_{\widehat{R}}(\mathfrak{q}^* \cap \widehat{R}) - \mathrm{m}_{\widehat{Q}}(\mathfrak{q}^* \cap \widehat{Q})) \\ &= \mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) - \mathrm{m}_{\widehat{Q}}(\mathfrak{q}^* \cap \widehat{Q}) \\ &\leq \mathbf{q}\mathrm{cmd}\,\varphi\psi,\end{aligned}$$

thus establishing (2). Again (1) is immediate by (6.3). This completes the proof. □

(6.17) **Theorem.** *Assume that ψ and φ are of finite G–dimension such that $\varphi\psi$ also is finite G–dimension, and assume that* $\operatorname{Spec}\widehat{S} \to \operatorname{Spec}\widehat{R}$ *is surjective. Then the following hold*

$$\text{(1)} \qquad \operatorname{depth}\varphi + \mathbf{q}\!\dim\psi \leq \mathbf{q}\!\dim\varphi\psi,$$

$$\text{(2)} \qquad \mathbf{q}\mathrm{cmd}\,\psi \leq \mathbf{q}\mathrm{cmd}\,\varphi\psi\,.$$

Proof. By (5.13) the existence of a $\mathfrak{p}^* \in \operatorname{Spec}\widehat{R}$ such that

$$\mathbf{q}\mathrm{cmd}\,\psi = \mathrm{m}_{\widehat{R}}(\mathfrak{p}^*) - \mathrm{m}_{\widehat{Q}}(\mathfrak{p}^* \cap \widehat{Q}),$$

is guaranteed. On the other hand the surjectivity of the spectra map $\operatorname{Spec}\widehat{S} \to \operatorname{Spec}\widehat{R}$ establishes the existence of a $\mathfrak{q}^* \in \operatorname{Spec}\widehat{S}$ realizing the equation $\mathfrak{q}^* \cap \widehat{R} = \mathfrak{p}^*$, and (5.11) yields $\mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) \geq \mathrm{m}_{\widehat{R}}(\mathfrak{p}^*)$ which makes it clear

$$\mathrm{m}_{\widehat{R}}(\mathfrak{p}^*) - \mathrm{m}_{\widehat{Q}}(\mathfrak{p}^* \cap \widehat{Q}) \leq \mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) - \mathrm{m}_{\widehat{Q}}(\mathfrak{p}^* \cap \widehat{Q}).$$

Hence by (5.13) we conclude

$$\mathrm{m}_{\widehat{S}}(\mathfrak{q}^*) - \mathrm{m}_{\widehat{Q}}(\mathfrak{p}^* \cap \widehat{Q}) \leq \mathbf{q}\mathrm{cmd}\,\varphi\psi,$$

and therefore

$$\mathbf{q}\mathrm{cmd}\,\psi \leq \mathbf{q}\mathrm{cmd}\,\varphi\psi,$$

proving (2). As before (1) is immediate using (6.3). The proof is now complete. □

(6.18) **Observation.** Applying the preceding results on the structure homomorphism $\eta : \mathbb{Z}_{(p)} \to R$, where $p = \operatorname{char}(R/\mathfrak{m})$, we get the following corollary, cf. [4, Cor. (4.3)].

(6.19) **Corollary.** *If* G–dim φ *is finite, then*

$$\begin{array}{ccccc} \operatorname{depth} R + \mathbf{q}\!\operatorname{dim} \varphi & \leq & \dim S & \leq & \dim R + \mathbf{q}\!\operatorname{dim} \varphi \\ \mathbf{q}\!\operatorname{cmd} \varphi & \leq & \operatorname{cmd} S & \leq & \operatorname{cmd} R + \mathbf{q}\!\operatorname{cmd} \varphi \end{array}$$

If furthermore Spec $\widehat{S} \to$ Spec $\widehat{R}$ *is surjective, then*

$$\begin{array}{ccccc} \dim R + \operatorname{depth} \varphi & \leq & \dim S & \leq & \dim R + \mathbf{q}\!\operatorname{dim} \varphi \\ \operatorname{cmd} R & \leq & \operatorname{cmd} S & & \end{array}$$

(6.20) ***Comment.*** We close this section with a version of the well–known relation, that for a local homomorphism $\varphi : R \to S$ one has $\dim S - \dim R \leq \dim(S/\mathfrak{m}S)$, cf. [22, Thm. 15.1.i] and [11, Thm. (2.4) and Thm. (2.7)].

(6.21) **Corollary.** *If* $\varphi : R \to S$ *is of finite G–dimension , then*

$$\dim S - \dim R \leq \mathbf{q}\!\operatorname{dim} \varphi \leq \dim_R(S/\mathfrak{m}S).$$

7. Quasi Cohen–Macaulay Homomorphisms

(7.1) ***Comment.*** In this section we investigate the class of local homomorphisms of finite G–dimension with trivial quasi Cohen–Macaulay defect, called *quasi Cohen–Macaulay homomorphisms*. As will become clear, this class of homomorphisms displays a remarkable rigidity with respect to composition and decomposition. Moreover we formulate some results on the ability of a local homomorphism to *ascent* and *descent* the Cohen–Macaulay property from the source to that of the target and vice versa.

(7.2) **Definition.** A local homomorphism φ is called *quasi Cohen–Macaulay* if G–dim φ is finite and $\mathbf{q}\!\operatorname{cmd} \varphi = 0$.

(7.3) **Proposition.** *Let* $\varphi : R \to S$ *be a surjective homomorphism of finite G–dimension . Then* φ *is quasi Cohen–Macaulay if and only if* $\ker \varphi$ *is quasi–perfect.*

Proof. By (6.8) we have $\mathbf{q}\!\operatorname{cmd} \varphi = \mathbf{q}\!\operatorname{imp}_R S$, thereby concluding the proof. □

(7.4) **Theorem.** *If* ψ *and* φ *are quasi Cohen–Macaulay homomorphisms such that* $\varphi\psi$ *also is of finite G–dimension , then* $\varphi\psi$ *is quasi Cohen–Macaulay.*

Proof. Immediate from (6.14). □

(7.5) **Theorem.** *Assume that* $\mathrm{G\text{--}dim}\,\varphi$ *is finite. If* ψ *is quasi Cohen–Macaulay such that* $\mathrm{G\text{--}dim}\,\varphi\psi$ *is finite, then the following formula holds*

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathbf{q}\mathrm{cmd}\,\varphi\psi\,.$$

In particular: If ψ *and* $\varphi\psi$ *is quasi Cohen–Macaulay, then* φ *quasi Cohen–Macaulay.*

Proof. A consequence of (6.14) in conjunction with (6.16). □

(7.6) **Theorem.** *Assume that* $\mathrm{G\text{--}dim}\,\psi$ *is finite. If* φ *is quasi Cohen–Macaulay such that* $\mathrm{G\text{--}dim}\,\varphi\psi$ *is finite, and* $\mathrm{Spec}\,\widehat{S} \to \mathrm{Spec}\,\widehat{R}$ *is surjective, then the following formula holds*

$$\mathbf{q}\mathrm{cmd}\,\psi = \mathbf{q}\mathrm{cmd}\,\varphi\psi\,.$$

In particular: If φ *and* $\varphi\psi$ *is quasi Cohen–Macaulay and* $\mathrm{Spec}\,\widehat{S} \to \mathrm{Spec}\,\widehat{R}$ *is surjective, then* ψ *is quasi Cohen–Macaulay.*

Proof. The conclusion follows from (6.14) in conjunction with (6.17). □

(7.7) ***Ascent–Descent Theorem***. Let $\varphi : R \to S$ be a local homomorphism. Then the following hold

- (**A**) If R is Cohen–Macaulay and φ is quasi Cohen–Macaulay, then S is Cohen–Macaulay.
- (**D**) If S is Cohen–Macaulay and $\mathrm{G\text{--}dim}\,\varphi$ is finite, then φ is quasi Cohen–Macaulay.

If furthermore the spectra map $\mathrm{Spec}\,\widehat{S} \to \mathrm{Spec}\,\widehat{R}$ is surjective one also has

- (**D′**) If S is Cohen–Macaulay and $\mathrm{G\text{--}dim}\,\varphi$ is finite, then φ is quasi Cohen–Macaulay, and R is Cohen–Macaulay.

Proof. Follows directly from (6.19). □

(7.8) **Theorem.** *Assume that* $\psi : Q \to R$ *and* $\varphi : R \to S$ *are of finite G–dimension such that* $\varphi\psi$ *also is of finite G–dimension . If* R *is Cohen–Macaulay, then the following formula holds*

$$\mathbf{q}\mathrm{cmd}\,\varphi\psi = \mathrm{cmd}\,S.$$

Proof. As R is Cohen–Macaulay we are informed by (7.7) that ψ is quasi Cohen–Macaulay hence $\mathbf{q}\mathrm{cmd}\,\varphi\psi = \mathbf{q}\mathrm{cmd}\,\varphi$ by (7.5). Now apply (6.19) to arrive at the conclusion $\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{cmd}\,S$ completing the proof. □

8. Compositions when φ is flat

(8.1) ***Comment.*** In this section we gather some results which will be used to prove the main theorem of the next section. The first result informs us, that whenever $\varphi : R \to S$ is of finite G–dimension and R admits a dualizing complex the quasi Cohen–Macaulay defect of φ is simply the Cohen–Macaulay defect of the base–changed dualizing complex. We also show, that when $\psi : Q \to R$ is of finite G–dimension and $\varphi : R \to S$ is flat, then the quasi Cohen–Macaulay defect is additive on the composite $\varphi\psi : Q \to S$.

(8.2) **Proposition.** *If* G–dim φ *is finite and* D' *is a dualizing complex for* $\widehat{R}$*, then*

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{cmd}_{\widehat{S}}(D' \otimes^{\mathbf{L}}_{\widehat{R}} \widehat{S}).$$

Proof. Using $D' \sim D^{\widehat{R}}$ we can without loss of generality assume that $D' = D^{\widehat{R}}$. As G–dim φ is finite, the complex $D^{\widehat{\varphi}}$ is dualizing for $\widehat{\varphi}$ and

$$D^{\widehat{\varphi}} \simeq (D^{\widehat{R}} \otimes^{\mathbf{L}}_{\widehat{R}} \widehat{S})^{\dagger}_{\widehat{S}}.$$

Hence we may compute as follows

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{amp}\,D^{\widehat{\varphi}} = \mathrm{amp}(D^{\widehat{R}} \otimes^{\mathbf{L}}_{\widehat{R}} \widehat{S})^{\dagger}_{\widehat{S}} = \mathrm{cmd}_{\widehat{S}}(D^{\widehat{R}} \otimes^{\mathbf{L}}_{\widehat{R}} \widehat{S}),$$

where the last equation is due to [15, (16.20)(c)], thus proving the proposition. □

(8.3) **Lemma.** *Let* φ *be a local homomorphism. Then*

(a)
$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathbf{q}\mathrm{cmd}\,\grave{\varphi}\,.$$

If furthermore G–dim φ *is finite and* D *is a dualizing complex for* R*, then*

(b)
$$\mathrm{cmd}_S(D \otimes^{\mathbf{L}}_R S) = \mathrm{cmd}_{\widehat{S}}(D \otimes^{\mathbf{L}}_R \widehat{S}).$$

Proof. The statement under (a) is clear from the definition. To prove (b) we proceed as follows. Let D be dualizing for R and note that we without loss of generality may assume $D = D^R$. As the functors $- \otimes^{\mathbf{L}}_S \widehat{S}$ and $- \otimes_S \widehat{S}$ are naturally isomorphic we may compute as follows

$$\begin{aligned}
\mathrm{cmd}_{\widehat{S}}(D^R \otimes^{\mathbf{L}}_R \widehat{S}) &= \mathrm{cmd}_{\widehat{S}}((D^R \otimes^{\mathbf{L}}_R S) \otimes_S \widehat{S}) \\
&= \mathrm{cmd}_S(D^R \otimes^{\mathbf{L}}_R S) + \mathrm{cmd}(\widehat{S}/\mathfrak{n}\widehat{S}) \\
&= \mathrm{cmd}_S(D^R \otimes^{\mathbf{L}}_R S).
\end{aligned}$$

Here the second equality is by (3.18)(c). The proof is now complete. □

(8.4) **Proposition.** *If* G–dim φ *is finite and* D *is dualizing for* R*, then*

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{cmd}_S(D \otimes^{\mathbf{L}}_R S).$$

Proof. Again we may assume $D = D^R$. By (8.3) we can reduce to the case of S being complete. Hence S admits a dualizing complex. Thus D^φ exists and is dualizing for φ. Therefore we have the following chain

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{amp}\, D^\varphi = \mathrm{amp}(D^R \otimes^{\mathbf{L}}_R S)^\dagger_S = \mathrm{cmd}_S(D^R \otimes^{\mathbf{L}}_R S).$$

Here we have used (5.4) and [15, (16.20)]. This proves the assertion. □

(8.5) **Proposition.** *If φ is flat, then*

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{cmd}(S/\mathfrak{m}S) = \mathrm{cmd}\, S - \mathrm{cmd}\, R.$$

Proof. As φ is flat the equation $\mathrm{cmd}(S/\mathfrak{m}S) = \mathrm{cmd}\, S - \mathrm{cmd}\, R$ is well–known, cf. (3.18). The closed fiber of $\widehat{\varphi}$ is isomorphic to the completion of that of φ, therefore they have the same Cohen–Macaulay defect. Thus we have reduced the problem in question to the case where R and S are complete. In particular D^R exists and using the fact that the functors $- \otimes^{\mathbf{L}}_R S$ and $- \otimes_R S$ are naturally isomorphic, we may compute as follows

$$\begin{aligned}
\mathbf{q}\mathrm{cmd}\,\varphi &= \mathrm{cmd}_S(D^R \otimes^{\mathbf{L}}_R S)\\
&= \mathrm{cmd}_S(D^R \otimes_R S)\\
&\overset{(a)}{=} \mathrm{cmd}_R\, D^R + \mathrm{cmd}(S/\mathfrak{m}S)\\
&= \mathrm{amp}\, R + \mathrm{cmd}(S/\mathfrak{m}S)\\
&= \mathrm{cmd}(S/\mathfrak{m}S).
\end{aligned}$$

Here (a) is due to (4.7). The proof is now complete. □

(8.6) **Theorem.** *Assume that $\psi : Q \to R$ and $\varphi : R \to S$ are local homomorphisms such that* G–dim ψ *is finite and φ is flat. Then the following holds*

(1) $$\mathbf{q}\mathrm{dim}\,\varphi\psi = \mathbf{q}\mathrm{dim}\,\varphi + \mathbf{q}\mathrm{dim}\,\psi,$$

(2) $$\mathbf{q}\mathrm{cmd}\,\varphi\psi = \mathbf{q}\mathrm{cmd}\,\varphi + \mathbf{q}\mathrm{cmd}\,\psi\,.$$

Proof. Using [8, Prop. (4.7)] we have G–dim $\varphi\psi$ is finite. As φ is flat, it follows that the completion $\widehat{\varphi} : \widehat{R} \to \widehat{S}$ is flat too. Since we have the following isomorphisms

$$\begin{aligned}
D^{\widehat{\varphi\psi}} &\simeq (D^{\widehat{Q}} \otimes^{\mathbf{L}}_{\widehat{Q}} \widehat{S})^\dagger_{\widehat{S}},\\
D^{\widehat{\psi}} &\simeq (D^{\widehat{Q}} \otimes^{\mathbf{L}}_{\widehat{Q}} \widehat{R})^\dagger_{\widehat{R}},
\end{aligned}$$

we obtain the following sequence

$$\begin{aligned}\mathbf{q}\mathrm{cmd}\,\varphi\psi &= \mathrm{cmd}_{\widehat{S}}(D^{\widehat{Q}} \otimes^{\mathbf{L}}_{\widehat{Q}} \widehat{S}) \\ &= \mathrm{cmd}_{\widehat{S}}((D^{\widehat{Q}} \otimes^{\mathbf{L}}_{\widehat{Q}} \widehat{R}) \otimes_{\widehat{R}} \widehat{S}) \\ &= \mathrm{cmd}_{\widehat{R}}(D^{\widehat{Q}} \otimes^{\mathbf{L}}_{\widehat{Q}} \widehat{R}) + \mathrm{cmd}(S/\mathfrak{m}S) \\ &= \mathbf{q}\mathrm{cmd}\,\psi + \mathbf{q}\mathrm{cmd}\,\varphi\,.\end{aligned}$$

Hence (2) is proven. The statement under (1) is now immediate, and the proof is complete. □

9. Localization and Formal Fibers

(9.1) ***Comment***. Let $\varphi : R \to S$ be a local homomorphism of finite flat dimension. This property localizes: If $\mathfrak{q}$ is a prime ideal of S, then the induced homomorphism $\varphi_{\mathfrak{q}} : R_{\mathfrak{q}\cap R} \to S_{\mathfrak{q}}$ is again of finite flat dimension. Turning to Gorenstein dimension, it is still not known whether the finiteness of G–dim φ implies the finiteness of G–dim $\varphi_{\mathfrak{q}}$; the best result is due to Avramov and Foxby, cf. [8, Prop. (4.5)].

(9.2) **Definition.** Let $\varphi : R \to S$ be a local homomorphism. If G–dim $\varphi_{\mathfrak{q}}$ is finite for every prime ideal $\mathfrak{q} \in \operatorname{Spec} S$, we say that φ is locally of finite G–dimension.

(9.3) **Proposition.** *Let $\varphi : R \to S$ be a local homomorphism. If* G–dim φ *is finite and R admits a dualizing complex, then there is an inequality*

$$\mathbf{q}\mathrm{cmd}\,\varphi_{\mathfrak{q}} \leq \mathbf{q}\mathrm{cmd}\,\varphi$$

for every $\mathfrak{q} \in \operatorname{Spec} S$.

Proof. As R admits a dualizing complex, R has Gorenstein formal fibers. Hence G–dim $\varphi_{\mathfrak{q}}$ is finite for every $\mathfrak{q} \in \operatorname{Spec} S$. For any $\mathfrak{p} \in \operatorname{Spec} R$ the $R_{\mathfrak{p}}$–complex $(D^R)_{\mathfrak{p}}$ is dualizing for $R_{\mathfrak{p}}$. So applying (8.4) we obtain

$$\mathbf{q}\mathrm{cmd}\,\varphi = \mathrm{cmd}_S(D^R \otimes^{\mathbf{L}}_R S),$$

but also

$$\begin{aligned}\mathbf{q}\mathrm{cmd}\,\varphi_{\mathfrak{q}} &= \mathrm{cmd}_{S_{\mathfrak{q}}}((D^R)_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} S_{\mathfrak{q}}) \\ &= \mathrm{cmd}_{S_{\mathfrak{q}}}(D^R \otimes^{\mathbf{L}}_R S)_{\mathfrak{q}},\end{aligned}$$

where $\mathfrak{p} = \mathfrak{q} \cap R$. The conclusion is now obvious. □

(9.4) **Lemma.** *Let $\kappa : \widehat{R} \to R'$ be a flat local homomorphism of complete rings, let $\mathfrak{p}'$ be a prime ideal of R', and let $\mathfrak{p}^* = \mathfrak{p} \cap \widehat{R}$. If the closed fiber of κ is regular, then the local ring $(k(\mathfrak{p}^*) \otimes_{\widehat{R}} R')_{\mathfrak{p}'}$ is a complete intersection.*

Proof. See [4, Lem. (5.5)]. □

(9.5) **Theorem.** *Let $\varphi : R \to S$ be a local homomorphism locally of finite G–dimension. If $\mathfrak{q} \in \operatorname{Spec} S$ and $\mathfrak{p} = \mathfrak{q} \cap R$, then there is an inequality*

$$\mathbf{q}\mathrm{cmd}\,\varphi_{\mathfrak{q}} + \mathrm{cmd}(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \mathbf{q}\mathrm{cmd}\,\varphi + \mathrm{cmd}(k(\mathfrak{p}) \otimes_R \widehat{R}).$$

Proof. Let $R \to R' \to \widehat{S}$ be a Cohen factorization of $\dot{\varphi}$. Then $\widehat{R} \to R' \to \widehat{S}$ is a Cohen factorization of $\widehat{\varphi}$, and we have the following commutative diagram

$$\begin{tikzcd}
R \arrow[rr, "\varphi"] \arrow[d, equal] & & S \arrow[d, "\sigma"] \\
R \arrow[r, "\dot{\varphi}"] \arrow[d, "\rho"'] & R' \arrow[r, "\varphi'"] \arrow[d, equal] & \widehat{S} \arrow[d, equal] \\
\widehat{R} \arrow[r, "\kappa"] & R' \arrow[r, "\varphi'"] & \widehat{S}.
\end{tikzcd}$$

For an arbitrary $\tilde{\mathfrak{q}} \in \operatorname{Spec}(k(\mathfrak{q}) \otimes_S \widehat{S})$ we define

$$\begin{array}{rclcl}
\mathfrak{q}^* & = & \tilde{\mathfrak{q}} \cap \widehat{S} & \in & \operatorname{Spec} \widehat{S}, \\
\mathfrak{q} & = & \mathfrak{q}^* \cap S & \in & \operatorname{Spec} S, \\
\mathfrak{p}' & = & \mathfrak{q}^* \cap R' & \in & \operatorname{Spec} R', \\
\mathfrak{p}^* & = & \mathfrak{p}' \cap \widehat{R} & \in & \operatorname{Spec} \widehat{R}.
\end{array}$$

Since the diagram is commutative we have $\sigma\varphi = \varphi'\kappa\rho$ and it follows that

$$\sigma_{\mathfrak{q}^*}\varphi_{\mathfrak{q}} = \varphi'_{\mathfrak{q}^*}\kappa_{\mathfrak{p}'}\rho_{\mathfrak{p}^*}.$$

The homomorphisms $\rho_{\mathfrak{p}^*} : R_{\mathfrak{p}^* \cap R} \to \widehat{R}_{\mathfrak{p}^*}$ and $\sigma_{\mathfrak{q}^*} : S_{\mathfrak{q}^* \cap S} \to \widehat{S}_{\mathfrak{q}^*}$ are flat, and $\kappa : \widehat{R} \to R'$ (the completion of $\dot{\varphi}$) is flat too.

As φ is locally of finite G–dimension, we are informed by [8, Prop. (4.7)] that G–dim $\sigma_{\mathfrak{q}^*}\varphi_{\mathfrak{q}}$ is finite. Hence by commutativity $\varphi'\kappa\rho$ is locally of finite G–dimension. Furthermore, since φ is locally of finite G–dimension, it follows that the surjective homomorphism φ' is locally of finite G–dimension, cf. [8, Thm. (4.3) and Finite homs. (4.4.4)]. All in all we may compute as follows

$$\begin{aligned}
\mathbf{q}\mathrm{cmd}\,\varphi_{\mathfrak{q}} + \mathbf{q}\mathrm{cmd}\,\sigma_{\mathfrak{q}^*} &\overset{(a)}{=} \mathbf{q}\mathrm{cmd}\,(\sigma_{\mathfrak{q}^*}\varphi_{\mathfrak{q}}) \\
&= \mathbf{q}\mathrm{cmd}\,(\varphi'_{\mathfrak{q}^*}\kappa_{\mathfrak{p}'}\rho_{\mathfrak{p}^*}) \\
&\overset{(b)}{\leq} \mathbf{q}\mathrm{cmd}\,\varphi'_{\mathfrak{q}^*} + \mathbf{q}\mathrm{cmd}\,\kappa_{\mathfrak{p}'} + \mathbf{q}\mathrm{cmd}\,\rho_{\mathfrak{p}^*}\,.
\end{aligned}$$

Here (a) holds by (8.6) and (b) holds by (6.14).

By (8.5) we conclude that for the flat homomorphism $\sigma_{\mathfrak{q}^*}$ we have

$$\mathbf{q}\mathrm{cmd}\,\sigma_{\mathfrak{q}^*} = \mathrm{cmd}(k(\mathfrak{q}) \otimes_S \widehat{S})_{\tilde{\mathfrak{q}}},$$

as $(k(\mathfrak{q}) \otimes_S \widehat{S})_{\tilde{\mathfrak{q}}}$ is isomorphic to the closed fiber of $\sigma_{\mathfrak{q}^*}$. The same argument applies to the flat homomorphism $\rho_{\mathfrak{q}^*}$ giving

$$\mathbf{q}\mathrm{cmd}\, \rho_{\mathfrak{q}^*} = \mathrm{cmd}(k(\mathfrak{p}) \otimes_R \widehat{R})_{\mathfrak{p}^*} \leq \mathrm{cmd}(k(\mathfrak{p}) \otimes_R \widehat{R}).$$

Since $\kappa : \widehat{R} \to R'$ is a flat homomorphism of complete rings with regular closed fiber, we are informed by (9.4) that in this case

$$\mathbf{q}\mathrm{cmd}\, \kappa_{\mathfrak{p}'} = \mathrm{cmd}(k(\mathfrak{p}^*) \otimes_{\widehat{R}} R')_{\mathfrak{p}'} = 0.$$

Furthermore the homomorphism $\varphi' : R' \to \widehat{S}$ of complete rings has finite G–dimension, thus (see (9.3) and (6.5))

$$\mathbf{q}\mathrm{cmd}\, \varphi'_{\mathfrak{q}^*} \leq \mathbf{q}\mathrm{cmd}\, \varphi' = \mathbf{q}\mathrm{cmd}\, \varphi,$$

and we may conclude

$$\begin{aligned} \mathbf{q}\mathrm{cmd}\, \varphi_{\mathfrak{q}} + \mathbf{q}\mathrm{cmd}\, \sigma_{\mathfrak{q}^*} &= \mathbf{q}\mathrm{cmd}\, \varphi_{\mathfrak{q}} + \mathrm{cmd}(k(\mathfrak{q}) \otimes_S \widehat{S})_{\tilde{\mathfrak{q}}} \\ &\leq \mathbf{q}\mathrm{cmd}\, \varphi'_{\mathfrak{q}^*} + \mathbf{q}\mathrm{cmd}\, \kappa_{\mathfrak{p}'} + \mathbf{q}\mathrm{cmd}\, \rho_{\mathfrak{p}^*} \\ &\leq \mathbf{q}\mathrm{cmd}\, \varphi + \mathrm{cmd}(k(\mathfrak{p}) \otimes_R \widehat{R}). \end{aligned}$$

After taking supremum over $\tilde{\mathfrak{q}} \in \mathrm{Spec}(k(\mathfrak{q}) \otimes_S \widehat{S})$ we arrive at the inequality

$$\mathbf{q}\mathrm{cmd}\, \varphi_{\mathfrak{q}} + \mathrm{cmd}(k(\mathfrak{q}) \otimes_S \widehat{S}) \leq \mathbf{q}\mathrm{cmd}\, \varphi + \mathrm{cmd}(k(\mathfrak{p}) \otimes_R \widehat{R}),$$

and the proof is complete. $\square$

References

[1] D. Apassov, *Almost Finite Modules,* Comm. in Algebra **27** (1999), 919–931.

[2] M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society. **94** (American Mathematical Society, Providence, R. I., 1969).

[3] L. L. Avramov and H.–B. Foxby, *Homological dimensions of unbounded complexes*, J. Pure. Appl. Algebra **71** (1991), 129–155.

[4] L. L. Avramov and H.–B. Foxby, *Cohen–Macaulay Properties of Ring Homomorphisms*, Adv. in Math. **133** (1998), 54–95.

[5] L. L. Avramov and H.–B. Foxby, "Cohen–Macaulay Properties of Ring Homomorphisms", unpublished, (1992).

[6] L. L. Avramov and H.–B. Foxby, *Locally Gorenstein homomorphisms*, Amer. J. Math. **114** (1992), 1007–1047.

[7] L. L. Avramov and H.–B. Foxby, *Grothendieck's localization problem*, in "Syzygies, Multiplicities, and Birational Algebra", Contemporary Mathematics, Vol. **159**, 1–13.

[8] L. L. Avramov and H.–B. Foxby, *Ring Homomorphisms and Finite Gorenstein Dimension*, Proc. London Math. Soc. (3) **75** (1997), 241–270.

[9] L. L. Avramov and H.–B. Foxby, "Codimension and finite projective dimension", unpublished, (1992).

[10] L. L. Avramov, H.–B. Foxby and S. Halperin, *Descent and Ascent of Local Properties along Homomorphisms of Finite Flat Dimension*, J. of Pure and Appl. Algebra. **38** (1985), 167–185.

[11] L. L. Avramov, H.–B. Foxby and B. Herzog, *Structure of Local Homomorphisms*, Journal of Algebra. **164** (1994), 124–145.

[12] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.

[13] L. W. Christensen, "Gorenstein Dimensions", Københavns Universitet, master thesis, (1996).

[14] L. W. Christensen, *Semi–dualizing Complexes and their Auslander Categories*, to appear in Trans. Amer. Math. Soc.

[15] H.–B. Foxby, "Hyperhomological Algebra & Commutative Rings", Københavns Universitet, manuscript, (1996).

[16] H.–B. Foxby, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977) no. 1, 5–19.

[17] H.–B. Foxby, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15**, no. 2, 149–172.

[18] H.–B. Foxby, *Quasi–perfect modules over Cohen–Macaulay Rings*, Math. Nachr. **66** (1975), 103–110.

[19] A. Grothendieck, "Éléments de géométrie algébrique" IV. Étude local des schémas et des morphismes de schémas, *Inst. Hautes Études Sci. Publ. Math.* **20, 24, 28, 32** (1964–1967).

[20] R. Hartshorne, "Residues and duality", Lecture notes in Mathematics no. 20 (Springer, Berlin, 1971).

[21] P. Jørgensen, *Gorenstein Homomorphisms of Non Commutative rings*, Journal of Algebra. **211** (1999), 240–267.

[22] H. Matsumura, "Commutative Ring Theory", Cambridge Univ. Press, Cambridge, (1986).

[23] S. McAdam and L. J. Ratliff, Jr., *Semi–Local Taut Rings*, Indiana Univ. Math. J. **26** (1977), 73–79.

[24] T. Ogoma, *Non–catenary pseudo–geometric local rings*, Jpn. J. Math. **6** (1980), 147–163.

[25] L. J. Ratliff, Jr., *Catenary rings and the altitude formula*, Amer. J. Math. **94** (1972), 458–466.

[26] N. Spaltenstein, *Resolutions of unbounded complexes*, Composito Math. **65** (1988), 121–154.

[27] C. A. Weibel, "An introduction to homological algebra", Cambridge Univ. Press, Cambridge, (1997).

Matematisk Afdeling, Universitetsparken 5, DK–2100 København Ø, Denmark.

E-mail address: `frankild@math.ku.dk`

Journal of Algebra
vol. **251,** 479–502 (2002)

RESTRICTED HOMOLOGICAL DIMENSIONS AND COHEN–MACAULAYNESS

LARS WINTHER CHRISTENSEN
HANS–BJØRN FOXBY
AND
ANDERS FRANKILD

Abstract. The classical homological dimensions—the projective, flat, and injective ones—are usually defined in terms of resolutions and then proved to be computable in terms of vanishing of appropriate derived functors. In this paper we define restricted homological dimensions in terms of vanishing of the same derived functors but over classes of test modules that are restricted to assure automatic finiteness over commutative Noetherian rings of finite Krull dimension. When the ring is local, we use a mixture of methods from classical commutative algebra and the theory of homological dimensions to show that vanishing of these functors reveals that the underlying ring is a Cohen–Macaulay ring—or at least close to be one.

Introduction

The first restricted dimension comes about like this: Let R be a commutative Noetherian ring; the flat dimension of an R–module M can then be computed by non-vanishing of Tor modules,

$$\operatorname{fd}_R M = \sup\{\, m \in \mathbb{N}_0 \mid \operatorname{Tor}_m^R(T, M) \neq 0 \text{ for some module } T \,\},$$

and hence we define a *restricted flat dimension* as

$$\mathsf{Rfd}_R M = \sup\{\, m \in \mathbb{N}_0 \mid \operatorname{Tor}_m^R(T, M) \neq 0 \text{ for some module } T \text{ with } \operatorname{fd}_R T < \infty \,\}.$$

(The flat dimension is sometimes called the *Tor–dimension* and the dimension defined above has similarly been referred to as the *restricted Tor–dimension*).

The restricted flat dimension is often finite: First, it follows from [5, Thm. 2.4] that $\mathsf{Rfd}_R M \leq \dim R$ for all R–modules M. Second, by our Theorem (2.5), for all R–modules M there is an inequality $\mathsf{Rfd}_R M \leq \operatorname{fd}_R M$ with equality if $\operatorname{fd}_R M < \infty$; we say that the restricted flat dimension is a *refinement* of the flat dimension.

Furthermore, the restricted flat dimension is also a refinement of other dimension concepts: H. Holm [24] has proved that our Rfd_R is a refinement of Enochs' and Jenda's *Gorenstein flat dimension* Gfd_R introduced in [15]. Moreover, our Theorem (2.8) shows that for finitely generated modules Rfd_R is a refinement of the *Cohen–Macaulay dimension* $\operatorname{CM-dim}_R$ of A. Gerko in [23] (and thus of Auslander's G–dimension [2], as well as the CI–dimension by Avramov, Gasharov, and Peeva [8] and the projective dimension).

The restricted flat dimension can by our Theorem (2.4.b) always be computed by the formula

$$\mathsf{Rfd}_R M = \sup\{\, \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R \,\}$$

where $\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ denotes the index of the first non-vanishing $\operatorname{Ext}^m_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}, M_{\mathfrak{p}})$ module. We refer to this as the *local depth of M at $\mathfrak{p}$*. The equation above is an extension of Chouinard's formula [11, Cor. 1.2] where M has finite flat dimension.

Partially supported by Danish Natural Science Research Council.

Moreover, Chouinard's formula generalizes the classical Auslander–Buchsbaum formula, $\mathrm{pd}_R M = \mathrm{depth}\, R - \mathrm{depth}_R M$ [4, Thm. 3.7], for finite modules of finite projective dimension over a local ring. It is, therefore, natural to ask when the restricted flat dimension satisfies a formula of this type. The answer, provided in Theorem (3.4), is that $\mathsf{Rfd}_R M = \mathrm{depth}\, R - \mathrm{depth}_R M$ for every finitely generated R–module M if and only if R is Cohen–Macaulay.

We proceed by asking the obvious question: what happens if one tests only by finitely generated modules of finite flat (or equivalently projective) dimension? This leads to the definition of a *small* restricted flat dimension, $\mathsf{rfd}_R M$, which turns out to satisfy a Chouinard-like formula

$$\mathsf{rfd}_R M = \sup\{\, \mathrm{depth}_R(\mathfrak{p}, R) - \mathrm{depth}_R(\mathfrak{p}, M) \mid \mathfrak{p} \in \mathrm{Spec}\, R \,\}, \tag{I.1}$$

where $\mathrm{depth}_R(\mathfrak{p}, M)$ is the index of the first non-vanishing $\mathrm{Ext}_R^m(R/\mathfrak{p}, M)$ module. This is called the $\mathfrak{p}$*–depth of* M (or grade of $\mathfrak{p}$ on M), and we refer to it as the *non-local depth of* M *at* $\mathfrak{p}$. We have always $\mathrm{depth}_R(\mathfrak{p}, M) \leq \mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$.

The next question is: when do the small and large restricted flat dimensions agree? We prove, in Theorem (3.2), that the two dimensions agree over a local ring R if and only if it is *almost Cohen–Macaulay* in the sense that $\dim R_\mathfrak{p} - \mathrm{depth}\, R_\mathfrak{p} \leq 1$ for all prime ideals $\mathfrak{p}$. These rings are studied in detail in section 3.

Following this pattern we introduce, in section 5, four dimensions modeled on the formulas for computing projective and injective dimension by vanishing of Ext–modules. For a number of reasons these dimensions do not behave as nicely as those based on vanishing of Tor–modules.

The *small restricted injective dimension*,

$$\mathsf{rid}_R N = \sup\{\, m \in \mathbb{N}_0 \mid \mathrm{Ext}_R^m(T, N) \neq 0 \text{ for some module } T \text{ with } \mathrm{pd}_R T < \infty \,\},$$

is a *finer* invariant than the injective dimension over any commutative ring in the sense that there is always an inequality $\mathsf{rid}_R N \leq \mathrm{id}_R N$. Furthermore, by Corollary (5.9) a local ring R is almost Cohen–Macaulay if and only if $\mathsf{rid}_R N = \mathrm{id}_R N$ for all R–modules of finite injective dimension, that is, if and only if rid_R is a refinement of id_R.

A formula dual to (I.1) is satisfied by this restricted injective dimension

$$\mathsf{rid}_R N = \sup\{\, \mathrm{depth}_R(\mathfrak{p}, R) - \mathrm{width}_R(\mathfrak{p}, N) \mid \mathfrak{p} \in \mathrm{Spec}\, R \,\}. \tag{I.2}$$

The $\mathfrak{p}$–width of N, $\mathrm{width}_R(\mathfrak{p}, N)$, is a notion dual to the $\mathfrak{p}$–depth; it is introduced and studied in section 4. For finite non-zero modules over a local ring (I.2) reduces to

$$\mathsf{rid}_R N = \mathrm{depth}\, R$$

and hence emerges as a generalization of Bass' celebrated formula [10, Lem. (3.3)].

The small and large restricted projective dimensions (with obvious definitions) are finer invariants than the usual projective dimension but only refinements for finitely generated modules. Still, they detect Cohen–Macaulayness of the underlying ring as proved in Theorem (5.22).

1. Prerequisites

Throughout this paper R is a non-trivial, commutative, and Noetherian ring. When R is *local*, $\mathfrak{m}$ denote its unique maximal ideal and k denotes its residue field $R/\mathfrak{m}$. For a prime ideal $\mathfrak{p} \in \mathrm{Spec}\, R$ the residue field of the local ring $R_\mathfrak{p}$ is denoted by $k(\mathfrak{p})$, i.e., $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$. As usual, the set of prime ideals containing an ideal $\mathfrak{a}$ is written $\mathrm{V}(\mathfrak{a})$.

By $\boldsymbol{x}$ we denote an sequence of elements from R, e.g., $\boldsymbol{x} = x_1, \dots, x_n$. Finitely generated modules are called *finite* modules.

In this paper definitions and results are formulated within the framework of the derived category $\mathcal{D}(R)$ of the category of R–modules, and although some arguments have a touch of classical commutative algebra, the proofs draw heavily on the theory of homological dimensions for complexes using the derived functors $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $- \otimes^{\mathbf{L}}_R -$. Throughout we use notation and results from [13] and [7]. However, in order to make the text readable we list the most needed facts below.

For an object X in $\mathcal{D}(R)$ (that is, a complex X of R–modules) the *supremum* $\sup X$ and the *infimum* $\inf X$ of $X \in \mathcal{D}(R)$ are the (possibly infinite) numbers $\sup\{\, \ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0 \,\}$ and $\inf\{\, \ell \in \mathbb{Z} \mid \mathrm{H}_\ell(X) \neq 0 \,\}$, respectively. (Here $\sup \varnothing = -\infty$ and $\inf \varnothing = \infty$, as usual.) The *full subcategories* $\mathcal{D}_-(R)$ and $\mathcal{D}_+(R)$ consist of complexes X with, respectively, $\sup X < \infty$ and $\inf X > -\infty$. We set $\mathcal{D}_\mathrm{b}(R) = \mathcal{D}_-(R) \cap \mathcal{D}_+(R)$. The full subcategory $\mathcal{D}_0(R)$ of $\mathcal{D}_\mathrm{b}(R)$ consists of X with $\mathrm{H}_\ell(X) = 0$ for $\ell \neq 0$. Since each R–module M can be considered as a complex concentrated in degree 0 (hence $M \in \mathcal{D}_0(R)$) and since each $X \in \mathcal{D}_0(R)$ is isomorphic (in $\mathcal{D}(R)$) to the module $\mathrm{H}_0(X)$, we identify $\mathcal{D}_0(R)$ with the category of R–modules. The full subcategory $\mathcal{D}^\mathrm{f}(R)$ of $\mathcal{D}(R)$ consists of complexes X with all the modules $\mathrm{H}_\ell(X)$ finite for $\ell \in \mathbb{Z}$. The superscript f is also used with the full subcategories; for example, $\mathcal{D}^\mathrm{f}_\mathrm{b}(R)$ consists of complexes X with $\mathrm{H}(X)$ finite in each degree and bounded.

(1.1) ***Depth***. If R is local the local *depth* $\operatorname{depth}_R Y$ of $Y \in \mathcal{D}_-(R)$ is the (possibly infinite) number $-\sup(\mathbf{R}\mathrm{Hom}_R(k, Y))$, cf. [19, Sec. 3]. For finite modules this agrees with the classical definition.

In the following, R is any (commutative Noetherian) ring and $\mathfrak{a}$ is an ideal.

The non-local $\mathfrak{a}$*–depth* $\operatorname{depth}_R(\mathfrak{a}, Y)$ is the number $-\sup(\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, Y))$ when $Y \in \mathcal{D}(R)$. The $\mathfrak{a}$–depth is an extension to complexes of a well-known invariant, the grade, for (finite) modules. In particular, $\operatorname{depth}_R(\mathfrak{a}, R)$ is the maximal length of an R–sequence in $\mathfrak{a}$.

Let $\boldsymbol{a} = a_1, \dots, a_t$ be a finite sequence of generators for $\mathfrak{a}$, and let $\mathrm{K}(\boldsymbol{a})$ be the Koszul complex. When $Y \in \mathcal{D}_-(R)$, by [25, 6.1] there is an equality

$$\operatorname{depth}_R(\mathfrak{a}, Y) = t - \sup(\mathrm{K}(\boldsymbol{a}) \otimes_R Y)\,. \tag{1.1.1}$$

For $Y \in \mathcal{D}_-(R)$ the relation to local depth is given by [14, Prop. 4.5]

$$\operatorname{depth}_R(\mathfrak{a}, Y) = \inf\{\, \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \,\}. \tag{1.1.2}$$

In particular, we have

$$\operatorname{depth}_R(\mathfrak{p}, Y) \leq \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p}\,. \tag{1.1.3}$$

From (1.1.2) it also follows that

$$\operatorname{depth}_R(\mathfrak{b}, Y) \geq \operatorname{depth}_R(\mathfrak{a}, Y) \geq -\sup Y. \tag{1.1.4}$$

The *Cohen–Macaulay defect* $\operatorname{cmd} R$ of a local ring R is the (always non-negative) difference $\dim R - \operatorname{depth} R$ between the Krull dimension and the depth. For a non-local ring R the Cohen–Macaulay defect is the supremum over the defects at all prime ideals $\mathfrak{p} \in \operatorname{Spec} R$.

(1.2) ***Width***. If R is local, then the (local) *width* $\operatorname{width}_R X$ of $X \in \mathcal{D}_+(R)$ is the number $\inf(X \otimes^{\mathbf{L}}_R k)$, cf. [31, Def. 2.1], and note that if $X \in \mathcal{D}^\mathrm{f}_+(R)$, then [18, Lem. 2.1] and

Nakayama's lemma give

$$\operatorname{width}_R X = \inf X \tag{1.2.1}$$

For any $X \in \mathcal{D}_+(R)$ [18, Lem. 2.1] give the next inequality

$$\operatorname{width}_{R_\mathfrak{p}} X_\mathfrak{p} \geq \inf X_\mathfrak{p} \geq \inf X. \tag{1.2.2}$$

(1.3) ***Homological dimensions***. The *projective*, *injective*, and *flat dimensions* are abbreviated as pd, id, and fd, respectively. The full subcategories $\mathcal{P}(R)$, $\mathcal{I}(R)$, and $\mathcal{F}(R)$ of $\mathcal{D}_\mathrm{b}(R)$ consist of complexes of finite, respectively, projective, injective, and flat dimension, cf. [13, 1.4]. For example, a complex belongs to $\mathcal{F}(R)$ if and only if it is isomorphic in $\mathcal{D}(R)$ to a bounded complex of flat modules. Again we use the superscript f to denote finite homology and the subscript 0 to denote modules. For example, $\mathcal{P}_0^\mathrm{f}(R)$ denotes the category of finite modules of finite projective dimension.

We close this section by summing up some results on bounds for these dimensions; they will be used extensively in the rest of the text.

By [28, Thm. (3.2.6)] and [9, Prop. 5.4] we have

$$\sup\{\, \operatorname{pd}_R M \mid M \in \mathcal{P}_0(R)\,\} = \dim R\,. \tag{1.3.1}$$

The next shrewd observation due to Auslander and Buchsbaum [5] is often handy.

(1.4) **Lemma.** *If R is local and $\dim R > 0$, then there exists a prime ideal $\mathfrak{p} \subset \mathfrak{m}$ such that* $\operatorname{depth} R_\mathfrak{p} = \dim R - 1$.

In particular, for any local ring we have

$$\sup\{\, \operatorname{depth} R_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\,\} = \begin{cases} \dim R & \textit{if } R \textit{ is Cohen–Macaulay; and} \\ \dim R - 1 & \textit{if } R \textit{ is not Cohen–Macaulay.} \end{cases} \quad \square$$

By [5, Prop. 2.8] and [26, Thm. 1] this implies that,

$$\operatorname{fd}_R M, \operatorname{id}_R N \leq \begin{cases} \dim R & \text{if } R \text{ is Cohen–Macaulay, and} \\ \dim R - 1 & \text{if } R \text{ is not Cohen–Macaulay} \end{cases} \tag{1.4.1}$$

for $M \in \mathcal{F}_0(R)$ and $N \in \mathcal{I}_0(R)$.

The classical Auslander–Buchsbaum formula extends to complexes [20, (0.1)]

(1.5) **Auslander–Buchsbaum formula.** *If R is local and $X \in \mathcal{P}^\mathrm{f}(R)$, then*

$$\operatorname{pd}_R X = \operatorname{depth} R - \operatorname{depth}_R X. \quad \square$$

Actually, it is a special case of the following [20, Lem. 2.1]

(1.6) **Theorem.** *Let R be local. If $X \in \mathcal{F}(R)$ and $Y \in \mathcal{D}_\mathrm{b}(R)$, then the next three equalities hold.*

(a) $$\operatorname{depth}_R(X \otimes^{\mathbf{L}}_R Y) = -\sup\,(X \otimes^{\mathbf{L}}_R k) + \operatorname{depth}_R Y.$$

(b) $$\operatorname{depth}_R X = -\sup\,(X \otimes^{\mathbf{L}}_R k) + \operatorname{depth} R.$$

(c) $$\operatorname{depth}_R(X \otimes^{\mathbf{L}}_R Y) = \operatorname{depth}_R X + \operatorname{depth}_R Y - \operatorname{depth} R. \quad \square$$

In [25, Thm. 4.1] it was demonstrated that it is sufficient to take Y bounded on the left, i.e., $Y \in \mathcal{D}_-(R)$. We treat dual versions of this theorem in section 4.

2. Tor–dimensions

This section is devoted to the Tor–dimensions: the flat dimension, the large restricted flat dimension, and the small restricted flat dimension. The second one was introduced in [21], and the proofs of the first four results can be found in [12, Chap. 5].

(2.1) **Definition.** The *large restricted flat dimension*, $\mathsf{Rfd}_R X$, of $X \in \mathcal{D}_+(R)$ is

$$\mathsf{Rfd}_R X = \sup\{\, \sup(T \otimes^{\mathbf{L}}_R X) \mid T \in \mathcal{F}_0(R)\,\}.$$

For an R–module M we get

$$\mathsf{Rfd}_R M = \sup\{\, m \in \mathbb{N}_0 \mid \ \mathrm{Tor}^R_m(T, M) \neq 0 \text{ for some } T \in \mathcal{F}_0(R)\,\}.$$

The latter expression explains the name, which is justified further by Proposition (2.2) below. The number $\mathsf{Rfd}_R X$ is sometimes referred to as the *restricted Tor–dimension*, and it is denoted $\mathrm{Td}_R X$ in [21] and [12].

(2.2) **Proposition.** *If $X \in \mathcal{D}_+(R)$, then*

$$\sup X \leq \mathsf{Rfd}_R X \leq \sup X + \dim R\,.$$

In particular, $\mathsf{Rfd}_R X > -\infty$ if (and only if) $\mathrm{H}(X) \neq 0$; and if $\dim R$ is finite, then $\mathsf{Rfd}_R X < \infty$ if (and only if) $X \in \mathcal{D}_{\mathrm{b}}(R)$. □

(2.3) **Proposition.** *For every $\mathfrak{p} \in \operatorname{Spec} R$ and $X \in \mathcal{D}_+(R)$ there is an inequality*

$$\mathsf{Rfd}_{R_\mathfrak{p}} X_\mathfrak{p} \leq \mathsf{Rfd}_R X. \quad \square$$

The equation (b) below is the *Ultimate Auslander–Buchsbaum Formula.*

(2.4) **Theorem.** *If $X \in \mathcal{D}_{\mathrm{b}}(R)$, then*

(a) $\mathsf{Rfd}_R X = \sup\{\, \sup(U \otimes^{\mathbf{L}}_R X) - \sup U \mid U \in \mathcal{F}(R) \ \wedge \ \mathrm{H}(U) \neq 0\,\}$

(b) $\mathsf{Rfd}_R X = \sup\{\, \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\,\}.$ □

The large restricted flat dimension is a refinement of the flat dimension, that is,

(2.5) **Theorem.** *For every complex $X \in \mathcal{D}_+(R)$ there is an inequality*

$$\mathsf{Rfd}_R X \leq \mathrm{fd}_R X,$$

and equality holds if $\mathrm{fd}_R X < \infty$. □

(2.6) ***Gorenstein flat dimension***. E. Enochs and O. Jenda have in [15] introduced the *Gorenstein flat dimension* $\mathrm{Gfd}_R M$ of any R–module M. H. Holm has studied this concept further in [24] and proved that $\mathrm{Gfd}_R M$ is a refinement of $\mathrm{fd}_R M$ and that $\mathsf{Rfd}_R M$ is a refinement of $\mathrm{Gfd}_R M$, that is, for any R–module M there is a chain of inequalities

$$\mathsf{Rfd}_R M \leq \mathrm{Gfd}_R M \leq \mathrm{fd}_R M$$

with equality to the left left of any finite number. Thus, in particular, if $\mathrm{Gfd}_R M < \infty$ then Theorem (2.4.b) yields the formula

$$\mathrm{Gfd}_R M = \sup\{\, \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R\,\}\,.$$

(2.7) ***Cohen–Macaulay dimension***. In [23] A. Gerko has defined the *CM–dimension* $\mathrm{CM\text{-}dim}_R M$ of any finite module M over a local ring R in such a way that the ring is Cohen–Macaulay if and only if $\mathrm{CM\text{-}dim}_R M < \infty$ for all finite M. Furthermore, this dimension is a refinement of the Auslander G–dimension $\operatorname{G-dim}_R$, and thereby of the projective one pd_R. On the other hand, the next result shows that Rfd_R is a refinement of $\operatorname{CM-dim}_R$.

(2.8) **Theorem.** *If R is local and M is a finite R–module, then $\mathsf{Rfd}_R\, M \le \operatorname{CM-dim}_R M$ with equality if $\operatorname{CM-dim}_R M < \infty$.*

Proof. It suffices to assume that $\operatorname{CM-dim}_R M$ is finite. By [23, Thm. 3.8, Prop. 3.10] we have then $\operatorname{CM-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$ and $\operatorname{CM-dim}_{R_\mathfrak{p}} M_\mathfrak{p} \le \operatorname{CM-dim}_R M$ for all $\mathfrak{p} \in \operatorname{Spec} R$. These results combined with Theorem (2.4.b) yield for a suitable $\mathfrak{p}$ that

$$\begin{aligned}\mathsf{Rfd}_R\, M &= \operatorname{depth} R_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = \operatorname{CM-dim}_{R_\mathfrak{p}} M_\mathfrak{p}\\ &\le \operatorname{CM-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M \le \mathsf{Rfd}_R\, M\,. \quad \square\end{aligned}$$

When testing flat dimension by non-vanishing of Tor modules, cf. (I.1), it is sufficient to use finite, even cyclic, test modules. It is natural to ask if something similar holds for the large restricted flat dimension. In general the answer is negative, and (3.2) tells us exactly when it is positive. But testing by only finite modules of finite flat dimension gives rise to a new invariant with interesting properties of its own, e.g., see (2.11.b).

(2.9) **Definition.** The *small restricted flat dimension*, $\mathsf{rfd}_R\, X$, of $X \in \mathcal{D}_+(R)$ is

$$\mathsf{rfd}_R\, X = \sup\{\, \sup(T \otimes^{\mathbf{L}}_R X) \mid T \in \mathcal{P}^{\mathrm{f}}_0(R)\,\}.$$

(2.10) **Observation.** Let $X \in \mathcal{D}_+(R)$. It is immediate from the definition that

(2.10.1) $$\sup X = \sup(R \otimes^{\mathbf{L}}_R X) \le \mathsf{rfd}_R\, X \le \mathsf{Rfd}_R\, X \le \sup X + \dim R,$$

cf. (2.2). In particular, $\mathsf{rfd}_R\, X > -\infty$ if (and only if) $\mathrm{H}(X) \neq 0$; and if $\dim R$ is finite, then $\mathsf{rfd}_R\, X < \infty$ if (and only if) $X \in \mathcal{D}_\mathrm{b}(R)$.

By the Ultimate Auslander–Buchsbaum Formula (2.4.b) the large restricted flat dimension is a supremum of differences of *local* depths; the next result shows that the small one is a supremum of differences of *non-local* depths.

(2.11) **Theorem.** *If $X \in \mathcal{D}_\mathrm{b}(R)$, then there are the next two equalities.*

(a) $$\mathsf{rfd}_R\, X = \sup\{\, \sup(U \otimes^{\mathbf{L}}_R X) - \sup U \mid U \in \mathcal{P}^{\mathrm{f}}(R) \ \wedge\ \mathrm{H}(U) \neq 0\,\}$$

(b) $$\mathsf{rfd}_R\, X = \sup\{\, \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X) \mid \mathfrak{p} \in \operatorname{Spec} R\,\}.$$

Proof. As it is immediate from the definition that $\mathsf{rfd}_R\, X$ is less than or equal to the first supremum, it suffices to prove the next two inequalities

$$\sup\{\, U \otimes^{\mathbf{L}}_R X - \sup U\,\} \le \sup\{\, \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X)\,\} \le \mathsf{rfd}_R\, X$$

where $U \in \mathcal{P}^{\mathrm{f}}(R)$, $\mathrm{H}(U) \neq 0$, and $\mathfrak{p} \in \operatorname{Spec} R$.

First, let $U \in \mathcal{P}^{\mathrm{f}}(R)$ with $\mathrm{H}(U) \neq 0$ be given; we then want to prove the existence of a prime ideal $\mathfrak{p}$ such that

$(*)$ $$\sup(U \otimes^{\mathbf{L}}_R X) - \sup U \le \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X).$$

We can assume that $\mathrm{H}(U \otimes^{\mathbf{L}}_R X) \neq 0$, otherwise $(*)$ holds for every $\mathfrak{p}$. Set $s = \sup(U \otimes^{\mathbf{L}}_R X)$, choose $\mathfrak{p}$ in $\operatorname{Ass}_R(\mathrm{H}_s(U \otimes^{\mathbf{L}}_R X))$, and choose by (1.1.2) a prime ideal $\mathfrak{q} \supseteq \mathfrak{p}$, such that $\operatorname{depth}_R(\mathfrak{p}, R) = \operatorname{depth} R_\mathfrak{q}$. The first equality in the computation below follows by [18, Lem. 2.1], the second by (1.6.a) and [6, Cor. 2.10.F and Prop. 5.5], and the third by (1.5); the last inequality is by (1.1.3) and [18, Lem. 2.1].

$$\begin{aligned}\sup\,(U \otimes^{\mathbf{L}}_R X) - \sup U &= -\operatorname{depth}_{R_\mathfrak{p}}(U_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} X_\mathfrak{p}) - \sup U\\ &= \operatorname{pd}_{R_\mathfrak{p}} U_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \sup U\\ &\le \operatorname{pd}_{R_\mathfrak{q}} U_\mathfrak{q} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \sup U\\ &= \operatorname{depth} R_\mathfrak{q} - \operatorname{depth}_{R_\mathfrak{q}} U_\mathfrak{q} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \sup U\\ &= \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{q}} U_\mathfrak{q} - \sup U\\ &\le \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X).\end{aligned}$$

Second, let $\mathfrak{p} \in \operatorname{Spec} R$ be given, and the task is to find a finite module T of finite projective dimension with

$$\operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X) \le \sup\,(T \otimes^{\mathbf{L}}_R X).$$

Set $d = \operatorname{depth}_R(\mathfrak{p}, R)$, choose a maximal R–sequence $\boldsymbol{x} = x_1, \dots, x_d$ in $\mathfrak{p}$, and set $T = R/(\boldsymbol{x})$. Then T belongs to $\mathcal{P}^{\mathrm{f}}_0(R)$ and the Koszul complex $\mathrm{K}(\boldsymbol{x})$ is its minimal free resolution. By (1.1.1) and (1.1.4) we now have the desired

$$\begin{aligned}\sup\,(T \otimes^{\mathbf{L}}_R X) = \sup\,(\mathrm{K}(\boldsymbol{x}) \otimes_R X) &= d - \operatorname{depth}_R((\boldsymbol{x}), X)\\ \ge d - \operatorname{depth}_R(\mathfrak{p}, X) &= \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, X) \quad \square\end{aligned}$$

(2.12) **Observation.** Let $X \in \mathcal{D}_\mathrm{b}(R)$. It follows from (2.11.b) that

$$\sup\{\, \operatorname{depth} R_\mathfrak{m} - \operatorname{depth}_{R_\mathfrak{m}} X_\mathfrak{m} \mid \mathfrak{m} \in \operatorname{Max} R \,\} \le \operatorname{\mathsf{rfd}}_R X;$$

and in view of (1.1.2) and (1.1.4) we also have

$$\operatorname{\mathsf{rfd}}_R X \le \sup\{\, \operatorname{depth} R_\mathfrak{m} \mid \mathfrak{m} \in \operatorname{Max}_R X \,\} + \sup X.$$

In particular: if R is local, then

(2.12.1) $$\operatorname{depth} R - \operatorname{depth}_R X \le \operatorname{\mathsf{rfd}}_R X \le \operatorname{depth} R + \sup X.$$

The example below shows that the two restricted flat dimensions may differ, even for finite modules over local rings, and it shows that the small restricted flat dimension can grow under localization. The latter, unfortunate, property is reflected in the non-local nature of the formula given in (2.11.b).

(2.13) **Example.** Let R be a local ring with $\dim R = 2$ and $\operatorname{depth} R = 0$. By (1.4) choose $\mathfrak{q} \in \operatorname{Spec} R$ with $\operatorname{depth} R_\mathfrak{q} = 1$, choose $x \in \mathfrak{q}$ such that the fraction $x/1$ is $R_\mathfrak{q}$–regular, and set $M = R/(x)$. It follows by (2.12.1) that $\operatorname{\mathsf{rfd}}_R M = 0$, but

$$\operatorname{\mathsf{Rfd}}_R M \ge \operatorname{\mathsf{Rfd}}_{R_\mathfrak{q}} M_\mathfrak{q} \ge \operatorname{\mathsf{rfd}}_{R_\mathfrak{q}} M_\mathfrak{q} \ge \operatorname{depth} R_\mathfrak{q} - \operatorname{depth}_{R_\mathfrak{q}} M_\mathfrak{q} = 1 - 0 > \operatorname{\mathsf{rfd}}_R M$$

by (2.3), (2.10.1), and (2.12.1).

The ring considered above is of Cohen–Macaulay defect two, so in a sense — to be made clear by (3.2) — the example is a minimal one.

(2.14) **Remark.** It is straightforward to see that the restricted flat dimensions of modules can be described in terms of resolutions: If M is any R–module, then $\operatorname{\mathsf{Rfd}}_R M$ [respectively, $\operatorname{\mathsf{rfd}}_R M$] is less than or equal to a non-negative integer g if and only if there is an exact sequence of modules

$$0 \to T_g \to \cdots \to T_n \to \cdots \to T_0 \to M \to 0$$

such that $\operatorname{Tor}_i^R(T,T_n)=0$ for all $i>0$, all $T\in\mathcal{F}_0(R)$ [respectively, all $T\in\mathcal{P}_0^{\mathrm{f}}(R)$], and all n, $0\le n\le g$.

3. Almost Cohen–Macaulay Rings

In this section we characterize the rings over which the small and large restricted flat dimensions agree for for all complexes. It is evident from (2.4.b), (2.11.b), and (1.1.2) that the two dimensions will agree if $\operatorname{depth}_R(\mathfrak{p},R)=\operatorname{depth}R_\mathfrak{p}$ for all $\mathfrak{p}\in\operatorname{Spec}R$, and in (3.2) we show that this condition is also necessary. These rings are called *almost Cohen–Macaulay,* and Lemma (3.1) explains why.

Next we consider the question: when do the restricted flat dimensions satisfy an Auslander–Buchsbaum equality? The answer, provided by (3.4), is that it happens if and only if the ring is Cohen–Macaulay.

(3.1) **Lemma.** *The following are equivalent.*

(i) $\operatorname{cmd}R\le 1$.

(ii) $\operatorname{depth}_R(\mathfrak{p},R)=\operatorname{depth}R_\mathfrak{p}$ *for all* $\mathfrak{p}\in\operatorname{Spec}R$.

(iii) $\mathfrak{p}\in\operatorname{Ass}_R\big(R/(\boldsymbol{x})\big)$ *whenever* $\mathfrak{p}\in\operatorname{Spec}R$ *and* $\boldsymbol{x}$ *is a maximal* R*–sequence in* $\mathfrak{p}$.

(iv) *For every* $\mathfrak{p}\in\operatorname{Spec}R$ *there exists* $M\in\mathcal{P}_0^{\mathrm{f}}(R)$ *with* $\mathfrak{p}\in\operatorname{Ass}_R M$.

(v) *For every* $\mathfrak{p}\in\operatorname{Spec}R$ *there exists* $X\in\mathcal{P}^{\mathrm{f}}(R)$ *with* $\mathfrak{p}\in\operatorname{Ass}_R(\mathrm{H}_{\sup X}(X))$.

Proof. Conditions (i) through (iii) are the equivalent conditions (3), (4), and (5) in [17, Prop. 3.3], and the implications (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) are obvious.

(v) $\Rightarrow$ (i): It suffices to assume that R is local. If $\dim R=0$ there is nothing to prove, so we assume that $\dim R>0$ and choose by (1.4) a prime ideal $\mathfrak{p}$, such that $\operatorname{depth}R_\mathfrak{p}=\dim R-1$. For $X\in\mathcal{P}^{\mathrm{f}}(R)$ with $\mathfrak{p}\in\operatorname{Ass}_R(\mathrm{H}_{\sup X}(X))$ (1.5) and [18, Lem. 2.1] yield

$$\begin{aligned}\operatorname{depth}R=\operatorname{pd}_R X+\operatorname{depth}_R X &\ge \operatorname{pd}_{R_\mathfrak{p}}X_\mathfrak{p}-\sup X\\ &=\operatorname{depth}R_\mathfrak{p}-\operatorname{depth}_{R_\mathfrak{p}}X_\mathfrak{p}-\sup X=\dim R-1\,. \quad\square\end{aligned}$$

(3.2) **Theorem.** *If* R *is local, then the following are equivalent.*

(i) $\operatorname{cmd}R\le 1$.

(ii) $\mathsf{rfd}_R X=\mathsf{Rfd}_R X$ *for all complexes* $X\in\mathcal{D}_+(R)$.

(iii) $\mathsf{rfd}_R M=\mathsf{Rfd}_R M$ *for all finite* R*–modules* M.

Proof. The second condition is, clearly, stronger than the third, so there are two implications to prove.

(i) $\Rightarrow$ (ii): If X is not bounded, then $\infty=\mathsf{rfd}_R X=\mathsf{Rfd}_R X$, cf. (2.10.1). Suppose $X\in\mathcal{D}_\mathrm{b}(R)$; the (in)equalities in the next computation follow by, respectively, (2.11.b), (3.1), (1.1.2), and (2.4.b).

$$\begin{aligned}\mathsf{rfd}_R X&=\sup\{\operatorname{depth}_R(\mathfrak{p},R)-\operatorname{depth}_R(\mathfrak{p},X)\mid\mathfrak{p}\in\operatorname{Spec}R\}\\ &=\sup\{\operatorname{depth}R_\mathfrak{p}-\operatorname{depth}_R(\mathfrak{p},X)\mid\mathfrak{p}\in\operatorname{Spec}R\}\\ &\ge\sup\{\operatorname{depth}R_\mathfrak{p}-\operatorname{depth}_{R_\mathfrak{p}}X_\mathfrak{p}\mid\mathfrak{p}\in\operatorname{Spec}R\}\\ &=\mathsf{Rfd}_R X.\end{aligned}$$

The opposite inequality always holds, cf. (2.10.1), whence equality holds.

$(iii) \Rightarrow (i)$: We can assume that $\dim R > 0$ and choose a prime ideal $\mathfrak{q}$ such that $\operatorname{depth} R_{\mathfrak{q}} = \dim R - 1$, cf. (1.4). Set $M = R/\mathfrak{q}$ and apply (2.12.1) and (2.4.b) to get

$$\operatorname{depth} R \geq \mathsf{rfd}_R M = \mathsf{Rfd}_R M \geq \operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} = \dim R - 1\,. \quad \square$$

Over almost Cohen–Macaulay rings it is, actually, sufficient to use special cyclic modules for testing the restricted flat dimensions:

(3.3) **Corollary.** *If* $\operatorname{cmd} R \leq 1$ *and* $X \in \mathcal{D}_+(R)$, *then*

$$\mathsf{rfd}_R X = \mathsf{Rfd}_R X = \sup\{\, \sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X) \mid \boldsymbol{x} \text{ is an } R\text{–sequence}\,\}.$$

Proof. If X is not bounded, then

$$\mathsf{rfd}_R X = \mathsf{Rfd}_R X = \infty = \sup(R \otimes^{\mathbf{L}}_R X).$$

For $X \in \mathcal{D}_{\mathrm{b}}(R)$ we have $\mathsf{rfd}_R X = \mathsf{Rfd}_R X$ by (3.2). The definition yields

$$\sup\{\, \sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X) \mid \boldsymbol{x} \text{ is an } R\text{–sequence}\,\} \leq \mathsf{rfd}_R X.$$

By (2.4.b) it is sufficient for each $\mathfrak{p} \in \operatorname{Spec} R$ to find an R–sequence $\boldsymbol{x}$ such that

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} \leq \sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X).$$

This is easy: let $\boldsymbol{x}$ be any maximal R–sequence in $\mathfrak{p}$, then, by (3.1), $\mathfrak{p}$ is associated to $R/(\boldsymbol{x})$, in particular, $\operatorname{depth}_{R_{\mathfrak{p}}}(R/(\boldsymbol{x}))_{\mathfrak{p}} = 0$, so by [18, Lem. 2.1] and (1.6.c) we have

$$\begin{aligned}\sup(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X) &\geq -\operatorname{depth}_{R_{\mathfrak{p}}}(R/(\boldsymbol{x}) \otimes^{\mathbf{L}}_R X)_{\mathfrak{p}}\\ &= \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}}(R/(\boldsymbol{x}))_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}\\ &= \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}}. \quad \square\end{aligned}$$

For finite modules the large restricted flat dimension is a refinement of the projective dimension, and over a local ring we, therefore, have $\mathsf{Rfd}_R M = \operatorname{depth} R - \operatorname{depth}_R M$ for $M \in \mathcal{P}^{\mathrm{f}}_0(R)$. Now we ask when such a formula holds for all finite modules:

(3.4) **Theorem.** *If R is local, then the following are equivalent.*

(i) *R is Cohen–Macaulay.*

(ii) $\mathsf{Rfd}_R X = \operatorname{depth} R - \operatorname{depth}_R X$ *for all complexes* $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

(iii) $\mathsf{rfd}_R M = \operatorname{depth} R - \operatorname{depth}_R M$ *for all finite R–modules M.*

Proof. $(i)\Rightarrow(ii)$: Let $\mathfrak{p}$ be a prime ideal. It follows by a complex version of [10, Lem. (3.1)], cf. [16, Chp. 13], that $\operatorname{depth}_R X \leq \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} + \dim R/\mathfrak{p}$. Thus we get the first inequality in the next chain.

$$\begin{aligned}\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth}_{R_{\mathfrak{p}}} X_{\mathfrak{p}} &\leq \operatorname{depth} R_{\mathfrak{p}} - (\operatorname{depth}_R X - \dim R/\mathfrak{p})\\ &\leq \dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} - \operatorname{depth}_R X\\ &\leq \dim R - \operatorname{depth}_R X\\ &= \operatorname{depth} R - \operatorname{depth}_R X.\end{aligned}$$

The desired equality now follows by (2.4.b).

$(ii) \Rightarrow (iii)$: Immediate as

$$\operatorname{depth} R - \operatorname{depth}_R M \leq \mathsf{rfd}_R M \leq \mathsf{Rfd}_R M$$

by (2.12.1) and (2.10.1).

(iii) ⇒ *(i)*: We assume that R is not Cohen–Macaulay and seek a contradiction. Set $d = \operatorname{depth} R$ and let $\boldsymbol{x} = x_1, \dots, x_d$ be a maximal R–sequence. Since R is not Cohen–Macaulay, the ideal generated by the sequence is not $\mathfrak{m}$–primary; that is, there exists a prime ideal $\mathfrak{p}$ such that $(\boldsymbol{x}) \subseteq \mathfrak{p} \subset \mathfrak{m}$. Set $M = R/\mathfrak{p}$, then $\operatorname{depth}_R M > 0$, but $\operatorname{depth}_R(\mathfrak{p}, M) = 0$ and $\operatorname{depth}_R(\mathfrak{p}, R) = d$, so by (2.11.b) we have

$$\operatorname{depth} R - \operatorname{depth}_R M < d = \operatorname{depth}_R(\mathfrak{p}, R) - \operatorname{depth}_R(\mathfrak{p}, M) \leq \mathsf{rfd}_R M,$$

and the desired contradiction has been obtained. □

(3.5) **Corollary.** *If R is a Cohen–Macaulay local ring and $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, then*

$$\mathsf{rfd}_R X = \mathsf{Rfd}_R X = \operatorname{depth} R - \operatorname{depth}_R X.$$

Proof. Immediate by (3.4) and (3.2). □

4. Width of Complexes

To study dual notions of the restricted flat dimensions we need to learn more about width of complexes; in particular, we need a non-local concept of width. Inspired by Iyengar's [25] approach to depth, we will introduce the non-local width by way of Koszul complexes. When $\boldsymbol{x}$ is a sequence of elements in R the Koszul complex $\mathrm{K}(\boldsymbol{x})$ consists of free modules, so the functors $- \otimes_R \mathrm{K}(\boldsymbol{x})$ and $- \otimes^{\mathbf{L}}_R \mathrm{K}(\boldsymbol{x})$ are naturally isomorphic, and we will not distinguish between them. The next result follows from the discussion [25, 1.1–1.3].

(4.1) **Lemma.** *Let $X \in \mathcal{D}(R)$ and let $\mathfrak{a}$ be an ideal in R. If $\boldsymbol{a} = a_1, \dots, a_t$ and $\boldsymbol{x} = x_1, \dots, x_u$ are two finite sequences of generators for $\mathfrak{a}$, then*

$$\inf (X \otimes_R \mathrm{K}(\boldsymbol{a})) = \inf (X \otimes_R \mathrm{K}(\boldsymbol{x})). \quad \square$$

The lemma shows that the next definition of a non-local width makes sense.

(4.2) **Definition.** Let $\mathfrak{a}$ be an ideal in R, and let $\boldsymbol{a} = a_1, \dots, a_t$ be a finite sequence of generators for $\mathfrak{a}$. For $X \in \mathcal{D}(R)$ we define the $\mathfrak{a}$*–width*, $\operatorname{width}_R(\mathfrak{a}, X)$, as

$$\operatorname{width}_R(\mathfrak{a}, X) = \inf (X \otimes_R \mathrm{K}(\boldsymbol{a})).$$

(4.3) **Observation.** A Koszul complex has infimum at least zero, so by [18, Lem. 2.1] there is always an inequality

$$\operatorname{width}_R(\mathfrak{a}, X) \geq \inf X. \tag{4.3.1}$$

If $\boldsymbol{a} = a_1, \dots, a_t$ generates a proper ideal $\mathfrak{a}$ in R, then $\inf \mathrm{K}(\boldsymbol{a}) = 0$ as $\mathrm{H}_0(\mathrm{K}(\boldsymbol{a})) \cong R/\mathfrak{a}$. For $X \in \mathcal{D}^{\mathrm{f}}_{+}(R)$ with $\mathrm{H}(X) \neq 0$ it then follows by [18, Lem. 2.1] that equality holds in (4.3.1) if and only if $\operatorname{Supp}_R(\mathrm{H}_{\inf X}(X)) \cap \mathrm{V}(\mathfrak{a}) \neq \emptyset$. In particular: if R is local, then

$$\operatorname{width}_R(\mathfrak{a}, X) = \inf X \tag{4.3.2}$$

for all $X \in \mathcal{D}^{\mathrm{f}}_{+}(R)$ and all proper ideals $\mathfrak{a}$.

(4.4) **Observation.** Let $X \in \mathcal{D}(R)$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in R and $\mathfrak{b} \supseteq \mathfrak{a}$, then it follows easily by the definition that $\operatorname{width}_R(\mathfrak{b}, X) \geq \operatorname{width}_R(\mathfrak{a}, X)$.

(4.5) ***Matlis Duality.*** If E be a faithfully injective R–module and $X \in \mathcal{D}(R)$ then

$$\sup (\mathbf{R}\mathrm{Hom}_R(X, E)) = -\inf (X) \quad \text{and} \quad \inf (\mathbf{R}\mathrm{Hom}_R(X, E)) = -\sup (X)\,.$$

Every ring R admits a faithfully injective module E, e.g., $E = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$. For any faithfully injective module E we use the notation $-^{\vee} = \mathbf{R}\mathrm{Hom}_R(-, E)$. If R is local, then $-^{\vee} = \mathrm{Hom}_R(-, \mathrm{E}_R(R/\mathfrak{m}))$ is known as the *Matlis duality functor*. Here we tacitly use that $\mathrm{Hom}_R(-, E)$ and $\mathbf{R}\mathrm{Hom}_R(-, E)$ are naturally isomorphic in $\mathcal{D}(R)$ and we do not distinguish between them.

The next results will spell out the expected relations between $\mathfrak{a}$–width and width over local rings as well as the behavior of $\mathfrak{a}$-width and $\mathfrak{a}$-depth under duality with respect to faithfully injective modules. But first we establish a useful lemma and a remark

(4.6) **Lemma.** *If $\boldsymbol{a} = a_1, \ldots, a_t$ generates the ideal $\mathfrak{a}$ in R and $X \in \mathcal{D}(R)$ then*

$$\mathrm{width}_R(\mathfrak{a}, X) = t + \inf\left(\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{a}), X)\right).$$

Proof. When Σ denotes the shift functor we have $X \otimes_R \mathrm{K}(\boldsymbol{a}) \cong \Sigma^t \mathrm{Hom}_R(\mathrm{K}(\boldsymbol{a}), X)$, and the desired equality follows

$$\mathrm{width}_R(\mathfrak{a}, X) = \inf\left(\Sigma^t \mathrm{Hom}_R(\mathrm{K}(\boldsymbol{a}), X)\right) = t + \inf\left(\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{a}), X)\right). \quad \square$$

(4.7) **Remark.** For $Z \in \mathcal{D}_+^{\mathrm{f}}(R)$ and $W \in \mathcal{D}(R)$ we have the following special case of the Hom-evaluation morphism [1, Thm. 1, p. 27]

$$Z \otimes_R^{\mathbf{L}} W^{\vee} \simeq \mathbf{R}\mathrm{Hom}_R(Z, W)^{\vee}$$

provided either $\mathrm{pd}_R Z < \infty$ or $W \in \mathcal{D}_-(R)$.

(4.8) **Proposition.** *If $X \in \mathcal{D}_+(R)$ and $Y \in \mathcal{D}_-(R)$, then*

$$\mathrm{depth}_R(\mathfrak{a}, X^{\vee}) = \mathrm{width}_R(\mathfrak{a}, X) \quad \textit{and} \quad \mathrm{width}_R(\mathfrak{a}, Y^{\vee}) = \mathrm{depth}_R(\mathfrak{a}, Y)\,.$$

Proof. By (1.1.1) and (4.6), the equalities follow from (4.7) and adjointness, respectively. $\square$

(4.9) **Proposition.** *If M is a finite R–module with support $\mathrm{V}(\mathfrak{a})$ and $X \in \mathcal{D}_+(R)$, then*

$$\mathrm{width}_R(\mathfrak{a}, X) = \inf\left(X \otimes_R^{\mathbf{L}} R/\mathfrak{a}\right) = \inf\left(X \otimes_R^{\mathbf{L}} M\right).$$

Proof. By [14, Prop. 4.5] we obtain

$$\mathrm{depth}_R(\mathfrak{a}, X^{\vee}) = -\sup\left(\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, X^{\vee})\right) = -\sup\left(\mathbf{R}\mathrm{Hom}_R(M, X^{\vee})\right).$$

Hence (4.8), the adjointness isomorphism, and (4.5) yield the desired assertions. $\square$

(4.10) **Corollary.** *If $\mathfrak{q} \subseteq \mathfrak{p}$ are prime ideals in R and $X \in \mathcal{D}_+(R)$, then*

$$\mathrm{width}_R(\mathfrak{q}, X) \leq \mathrm{width}_{R_\mathfrak{p}}(\mathfrak{q}_\mathfrak{p}, X_\mathfrak{p})\,. \quad \square$$

(4.11) **Corollary.** *If R is local and $X \in \mathcal{D}_+(R)$, then* $\mathrm{width}_R(\mathfrak{m}, X) = \mathrm{width}_R X$. $\square$

(4.12) **Corollary.** *For $\mathfrak{p} \in \mathrm{Spec}\, R$ and $X \in \mathcal{D}_+(R)$ there is an inequality*

$$\mathrm{width}_R(\mathfrak{p}, X) \leq \mathrm{width}_{R_\mathfrak{p}} X_\mathfrak{p}. \quad \square$$

Finally, we want to dualize (1.6). The first result in this direction is [31, Lem. 2.6] which is stated just below (cf. also [1, Thm. 1(2), p. 27]).

(4.13) **Theorem.** *If R is local, $X \in \mathcal{D}_-(R)$, and $Y \in \mathcal{I}(R)$, then the following hold.*

(a) $$\operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \inf(\mathbf{R}\mathrm{Hom}_R(k,Y)) + \operatorname{depth}_R X.$$

(b) $$\operatorname{width}_R Y = \inf(\mathbf{R}\mathrm{Hom}_R(k,Y)) + \operatorname{depth} R.$$

(c) $$\operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \operatorname{width}_R Y + \operatorname{depth}_R X - \operatorname{depth} R. \quad \square$$

If the hypothesis $(X,Y) \in \mathcal{D}_-(R) \times \mathcal{I}(R)$ above is replaced by the hypothesis $(X,Y) \in \mathcal{P}(R) \times \mathcal{D}_+(R)$ we get a corresponding result which is stated below. (By the Auslander–Buchsbaum formula (1.5) part (b) below can be written exactly like (c) above.)

(4.14) **Theorem.** *Let R be local and $Y \in \mathcal{D}_+(R)$. If $X \in \mathcal{P}(R)$, then*

(a) $$\operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \operatorname{width}_R Y - \sup(X \otimes^{\mathbf{L}}_R k).$$

In particular: if $X \in \mathcal{P}^{\mathrm{f}}(R)$, then

(b) $$\operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \operatorname{width}_R Y - \operatorname{pd}_R X.$$

Proof. Let P be a bounded projective resolution of X, and let K be the Koszul complex on a sequence of generators for the maximal ideal $\mathfrak{m}$. It is straightforward to check that

$(*)$ $$\mathrm{Hom}_R(P,Y) \otimes_R K \cong \mathrm{Hom}_R(P, Y \otimes_R K);$$

it uses that K is a bounded complex of finite free modules, and that tensoring by finite modules commutes with direct products. We can assume that $Y_\ell = 0$ for $\ell \ll 0$, the same then holds for $Y \otimes_R K$ and $\mathrm{Hom}_R(P, Y \otimes_R K)$. By $(*)$ the spectral sequence corresponding to the double complex $\mathrm{Hom}_R(P, Y \otimes_R K)$ converges to $\mathrm{H}(\mathrm{Hom}_R(P,Y) \otimes_R K)$. Filtrating by columns, cf. [30, Def. 5.6.1] we may write

$$E^2_{pq} = \mathrm{H}_p(\mathrm{Hom}_R(P, \mathrm{H}_q(Y \otimes_R K))),$$

as P is a complex of projectives. Evoking the fact that the homology module $\mathrm{H}_q(Y \otimes_R K)$ is a vector space over k, we can compute E^2_{pq} as follows

$(**)$ $$\begin{aligned} E^2_{pq} &= \mathrm{H}_p(\mathrm{Hom}_R(P, \mathrm{Hom}_k(k, \mathrm{H}_q(Y \otimes_R K)))) \\ &= \mathrm{H}_p(\mathrm{Hom}_k(P \otimes_R k, \mathrm{H}_q(Y \otimes_R K))) \\ &= \mathrm{Hom}_k(\mathrm{H}_{-p}(P \otimes_R k), \mathrm{H}_q(Y \otimes_R K)). \end{aligned}$$

In $\mathcal{D}(R)$ there are isomorphisms

$(*{*}*)$ $$P \otimes_R k \simeq X \otimes^{\mathbf{L}}_R k \quad \text{and} \quad \mathrm{Hom}_R(P,Y) \otimes_R K \simeq \mathbf{R}\mathrm{Hom}_R(X,Y) \otimes_R K.$$

If $\mathrm{H}(P \otimes_R k) = 0$ or $\mathrm{H}(Y \otimes_R K) = 0$ (i.e., $\sup(X \otimes^{\mathbf{L}}_R k) = -\infty$ or $\operatorname{width}_R Y = \infty$), then $E^2_{pq} = 0$ for all p and q, so also $\mathrm{H}(\mathrm{Hom}_R(P,Y) \otimes_R K)$ vanishes making $\operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \infty$. Otherwise, it is easy to see from $(**)$ that

$$\begin{aligned} &E^2_{pq} = 0 \quad \text{for } -p > \sup(P \otimes_R k) \quad \text{or} \quad q < \inf(Y \otimes_R K)\,; \quad \text{and} \\ &E^2_{pq} \neq 0 \quad \text{for } -p = \sup(P \otimes_R k) \quad \text{and} \quad q = \inf(Y \otimes_R K)\,. \end{aligned}$$

A standard "corner" argument now shows that

$$\inf(\mathrm{Hom}_R(P,Y) \otimes_R K) = \inf(Y \otimes_R K) - \sup(P \otimes_R k);$$

and by $(*{*}*)$, (4.2), and (4.12) this is the desired equality (a).

Part (b) follows from (a) in view of [6, Cor. 2.10.F and Prop. 5.5]. $\square$

The difference between the proofs of the last two theorems is, basically, that between k and K: In [31] Yassemi uses the k–structure of the complex $\mathbf{R}\mathrm{Hom}_R(k, X)$, and to get in a position to do so he needs the power of evaluation-morphisms, cf. [6, Lem. 4.4]. The same procedure could be applied to produce (4.14.b), but not part (a). The proof of (4.14) uses only the k–structure of the homology modules $\mathrm{H}_\ell(Y \otimes_R K)$, where K is the Koszul complex on a generating sequence for $\mathfrak{m}$, and the simple structure of the Koszul complex allows us to avoid evaluation-morphisms, cf. $(*)$ in the proof.

5. Ext–dimensions

Restricted injective dimensions are analogues to the restricted flat ones. When it comes to generalizing Bass' formula [10, Lem. (3.3)] the small restricted injective dimension is the more interesting, and we start with that one.

Furthermore, we study restricted projective dimensions, and for all four Ext–dimensions we examine their ability to detect (almost) Cohen–Macaulayness of the underlying ring.

(5.1) **Definition.** The *small restricted injective dimension*, $\mathsf{rid}_R\, Y$, of $Y \in \mathcal{D}_-(R)$ is

$$\mathsf{rid}_R\, Y = \sup\{\, -\inf(\mathbf{R}\mathrm{Hom}_R(T, Y)) \mid T \in \mathcal{P}_0^{\mathrm{f}}(R)\,\}.$$

For an R–module N the definition reads

$$\mathsf{rid}_R\, N = \sup\{\, m \in \mathbb{N}_0 \mid \ \mathrm{Ext}_R^m(T, N) \neq 0 \text{ for some } T \in \mathcal{P}_0^{\mathrm{f}}(R)\,\}.$$

(5.2) **Observation.** Let $Y \in \mathcal{D}_-(R)$. It is immediate from the definition that

$$-\inf Y = -\inf(\mathbf{R}\mathrm{Hom}_R(R, Y)) \leq \mathsf{rid}_R\, Y;$$

and for $T \in \mathcal{P}_0(R)$ we have

$$-\inf(\mathbf{R}\mathrm{Hom}_R(T, Y)) \leq -\inf Y + \mathrm{pd}_R\, T,$$

cf. [13, (1.4.3)], so by (1.3.1) there are always inequalities

$$-\inf Y \leq \mathsf{rid}_R\, Y \leq -\inf Y + \dim R. \tag{5.2.1}$$

In particular, $\mathsf{rid}_R\, Y > -\infty$ if (and only if) $\mathrm{H}(Y) \neq 0$; and if $\dim R$ is finite, then $\mathsf{rid}_R\, Y < \infty$ if (and only if) $Y \in \mathcal{D}_\mathrm{b}(R)$.

The next two results are parallel to (2.11.b) and (3.3); the key is the duality expressed by the first equation in (5.3), and it essentially hinges on (4.7).

(5.3) **Proposition.** *If* $Y \in \mathcal{D}_\mathrm{b}(R)$, *then*

(a) $\mathsf{rid}_R\, Y = \mathsf{rfd}_R\, Y^\vee$

(b) $\mathsf{rid}_R\, Y = \sup\{\, -\sup U - \inf(\mathbf{R}\mathrm{Hom}_R(U, Y)) \mid U \in \mathcal{P}^{\mathrm{f}}(R) \ \wedge\ \mathrm{H}(U) \neq 0\,\}$

(c) $\mathsf{rid}_R\, Y = \sup\{\, \mathrm{depth}_R(\mathfrak{p}, R) - \mathrm{width}_R(\mathfrak{p}, Y) \mid \mathfrak{p} \in \mathrm{Spec}\, R\,\}$.

Proof. It follows by (4.7) that

$$\sup(T \otimes_R^{\mathbf{L}} Y^\vee) = \sup(\mathbf{R}\mathrm{Hom}_R(T, Y)^\vee) = -\inf(\mathbf{R}\mathrm{Hom}_R(T, Y))$$

for $T \in \mathcal{P}_0^{\mathrm{f}}(R)$, and (a) is proved. Now, (b) follows by (2.11.a), and (c) is a consequence of (4.8) and (2.11.b). □

(5.4) **Proposition.** *If* $\mathrm{cmd}\, R \leq 1$ *and* $Y \in \mathcal{D}_\mathrm{b}(R)$, *then*

$$\mathsf{rid}_R\, Y = \sup\{\, -\inf(\mathbf{R}\mathrm{Hom}_R(R/(\boldsymbol{x}), Y)) \mid \boldsymbol{x} \text{ is an } R\text{–sequence}\,\}.$$

Proof. Use (4.7), (5.3.a), and (3.3). □

(5.5) **Corollary.** *If R is local, $Y \in \mathcal{D}_{-}^{\mathrm{f}}(R)$, and $N \neq 0$ is an R–modules, then*

(a) $$\operatorname{rid}_R Y = \operatorname{depth} R - \inf Y$$

(b) $$\operatorname{rid}_R N = \operatorname{depth} R\,.$$

Proof. Use also (5.3.c), (4.3.2), and (1.1.4). □

(5.6) **Corollary.** *If $\operatorname{cmd} R \leq 1$ then $\operatorname{rid}_{R_\mathfrak{p}} Y_\mathfrak{p} \leq \operatorname{rid}_R Y$ for $Y \in \mathcal{D}_{-}(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. Apply also (3.1) and (4.10). □

(5.7) **Remark.** Just like the small restricted flat dimension, the small restricted injective dimension can grow under localization: Let R and $\mathfrak{q}$ be as in (2.13); for every finite R–module N with $\mathfrak{q} \in \operatorname{Supp}_R N$ (e.g., $N = R/\mathfrak{q}$) we then have

$$\operatorname{rid}_{R_\mathfrak{q}} N_\mathfrak{q} = \operatorname{depth} R_\mathfrak{q} = 1 > 0 = \operatorname{depth} R = \operatorname{rid}_R N.$$

It is well-known that a local ring must be Cohen–Macaulay in order to allow a non-zero finite module of finite injective dimension (this is the Bass conjecture proved by Peskine and Szpiro [27] and Roberts [29]), so it follows by the next result that the original Bass formula (I.4) is contained in (5.5.b).

(5.8) **Proposition.** *For every complex $Y \in \mathcal{D}_{-}(R)$ there is an inequality*

$$\operatorname{rid}_R Y \leq \operatorname{id}_R Y,$$

and equality holds if $\operatorname{id}_R Y < \infty$ and $\operatorname{cmd} R \leq 1$.

Proof. Since the inequality is immediate and equality holds if $Y \simeq 0$, we assume that $\operatorname{id}_R Y = n \in \mathbb{Z}$. By [6, Prop. 5.3.I] there exists then $\mathfrak{p} \in \operatorname{Spec} R$ such that $\mathrm{H}_{-n}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), Y_\mathfrak{p}))$ is non-trivial. Choose next, by (3.1), a module $T \in \mathcal{P}_0^{\mathrm{f}}(R)$ with $\mathfrak{p} \in \operatorname{Ass}_R T$. The short exact sequence

$$0 \to R/\mathfrak{p} \to T \to C \to 0,$$

induces a long exact homology sequence which shows that $\mathrm{H}_{-n}(\mathbf{R}\mathrm{Hom}_R(T, Y)) \neq 0$. Thus

$$\operatorname{rid}_R Y \geq -\inf\left(\mathbf{R}\mathrm{Hom}_R(T, Y)\right) \geq n\,. \quad \square$$

(5.9) **Corollary.** *For a local ring R the next three conditions are equivalent.*

(i) $\operatorname{cmd} R \leq 1$.

(ii) $\operatorname{rid}_R Y = \operatorname{id}_R Y$ *for all complexes $Y \in \mathcal{I}(R)$.*

(iii) $\operatorname{rid}_R M = \operatorname{id}_R M$ *for all R–modules of finite injective dimension.*

Proof. By the proposition, (i) implies (ii) which is stronger than (iii). To see that (iii) implies (i) we may assume that $\dim R > 0$. Choose an R–module M with $\operatorname{id}_R M = \dim R - 1$ and a finite R–module T of finite projective dimension such that $\operatorname{rid}_R M = -\inf\left(\mathbf{R}\mathrm{Hom}_R(T, M)\right)$. Now [6, Thm. 2.4.P] and (1.5) yield

$$\dim R - 1 = -\inf\left(\mathbf{R}\mathrm{Hom}_R(T, M)\right) \leq \operatorname{pd}_R T \leq \operatorname{depth} R\,. \quad \square$$

(5.10) **Definition.** The *large restricted injective dimension*, $\operatorname{Rid}_R Y$, of $Y \in \mathcal{D}_{-}(R)$ is

$$\operatorname{Rid}_R Y = \sup\{\, -\inf\left(\mathbf{R}\mathrm{Hom}_R(T, Y)\right) \mid T \in \mathcal{P}_0(R)\,\}.$$

For an R–module M the definition reads

$$\mathsf{Rid}_R\, M = \sup\{\, m \in \mathbb{N}_0 \mid \operatorname{Ext}_R^m(T,M) \neq 0 \text{ for some } T \in \mathcal{P}_0(R)\,\}. \tag{5.10.1}$$

(5.11) **Observation.** Let $Y \in \mathcal{D}_-(R)$. As in (5.2) we see that there are inequalities

$$-\inf Y \leq \mathsf{rid}_R\, Y \leq \mathsf{Rid}_R\, Y \leq -\inf Y + \dim R. \tag{5.11.1}$$

In particular, $\mathsf{Rid}_R\, Y > -\infty$ if (and only if) $\mathrm{H}(Y) \neq 0$; and if $\dim R$ is finite, then $\mathsf{Rid}_R\, Y < \infty$ if (and only if) $Y \in \mathcal{D}_{\mathrm{b}}(R)$.

(5.12) **Remark.** Below we show that the large injective dimension is a refinement of the injective dimension, at least over almost Cohen–Macaulay rings. For a complex Y of finite injective dimension over such a ring we, therefore, have

$$\mathsf{Rid}_R\, Y = \sup\{\, \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} Y_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\,\},$$

by the extension to complexes [31, Thm. 2.10] of Chouinard's formula (I.3). It is, however, easy to see that this formula fails in general. Let R be local and not Cohen–Macaulay, and let T be a module with $\operatorname{pd}_R T = \dim R$, cf. [9, Prop. 5.4]. By [6, 2.4.P] there is then an R–module N with $-\inf(\mathbf{R}\mathrm{Hom}_R(T,N)) = \dim R$, so $\mathsf{Rid}_R\, N \geq \dim R$ but by (4.3.1) and (1.4) we have

$$\sup\{\, \operatorname{depth} R_{\mathfrak{p}} - \operatorname{width}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\,\} \leq \sup\{\, \operatorname{depth} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\,\} = \dim R - 1\,.$$

(5.13) **Proposition.** *For every complex $Y \in \mathcal{D}_-(R)$ there is an inequality*

$$\mathsf{Rid}_R\, Y \leq \operatorname{id}_R Y,$$

and equality holds if $\operatorname{id}_R Y < \infty$ and $\operatorname{cmd} R \leq 1$.

Proof. The inequality is immediate; apply (5.8) and (5.11.1) to complete the proof. □

(5.14) **Definition.** The *large restricted projective dimension*, $\mathsf{Rpd}_R\, X$ of $X \in \mathcal{D}_+(R)$ is

$$\mathsf{Rpd}_R\, X = \sup\{\, -\inf(\mathbf{R}\mathrm{Hom}_R(X,T)) \mid T \in \mathcal{I}_0(R)\,\}.$$

For an R–module M the definition reads

$$\mathsf{Rpd}_R\, M = \sup\{\, m \in \mathbb{N}_0 \mid \operatorname{Ext}_R^m(M,T) \neq 0 \text{ for some } T \in \mathcal{I}_0(R)\,\}.$$

(5.15) **Observation.** For $X \in \mathcal{D}_+(R)$ there are inequalities

$$\sup X \leq \mathsf{Rpd}_R\, X \leq \sup X + \dim R \tag{5.15.1}$$

by (4.5), cf. [13, (1.4.2)], and (1.4.1). In particular, $\mathsf{Rpd}_R\, X > -\infty$ if (and only if) $\mathrm{H}(X) \neq 0$; and if $\dim R$ is finite, then $\mathsf{Rpd}_R\, X < \infty$ if (and only if) $X \in \mathcal{D}_{\mathrm{b}}(R)$.

(5.16) **Lemma.** *If $X \in \mathcal{D}_{\mathrm{b}}(R)$ then $\mathsf{Rfd}_R\, X \leq \mathsf{Rpd}_R\, X$ with equality when $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.*

Proof. For an R–module T of finite flat dimension, the Matlis dual $T^\vee$ is a module of finite injective dimension. By adjointness $\sup(T \otimes^{\mathbf{L}}_R X)$ equals $-\inf(\mathbf{R}\mathrm{Hom}_R(X,T^\vee))$ and (a) follows.

To show (b) let next T denote an R–module of finite injective dimension. Then $T^\vee$ is a module of finite flat dimension, and the desired equality follows since $\inf(\mathbf{R}\mathrm{Hom}_R(X,T))$ equals $-\sup(X \otimes^{\mathbf{L}}_R T^\vee)$ by (4.7). □

(5.17) **Observation.** The inequality $\mathsf{Rpd}_R\, X \leq \mathrm{pd}_R\, X$ holds for every $X \in \mathcal{D}_+(R)$. If R is local and $X \in \mathcal{P}^{\mathrm{f}}(R)$, then (2.5), [13, (1.4.4)], and (5.16) yield

$$\mathrm{pd}_R\, X = \mathrm{fd}_R\, X = \mathsf{Rfd}_R\, X = \mathsf{Rpd}_R\, X\,.$$

(5.18) **Remark.** One can prove that Rpd_R is a refinement of pd_R when R is a Cohen–Macaulay local ring with a dualizing module. If R is not Cohen–Macaulay, then $\mathsf{Rpd}_R\, M \leq \dim R - 1$ for every R–module M by (1.4.1), but there exists an R–module M with $\mathrm{pd}_R\, M = \dim R$ (by [9, Prop. 5.4]).

A small restricted projective dimension based on the expression

$$\sup\{\, -\inf\left(\mathbf{R}\mathrm{Hom}_R(X,T)\right) \mid T \in \mathcal{I}_0^{\mathrm{f}}(R)\,\}$$

would be trivial over non-Cohen–Macaulay rings as they do not allow non-zero finite modules of finite injective dimension. Inspired by (2.11.a) and (5.3.b) we instead make the following:

(5.19) **Definition.** The *small restricted projective dimension* $\mathsf{rpd}_R\, X$ of $X \in \mathcal{D}_+(R)$ is

$$\mathsf{rpd}_R\, X = \sup\{\, \inf U - \inf\left(\mathbf{R}\mathrm{Hom}_R(X,U)\right) \mid U \in \mathcal{I}^{\mathrm{f}}(R) \ \wedge\ \mathrm{H}(U) \neq 0\,\}.$$

It should be noted that the supremum above is of a non-empty set as any (commutative Noetherian) ring admits a $U \in \mathcal{I}^{\mathrm{f}}(R)$ with $\mathrm{H}(U) \neq 0$; for example, $U = \mathrm{K}(\boldsymbol{x})^{\vee}$ when $\boldsymbol{x}$ is a sequence of generators of a maximal ideal.

(5.20) **Lemma.** *If R is local and $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, then*

$$\inf U - \inf\left(\mathbf{R}\mathrm{Hom}_R(X,U)\right) = \operatorname{depth} R - \operatorname{depth}_R X$$

for every $U \in \mathcal{I}^{\mathrm{f}}(R)$ with $\mathrm{H}(U) \neq 0$. In particular,

$$\mathsf{rpd}_R\, X = \operatorname{depth} R - \operatorname{depth}_R X.$$

Proof. For X and U as above $\mathbf{R}\mathrm{Hom}_R(X,U)$ is in $\mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, and the first equality follows by applying (1.2.1) twice and then (4.13.c). ☐

(5.21) **Observation.** If $X \in \mathcal{D}_+(R)$ then $\mathsf{rpd}_R\, X \leq \mathrm{pd}_R\, X$, and if R is local and $X \in \mathcal{P}^{\mathrm{f}}(R)$ then equality holds by (5.20).

(5.22) **Theorem.** *If R is local, then the following are equivalent.*

- (i) *R is Cohen–Macaulay.*
- (ii) $\mathsf{rpd}_R\, X = \mathsf{Rpd}_R\, X$ *for all complexes $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$.*
- (iii) $\mathsf{Rpd}_R\, M = \operatorname{depth} R - \operatorname{depth}_R M$ *for all finite R–modules M.*

Proof. $(i) \Rightarrow (ii)$ follows from (5.16), (3.4), and (5.20). $(ii) \Rightarrow (iii)$ is immediate by (5.20), while $(iii) \Rightarrow (i)$ results from (5.16), and (3.4). ☐

(5.23) **Corollary.** *If R is a Cohen–Macaulay local ring and $X \in \mathcal{D}_{\mathrm{b}}^{\mathrm{f}}(R)$, then*

$$\mathsf{rpd}_R\, X = \mathsf{Rpd}_R\, X = \operatorname{depth} R - \operatorname{depth}_R X. \quad \square$$

(5.24) ***Other Ext–dimensions***. The restrictions on the test modules T in the definitions of the restricted Ext–dimension have been made such that (among other things) these dimensions are always finite for non-zero modules over (e.g.) local rings. In the case of modules, we also consider the next two alternative projective dimensions

$$\mathsf{Apd}_R\, M = \sup\{\, m \in \mathbb{N}_0 \mid \operatorname{Ext}_R^m(M,T) \neq 0 \text{ for some } T \in \mathcal{P}_0(R)\,\}$$
$$\mathsf{apd}_R\, M = \sup\{\, m \in \mathbb{N}_0 \mid \operatorname{Ext}_R^m(M,T) \neq 0 \text{ for some } T \in \mathcal{P}_0^{\mathrm{f}}(R)\,\}\,.$$

Even when the ring is local these two numbers are not always finite (the ring is Gorenstein if and only if they are always finite). However, it is easy to verify that the dimension Apd_R is always a refinement of pd_R and that apd_R is a refinement of pd_R over finite modules. Actually, it is proved in [24] that Apd_R is a refinement of the Gorenstein projective dimension Gpd_R, and in [3, Thm. 4.13] that apd_R is, for finite modules, a refinement of Auslander's G–dimension G-dim_R.

Moreover, if R is a complete local ring and $M \in \mathcal{P}_0(R)$, then it is proved in [22] that the Auslander–Buchsbaum Formula holds (without finiteness condition on M), that is,

$$\mathsf{apd}_R\, M = \operatorname{depth} R - \operatorname{depth}_R M\,.$$

Acknowledgments

We wish to thank professor Peter Jørgensen for stimulating and helpful discussions concerning the material presented here.

References

1. Dmitri Apassov, *Complexes and differential graded modules*, Ph.D. thesis, Lund University, 1999.
2. Maurice Auslander, *Anneaux de Gorenstein, et torsion en algèbre commutative*, Secrétariat mathématique, Paris, 1967, Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, 1966/67. Texte rédigé, d'après des exposés de Maurice Auslander, par Marquerite Mangeney, Christian Peskine et Lucien Szpiro. École Normale Supérieure de Jeunes Filles.
3. Maurice Auslander and Mark Bridger, *Stable module theory*, American Mathematical Society, Providence, R.I., 1969, Memoirs of the American Mathematical Society, No. 94.
4. Maurice Auslander and David A. Buchsbaum, *Homological dimension in local rings*, Trans. Amer. Math. Soc. **85** (1957), 390–405.
5. ———, *Homological dimension in Noetherian rings. II*, Trans. Amer. Math. Soc. **88** (1958), 194–206.
6. Luchezar L. Avramov and Hans-Bjørn Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), no. 2-3, 129–155.
7. ———, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270.
8. Luchezar L. Avramov, Vesselin N. Gasharov, and Irena V. Peeva, *Complete intersection dimension*, Inst. Hautes Études Sci. Publ. Math. (1997), no. 86, 67–114 (1998).
9. Hyman Bass, *Injective dimension in Noetherian rings*, Trans. Amer. Math. Soc. **102** (1962), 18–29.
10. ———, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.
11. Leo G. Chouinard, II, *On finite weak and injective dimension*, Proc. Amer. Math. Soc. **60** (1976), 57–60 (1977).
12. Lars Winther Christensen, *Gorenstein Dimensions*, Springer-Verlag, Berlin, 2000, Lecture Notes in Mathematics 1747.
13. ———, *Semi–dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353** (2001), 1839–1883.
14. ———, *Sequences for complexes*, Math. Scandinavica **89** (2001), 1–20.
15. Edgar E. Enochs and Overtoun M. G. Jenda, *Gorenstein injective and flat dimensions*, Math. Japon. **44** (1996), no. 2, 261–268.
16. Hans-Bjørn Foxby, *Hyperhomological algebra & commutative rings*, notes in preparation.
17. ———, *n-Gorenstein rings*, Proc. Amer. Math. Soc. **42** (1974), 67–72.
18. ———, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), no. 1, 5–19.
19. ———, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), no. 2, 149–172.
20. ———, *Homological dimensions of complexes of modules*, Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, 32ème année (Paris, 1979), Springer, Berlin, 1980, pp. 360–368.
21. ———, *Auslander–Buchsbaum equalities*, Amer. Math. Soc. 904th Meeting, Kent State Univ., Amer. Math. Soc., Providence, RI, 1995, pp. 759–760.
22. Anders Frankild, *Vanishing of local homology*, in preparation.
23. A. A. Gerko, *On homological dimensions*, in preparation, in Russian.
24. Henrik Holm, *Gorenstein homological dimensions*, manuscript, 2000.
25. Srikanth Iyengar, *Depth for complexes, and intersection theorems*, Math. Z. **230** (1999), no. 3, 545–567.
26. Eben Matlis, *Applications of duality*, Proc. Amer. Math. Soc. **10** (1959), 659–662.
27. Christian Peskine and Lucien Szpiro, *Dimension projective finie et cohomologie locale. Applications à la démonstration de conjectures de M. Auslander, H. Bass et A. Grothendieck*, Inst. Hautes Études Sci. Publ. Math. (1973), no. 42, 47–119.
28. Michel Raynaud and Laurent Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.
29. Paul Roberts, *Le théorème d'intersection*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987), no. 7, 177–180.
30. Charles A. Weibel, *An introduction to homological algebra*, Cambridge University Press, Cambridge, 1994.
31. Siamak Yassemi, *Width of complexes of modules*, Acta Math. Vietnam. **23** (1998), no. 1, 161–169.

LWC: Cryptomathic A/S, Christians Brygge 28, 2, DK–1559 København V, Denmark.

HBF&AF: Matematisk Afdeling, Universitetsparken 5, DK–2100 København Ø, Denmark.

E-mail address: `lars.winther@cryptomathic.com foxby@math.ku.dk frankild@math.ku.dk`

MATH. Z.
(to appear)

VANISHING OF LOCAL HOMOLOGY

ANDERS FRANKILD

ABSTRACT. We study the vanishing properties of local homology of complexes of modules without assuming that its homology is artinian. Using vanishing results for local homology and cohomology we prove new vanishing results for Ext- and Tor-modules.

INTRODUCTION

The main objective of this paper is to study vanishing properties of local homology without any assumption on "artinianness" and to derive some new vanishing results for Ext- and Tor-modules.

Local cohomology and homology. Let R be a commutative, noetherian ring. Pick an ideal $\mathfrak{a}$ in R, an R–module M, and consider the section functor

$$\Gamma_{\mathfrak{a}}(M) = \varinjlim \operatorname{Hom}_R(R/\mathfrak{a}^n, M).$$

Right deriving $\Gamma_{\mathfrak{a}}(-)$ we obtain the famous local cohomology functors denoted $\mathrm{H}^i_{\mathfrak{a}}(-)$; it is safe to say that these are of great importance in algebra.

Dually, when $\mathfrak{a}$ and M are as above, consider the $\mathfrak{a}$–adic completion functor

$$\Lambda^{\mathfrak{a}}(M) = \varprojlim (R/\mathfrak{a}^n \otimes_R M).$$

Left deriving $\Lambda^{\mathfrak{a}}(-)$ we obtain the so–called local homology functors denoted $\mathrm{H}_i^{\mathfrak{a}}(-)$.

While the local cohomology functors are studied in great detail not so much is known about the local homology functors.

The local homology functors were first studied by Matlis [16] and [17] for ideals $\mathfrak{a}$ generated by a regular sequence. Then came the work of Greenlees and May [14], and Lipman, López, and Tarrío [1], showing the existence of a strong connection between local homology and local cohomology. This connection is particularly clear (see [1]) when formulated within the derived category $\mathsf{D}(R)$ of the category of R–modules, that is, before passing to homology.

The Čech complex. Already in the early days of local cohomology it was shown that the right derived functors of $\Gamma_{\mathfrak{a}}(-)$ can be computed via the Čech complex on a set of generators for the ideal $\mathfrak{a}$. To be more precise: If X is a complex of R–modules, $\mathrm{C}(\boldsymbol{a})$ the Čech complex on a set $\boldsymbol{a}$ of generators of $\mathfrak{a}$, and $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$ is the right derived section functor, then

$$\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \mathrm{C}(\boldsymbol{a}) \otimes^{\mathbf{L}}_R X,$$

where $\simeq$ denotes an isomorphism in $\mathsf{D}(R)$.

1991 *Mathematics Subject Classification.* 13C12, 13D07, 13D45.

Key words and phrases. Local homology, local cohomology, derived completeness, derived torsion, vanishing of Ext– and Tor–modules.

In [14] and [1] is was shown that the left derived functor of $\Lambda^{\mathfrak{a}}(-)$ can also be computed via the Čech complex: If X and $\mathrm{C}(\boldsymbol{a})$ are as above, and $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ is the left derived completion functor, then

$$\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{C}(\boldsymbol{a}), X).$$

This isomorphism will play a major role in this paper.

Vanishing properties. Recall the following well known vanishing result for local cohomology, extended to complexes. If $Y \in \mathsf{D}(R)$ is a complex bounded to the left, then

$$\inf\{\, i \in \mathbb{Z} \mid \mathrm{H}^i_{\mathfrak{a}}(Y) \neq 0 \,\} = \operatorname{depth}_R(\mathfrak{a}, Y), \tag{1}$$

where $\operatorname{depth}_R(\mathfrak{a}, Y)$ denotes the index of the first non–vanishing $\mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, Y))$ module; this number is called the $\mathfrak{a}$*–depth* of Y or the grade of $\mathfrak{a}$ on Y (see e.g. [15, sec. 2]). Note that when Y is just an ordinary module $\mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, Y)) \cong \mathrm{Ext}^i_R(R/\mathfrak{a}, Y)$.

Turning to local homology we prove a dual result. If $X \in \mathsf{D}(R)$ is a complex bounded to the right, then

$$\inf\{\, i \in \mathbb{Z} \mid \mathrm{H}^{\mathfrak{a}}_i(X) \neq 0 \,\} = \operatorname{width}_R(\mathfrak{a}, X), \tag{2}$$

where $\operatorname{width}_R(\mathfrak{a}, X)$ denotes the index of the first non–vanishing $\mathrm{H}_i(R/\mathfrak{a} \otimes^{\mathbf{L}}_R X)$ module; this number is called the $\mathfrak{a}$*–width* of X, and is a dual notion of $\mathfrak{a}$–depth. The $\mathfrak{a}$–width is introduced in [6, sec. 4]. Note that when X is just an ordinary module $\mathrm{H}_i(R/\mathfrak{a} \otimes^{\mathbf{L}}_R X) \cong \mathrm{Tor}^R_i(R/\mathfrak{a}, X)$.

This was shown by Simon in [21] for modules. Even in the module case our proof is simpler than Simon's, and uses some of the modern tools made available in [1] and [14]. Furthermore, we compute an upper bound for the number $\sup\{\, i \in \mathbb{Z} \mid \mathrm{H}^{\mathfrak{a}}_i(X) \neq 0 \,\}$ when X is bounded, (see (2.12)).

Local homology and artinianness. In [23], and recently in [7], the interplay between local homology and artinian modules is studied. It is shown that local homology and artinian modules admit a theory parallel to the theory for local cohomology and finitely generated modules (see also (2.10)).

Vanishing for Ext- and Tor-modules. Combining (1) and (2) with the notion of *restricted homological dimensions*, introduced in [6], we obtain new vanishing results of Ext– and Tor–modules (see corollaries (3.7), (3.10), and (3.13)). For example, we show the following: If R is a local ring, M a non–trivial *complete* module, and T a module of finite projective dimension, then

$$\begin{aligned} &\mathrm{Ext}^i_R(T, M) = 0 \quad \text{for} \quad i > \operatorname{depth} R - \operatorname{depth}_R T, \text{ and} \\ &\mathrm{Ext}^i_R(T, M) \neq 0 \quad \text{for} \quad i = \operatorname{depth} R - \operatorname{depth}_R T. \end{aligned} \tag{3}$$

When T is a *finitely generated* module of finite projective dimension this is well documented (recall that by the Auslander–Buchsbaum formula the projective dimension of T equals $\operatorname{depth} R - \operatorname{depth}_R T$). But in view of (3), it seems as if the number $\operatorname{depth} R - \operatorname{depth}_R T$ yields an important invariant for *every* module of finite projective dimension.

Recall that the *finitistic projective dimension* of a commutative noetherian ring R equals $\dim R$, the *Krull dimension* of R (see [18, thm. (3.2.6)] and [3, prop. 5.4]) (This concept should not be confused with the finitistic global dimension of a ring as studied by Auslander and Buchsbaum, where only finitely generated modules were taken into account). Thus, it follows from (3) that over *complete local, non-Cohen–Macaulay rings*, we cannot measure projective dimension using complete modules, and hence finitely generated ones, as test modules.

(0.1) **Synopsis.** This paper is organized as follows: Section 1 is a brief recap on notation; also, we list a couple of results needed for later use. Section 2 contains the vanishing result for local homology, while section 3 is devoted to the vanishing results for Ext– and Tor– modules.

1. Some Notation and Preparatory Results

(1.1) **Blanket setup.** Throughout, R will denote a non–trivial, commutative, noetherian ring. When R is *local*, $\mathfrak{m}$ will denote its maximal ideal, and $k = R/\mathfrak{m}$ will denote the residue class field. The injective hull of k is denoted $\mathrm{E}_R(k)$.

If $\mathfrak{a}$ is an ideal in R, then $\widehat{R_{\mathfrak{a}}}$ denotes the ring completed in the $\mathfrak{a}$–adic topology (if R is local, then $\widehat{R}$ denotes the ring completed in the $\mathfrak{m}$–adic topology).

Finitely generated modules are called *finite*.

(1.2) **The derived category.** In this paper definitions and results are formulated within the universe of the derived category $\mathsf{D}(R)$ of the category of R–modules. Recall that the objects in $\mathsf{D}(R)$ are complexes of R–modules (see [24]).

The symbol $\simeq$ will denote an isomorphism in $\mathsf{D}(R)$; recall that a morphism of R–complexes represents an isomorphism in $\mathsf{D}(R)$ exactly when the morphism is an quasi–isomorphism.

For an object X in $\mathsf{D}(R)$ the *supremum* denoted $\sup X$ and the *infimum* denoted $\inf X$ are the (possibly infinite) numbers $\sup\{\, l \in \mathbb{Z} \mid \mathrm{H}_l(X) \neq 0\,\}$ and $\inf\{\, l \in \mathbb{Z} \mid \mathrm{H}_l(X) \neq 0\,\}$. As usual we operate with the convention $\sup\emptyset = -\infty$ and $\inf\emptyset = \infty$.

The full subcategories $\mathsf{D}_-(R)$ and $\mathsf{D}_+(R)$ consist of complexes X with, respectively, $\sup X < \infty$ and $\inf X > -\infty$. We set $\mathsf{D}_\mathrm{b}(R) = \mathsf{D}_-(R) \cap \mathsf{D}_+(R)$. Genuine modules can be identified with the objects in the full subcategory of $\mathsf{D}_\mathrm{b}(R)$ for which $\mathrm{H}_l(X) = 0$ for $l \neq 0$, and we use the symbol $\mathsf{D}_0(R)$ to denote them. The full subcategory $\mathsf{D}^\mathrm{f}(R)$ of $\mathsf{D}(R)$ will denote complexes X for which $\mathrm{H}_l(X)$ is a finite module in every degree. The superscript f can be combined with full subcategories; for example, $\mathsf{D}^\mathrm{f}_\mathrm{b}(R)$ consists of complexes with bounded homology which is finite in every degree. The full subcategory $\mathsf{D}^\mathrm{art}(R)$ of $\mathsf{D}(R)$ will denote complexes X for which $\mathrm{H}_l(X)$ is an artinian module in every degree.

(1.3) **Bounds.** Here we list some standard results (see [9, lem. 2.1]). If $X \in \mathsf{D}_+(R)$ and $Y \in \mathsf{D}_-(R)$, then $\mathbf{R}\mathrm{Hom}_R(X,Y) \in \mathsf{D}_-(R)$ and there is an inequality:

(1.3.1) $$\sup(\mathbf{R}\mathrm{Hom}_R(X,Y)) \leq \sup Y - \inf X,$$

which is an equality if $i = \inf X$ and $s = \sup Y$ are finite and $\mathrm{Hom}_R(\mathrm{H}_i(X), \mathrm{H}_s(Y)) \neq 0$.

If $X, Y \in \mathsf{D}_+(R)$, then $X \otimes^{\mathbf{L}}_R Y \in \mathsf{D}_+(R)$ and there is an inequality:

$$\inf (X \otimes^{\mathbf{L}}_R Y) \geq \inf X + \inf Y. \tag{1.3.2}$$

which is an equality if $i = \inf X$ and $j = \inf Y$ are finite and $\mathrm{H}_i(X) \otimes_R \mathrm{H}_j(Y) \neq 0$.

(1.4) **Depth.** For a local ring R the (local) *depth* of a complex $Y \in \mathsf{D}_-(R)$ is defined as the number

$$\operatorname{depth}_R Y = -\sup (\mathbf{R}\mathrm{Hom}_R(k, Y)),$$

which may be infinite (see [10, sec. 3]). When Y is a module the definition reads

$$\operatorname{depth}_R Y = \inf\{\, i \mid \mathrm{Ext}^i_R(k, Y) \neq 0 \,\},$$

and for finite modules this agrees with the classical definition, that is, equals the unique maximal length of a regular Y–sequence in $\mathfrak{m}$.

Suppose that $\mathfrak{a}$ is an ideal in R (which is not necessarily local) and let $\boldsymbol{a} = a_1, \dots, a_t$ be a set of generators for $\mathfrak{a}$. In [15, sec. 2] the non–local $\mathfrak{a}$–*depth* of $Y \in \mathsf{D}(R)$ is defined as the number

$$\operatorname{depth}_R(\mathfrak{a}, Y) = -\sup (\mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{a}), Y)), \tag{1.4.1}$$

where $\mathrm{K}(\boldsymbol{a})$ is the *Koszul complex* on $\boldsymbol{a}$ (see (2.1)); it is, of course, independent of the particular choice of $\boldsymbol{a}$. By [5, prop. 4.5] the non–local $\mathfrak{a}$-depth of $Y \in \mathsf{D}_-(R)$ may also be computed as

$$\operatorname{depth}_R(\mathfrak{a}, Y) = -\sup (\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, Y)).$$

Hence, when R is local, $\mathfrak{a} = \mathfrak{m}$, and $Y \in \mathsf{D}_-(R)$ we have the expected equality

$$\operatorname{depth}_R(\mathfrak{m}, Y) = \operatorname{depth}_R Y. \tag{1.4.2}$$

Observe, when R is local and $Y \in \mathsf{D}_-(R)$ we have the estimate

$$\operatorname{depth}_R Y \geq -\sup Y. \tag{1.4.3}$$

Finally, if $\mathrm{H}(Y) \neq 0$, we may conclude, using (1.3.1) that

$$\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{H}_{\sup Y}(Y)) \iff \operatorname{depth}_{R_\mathfrak{p}} Y_\mathfrak{p} = -\sup Y; \tag{1.4.4}$$

here, of course, R need not be local.

(1.5) **Width.** Let R be local. Then the *width* of a complex $X \in \mathsf{D}_+(R)$ is defined by:

$$\operatorname{width}_R X = \inf (X \otimes^{\mathbf{L}}_R k),$$

see [26, def. 2.1]. In particular, if $X \in \mathsf{D}^{\mathrm{f}}_+(R)$, then

$$\operatorname{width}_R X = \inf X \tag{1.5.1}$$

according to (1.3.2) and Nakayama's lemma.

(1.6) **Homological dimensions.** The *projective*, *injective* and *flat* dimensions of complexes are abbreviated pd_R, id_R and fd_R. The full subcategories $\mathsf{P}(R)$, $\mathsf{I}(R)$ and $\mathsf{F}(R)$ of $\mathsf{D}_\mathrm{b}(R)$ consist of complexes of finite projective, injective and flat dimension. For example, a complex sits inside $\mathsf{P}(R)$ if it is isomorphic (in $\mathsf{D}(R)$) to a bounded complex of projectives. Again the superscript f is used to denote finite homology and the subscript 0 is used to denote modules; for example $\mathsf{P}_0^\mathrm{f}(R)$ denotes the category of finite modules of finite projective dimension.

Recall that $\mathsf{P}(R) \subseteq \mathsf{F}(R)$ and that $\mathsf{I}_0^\mathrm{f}(R)$ is trivial unless R is Cohen–Macaulay.

(1.7) **Restricted homological dimensions.** The classical approach to homological dimensions is usually to define them in terms of resolutions and then prove them to be computable in terms of vanishing of appropriate derived functors.

Focusing on the same derived functors but now over a restricted class of test modules to assure automatic finiteness, [6] introduces the notion of *restricted* homological dimension; they are defined solely in terms of vanishing of derived functors (see below).

As these dimensions play a central role in section 3 we will here recall the definitions as well as a couple of their most basic properties for later use.

The *restricted projective dimension* and the *small restricted projective dimension* of $X \in \mathsf{D}_+(R)$ are defined respectively as:

$$\begin{aligned}\mathsf{R}\mathrm{pd}_R\, X &= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(X,T)) \mid T \in \mathsf{I}_0(R)\},\\ \mathsf{r}\mathrm{pd}_R\, X &= \sup\{\inf U - \inf(\mathbf{R}\mathrm{Hom}_R(X,U)) \mid U \in \mathsf{I}^\mathrm{f}(R) \ \wedge\ \mathrm{H}(U) \neq 0\}.\end{aligned}$$

The *restricted injective dimension* and the *small restricted injective dimension* of $Y \in \mathsf{D}_-(R)$ are defined respectively as:

$$\begin{aligned}\mathsf{R}\mathrm{id}_R\, Y &= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(T,Y)) \mid T \in \mathsf{P}_0(R)\},\\ \mathsf{r}\mathrm{id}_R\, Y &= \sup\{-\inf(\mathbf{R}\mathrm{Hom}_R(T,Y)) \mid T \in \mathsf{P}_0^\mathrm{f}(R)\}.\end{aligned}$$

The *restricted flat dimension* and the *small restricted flat dimension* of $X \in \mathsf{D}_+(R)$ are defined respectively as:

$$\begin{aligned}\mathsf{R}\mathrm{fd}_R\, X &= \sup\{\sup(T \otimes_R^{\mathbf{L}} X) \mid T \in \mathsf{F}_0(R)\},\\ \mathsf{r}\mathrm{fd}_R\, X &= \sup\{\sup(T \otimes_R^{\mathbf{L}} X) \mid T \in \mathsf{P}_0^\mathrm{f}(R)\}.\end{aligned}$$

Note that the restricted homological dimensions all are finite, if R is of finite Krull dimension.

For any $X \in \mathsf{D}_\mathrm{b}(R)$ we have

$$\mathsf{R}\mathrm{fd}_R\, X \leq \mathsf{R}\mathrm{pd}_R\, X, \tag{1.7.1}$$

by [6, lem. (5.16)] and if, in addition, R is local we have

$$\operatorname{depth} R - \operatorname{depth}_R X \leq \mathsf{r}\mathrm{fd}_R\, X \leq \operatorname{depth} R + \sup X, \tag{1.7.2}$$

by [6, (2.12.1)].

(1.8) **Auslander–Buchsbaum formulae.** In section 3 the Auslander–Buchsbaum formula and a dual version of it will also play a central role. We choose to list them here for the benefit of the reader.

Let R be local. If $X \in \mathsf{F}(R)$ and $Y \in \mathsf{D}_{\mathrm{b}}(R)$, then by [11, lem. 2.1]

(a) $\qquad \operatorname{depth}_R(X \otimes^{\mathbf{L}}_R Y) = -\sup(X \otimes^{\mathbf{L}}_R k) + \operatorname{depth}_R Y.$

(b) $\qquad \operatorname{depth}_R(X \otimes^{\mathbf{L}}_R Y) = \operatorname{depth}_R X + \operatorname{depth}_R Y - \operatorname{depth} R.$

If $X \in \mathsf{D}_-(R)$ and $Y \in \mathsf{I}(R)$, then by [26, lem. 2.6]

(c) $\qquad \operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X, Y)) = \operatorname{width}_R Y + \operatorname{depth}_R X - \operatorname{depth} R.$

And finally, if $Y \in \mathsf{D}_+(R)$, $X \in \mathsf{P}(R)$ and $X' \in \mathsf{P}^{\mathrm{f}}(R)$, then by [6, thm. (4.13)]

(d) $\qquad \operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X, Y)) = \operatorname{width}_R Y - \sup(X \otimes^{\mathbf{L}}_R k).$

(e) $\qquad \operatorname{width}_R(\mathbf{R}\mathrm{Hom}_R(X', Y)) = \operatorname{width}_R Y + \operatorname{depth}_R X' - \operatorname{depth} R.$

2. Vanishing of Local Homology

In this section we prove the announced vanishing result for local homology.

(2.1) **Koszul and Čech complexes.** Let x be an element in R. The complex $\mathrm{K}(x) = 0 \to R \xrightarrow{x} R \to 0$ concentrated in degrees 1 and 0 denotes the *Koszul complex* on x, while the complex $\mathrm{C}(x) = 0 \to R \xrightarrow{\rho_x} R_x \to 0$ concentrated in degrees 0 and -1 denotes the *Čech complex* on x.

For a set $\boldsymbol{x} = x_1, \dots, x_n$ of elements in R we define $\mathrm{K}(\boldsymbol{x}) = \mathrm{K}(x_1) \otimes_R \cdots \otimes_R \mathrm{K}(x_n)$ and $\mathrm{C}(\boldsymbol{x}) = \mathrm{C}(x_1) \otimes_R \cdots \otimes_R \mathrm{C}(x_n)$. Note that the two pairs of functors $- \otimes_R \mathrm{K}(\boldsymbol{x})$ and $- \otimes^{\mathbf{L}}_R \mathrm{K}(\boldsymbol{x})$, and $- \otimes_R \mathrm{C}(\boldsymbol{x})$ and $- \otimes^{\mathbf{L}}_R \mathrm{C}(\boldsymbol{x})$ are naturally equivalent, as $\mathrm{K}(\boldsymbol{x})$ is a bounded complex of free modules, while $\mathrm{C}(\boldsymbol{x})$ is a bounded complex of flats.

Let M be an R–module and consider the complex $\mathrm{K}(\boldsymbol{x}) \otimes_R M$. Then it is well–know that the ideal $(\boldsymbol{x})$ annihilates the homology of $\mathrm{K}(\boldsymbol{x}) \otimes_R M$, that is, $(\boldsymbol{x})\,\mathrm{H}_l(\mathrm{K}(\boldsymbol{x}) \otimes_R M) = 0$ for all $l \in \mathbb{Z}$. This result can, of course, be extended to complexes, that is, replace M with any R–complex X.

(2.2) **The right derived section functor.** Let $\mathfrak{a}$ be an ideal in R and M an R–module. The *section functor with support in* $\mathrm{V}(\mathfrak{a})$ applied to M is defined by

$$\Gamma_{\mathfrak{a}}(M) = \varinjlim \mathrm{Hom}_R(R/\mathfrak{a}^n, M)$$

and it is well–known that $\Gamma_{\mathfrak{a}}(-)$ is a left exact, covariant module functor, see [4, chap. 1]. The right derived functors of the section functor, denoted $\mathrm{H}^i_{\mathfrak{a}}(-)$, are the famous local cohomology functors.

Suppose that $Y \in \mathsf{D}(R)$ and let $Y \xrightarrow{\simeq} I$ be a K–injective resolution of Y (see [13] and [22]). The *right derived section functor* of the complex Y is defined as

$$\mathbf{R}\Gamma_{\mathfrak{a}}(Y) = \Gamma_{\mathfrak{a}}(I).$$

The right derived section functor can be computed via Čech complexes. If $\boldsymbol{a} = a_1, \dots, a_t$ is a set of generators of $\mathfrak{a}$ and $\mathrm{C}(\boldsymbol{a})$ the corresponding Čech complex, then for every $Y \in \mathsf{D}(R)$ we have

$$\mathbf{R}\Gamma_{\mathfrak{a}}(Y) \simeq \mathrm{C}(\boldsymbol{a}) \otimes_R Y,$$

by [19, thm. 1.1(iv)].

(2.3) **The left derived completion functor.** Let $\mathfrak{a}$ be an ideal in R and M an R–module. The $\mathfrak{a}$*–adic completion functor* applied to M is defined by

$$\Lambda^{\mathfrak{a}}(M) = \varprojlim (R/\mathfrak{a}^n \otimes_R M)$$

Suppose that $X \in \mathsf{D}(R)$ and let $P \xrightarrow{\simeq} X$ be a K–projective resolution of X (see [13] and [22]). The *left derived completion functor* of the complex X is defined as

$$\mathbf{L}\Lambda^{\mathfrak{a}}(X) = \Lambda^{\mathfrak{a}}(P).$$

The left derived completion functor can also be computed via Čech complexes. If $\mathfrak{a}$ and $\mathrm{C}(\boldsymbol{a})$ are as above, then for every $X \in \mathsf{D}(R)$ we have

$$\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{C}(\boldsymbol{a}), X),$$

by [1, $(0.3)_{\text{aff}}$, p.4].

(2.4) **Vanishing of local cohomology.** In this paragraph we recall Grothendieck's important vanishing results for local cohomology. Assume that $Y \in \mathsf{D}_-(R)$ and $X \in \mathsf{D}_+(R)$. Then

$$-\sup \mathbf{R}\Gamma_{\mathfrak{a}}(Y) = \operatorname{depth}_R(\mathfrak{a}, Y), \tag{2.4.1}$$

$$-\inf \mathbf{R}\Gamma_{\mathfrak{a}}(X) \le \dim_R X, \tag{2.4.2}$$

and equality holds if, in addition, R is local, $\mathfrak{a} = \mathfrak{m}$, and $\mathrm{H}_l(Y)$ is finite for every $l \in \mathbb{Z}$ (see [12, prop. 7.10 , thm. 7.8 and cor. 8.29], [15, thm. 6.1], and [4, chap. 6]). Here the Krull dimension of any complex Z is defined as

$$\dim_R Z = \sup\{ \dim(R/\mathfrak{p}) - \inf Z_{\mathfrak{p}} \mid \mathfrak{p} \in \mathsf{Spec}\,(R) \}.$$

(2.5) **Associativity and $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$.** Let X and Y be complexes in $\mathsf{D}(R)$. Observe that the isomorphism $\mathbf{R}\Gamma_{\mathfrak{a}}(Y) \simeq \mathrm{C}(\boldsymbol{a}) \otimes^{\mathbf{L}}_R Y$ immediately gives

$$\mathbf{R}\Gamma_{\mathfrak{a}}(Y \otimes^{\mathbf{L}}_R X) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(Y) \otimes^{\mathbf{L}}_R X,$$

using associativity of the derived tensor product functor.

(2.6) **Adjointness and $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$.** Let X and Y be complexes in $\mathsf{D}(R)$. Observe that the isomorphism $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{C}(\boldsymbol{a}), X)$ immediately gives

$$\mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\mathrm{Hom}_R(X,Y)) \simeq \mathbf{R}\mathrm{Hom}_R(X, \mathbf{L}\Lambda^{\mathfrak{a}}(Y)) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(X), Y),$$

using adjointness of the derived tensor product functor and the derived homomorphism functor.

(2.7) **Proposition.** *Let $\mathfrak{a}$ be an ideal in R and $X \in \mathsf{D}^{\mathrm{f}}_+(R)$. Then there is an isomorphism in $\mathsf{D}(R)$*

$$\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq X \otimes_R \widehat{R_{\mathfrak{a}}}.$$

In particular, if R is local and $\mathfrak{a} = \mathfrak{m}$, then

$$\mathbf{L}\Lambda^{\mathfrak{m}}(X) \simeq X \otimes_R \widehat{R}.$$

Proof. Note that since $X \in \mathsf{D}^{\mathrm{f}}_{+}(R)$ it admits a K–projective resolution $P \xrightarrow{\simeq} X$ of the form $P = \cdots \to P_n \to \cdots \to P_m \to 0$ where P_k are finite free R–modules. By definition $\mathbf{L}\Lambda^{\mathfrak{a}}(X) = \Lambda^{\mathfrak{a}}(P)$ and since each module P_k in P is finite free we have $\Lambda^{\mathfrak{a}}(P_k) \cong P_k \otimes_R \widehat{R_{\mathfrak{a}}}$ making $\Lambda^{\mathfrak{a}}(P)$ isomorphic to $P \otimes_R \widehat{R_{\mathfrak{a}}}$ which, in turn, is isomorphic to $X \otimes_R \widehat{R_{\mathfrak{a}}}$. □

(2.8) **Finite homology.** Note that from proposition (2.7), we see that if R is $\mathfrak{a}$–adically complete, then $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ is equivalent to the identity functor on the full subcategory $\mathsf{D}^{\mathrm{f}}_{+}(R)$.

(2.9) **Width of complexes.** To facilitate the discussion on the vanishing of local homology we allow ourselves to recall the definition of the *non–local width* of complexes; a notion introduced in [6, sec. 4].

Let $\boldsymbol{a} = a_1, \dots, a_t$ be a set of generators for the ideal $\mathfrak{a}$. For $X \in \mathsf{D}(R)$ the $\mathfrak{a}$*–width* of X is defined by:

$$\operatorname{width}_R(\mathfrak{a}, X) = \inf\,(X \otimes_R \mathrm{K}(\boldsymbol{a})).$$

If R is local, then

$$\operatorname{width}_R(\mathfrak{m}, X) = \operatorname{width}_R X \tag{2.9.1}$$

for $X \in \mathsf{D}_{+}(R)$ (see [6, cor. (4.11)]), and it is easy to see that

$$\operatorname{width}_R(\mathfrak{a}, X) = \inf X \tag{2.9.2}$$

for all $X \in \mathsf{D}^{\mathrm{f}}_{+}(R)$ and all proper ideals $\mathfrak{a}$ (see [6, (4.3)]).

(2.10) **Artinian homology and** $\inf \mathbf{L}\Lambda^{\mathfrak{a}}(-)$**.** Let R be a *complete local* ring, and let $\mathfrak{a}$ be an ideal in R. For any complex $X \in \mathsf{D}(R)$ we let $X^{\vee} = \operatorname{Hom}_R(X, \mathrm{E}_R(k))$; it is called the Matlis dual of X.

As $X \in \mathsf{D}^{\mathrm{art}}_{\mathrm{b}}(R)$ it follows that $X^{\vee\vee} \simeq X$ in $\mathsf{D}(R)$ which enables us to perform the following computation,

$$\mathbf{L}\Lambda^{\mathfrak{a}}(X) \overset{(a)}{\simeq} \mathbf{R}\mathrm{Hom}_R(\mathrm{C}(\boldsymbol{a}) \otimes^{\mathbf{L}}_R X^{\vee}, \mathrm{E}_R(k)) \overset{(b)}{\simeq} \mathbf{R}\Gamma_{\mathfrak{a}}(X^{\vee})^{\vee}.$$

Here (a) by adjointness, and (b) is by (2.5).

Since $\mathrm{E}_R(k)$ is faithfully injective we have $\inf Z^{\vee} = -\sup Z$ for any $Z \in \mathsf{D}(R)$ allowing us to perform the following computation,

$$\begin{aligned} \inf \mathbf{L}\Lambda^{\mathfrak{a}}(X) &= \inf \mathbf{R}\Gamma_{\mathfrak{a}}(X^{\vee})^{\vee} \;= -\sup \mathbf{R}\Gamma_{\mathfrak{a}}(X^{\vee}) \\ &\overset{(c)}{=} \operatorname{depth}_R(\mathfrak{a}, X^{\vee}) \overset{(d)}{=} \operatorname{width}_R(\mathfrak{a}, X), \end{aligned}$$

where (c) by (2.4.1), and (d) is by [6, prop. (4.8)].

In [7] and [23] the interplay between artinian modules and local homology is studied in detail.

(2.11) **Theorem.** *Let $\mathfrak{a}$ be an ideal in R and $X \in \mathsf{D}_{+}(R)$. Then there is an equality*

$$\inf \mathbf{L}\Lambda^{\mathfrak{a}}(X) = \operatorname{width}_R(\mathfrak{a}, X).$$

Proof. First we show that

$$\operatorname{width}_R(\mathfrak{a}, X) = \operatorname{width}_R(\mathfrak{a}, \mathbf{L}\Lambda^{\mathfrak{a}}(X)).$$

To this end, let $\boldsymbol{a} = a_1, \dots, a_t$ be a set of generators of $\mathfrak{a}$. As the homology modules of $\mathrm{K}(\boldsymbol{a})$ are annihilated by $\mathfrak{a}$, it follows that

$$\mathrm{K}(\boldsymbol{a}) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(\mathrm{K}(\boldsymbol{a})) \simeq \mathrm{C}(\boldsymbol{a}) \otimes^{\mathbf{L}}_R \mathrm{K}(\boldsymbol{a}).$$

This enables us to perform the following computation,

$$\begin{aligned}
\mathrm{width}_R(\mathfrak{a}, \mathbf{L}\Lambda^{\mathfrak{a}}(X)) &= \inf\left(\mathrm{K}(\boldsymbol{a}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(\mathrm{C}(\boldsymbol{a}), X)\right) \\
&\overset{(a)}{=} \inf\left(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{a}), \mathrm{C}(\boldsymbol{a})), X)\right) \\
&\overset{(b)}{=} \inf\left(\mathbf{R}\mathrm{Hom}_R(\mathcal{S}^{-t}(\mathrm{K}(\boldsymbol{a}) \otimes^{\mathbf{L}}_R \mathrm{C}(\boldsymbol{a})), X)\right) \\
&= t + \inf\left(\mathrm{Hom}_R(\mathrm{K}(\boldsymbol{a}), X)\right) \\
&\overset{(c)}{=} \mathrm{width}_R(\mathfrak{a}, X).
\end{aligned}$$

Here (a) is due to [2, thm. 1, p.27], and (b) follows from the fact

$$\mathrm{Hom}_R(\mathrm{K}(a_1, \dots, a_t), X) \cong \mathcal{S}^{-t}(X \otimes_R \mathrm{K}(a_1, \dots, a_t)),$$

which is straightforward to check whereas (c) follows from [6, lem. (4.6)].

By $\mathcal{S}^n Z$ we denote the n'th shift (or suspension) of an R–complex Z, that is, the i'th module of $\mathcal{S}^n Z$ is $(\mathcal{S}^n Z)_i = Z_{i-n}$ and the i'th differential is $\partial_i^{\mathcal{S}^n Z} = (-1)^n \partial_{i-n}^Z$.

Thus, it remains to establish

$$\mathrm{width}_R(\mathfrak{a}, \mathbf{L}\Lambda^{\mathfrak{a}}(X)) = \inf \mathbf{L}\Lambda^{\mathfrak{a}}(X).$$

To this end, let $P \xrightarrow{\simeq} X$ be a K–projective resolution of X. As $X \in \mathsf{D}_+(R)$ we can actually take P to be a complex of projectives bounded to the right. By definition $\mathbf{L}\Lambda^{\mathfrak{a}}(X) = \Lambda^{\mathfrak{a}}(P)$ and by [20, prop. 1.4] we are informed that for $l \in \mathbb{Z}$ either $\mathrm{H}_l(\Lambda^{\mathfrak{a}}(P)) \neq \mathfrak{a}\,\mathrm{H}_l(\Lambda^{\mathfrak{a}}(P))$ or $\mathrm{H}_l(\Lambda^{\mathfrak{a}}(P)) = 0$.

The case $\mathrm{H}(\Lambda^{\mathfrak{a}}(P)) = 0$: Here the claim is trivial.

The case $\mathrm{H}(\Lambda^{\mathfrak{a}}(P)) \neq 0$: Let $i = \inf \Lambda^{\mathfrak{a}}(P)$. Then $\mathrm{H}_i(\Lambda^{\mathfrak{a}}(P)) \neq 0$ and thus $(R/\mathfrak{a}) \otimes_R \mathrm{H}_i(\Lambda^{\mathfrak{a}}(P)) \neq 0$ so by (1.3.2) we conclude

$$\mathrm{width}_R(\mathfrak{a}, \mathbf{L}\Lambda^{\mathfrak{a}}(X)) = \inf\left(\mathrm{K}(\boldsymbol{a}) \otimes_R \mathbf{L}\Lambda^{\mathfrak{a}}(X)\right) = \inf \mathbf{L}\Lambda^{\mathfrak{a}}(X).$$

□

(2.12) **Bounds for** $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(-)$**.** Suppose $\boldsymbol{a} = a_1, \dots, a_t$ is a set of generators for the ideal $\mathfrak{a}$. As

$$\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathbf{R}\mathrm{Hom}_R(\mathrm{C}(\boldsymbol{a}), X)$$

we immediately get, using (1.3.1) and (2.4.2), that

$$\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X) \leq \sup X + \dim R.$$

But we can do better than that. By [1, p. 6, cor., part (iii) and (iv)] we have the following isomorphisms in $\mathsf{D}(R)$:

$$\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(\mathbf{R}\Gamma_{\mathfrak{a}}(X)), \tag{2.12.1}$$

$$\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(\mathbf{L}\Lambda^{\mathfrak{a}}(X)). \tag{2.12.2}$$

Thus, combining (1.3.1), (2.4.2), and (2.4.1) with (2.12.1) we obtain

$$\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X) \leq -\mathrm{depth}_R(\mathfrak{a}, X) + \dim R.$$

We end this observation with a conjecture. Suppose that R is *local*, and, for the sake of simplicity, let $X, Y \in \mathsf{D}_{\mathrm{b}}(R)$. Since $\mathfrak{a}$–depth and $\mathfrak{a}$–width are dual notions (see [6, sec. 4]), and since we have the (in)equalities

$$\begin{aligned} \inf \mathbf{L}\Lambda^{\mathfrak{a}}(X) &= \operatorname{width}_R(\mathfrak{a}, X), \\ -\sup \mathbf{R}\Gamma_{\mathfrak{a}}(Y) &= \operatorname{depth}_R(\mathfrak{a}, Y), \\ -\inf \mathbf{R}\Gamma_{\mathfrak{a}}(Y) &\leq \dim_R Y, \end{aligned}$$

it is natural to suspect that a good bound for $\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X)$ should involve an invariant dual to that of $\dim_R Y$.

A natural candidate is the so–called *magnitude* of X, denoted $\operatorname{mag}_R X$, defined by Yassemi in [25] for modules. The magnitude can be extended to complexes in a obvious way. Following this idea one could expect

$$\sup \mathbf{L}\Lambda^{\mathfrak{a}}(X) \leq \operatorname{mag}_R X$$

for any bounded (to the left) complex X.

3. Vanishing Results for Ext– and Tor–Modules

In this section we prove the announced vanishing results for Ext– and Tor– modules. This is done in corollaries (3.7), (3.10), and (3.13). Essential to the proofs are the following two observations:

- If M is an R–module such that $\Lambda^{\mathfrak{m}}(M) \cong M$, then $\mathbf{L}\Lambda^{\mathfrak{m}}(M) \simeq M$ in $\mathsf{D}(R)$.
- If M is an R–module such that $\Gamma_{\mathfrak{m}}(M) \cong M$, then $\mathbf{R}\Gamma_{\mathfrak{m}}(M) \simeq M$ in $\mathsf{D}(R)$.

Combining these observations with the vanishing result for local cohomology (see (2.4.1)), local homology (see theorem (2.11)) and the restricted homological dimensions, reviewed in (1.7), produce the claimed results for Ext– and Tor–modules.

(3.1) **Artinian homology.** Let $\mathfrak{a}$ be an ideal in R and M an R–module. If $\Gamma_{\mathfrak{a}}(M) \cong M$ then M is called an $\mathfrak{a}$–torsion module.

Every such module admits an injective resolution in which each term is an $\mathfrak{a}$–torsion module (see [4, cor. 2.1.6]).

Let $Y \in \mathsf{D}_{-}(R)$, and suppose that for all $l \in \mathbb{Z}$ the homology module $\mathrm{H}_l(Y)$ is a $\mathfrak{a}$–torsion module. Then by a complex version of [4, cor. 2.1.6], Y admits an injective resolution in which each term is an $\mathfrak{a}$–torsion module.

Consequently, if $Y \in \mathsf{D}_{-}(R)$ has artinian homology, the homology modules of Y are $\mathfrak{m}$–torsion modules forcing $\mathbf{R}\Gamma_{\mathfrak{m}}(Y) \simeq Y$.

In other words: Let $\mathsf{D}_{-}^{\mathrm{art}}(R)$ denote the full category of R–complexes bounded to the left with artinian homology. Then $\mathbf{R}\Gamma_{\mathfrak{m}}(-)$ is equivalent to the identity functor on $\mathsf{D}_{-}^{\mathrm{art}}(R)$.

(3.2) **Artinian modules and depth.** Let M be a non–zero artinian R–module. Since $\mathfrak{m} \in \operatorname{Ass}_R M$ we are informed by (1.4.4) that $\operatorname{depth}_R M = 0$.

Consequently, if $X \in \mathsf{D}_{-}^{\mathrm{art}}(R)$ and $\mathrm{H}(X) \neq 0$, then $\mathfrak{m} \in \operatorname{Ass}_R \mathrm{H}_{\sup X}(X)$ and (1.4.4) yields $\operatorname{depth}_R X = -\sup X$.

(3.3) **Complete modules.** Let M be an R–module which is $\mathfrak{a}$–adically complete. It is natural to ask if $\mathbf{L}\Lambda^{\mathfrak{a}}(M)$ is isomorphic to M in $\mathsf{D}(R)$? This is indeed the case. To see this we record the following useful fact due to Simon (see [20, prop. 2.5]):
Every $\mathfrak{a}$–adic complete module M admits a flat resolution consisting of flat $\mathfrak{a}$–adic complete modules.

Moreover, by [20, sec. 5] we can actually use *flat resolutions* of modules to compute $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$, that is, if $F \xrightarrow{\simeq} M$ a flat resolution of M, then $\mathbf{L}\Lambda^{\mathfrak{a}}(M) = \Lambda^{\mathfrak{a}}(F)$.

By the above result we may choose a flat resolution $F \xrightarrow{\simeq} X$ of the $\mathfrak{a}$–adically complete module M of the form $F = \cdots \to F_n \to \cdots \to F_m \to 0$, where $\Lambda^{\mathfrak{a}}(F_i) \cong F_i$ for all $i \in \mathbb{Z}$. Consequently, we have $\mathbf{L}\Lambda^{\mathfrak{a}}(M) = \Lambda^{\mathfrak{a}}(F) \simeq M$ in $\mathsf{D}(R)$.

(3.4) **Derived completeness and torsion.** Let $\mathfrak{a}$ be an ideal in R and $X, Y \in \mathsf{D}_{\mathrm{b}}(R)$. We say that X is a *derived $\mathfrak{a}$–complete* complex if the canonical map,

$$X \longrightarrow \mathbf{L}\Lambda^{\mathfrak{a}}(X),$$

is an isomorphism. Moreover, we say that Y is a *derived $\mathfrak{a}$–torsion* complex if the canonical map,

$$\mathbf{R}\Gamma_{\mathfrak{a}}(Y) \longrightarrow Y,$$

is an isomorphism. See also the beautiful work of Dwyer and Greenlees [8].

Now suppose that $X \in \mathsf{D}^{\mathrm{f}}_{+}(R)$. Then by (2.9.2) we have $\operatorname{width}_R X = \inf X$. Complexes which are derived $\mathfrak{a}$–complete obey the same equality; a consequence of theorem (2.11). To see this just note that if $X \in \mathsf{D}_{+}(R)$ and $X \xrightarrow{\simeq} \mathbf{L}\Lambda^{\mathfrak{a}}(X)$, then

$$\inf X = \inf \mathbf{L}\Lambda^{\mathfrak{a}}(X) = \operatorname{width}_R(\mathfrak{a}, X). \tag{3.4.1}$$

Now suppose that $Y \in \mathsf{D}^{\mathrm{art}}_{-}(R)$. Then by (3.2) we have $\operatorname{depth}_R Y = -\sup Y$. Complexes which are derived $\mathfrak{a}$–torsion obey the same equality; a consequence of (2.4.1). To see this just note that if $Y \in \mathsf{D}_{-}(R)$ and $\mathbf{R}\Gamma_{\mathfrak{a}}(Y) \xrightarrow{\simeq} Y$ then

$$-\sup Y = -\sup \mathbf{R}\Gamma_{\mathfrak{a}}(Y) = \operatorname{depth}_R(\mathfrak{a}, Y). \tag{3.4.2}$$

(3.5) **Lemma.** *Let R be a local ring, and let $X \in \mathsf{D}_{\mathrm{b}}(R)$ be a non–trivial derived $\mathfrak{m}$–complete complex. Then*

$$\operatorname{\mathsf{rid}}_R X = \operatorname{depth} R - \inf X.$$

Proof. Take $T \in \mathsf{P}^{\mathrm{f}}_0(R)$ and consider the number $-\inf \mathbf{R}\mathrm{Hom}_R(T, X)$. We may now perform the following computation,

$$\begin{aligned}
-\inf \mathbf{R}\mathrm{Hom}_R(T, X) &\overset{(a)}{=} -\inf \mathbf{L}\Lambda^{\mathfrak{m}}(\mathbf{R}\mathrm{Hom}_R(T, X)) \\
&\overset{(b)}{=} -\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(T, X) \\
&\overset{(c)}{=} -\operatorname{width}_R X - \operatorname{depth}_R T + \operatorname{depth} R \\
&\overset{(d)}{=} -\operatorname{depth}_R T + \operatorname{depth} R - \inf X.
\end{aligned}$$

Here (a) is by (2.6), (b) by theorem (2.11), (c) by (1.8)(e), and finally (d) is by (3.4.1). By (1.4.3) we have the estimate

$$\sup\{-\operatorname{depth}_R T \mid T \in \mathsf{P}_0^{\mathrm{f}}(R)\} \leq 0.$$

However, let $\boldsymbol{x} = x_1, \ldots, x_n$ be a maximal R–sequence in $\mathfrak{m}$. Then $R/\boldsymbol{x} \in \mathsf{P}_0^{\mathrm{f}}(R)$ and $\operatorname{depth}_R R/\boldsymbol{x} = 0$ forcing $\mathsf{rid}_R X = \operatorname{depth} R - \inf X$. □

(3.6) **Theorem.** *Let R be a local ring, and let $X \in \mathsf{D}_{\mathrm{b}}(R)$ be a non–trivial derived $\mathfrak{m}$–complete complex. Then*

$$\mathsf{Rid}_R X = \mathsf{rid}_R X = \operatorname{depth} R - \inf X.$$

Proof. The second equality is lemma (3.5).

To show the first one we proceed as follows: Take $T \in \mathsf{P}_0(R)$ and consider the number $-\inf(\mathbf{R}\mathrm{Hom}_R(T, X))$. We may now perform the following computation,

$$\begin{aligned}
-\inf(\mathbf{R}\mathrm{Hom}_R(T,X)) &\overset{(a)}{=} -\inf(\mathbf{L}\Lambda^{\mathfrak{m}}(\mathbf{R}\mathrm{Hom}_R(T,X))) \\
&\overset{(b)}{=} -\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(T,X) \\
&\overset{(c)}{=} \sup(T \otimes^{\mathbf{L}}_R k) - \operatorname{width}_R X \\
&\overset{(d)}{=} \sup(T \otimes^{\mathbf{L}}_R k) - \inf X \\
&\overset{(e)}{=} -\operatorname{depth}_R T + \operatorname{depth} R - \inf X \\
&\leq \operatorname{depth} R - \inf X.
\end{aligned}$$

Here (a) is by (2.6), (b) by theorem (2.11), (c) by (1.8)(d), (d) by (3.4.1), and finally (e) is by (1.8)(a)&(b). The computation shows $\mathsf{Rid}_R X \leq \operatorname{depth} R - \inf X$, and the assertion now follows since $\operatorname{depth} R - \inf X = \mathsf{rid}_R X \leq \mathsf{Rid}_R X \leq \operatorname{depth} R - \inf X$, where the first equality follows by lemma (3.5). □

(3.7) **Corollary.** *Let R be a local ring. If M is an non-trivial R–module such that $\Lambda^{\mathfrak{m}}(M) \cong M$ and T is an R–module of finite projective dimension, then*

$$\begin{aligned}
&\mathrm{Ext}^i_R(T,M) = 0 \quad \textit{for} \quad i > \operatorname{depth} R - \operatorname{depth}_R T, \textit{ and} \\
&\mathrm{Ext}^i_R(T,M) \neq 0 \quad \textit{for} \quad i = \operatorname{depth} R - \operatorname{depth}_R T. \quad \square
\end{aligned}$$

(3.8) **Corollary.** *Let R is a complete local ring. If M is a non–trivial finite R–module and T is an R–module of finite projective dimension, then*

$$\begin{aligned}
&\mathrm{Ext}^i_R(T,M) = 0 \quad \textit{for} \quad i > \operatorname{depth} R - \operatorname{depth}_R T, \textit{ and} \\
&\mathrm{Ext}^i_R(T,M) \neq 0 \quad \textit{for} \quad i = \operatorname{depth} R - \operatorname{depth}_R T. \quad \square
\end{aligned}$$

(3.9) **Theorem.** *Let R be a local ring, and let $X \in \mathsf{D}_{\mathrm{b}}(R)$ be a non–trivial derived $\mathfrak{m}$–torsion complex. Then*

$$\mathsf{Rfd}_R X = \mathsf{rfd}_R X = \operatorname{depth} R + \sup X.$$

Proof. Take $T \in \mathsf{F}_0(R)$ and consider the number $\sup(T \otimes^{\mathbf{L}}_R X)$. We may now perform the following computation,

$$\begin{aligned}
\sup(T \otimes^{\mathbf{L}}_R X) &\overset{(a)}{=} \sup(\mathbf{R}\Gamma_{\mathfrak{m}}(T \otimes^{\mathbf{L}}_R X)) \\
&\overset{(b)}{=} -\operatorname{depth}_R(T \otimes^{\mathbf{L}}_R X) \\
&\overset{(c)}{=} -\operatorname{depth}_R T - \operatorname{depth}_R X + \operatorname{depth} R \\
&\overset{(d)}{=} -\operatorname{depth}_R T + \sup X + \operatorname{depth} R \\
&\leq \sup X + \operatorname{depth} R.
\end{aligned}$$

Here (a) is by (2.5), (b) by (2.4.1), (c) by (1.8)(b), and finally (d) is by (3.4.2). The computation shows $\mathsf{Rfd}_R X \leq \operatorname{depth} R + \sup X$, and the assertion now follows since we by (3.4.2) and (1.4.2) may conclude $\operatorname{depth} R + \sup X = \operatorname{depth} R - \operatorname{depth}_R X$. However, from (1.7.2) we see that this difference between depths is less than or equal to $\mathsf{rfd}_R X$, consequently $\operatorname{depth} R + \sup X \leq \mathsf{rfd}_R X \leq \mathsf{Rfd}_R X \leq \operatorname{depth} R + \sup X$. □

(3.10) **Corollary.** *Let R be a local ring. If M is a non–trivial R–module such that $\Gamma_{\mathfrak{m}}(M) \cong M$ and T is an R–module of finite flat dimension, then*

$$\begin{aligned}
&\operatorname{Tor}_i^R(T, M) = 0 \quad \textit{for} \quad i > \operatorname{depth} R - \operatorname{depth}_R T, \textit{ and} \\
&\operatorname{Tor}_i^R(T, M) \neq 0 \quad \textit{for} \quad i = \operatorname{depth} R - \operatorname{depth}_R T. \quad \square
\end{aligned}$$

(3.11) **Lemma.** *Let R be a local ring, and let $X \in \mathsf{D}_{\mathrm{b}}(R)$ be a non–trivial derived $\mathfrak{m}$–torsion complex. Then*

$$\mathsf{rpd}_R X = \operatorname{depth} R + \sup X.$$

Proof. Take $U \in \mathsf{I}^{\mathrm{f}}(R)$ with $\mathrm{H}(U) \neq 0$ and consider the number $\inf U - \inf \mathbf{R}\mathrm{Hom}_R(X, U)$. We may now perform the following computation,

$$\begin{aligned}
\inf U - \inf \mathbf{R}\mathrm{Hom}_R(X, U) &\overset{(a)}{=} \inf U - \inf \mathbf{L}\Lambda^{\mathfrak{m}}(\mathbf{R}\mathrm{Hom}_R(X, U)) \\
&\overset{(b)}{=} \inf U - \operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(X, U) \\
&\overset{(c)}{=} \inf U + \operatorname{depth} R - \operatorname{depth}_R X - \operatorname{width}_R U \\
&\overset{(d)}{=} \inf U + \operatorname{depth} R - \operatorname{depth}_R X - \inf U \\
&= \operatorname{depth} R - \operatorname{depth}_R X \\
&\overset{(e)}{=} \operatorname{depth} R + \sup X.
\end{aligned}$$

Here (a) is by (2.6), (b) by theorem (2.11), (c) by (1.8)(c), (d) by (1.5.1), and finally (e) is by (3.4.2) and (1.4.2). The computation shows that $\mathsf{rpd}_R X = \operatorname{depth} R + \sup X$. □

(3.12) **Theorem.** *Let R be a local ring, and let $X \in \mathsf{D}_{\mathrm{b}}(R)$ be a non–trivial derived $\mathfrak{m}$–torsion complex. Then*

$$\mathsf{Rpd}_R X = \mathsf{rpd}_R X = \operatorname{depth} R + \sup X.$$

Proof. The second equality is lemma (3.11).

To show the first one we proceed as follows: Take $T \in \mathsf{I}_0(R)$ and consider the number $-\inf \mathbf{R}\mathrm{Hom}_R(X,T)$. We may now perform the following computation,

$$\begin{aligned}
-\inf \mathbf{R}\mathrm{Hom}_R(X,T) &\overset{(a)}{=} -\inf \mathbf{L}\Lambda^{\mathfrak{m}}(\mathbf{R}\mathrm{Hom}_R(X,T)) \\
&\overset{(b)}{=} -\operatorname{width}_R \mathbf{R}\mathrm{Hom}_R(X,T) \\
&\overset{(c)}{=} \operatorname{depth} R - \operatorname{depth}_R X - \operatorname{width}_R T \\
&\overset{(d)}{=} \operatorname{depth} R + \sup X - \operatorname{width}_R T \\
&\leq \operatorname{depth} R + \sup X.
\end{aligned}$$

Here (a) is by (2.6), (b) is by theorem (2.11), (c) is by (1.8)(c), and finally (d) is by (3.4.2) and (1.4.2). The computation shows $\mathsf{Rpd}_R X \leq \operatorname{depth} R + \sup X$. Now theorem (3.9) yields $\operatorname{depth} R + \sup X = \mathsf{Rfd}_R X$, making us able to perform the following computation,

$$\operatorname{depth} R + \sup X \overset{(e)}{=} \mathsf{rpd}_R X \overset{(f)}{=} \mathsf{Rfd}_R X \overset{(g)}{\leq} \mathsf{Rpd}_R X \leq \operatorname{depth} R + \sup X,$$

where (e) is by lemma (3.11), and (f) is by theorem (3.9), and finally (g) is by (1.7.1). □

(3.13) **Corollary.** *Let R be a local ring. If M is a non–trivial R–module such that $\Gamma_{\mathfrak{m}}(M) \cong M$ and T is an R–module of finite injective dimension, then*

$$\begin{aligned}
&\mathrm{Ext}^i_R(M,T) = 0 \quad \textit{for} \quad i > \operatorname{depth} R - \operatorname{width}_R T, \textit{ and} \\
&\mathrm{Ext}^i_R(M,T) \neq 0 \quad \textit{for} \quad i = \operatorname{depth} R - \operatorname{width}_R T. \quad \square
\end{aligned}$$

Acknowledgments

This paper will be included in the author's Ph.D. thesis. The author expresses his deep gratitude to his advisor, Professor Hans-Bjørn Foxby. The author also wish to thank Professors Peter Schenzel, Peter Jørgensen, Srikanth Iyengar and Lars Winther Christensen for fruitful discussions concerning this paper.

References

1. Leovigildo Alonso Tarrío, Ana Jeremías López, and Joseph Lipman, *Local homology and cohomology on schemes*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 1, 1–39. MR **98d:**14028
2. Dmitri Apassov, *Complexes and differential graded modules*, Ph.D. thesis, Lund University, 1999.
3. Hyman Bass, *Injective dimension in Noetherian rings*, Trans. Amer. Math. Soc. **102** (1962), 18–29.
4. M. P. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge University Press, Cambridge, 1998. MR **99h:**13020
5. Lars Winther Christensen, *Sequences for complexes*, Math. Scandinavica **89** (2001), 1–20.

6. Lars Winther Christensen, Hans-Bjørn Foxby, and Anders Frankild, *Restricted Homological Dimensions and Cohen–Macaulayness*, J. Algebra **251** (2002), 479–502.
7. Nguyen Tu Cuong and Tran Tuan Nam, *The I-adic completion and local homology for Artinian modules*, Math. Proc. Cambridge Philos. Soc. **131** (2001), no. 1, 61–72. MR **2002f:**13035
8. W. G. Dwyer and J. P. C. Greenlees, *Complete modules and torsion modules*, Amer. J. Math. **124** (2002), no. 1, 199–220. MR 1 879 003
9. Hans-Bjørn Foxby, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), no. 1, 5–19.
10. ———, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), no. 2, 149–172.
11. ———, *Homological dimensions of complexes of modules*, Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin, 32ème année (Paris, 1979), Springer, Berlin, 1980, pp. 360–368.
12. ———, *A homological theory for complexes of modules*, Preprint Series 1981, nos. 19a&b, Matematisk Institut, Københavns Universitet, 1981.
13. S. I. Gelfand and Yu. I. Manin, *Homological algebra*, Springer-Verlag, Berlin, 1999, Translated from the 1989 Russian original by the authors, Reprint of the original English edition from the series Encyclopaedia of Mathematical Sciences [*Algebra, V*, Encyclopaedia Math. Sci., 38, Springer, Berlin, 1994; MR 95g:18007]. MR **2000b:**18016
14. J. P. C. Greenlees and J. P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), no. 2, 438–453. MR **93h:**13009
15. Srikanth Iyengar, *Depth for complexes, and intersection theorems*, Math. Z. **230** (1999), no. 3, 545–567.
16. Eben Matlis, *The Koszul complex and duality*, Comm. Algebra **1** (1974), 87–144.
17. ———, *The higher properties of R-sequences*, J. Algebra **50** (1978), no. 1, 77–112.
18. Michel Raynaud and Laurent Gruson, *Critères de platitude et de projectivité. Techniques de "platification" d'un module*, Invent. Math. **13** (1971), 1–89.
19. Peter Schenzel, *Proregular sequences, local cohomology, and completions*, manuscript, 1999.
20. Anne-Marie Simon, *Some homological properties of complete modules*, Math. Proc. Camb. Phil. Soc **108** (1990), 231–246.
21. ———, *Adic-completion and some dual homological results*, Publ. Mat. **36** (1992), no. 2B, 965–979 (1993). MR **94a:**13013
22. Nicolas Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154.
23. Zhong Ming Tang, *Local homology theory for Artinian modules*, Comm. Algebra **22** (1994), no. 5, 1675–1684. MR **94m:**13024
24. Charles A. Weibel, *An introduction to homological algebra*, Cambridge University Press, Cambridge, 1994.
25. Siamak Yassemi, *Magnitude of modules*, Comm. Algebra **23** (1995), no. 11, 3993–4008.
26. ———, *Width of complexes of modules*, Acta Math. Vietnam. **23** (1998), no. 1, 161–169.

Matematisk afdeling, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark

E-mail address: `frankild@math.ku.dk`

Journal of Pure and Applied Algebra
vol. **174**, 135-147 (2002)

FOXBY EQUIVALENCE, COMPLETE MODULES, AND TORSION MODULES

ANDERS FRANKILD AND PETER JØRGENSEN

Section 1 of this manuscript takes its idea from classical Foxby equivalence for noetherian, local, commutative rings (see [3]), and generalizes it to an equivalence theory for derived categories over Differential Graded Algebras (henceforth abbreviated DGAs). Section 2 shows some simple properties of the new equivalence, and section 3 shows that both classical Foxby equivalence, and the Morita equivalence for complete modules and torsion modules developed by Dwyer and Greenlees in [8] arise as special cases. It also shows that a new instance of our theory which one can reasonably call "Matlis equivalence" gives a new characterization of Gorenstein rings.

1. Generalized Foxby equivalence

This section starts with a very general result in theorem (1.1), and then immediately proceeds to look at DGAs. In (1.5), we give the equivalence for derived categories over DGAs mentioned in the introduction.

(1.1) **Theorem.** *Consider categories C, D and an adjoint pair of functors (F, G),*

$$\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}.$$

Denote unit and counit of the adjunction by η and ϵ. Define full subcategories of C and D,

$$\mathcal{A} = \{A \in \mathsf{C} \mid \eta_A \text{ is an isomorphism}\},$$
$$\mathcal{B} = \{B \in \mathsf{D} \mid \epsilon_B \text{ is an isomorphism}\}.$$

Then the functors F and G restrict to a pair of quasi-inverse equivalences of categories,

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}.$$

Proof. This is an easy exercise in adjoint functors. □

1991 *Mathematics Subject Classification.* 16E45, 18E30, 13D05.

Key words and phrases. Adjoint functors, Foxby equivalence, Auslander class, Bass class, Morita equivalence, complete module, torsion module, Gorenstein ring.

(1.2) **Definition (Auslander and Bass classes).** In the situation of theorem (1.1), we call $\mathcal{A}$ the Auslander class, and $\mathcal{B}$ the Bass class. The name "Auslander class" (strictly speaking, "Auslander category") is due to [3], while "Bass class" is due to [6] and [7].

(1.3) **Setup.** In the rest of this section and the next, R and S are DGAs, and ${}_{R,S}M$ is a DG-R-left-S-left-module. (We use subscripts on DG-modules to indicate their structures, as is custom in non-commutative ring theory.)

(1.4) **Tensor and Hom over a DGA.** ¿From [10, sec. 6.1] we know that there is an adjoint pair of functors between homotopy categories of DG-modules,

$$\mathsf{K}(R^{\mathrm{opp}}) \underset{\mathrm{Hom}_S(M,-)}{\overset{-\otimes_R M}{\rightleftarrows}} \mathsf{K}(S). \tag{1.4.1}$$

Here R^{opp} is the opposite DGA of R, whose multiplication is given by $s \cdot_{\mathrm{opp}} r = (-1)^{|r||s|} rs$, where $|r|$ denotes the degree of the homogeneous element r. We identify DG-R^{opp}-left-modules with DG-R-right-modules, so $\mathsf{K}(R^{\mathrm{opp}})$ is identified with the homotopy category of DG-R-right-modules.

The unit η of the adjunction (1.4.1) is given by

$$\mathrm{id}_{\mathsf{K}(R^{\mathrm{opp}})}(L) \xrightarrow{\eta_L} \mathrm{Hom}_S(M, L \otimes_R M), \quad \eta_L(\ell) = (m \mapsto \ell \otimes m),$$

and the counit ϵ is given by

$$\mathrm{Hom}_S(M,N) \otimes_R M \xrightarrow{\epsilon_N} \mathrm{id}_{\mathsf{K}(S)}(N), \quad \epsilon_N(\mu \otimes m) = \mu(m).$$

Since all modules have K-projective and K-injective resolutions by [10, secs. 3.1 and 3.2], we can get $- \overset{\mathrm{L}}{\otimes}_R M$ from $- \otimes_R M$ by using a K-projective resolution in the first variable, and we can get $\mathrm{RHom}_S(M,-)$ from $\mathrm{Hom}_S(M,-)$ by using a K-injective resolution in the second variable. The adjointness described above is inherited by the derived functors in a straightforward way.

(1.5) **Generalized Foxby equivalence.** By (1.4) we have an adjoint pair of derived functors between derived categories of DG-modules,

$$\mathsf{D}(R^{\mathrm{opp}}) \underset{\mathrm{RHom}_S(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathsf{D}(S).$$

Theorem (1.1) now says: Denoting unit and counit of the adjunction by η and ϵ, there are Auslander and Bass classes,

$$\begin{aligned} \mathcal{A}_M(R^{\mathrm{opp}}) &= \{L \in \mathsf{D}(R^{\mathrm{opp}}) \mid \eta_L \text{ is an isomorphism}\}, \\ \mathcal{B}_M(S) &= \{N \in \mathsf{D}(S) \mid \epsilon_N \text{ is an isomorphism}\}, \end{aligned}$$

and the functors $- \overset{\mathrm{L}}{\otimes}_R M$ and $\mathrm{RHom}_S(M,-)$ restrict to a pair of quasi-inverse equivalences of categories,

$$\mathcal{A}_M(R^{\mathrm{opp}}) \underset{\mathrm{RHom}_S(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathcal{B}_M(S).$$

2. Size of Auslander and Bass classes

This section continues to work under setup (1.3).

It starts in (2.1) by recalling, among other things, the so-called evaluation morphisms ω and θ from [2], and in lemma (2.2) rewrites unit and counit of the adjoint pair $(- \overset{\mathrm{L}}{\otimes}_R M, \mathrm{RHom}_S(M,-))$ in terms of ω and θ. This is used in theorem (2.4) which under certain conditions characterizes objects in the Auslander class by ω being an isomorphism, and objects in the Bass class by θ being an isomorphism. Lemma (2.2) and theorem (2.4) follow ideas from [3, pf. of thm. (3.2)]. The section ends by deriving corollaries (2.5) and (2.7) which state under appropriate conditions that the Auslander and Bass classes contain "many" DG-modules.

(2.1) **Some morphisms.** In [2, 4.3] two so-called evaluation morphisms are considered for complexes of modules over rings. The same method gives morphisms for DG-modules over DGAs as follows: If

$${}_TF_R,\ {}_{U,S}A,\ {}_{R,S}B$$

are DG-modules with structures as indicated, then there is a natural morphism of DG-T-left-U-right-modules

$${}_TF_R \otimes_R \mathrm{Hom}_S({}_{U,S}A, {}_{R,S}B) \xrightarrow{\omega} \mathrm{Hom}_S({}_{U,S}A, {}_TF_R \otimes_R {}_{R,S}B)$$

given by

$$(\omega(f \otimes \alpha))(a) = f \otimes \alpha(a).$$

Moreover, if F can be resolved by a DG-T-left-R-right-module which is K-flat over R, and A can be resolved by a DG-U-left-S-left-module which is K-projective over S, then ω induces a natural morphism of derived functors,

$${}_TF_R \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_S({}_{U,S}A, {}_{R,S}B) \xrightarrow{\omega} \mathrm{RHom}_S({}_{U,S}A, {}_TF_R \overset{\mathrm{L}}{\otimes}_R {}_{R,S}B).$$

Note that it is not necessary that F and A have structures over T and U. That is, omitting T or U or both, there are still morphisms given by the same prescriptions.

And if

$${}_{R,T}A,\ {}_{R,S}B,\ {}_SI_U$$

are DG-modules with structures as indicated, then there is a natural morphism of DG-T-left-U-right-modules

$$\mathrm{Hom}_S({}_{R,S}B, {}_SI_U) \otimes_R {}_{R,T}A \xrightarrow{\theta} \mathrm{Hom}_S(\mathrm{Hom}_R({}_{R,T}A, {}_{R,S}B), {}_SI_U)$$

given by

$$(\theta(\beta \otimes a))(\alpha) = (-1)^{|a||\alpha|}\beta\alpha(a).$$

And if A can be resolved by a DG-R-left-T-left-module which is K-projective over R, and I can be resolved by a DG-S-left-U-right-module which is K-injective over S, then θ induces a natural morphism of derived functors,

$$\mathrm{RHom}_S({}_{R,S}B, {}_SI_U) \overset{\mathrm{L}}{\otimes}_R {}_{R,T}A \xrightarrow{\theta} \mathrm{RHom}_S(\mathrm{RHom}_R({}_{R,T}A, {}_{R,S}B), {}_SI_U).$$

Note that again, T or U or both could be omitted in both morphisms.

We also need some morphisms which sometimes exist in the situation of setup (1.3): Suppose that M can be resolved by a DG-R-left-S-left-module P which is K-projective over S. Then we have $\mathrm{Hom}_S(P, P) \cong \mathrm{RHom}_S(M, M)$, and the morphism $R \longrightarrow \mathrm{Hom}_S(P, P)$ given by $r \mapsto (p \mapsto rp)$ gives a canonical morphism in the derived category of DG-R-left-R-right-modules,

$$R \xrightarrow{\rho} \mathrm{RHom}_S(M, M).$$

Similarly, if M can be resolved by a DG-R-left-S-left-module which is K-projective over R, then we get a canonical morphism in the derived category of DG-S-left-S-right-modules,

$$S \xrightarrow{\sigma} \mathrm{RHom}_R(M, M).$$

The following lemma is an abstraction of part of [3, proof of thm. (3.2)].

(2.2) **Lemma.** (1): *Suppose that M can be resolved by a DG-R-left-S-left-module which is K-projective over S. For any DG-R-right-module, L, there is a commutative diagram,*

$$\begin{tikzcd}
L \arrow[r, "\varphi"] \arrow[drr, "\eta_L"'] & L \overset{\mathrm{L}}{\otimes}_R R \arrow[r, "1_L \overset{\mathrm{L}}{\otimes}_R \rho"] & L \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_S(M, M) \arrow[d, "\omega"] \\
 & & \mathrm{RHom}_S(M, L \overset{\mathrm{L}}{\otimes}_R M),
\end{tikzcd}$$

where φ is the canonical isomorphism, where η_L is the unit of the adjoint pair $(- \overset{\mathrm{L}}{\otimes}_R M, \mathrm{RHom}_S(M, -))$ evaluated on L, and where ω comes from (2.1).

(2): *Suppose that M can be resolved by a DG-R-left-S-left-module which is K-projective over R. For any DG-S-left-module, N,* *there is a commutative diagram,*

$$\begin{tikzcd}
\mathrm{RHom}_S(M,N) \overset{\mathrm{L}}{\otimes}_R M \arrow[d, "\theta"'] \arrow[drr, "\epsilon_N"] & & \\
\mathrm{RHom}_S(\mathrm{RHom}_R(M,M),N) \arrow[r, "\mathrm{RHom}_S(\sigma,1_N)"'] & \mathrm{RHom}_S(S,N) \arrow[r, "\psi"'] & N,
\end{tikzcd}$$

where ψ *is the canonical isomorphism, where* ϵ_N *is the counit of the adjoint pair* $(- \overset{\mathrm{L}}{\otimes}_R M, \mathrm{RHom}_S(M,-))$ *evaluated on* N*, and where* θ *comes from* (2.1).

Proof. The proofs of (1) and (2) are similar, so we only show (1).

Replace L by a K-flat resolution (this is always possible, e.g. by replacing L by a K-projective resolution), and replace M by a resolution which is a DG-R-left-S-left-module that is K-projective over S. This enables us to write Hom_S and $\otimes_R$ rather than RHom_S and $\overset{\mathrm{L}}{\otimes}_R$.

Now let ℓ be in L, and consider the composition of morphisms appearing in the lemma, evaluated on ℓ:

$$(\omega \circ (1_L \otimes \rho) \circ \varphi)(\ell) = (\omega \circ (1_L \otimes \rho))(\ell \otimes 1_R) = \omega(\ell \otimes \mathrm{id}_M) = (m \mapsto \ell \otimes m).$$

This is indeed $\eta_L(\ell)$, as one sees in (1.4). □

(2.3) **Conditions.** Here are two conditions which can be imposed on M:

(1): M can be resolved by a DG-R-left-S-left-module which is K-projective over S, and the canonical map $R \xrightarrow{\rho} \mathrm{RHom}_S(M,M)$ is an isomorphism.

(2): M can be resolved by a DG-R-left-S-left-module which is K-projective over R, and the canonical map $S \xrightarrow{\sigma} \mathrm{RHom}_R(M,M)$ is an isomorphism.

The following theorem is proved essentially in the same way as [3, thm. (3.2)].

(2.4) **Theorem.** (1): *Suppose that* M *satisfies condition* (2.3)(1). *Then*

$$\mathcal{A}_M(R^{\mathrm{opp}}) = \left\{ L \in \mathsf{D}(R^{\mathrm{opp}}) \;\middle|\; \begin{array}{l} L \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_S(M,M) \xrightarrow{\omega} \\ \mathrm{RHom}_S(M, L \overset{\mathrm{L}}{\otimes}_R M) \\ \textit{is an isomorphism} \end{array} \right\}.$$

(2): *Suppose that* M *satisfies condition* (2.3)(2). *Then*

$$\mathcal{B}_M(S) = \left\{ N \in \mathsf{D}(S) \;\middle|\; \begin{array}{l} \mathrm{RHom}_S(M,N) \overset{\mathrm{L}}{\otimes}_R M \xrightarrow{\theta} \\ \mathrm{RHom}_S(\mathrm{RHom}_R(M,M),N) \\ \textit{is an isomorphism} \end{array} \right\}.$$

Proof. Again, the proofs of (1) and (2) are similar, so we only show (1).

Condition (2.3)(1) says that M can be resolved by a DG-R-left-S-left-module which is K-projective over S, so we are in the situation of lemma (2.2)(1). The composition $\omega \circ (1_L \overset{\mathrm{L}}{\otimes}_R \rho) \circ \varphi$ in the lemma equals η_L, so by (1.5) the DG-module L is in $\mathcal{A}_M(R^{\mathrm{opp}})$ precisely when $\omega \circ (1_L \overset{\mathrm{L}}{\otimes}_R \rho) \circ \varphi$ is an isomorphism.

But since $R \xrightarrow{\rho} \mathrm{RHom}_S(M, M)$ is an isomorphism by condition (2.3)(1), both maps φ and $1_L \overset{\mathrm{L}}{\otimes}_R \rho$ are isomorphisms. Hence $\omega \circ (1_L \overset{\mathrm{L}}{\otimes}_R \rho) \circ \varphi$ is an isomorphism precisely when ω is. □

In the following corollary, $(-)^\natural$ denotes the functor which forgets differentials. It sends DGAs and DG-modules to graded algebras and graded modules.

(2.5) **Corollary.** (1): *Suppose that M satisfies condition* (2.3)(1). *Suppose moreover that when we forget the R-structure on M, we can resolve M by a K-projective DG-S-left-module, A, so that $({}_SA)^\natural$ is a direct summand in a finite coproduct of shifts of $S^\natural$.*

Then the Auslander class $\mathcal{A}_M(R^{\mathrm{opp}})$ is all of $\mathsf{D}(R^{\mathrm{opp}})$.

(2): *Suppose that M satisfies condition* (2.3)(2). *Suppose moreover that when we forget the S-structure on M, we can resolve M by a K-projective DG-R-left-module, B, so that $({}_RB)^\natural$ is a direct summand in a finite coproduct of shifts of $R^\natural$.*

Then the Bass class $\mathcal{B}_M(S)$ is all of $\mathsf{D}(S)$.

Proof. Again the proofs of (1) and (2) are similar, so we only show (1).

Theorem (2.4)(1) implies that to prove the corollary's claim that $\mathcal{A}_M(R^{\mathrm{opp}})$ is all of $\mathsf{D}(R^{\mathrm{opp}})$, we must show that

$$L \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_S(M, M) \xrightarrow{\omega} \mathrm{RHom}_S(M, L \overset{\mathrm{L}}{\otimes}_R M)$$

is an isomorphism for any L.

Now, to see whether ω is an isomorphism, there is no need to remember the R-structure on the M's appearing in the first variable of the RHom's. Hence we can use the DG-S-left-module ${}_SA$ which is a K-projective resolution of ${}_SM$ to compute the two RHom's. But when ${}_SA$ has the special form required in the corollary, ω is an isomorphism by [1, sec. 1, thm. 2]. (Note that [1] actually requires $A^\natural$ itself to be a finite coproduct of shifts of $S^\natural$, but gives a proof which also applies to direct summands.) □

(2.6) **Definition.** If Q is a DGA, then we define two classes of DG-Q-left-modules by

$$\mathcal{F}(Q) = \left\{ L \in \mathsf{D}(Q) \;\middle|\; \begin{array}{l} L \text{ is isomorphic in } \mathsf{D}(Q) \text{ to a} \\ K\text{-flat left-bounded DG-module} \end{array} \right\}$$

and

$$\mathcal{I}(Q) = \left\{ N \in \mathsf{D}(Q) \;\middle|\; \begin{array}{l} N \text{ is isomorphic in } \mathsf{D}(Q) \text{ to a} \\ K\text{-injective right-bounded DG-module} \end{array} \right\}.$$

In the following, $\Sigma^i X$ denotes the i'th suspension of the DG-module X, so $(\Sigma^i X)_j = X_{j-i}$.

(2.7) **Corollary.** (1): *Suppose that M satisfies condition* (2.3)(1). *Suppose moreover the following:*

- *R and S are non-negatively graded.*
- *$\mathrm{H}_0 S$ is left-noetherian, and each $\mathrm{H}_i S$ is finitely generated from the left over $\mathrm{H}_0 S$.*
- *$\mathrm{H}M$ is bounded, and each $\mathrm{H}_i M$ is finitely generated over $\mathrm{H}_0 S$.*

Then

$$\mathcal{F}(R^{\mathrm{opp}}) \subseteq \mathcal{A}_M(R^{\mathrm{opp}}).$$

(2): *Suppose that M satisfies condition* (2.3)(2). *Suppose moreover the following:*

- *R and S are non-negatively graded.*
- *$\mathrm{H}_0 R$ is left-noetherian, and each $\mathrm{H}_i R$ is finitely generated from the left over $\mathrm{H}_0 R$.*
- *$\mathrm{H}M$ is bounded, and each $\mathrm{H}_i M$ is finitely generated over $\mathrm{H}_0 R$.*

Then

$$\mathcal{I}(S) \subseteq \mathcal{B}_M(S).$$

Proof. Again, the proofs of (1) and (2) are similar, so we only show (1).

Theorem (2.4) implies that to prove the corollary's claim that $\mathcal{F}(R^{\mathrm{opp}})$ is contained in $\mathcal{A}_M(R^{\mathrm{opp}})$, we must show that

$$L \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_S(M, M) \xrightarrow{\omega} \mathrm{RHom}_S(M, L \overset{\mathrm{L}}{\otimes}_R M)$$

is an isomorphism when L is in $\mathcal{F}(R^{\mathrm{opp}})$.

Now, to see whether ω is an isomorphism, there is no need to remember the R-structure on the M's appearing in the first variable of the RHom's. Hence we can replace these M's by any DG-S-left-module ${}_S P$ which is isomorphic to ${}_S M$ in $\mathsf{D}(S)$. Inserting the bulleted assumptions on S and M into [4, thm. 10.1.5] gives that we can choose an ${}_S P$ which is semi-free and in particular K-projective, and has

$$({}_S P)^\natural = \bigoplus_{j \geq i} \Sigma^j (S^\natural)^{\gamma_j}$$

for certain finite numbers i and γ_j.

We can also replace the M's appearing in the second variable of the RHom's by any quasi-isomorphic DG-R-left-S-left-module ${}_{R,S}B$. And B can be chosen left-bounded: Since R and S are both non-negatively graded, it makes sense to truncate DG-R-left-S-left-modules, and since $\mathrm{H}M$ is bounded, and so in particular left-bounded, we can truncate M to the left to get a left-bounded B.

Finally, when L_R is in $\mathcal{F}(R^{\mathrm{opp}})$, we can replace L_R by a K-flat left-bounded DG-R-right-module F_R which is isomorphic to L_R in $\mathsf{D}(R^{\mathrm{opp}})$.

So what we need to see is in fact that

$$F \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_S(P, B) \xrightarrow{\omega} \mathrm{RHom}_S(P, F \overset{\mathrm{L}}{\otimes}_R B)$$

is an isomorphism. But P is K-projective and F is K-flat, so this is represented by

$$F \otimes_R \mathrm{Hom}_S(P, B) \xrightarrow{\omega} \mathrm{Hom}_S(P, F \otimes_R B).$$

And P, B, and F being as they are, this is an isomorphism by [1, sec. 1, thm. 2]. $\square$

3. Applications of the theory

This section describes three concrete instances of the theory from section 1:

In (3.1) it is shown that our theory contains classical Foxby equivalence over noetherian, local, commutative rings, as known from [3], and corollary (2.7) is used to recover the previously known results that the Auslander class contains all bounded complexes of flat modules, while the Bass class contains all bounded complexes of injective modules.

In (3.2) it is shown that our theory contains the Morita equivalence for complete modules and torsion modules developed by Dwyer and Greenless in [8], and corollary (2.5) is used to recover the previously known result that a certain Auslander class contains all complexes.

Finally, in (3.3) to (3.5), we consider a new instance of our theory, where the dualizing complex from classical Foxby equivalence is replaced with $\mathrm{E}(k)$, the injective hull of the residue class field k. This theory turns out to be able to detect Gorensteinness in the same way as classical Foxby equivalence, namely by k being in the Auslander and Bass classes (see [6, (3.1.12) and (3.2.10)]).

(3.1) **Classical Foxby equivalence.** Classical Foxby equivalence in the setup of [3, sec. 3] is a special case of the theory of section 1: Let R be a noetherian commutative ring, viewed as a DGA concentrated in degree zero, and let S equal R. Let M be a dualizing complex over R, that is, M is a bounded complex of injective modules with finitely generated homology, so that the canonical morphism $R \longrightarrow \mathrm{RHom}_R(M, M)$ is an isomorphism. Clearly, M is a DG-R-left-S-left-module.

So (1.5) applies, and since R^{opp} equals R, the adjoint pair of (1.5) is

$$\mathsf{D}(R) \underset{\mathrm{RHom}_R(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathsf{D}(R),$$

and this is simply the pair of functors from the classical Foxby equivalence theorem, [3, thm. (3.2)]. Also, our Auslander and Bass classes,

$$\mathcal{A}_M(R) = \{L \in \mathsf{D}(R) \mid \eta_L \text{ is an isomorphism}\},$$
$$\mathcal{B}_M(R) = \{N \in \mathsf{D}(R) \mid \epsilon_N \text{ is an isomorphism}\},$$

are simply the classes $\mathbf{A}(R)$ and $\mathbf{B}(R)$ of [3, def. (3.1)], except that we have avoided the (unnecessary) boundedness conditions in [3]. Our equivalence result (1.5) essentially specializes to the equivalence theorem [3, thm. (3.2)].

Moreover, the conditions of corollary (2.7)(1) hold. First, condition (2.3)(1) holds: Since R is a noetherian, commutative ring and S equals R, we can resolve M by a DG-R-left-S-left-module which is K-projective over S simply by resolving it by a K-projective resolution of M as an R-complex. And we have that the canonical morphism $R \xrightarrow{\rho} \mathrm{RHom}_S(M, M)$ is a quasi-isomorphism by assumption on M. Secondly, the three itemized requirements in corollary (2.7)(1) are immediate by the assumptions on R, S, and M.

So corollary (2.7)(1) says that $\mathcal{A}_M(R)$ contains $\mathcal{F}(R)$. In particular, $\mathcal{A}_M(R)$ contains all bounded complexes of flat modules.

Symmetrically, corollary (2.7)(2) says that $\mathcal{B}_M(R)$ contains $\mathcal{I}(R)$. In particular, $\mathcal{B}_M(R)$ contains all bounded complexes of injective modules.

Note that the above way of viewing classical Foxby equivalence also applies to the more general Foxby equivalence theory with semi-dualizing complexes constructed in [7, sec. 4].

(3.2) **Dwyer and Greenlees' theory.** Dwyer and Greenless' Morita equivalence theory from [8] which generalizes Rickard's theory from [12] is a special case of the theory of section 1: Let S be any ring, viewed as a DGA concentrated in degree zero, and let M be a perfect complex of S-left-modules, that is, a bounded complex of finitely generated projective S-left-modules. Set R equal to $\mathrm{Hom}_S(M, M)$. It is not difficult to check that this is a DGA, that M acquires the structure of DG-R-left-module, and that this structure is compatible with the S-structure of M, so that M is in fact a DG-R-left-S-left-module, ${}_{R,S}M$.

So (1.5) applies, and we get quasi-inverse equivalences between the Auslander and Bass classes,

$$\mathcal{A}_M(R^{\mathrm{opp}}) \underset{\mathrm{RHom}_S(M,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_R M}{\rightleftarrows}} \mathcal{B}_M(S).$$

Moreover, the conditions of corollary (2.5)(1) hold. First, condition (2.3)(1) holds: ${}_{R,S}M$ is a resolution of itself which is K-projective over S, because we have started with an M which is perfect over S. Also, the canonical morphism $R \xrightarrow{\rho} \mathrm{RHom}_S(M, M)$ is an isomorphism since we have in effect defined R to be $\mathrm{RHom}_S(M, M)$. Secondly, when we forget the R-structure on M, we are left with ${}_SM$, that is, the original M over S, which is perfect. Hence $({}_SM)^\natural$ is clearly a direct summand in a finite coproduct of shifts of $S^\natural$.

So corollary (2.5)(1) says $\mathcal{A}_M(R^{\mathrm{opp}}) = \mathsf{D}(R^{\mathrm{opp}})$.

But then, the above diagram is identical to the right half of the following diagram from Dwyer and Greenlees' Morita theorem, [8, thm. 2.1]:

$$\begin{tikzcd} \mathbf{A}_{\mathrm{comp}} \arrow[r, shift left, "E"] & \mathsf{mod}\text{-}\mathcal{E} \arrow[l, shift left, "C"] \arrow[r, shift left, "T"] & \mathbf{A}_{\mathrm{tors}} \arrow[l, shift left, "E"] \end{tikzcd}$$

([8] denotes our R by $\mathcal{E}$, and our $\mathsf{D}(R^{\mathrm{opp}})$ by $\mathsf{mod}\text{-}\mathcal{E}$). This can be seen by checking:

- Our functors $- \overset{\mathrm{L}}{\otimes}_R M$ and $\mathrm{RHom}_S(M, -)$ are the same as the functors T and E from [8] (this is trivial).
- The Bass class $\mathcal{B}_M(S)$ equals $\mathbf{A}_{\mathrm{tors}}$ (this is done in [8, thm. 2.1]).

Note that by replacing M by $\mathrm{Hom}_S(M, S)$, our theory can be specialized to the other half of [8, thm. 2.1].

(3.3) **The Auslander and Bass classes for** $\mathrm{E}(k)$**.** Let R be a noetherian, local, commutative ring, with maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$, and let $\mathrm{E}(k)$ denote the injective hull of k. We want to consider our theory with $S = R$ and $M = \mathrm{E}(k)$. In this setup it turns out that the corresponding Auslander and Bass classes contain k precisely when R is Gorenstein.

Recall that the same statement is true for the Auslander and Bass classes of classical Foxby equivalence where M is the dualizing complex D (see [6, (3.1.12) and (3.2.10)]). However, not all commutative, local, noetherian rings admit a dualizing complex.

Note that, since the duality theory involving the functor $\mathrm{RHom}_R(-, \mathrm{E}(k))$ is just classical Matlis duality, it seems reasonable that one should call the theory treated in this and the next two paragraphs "Matlis equivalence".

(3.4) **Lemma.** *Let R be as in* (3.3)*. Then the following statements are equivalent:*

(1): *R is Gorenstein.*
(2): $\mathrm{RHom}_R(\mathrm{E}(k), k) \cong \Sigma^{-d}k$ *for some d.*
(3): $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong \Sigma^{d}k$ *for some d.*

If the equivalent statements hold, then $d = \dim R$.

Proof. Let $(-)^\vee = \mathrm{RHom}_R(-, \mathrm{E}(k))$ denote the Matlis duality functor.

It is not difficult to see that each of the numbered statements is equivalent to the same statement for $\widehat{R}$, the completion of R in the $\mathfrak{m}$-adic topology. For this, one uses that the artinian R-module $\mathrm{E}(k)$ can be viewed as an $\widehat{R}$-module which satisfies the isomorphisms of $\widehat{R}$-modules $\widehat{R} \otimes_R \mathrm{E}(k) \cong \mathrm{E}_{\widehat{R}}(k) \cong \mathrm{E}(k)$, see [5, ex. 3.2.14]. Hence, we can suppose that R is complete.

(1) $\Leftrightarrow$ (2): There are isomorphisms

$$\mathrm{RHom}_R(\mathrm{E}(k), k) \cong \mathrm{RHom}_R(k^\vee, \mathrm{E}(k)^\vee) \cong \mathrm{RHom}_R(k, R) \tag{3.4.1}$$

by Matlis duality (see [11, thm. 18.6]). But R is Gorenstein precisely if $\mathrm{RHom}_R(k, R)$ is isomorphic to $\Sigma^{-d}k$ for some d, by [11, thm. 18.1]. So the result follows.

(2) $\Leftrightarrow$ (3): There are isomorphisms

$$\begin{aligned}
\mathrm{RHom}_R(\mathrm{E}(k), k) &\cong \mathrm{RHom}_R(\mathrm{E}(k), k^\vee) \\
&= \mathrm{RHom}_R(\mathrm{E}(k), \mathrm{RHom}_R(k, \mathrm{E}(k))) \\
&\overset{(a)}{\cong} \mathrm{RHom}_R(k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k), \mathrm{E}(k)) \\
&= (k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k))^\vee,
\end{aligned}$$

where "(a)" is by adjointness. This shows (3) $\Rightarrow$ (2). And taking Matlis duals gives

$$\mathrm{RHom}_R(\mathrm{E}(k), k)^\vee \cong (k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k))^{\vee\vee} \cong k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k), \tag{3.4.2}$$

where the second "$\cong$" is by Matlis duality, because $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)$ has artinian homology. This shows (2) $\Rightarrow$ (3).

Finally, in case the numbered conditions hold so R is Gorenstein, we know $\mathrm{RHom}_R(k, R) \cong \Sigma^{-\dim R}k$, again by [11, thm. 18.1]. Hence (3.4.1) proves $\mathrm{RHom}_R(\mathrm{E}(k), k) \cong \Sigma^{-\dim R}k$, and (3.4.2) proves $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong (\Sigma^{-\dim R}k)^\vee \cong \Sigma^{\dim R}k$. So we conclude $d = \dim R$. $\square$

(3.5) **Theorem (Gorenstein sensitivity).** *Let R be as in* (3.3). *Then the following statements are equivalent:*

- (1): *R is Gorenstein.*
- (2): *$k \in \mathcal{A}_{\mathrm{E}(k)}(R)$.*
- (3): *$k \in \mathcal{B}_{\mathrm{E}(k)}(R)$.*

Proof. (1) $\Rightarrow$ (2). When R is Gorenstein, we have $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong \Sigma^{\dim R}k$ by lemma (3.4)(3). Hence

$$\mathrm{RHom}_R(\mathrm{E}(k), k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)) \cong \mathrm{RHom}_R(\mathrm{E}(k), \Sigma^{\dim R}k) \cong k,$$

where the second "$\cong$" uses lemma (3.4)(2).

To see $k \in \mathcal{A}_{\mathrm{E}(k)}(R)$ we must see that the unit of the adjoint pair $(- \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k), \mathrm{RHom}_R(\mathrm{E}(k), -))$ evaluated on k is an isomorphism, that is, that $k \xrightarrow{\eta_k} \mathrm{RHom}_R(\mathrm{E}(k), k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k))$ is an isomorphism. This is the same as seeing that its homology $\mathrm{H}\eta_k$ is an isomorphism. But by the above computation, both source and target of η_k have homology given by k in degree 0, and 0 in all other degrees, so since k is a simple module, it suffices to see that $\mathrm{H}_0\eta_k$ is non-zero.

For this, replace $\mathrm{E}(k)$ by a free resolution F. Then η_k is represented by the chain map $k \longrightarrow \mathrm{Hom}_R(F, k \otimes_R F)$ given by $x \mapsto (f \mapsto x \otimes f)$. In particular we have $1_k \mapsto (f \mapsto 1_k \otimes f)$. Now, 1_k is a cycle in the complex k, so represents an element in homology. Its image under $\mathrm{H}_0\eta_k$ is represented by the cycle $f \mapsto 1_k \otimes f$ in the complex $\mathrm{Hom}_R(F, k \otimes_R F)$. Hence to see that $\mathrm{H}_0\eta_k$ is non-zero, all we need to see is that the cycle $f \mapsto 1_k \otimes f$ is not a boundary. But the boundaries in a Hom complex are exactly the null homotopic chain maps, so we must check that $f \mapsto 1_k \otimes f$ is not null homotopic.

But if it were null homotopic, then it would remain so upon tensoring with k. That is, $k \otimes_R F \longrightarrow k \otimes_R k \otimes_R F$ given by $y \otimes f \mapsto y \otimes 1_k \otimes f$ would be null homotopic. But using $k \otimes_R k \cong k$, this map can be identified with the identity on $k \otimes_R F$, hence cannot be null homotopic because $k \otimes_R F \cong k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)$ has non-vanishing homology by lemma (3.4).

(1) $\Rightarrow$ (3). This is seen by a computation similar to the one above.

(2) $\Rightarrow$ (1). If $k \in \mathcal{A}_{\mathrm{E}(k)}(R)$ then we have

$$k \xrightarrow{\cong} \mathrm{RHom}_R(\mathrm{E}(k), k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)).$$

And it is easy to see that the maximal ideal $\mathfrak{m}$ in R annihilates the modules in a suitable representative of $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)$, so

$$k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong \bigoplus_{i \in I} \Sigma^{\beta_i} k. \tag{3.5.1}$$

Combining these gives $k \cong \mathrm{RHom}_R(\mathrm{E}(k), \bigoplus_{i \in I} \Sigma^{\beta_i} k)$.

Suppose that $\bigoplus_{i \in I} \Sigma^{\beta_i} k$ contained more than one summand, so was equal to $\Sigma^{\beta_1} k \oplus \Sigma^{\beta_2} k \oplus (\bigoplus_{i \in I'} \Sigma^{\beta_i} k)$. Then we would have

$$k \cong \mathrm{RHom}_R(\mathrm{E}(k), \Sigma^{\beta_1} k) \oplus \mathrm{RHom}_R(\mathrm{E}(k), \Sigma^{\beta_2} k) \oplus \mathrm{RHom}_R(\mathrm{E}(k), \textstyle\bigoplus_{i \in I'} \Sigma^{\beta_i} k). \tag{3.5.2}$$

However, using $\widehat{R} \otimes_R \mathrm{E}(k) \cong \mathrm{E}_{\widehat{R}}(k) \cong \mathrm{E}(k)$ again, it is not difficult to see

$$\mathrm{RHom}_R(\mathrm{E}(k), k) \cong \mathrm{RHom}_{\widehat{R}}(\mathrm{E}_{\widehat{R}}(k), k),$$

and by Matlis duality, this is again $\mathrm{RHom}_{\widehat{R}}(k, \widehat{R})$ which is non-zero. As k is an indecomposable object in $\mathsf{D}(R)$, this gives a contradiction with equation (3.5.2), and thus there can only be one summand in (3.5.1), so $k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong \Sigma^{\beta_j} k$. By lemma (3.4) we get that R is Gorenstein.

(3) $\Rightarrow$ (1). If $k \in \mathcal{B}_{\mathrm{E}(k)}(R)$ then we have

$$\mathrm{RHom}_R(\mathrm{E}(k), k) \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \xrightarrow{\cong} k.$$

Again it is easy to see that $\mathfrak{m}$ annihilates the modules in a suitable representative of $\mathrm{RHom}_R(\mathrm{E}(k), k)$. Thus

$$\mathrm{RHom}_R(\mathrm{E}(k), k) \cong \bigoplus_{i \in I} \Sigma^{\beta_i} k.$$

Combining these gives $(\bigoplus_{i\in I} \Sigma^{\beta_i} k) \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong \bigoplus_{i\in I}(\Sigma^{\beta_i} k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k)) \cong k$.

Again, using that k is an indecomposable object in $\mathsf{D}(R)$, the only possibility is that there is only one summand, so $\Sigma^{\beta_j} k \overset{\mathrm{L}}{\otimes}_R \mathrm{E}(k) \cong k$. By lemma (3.4)) we get that R is Gorenstein. $\square$

References

[1] D. Apassov, Homological dimensions over differential graded rings, preprint (1999).

[2] L. L. Avramov and H.-B. Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), 129–155.

[3] L. L. Avramov and H.-B. Foxby, *Ring homomorphisms and finite Gorenstein dimension,* Proc. London Math. Soc. (3) **75** (1997), 241–270.

[4] ———, ———, and S. Halperin, Differential graded homological algebra, in preparation.

[5] W. Bruns and J. Herzog, "Cohen-Macaulay rings", Cambridge Stud. Adv. Math., Vol. 39, Cambridge University Press, Cambridge, 1998, revised edition.

[6] L. W. Christensen, "Gorenstein Dimensions", Lecture Notes in Math., Vol. 1747, Springer, Berlin, 2000.

[7] L. W. Christensen, *Semi-dualizing complexes and their Auslander categories,* Trans. Amer. Math. Soc. **353** (2001), 1839–1883.

[8] W. G. Dwyer and J. P. C. Greenlees, Complete modules and torsion modules, preprint (2000).

[9] A. Frankild and P. Jørgensen, Homological identities for Differential Graded Algebras, in preparation.

[10] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 63–102.

[11] H. Matsumura, "Commutative ring theory", Cambridge Stud. Adv. Math., Vol. 8, Cambridge University Press, Cambridge, 1989, second edition.

[12] J. Rickard, *Morita theory for derived categories,* J. London Math. Soc. (2) **39** (1989), 436–456.

Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, 2100 København Ø, DK–Danmark

E-mail address: `frankild@math.ku.dk, popjoerg@math.ku.dk`

Math. Scand.
(to appear)

AFFINE EQUIVALENCE AND GORENSTEINNESS

ANDERS FRANKILD AND PETER JØRGENSEN

0. Introduction

(0.1) **Background.** For a commutative, local, noetherian ring R and an object X in $\mathsf{D}(R)$, the derived category of R, one can consider the adjoint pair of covariant functors

$$X \overset{\mathrm{L}}{\otimes}_R - \quad \text{and} \quad \mathrm{RHom}_R(X,-), \tag{0.1.1}$$

and the contravariant functor

$$\mathrm{RHom}_R(-,X). \tag{0.1.2}$$

It is familiar that for certain X's, these functors restrict to quasi-inverse equivalences between suitable full subcategories of $\mathsf{D}(R)$,

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "X \overset{\mathrm{L}}{\otimes}_R -"] & \mathcal{B} \arrow[l, shift left, "\mathrm{RHom}_R(X,-)"] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "\mathrm{RHom}_R(-,X)"] & \mathcal{D}. \arrow[l, shift left, "\mathrm{RHom}_R(-,X)"] \end{tikzcd}$$

Important examples of this abound in the literature:

X	Equivalence theory based on $\mathrm{RHom}_R(-,X)$	Equivalence theory based on $X \overset{\mathrm{L}}{\otimes}_R -$ and $\mathrm{RHom}_R(X,-)$
D	Grothendieck/Hartshorne [12]	Foxby [2]
$\mathrm{E}(k)$	Matlis [14]	F+J [11]
$\mathrm{R}\Gamma_{\mathfrak{a}}(D)$	Hartshorne [13]	–
R	Foxby/Yassemi [17]	Trivial
$\mathrm{R}\Gamma_{\mathfrak{a}}(R)$	–	Dwyer/Greenlees [7]

The first three X's in the diagram are:

- D is a dualizing complex for R.
- $\mathrm{E}(k)$ is the injective hull of R's residue class field k.
- $\mathrm{R}\Gamma_{\mathfrak{a}}(D)$ is obtained by taking the right derived section functor $\mathrm{R}\Gamma_{\mathfrak{a}}$ with respect to the ideal $\mathfrak{a}$ in R, and applying it to D.

1991 *Mathematics Subject Classification.* 13D07, 13D45.

Key words and phrases. Foxby equivalence, Auslander class, Bass class, complete module, torsion module, Gorenstein ring.

The purpose of this text is to study the two theories missing from the diagram. In fact, these theories will contain the other theories in the upper right and lower left quadrants of the diagram as special cases.

(0.2) **This text.** A central point of section (0.1)'s diagram is that the existing equivalence theories in the upper right and lower left quadrants can recognize when the ring R is Gorenstein. They do this by the sizes of the full subcategories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ in equations (0.1.1) and (0.1.2), which (in suitable senses) are maximal exactly when R is Gorenstein. These results are known as "Gorenstein theorems", see [6, (2.3.14), (3.1.12), and (3.2.10)] and [11, thm. (3.5)], and live in the world of "Foxby equivalence" which deals with equivalences of categories induced by functors such as $X \overset{\mathrm{L}}{\otimes}_R -$ and $\mathrm{RHom}_R(X, -)$, see [11].

Given this, and given that the two theories missing from section (0.1)'s diagram fall in the upper right and lower left quadrants, a reasonable question is: Can the missing theories also recognize Gorenstein rings? We show in our main result, theorem (2.2), that the answer is yes. Thus, we fill in the blanks in section (0.1)'s diagram by studying the missing theories and showing that they are ring theoretically interesting.

To be specific, the theories missing from section (0.1)'s diagram are based on the functors

$$\mathrm{R}\Gamma_{\mathfrak{a}}(D) \overset{\mathrm{L}}{\otimes}_R - \quad \text{and} \quad \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D), -), \tag{0.2.1}$$

respectively

$$\mathrm{RHom}_R(-, \mathrm{R}\Gamma_{\mathfrak{a}}(R)), \tag{0.2.2}$$

and we prove in theorem (2.2) that the subcategories between which these functors induce equivalences are maximal exactly when R is Gorenstein. Note that $\mathrm{RHom}_R(-, \mathrm{R}\Gamma_{\mathfrak{a}}(R))$ equals $\mathrm{RHom}_R(-, \mathrm{C}(\mathfrak{a}))$ where $\mathrm{C}(\mathfrak{a})$ is the Čech complex of $\mathfrak{a}$, cf. remark (1.2).

We will not reproduce theorem (2.2) in this introduction. However, in the special case $\mathfrak{a} = 0$, the theorem gives corollary (2.4) which is the following improved version of the above mentioned Gorenstein theorems from [6]:

Corollary. *Let R be a commutative, local, noetherian ring with residue class field k. Now the following conditions are equivalent:*

(1) *R is Gorenstein.*
(2) *The biduality morphism*

$$X \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X, R), R)$$

is an isomorphism for $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

If R has a dualizing complex D, then the above conditions are also equivalent to:

(3) *$k \in \mathcal{A}_D$.*
(4) *$\mathcal{A}_D = \mathsf{D}(R)$.*
(5) *$k \in \mathcal{B}_D$.*
(6) *$\mathcal{B}_D = \mathsf{D}(R)$.*

The notation employed here is: $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ is the derived category of bounded complexes with finitely generated homology, and $\mathcal{A}_D$ and $\mathcal{B}_D$ are the so-called

Auslander and Bass classes of D which are, in a sense, the largest full subcategories of $\mathsf{D}(R)$ between which $D \overset{\mathrm{L}}{\otimes}_R -$ and $\mathrm{RHom}_R(D,-)$ induce equivalences. See [11, (1.5)] (or (1.1) below with $\mathfrak{a} = 0$) for the technical definition of $\mathcal{A}_D$ and $\mathcal{B}_D$.

Another special case of theorem (2.2) is $\mathfrak{a} = \mathfrak{m}$ where $\mathfrak{m}$ is R's maximal ideal; this is given in corollary (2.6) which contains the Gorenstein theorem [11, thm. (3.5)]. The corollary states the following:

Corollary. *Let R be a commutative, local, noetherian ring with maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$, and let $\mathrm{C}(\mathfrak{m})$ be the Čech complex of $\mathfrak{m}$. Now the following conditions are equivalent:*

(1) *R is Gorenstein.*
(2) *The standard morphism*

$$X \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{C}(\mathfrak{m}), \mathrm{C}(\mathfrak{m})) \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X, \mathrm{C}(\mathfrak{m})), \mathrm{C}(\mathfrak{m}))$$

is an isomorphism for $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

If R has a dualizing complex D, and $\mathrm{E}(k)$ denotes the injective hull of k, then the above conditions are also equivalent to:

(3) *$k \in \mathcal{A}_{\mathrm{E}(k)}$.*
(4) *$\mathcal{A}_{\mathrm{E}(k)} = \mathbf{A}^{\mathfrak{m}}_{\mathrm{comp}}$.*
(5) *$k \in \mathcal{B}_{\mathrm{E}(k)}$.*
(6) *$\mathcal{B}_{\mathrm{E}(k)} = \mathbf{A}^{\mathrm{tors}}_{\mathfrak{m}}$.*

Here $\mathcal{A}_{\mathrm{E}(k)}$ and $\mathcal{B}_{\mathrm{E}(k)}$ are the Auslander and Bass classes of $\mathrm{E}(k)$ which are defined in a way analogous to $\mathcal{A}_D$ and $\mathcal{B}_D$ above, see [11, (3.3)], and $\mathbf{A}^{\mathfrak{m}}_{\mathrm{comp}}$ and $\mathbf{A}^{\mathrm{tors}}_{\mathfrak{m}}$ are the categories of so-called derived complete and derived torsion complexes with respect to $\mathfrak{m}$, see [7] or remark (1.2) below.

Observe that part (2) of the corollary gives a new, simple way of characterizing Gorenstein rings. In fact, $\mathrm{RHom}_R(\mathrm{C}(\mathfrak{m}), \mathrm{C}(\mathfrak{m}))$ is $\widehat{R}$, the $\mathfrak{m}$-adic completion of R, by lemma (1.9), so part (2) of the corollary is even simpler than it first appears.

(0.3) **Remarks.** The title of this text is chosen for the following reason: Hartshorne in [13] considers an instance of the contravariant equivalence theory based on $\mathrm{R}\Gamma_{\mathfrak{a}}(D)$, that is, on the functor $\mathrm{RHom}_R(-, \mathrm{R}\Gamma_{\mathfrak{a}}(D))$. He calls it "affine duality". It hence seems natural that we should call the covariant equivalence theory based on $\mathrm{R}\Gamma_{\mathfrak{a}}(D)$, that is, on the functors from (0.2.1), "affine equivalence", whence our title.

Note that the equivalence theories based on the functors (0.2.1) and (0.2.2) contain a number of the other theories in section (0.1)'s diagram as special cases: When R has a dualizing complex D, the theories with $X = D$ and $X = \mathrm{E}(k)$ in the upper portion of the diagram can be obtained from the theories with $X = \mathrm{R}\Gamma_{\mathfrak{a}}(D)$; namely, $D \cong \mathrm{R}\Gamma_0(D)$ and $\mathrm{E}(k) \cong \mathrm{R}\Gamma_{\mathfrak{m}}(D)$. Similarly, the theories with $X = R$ in the lower portion of the diagram can be obtained from the theories with $X = \mathrm{R}\Gamma_{\mathfrak{a}}(R)$; namely, $R \cong \mathrm{R}\Gamma_0(R)$. Of course, this is the reason theorem (2.2) contains as a special case corollary (2.4).

(0.4) **Synopsis.** The text is organized as follows: After this introduction comes section 1 which gives a number of ways of characterizing Gorenstein rings, plus a number of results about the derived section and completion functors, $\mathrm{R}\Gamma_{\mathfrak{a}}$ and $\mathrm{L}\Lambda^{\mathfrak{a}}$. Finally comes section 2 which gives our main result, theorem (2.2), and concludes with some special cases in corollaries (2.4) and (2.6).

(0.5) **Notation.** First note that all our results are formulated in the derived category, $\mathsf{D}(R)$. We use the hyperhomological notation set up in [9, sec. 2], with a single exception: We denote isomorphisms in $\mathsf{D}(R)$ by "$\cong$" rather than by "$\simeq$".

One very important tool is a number of so-called standard homomorphisms between derived functors. These are treated in [9, sec. 2], and another reference is [6, (A.4)].

Apart from the material covered in [9, sec. 2], we make extensive use of the right derived section functor $\mathrm{R}\Gamma_{\mathfrak{a}}$ and the left derived completion functor $\mathrm{L}\Lambda^{\mathfrak{a}}$. They are defined as follows:

When $\mathfrak{a}$ is an ideal in R, the section functor with respect to $\mathfrak{a}$ is defined on modules by

$$\Gamma_{\mathfrak{a}}(-) = \operatorname{colim}_n \operatorname{Hom}_R(R/\mathfrak{a}^n, -).$$

It is left exact, and has a right derived functor $\mathrm{R}\Gamma_{\mathfrak{a}}$ which lives on $\mathsf{D}(R)$. Similarly, the completion functor with respect to $\mathfrak{a}$ is defined on modules by

$$\Lambda^{\mathfrak{a}}(-) = \lim_n (R/\mathfrak{a}^n \otimes_R -).$$

It has a left derived functor $\mathrm{L}\Lambda^{\mathfrak{a}}$ which also lives on $\mathsf{D}(R)$.

A salient fact is that $(\mathrm{R}\Gamma_{\mathfrak{a}}, \mathrm{L}\Lambda^{\mathfrak{a}})$ is an adjoint pair. For this and other properties, see [1].

(0.6) **Setup.** Throughout the text, R is a commutative, local, noetherian ring with maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$, and $\mathfrak{a}$ is an ideal in R generated by $\boldsymbol{a} = (a_1, \ldots, a_n)$. The $\mathfrak{a}$-adic completion of R is denoted $\widehat{R_{\mathfrak{a}}}$. The Koszul complex on $\boldsymbol{a}$ is denoted $\mathrm{K}(\boldsymbol{a})$; it is a bounded complex of finitely generated free modules. The Čech complex of $\mathfrak{a}$ (also known as the stable Koszul complex of $\mathfrak{a}$) is denoted $\mathrm{C}(\mathfrak{a})$; it is a bounded complex of flat modules. See [4, chp. 5] for a brushup on Koszul and Čech complexes.

1. Preparatory results

(1.1) **Affine equivalence.** Suppose that R has a dualizing complex D. As described in the introduction, we shall consider the adjoint pair of functors

$$\mathsf{D}(R) \underset{\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D),-)}{\overset{\mathrm{R}\Gamma_{\mathfrak{a}}(D)\overset{\mathrm{L}}{\otimes}_R -}{\rightleftarrows}} \mathsf{D}(R).$$

Let us sum up the main content of Foxby equivalence as introduced in [11, (1.5)] in this situation: Letting η be the unit and ϵ the counit of the adjunction, and defining the *Auslander class* by

$$\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)} = \left\{ X \;\middle|\; \begin{array}{l} \eta_X : X \longrightarrow \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D), \mathrm{R}\Gamma_{\mathfrak{a}}(D) \overset{\mathrm{L}}{\otimes}_R X) \\ \text{is an isomorphism} \end{array} \right\}$$

and the *Bass class* by

$$\mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)} = \left\{ Y \;\middle|\; \begin{array}{l} \epsilon_Y : \mathrm{R}\Gamma_{\mathfrak{a}}(D) \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D), Y) \longrightarrow Y \\ \text{is an isomorphism} \end{array} \right\},$$

there are quasi-inverse equivalences of categories between the Auslander and Bass classes,

$$\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)} \underset{\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D),-)}{\overset{\mathrm{R}\Gamma_{\mathfrak{a}}(D)\overset{\mathrm{L}}{\otimes}_R -}{\rightleftarrows}} \mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}. \tag{1.1.1}$$

Our main result, theorem (2.2), characterizes Gorenstein rings in terms of maximality of $\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$ and $\mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$.

(1.2) **Remark.** In [7] is considered the following situation: Given a ring, S, and a bounded complex of finitely generated projective S-left-modules, A, one can construct the endomorphism Differential Graded Algebra, $\mathcal{E} = \mathrm{Hom}_S(A, A)$, and A becomes a Differential Graded $\mathcal{E}$-left-module whose $\mathcal{E}$-structure is compatible with its S-structure. Likewise, the complex $A^\sharp = \mathrm{Hom}_S(A, S)$ is a bounded complex of finitely generated projective S-right-modules, and becomes a Differential Graded $\mathcal{E}$-right-module whose $\mathcal{E}$-structure is compatible with its S-structure. Moreover, there are two full subcategories $\mathbf{A}_{\mathrm{comp}}$ and $\mathbf{A}^{\mathrm{tors}}$ of $\mathsf{D}(S)$, and a diagram

$$\mathbf{A}_{\mathrm{comp}} \underset{\mathrm{RHom}_{\mathcal{E}^{\mathrm{opp}}}(A^\sharp,-)}{\overset{A^\sharp\overset{\mathrm{L}}{\otimes}_S -}{\rightleftarrows}} \mathsf{D}(\mathcal{E}^{\mathrm{opp}}) \underset{\mathrm{RHom}_S(A,-)}{\overset{-\overset{\mathrm{L}}{\otimes}_{\mathcal{E}} A}{\rightleftarrows}} \mathbf{A}^{\mathrm{tors}} \tag{1.2.1}$$

where each half is a pair of quasi-inverse equivalences of categories. Note that we write $\mathcal{E}^{\mathrm{opp}}$ for the opposite algebra of $\mathcal{E}$ and $\mathsf{D}(\mathcal{E}^{\mathrm{opp}})$ for the derived category of Differential Graded $\mathcal{E}^{\mathrm{opp}}$-left-modules which is equivalent to the derived category of Differential Graded $\mathcal{E}$-right-modules.

In this text, we use the following special case, based on the data from setup (0.6): The ring S is R, and the complex A is $\mathrm{K}(\boldsymbol{a})$. We then write $\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}$ for $\mathbf{A}_{\mathrm{comp}}$, and $\mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}$ for $\mathbf{A}^{\mathrm{tors}}$. By [7, proof of 4.3] and [7, prop. 6.10] we have

$$\mathrm{K}(\boldsymbol{a})^\sharp \overset{\mathrm{L}}{\otimes}_{\mathcal{E}} \mathrm{K}(\boldsymbol{a}) \cong \mathrm{Cell}_{\mathrm{K}(\boldsymbol{a})}(R) \cong \mathrm{C}(\mathfrak{a}),$$

so the composite of the two upper functors in diagram (1.2.1) is

$$\begin{aligned} (\mathrm{K}(\boldsymbol{a})^\sharp \overset{\mathrm{L}}{\otimes}_R -) \overset{\mathrm{L}}{\otimes}_{\mathcal{E}} \mathrm{K}(\boldsymbol{a}) &\simeq (\mathrm{K}(\boldsymbol{a})^\sharp \overset{\mathrm{L}}{\otimes}_{\mathcal{E}} \mathrm{K}(\boldsymbol{a})) \overset{\mathrm{L}}{\otimes}_R - \\ &\simeq \mathrm{C}(\mathfrak{a}) \overset{\mathrm{L}}{\otimes}_R -, \end{aligned} \tag{1.2.2}$$

where "$\simeq$" signifies an equivalence of functors, and where the first "$\simeq$" is by associativity of tensor products, see [3, sec. 4.4]. Similarly, the composite of the two lower functors is

$$\begin{aligned}
&\mathrm{RHom}_{\mathcal{E}^{\mathrm{opp}}}(\mathrm{K}(\boldsymbol{a})^\sharp, \mathrm{RHom}_R(\mathrm{K}(\boldsymbol{a}), -)) \\
&\quad \simeq \mathrm{RHom}_R(\mathrm{K}(\boldsymbol{a})^\sharp \overset{\mathrm{L}}{\otimes}_{\mathcal{E}} \mathrm{K}(\boldsymbol{a}), -) \\
&\quad \simeq \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}), -), \qquad (1.2.3)
\end{aligned}$$

where the first "$\simeq$" is by adjointness, see [3, sec. 4.4]. Note that these equivalences are valid as equivalences of functors defined on the *entire* derived category $\mathsf{D}(R)$.

Diagram (1.2.1) shows that the essential image of the functor $\mathrm{K}(\boldsymbol{a})^\sharp \overset{\mathrm{L}}{\otimes}_R -$ defined on $\mathsf{D}(R)$ is all of $\mathsf{D}(\mathcal{E}^{\mathrm{opp}})$. (The *essential image* of a functor is the closure of the functor's image under isomorphisms.) In turn, equation (1.2.2) therefore shows that the essential image of the functor $\mathrm{C}(\mathfrak{a}) \overset{\mathrm{L}}{\otimes}_R -$, defined on $\mathsf{D}(R)$, equals the essential image of the functor $- \overset{\mathrm{L}}{\otimes}_{\mathcal{E}} \mathrm{K}(\boldsymbol{a})$, defined on all of $\mathsf{D}(\mathcal{E}^{\mathrm{opp}})$, and this image is $\mathbf{A}_{\mathfrak{a}}^{\mathrm{tors}}$ by diagram (1.2.1). A similar argument with equation (1.2.3) shows that the essential image of the functor $\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}), -)$, defined on $\mathsf{D}(R)$, equals $\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}$.

Note that by [16, thm. 1.1(iv)] and [1, $(0.3)_{\mathrm{aff}}$, p. 4] there are natural equivalences of functors on $\mathsf{D}(R)$,

$$\mathrm{R}\Gamma_{\mathfrak{a}}(-) \simeq \mathrm{C}(\mathfrak{a}) \overset{\mathrm{L}}{\otimes}_R - \quad \text{and} \quad \mathrm{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}), -), \tag{1.2.4}$$

so the above can also be phrased: The essential image of $\mathrm{R}\Gamma_{\mathfrak{a}}$ is $\mathbf{A}_{\mathfrak{a}}^{\mathrm{tors}}$, and the essential image of $\mathrm{L}\Lambda^{\mathfrak{a}}$ is $\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}$.

Note also the following special case of the first of equations (1.2.4),

$$\mathrm{R}\Gamma_{\mathfrak{a}}(R) \cong \mathrm{C}(\mathfrak{a}) \overset{\mathrm{L}}{\otimes}_R R \cong \mathrm{C}(\mathfrak{a}). \tag{1.2.5}$$

Computations (1.2.2) and (1.2.3) also show that ignoring the middle part of diagram (1.2.1) leaves the pair of quasi-inverse equivalences of categories

$$\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}} \underset{\mathrm{L}\Lambda^{\mathfrak{a}}(-) \simeq \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}), -)}{\overset{\mathrm{R}\Gamma_{\mathfrak{a}}(-) \simeq \mathrm{C}(\mathfrak{a}) \overset{\mathrm{L}}{\otimes}_R -}{\rightleftarrows}} \mathbf{A}_{\mathfrak{a}}^{\mathrm{tors}}. \tag{1.2.6}$$

In particular, $X \in \mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}$ gives

$$X \xrightarrow{\cong} \mathrm{L}\Lambda^{\mathfrak{a}} \mathrm{R}\Gamma_{\mathfrak{a}} X \xrightarrow{\cong} \mathrm{L}\Lambda^{\mathfrak{a}} X \tag{1.2.7}$$

where the first isomorphism is the unit of the adjunction in diagram (1.2.6), and the second is by [1, p. 6, cor., part (iii)]. Similarly, $Y \in \mathbf{A}_{\mathfrak{a}}^{\mathrm{tors}}$ gives

$$\mathrm{R}\Gamma_{\mathfrak{a}} Y \xrightarrow{\cong} \mathrm{R}\Gamma_{\mathfrak{a}} \mathrm{L}\Lambda^{\mathfrak{a}} Y \xrightarrow{\cong} Y \tag{1.2.8}$$

where the first isomorphism is by [1, p. 6, cor., part (iv)], and the second is the counit of the adjunction in diagram (1.2.6).

(1.3) **Lemma.** *R is Gorenstein if and only if $\widehat{R_{\mathfrak{a}}}$ is Gorenstein.*

Proof. The canonical homomorphism $R \longrightarrow \widehat{R_{\mathfrak{a}}}$ is flat and local by [15, p. 63, (3) and (4)]. We also have

$$\widehat{R_{\mathfrak{a}}}/\mathfrak{m}\widehat{R_{\mathfrak{a}}} \cong \widehat{R_{\mathfrak{a}}} \otimes_R R/\mathfrak{m} = \widehat{R_{\mathfrak{a}}} \otimes_R k \cong k,$$

where the last "$\cong$" is because k is complete in any $\mathfrak{a}$-adic topology, so $\widehat{R_{\mathfrak{a}}}/\mathfrak{m}\widehat{R_{\mathfrak{a}}}$ is Gorenstein. Hence R and $\widehat{R_{\mathfrak{a}}}$ are Gorenstein simultaneously by [5, cor. 3.3.15]. □

(1.4) **Lemma.** *R is Gorenstein if and only if*

$$\mathrm{RHom}_R(\mathrm{RHom}_R(k,R),R) \cong k. \tag{1.4.1}$$

Proof. If R is Gorenstein, then we have $\mathrm{RHom}_R(\mathrm{RHom}_R(k,R),R) \xleftarrow{\cong} k$ via the biduality morphism, see [6, thm. (2.3.14)].

Conversely, suppose that (1.4.1) holds. It is easy to see in general that $\mathrm{RHom}_R(k,R)$ can be represented by a complex where the modules are annihilated by $\mathfrak{m}$. So $\mathrm{RHom}_R(k,R)$ is really just a complex over the field $k = R/\mathfrak{m}$. Hence we can use [6, (A.7.9.3)] with $V = \mathrm{RHom}_R(k,R)$ and $Y = R$ to get

$$\sup \mathrm{RHom}_R(\mathrm{RHom}_R(k,R),R) = \sup \mathrm{RHom}_R(k,R) - \inf \mathrm{RHom}_R(k,R).$$

In the present situation, the left hand side is zero by equation (1.4.1). Hence $\sup \mathrm{RHom}_R(k,R) = \inf \mathrm{RHom}_R(k,R)$, so $\mathrm{RHom}_R(k,R)$ only has homology in a single degree, so only a single $\mathrm{Ext}^i_R(k,R)$ is non-zero. This implies R Gorenstein by [15, thm. 18.1].

□

(1.5) **Lemma.** *R is Gorenstein if and only if*

$$\mathrm{RHom}_R(\mathrm{RHom}_R(k,R),\widehat{R_{\mathfrak{a}}}) \cong k.$$

in $\mathsf{D}(\widehat{R_{\mathfrak{a}}})$.

Proof. We start with a computation in $\mathsf{D}(\widehat{R_{\mathfrak{a}}})$,

$$\begin{aligned}
&\mathrm{RHom}_{\widehat{R_{\mathfrak{a}}}}(\mathrm{RHom}_{\widehat{R_{\mathfrak{a}}}}(k,\widehat{R_{\mathfrak{a}}}),\widehat{R_{\mathfrak{a}}}) \\
&\stackrel{(a)}{\cong} \mathrm{RHom}_{\widehat{R_{\mathfrak{a}}}}(\mathrm{RHom}_{\widehat{R_{\mathfrak{a}}}}(k \overset{\mathrm{L}}{\otimes}_R \widehat{R_{\mathfrak{a}}},\widehat{R_{\mathfrak{a}}}),\widehat{R_{\mathfrak{a}}}) \\
&\stackrel{(b)}{\cong} \mathrm{RHom}_{\widehat{R_{\mathfrak{a}}}}(\mathrm{RHom}_R(k,\mathrm{RHom}_{\widehat{R_{\mathfrak{a}}}}(\widehat{R_{\mathfrak{a}}},\widehat{R_{\mathfrak{a}}})),\widehat{R_{\mathfrak{a}}}) \\
&\cong \mathrm{RHom}_{\widehat{R_{\mathfrak{a}}}}(\mathrm{RHom}_R(k,\widehat{R_{\mathfrak{a}}}),\widehat{R_{\mathfrak{a}}}) \\
&\cong \mathrm{RHom}_{\widehat{R_{\mathfrak{a}}}}(\mathrm{RHom}_R(k,R \overset{\mathrm{L}}{\otimes}_R \widehat{R_{\mathfrak{a}}}),\widehat{R_{\mathfrak{a}}}) \\
&\stackrel{(c)}{\cong} \mathrm{RHom}_{\widehat{R_{\mathfrak{a}}}}(\mathrm{RHom}_R(k,R) \overset{\mathrm{L}}{\otimes}_R \widehat{R_{\mathfrak{a}}},\widehat{R_{\mathfrak{a}}}) \\
&\stackrel{(d)}{\cong} \mathrm{RHom}_R(\mathrm{RHom}_R(k,R),\mathrm{RHom}_{\widehat{R_{\mathfrak{a}}}}(\widehat{R_{\mathfrak{a}}},\widehat{R_{\mathfrak{a}}})) \\
&\cong \mathrm{RHom}_R(\mathrm{RHom}_R(k,R),\widehat{R_{\mathfrak{a}}}).
\end{aligned}$$

Here "(a)" is because $k \overset{\mathrm{L}}{\otimes}_R \widehat{R_{\mathfrak{a}}}$ is $\widehat{k_{\mathfrak{a}}}$ which is just k since k is complete in any $\mathfrak{a}$-adic topology. "(b)" and "(d)" are by adjointness, [6, (A.4.21)]. "(c)" is by [6, (A.4.23)] because we have $k \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and $R \in \mathsf{D}_{\mathrm{b}}(R)$, while $\widehat{R_{\mathfrak{a}}}$ is a bounded complex of flat modules. Observe that both "(b)", "(c)", and "(d)" are proved using the standard homomorphisms mentioned in the introduction. The remaining isomorphisms follow from

$$\mathrm{RHom}_{R^{\widehat{}}_{\mathfrak{a}}}(R^{\widehat{}}_{\mathfrak{a}}, R^{\widehat{}}_{\mathfrak{a}}) \cong R^{\widehat{}}_{\mathfrak{a}} \quad \text{and} \quad R \overset{\mathrm{L}}{\otimes}_R R^{\widehat{}}_{\mathfrak{a}} \cong R^{\widehat{}}_{\mathfrak{a}}.$$

Now, R is Gorenstein if and only if $R^{\widehat{}}_{\mathfrak{a}}$ is Gorenstein by lemma (1.3). By lemma (1.4) applied to $R^{\widehat{}}_{\mathfrak{a}}$ this amounts to

$$\mathrm{RHom}_{R^{\widehat{}}_{\mathfrak{a}}}(\mathrm{RHom}_{R^{\widehat{}}_{\mathfrak{a}}}(k, R^{\widehat{}}_{\mathfrak{a}}), R^{\widehat{}}_{\mathfrak{a}}) \cong k.$$

And by the above computation, this is equivalent to

$$\mathrm{RHom}_R(\mathrm{RHom}_R(k, R), R^{\widehat{}}_{\mathfrak{a}}) \cong k$$

in $\mathsf{D}(R^{\widehat{}}_{\mathfrak{a}})$. □

(1.6) **Proposition.** *If R has a dualizing complex D, then*

$$\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)} \subseteq \mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}} \quad \textit{and} \quad \mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)} \subseteq \mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}.$$

Proof. We only prove the first inclusion, as the proof of the second is similar.

Let $X \in \mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$ be given. Then X is the image under $\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D), -)$ of some $Y \in \mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$, by diagram (1.1.1). Hence

$$\begin{aligned} X &\cong \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D), Y) \\ &\overset{(a)}{\cong} \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}) \overset{\mathrm{L}}{\otimes}_R D, Y) \\ &\overset{(b)}{\cong} \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}), \mathrm{RHom}_R(D, Y)) \\ &\overset{(c)}{\cong} \mathrm{L}\Lambda^{\mathfrak{a}}(\mathrm{RHom}_R(D, Y)), \end{aligned}$$

where "(a)" and "(c)" are by equations (1.2.4), and where "(b)" is by adjointness, [6, (A.4.21)].

So X is in the essential image of $\mathrm{L}\Lambda^{\mathfrak{a}}$, hence X is in $\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}$ by remark (1.2). □

(1.7) **Lemma.** *We have $k \in \mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}$ and $k \in \mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}$.*

Proof. To prove the first statement, consider

$$\mathrm{L}\Lambda^{\mathfrak{a}}(k) \cong k \overset{\mathrm{L}}{\otimes}_R R^{\widehat{}}_{\mathfrak{a}} \cong k$$

where the first "$\cong$" is by [10, prop. (2.7)] and the second "$\cong$" is because k is complete in any $\mathfrak{a}$-adic topology. This shows that k is in the essential image of $\mathrm{L}\Lambda^{\mathfrak{a}}$, whence it is in $\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}$ by remark (1.2).

To prove the second statement, note that by [4, cor. 2.1.6] there is an injective resolution I of k in which each I_i satisfies that each of its elements is annihilated by some power $\mathfrak{m}^n$, and hence also by some power $\mathfrak{a}^n$. This gives $\Gamma_{\mathfrak{a}}(I) \cong I$, and therefore

$$\mathrm{R}\Gamma_{\mathfrak{a}}(k) \cong \Gamma_{\mathfrak{a}}(I) \cong I \cong k.$$

This shows that k is in the essential image of $\mathrm{R}\Gamma_{\mathfrak{a}}$, whence it is in $\mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}$ by remark (1.2). □

(1.8) **Lemma.** *If R has a dualizing complex D, then*

(1) *For $X \in \mathsf{D}^{\mathrm{f}}_{+}(R) \cap \mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}$ we have*

$$\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D), \mathrm{R}\Gamma_{\mathfrak{a}}(D) \overset{\mathrm{L}}{\otimes}_R X) \cong \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R X).$$

(2) *For $Y \in \mathsf{D}_{-}(R) \cap \mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}$ we have*

$$\mathrm{R}\Gamma_{\mathfrak{a}}(D) \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D), Y) \cong D \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(D, Y).$$

Proof. We only prove (1), as the proof of (2) is similar:

$$\begin{aligned}
&\mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D), \mathrm{R}\Gamma_{\mathfrak{a}}(D) \overset{\mathrm{L}}{\otimes}_R X)\\
&\overset{(a)}{\cong} \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D), \mathrm{R}\Gamma_{\mathfrak{a}}(D \overset{\mathrm{L}}{\otimes}_R X))\\
&\overset{(b)}{\cong} \mathrm{RHom}_R(D, \mathrm{L}\Lambda^{\mathfrak{a}}\,\mathrm{R}\Gamma_{\mathfrak{a}}(D \overset{\mathrm{L}}{\otimes}_R X))\\
&\overset{(c)}{\cong} \mathrm{RHom}_R(D, \mathrm{L}\Lambda^{\mathfrak{a}}(D \overset{\mathrm{L}}{\otimes}_R X))\\
&\overset{(d)}{\cong} \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R X \overset{\mathrm{L}}{\otimes}_R \widehat{R_{\mathfrak{a}}})\\
&\overset{(e)}{\cong} \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R \mathrm{L}\Lambda^{\mathfrak{a}}(X))\\
&\overset{(f)}{\cong} \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R X),
\end{aligned}$$

where "(a)" follows from (1.2.4) by an easy computation, "(b)" is by [1, $(0.3)_{\mathrm{aff}}$, p. 4], "(c)" is by [1, p. 6, cor., part (iii)], "(d)" and "(e)" are by [10, prop. (2.7)], and "(f)" is by equation (1.2.7). □

(1.9) **Lemma.** *We have* $\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}), \mathrm{C}(\mathfrak{a})) \cong \widehat{R_{\mathfrak{a}}}$ *in* $\mathsf{D}(R)$.

Proof. This is a computation,

$$\begin{aligned}
\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}), \mathrm{C}(\mathfrak{a})) &\overset{(a)}{\cong} \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(R), \mathrm{R}\Gamma_{\mathfrak{a}}(R))\\
&\overset{(b)}{\cong} \mathrm{RHom}_R(R, \mathrm{L}\Lambda^{\mathfrak{a}}\,\mathrm{R}\Gamma_{\mathfrak{a}}(R))\\
&\cong \mathrm{L}\Lambda^{\mathfrak{a}}\,\mathrm{R}\Gamma_{\mathfrak{a}}(R)\\
&\overset{(c)}{\cong} \mathrm{L}\Lambda^{\mathfrak{a}}(R)\\
&\overset{(d)}{\cong} R \overset{\mathrm{L}}{\otimes}_R \widehat{R_{\mathfrak{a}}}\\
&\cong \widehat{R_{\mathfrak{a}}},
\end{aligned}$$

where "(a)" is by equation (1.2.5), "(b)" is by [1, $(0.3)_{\mathrm{aff}}$, p. 4], "(c)" is by [1, p. 6, cor., part (iii)], and "(d)" is by [10, prop. (2.7)]. □

2. The parametrized Gorenstein theorem

(2.1) **Remark.** Theorem (2.2) below is our main result. Among other things, it considers complexes X for which the standard morphism

$$X \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}), \mathrm{C}(\mathfrak{a})) \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X, \mathrm{C}(\mathfrak{a})), \mathrm{C}(\mathfrak{a}))$$

from [6, (A.4.24)] is an isomorphism. Note that by lemma (1.9) we have

$$\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a})) \cong \widehat{R_{\mathfrak{a}}},$$

so the X's in question have the property that there is an isomorphism

$$X \overset{\mathrm{L}}{\otimes}_R \widehat{R_{\mathfrak{a}}} \cong \mathrm{RHom}_R(\mathrm{RHom}_R(X,\mathrm{C}(\mathfrak{a})),\mathrm{C}(\mathfrak{a})).$$

(2.2) **The parametrized Gorenstein theorem.** *Recall from setup* (0.6) *that R is a commutative, local, noetherian ring which has residue class field k and contains the ideal $\mathfrak{a}$, and that $\mathrm{C}(\mathfrak{a})$ denotes the Čech complex of $\mathfrak{a}$. Now the following conditions are equivalent:*

(1) *R is Gorenstein.*
(2) *The standard morphism*

$$X \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a})) \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X,\mathrm{C}(\mathfrak{a})),\mathrm{C}(\mathfrak{a}))$$

is an isomorphism for $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

If R has a dualizing complex D, then the above conditions are also equivalent to the following, where we remind the reader that $\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$ and $\mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$ were defined in (1.1)*:*

(3) *$k \in \mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$.*
(4) *$\mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)} = \mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}$.*
(5) *$k \in \mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$.*
(6) *$\mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)} = \mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}}$.*

Proof. We show this by showing the following implications:

$$\begin{tikzcd}
 & & (2) \arrow[d, Leftrightarrow] & & \\
(4) \arrow[dr, Rightarrow] & & (1) \arrow[ll, Rightarrow] \arrow[rr, Rightarrow] & & (6) \arrow[dl, Rightarrow] \\
 & (3) \arrow[ur, Rightarrow] & & (5) \arrow[ul, Rightarrow] &
\end{tikzcd}$$

(1) $\Leftrightarrow$ (2). We start by considering the chain of morphisms

$$\begin{aligned}
&X \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a}))\\
&\quad\xrightarrow{\epsilon} \mathrm{RHom}_R(\mathrm{RHom}_R(X,R),R) \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a}))\\
&\quad\xrightarrow{\alpha} \mathrm{RHom}_R(\mathrm{RHom}_R(X,R),R \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a})))\\
&\quad\xrightarrow{\cong} \mathrm{RHom}_R(\mathrm{RHom}_R(X,R),\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a})))\\
&\quad\xleftarrow{\cong} \mathrm{RHom}_R(\mathrm{RHom}_R(X,R) \overset{\mathrm{L}}{\otimes}_R \mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a}))\\
&\quad\xleftarrow{\cong} \mathrm{RHom}_R(\mathrm{RHom}_R(X,R \overset{\mathrm{L}}{\otimes}_R \mathrm{C}(\mathfrak{a})),\mathrm{C}(\mathfrak{a}))\\
&\quad\xleftarrow{\cong} \mathrm{RHom}_R(\mathrm{RHom}_R(X,\mathrm{C}(\mathfrak{a})),\mathrm{C}(\mathfrak{a})),
\end{aligned} \tag{2.2.1}$$

where ϵ is $\delta \overset{\mathrm{L}}{\otimes}_R 1_{\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a}))}$ with

$$X \xrightarrow{\delta} \mathrm{RHom}_R(\mathrm{RHom}_R(X,R),R)$$

being the biduality morphism from [6, def. (2.1.3)], and where the other arrows are either induced by the standard morphisms from [6, sec. (A.4)] or induced by the identifications

$$R \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a})) \xrightarrow{\cong} \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a}))$$

and

$$R \overset{\mathrm{L}}{\otimes}_R \mathrm{C}(\mathfrak{a}) \xrightarrow{\cong} \mathrm{C}(\mathfrak{a}).$$

For $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, the morphisms in (2.2.1) marked "$\cong$" are isomorphisms; this is clear except for the one second to last, for which it follows from [6, (A.4.23)] because $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and $R \in \mathsf{D}_{\mathrm{b}}(R)$, while $\mathrm{C}(\mathfrak{a})$ is a bounded complex of flat modules.

As one can check, the morphisms in (2.2.1) combine simply to give the standard morphism

$$X \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a})) \xrightarrow{\theta} \mathrm{RHom}_R(\mathrm{RHom}_R(X,\mathrm{C}(\mathfrak{a})),\mathrm{C}(\mathfrak{a}))$$

from [6, (A.4.24)].

Now suppose that (1) holds, that is, R is Gorenstein, and let $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ be given. Then ϵ an isomorphism, since already the biduality morphism δ is an isomorphism [6, thm. (2.3.14)(iii')]. And α is an isomorphism by [6, (A.4.23)] because we have $\mathrm{RHom}_R(X,R) \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ by [6, thm. (2.3.14)(iii')], and clearly have $R \in \mathsf{D}_{\mathrm{b}}(R)$, while $\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a}))$ is isomorphic to a bounded complex of flat modules by lemma (1.9). Hence θ is an isomorphism, so (2) holds.

Conversely, suppose that (2) holds, that is, θ is an isomorphism for each $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. Letting X be k gives

$$\begin{aligned} k &\overset{(a)}{\cong} k \overset{\mathrm{L}}{\otimes}_R \widehat{R_{\mathfrak{a}}} \\ &\overset{(b)}{\cong} k \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a})) \\ &\xrightarrow{\cong} \mathrm{RHom}_R(\mathrm{RHom}_R(k,\mathrm{C}(\mathfrak{a})),\mathrm{C}(\mathfrak{a})) \end{aligned}$$

where "(a)" is because k is complete in any $\mathfrak{a}$-adic topology, and "(b)" is by lemma (1.9). Now, the second half of the chain of isomorphisms (2.2.1) read backwards is

$$\begin{aligned} &\mathrm{RHom}_R(\mathrm{RHom}_R(X,\mathrm{C}(\mathfrak{a})),\mathrm{C}(\mathfrak{a})) \\ &\quad\xrightarrow{\cong} \mathrm{RHom}_R(\mathrm{RHom}_R(X, R \overset{\mathrm{L}}{\otimes}_R \mathrm{C}(\mathfrak{a})),\mathrm{C}(\mathfrak{a})) \\ &\quad\xrightarrow{\cong} \mathrm{RHom}_R(\mathrm{RHom}_R(X,R) \overset{\mathrm{L}}{\otimes}_R \mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a})) \\ &\quad\xrightarrow{\cong} \mathrm{RHom}_R(\mathrm{RHom}_R(X,R),\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a}))). \end{aligned}$$

By lemma (1.9) we again have

$$\begin{aligned} &\mathrm{RHom}_R(\mathrm{RHom}_R(X,R),\mathrm{RHom}_R(\mathrm{C}(\mathfrak{a}),\mathrm{C}(\mathfrak{a}))) \\ &\quad\cong \mathrm{RHom}_R(\mathrm{RHom}_R(X,R),\widehat{R_{\mathfrak{a}}}). \end{aligned}$$

Setting $X = k$ and combining the three previous computations says

$$k \cong \mathrm{RHom}_R(\mathrm{RHom}_R(k,R), \widehat{R_{\mathfrak{a}}}),$$

whence R is Gorenstein by lemma (1.5), so (1) holds.

(1) $\Rightarrow$ (4). When R is Gorenstein, then the dualizing complex D is a shift of R by [6, thm. (A.8.3)], so we can assume $D = R$. But then $\mathrm{R}\Gamma_{\mathfrak{a}}(D) = \mathrm{R}\Gamma_{\mathfrak{a}}(R) \cong \mathrm{C}(\mathfrak{a})$ by equation (1.2.5), so the functors in diagram (1.1.1) are equivalent to the functors in diagram (1.2.6). But this certainly shows $\mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}} \subseteq \mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$ and $\mathbf{A}^{\mathrm{tors}}_{\mathfrak{a}} \subseteq \mathcal{B}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$, and the reverse inclusions are by proposition (1.6).

(4) $\Rightarrow$ (3). This is clear since $k \in \mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}$ by lemma (1.7).

(3) $\Rightarrow$ (1). Suppose that $k \in \mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$ holds. It is easy to see in general that $D \overset{\mathrm{L}}{\otimes}_R k$ can be represented by a complex where all the modules are annihilated by R's maximal ideal $\mathfrak{m}$. So $D \overset{\mathrm{L}}{\otimes}_R k$ is really just a complex over the field $k = R/\mathfrak{m}$, hence satisfies $\mathrm{RHom}_k(k, D \overset{\mathrm{L}}{\otimes}_R k) \cong D \overset{\mathrm{L}}{\otimes}_R k$. This observation gives the first "$\cong$" in

$$\begin{aligned}\mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R k) &\cong \mathrm{RHom}_R(D, \mathrm{RHom}_k(k, D \overset{\mathrm{L}}{\otimes}_R k)) \\ &\cong \mathrm{RHom}_k(D \overset{\mathrm{L}}{\otimes}_R k, D \overset{\mathrm{L}}{\otimes}_R k),\end{aligned}$$

where the second "$\cong$" is by adjointness, [6, (A.4.21)]. However, since $D \overset{\mathrm{L}}{\otimes}_R k$ is a complex over k, we can use [6, (A.7.9.3)] with $V = Y = D \overset{\mathrm{L}}{\otimes}_R k$ to get

$$\begin{aligned}\sup \mathrm{RHom}_k(D \overset{\mathrm{L}}{\otimes}_R k, D \overset{\mathrm{L}}{\otimes}_R k) &= \sup \mathrm{RHom}_k(k, D \overset{\mathrm{L}}{\otimes}_R k) - \inf(D \overset{\mathrm{L}}{\otimes}_R k) \\ &= \sup(D \overset{\mathrm{L}}{\otimes}_R k) - \inf(D \overset{\mathrm{L}}{\otimes}_R k).\end{aligned}$$

Combining the equations gives

$$\sup \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R k) = \sup(D \overset{\mathrm{L}}{\otimes}_R k) - \inf(D \overset{\mathrm{L}}{\otimes}_R k). \tag{2.2.2}$$

Now, in the present situation, $k \in \mathcal{A}_{\mathrm{R}\Gamma_{\mathfrak{a}}(D)}$ gives the first isomorphism in

$$\begin{aligned}k &\xrightarrow{\cong} \mathrm{RHom}_R(\mathrm{R}\Gamma_{\mathfrak{a}}(D), \mathrm{R}\Gamma_{\mathfrak{a}}(D) \overset{\mathrm{L}}{\otimes}_R k) \\ &\cong \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R k),\end{aligned} \tag{2.2.3}$$

and the second isomorphism is by lemma (1.8)(1), which applies because $k \in \mathbf{A}^{\mathfrak{a}}_{\mathrm{comp}}$ by lemma (1.7). This says that the left hand side of equation (2.2.2) is zero, so $\sup(D \overset{\mathrm{L}}{\otimes}_R k) = \inf(D \overset{\mathrm{L}}{\otimes}_R k)$, so $D \overset{\mathrm{L}}{\otimes}_R k$ only has homology in a single degree. By [6, eq. (A.7.4.1)] this says that D has finite projective dimension, so D is a non-zero complex in $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ with finite injective and projective dimensions. Hence R is Gorenstein by [8, prop. 2.10].

(1) $\Rightarrow$ (6), (6) $\Rightarrow$ (5), and (5) $\Rightarrow$ (1): These are proved by arguments dual to the ones given for (1) $\Rightarrow$ (4), (4) $\Rightarrow$ (3), and (3) $\Rightarrow$ (1). $\square$

(2.3) **Remark.** The reason that we refer to (2.2) as "The parametrized Gorenstein theorem" is that it is parametrized by the ideal $\mathfrak{a}$, and generalizes a number of "Gorenstein theorems" from the literature, as shown below.

(2.4) **Corollary.** *Recall from setup* (0.6) *that R is a commutative, local, noetherian ring with residue class field k. Now the following conditions are equivalent:*

(1) *R is Gorenstein.*
(2) *The biduality morphism*

$$X \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X, R), R)$$

is an isomorphism for $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

If R has a dualizing complex D, then the above conditions are also equivalent to:

(3) *$k \in \mathcal{A}_D$.*
(4) *$\mathcal{A}_D = \mathsf{D}(R)$.*
(5) *$k \in \mathcal{B}_D$.*
(6) *$\mathcal{B}_D = \mathsf{D}(R)$.*

Proof. Immediate from theorem (2.2) by setting $\mathfrak{a} = 0$. $\square$

(2.5) **Remark.** Note that corollary (2.4) contains several of the "Gorenstein theorems" from [6], namely, [6, (2.3.14) and (3.1.12), and (3.2.10)]. In fact, corollary (2.4) improves these results, since our classes $\mathcal{A}_D$ and $\mathcal{B}_D$ avoid the boundedness restrictions imposed in [6].

(2.6) **Corollary.** *Recall from setup* (0.6) *that R is a commutative, local, noetherian ring with maximal ideal $\mathfrak{m}$ and residue class field k, and that* $\mathrm{C}(\mathfrak{m})$ *denotes the Čech complex of $\mathfrak{m}$. Now the following conditions are equivalent:*

(1) *R is Gorenstein.*
(2) *The standard morphism*

$$X \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(\mathrm{C}(\mathfrak{m}), \mathrm{C}(\mathfrak{m})) \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_R(X, \mathrm{C}(\mathfrak{m})), \mathrm{C}(\mathfrak{m}))$$

is an isomorphism for $X \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

If R has a dualizing complex D, and $\mathrm{E}(k)$ *denotes the injective hull of k, then the above conditions are also equivalent to:*

(3) *$k \in \mathcal{A}_{\mathrm{E}(k)}$.*
(4) *$\mathcal{A}_{\mathrm{E}(k)} = \mathbf{A}^{\mathfrak{m}}_{\mathrm{comp}}$.*
(5) *$k \in \mathcal{B}_{\mathrm{E}(k)}$.*
(6) *$\mathcal{B}_{\mathrm{E}(k)} = \mathbf{A}^{\mathrm{tors}}_{\mathfrak{m}}$.*

Proof. Immediate from theorem (2.2) by setting $\mathfrak{a} = \mathfrak{m}$, since if D is a dualizing complex, shifted so that its leftmost homology module sits in degree $\dim R$, then $\mathrm{R}\Gamma_{\mathfrak{m}}(D) \cong \mathrm{E}(k)$ by the local duality theorem, [8, p. 155]. $\square$

(2.7) **Remark.** Note that if R has a dualizing complex, then corollary (2.6) implies the "Gorenstein sensitivity theorem" [11, thm. (3.5)]. Note also that part (2) of the corollary gives a new characterization of Gorenstein rings.

References

[1] L. Alonso Tarrío, A. Jeremías López, and J. Lipman, *Local homology and cohomology on schemes*, Ann. Sci. École Norm. Sup. (4) **30** (1997), 1–39.
[2] L. L. Avramov and H.-B. Foxby, *Ring homomorphisms and finite Gorenstein dimension,* Proc. London Math. Soc. (3) **75** (1997), 241–270.
[3] ———, ———, and S. Halperin, Differential graded homological algebra, in preparation.
[4] M. P. Brodmann and R. Y. Sharp, "Local cohomology: An algebraic introduction with geometric applications", Cambridge Stud. Adv. Math., Vol. 60, Cambridge University Press, Cambridge, 1998.
[5] W. Bruns and J. Herzog, "Cohen-Macaulay rings", Cambridge Stud. Adv. Math., Vol. 39, Cambridge University Press, Cambridge, 1998, revised edition.
[6] L. W. Christensen, "Gorenstein Dimensions", Lecture Notes in Math., Vol. 1747, Springer, Berlin, 2000.
[7] W. G. Dwyer and J. P. C. Greenlees, *Complete modules and torsion modules*, Amer. J. Math. **124** (2002), no. 1, 199–220.
[8] H.-B. Foxby, *Bounded complexes of flat modules*, J. Pure Appl. Algebra **15** (1979), 149–172.
[9] A. Frankild, *Quasi Cohen-Macaulay properties of local homomorphisms*, J. Algebra **235** (2001), 214–242.
[10] A. Frankild, *Vanishing of local homology*, Math. Z. (to appear).
[11] A. Frankild and P. Jørgensen, *Foxby equivalence, complete modules, and torsion modules*, J. Pure Appl. Algebra **174** (2002), 135–147.
[12] A. Grothendieck, "Residues and duality", Lecture Notes in Math., Vol. 20, Springer, Berlin, 1966. Notes by R. Hartshorne.
[13] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9** (1970), 145–164.
[14] E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528.
[15] H. Matsumura, "Commutative ring theory", Cambridge Stud. Adv. Math., Vol. 8, Cambridge University Press, Cambridge, 1989, second edition.
[16] P. Schenzel, Proregular sequences, local cohomology, and completion, preprint (1998).
[17] S. Yassemi, *G-dimension*, Math. Scand. **77** (1995), 161–174.

Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, 2100 København Ø, DK–Danmark

E-mail address: `frankild@math.ku.dk`

Danish National Library of Science and Medicine, Nørre Allé 49, 2200 København N, DK–Denmark

E-mail address: `pej@dnlb.dk`

Israel Journal of Math.
(to appear)

GORENSTEIN DIFFERENTIAL GRADED ALGEBRAS

ANDERS FRANKILD AND PETER JØRGENSEN

Abstract. We propose a definition of Gorenstein Differential Graded Algebra. In order to give examples, we introduce the technical notion of Gorenstein morphism. This enables us to prove the following:

Theorem. Let A be a noetherian local commutative ring, let L be a bounded complex of finitely generated projective A-modules which is not homotopy equivalent to zero, and let $\mathcal{E} = \operatorname{Hom}_A(L, L)$ be the endomorphism Differential Graded Algebra of L. Then $\mathcal{E}$ is a Gorenstein Differential Graded Algebra if and only if A is a Gorenstein ring.

Theorem. Let A be a noetherian local commutative ring with a sequence of elements $\boldsymbol{a} = (a_1, \ldots, a_n)$ in the maximal ideal, and let $\mathrm{K}(\boldsymbol{a})$ be the Koszul complex of $\boldsymbol{a}$. Then $\mathrm{K}(\boldsymbol{a})$ is a Gorenstein Differential Graded Algebra if and only if A is a Gorenstein ring.

Theorem. Let A be a noetherian local commutative ring containing a field k, and let X be a simply connected topological space with $\dim_k \mathrm{H}_*(X; k) < \infty$, which has Poincaré duality over k. Let $\mathrm{C}^*(X; A)$ be the singular cochain Differential Graded Algebra of X with coefficients in A. Then $\mathrm{C}^*(X; A)$ is a Gorenstein Differential Graded Algebra if and only if A is a Gorenstein ring.

The second of these theorems is a generalization of a result by Avramov and Golod from [4].

0. Introduction

Some parts of the homological theory of Differential Graded Algebras can be viewed as a generalization of the homological theory of rings. One of the central notions of this last theory is that of Gorenstein rings. Hence it is natural to seek to define Gorenstein Differential Graded Algebras.

We propose such a definition, and give criteria for when some naturally occuring Differential Graded Algebras (abbreviated DGAs henceforth) are Gorenstein in our sense.

1991 *Mathematics Subject Classification.* 16E45.

Key words and phrases. Differential Graded Algebra, Gorenstein condition, duality for Differential Graded modules, Gorenstein morphism, endomorphism Differential Graded Algebra, Koszul complex, singular cochain Differential Graded Algebra of a topological space.

(0.1) **Background.** The ring theoretical idea lying behind our definition of Gorenstein DGAs is the following: If A is a noetherian local commutative ring, then A is a Gorenstein ring precisely if the functor $\mathrm{RHom}_A(-, A)$ gives a duality, that is, a pair of quasi-inverse contravariant equivalences of categories,

$$\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A) \underset{\mathrm{RHom}_A(-,A)}{\overset{\mathrm{RHom}_A(-,A)}{\rightleftarrows}} \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A),$$

where $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$ is the derived category of bounded complexes of finitely generated A-modules, see [6, thm. (2.3.14)]. For this to happen is equivalent to the following two conditions:

- There is a natural isomorphism
$$M \longrightarrow \mathrm{RHom}_A(\mathrm{RHom}_A(M, A), A)$$
for M in $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$.
- $\mathrm{RHom}_A(-, A)$ sends $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$ to $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$.

(0.2) **Gorenstein DGAs.** In section 2 these conditions are what we shall use as direct inspiration for our definition of Gorenstein DGAs, given as definition (2.1) below. The definition has two parts, [G1] and [G2], which generalize the two above conditions directly, using derived categories of Differential Graded modules (abbreviated DG-modules henceforth).

To show right away that our definition of Gorenstein DGAs is reasonable, section 2 continues by considering some ordinary rings as DGAs concentrated in degree zero, showing that they are Gorenstein DGAs precisely when they are Gorenstein rings in the approprate classical sense (propositions (2.5) and (2.6)).

(0.3) **Gorenstein morphisms of DGAs.** In section 3 we introduce in definition (3.4) the key technical tool of Gorenstein morphisms of DGAs. These morphisms are modeled on Gorenstein homomorphisms from ring theory (see [2], [3], and [13]).

The purpose of considering Gorenstein morphisms is to get a practical tool which will enable us to determine in section 4 when some DGAs occuring in nature are Gorenstein DGAs.

The two main results on Gorenstein morphisms are:

- Theorem (3.6) (Ascent): Let $R \longrightarrow S$ be a Gorenstein morphism of DGAs. If R is a Gorenstein DGA, then S is also a Gorenstein DGA.
- Proposition (3.10) (Partial Descent): Let Q be a local commutative DGA with residue class field k, and let $Q \longrightarrow T$ be a (nice) Gorenstein morphism of DGAs. If T is a

Gorenstein DGA, then Q satisfies the Gorenstein condition $\dim_k \operatorname{Ext}_Q(k,Q) = 1$ from [2, sec. 3].

We conjecture in (3.7) that "Descent" holds in full generality, that is, if $R \longrightarrow S$ is a Gorenstein morphism of DGAs and S is a Gorenstein DGA, then R is a Gorenstein DGA, but are unable to prove this.

(0.4) **Examples.** In section 4 we determine when three naturally occuring types of DGAs, namely endomorphism DGAs of perfect complexes of modules, Koszul complexes, and singular cochain DGAs of topological spaces with Poincaré duality, are Gorenstein.

Let A be a noetherian local commutative ring, let L be a bounded complex of finitely generated projective modules (i.e. L is a so-called perfect complex) which is not homotopy equivalent to zero, and let $\mathcal{E} = \operatorname{Hom}_A(L,L)$ be the endomorphism DGA of L, see setup (4.1). We show the following "Ascent-Descent theorem",

- Theorem (4.5): A is a Gorenstein ring $\Leftrightarrow$ $\mathcal{E}$ is a Gorenstein DGA.

Also, let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a sequence of elements in the maximal ideal of A, and consider the corresponding Koszul complex $\mathrm{K}(\boldsymbol{a})$ which is a DGA, see setup (4.6). We show

- Theorem (4.9): A is a Gorenstein ring $\Leftrightarrow$ $\mathrm{K}(\boldsymbol{a})$ is a Gorenstein DGA.

Finally, suppose that A contains a field k. Let X be a simply connected topological space with $\dim_k \mathrm{H}_*(X;k) < \infty$, which has Poincaré duality over k, meaning that there is an isomorphism of graded $\mathrm{H}^*(X;k)$-modules

$$\mathrm{H}^*(X;k)' \cong \Sigma^d \, \mathrm{H}^*(X;k)$$

for some d, where the prime denotes dualization with respect to k, see setup (4.11). Consider $\mathrm{C}^*(X;A)$, the singular cochain DGA of X with coefficients in A, see paragraph (4.13). We show

- Theorem (4.16): A is a Gorenstein ring $\Leftrightarrow$ $\mathrm{C}^*(X;A)$ is a Gorenstein DGA.

Theorem (4.9) is a generalization of a result by Avramov and Golod from [4], which is confined to the case where $\boldsymbol{a}$ is a minimal set of generators for the maximal ideal of A.

(0.5) **Perspectives.** In the literature, there are several papers which consider Gorenstein conditions for *augmented* DGAs. In [9], Félix, Halperin, and Thomas consider augmented cochain DGAs; in [2], Avramov and Foxby consider augmented chain DGAs; and in the recent [8], Dwyer, Greenlees, and Iyengar consider more general augmented DGAs.

In a subsequent paper with Iyengar [11], we will show for most of the DGAs in question that the Gorenstein conditions from [2] and [9] coincide with our notion of Gorenstein DGA.

Note, however, that our setup differs from that of [2], [8], and [9], in that we do not use augmentations or other auxiliary data to define Gorenstein DGAs.

Also, we make it a point not to work only with chain or cochain DGAs, but rather to give a definition of Gorenstein DGA which is left/right symmetric.

Indeed, we shall see in section 4 that our theory can be applied to endomorphism DGAs which in general have no canonical augmentation, nor satisfy being either chain or cochain DGAs.

(0.6) **Acknowledgement.** This paper owes a great debt to [21] which was the first paper to introduce dualizing complexes in a non-commutative situation, and hence the first paper that had to deal with such ensuing complications as left-, right-, and bi-structures of modules and functors.

Another paper we should mention is [12] in which duality over DGAs is employed to prove an existence result for dualizing complexes over rings.

We thank professors Amnon Yekutieli and Srikanth Iyengar for their interest in this project.

The diagrams were typeset with Paul Taylor's `diagrams.tex`.

1. Notation and terminology

The purpose of this section is to fix the notation and terminology we shall use. For more details, see [14] or [16, part III].

(1.1) **DGAs.** A Differential Graded Algebra (DGA) R over the commutative ground ring $\Bbbk$ is a graded algebra $\{R_i\}_{i\in\mathbb{Z}}$ over $\Bbbk$ which is equipped with a $\Bbbk$-linear differential $\partial^R : R_* \longrightarrow R_{*-1}$ with square zero, satisfying the Leibnitz rule

$$\partial^R(rs) = \partial^R(r)s + (-1)^{|r|}r\partial^R(s)$$

when r is a graded element of degree $|r|$. Note that we almost exclusively employ homological notation, that is, lower indices and differentials of degree -1, and that we observe the Koszul sign convention of introducing a sign $(-1)^{mn}$ when graded objects of degrees m and n are interchanged.

The opposite DGA of R is denoted R^{opp} and is the same as R except that the product is changed to

$$r \overset{\mathrm{opp}}{\cdot} s = (-1)^{|r||s|}sr.$$

A morphism of DGAs over $\Bbbk$ is a morphism of graded algebras over $\Bbbk$ which is compatible with the differentials.

(1.2) **Notation.** In the rest of this section, R and S denote DGAs over the commutative ground ring $\Bbbk$.

Both in this section and in the rest of the paper, we will often suppres the ground ring $\Bbbk$ from the formulation of the results. If no canonical ground ring is present, then one can simply use $\Bbbk = \mathbb{Z}$.

(1.3) **DG-modules.** A Differential Graded R-left-module (DG-R-left-module) M is a graded left-module $\{M_i\}_{i\in\mathbb{Z}}$ over R (viewed as a graded algebra), which is equipped with a $\Bbbk$-linear differential $\partial^M : M_* \longrightarrow M_{*-1}$ with square zero, satisfying the Leibnitz rule

$$\partial^M(rm) = \partial^R(r)m + (-1)^{|r|} r\partial^M(m)$$

when r is a graded element of R of degree $|r|$.

DG-R-right-modules are defined similarly. Often we identify DG-R-right-modules with DG-R^{opp}-left-modules.

Note that we can also consider DG-modules having more than one DG-module structure, for instance DG-R-left-R-right-modules which would typically be denoted by ${}_RM_R$, or DG-R-left-S-right-modules which would typically be denoted by ${}_RN_S$. In such cases, all the different structures are required to be compatible; for a DG-R-left-R-right-module, compatibility means that the rule $(r_1m)r_2 = r_1(mr_2)$ holds. An example of a DG-R-left-R-right-module is R itself.

For a DG-module M we define the i'th suspension by

$$(\Sigma^i M)_j = M_{j-i}, \quad \partial_j^{\Sigma^i M} = (-1)^i \partial_{j-i}^M.$$

For each type of DG-modules (for instance, DG-R-left-modules or DG-R-left-R-right-modules), there is a notion of morphism. A morphism is a homomorphism of graded modules which is compatible with the differentials. Accordingly, each type of DG-modules forms an abelian category.

(1.4) **Homology.** A DG-module M is in particular a complex, so has homology which we denote $\mathrm{H}(M)$ or $\mathrm{H}M$.

The product in R induces a product in $\mathrm{H}R$ which becomes a graded algebra, and the action of R on a DG-module M induces an action of $\mathrm{H}R$ on $\mathrm{H}M$ which becomes a graded $\mathrm{H}R$-module.

(1.5) **Quasi-isomorphisms and derived categories.** If a morphism of DG-modules $M \longrightarrow N$ induces an isomorphism in homology $\mathrm{H}M \xrightarrow{\cong} \mathrm{H}N$, then the morphism is called a quasi-isomorphism, and is denoted $M \xrightarrow{\simeq} N$.

If we take one of the abelian categories of DG-modules introduced above and (formally) invert the quasi-isomorphisms, then we get the corresponding derived category of DG-modules which is a triangulated category. The derived category of DG-R-left-modules is denoted $\mathsf{D}(R)$.

Observe that when we identify DG-R-right-modules with DG-R^{opp}-left-modules, then we also identify the derived category of DG-R-right-modules with $\mathsf{D}(R^{\mathrm{opp}})$.

If $R \longrightarrow S$ is a morphism of DGAs over $\Bbbk$ which is moreover a quasi-isomorphism, then the derived categories $\mathsf{D}(R)$ and $\mathsf{D}(S)$ are equivalent as triangulated categories; see [16, III.4.2]. This obviously extends: If R and S are connected by a sequence of morphisms of DGAs over $\Bbbk$ all of which are quasi-isomorphisms,

$$R \xrightarrow{\simeq} T_1 \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} T_n \xleftarrow{\simeq} S,$$

then $\mathsf{D}(R)$ and $\mathsf{D}(S)$ are equivalent as triangulated categories. In this situation, R and S are called equivalent by a series of quasi-isomorphisms, or just equivalent, and are indistinguishable for homological purposes. In particular, R is a Gorenstein DGA in the sense of this paper if and only if S is a Gorenstein DGA, cf. paragraph (2.2).

(1.6) **Hom and Tensor.** On the abelian categories of DG-modules, we can define the functors Hom and $\otimes$:

If M and N are DG-R-left-modules, then $\mathrm{Hom}_R(M, N)$ is defined in a classical way, as the total complex of a certain double complex. The totaling is done taking products along diagonals.

Similarly, if A is a DG-R-right-module and B is a DG-R-left-module, then $A \otimes_R B$ is defined as the total complex of a certain double complex. The totaling is done taking coproducts along diagonals.

Note that extra structures on M, N, A, and B are inherited by Hom_R and $\otimes_R$. For instance, if ${}_RM$ is a DG-R-left-module and ${}_RN_S$ is a DG-R-left-S-right-module, then $\mathrm{Hom}_R({}_RM, {}_RN_S)$ is a DG-S-right-module.

(1.7) **Derived Hom and Tensor.** On the derived categories of DG-modules, we can define the functors right-derived Hom, denoted RHom, and left-derived $\otimes$, denoted $\overset{\mathrm{L}}{\otimes}$. The way to do this is to use appropriate resolutions:

Let P, I, and F be DG-R-left-modules. Then P is called K-projective, I is called K-injective, and F is called K-flat if the functors $\mathrm{Hom}_R(P, -)$, $\mathrm{Hom}_R(-, I)$, and $- \otimes_R F$ send quasi-isomorphisms to quasi-isomorphisms. By adjointness, a K-projective DG-module is also K-flat.

Now let M, N, and B be DG-R-left-modules. Then K-projective, K-injective, and K-flat resolutions of M, N, and B are quasi-isomorphisms of DG-R-left-modules $P \xrightarrow{\simeq} M$, $N \xrightarrow{\simeq} I$, and $F \xrightarrow{\simeq} B$ so that P is K-projective, I is K-injective, and F is K-flat. Such resolutions always exist; see [14] or [16, part III]. The original construction of such "unbounded" resolutions is due to [5] and [19].

With the resolutions, we can define $\mathrm{RHom}_R(M, N)$ as $\mathrm{Hom}_R(P, N)$ or $\mathrm{Hom}_R(M, I)$, and when A is a DG-R-right-module, we can define $A \overset{\mathrm{L}}{\otimes}_R B$ as $A \otimes_R F$. We could also define $A \overset{\mathrm{L}}{\otimes}_R B$ as $G \otimes_R B$, where $G \xrightarrow{\simeq} A$ is a K-flat resolution of A.

These definitions turn out to give well-defined functors on derived categories of DG-R-left- and DG-R-right-modules.

Extra structures on M, N, A, and B are inherited by RHom_R and $\overset{\mathrm{L}}{\otimes}_R$, but complications may arise: For instance, while it is always true that $\mathrm{RHom}_R({}_RM, {}_RN_S)$ is in the derived category of DG-S-right-modules, if we want to compute it as $\mathrm{Hom}_R(M, I)$ then we need a quasi-isomorphism ${}_RN_S \xrightarrow{\simeq} {}_RI_S$ of DG-R-left-S-right-modules so that ${}_RI$ is a K-injective DG-R-left-module. The existence of a resolution such as I is not guaranteed by [14] and [16, part III] (but see the next paragraph).

(1.8) **Existence of resolutions.** As we said in the previous paragraph, in case of DG-modules with two or more structures, existence of resolutions is a potential problem. We will comment on one important instance: Existence of a resolution of ${}_RR_R$ which is K-injective from the left and from the right, for the purpose of defining the functors $\mathrm{RHom}_R(-, {}_RR_R)$ and $\mathrm{RHom}_{R^{\mathrm{opp}}}(-, {}_RR_R)$ which play a large role in this paper.

Now, we can always define the derived functors $\mathrm{RHom}_R(-, {}_RR_R)$ and $\mathrm{RHom}_{R^{\mathrm{opp}}}(-, {}_RR_R)$, for we can simply use K-projective resolutions of the DG-R-left- and DG-R-right-modules in the first variables. However, it is valuable for computations (e.g. with biduality morphisms) also to be able to use a resolution in the second variable. To be precise, what we want is a quasi-isomorphism ${}_RR_R \xrightarrow{\simeq} {}_RI_R$ where ${}_RI$ and I_R are K-injective. This will give $\mathrm{RHom}_R(-, {}_RR_R) \simeq \mathrm{Hom}_R(-, {}_RI_R)$ and $\mathrm{RHom}_{R^{\mathrm{opp}}}(-, {}_RR_R) \simeq \mathrm{Hom}_{R^{\mathrm{opp}}}(-, {}_RI_R)$.

It is not clear how to get such an I, except in one case: If R itself is K-flat over the ground ring $\Bbbk$. In this case, we take the DG-R-left-R-right-module ${}_RR_R$ and view it as a DG-left-module over $R \otimes_{\Bbbk} R^{\mathrm{opp}}$, the "enveloping" DGA. It then has a K-injective resolution ${}_{R\otimes_{\Bbbk}R^{\mathrm{opp}}}R \xrightarrow{\simeq} {}_{R\otimes_{\Bbbk}R^{\mathrm{opp}}}I$. We can view this as a quasi-isomorphism of

DG-R-left-R-right-modules ${}_R R_R \xrightarrow{\simeq} {}_R I_R$, and here ${}_R I$ and I_R turn out to be K-injective. For ${}_R I$, this follows from the computation

$$\operatorname{Hom}_R(-, {}_R I) \simeq \operatorname{Hom}_{R \otimes_{\Bbbk} R^{\text{opp}}}(- \otimes_{\Bbbk} R, {}_{R \otimes_{\Bbbk} R^{\text{opp}}} I),$$

which shows that $\operatorname{Hom}_R(-, {}_R I)$ is the composition of the functors $- \otimes_{\Bbbk} R$ and $\operatorname{Hom}_{R \otimes_{\Bbbk} R^{\text{opp}}}(-, {}_{R \otimes_{\Bbbk} R^{\text{opp}}} I)$, both of which send quasi-isomorphisms to quasi-isomorphisms, the first because R is K-flat over $\Bbbk$, the second because ${}_{R \otimes_{\Bbbk} R^{\text{opp}}} I$ is K-injective over $R \otimes_{\Bbbk} R^{\text{opp}}$.

In general, R is not K-flat over $\Bbbk$. However, by [15, lem. 3.2(a)] there always exists a morphism $\widetilde{R} \longrightarrow R$ of DGAs over $\Bbbk$ which is a quasi-isomorphism, so that $\widetilde{R}$ *is* K-flat over $\Bbbk$.

In other words, if we are willing to replace our DGA with a quasi-isomorphic DGA, then we can always assume that there is a resolution ${}_R R_R \xrightarrow{\simeq} {}_R I_R$ so that ${}_R I$ and I_R are K-injective. Let us remind the reader from paragraph (1.5) that quasi-isomorphic DGAs are indistinguishable for homological purposes.

(1.9) **Definition (The category fin).** Suppose that $\mathrm{H}_0 R$ is a noetherian ring. Then by $\mathsf{fin}(R)$ we denote the full subcategory of the derived category $\mathsf{D}(R)$ which consists of DG-modules M so that $\mathrm{H}M$ is bounded, and so that each $\mathrm{H}_i M$ is finitely generated as an $\mathrm{H}_0 R$-module.

(1.10) **The centre.** A graded element c in a graded algebra H (which could be a DGA) is called central if it satisfies $cd = (-1)^{|c||d|} dc$ for all graded elements d. An arbitrary element in H is called central if all its graded components are central. The centre of H is the set of all central elements, and H is called commutative if all its elements are central.

(1.11) **DG-modules in the ring case.** Note that an ordinary ring A can be viewed as a DGA concentrated in degree zero. A DG-module over A (when A is viewed as a DGA) is then the same thing as a complex of modules over A (when A is viewed as a ring); the various derived categories of DG-A-modules are the same as the various ordinary derived categories over A; and the derived functors of Hom and $\otimes$ of DG-A-modules are the ordinary RHom_A and $\overset{\mathrm{L}}{\otimes}_A$. When A is noetherian, the category $\mathsf{fin}(A)$ equals $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$, the derived category of complexes with bounded, finitely generated homology.

2. Gorenstein DGAs

This section defines our notion of Gorenstein DGA, and shows that it behaves sensibly when specialized to some important types of ordinary rings.

(2.1) **Definition (Gorenstein DGAs).** Let R be a DGA for which $\mathrm{H}_0 R$ is a noetherian ring. We call R a $\mathsf{Gorenstein\ DGA}$ if it satisfies:

[G1]: For M in $\mathsf{fin}(R)$ and N in $\mathsf{fin}(R^{\mathrm{opp}})$ the following biduality morphisms are isomorphisms,

$$M \longrightarrow \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(M, {}_R R_R), {}_R R_R),$$
$$N \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_{R^{\mathrm{opp}}}(N, {}_R R_R), {}_R R_R).$$

[G2]: The functor $\mathrm{RHom}_R(-, {}_R R_R)$ maps $\mathsf{fin}(R)$ to $\mathsf{fin}(R^{\mathrm{opp}})$, and the functor $\mathrm{RHom}_{R^{\mathrm{opp}}}(-, {}_R R_R)$ maps $\mathsf{fin}(R^{\mathrm{opp}})$ to $\mathsf{fin}(R)$.

(2.2) **Invariance under quasi-isomorphism.** Note that conditions [G1] and [G2] only concern functors on derived categories. So if two DGAs are equivalent, then they are Gorenstein simultaneously.

(2.3) **Realizing the biduality morphisms.** From paragraph (1.8) we know that after replacing R by an equivalent DGA, we can assume that there exists a resolution ${}_R R_R \xrightarrow{\simeq} {}_R I_R$ so that ${}_R I$ and I_R are K-injective. Hence the biduality morphisms from condition [G1] can be realized as concrete biduality morphisms

$$M \longrightarrow \mathrm{Hom}_{R^{\mathrm{opp}}}(\mathrm{Hom}_R(M, {}_R I_R), {}_R I_R),$$
$$N \longrightarrow \mathrm{Hom}_R(\mathrm{Hom}_{R^{\mathrm{opp}}}(N, {}_R I_R), {}_R I_R).$$

(2.4) **Duality.** It is clear that if R is a Gorenstein DGA, then there is a duality, that is, a pair of quasi-inverse contravariant equivalences of categories,

$$\mathsf{fin}(R) \underset{\mathrm{RHom}_{R^{\mathrm{opp}}}(-,R)}{\overset{\mathrm{RHom}_R(-,R)}{\rightleftarrows}} \mathsf{fin}(R^{\mathrm{opp}}).$$

(2.5) **Proposition (Commutative rings).** *Let A be a noetherian commutative ring of finite Krull dimension. Then the following conditions are equivalent:*

(1): *When A is viewed as a DGA concentrated in degree zero, it is a Gorenstein DGA.*
(2): *The injective dimension* $\mathrm{id}_A(A)$ *is finite.*
(3): *For each prime ideal $\mathfrak{p}$ in A, the localization $A_{\mathfrak{p}}$ is a noetherian local commutative Gorenstein ring.*

Proof. It is well-known that (2) and (3) are equivalent.

(1) $\Rightarrow$ (3). We show that condition [G2] implies (3): Let $\mathfrak{p}$ be a prime ideal in A. It is clear that $A/\mathfrak{p}$ is in $\mathsf{fin}(A)$, so $\mathrm{RHom}_A(A/\mathfrak{p}, A)$ is in $\mathsf{fin}(A^{\mathrm{opp}}) = \mathsf{fin}(A)$ by condition [G2]. Localizing in $\mathfrak{p}$, we have

that $\mathrm{RHom}_A(A/\mathfrak{p},A)_\mathfrak{p}$ is in $\mathsf{fin}(A_\mathfrak{p})$. However, [6, lem. (A.4.5)] gives the first $\cong$ in

$$\begin{aligned}\mathrm{RHom}_A(A/\mathfrak{p},A)_\mathfrak{p} &\cong \mathrm{RHom}_{A_\mathfrak{p}}((A/\mathfrak{p})_\mathfrak{p},A_\mathfrak{p})\\ &\cong \mathrm{RHom}_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p},A_\mathfrak{p}),\end{aligned}$$

so we have

$$\mathrm{RHom}_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p},A_\mathfrak{p}) \in \mathsf{fin}(A_\mathfrak{p}).$$

In particular, $\mathrm{RHom}_{A_\mathfrak{p}}(A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p},A_\mathfrak{p})$ has bounded homology. As $A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ is the residue class field of the noetherian local commutative ring $A_\mathfrak{p}$, this proves that $A_\mathfrak{p}$ is Gorenstein by [17, thm. 18.2].

(2) $\Rightarrow$ (1). We must see that conditions [G1] and [G2] hold. Note that since A is commutative, the two halves of condition [G1] are equivalent, and the two halves of condition [G2] are equivalent.

[G1]: Since we have $\mathrm{id}_A(A) < \infty$, there exists an injective resolution $A \xrightarrow{\simeq} I$ so that I is bounded. Also, any M in $\mathsf{fin}(A)$ has homology which is bounded to the right and consists of finitely generated A-modules. This shows that the biduality morphisms in condition [G1] are isomorphisms by [6, (A.4.24)].

[G2]: Given M in $\mathsf{fin}(A)$ we have $\mathrm{RHom}_A(M,A) \cong \mathrm{Hom}_A(P,A) \cong \mathrm{Hom}_A(M,I)$, where I is the resolution from above and $P \xrightarrow{\simeq} M$ is a projective resolution which can be chosen to consist of finitely generated projective A-modules. Now $\mathrm{Hom}_A(M,I)$ has bounded homology (because M has bounded homology while I is bounded), and $\mathrm{Hom}_A(P,A)$ has finitely generated homology modules (because P consists of finitely generated projective modules). Hence we have $\mathrm{RHom}_A(M,A) \in \mathsf{fin}(A)$, so [G2] holds. □

(2.6) **Proposition (Non-commutative local rings).** *Let k be a field, and A a k-algebra which is noetherian semilocal PI (see* [20]*). Then the following conditions are equivalent:*

- (1): *When A is viewed as a DGA concentrated in degree zero, it is a Gorenstein DGA.*
- (2): *The injective dimensions $\mathrm{id}_A(A)$ and $\mathrm{id}_{A^{\mathrm{opp}}}(A)$ are finite.*

Proof. (1) $\Rightarrow$ (2). We show that condition [G2] implies (2): Let $\mathrm{J}(A)$ be the Jacobson radical of A, and let A_0 be $A/\mathrm{J}(A)$. Clearly, A_0 can be viewed either as a DG-A-left-module and as such it is in $\mathsf{fin}(A)$, or as a DG-A-right-module and as such it is in $\mathsf{fin}(A^{\mathrm{opp}})$. So condition [G2] implies that $\mathrm{RHom}_A(A_0,A)$ and $\mathrm{RHom}_{A^{\mathrm{opp}}}(A_0,A)$ have bounded homology. By [20, prop. 5.7(1)] this says $\mathrm{id}_A(A) < \infty$ and $\mathrm{id}_{A^{\mathrm{opp}}}(A) < \infty$.

(2) $\Rightarrow$ (1). This is completely analogous to the proof of (2) $\Rightarrow$ (1) in proposition (2.5). □

We suspect that proposition (2.6) is far from optimal. In fact, we propose the following conjecture:

(2.7) **Conjecture (Non-commutative rings).** The conditions in proposition (2.6) are equivalent for all noetherian rings of finite left- and right-Krull dimension.

Proposition (2.5) makes the conjecture seem reasonable. See [18, chp. 6] for the definition of Krull dimension over non-commutative rings.

3. Gorenstein morphisms of DGAs

This section defines what we call Gorenstein morphisms of DGAs, in order to make us able to give examples of Gorenstein DGAs. We show that these morphisms are capable of transporting Gorenstein properties back and forth between source and target, in a way analogous to ring theory.

The following is a generalization of finite ring homomorphisms of finite flat dimension:

(3.1) **Definition (Finite morphisms).** Let R and S be DGAs for which H_0R and H_0S are noetherian rings, and let $R \xrightarrow{\rho} S$ be a morphism of DGAs. We call ρ a $\mathsf{finite\ morphism}$ if it satisfies:

- The functor ${}_SS_R \overset{\mathrm{L}}{\otimes}_R - : \mathsf{D}(R) \longrightarrow \mathsf{D}(S)$ sends $\mathsf{fin}(R)$ to $\mathsf{fin}(S)$.
- The functor $- \overset{\mathrm{L}}{\otimes}_R {}_RS_S : \mathsf{D}(R^{\mathrm{opp}}) \longrightarrow \mathsf{D}(S^{\mathrm{opp}})$ sends $\mathsf{fin}(R^{\mathrm{opp}})$ to $\mathsf{fin}(S^{\mathrm{opp}})$.
- The functor $\rho^* : \mathsf{D}(S) \longrightarrow \mathsf{D}(R)$, restricting scalars from S to R, satisfies
$$M \in \mathsf{fin}(S) \Leftrightarrow \rho^* M \in \mathsf{fin}(R).$$
- The functor $\rho^* : \mathsf{D}(S^{\mathrm{opp}}) \longrightarrow \mathsf{D}(R^{\mathrm{opp}})$, restricting scalars from S to R, satisfies
$$M \in \mathsf{fin}(S^{\mathrm{opp}}) \Leftrightarrow \rho^* M \in \mathsf{fin}(R^{\mathrm{opp}}).$$

(Note that ρ^* is used to denote the functor which restricts scalars from S to R both on DG-S-left-modules and on DG-S-right-modules.)

(3.2) **Finite morphisms in the ring case.** If $A \xrightarrow{\varphi} B$ is a homomorphism of noetherian local commutative rings, then we can view φ as a morphism of DGAs. As such it is finite precisely if B, viewed as an A-module, is finitely generated and of finite flat dimension.

Let us next generalize finite Gorenstein homomorphisms of rings:

Reading [3, lem. (6.5), (7.7.1), and thm. (7.8)] one can see that if A and B are noetherian local commutative rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, and $A \xrightarrow{\varphi} B$ is a local homomorphism so that B viewed as an A-module is finitely generated and of finite flat dimension, then φ is "Gorenstein at $\mathfrak{n}$" in the sense of [3] precisely if $\mathrm{RHom}_A(B, A)$ is isomorphic to $\Sigma^{-n}B$ for some n. We shall attempt in definition (3.4) to generalize this to the world of non-commutative DGAs.

Already the non-commutativity makes a refinement necessary as also observed in [13], since in a non-commutative situation the complexes $\mathrm{RHom}_A(B, A)$ and $\mathrm{RHom}_{A^{\mathrm{opp}}}(B, A)$ have different structures (the first has B-left-A-right-structure, the second has B-right-A-left-structure). Hence the technical nature of the following two paragraphs.

(3.3) **Induced morphisms.** Let $R \xrightarrow{\rho} S$ be a morphism of DGAs. Given a morphism ${}_SS_R \xrightarrow{\alpha} \mathrm{RHom}_R({}_RS_S, \Sigma^n({}_RR_R))$ and a DG-S-left-module M we can consider

$$\begin{aligned} &\rho^* \mathrm{RHom}_S(M, {}_SS_S) \\ &\qquad = \quad \mathrm{RHom}_S(M, {}_SS_R) \\ &\qquad \xrightarrow{\mathrm{RHom}_S(M,\alpha)} \quad \mathrm{RHom}_S(M, \mathrm{RHom}_R({}_RS_S, \Sigma^n({}_RR_R))) \\ &\qquad \xrightarrow{\text{adjointness}} \quad \mathrm{RHom}_R({}_RS_S \overset{\mathrm{L}}{\otimes}_S M, \Sigma^n({}_RR_R)) \\ &\qquad = \quad \mathrm{RHom}_R(\rho^* M, \Sigma^n({}_RR_R)). \end{aligned}$$

In short, this gives an induced morphism

$$\rho^* \mathrm{RHom}_S(M, {}_SS_S) \longrightarrow \mathrm{RHom}_R(\rho^* M, \Sigma^n({}_RR_R)), \tag{1}$$

which is an isomorphism if α is an isomorphism.

Similarly, given a morphism ${}_RS_S \xrightarrow{\beta} \mathrm{RHom}_{R^{\mathrm{opp}}}({}_SS_R, \Sigma^n({}_RR_R))$ and a DG-S-right-module N there is an induced morphism

$$\rho^* \mathrm{RHom}_{S^{\mathrm{opp}}}(N, {}_SS_S) \longrightarrow \mathrm{RHom}_{R^{\mathrm{opp}}}(\rho^* N, \Sigma^n({}_RR_R)), \tag{2}$$

which is an isomorphism if β is an isomorphism.

(3.4) **Definition (Gorenstein morphisms).** Let R and S be DGAs for which H_0R and H_0S are noetherian, and let $R \xrightarrow{\rho} S$ be a finite morphism of DGAs. We call ρ a Gorenstein morphism if it satisfies:

- (1): There are isomorphisms
  - (a): ${}_SS_R \xrightarrow{\alpha} \mathrm{RHom}_R({}_RS_S, \Sigma^n({}_RR_R))$.
  - (b): ${}_RS_S \xrightarrow{\beta} \mathrm{RHom}_{R^{\mathrm{opp}}}({}_SS_R, \Sigma^n({}_RR_R))$.
- (2): The isomorphisms α and β are compatible in the following sense:

(a): For each DG-S-left-module M the following square is commutative,

$$\begin{tikzcd}
\rho^* M \arrow[r, "\rho^* t"] \arrow[d, "s"'] & \rho^* \operatorname{RHom}_{S^{\mathrm{opp}}}(\operatorname{RHom}_S(M, {}_S S_S), {}_S S_S) \arrow[d, "\cong", "b"'] \\
\operatorname{RHom}_{R^{\mathrm{opp}}}(\operatorname{RHom}_R(\rho^* M, \Sigma^n({}_R R_R)), \Sigma^n({}_R R_R)) \arrow[r, "\cong", "a"'] & \operatorname{RHom}_{R^{\mathrm{opp}}}(\rho^* \operatorname{RHom}_S(M, {}_S S_S), \Sigma^n({}_R R_R)),
\end{tikzcd}$$

where s and t are biduality morphisms as in condition [G1], and where a and b are induced by α and β as explained in paragraph (3.3).

(b): For each DG-S-right-module N there is a commutative square constructed like the one above.

(3.5) **Gorenstein morphisms in the ring case.** Note from the observations before paragraph (3.3) that if A and B are noetherian local commutative rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, and $A \xrightarrow{\varphi} B$ is a local ring homomorphism so that B viewed as an A-module is finitely generated and of finite flat dimension, then φ is a Gorenstein morphism of DGAs in the sense of definition (3.4) precisely if φ is "Gorenstein at $\mathfrak{n}$" in the sense of [3].

(3.6) **Theorem (Ascent).** *Let R and S be DGAs for which $\mathrm{H}_0 R$ and $\mathrm{H}_0 S$ are noetherian, and let $R \xrightarrow{\rho} S$ be a finite Gorenstein morphism of DGAs. Then*

$$R \textit{ is a Gorenstein DGA } \Rightarrow S \textit{ is a Gorenstein DGA.}$$

Proof. We prove the theorem by showing that condition [G1] for R implies condition [G1] for S, and that condition [G2] for R implies condition [G2] for S.

[G1]: Let us assume condition [G1] for R. In the first half of condition [G1] for S we are given M in $\mathsf{fin}(S)$ and must show that the biduality morphism

$$M \xrightarrow{t} \operatorname{RHom}_{S^{\mathrm{opp}}}(\operatorname{RHom}_S(M, {}_S S_S), {}_S S_S)$$

is an isomorphism. This is equivalent to showing that $\rho^* t$ is an isomorphism, because both things amount to seeing that t becomes bijective when the homology functor H is applied to it.

But $\rho^* t$ is one of the morphisms in the diagram in definition (3.4), part (2)(a), and if we can prove that the other arrows in the diagram are isomorphisms, then it follows that $\rho^* t$ is too. The only of the diagram's other arrows which is not a priori an isomorphism is s. And in the situation at hand, ρ is a finite morphism so $\rho^* M$ is in $\mathsf{fin}(R)$, and condition [G1] for R then says that s is an isomorphism.

The second half of condition [G1] for S is proved in a symmetrical way.

[G2]: Let us assume condition [G2] for R. In the first half of condition [G2] for S we must show $\mathrm{RHom}_S(M, {}_SS_S) \in \mathsf{fin}(S^{\mathrm{opp}})$ for M in $\mathsf{fin}(S)$. Since ρ is a finite morphism, this is equivalent to showing $\rho^* \,\mathrm{RHom}_S(M, {}_SS_S) \in \mathsf{fin}(R^{\mathrm{opp}})$, and since ρ is a Gorenstein morphism, paragraph (3.3) implies that this is the same as showing

$$\mathrm{RHom}_R(\rho^* M, \Sigma^n({}_RR_R)) \in \mathsf{fin}(R^{\mathrm{opp}}). \tag{3}$$

But as ρ is a finite morphism, $\rho^* M$ is in $\mathsf{fin}(R)$, and condition [G2] for R then says that (3) holds.

The second half of condition [G2] for S is proved in a symmetrical way. □

Theorem (3.6) says that Gorenstein morphisms transfer the Gorenstein property of DGAs in the direction of the morphism. This is analogous to the situation in commutative ring theory, see [3, (7.7.2)], and non-commutative ring theory, see [13, thm. 4.7].

Moreover, in view of [3, (7.7.2)] and [13, thm. 4.7], we venture the following conjecture that Gorenstein morphisms also transfer the Gorenstein property of DGAs in the direction opposite to the morphism:

(3.7) **Conjecture (Descent).** In the situation of theorem (3.6), we have

$$S \text{ is a Gorenstein DGA } \Rightarrow R \text{ is a Gorenstein DGA.}$$

Unfortunately, we are unable to prove conjecture (3.7). As consolation, we aim for proposition (3.10) below.

(3.8) **Local commutative DGAs.** Let Q be a DGA. Then Q is called local commutative if it satisfies:

- $Q_i = 0$ for $i < 0$.
- Q is commutative and the commutative ring $\mathrm{H}_0 Q$ is noetherian and local.
- Viewed as a DG-Q-module, Q is in $\mathsf{fin}(Q)$.

In this case, the residue class field k of $\mathrm{H}_0 Q$ is also called the residue class field of Q. It is easy to see that one can get a DG-Q-module by placing k in degree zero, and zero in all other degrees. This DG-module is again denoted k.

Note that an ordinary noetherian local commutative ring placed in degree zero is a local commutative DGA.

(3.9) **Morphisms with image in the centre.** Let Q and T be DGAs with Q commutative, and let $Q \xrightarrow{\varphi} T$ be a morphism of

DGAs with image inside the centre of T (see paragraph (1.10)). Now φ makes it possible to view T as a DG-Q-left-Q-right-module in a way which is compatible with the structure of T as DG-T-left-T-right-module. In other words, T can be viewed as a DG-module with structure ${}_{Q,T}T_{Q,T}$.

Note that the Q-left- and Q-right-structures of ${}_{Q,T}T_{Q,T}$ are equivalent in the sense that $qt = (-1)^{|q||t|}tq$ holds for graded elements q and t in Q and T. In fact, Q behaves almost like a commutative ring of scalars, so we will frequently omit the subscripts indicating Q-structures.

(3.10) **Proposition (Partial Descent).** *Suppose given the following data:*

(1): *Q is a local commutative DGA with residue class field k.*
(2): *T is a DGA with H_0T noetherian.*
(3): *$Q \xrightarrow{\varphi} T$ is a finite Gorenstein morphism of DGAs which has image inside the centre of T.*
(4): *There is a K-projective resolution of T viewed as a DG-Q-module, $P \xrightarrow{\simeq} T$, so that P is minimal over Q, i.e. $P \otimes_Q k$ has zero differential.*
(5): *We have $T \overset{\mathrm{L}}{\otimes}_Q k \ncong 0$.*

Then

$$T \text{ satisfies condition } [\mathrm{G2}] \Rightarrow \dim_k \mathrm{Ext}_Q(k,Q) = 1.$$

Remark. The right hand side in the implication is the Gorenstein condition from [2, sec. 3].

Proof. The DG-Q-module k is clearly in $\mathsf{fin}(Q)$. So ${}_TT \overset{\mathrm{L}}{\otimes}_Q k$ is in $\mathsf{fin}(T)$ since φ is a finite morphism, so condition [G2] on T implies that $\mathrm{RHom}_T({}_TT \overset{\mathrm{L}}{\otimes}_Q k, {}_TT_T)$ is in $\mathsf{fin}(T^{\mathrm{opp}})$. Applying φ^*, the functor which restricts scalars from T to Q, gives that $\varphi^* \mathrm{RHom}_T({}_TT \overset{\mathrm{L}}{\otimes}_Q k, {}_TT_T)$ is in $\mathsf{fin}(Q)$, again since φ is a finite morphism. Writing this in a simpler way, we get

$$\mathrm{RHom}_T({}_TT \overset{\mathrm{L}}{\otimes}_Q k, {}_TT) \in \mathsf{fin}(Q). \tag{4}$$

However, φ is a Gorenstein morphism, so by paragraph (3.3) there is an isomorphism

$$\varphi^* \mathrm{RHom}_T({}_TT \overset{\mathrm{L}}{\otimes}_Q k, {}_TT_T) \xrightarrow{\cong} \mathrm{RHom}_Q(\varphi^*({}_TT \overset{\mathrm{L}}{\otimes}_Q k), \Sigma^n Q),$$

and writing this in a simpler way gives

$$\mathrm{RHom}_T({}_TT \overset{\mathrm{L}}{\otimes}_Q k, {}_TT) \xrightarrow{\cong} \mathrm{RHom}_Q(T \overset{\mathrm{L}}{\otimes}_Q k, \Sigma^n Q). \tag{5}$$

Now, we have

$$T \overset{\mathrm{L}}{\otimes}_Q k \cong P \otimes_Q k \cong \coprod_{i \in I} \Sigma^{\beta_i} k$$

where the second $\cong$ is because P is minimal over Q. Here the β_i are integers and I is a non-empty index set since $T \overset{\mathrm{L}}{\otimes}_Q k \not\cong 0$. Substituting this into the right hand side of (5) gives

$$\begin{aligned} \mathrm{RHom}_T({}_T T \overset{\mathrm{L}}{\otimes}_Q k, {}_T T) &\xrightarrow{\cong} \mathrm{RHom}_Q(\coprod_{i \in I} \Sigma^{\beta_i} k, \Sigma^n Q) \\ &\cong \prod_{i \in I} \Sigma^{n-\beta_i} \mathrm{RHom}_Q(k, Q). \end{aligned}$$

By equation (4) the left hand side is in $\mathsf{fin}(Q)$, so the same must hold for the right hand side.

But then $\mathrm{RHom}_Q(k, Q)$ itself is certainly in $\mathsf{fin}(Q)$, so in particular $\mathrm{RHom}_Q(k, Q)$ has bounded homology, and by [2, thm. (3.1)] this implies $\dim_k \mathrm{Ext}_Q(k, Q) = 1$. □

There are examples of Gorenstein morphisms occuring in nature. Indeed, in the next section they are our chief tool to show that some DGAs occuring in nature are Gorenstein. The following lemma gives a way to obtain Gorenstein morphisms.

(3.11) **Lemma (A way to obtain Gorenstein morphisms).** *Suppose given the following data:*

(1): *Q is a commutative DGA with $\mathrm{H}_0 Q$ noetherian.*
(2): *T is a DGA with $\mathrm{H}_0 T$ noetherian.*
(3): *$Q \xrightarrow{\varphi} T$ is a finite morphism of DGAs which has image inside the centre of T.*
(4): *There is an isomorphism in the derived category of DG-T-left-T-right-modules,*

$${}_T T_T \xrightarrow{\gamma} \mathrm{RHom}_Q({}_T T_T, \Sigma^n Q).$$

Then φ is a Gorenstein morphism of DGAs.

Proof. Restricting the right-structure from T to Q, the morphism γ restricts to an isomorphism

$${}_T T_Q \xrightarrow{\alpha} \mathrm{RHom}_Q({}_Q T_T, \Sigma^n({}_Q Q_Q))$$

as in definition (3.4)(1)(a), and restricting the left-structure from T to Q, the morphism γ restricts to an isomorphism

$${}_Q T_T \xrightarrow{\beta} \mathrm{RHom}_{Q^{\mathrm{opp}}}({}_T T_Q, \Sigma^n({}_Q Q_Q))$$

as in (3.4)(1)(b).

To prove the lemma, we must see that the α and β so obtained are compatible in the sense of (3.4)(2). So we must see that the diagrams from (3.4)(2), with Q and T in place of R and S, are commutative. This is easy, but tedious: Take for instance the diagram from (3.4)(2)(a). To see that it is commutative, we need to replace the various modules with suitable resolutions so that the derived Hom's become ordinary Hom's, the derived tensors become ordinary tensors, and the morphisms can be computed explicitly. A good choice is to start by picking a K-injective resolution $Q \xrightarrow{\simeq} I$ and go on by using ${}_TJ_T = \mathrm{Hom}_Q({}_TT_T, \Sigma^n I)$ as a resolution of ${}_TT_T$. The DG-module ${}_TJ_T$ has the virtue of being isomorphic to $\mathrm{RHom}_Q({}_TT_T, \Sigma^n Q)$, while being K-injective from the left and K-injective from the right. Having introduced resolutions, the computation to check commutativity is a matter of patience. $\square$

4. Examples: Endomorphism DGAs, Koszul complexes, and singular cochain DGAs of topological spaces

This section considers three types of DGAs: Endomorphism DGAs of perfect complexes of modules, Koszul complexes, and singular cochain DGAs of topological spaces with Poincaré duality, all over noetherian local commutative rings. Using the theory of section 3, we give complete criteria for when these are Gorenstein DGAs: They are so if and only if the base ring is Gorenstein. The proofs work by showing that a suitable morphism from the base ring to the DGA in question is a Gorenstein morphism.

Endomorphism DGAs. The following paragraph recapitulates the definition of endomorphism DGAs of perfect complexes of modules; see [7] for more details.

(4.1) **Setup.** In paragraphs (4.1) to (4.5) we consider the following situation: A is a noetherian local commutative ring, L is a bounded complex of finitely generated projective A-modules which is not homotopy equivalent to zero, and we look at $\mathcal{E} = \mathrm{Hom}_A(L, L)$.

A priori, $\mathcal{E}$ is just a complex of A-modules. However, there is a multiplication on $\mathcal{E}$ given by composition: An element ϵ in $\mathcal{E}_i$ is an A-linear map $L \xrightarrow{\epsilon} \Sigma^{-i} L$. If we also have an element ϵ' in $\mathcal{E}_j$, then we define the product $\epsilon\epsilon'$ as the composition $\Sigma^{-j}(\epsilon) \circ \epsilon'$ which is an A-linear map $L \xrightarrow{\epsilon\epsilon'} \Sigma^{-(i+j)} L$, that is, an element in $\mathcal{E}_{i+j}$. It is not hard to check that with this multiplication, $\mathcal{E}$ is a DGA.

The complex L becomes a DG-$\mathcal{E}$-left-module with scalar multiplication $\epsilon\ell = \epsilon(\ell)$ for ϵ in $\mathcal{E}$ and ℓ in L. The $\mathcal{E}$-structure on L is compatible with the A-structure, so L is a DG-A-left-$\mathcal{E}$-left-module,

${}_{A,\mathcal{E}}L$. Moreover, the identification map is an isomorphism of DG-$\mathcal{E}$-left-$\mathcal{E}$-right-modules,

$$ {}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}} \xrightarrow{\cong} \operatorname{Hom}_A({}_{A,\mathcal{E}}L, {}_{A,\mathcal{E}}L). $$

(4.2) **Remark.** Note that $\mathcal{E}$ and its homology $\mathrm{H}\mathcal{E}$ are usually far from commutative. For instance, if L is the projective resolution of a finitely generated A-module of finite projective dimension, M, then we have $\mathrm{H}_0\mathcal{E} \cong \operatorname{End}_A(M)$. Also, $\mathcal{E}$ usually has non-zero homology both in positive and negative degrees.

(4.3) **The morphism $\varphi_{\mathcal{E}}$.** Since A is commutative, each element a in A gives a chain map $L \xrightarrow{a\cdot} L$ which is just multiplication by a. This chain map is an element in the degree 0 component of $\operatorname{Hom}_A(L, L)$. In other words, it is an element in $\operatorname{Hom}_A(L, L)_0 = \mathcal{E}_0$. One checks easily that this gives a morphism of DGAs,

$$ A \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}, \qquad a \longmapsto (L \xrightarrow{a\cdot} L). $$

Here are three useful observations:

- The ring $\mathrm{H}_0\mathcal{E}$ is noetherian since it is a finitely generated A-module.
- The morphism $\varphi_{\mathcal{E}}$ has image inside the centre of $\mathcal{E}$, because an element ϵ in $\mathcal{E}_i$ is an A-linear map $L \xrightarrow{\epsilon} \Sigma^{-i}L$ whence
$$ \begin{aligned} (\varphi_{\mathcal{E}}(a)\epsilon)(\ell) &= (\Sigma^{-i}(\varphi_{\mathcal{E}}(a)) \circ \epsilon)(\ell) = a\epsilon(\ell) \\ &= \epsilon(a\ell) = (\epsilon \circ \varphi_{\mathcal{E}}(a))(\ell) = (\epsilon\varphi_{\mathcal{E}}(a))(\ell) \end{aligned} $$
so $\varphi_{\mathcal{E}}(a)\epsilon = \epsilon\varphi_{\mathcal{E}}(a)$.
- The morphism $\varphi_{\mathcal{E}}$ is a finite morphism of DGAs since L and hence $\mathcal{E}$ are bounded complexes of finitely generated projective A-modules.

(4.4) **Lemma ($\varphi_{\mathcal{E}}$ is Gorenstein).** *The morphism $A \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}$ is a Gorenstein morphism.*

Proof. We shall use lemma (3.11) with the morphism $Q \xrightarrow{\varphi} T$ equal to $A \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}$. The observations in paragraph (4.3) show that the lemma's conditions (1) to (3) hold. If we can show that the lemma's condition (4) also holds, then the lemma gives our desired conclusion, that $\varphi_{\mathcal{E}}$ is Gorenstein.

So we must find an isomorphism ${}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}} \xrightarrow{\gamma} \operatorname{RHom}_A({}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}}, \Sigma^n A)$ in the derived category of DG-$\mathcal{E}$-left-$\mathcal{E}$-right-modules. Since $\mathcal{E}$ is K-projective over A, we have $\operatorname{RHom}_A({}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}}, \Sigma^n A) \cong \operatorname{Hom}_A({}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}}, \Sigma^n A)$, so it is enough to find a quasi-isomorphism in the abelian category of DG-$\mathcal{E}$-left-$\mathcal{E}$-right-modules,

$$ {}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}} \xrightarrow{\simeq} \operatorname{Hom}_A({}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}}, \Sigma^n A). $$

However, ${}_{A,\mathcal{E}}L$ is a bounded complex of finitely generated projective A-modules, so by [1, sec. 1, thms. 1 and 2] the two so-called evaluation morphisms appearing as the last two arrows in the following diagram are isomorphisms,

$$\begin{tikzcd}
\operatorname{Hom}_A({}_{A,\mathcal{E}}L, {}_{A,\mathcal{E}}L) \arrow[d, "\cong"'] & \\
\operatorname{Hom}_A({}_{A,\mathcal{E}}L, A \otimes_A {}_{A,\mathcal{E}}L) & \\
\operatorname{Hom}_A({}_{A,\mathcal{E}}L, A) \otimes_A {}_{A,\mathcal{E}}L \arrow[u, "\cong"] \arrow[r, "\cong"] & \operatorname{Hom}_A(\operatorname{Hom}_A({}_{A,\mathcal{E}}L, {}_{A,\mathcal{E}}L), A).
\end{tikzcd}$$

Substituting ${}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}} \cong \operatorname{Hom}_A({}_{A,\mathcal{E}}L, {}_{A,\mathcal{E}}L)$ twice, this gives an isomorphism

$${}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}} \xrightarrow{\cong} \operatorname{Hom}_A({}_{\mathcal{E}}\mathcal{E}_{\mathcal{E}}, A),$$

which is in particular a quasi-isomorphism, as desired. $\square$

(4.5) **Theorem (Ascent-Descent for endomorphism DGAs).** *In the situation of setup* (4.1), *we have*

$$A \text{ is a Gorenstein ring } \Leftrightarrow \ \mathcal{E} \text{ is a Gorenstein DGA.}$$

Proof. $\Rightarrow$: We will use theorem (3.6) with $R \xrightarrow{\rho} S$ equal to $A \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}$ to see this.

Paragraph (4.3) and lemma (4.4) say that the hypotheses of theorem (3.6) hold, so the theorem applies.

Now, A is noetherian commutative and has finite Krull dimension. So if A is a Gorenstein ring then proposition (2.5) says that A viewed as a DGA is a Gorenstein DGA. And theorem (3.6) then implies that $\mathcal{E}$ is a Gorenstein DGA.

$\Leftarrow$: We will use proposition (3.10) with $Q \xrightarrow{\varphi} T$ equal to $A \xrightarrow{\varphi_{\mathcal{E}}} \mathcal{E}$ to see this.

Indeed, proposition (3.10) applies: The proposition's condition (1) clearly holds, and conditions (2) and (3) hold by paragraph (4.3) and lemma (4.4). Condition (4) holds since $\mathcal{E}$ is a bounded complex of finitely generated projective A-modules, hence has homology which is bounded and finitely generated over A, hence has a minimal projective resolution over A. Condition (5) holds since L is not homotopy equivalent to zero whence the homology of $\mathcal{E}$ cannot be zero, so $\mathcal{E} \overset{\mathrm{L}}{\otimes}_A k \not\cong 0$.

So if $\mathcal{E}$ is a Gorenstein DGA, then proposition (3.10) gives

$$\dim_k \operatorname{Ext}_A(k, A) = 1$$

which implies that A is a Gorenstein ring by [17, thm. 18.1]. □

Koszul complexes. The following paragraph recapitulates the definition of Koszul complexes.

(4.6) **Setup.** In paragraphs (4.6) to (4.10) we consider the following situation: A is a noetherian local commutative ring, and $\boldsymbol{a} = (a_1, \ldots, a_n)$ is a sequence of elements in the maximal ideal of A.

We can construct the so-called Koszul complex $\mathrm{K}(\boldsymbol{a})$ of $\boldsymbol{a}$ which is a DGA: As a graded algebra, $\mathrm{K}(\boldsymbol{a})$ is simply the exterior algebra $\bigwedge F$ on the free module $F = Ae_1 \oplus \cdots \oplus Ae_n$. To get a DGA, we introduce the differential

$$\partial_j^{\mathrm{K}(\boldsymbol{a})}(e_{s_1} \wedge \cdots \wedge e_{s_j}) = \sum_i (-1)^{i+1} a_{s_i} e_{s_1} \wedge \cdots \wedge \widehat{e_{s_i}} \wedge \cdots \wedge e_{s_j},$$

where the hat indicates that e_{s_i} is left out of the wedge product.

(4.7) **The morphism $\varphi_{\mathrm{K}(\boldsymbol{a})}$.** There is a morphism of DGAs

$$A \xrightarrow{\varphi_{\mathrm{K}(\boldsymbol{a})}} \mathrm{K}(\boldsymbol{a})$$

given by noting that the degree zero component of $\mathrm{K}(\boldsymbol{a})$ is A itself. As in paragraph (4.3), here are three useful observations:

- The ring $\mathrm{H}_0\mathrm{K}(\boldsymbol{a})$ is noetherian since it is a finitely generated A-module. Also, $\mathrm{K}(\boldsymbol{a})$ is a commutative DGA.
- The morphism $\varphi_{\mathrm{K}(\boldsymbol{a})}$ has image inside the centre of $\mathrm{K}(\boldsymbol{a})$, since the centre is all of $\mathrm{K}(\boldsymbol{a})$.
- The morphism $\varphi_{\mathrm{K}(\boldsymbol{a})}$ is a finite morphism of DGAs since $\mathrm{K}(\boldsymbol{a})$ is a bounded complex of finitely generated projective A-modules.

(4.8) **Lemma ($\varphi_{\mathrm{K}(\boldsymbol{a})}$ is Gorenstein).** *The morphism $A \xrightarrow{\varphi_{\mathrm{K}(\boldsymbol{a})}} \mathrm{K}(\boldsymbol{a})$ is a Gorenstein morphism.*

Proof. Like the proof of lemma (4.4), this is based on lemma (3.11), and again, what we need is to show that the lemma's condition (4) holds. So we need to find an isomorphism $\mathrm{K}(\boldsymbol{a}) \xrightarrow{\gamma} \mathrm{RHom}_A(\mathrm{K}(\boldsymbol{a}), \Sigma^n A)$ (by commutativity of $\mathrm{K}(\boldsymbol{a})$ we need not worry about left- and right-structures here). Since $\mathrm{K}(\boldsymbol{a})$ is K-projective over A we have $\mathrm{RHom}_A(\mathrm{K}(\boldsymbol{a}), \Sigma^n A) \cong \mathrm{Hom}_A(\mathrm{K}(\boldsymbol{a}), \Sigma^n A)$, so it is enough to find a quasi-isomorphism

$$\mathrm{K}(\boldsymbol{a}) \xrightarrow{\simeq} \mathrm{Hom}_A(\mathrm{K}(\boldsymbol{a}), \Sigma^n A).$$

However, the degree n component of $\mathrm{K}(\boldsymbol{a})$ is $\bigwedge^n F$ which is A itself, so the projection of $\mathrm{K}(\boldsymbol{a})$ onto its degree n component has the form $\mathrm{K}(\boldsymbol{a}) \xrightarrow{\pi} \Sigma^n A$. It is now easy to check that there is an isomorphism

$$\mathrm{K}(\boldsymbol{a}) \xrightarrow{\cong} \mathrm{Hom}_A(\mathrm{K}(\boldsymbol{a}), \Sigma^n A), \qquad k \longmapsto (\ell \longmapsto \pi(k \wedge \ell)),$$

which is in particular a quasi-isomorphism, as desired. □

(4.9) **Theorem (Ascent-Descent for Koszul complexes).** *In the situation of setup* (4.6), *we have*

$$A \textit{ is a Gorenstein ring } \Leftrightarrow \ \mathrm{K}(\boldsymbol{a}) \textit{ is a Gorenstein DGA.}$$

Proof. This is almost verbatim to the proof of theorem (4.5), in that we apply theorem (3.6) and proposition (3.10) to the finite Gorenstein morphism $A \xrightarrow{\varphi_{\mathrm{K}(\boldsymbol{a})}} \mathrm{K}(\boldsymbol{a})$. This gives the implications $\Rightarrow$ and $\Leftarrow$. □

(4.10) **Relation to a result by Avramov-Golod.** Avramov and Golod proved in [4] a result which can be stated in the language of Avramov and Foxby [2] as follows: Let A be a noetherian local commutative ring, let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a minimal system of generators of the maximal ideal of A, and let $\mathrm{K}(\boldsymbol{a})$ be the Koszul complex of $\boldsymbol{a}$. Then

$$A \text{ is a Gorenstein ring} \Leftrightarrow \dim_k \mathrm{Ext}_{\mathrm{K}(\boldsymbol{a})}(k, \mathrm{K}(\boldsymbol{a})) = 1, \tag{6}$$

where k is residue class field of $\mathrm{K}(\boldsymbol{a})$.

From theorem (4.9) follows the more general statement that (6) holds for the Koszul complex $\mathrm{K}(\boldsymbol{a})$ on *any* sequence $\boldsymbol{a}$ of elements in the maximal ideal of A. This is because the right hand sides of the bi-implications in theorem (4.9) respectively equation (6) are equivalent, as proved in [11, thm. I].

Equation (6) could also be proved for the Koszul complex $\mathrm{K}(\boldsymbol{a})$ on any sequence $\boldsymbol{a}$ in the maximal ideal of A by using [2, thm. (3.1)].

Singular cochain DGAs of topological spaces. In this subsection we break the habit of the rest of the paper and switch to cohomological notation, that is, upper indices on graded objects and differentials of degree $+1$.

The following three paragraphs give some facts on singular cochain DGAs of topological spaces; see [9] and [10] for more details.

(4.11) **Setup.** In paragraphs (4.11) to (4.16) we consider the following situation: A is a noetherian local commutative ring which contains a field k, and X is a simply connected topological space with $\dim_k \mathrm{H}_*(X; k) < \infty$, which has Poincaré duality over k, in the sense that there is an isomorphism of graded $\mathrm{H}^*(X;k)$-modules

$$\mathrm{H}^*(X;k)' \cong \Sigma^d \, \mathrm{H}^*(X;k) \tag{7}$$

for some d, where the prime denotes dualization with respect to k.

Note that the isomorphism (7) implies

$$\mathrm{H}^d(X;k) \cong k \quad \text{and} \quad \mathrm{H}^i(X;k) = 0 \text{ for } i > d. \tag{8}$$

An important object is $\mathrm{C}^*(X;k)$, the singular cochain DGA of X with coefficients in k, which can be defined as

$$\mathrm{C}^*(X;k) = \mathrm{Hom}_k(\mathrm{C}_*(X;k), k),$$

where $\mathrm{C}_*(X;k)$ is the singular chain complex of X with coefficients in k. The multiplication which turns $\mathrm{C}^*(X;k)$ into a DGA is cup product, which is defined using the Alexander-Whitney map on $\mathrm{C}_*(X;k)$.

The singular cohomology $\mathrm{H}^*(X;k)$ is defined as the cohomology algebra of $\mathrm{C}^*(X;k)$. See e.g. [10, chp. 5] for details on $\mathrm{C}^*(X;k)$ and $\mathrm{H}^*(X;k)$.

(4.12) **Introducing S.** By the "free model" construction employed in [9, proof of thm. 3.6], we have that $\mathrm{C}^*(X;k)$ is equivalent to some R which is a DGA over k with $R^0 = k$ and $R^1 = 0$, and with each R^i finite dimensional over k.

Next, by the method employed in [10, ex. 6, p. 146], there exists a DG-ideal I in R so that the canonical surjection $R \longrightarrow R/I$ is a quasi-isomorphism, and so that the right-most non-vanishing component of $S = R/I$ has the same degree as R's right-most non-vanishing cohomology; namely, degree d (see equation (8)). So S looks like

$$\cdots \rightarrow 0 \rightarrow k \rightarrow 0 \rightarrow S^2 \rightarrow \cdots \rightarrow S^d \rightarrow 0 \rightarrow \cdots,$$

with each S^i finite dimensional over k. Note that S^0 is central in S because R^0 is central in R.

To sum up, $\mathrm{C}^*(X;k)$ is equivalent to R which is again equivalent to S.

(4.13) **Remarks on $\mathrm{C}^*(X;A)$.** Our main object of interest in this part of the paper is $\mathrm{C}^*(X;A)$, the singular cochain DGA of X with coefficients in A, which can be defined as

$$\mathrm{C}^*(X;A) = \mathrm{Hom}_k(\mathrm{C}_*(X;k), A).$$

Again, the multiplication is cup product, defined using the Alexander-Whitney map on $\mathrm{C}_*(X;k)$.

The purpose of the following paragraphs is to show that $\mathrm{C}^*(X;A)$ is a Gorenstein DGA if and only if A is a Gorenstein ring.

However, it will be an advantage not to work with $\mathrm{C}^*(X;A)$ itself but rather with an equivalent DGA which is more tractable: First, there is an evaluation morphism

$$\mathrm{Hom}_k(\mathrm{C}_*(X;k),k)\otimes_k A \longrightarrow \mathrm{Hom}_k(\mathrm{C}_*(X;k),k\otimes_k A)$$

which is a quasi-isomorphism because the homology

$$\mathrm{H}(\mathrm{C}_*(X;k)) = \mathrm{H}_*(X;k)$$

is finite dimensional over k, see [1, sec. 1, thm. 2]. This can also be read

$$\mathrm{C}^*(X;k)\otimes_k A \xrightarrow{\simeq} \mathrm{C}^*(X;A),$$

and it is not hard to check that this is a morphism of DGAs.

Secondly, by paragraph (4.12), we have that $\mathrm{C}^*(X;k)$ is equivalent to the DGA called S.

To sum up, we have that $\mathrm{C}^*(X;A)$ is equivalent to $\mathrm{C}^*(X;k)\otimes_k A$, and as A is flat over k, this is again equivalent to $S\otimes_k A$.

(4.14) **The morphism φ_S.** There is a morphism of DGAs

$$A \xrightarrow{\varphi_S} S\otimes_k A, \qquad a \longmapsto 1_S\otimes a.$$

As in paragraphs (4.3) and (4.7), here are three useful observations:

- The ring $\mathrm{H}^0(S\otimes_k A)$ is noetherian because we have
$$\mathrm{H}^0(S\otimes_k A)\cong \mathrm{H}^0(\mathrm{C}^*(X;A))\cong A,$$
where the second $\cong$ holds because X is connected.
- The morphism φ_S has image inside $S^0\otimes_k A$, and as $S^0 = k$ is central in S and A is commutative, φ_S has image inside the centre of $S\otimes_k A$.
- The morphism φ_S is a finite morphism of DGAs since S is finite dimensional over k, whence $S\otimes_k A$ is a bounded complex of finitely generated projective A-modules.

(4.15) **Lemma (φ_S is Gorenstein).** *The morphism $A \xrightarrow{\varphi_S} S\otimes_k A$ is a Gorenstein morphism.*

Proof. Let us start with some computations. Since we have $\mathrm{H}(S)\cong \mathrm{H}(\mathrm{C}^*(X;k))\cong \mathrm{H}^*(X;k)$, the Poincaré duality isomorphism (7) gives an isomorphism of graded $\mathrm{H}(S)$-modules

$$\Sigma^{-d}(\mathrm{H}(S)')\cong \mathrm{H}(S).$$

This means that $\Sigma^{-d}(\mathrm{H}(S)')$ is free; in other words, there is an element ξ in $(\Sigma^{-d}(\mathrm{H}(S)'))^0$ so that

$$\mathrm{H}(S)\ni c \longmapsto c\xi \in \Sigma^{-d}(\mathrm{H}(S)') \tag{9}$$

is an isomorphism.

Actually, ξ is a linear form $\mathrm{H}^d(S) \xrightarrow{\xi} k$. As S^d is the right-most non-vanishing component of S, we have a surjection $S^d \longrightarrow \mathrm{H}^d(S)$, and we can define an linear form $S^d \xrightarrow{\Xi} k$ as the composition

$$S^d \longrightarrow \mathrm{H}^d(S) \xrightarrow{\xi} k.$$

This can again be viewed as an element Ξ in $(\Sigma^{-d}(S'))^0$.

Now, Ξ is an element in the DG-S-left-S-right-module $\Sigma^{-d}(S')$ which satisfies $\Xi s = (-1)^{|\Xi||s|} s\Xi$ for any graded element s in S, as one proves easily using that Ξ is induced by ξ which is defined on the commutative graded algebra $\mathrm{H}(S) \cong \mathrm{H}^*(X;k)$. It is also mapped to zero by the differential of $\Sigma^{-d}(S')$. Hence we can define a morphism of DG-S-left-S-right-modules δ by

$$S \ni s \xrightarrow{\delta} s\Xi \in \Sigma^{-d}(S').$$

The cohomology of δ is easily seen to be the isomorphism (9), so we have that δ is a quasi-isomorphism, hence an isomorphism in the derived category of DG-S-left-S-right-modules.

Denoting source and target differently, δ reads

$$_{S}S_{S} \xrightarrow{\delta} \mathrm{RHom}_k({}_{S}S_{S}, \Sigma^{-d}k). \tag{10}$$

We shall use this isomorphism below. (Note that by lemma (3.11), the existence of δ actually shows that the canonical morphism $k \longrightarrow S$ is Gorenstein.)

Now for the proof proper: To show that

$$A \xrightarrow{\varphi_S} S \otimes_k A$$

is Gorenstein, we shall use lemma (3.11) with $Q \xrightarrow{\varphi} T$ equal to $A \xrightarrow{\varphi_S} S \otimes_k A$. The observations in paragraph (4.14) show that the lemma's conditions (1) to (3) hold, so we must show that the lemma's condition (4) also holds.

This condition requires a certain isomorphism in the derived category of DG-$(S \otimes_k A)$-left-$(S \otimes_k A)$-right-modules, which we obtain as follows using the isomorphism δ from equation (10):

$$\begin{aligned}
S \otimes_k A &\overset{\delta\otimes A}{\cong} \mathrm{RHom}_k(S, \Sigma^{-d}k) \otimes_k A \\
&\overset{\text{(a)}}{\cong} \mathrm{RHom}_k(S, \Sigma^{-d}k \otimes_k A) \\
&\cong \mathrm{RHom}_k(S, \Sigma^{-d}A) \\
&\cong \mathrm{RHom}_k(S, \mathrm{RHom}_A(A, \Sigma^{-d}A)) \\
&\overset{\text{(b)}}{\cong} \mathrm{RHom}_A(S \overset{\mathrm{L}}{\otimes}_k A, \Sigma^{-d}A) \\
&\cong \mathrm{RHom}_A(S \otimes_k A, \Sigma^{-d}A),
\end{aligned}$$

where (a) is an evaluation morphism which is a quasi-isomorphism and hence an isomorphism in the derived category, because the homology $\mathrm{H}(S) \cong \mathrm{H}(\mathrm{C}^*(X;k)) = \mathrm{H}^*(X;k)$ is finite dimensional over k, see [1, sec. 1, thm. 2], and where (b) is an adjunction isomorphism. $\square$

(4.16) **Theorem (Ascent-Descent for singular cochain DGAs).** *In the situation of setup* (4.11), *we have*

$$A \text{ is a Gorenstein ring } \Leftrightarrow \ \mathrm{C}^*(X;A) \text{ is a Gorenstein DGA.}$$

Proof. This is almost verbatim to the proof of theorem (4.5): Observe that as $\mathrm{C}^*(X;A)$ is equivalent to $S \otimes_k A$ by paragraph (4.13), paragraph (2.2) implies that it is enough to show

$$A \text{ is a Gorenstein ring } \Leftrightarrow \ S \otimes_k A \text{ is a Gorenstein DGA.}$$

For this, we apply theorem (3.6) and proposition (3.10) to the finite Gorenstein morphism $A \xrightarrow{\varphi_S} A \otimes_k S$. This gives the implications $\Rightarrow$ and $\Leftarrow$. $\square$

References

[1] D. Apassov, *Homological dimensions over differential graded rings*, pp. 25–39 in "Complexes and Differential Graded Modules", ph.d. thesis, Lund University, 1999.

[2] L. L. Avramov and H.-B. Foxby, *Locally Gorenstein homomorphisms*, Amer. J. Math. **114** (1992), 1007–1047.

[3] ——— and ———, *Ring homomorphisms and finite Gorenstein dimension,* Proc. London Math. Soc. (3) **75** (1997), 241–270.

[4] ——— and E. S. Golod, *Homology algebra of the Koszul complex of a local Gorenstein ring*, Math. Notes **9** (1971), 30–32.

[5] M. Bökstedt and A. Neeman, *Homotopy limits in triangulated categories*, Comp. Math. **86** (1993), 209–234.

[6] L. W. Christensen, "Gorenstein Dimensions", Lecture Notes in Math., Vol. 1747, Springer, Berlin, 2000.

[7] W. G. Dwyer and J. P. C. Greenlees, Complete modules and torsion modules, preprint (2000).

[8] ———, ———, and S. Iyengar, Duality in algebra and topology, preprint (2002).

[9] Y. Félix, S. Halperin, and J.-C. Thomas, *Gorenstein spaces*, Adv. Math. **71** (1988), 92–112.

[10] ———, ———, and ———, "Rational Homotopy Theory", Grad. Texts in Math., Vol. 205, Springer, Berlin, 2001.

[11] A. Frankild, S. Iyengar, and P. Jørgensen, Dualizing DG modules and Gorenstein DG algebras, preprint (2002).

[12] V. Hinich, *Rings with approximation property admit a dualizing complex*, Math. Nachr. **163** (1993), 289–296.

[13] P. Jørgensen, *Gorenstein homomorphisms of noncommutative rings*, J. Algebra **211** (1999), 240–267.

[14] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 63–102.

[15] ______, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra **136** (1999), 1–56.

[16] I. Kriz and J. P. May, *Operads, algebras, modules and motives*, Astérisque **233** (1995).

[17] H. Matsumura, "Commutative ring theory", Cambridge Stud. Adv. Math., Vol. 8, Cambridge University Press, Cambridge, 1989, second edition.

[18] J. C. McConnell and J. C. Robson, "Noncommutative Noetherian Rings", Wiley, New York, 1987.

[19] N. Spaltenstein, *Resolutions of unbounded complexes,* Comp. Math. **65** (1988), 121-154.

[20] Q. S. Wu and J. J. Zhang, *Some homological invariants of local PI algebras*, J. Algebra **225** (2000), 904–935.

[21] A. Yekutieli, *Dualizing complexes over noncommutative graded algebras*, J. Algebra **153** (1992), 41–84.

Frankild: Matematisk Afdeling, Universitetsparken 5, 2100 København Ø, DK–Denmark

E-mail address: frankild@math.ku.dk

Jørgensen: Danish National Library of Science and Medicine, Nørre Allé 49, 2200 København N, DK–Denmark

E-mail address: pej@dnlb.dk

DUALIZING DG-MODULES FOR DIFFERENTIAL GRADED ALGEBRAS

ANDERS FRANKILD AND PETER JØRGENSEN

Abstract. Over a general Differential Graded Algebra, we introduce the notion of dualizing DG-module. We prove that for certain types of rings, viewed as Differential Graded Algebras concentrated in degree zero, "dualizing DG-module" simply means "dualizing complex". Using dualizing DG-modules, we develop a Foxby equivalence theory which can detect the Gorenstein property of a Differential Graded Algebra (in the sense of [15, def. (1.1)]).

0. Introduction

(0.1) **Generalities.** This manuscript is a counterpart to [15]. Like [15], it deals with Differential Graded Algebras (abbreviated DGAs). (For a thorough introduction to the theory of DGAs we refer the reader to [6] and [19].)

In [15, def. (1.1)] we gave a definition of "Gorenstein DGA", reproduced in paragraph (1.3) below. If R is a DGA, our Gorenstein condition requires that R is a sensible "dualizing object" over itself, in the sense that there are quasi-inverse contravariant equivalences of categories,

$$\mathsf{fin}(R) \underset{\mathrm{RHom}_{R^{\mathrm{opp}}}(-,R)}{\overset{\mathrm{RHom}_R(-,R)}{\rightleftarrows}} \mathsf{fin}(R^{\mathrm{opp}}),$$

where $\mathsf{fin}(R)$ and $\mathsf{fin}(R^{\mathrm{opp}})$ are suitably defined categories of "finite" DG-R-left- and DG-R-right-modules. (R^{opp} is the opposite DGA of R, and we identify DG-R-right-modules with DG-R^{opp}-left-modules.)

In this manuscript, we will do something more general: We shall ask for DG-modules D with duality properties resembling the ones enjoyed by R itself when R is Gorenstein. We define such D's in definition (1.1), calling them *dualizing DG-modules.*

Of course, a dualizing DG-module D must provide quasi-inverse contravariant equivalences of categories,

$$\mathsf{fin}(R) \underset{\mathrm{RHom}_{R^{\mathrm{opp}}}(-,D)}{\overset{\mathrm{RHom}_R(-,D)}{\rightleftarrows}} \mathsf{fin}(R^{\mathrm{opp}}), \qquad (0.1.1)$$

1991 *Mathematics Subject Classification.* 16E45.

Key words and phrases. Dualizing DG-module, Gorenstein DGA, Foxby equivalence, Auslander class, Bass class, grade, reflexive DG-module.

but we require more of it. Namely, by [14, (1.5)] or paragraph (2.1) below, one can also use an object such as D to get quasi-inverse covariant equivalences of categories,

$$\mathcal{A}(R) \underset{\mathrm{RHom}_R(D,-)}{\overset{D\overset{\mathrm{L}}{\otimes}_R -}{\rightleftarrows}} \mathcal{B}(R),$$

called *Foxby equivalence*, and we also place conditions on D ensuring that Foxby equivalence with respect to D is well-behaved. Here $\mathcal{A}(R)$ and $\mathcal{B}(R)$ are so-called *Auslander* and *Bass classes*, defined suitably (see [14, sec. 1] or paragraph (2.1) below).

(0.2) **Main results.** The setup we have sketched is heavily inspired by the theory of dualizing complexes from commutative ring theory, as expounded in [8, chp. 3]. And indeed, we are able to prove a number of results generalizing the central parts of that theory:

First, (1.4) gives that R is a Gorenstein DGA precisely if it is a dualizing DG-module for itself.

Secondly, theorems (2.4) and (2.8) say that a dualizing DG-module gives a good Foxby equivalence theory.

Thirdly, and most interestingly, our main result (2.19) is a "Gorenstein Theorem". It states that maximality of the Auslander and Bass classes with respect to a dualizing DG-module, D, captures the Gorenstein property of a DGA, R. Specifically, the following three conditions are equivalent under weak assumptions on R:

(1) R is a Gorenstein DGA.

(2) $\begin{cases} \mathcal{A}^{\mathrm{f}}(R) = \mathsf{fin}(R) \quad \text{and} \\ \mathcal{A}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}}). \end{cases}$

(3) $\begin{cases} \mathcal{B}^{\mathrm{f}}(R) = \mathsf{fin}(R) \quad \text{and} \\ \mathcal{B}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}}). \end{cases}$

Here $\mathcal{A}^{\mathrm{f}}$ and $\mathcal{B}^{\mathrm{f}}$ are versions of $\mathcal{A}$ and $\mathcal{B}$ which are suitably cut down with fin.

(0.3) **The ring case.** The theory of this manuscript is new, even when applied to a DGA which is just a non-commutative ring (placed in degree zero).

In the ring case, "DG-module" means "complex of modules", "fin" means "$\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}$" (the derived category of complexes with bounded, finitely generated homology), and in many cases, "dualizing DG-module" turns out to mean "dualizing complex".

Our theory can be applied successfully to at least two types of rings:

First, noetherian local commutative rings of finite finitistic flat dimension. Here theorem (1.7) proves that "dualizing DG-module" means "dualizing complex", and paragraph (3.1) shows that the technical requirements of our theory are satisfied. The theory specializes to a version of the theory explained in [8, chp. 3].

Secondly, non-commutative noetherian local PI algebras over a field (see [20]). Here one uses theorem (1.9) and paragraph (3.2) instead of (1.7) and (3.1). The theory is new here, and, we think, interesting. Its main result, theorem (2.19), gives the following new characterization of certain non-commutative Gorenstein rings (using [15, prop. (1.6)]):

Gorenstein Theorem for non-commutative rings. *Let A be a noetherian local PI algebra over a field, and suppose that A has the dualizing complex D. Then the following conditions are equivalent:*

$$\text{(1)} \qquad \mathrm{id}_A(A) < \infty \text{ and } \mathrm{id}_{A^{\mathrm{opp}}}(A) < \infty.$$

$$\text{(2)} \qquad \begin{cases} \mathcal{A}^{\mathrm{f}}(A) = \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A) \quad \text{and} \\ \mathcal{A}^{\mathrm{f}}(A^{\mathrm{opp}}) = \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A^{\mathrm{opp}}). \end{cases}$$

$$\text{(3)} \qquad \begin{cases} \mathcal{B}^{\mathrm{f}}(A) = \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A) \quad \text{and} \\ \mathcal{B}^{\mathrm{f}}(A^{\mathrm{opp}}) = \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A^{\mathrm{opp}}). \end{cases}$$

(0.4) **Note to the ring theorist.** Let us make the point of paragraph (0.3) again, in an even stronger form.

Of course, the theory of this manuscript can be applied to a DGA which is just a ring, as indicated in paragraph (0.3).

However, had we been interested in developing a theory such as the one described here for rings only (without ever mentioning general DGAs) we would have found that we had to go through exactly the same motions, all the way down to the typography. We would have had to use the same RHom's and $\overset{\mathrm{L}}{\otimes}$'s and so on, only over the ring in question, and not over a general DGA.

So instead of *applying* the theory of this manuscript to a DGA which is just a ring, one can do something else: Read the paper *pretending that R, our standing DGA, is just a ring.* The two approaches are completely equivalent.

In other words, the ring theoretically minded reader can read the paper without ever worrying about DGAs.

Specifically, as described in (0.3), this works nicely with both noetherian local commutative rings of finite finitistic flat dimension and noetherian local PI algebras over fields.

Let us now comment on the obvious question: Do dualizing DG-modules exist in natural situations? The answer is yes:

(0.5) **Existence.** In [11] we will show that, if A is a noetherian commutative ring admitting a dualizing complex, then:

- The Koszul complex $\mathrm{K}(\boldsymbol{a})$ on a sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$ of elements in A admits a dualizing DG-module. (Note that by [21, exer. 4.5.1] the Koszul complex is a commutative DGA.)

- The endomorphism DGA, $\mathcal{E} = \mathrm{Hom}_A(L, L)$, where L is a bounded complex of finitely generated projective A-modules which is not exact, admits a dualizing DG-module. (For details on $\mathcal{E}$ we refer the reader to [9] and [12, rem. (1.2)].)

If, in addition, A is local, and if A' denotes another noetherian local commutative ring, and $A' \xrightarrow{\alpha'} A$ is a local homomorphism so that A has finite flat dimension when viewed as a A'-module, then:

- The fibre $F(\alpha')$ of α' (which is a DGA, see [7, 3.7]) admits a dualizing DG-module.

Moreover, let k be a field, and let X be a topological space for which

- $\mathrm{H}^0(X; k) = k$.
- $\mathrm{H}^1(X; k) = 0$.
- $\mathrm{H}^*(X; k)$ is finite dimensional over k.

Consider $\mathrm{C}^*(X; k)$, the cochain DGA of X with coefficients in k. It is also shown in [11] that:

- The cochain DGA $\mathrm{C}^*(X; k)$ admits a dualizing DG-module.

(0.6) **Weak dualizing DG-modules.** An aspect of the theory below is that some of the results (theorem (2.4) and proposition (2.11)) work for what we call weak dualizing DG-modules. These are objects satisfying only part of definition (1.1), in that they lack the "finiteness" condition called [D4].

Such objects exist: Let A be a noetherian local commutative ring with dualizing complex D, let $\mathfrak{a}$ be an ideal of A so that A is $\mathfrak{a}$-adically complete, and let $\mathrm{R}\Gamma_{\mathfrak{a}}$ be the right derived section functor. Then $\mathrm{R}\Gamma_{\mathfrak{a}}D$ is a weak dualizing DG-module over A, viewed as a DGA concentrated in degree zero.

Weak dualizing DG-modules of this type are also put to use in [12] to prove a "parametrized Gorenstein Theorem".

(0.7) **Synopsis.** This manuscript is organized as follows:

After this synopsis, the introduction ends with a few definitions.

Section 1 defines dualizing DG-modules, and remarks on their connection to duality and Gorenstein DGAs. It then considers DGAs which are just rings placed in degree zero, and proves in theorems (1.7) and (1.9) that for some classes of rings, "dualizing DG-module" simply means "dualizing complex".

Section 2 contains the bulk of our theory. It first sums up Foxby equivalence with respect to a weak dualizing DG-module in paragraph (2.1). It goes on to define versions of the Auslander and Bass classes which are suitably cut down with fin, and sets up a stronger Foxby equivalence for these than the one set up in paragraph (2.1). This is done in theorems (2.4) and (2.8). Finally, the cut down classes are used to get our main result, the Gorenstein Theorem (2.19).

Section 3 deals with a technical condition on DGAs which we call [Grade]. The condition is needed to get section 2 to work. Section 3 proves that four natural types of DGAs satisfy [Grade]: Noetherian local commutative rings, non-commutative noetherian local PI algebras over fields, Koszul complexes, and endomorphism DGAs.

(0.8) **Setup.** Throughout the manuscript, R denotes a DGA for which $\mathrm{H}_0 R$ is a noetherian ring.

(0.9) **Definition (The category fin).** By $\mathsf{fin}(R)$ we denote the full triangulated subcategory of the derived category of DG-R-left-modules, $\mathsf{D}(R)$, consisting of M's so that the homology $\mathrm{H}M$ is bounded, and so that each $\mathrm{H}_i M$ is finitely generated as a module over $\mathrm{H}_0 R$.

(0.10) **Opposite DGAs.** By R^{opp} we denote the opposite DGA of R, whose product is defined as $s \cdot r = (-1)^{|r||s|} rs$ for graded elements r and s. The purpose of R^{opp} is that we can identify DG-R-right-modules with DG-R^{opp}-left-modules. So for instance, $\mathsf{D}(R^{\mathrm{opp}})$ is the derived category of DG-R-right-modules.

(0.11) **DG-modules in the ring case.** Note that any ring A can be viewed as a DGA concentrated in degree zero, and that $\mathrm{H}_0 A \cong A$. A DG-A-module is then the same thing as a complex of A-modules, and the various derived categories of DG-A-modules are the same as the various ordinary derived categories over A. Moreover, still viewing A as a DGA, the derived functors of Hom_A and $\otimes_A$ coincide with the ordinary RHom_A and $\overset{\mathrm{L}}{\otimes}_A$. Note that when A is noetherian the category $\mathsf{fin}(A)$ is $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$, the derived category of complexes with bounded, finitely generated homology.

(0.12) **Acknowledgement.** This manuscript owes a great debt to [24]. This was the first paper to introduce dualizing complexes in a non-commutative situation, and hence the first paper that had to deal with such ensuing complications as left-, right-, and bi-structures of modules and functors.

Note that [1] contains another definition than ours of dualizing DG-modules.

1. Dualizing DG-Modules for Differential Graded Algebras

This section defines dualizing DG-modules (definition (1.1)). In (1.2) we remark that a dualizing DG-module gives a duality between $\mathsf{fin}(R)$ and $\mathsf{fin}(R^{\text{opp}})$, and in (1.4) we note the easy, but important fact that for R itself to be a dualizing DG-module is equivalent to R being a Gorenstein DGA.

We end the section by considering rings, viewed as DGAs concentrated in degree zero. It turns out that for these, "dualizing DG-module" is sometimes synonymous with "dualizing complex". After recalling in (1.5) the definition of dualizing complexes for rings, we prove this synonymity in theorem (1.7) for noetherian commutative rings of finite finitistic flat dimension, and in theorem (1.9) for non-commutative noetherian semi-local PI algebras over fields.

(1.1) **Definition (Dualizing DG-modules).** Let ${}_RD_R$ be a DG-R-left-R-right-module. (We indicate DG-R-left- and DG-R-right-module structures with subscripts.) We call ${}_RD_R$ a weak dualizing DG-module for R if it satisfies:

[D1]: There are quasi-isomorphisms of DG-R-left-R-right-modules $P \xrightarrow{\simeq} D$ and $D \xrightarrow{\simeq} I$ such that ${}_RP$ and P_R are K-projective and ${}_RI$ and I_R are K-injective.

[D2]: The following canonical morphisms in the derived category of DG-R-left-R-right-modules are isomorphisms,

$$R \xrightarrow{\rho} \mathrm{RHom}_R(D, D),$$
$$R \xrightarrow{\rho^{\text{opp}}} \mathrm{RHom}_{R^{\text{opp}}}(D, D).$$

[D3]: For $M \in \mathsf{fin}(R)$ and $N \in \mathsf{fin}(R^{\text{opp}})$ and ${}_RL_R$ equal to either ${}_RR_R$ or ${}_RD_R$, the following evaluation morphisms are isomorphisms:

$$\mathrm{RHom}_{R^{\text{opp}}}(L,D) \overset{\mathrm{L}}{\otimes}_R M \longrightarrow \mathrm{RHom}_{R^{\text{opp}}}(\mathrm{RHom}_R(M, L), D),$$
$$N \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(L, D) \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_{R^{\text{opp}}}(N, L), D).$$

We call D a dualizing DG-module for R if it also satisfies:

[D4]: The functor $\mathrm{RHom}_R(-, D)$ maps $\mathsf{fin}(R)$ to $\mathsf{fin}(R^{\text{opp}})$, and the functor $\mathrm{RHom}_{R^{\text{opp}}}(-, D)$ maps $\mathsf{fin}(R^{\text{opp}})$ to $\mathsf{fin}(R)$.

For a list of natural DGAs having dualizing DG-modules, see paragraph (0.5). Note that with a few twists, the definition resembles the definition of dualizing complexes in ring theory, see (1.5) below.

(1.2) **Duality between $\mathsf{fin}(R)$ and $\mathsf{fin}(R^{\mathrm{opp}})$.** Suppose R admits a dualizing DG-module D. Then by conditions [D3] and [D4] we have a duality between $\mathsf{fin}(R)$ and $\mathsf{fin}(R^{\mathrm{opp}})$. To be precise, the diagram

$$\mathsf{fin}(R) \underset{\mathrm{RHom}_{R^{\mathrm{opp}}}(-,D)}{\overset{\mathrm{RHom}_R(-,D)}{\rightleftarrows}} \mathsf{fin}(R^{\mathrm{opp}})$$

yields the claimed duality.

This is analogous to classical ring theory, where R is a ring, $\mathsf{fin}(R) = \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ and $\mathsf{fin}(R^{\mathrm{opp}}) = \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(R^{\mathrm{opp}})$, and D a dualizing complex.

(1.3) **Gorenstein DGAs.** Recall the standing convention from setup (0.8) that R denotes a DGA for which H_0R is noetherian. Recall from [15, def. (1.1)] that R is called a *Gorenstein DGA* if it satisfies:

[G1]: There is a quasi-isomorphism of DG-R-left-R-right-modules $R \xrightarrow{\simeq} I$ where ${}_RI$ and I_R are K-injective.

[G2]: For $M \in \mathsf{fin}(R)$ and $N \in \mathsf{fin}(R^{\mathrm{opp}})$ the following evaluation morphisms are isomorphisms:

$$\mathrm{RHom}_{R^{\mathrm{opp}}}(R,R) \overset{\mathrm{L}}{\otimes}_R M \longrightarrow \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(M,R),R),$$
$$N \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(R,R) \longrightarrow \mathrm{RHom}_R(\mathrm{RHom}_{R^{\mathrm{opp}}}(N,R),R).$$

[G3]: The functor $\mathrm{RHom}_R(-,R)$ maps $\mathsf{fin}(R)$ to $\mathsf{fin}(R^{\mathrm{opp}})$, and the functor $\mathrm{RHom}_{R^{\mathrm{opp}}}(-,R)$ maps $\mathsf{fin}(R^{\mathrm{opp}})$ to $\mathsf{fin}(R)$.

(1.4) **Dualizing DG-modules and Gorensteinness.** Observe that R is a Gorenstein DGA (in the sense of [15, def. (1.1)]) exactly when ${}_RR_R$ is a dualizing DG-module for R. This is analogous to classical ring theory.

(1.5) **Dualizing complexes for rings.** Let A be a noetherian ring. Recall from [25, def. 1.1] that a complex of A-left-A-right-modules D is called a *dualizing complex* for A if:

[D1′]: There exist quasi-isomorphisms of complexes of A-left-A-right-modules $P \xrightarrow{\simeq} D$ and $D \xrightarrow{\simeq} I$ such that ${}_AP$ and P_A are K-projective and ${}_AI$ and I_A are K-injective.

[D2′]: The following canonical morphisms in the derived category of A-left-A-right-modules are isomorphisms,

$$A \xrightarrow{\rho} \mathrm{RHom}_A(D,D),$$
$$A \xrightarrow{\rho^{\mathrm{opp}}} \mathrm{RHom}_{A^{\mathrm{opp}}}(D,D).$$

[D3′]: The injective dimensions $\mathrm{id}_A(D)$ and $\mathrm{id}_{A^{\mathrm{opp}}}(D)$ are finite.

[D4′]: We have ${}_AD \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$ and $D_A \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A^{\mathrm{opp}})$.

Condition [D1′] is usually automatically satisfied and hence omitted. However, we choose to state it here in order to make the proof of the next theorem more clear.

(1.6) **Symmetric A-left-A-right-modules.** In the next result, A is a commutative ring and $D \in \mathsf{D}(A)$ is a complex of A-left-modules which we view as a complex of A-left-A-right-modules via the A-right-multiplication defined by $d \cdot a = ad$ on each module in the complex. In other words, each module in the complex becomes a symmetric A-left-A-right-module.

(1.7) **Theorem.** *Let A be a noetherian commutative ring of finite finitistic flat dimension (or equivalently finite finitistic injective dimension), and let $D \in \mathsf{D}(A)$. Then D is a dualizing DG-module for A, viewed as a DGA concentrated in degree zero, if and only if D is a dualizing complex for the ring A.*

Proof. Since the A-right-structure of D is induced by the A-left-structure of D, both [D1] and [D1′] automatically hold. Also, all statements on D from the right follow from the corresponding statements on D from the left. This applies to both conditions [D2] to [D4] and conditions [D2′] to [D4′].

Assume that [D2′] to [D4′] hold. We need to check conditions [D2] to [D4].

Condition [D2]: Follows from [D2′].

Condition [D3]: This is a consequence of [D3′] and the standard isomorphism [8, (A.4.24)].

Condition [D4]: If M is in $\mathsf{fin}(A) = \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$, then M has a projective resolution $P \xrightarrow{\simeq} M$ where P is right-bounded and consists of finitely generated modules. Then $\mathrm{RHom}_A(M, D) \cong \mathrm{Hom}_A(P, D)$ certainly has finitely generated homology by [D4′] and has bounded homology since the injective dimension of D is finite by [D3′].

Now assume that [D2] to [D4] hold. We need to check conditions [D2′] to [D4′].

Condition [D2′]: Follows from [D2].

Condition [D3′]: This is a consequence of [D4]. For any prime ideal $\mathfrak{p}$ we have $A/\mathfrak{p} \in \mathsf{fin}(A)$. Therefore $\mathrm{RHom}_A(A/\mathfrak{p}, D) \in \mathsf{fin}(A)$ by [D4], and hence by localization

$$\mathrm{RHom}_A(A/\mathfrak{p}, D)_{\mathfrak{p}} \cong \mathrm{RHom}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}, D_{\mathfrak{p}}) \in \mathsf{fin}(A_{\mathfrak{p}}).$$

But this means that $\mathrm{id}_{A_{\mathfrak{p}}}(D_{\mathfrak{p}})$ is finite by [8, (A.5.7.4)], since $D_{\mathfrak{p}} \in \mathsf{fin}(A_{\mathfrak{p}})$. Thus

$$\begin{aligned} \mathrm{id}_{A_{\mathfrak{p}}}(D_{\mathfrak{p}}) &= -\inf\{i \mid \mathrm{H}_i(D_{\mathfrak{p}}) \neq 0\} + \operatorname{depth} A_{\mathfrak{p}} \\ &\leq -\inf\{i \mid \mathrm{H}_i(D) \neq 0\} + \mathrm{FFD}(A), \end{aligned}$$

by [10, cor. 4.3(1)] and [3, thm. 1.4]. Here $\mathrm{FFD}(A)$ denotes the finitistic flat dimension of A.

Now let M be any finitely generated module, and let $\mathfrak{p}$ be any prime ideal. Then we have the following chain,

$$\begin{aligned} & -\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_A(M,D)_{\mathfrak{p}}) \neq 0\} \\ & \qquad = -\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}},D_{\mathfrak{p}})) \neq 0\} \\ & \qquad \leq \mathrm{id}_{A_{\mathfrak{p}}}(D_{\mathfrak{p}}) \\ & \qquad \leq -\inf\{\, i \mid \mathrm{H}_i(D) \neq 0\} + \mathrm{FFD}(A). \end{aligned}$$

Thus

$$\begin{aligned} & -\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_A(M,D)) \neq 0\} \\ & \qquad \leq -\inf\{\, i \mid \mathrm{H}_i(D) \neq 0\} + \mathrm{FFD}(A), \end{aligned}$$

and this shows that $\mathrm{id}_A(D)$ is finite.

Condition [D4′]: This follows from [D4] by inserting A into the functor $\mathrm{RHom}_A(-,D)$.

□

(1.8) **The non-commutative version of** (1.7)**.** Using essentially the same proof as above, only now using [23, prop. 5.7(1)] to show that [D3′] follows from [D4], we get the following result.

(1.9) **Theorem.** *Let A be a noetherian semi-local PI algebra over the field k, and let D be an object in $\mathsf{D}(A \otimes_k A^{\mathrm{opp}})$. Then D is a dualizing DG-module for A, viewed as a DGA concentrated in degree zero, if and only if D is a dualizing complex for the algebra A.*

2. Foxby equivalence and Gorensteinness

This is the main section. We start in (2.1) by summing up Foxby equivalence between the Auslander class $\mathcal{A}$ and the Bass class $\mathcal{B}$, with respect to a weak dualizing DG-module.

In definition (2.3) we define smaller versions of $\mathcal{A}$ and $\mathcal{B}$ which are suitably cut down with fin. We denote these versions of the classes by $\mathcal{A}^{\mathrm{f}}$ and $\mathcal{B}^{\mathrm{f}}$.

Two important results are now possible: Theorems (2.4) and (2.8). They show that $\mathcal{A}^{\mathrm{f}}$ and $\mathcal{B}^{\mathrm{f}}$ enjoy a stronger Foxby equivalence theory than the one set up in paragraph (2.1). To get theorem (2.8) to work, we need to introduce in definition (2.5) a technical condition, [Grade], on DGAs.

Next, some technicalities. In definition (2.9), we introduce the class $\mathcal{R}$ of reflexive DG-R-modules. These are the modules which dualize well with respect to R itself, and are a key technical tool of this section. Proposition (2.11) proves an important property: We have $\mathcal{A}(R) \cap \mathsf{fin}(R) = \mathcal{R}(R) \cap \mathsf{fin}(R)$ when $\mathcal{A}$ is the Auslander class with respect to any weak dualizing DG-module for R. Hence $\mathcal{R}$ can be used to recognize the size of the finite part of $\mathcal{A}$. There is also a smaller version of $\mathcal{R}$ which is suitably cut down with fin, denoted by $\mathcal{R}^{\mathrm{f}}$. It satisfies $\mathcal{A}^{\mathrm{f}}(R) = \mathcal{R}^{\mathrm{f}}(R)$ by proposition (2.17) when $\mathcal{A}^{\mathrm{f}}$ is the cut down Auslander class with respect to any dualizing DG-module for R.

Finally, the main result of this manuscript, the Gorenstein Theorem (2.19). Using the classes $\mathcal{R}^{\mathrm{f}}$, $\mathcal{A}^{\mathrm{f}}$ and $\mathcal{B}^{\mathrm{f}}$ it states under weak conditions on R that Gorensteinness of R is equivalent to either maximality of the classes $\mathcal{R}^{\mathrm{f}}$, or maximality of the classes $\mathcal{A}^{\mathrm{f}}$, or maximality of the classes $\mathcal{B}^{\mathrm{f}}$.

(2.1) **Foxby equivalence.** Let D be a weak dualizing DG-module for R. Then $(D \overset{\mathrm{L}}{\otimes}_R -, \mathrm{RHom}_R(D,-))$ is an adjoint pair of functors. Let η denote the unit and ε the counit of the adjoint pair. Then we may define the so-called *Auslander* and *Bass classes* with respect to D in terms of η and ε being isomorphisms. To be precise, we let

$$\mathcal{A}_D(R) = \left\{ X \in \mathsf{D}(R) \;\middle|\; \begin{array}{l} \eta_X \,:\, X \longrightarrow \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R X) \\ \text{is an isomorphism} \end{array} \right\}$$

and

$$\mathcal{B}_D(R) = \left\{ Y \in \mathsf{D}(R) \;\middle|\; \begin{array}{l} \varepsilon_Y \,:\, D \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(D,Y) \longrightarrow Y \\ \text{is an isomorphism} \end{array} \right\}.$$

These are clearly full triangulated subcategories of $\mathsf{D}(R)$. Here $\mathcal{A}_D(R)$ is called the Auslander class (with respect to D), and $\mathcal{B}_D(R)$ is called the Bass class (with respect to D).

In any given situation, we only consider these classes with respect to a specific weak dualizing DG-module D, so we shall omit the subscript D, denoting the classes by $\mathcal{A}(R)$ and $\mathcal{B}(R)$.

There are of course corresponding definitions for Auslander and Bass classes of DG-R-right-modules, denoted $\mathcal{A}(R^{\mathrm{opp}})$ and $\mathcal{B}(R^{\mathrm{opp}})$.

There are now quasi-inverse equivalences between the Auslander and Bass classes. For DG-R-left-modules they read:

$$\mathcal{A}(R) \underset{\mathrm{RHom}_R(D,-)}{\overset{D \overset{\mathrm{L}}{\otimes}_R -}{\rightleftarrows}} \mathcal{B}(R).$$

This is called *Foxby equivalence.*

There are corresponding quasi-inverse equivalences for DG-R-right-modules.

We refer the reader to [14] for more details on these quasi-inverse equivalences, which are instances of *generalized Foxby equivalence.*

(2.2) **Size of $\mathcal{A}$ and $\mathcal{B}$.** Note from condition [D3] on D that we have ${}_RR \in \mathcal{A}(R)$, and hence also ${}_RD \cong D \overset{\mathrm{L}}{\otimes}_R {}_RR \in \mathcal{B}(R)$. So any object built from finitely many copies of suspensions of ${}_RR$ is in $\mathcal{A}(R)$, and any object built from finitely many copies of suspensions of ${}_RD$ is in $\mathcal{B}(R)$.

Of course, the corresponding statements also hold for $\mathcal{A}(R^{\mathrm{opp}})$ and $\mathcal{B}(R^{\mathrm{opp}})$.

If R is a noetherian commutative ring, and the dualizing DG-module D is a dualizing complex, even more is true: All bounded complexes of flat modules are in $\mathcal{A}(R)$, and all bounded complexes of injective modules are in $\mathcal{B}(R)$. This follows from the proof of [5, thm. (3.2)].

(2.3) **Definition ((Finite) Auslander and Bass classes).** Let D be a weak dualizing DG-module for R. For DG-R-left-modules we put

$$\mathcal{A}^{\mathrm{f}}_D(R) = \left\{ X \in \mathcal{A}(R) \;\middle|\; \begin{array}{l} X \in \mathsf{fin}(R) \text{ and} \\ D \overset{\mathrm{L}}{\otimes}_R X \in \mathsf{fin}(R) \end{array} \right\}$$

and

$$\mathcal{B}^{\mathrm{f}}_D(R) = \left\{ Y \in \mathcal{B}(R) \;\middle|\; \begin{array}{l} Y \in \mathsf{fin}(R) \text{ and} \\ \mathrm{RHom}_R(D,Y) \in \mathsf{fin}(R) \end{array} \right\}.$$

Again, in any given situation, we only consider these classes with respect to a specific weak dualizing DG-module D, so we shall omit the subscript D, denoting the classes by $\mathcal{A}^{\mathrm{f}}(R)$ and $\mathcal{B}^{\mathrm{f}}(R)$.

There are of course corresponding classes of DG-R-right-modules, denoted $\mathcal{A}^{\mathrm{f}}(R^{\mathrm{opp}})$ and $\mathcal{B}^{\mathrm{f}}(R^{\mathrm{opp}})$.

(2.4) **Theorem ((Finite) Foxby equivalence part I).** *Let D be a weak dualizing DG-module for R. Then there are the following quasi-inverse equivalences,*

$$\begin{tikzcd} \mathcal{A}^{\mathrm{f}}(R) \arrow[r, shift left, "D\overset{\mathrm{L}}{\otimes}_R -"] & \mathcal{B}^{\mathrm{f}}(R). \arrow[l, shift left, "\mathrm{RHom}_R(D,-)"] \end{tikzcd}$$

There are of course corresponding quasi-inverse equivalences between $\mathcal{A}^{\mathrm{f}}(R^{\mathrm{opp}})$ and $\mathcal{B}^{\mathrm{f}}(R^{\mathrm{opp}})$.

Proof. The definitions make it clear that the functors land in the claimed categories, so the diagrams exist. As everything is left/right-symmetric, it is enough to consider the statement for $\mathcal{A}^{\mathrm{f}}(R)$ and $\mathcal{B}^{\mathrm{f}}(R)$.

The adjoint pair $(D \overset{\mathrm{L}}{\otimes}_R -, \mathrm{RHom}_R(D,-))$ defined on all of $\mathsf{D}(R)$ has unit, denoted η, and counit, denoted ε, which are natural transformations,

$$1_{\mathsf{D}(R)} \xrightarrow{\eta} \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R -)$$

and

$$D \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(D,-) \xrightarrow{\varepsilon} 1_{\mathsf{D}(R)}.$$

Restricting the functors to $\mathcal{A}^{\mathrm{f}}(R)$ and $\mathcal{B}^{\mathrm{f}}(R)$, the natural transformations also restrict, and since $\mathcal{A}^{\mathrm{f}}(R) \subseteq \mathcal{A}(R)$ and $\mathcal{B}^{\mathrm{f}}(R) \subseteq \mathcal{B}(R)$, unit and counit become natural equivalences. So the restricted functors satisfy

$$1_{\mathcal{A}^{\mathrm{f}}(R)} \simeq \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R -)$$

and

$$D \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(D, -) \simeq 1_{\mathcal{B}^{\mathrm{f}}(R)},$$

thereby proving the result. $\square$

(2.5) **Definition (Grade).** We say that R satisfies [Grade] if the following hold:

- $M \in \mathsf{fin}(R)$ and $M \not\cong 0 \Longrightarrow \mathrm{RHom}_R(M, R) \not\cong 0$ and
- $N \in \mathsf{fin}(R^{\mathrm{opp}})$ and $N \not\cong 0 \Longrightarrow \mathrm{RHom}_{R^{\mathrm{opp}}}(N, R) \not\cong 0$.

(2.6) **Remark.** In section 3 we show that many natural DGAs satisfy [Grade]. The condition is so named because it ensures that for a module $M \in \mathsf{fin}(R)$, the number

$$-\sup\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(M, R)) \neq 0 \,\},$$

traditionally known as $\mathrm{grade}_R(M)$ (when R is a ring), is not ∞.

(2.7) **Lemma.** *Let R satisfy* [Grade]*, and let D be a dualizing DG-module for R. Let $M \xrightarrow{\mu} N$ be a morphism in $\mathsf{fin}(R)$. Then*

$$\left.\begin{array}{r} D \overset{\mathrm{L}}{\otimes}_R \mu \text{ is an isomorphism or} \\ \mathrm{RHom}_R(D, \mu) \text{ is an isomorphism} \end{array}\right\} \Rightarrow \quad \mu \text{ is an isomorphism.}$$

There are of course corresponding results for morphisms in $\mathsf{fin}(R^{\mathrm{opp}})$.

Proof. By a standard argument, it is sufficient to see that if $C \in \mathsf{fin}(R)$ has $D \overset{\mathrm{L}}{\otimes}_R C \cong 0$ or $\mathrm{RHom}_R(D, C) \cong 0$, then $C \cong 0$.

The case $D \overset{\mathrm{L}}{\otimes}_R C \cong 0$: We may compute as follows,

$$\begin{aligned} 0 &\cong \mathrm{RHom}_R(D \overset{\mathrm{L}}{\otimes}_R C, D) \\ &\overset{(a)}{\cong} \mathrm{RHom}_R(C, \mathrm{RHom}_R(D, D)) \\ &\cong \mathrm{RHom}_R(C, R), \end{aligned}$$

where (a) is by adjointness, whence $C \cong 0$ by [Grade].

The case $\mathrm{RHom}_R(D, C) \cong 0$: Start by writing $X = \mathrm{RHom}_R(C, D)$. Since X is the dual of C under the duality described in (1.2), we know that $X \in \mathsf{fin}(R^{\text{opp}})$ and $C \cong \mathrm{RHom}_{R^{\text{opp}}}(X, D)$. This enables us to perform the following computations,

$$\begin{aligned} 0 &\cong \mathrm{RHom}_R(D, C) \\ &\cong \mathrm{RHom}_R(D, \mathrm{RHom}_{R^{\text{opp}}}(X, D)) \\ &\overset{(a)}{\cong} \mathrm{RHom}_{R^{\text{opp}}}(X, \mathrm{RHom}_R(D, D)) \\ &\overset{(b)}{\cong} \mathrm{RHom}_{R^{\text{opp}}}(X, R). \end{aligned}$$

Here (a) is due to the so-called swap isomorphism, and (b) is due to [D2]. So [Grade] forces $X \cong 0$, hence $C \cong 0$. $\square$

(2.8) **Theorem ((Finite) Foxby equivalence part II).** *Let R satisfy* [Grade] *and let D be a dualizing DG-module for R. Then*

- $\left.\begin{array}{r} X \in \mathsf{fin}(R) \text{ and} \\ D \overset{\mathrm{L}}{\otimes}_R X \in \mathcal{B}^{\mathrm{f}}(R) \end{array}\right\} \Rightarrow X \in \mathcal{A}^{\mathrm{f}}(R).$
- $\left.\begin{array}{r} Y \in \mathsf{fin}(R) \text{ and} \\ \mathrm{RHom}_R(D, Y) \in \mathcal{A}^{\mathrm{f}}(R) \end{array}\right\} \Rightarrow Y \in \mathcal{B}^{\mathrm{f}}(R).$

There are of course corresponding results for DG-R-right-modules.

Proof. By symmetry it is enough to prove the statements for DG-R-left-modules.

Suppose $X \in \mathsf{fin}(R)$ and $D \overset{\mathrm{L}}{\otimes}_R X \in \mathcal{B}^{\mathrm{f}}(R)$. The latter fact means that $D \overset{\mathrm{L}}{\otimes}_R X \in \mathsf{fin}(R)$ and $\mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R X) \in \mathsf{fin}(R)$, and that the following morphism is in fact an isomorphism,

$$D \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R X) \xrightarrow{\varepsilon} D \overset{\mathrm{L}}{\otimes}_R X,$$

where ε denotes the counit of the adjoint pair $(D \overset{\mathrm{L}}{\otimes}_R -, \mathrm{RHom}_R(D, -))$ applied to the object $D \overset{\mathrm{L}}{\otimes}_R X$.

However, by adjoint functor theory there is a commutative diagram

$$\begin{tikzcd} D \overset{\mathrm{L}}{\otimes}_R X \arrow[d, "D \overset{\mathrm{L}}{\otimes}_R \eta"'] \arrow[dr, equal] & \\ D \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R X) \arrow[r, "\varepsilon"'] & D \overset{\mathrm{L}}{\otimes}_R X, \end{tikzcd}$$

where η denotes the unit of the adjoint pair $(D \overset{\mathrm{L}}{\otimes}_R -, \mathrm{RHom}_R(D, -))$ applied to the object X. So $D \overset{\mathrm{L}}{\otimes}_R \eta$ is an isomorphism. Evoking lemma (2.7) we conclude that η is an isomorphism. Thus $X \in \mathcal{A}(R)$, and since we have both $X \in \mathsf{fin}(R)$ and $D \overset{\mathrm{L}}{\otimes}_R X \in \mathsf{fin}(R)$, we finally get $X \in \mathcal{A}^{\mathrm{f}}(R)$.

The second implication is proved by an entirely analogous argument: Suppose $Y \in \mathsf{fin}(R)$ and $\mathrm{RHom}_R(D,Y) \in \mathcal{A}^{\mathrm{f}}(R)$. This latter fact means that $\mathrm{RHom}_R(D,Y) \in \mathsf{fin}(R)$ and $D \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(D,Y) \in \mathsf{fin}(R)$, and that the following morphism is in fact an isomorphism,

$$\mathrm{RHom}_R(D,Y) \xrightarrow{\quad\eta\quad} \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(D,Y)),$$

where η denotes the unit of the adjoint pair $(D \overset{\mathrm{L}}{\otimes}_R -, \mathrm{RHom}_R(D,-))$ applied to the object $\mathrm{RHom}_R(D,Y)$.

Again, by adjoint functor theory there is a commutative diagram

$$\begin{tikzcd} \mathrm{RHom}_R(D,Y) \arrow[r, "\eta"] \arrow[dr, equal] & \mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R \mathrm{RHom}_R(D,Y)) \arrow[d, "\mathrm{RHom}_R(D,\varepsilon)"] \\ & \mathrm{RHom}_R(D,Y), \end{tikzcd}$$

where ε denotes the counit of the adjoint pair $(D \overset{\mathrm{L}}{\otimes}_R -, \mathrm{RHom}_R(D,-))$ applied to the object Y. So $\mathrm{RHom}_R(D,\varepsilon)$ is an isomorphism. Again evoking lemma (2.7), we learn that ε is an isomorphism. Thus $Y \in \mathcal{B}(R)$, and since we have both $Y \in \mathsf{fin}(R)$ and $\mathrm{RHom}_R(D,Y) \in \mathsf{fin}(R)$ we finally get $Y \in \mathcal{B}^{\mathrm{f}}(R)$. $\square$

(2.9) **Definition (Reflexive DG-R-modules).** We can consider DG-R-left-modules M for which the evaluation morphism

$$\mathrm{RHom}_{R^{\mathrm{opp}}}(R,R) \overset{\mathrm{L}}{\otimes}_R M \longrightarrow \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(M,R),R)$$

is an isomorphism, and we will call a DG-R-left-module with this property $\mathsf{reflexive}$. The full subcategory consisting of reflexive DG-R-left-modules is denoted $\mathcal{R}(R)$.

There is of course a corresponding class of reflexive DG-R-right-modules, denoted $\mathcal{R}(R^{\mathrm{opp}})$.

(2.10) **Remark.** The evaluation morphism appearing in the definition of $\mathcal{R}(R)$ is the first morphism from condition [G2]. Hence condition [G2] for R can be phrased as follows:

- $\mathcal{R}(R) \supseteq \mathsf{fin}(R)$ and
- $\mathcal{R}(R^{\mathrm{opp}}) \supseteq \mathsf{fin}(R^{\mathrm{opp}})$.

(2.11) **Proposition.** *Let D be a weak dualizing DG-module for R. Then*

$$\mathcal{A}(R) \cap \mathsf{fin}(R) = \mathcal{R}(R) \cap \mathsf{fin}(R).$$

There is of course a corresponding equality for DG-R-right-modules.

Proof. We only need to prove the statement for DG-R-left-modules as the statement for DG-R-right-modules follows by symmetry. Consider the following diagram where ${}_RM \in \mathsf{fin}(R)$,

$$\begin{tikzcd}
M \arrow[rr, "\cong"] \arrow[dd, "\eta_M"'] & & \mathrm{RHom}_{R^{\mathrm{opp}}}(R,R) \overset{\mathrm{L}}{\otimes}_R M \arrow[d, "\theta_1"] \\
& & \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(M,R),R) \arrow[d, "\cong"', "\mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(M,R),\rho)"] \\
\mathrm{RHom}_R(D, D \overset{\mathrm{L}}{\otimes}_R M) \arrow[d, "\cong"] & & \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(M,R),\mathrm{RHom}_R(D,D)) \arrow[d, "\cong"', "s"] \\
\mathrm{RHom}_R(D, \mathrm{RHom}_{R^{\mathrm{opp}}}(R,D) \overset{\mathrm{L}}{\otimes}_R M) \arrow[rr, "\cong", "\mathrm{RHom}_R(D,\theta_2)"'] & & \mathrm{RHom}_R(D, \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(M,R),D)).
\end{tikzcd}$$

All derived functors exist because we can use the resolutions P and I from [D1] to:

- replace the functor $\mathrm{RHom}_R(D,-)$ with $\mathrm{Hom}_R(P,-)$.
- replace the functor $\mathrm{RHom}_R(-,D)$ with $\mathrm{Hom}_R(-,I)$.
- replace the functor $\mathrm{RHom}_R(-,R)$ with $\mathrm{Hom}_R(-,\mathrm{Hom}_R(P,I))$ when needed. (Note that $\mathrm{Hom}_R(P,I)$ is right-K-injective.)
- replace M with a left-K-projective resolution Q.

Let us take a look at the morphisms in the diagram starting with η_M and proceeding counterclockwise:

- the morphism η_M is the unit of the adjoint pair $(D \overset{\mathrm{L}}{\otimes}_R -, \mathrm{RHom}_R(D,-))$ applied to M.
- the next morphism is an isomorphism since it is induced by the canonical identification $D \cong \mathrm{RHom}_{R^{\mathrm{opp}}}(R,D)$.
- the morphism $\mathrm{RHom}_R(D,\theta_2)$ is obtained from the first evaluation morphism from [D3] which we here denote θ_2. Since θ_2 is an isomorphism, so is $\mathrm{RHom}_R(D,\theta_2)$.
- the morphism s is the so-called swap isomorphism.
- the morphism $\mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(M,R),\rho)$ is an isomorphism since it is induced by the isomorphism $R \xrightarrow{\rho} \mathrm{RHom}_R(D,D)$ from [D2].
- the morphism θ_1 is the first evaluation morphism from condition [G2].
- the last morphism is an isomorphism since it is induced by the canonical identification $R \cong \mathrm{RHom}_{R^{\mathrm{opp}}}(R,R)$.

It is now easy but tedious to check that the diagram is commutative, hence θ_1 and η are isomorphisms simultaneously. □

(2.12) **Definition ((Finite) Reflexive DG-modules).** We let

$$\mathcal{R}^{\mathrm{f}}(R) = \left\{ X \in \mathcal{R}(R) \;\middle|\; \begin{array}{l} X \in \mathsf{fin}(R) \text{ and} \\ \mathrm{RHom}_R(X,R) \in \mathsf{fin}(R^{\mathrm{opp}}) \end{array} \right\}.$$

We call the DG-modules in $\mathcal{R}^{\mathrm{f}}(R)$ finite reflexive.

There is of course a corresponding class of finite reflexive DG-R-right-modules, denoted $\mathcal{R}^{\mathrm{f}}(R^{\mathrm{opp}})$.

(2.13) **Remark.** Observe from remark (2.10) and definition (2.12) that conditions [G2] and [G3] together are equivalent to:

- $\mathcal{R}^{\mathrm{f}}(R) = \mathsf{fin}(R)$ and
- $\mathcal{R}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}})$.

(2.14) **Definition.** Let D be a dualizing DG-module for R. We define the following auxiliary classes:

$$\mathcal{X}(R) = \mathsf{fin}(R) \cap \{\, X \in \mathsf{D}(R) \mid D \overset{\mathrm{L}}{\otimes}_R X \in \mathsf{fin}(R) \,\},$$
$$\mathcal{Z}(R) = \mathsf{fin}(R) \cap \{\, Z \in \mathsf{D}(R) \mid \mathrm{RHom}_R(Z, R) \in \mathsf{fin}(R^{\mathrm{opp}}) \,\},$$

and

$$\mathcal{Y}(R) = \left\{ Y \in \mathsf{D}(R) \;\middle|\; \begin{array}{l} Y \cong \mathrm{RHom}_{R^{\mathrm{opp}}}(A, D) \text{ for an } A \in \mathsf{fin}(R^{\mathrm{opp}}) \\ \text{with } \mathrm{RHom}_{R^{\mathrm{opp}}}(D, A) \in \mathsf{fin}(R^{\mathrm{opp}}) \end{array} \right\}.$$

There are of course corresponding classes of DG-R-right-modules.

(2.15) **Remark.** The point of $\mathcal{X}$ and $\mathcal{Z}$ is that we have

$$\mathcal{A}^{\mathrm{f}}(R) = \mathcal{A}(R) \cap \mathcal{X}(R) \qquad \text{and} \qquad \mathcal{R}^{\mathrm{f}}(R) = \mathcal{R}(R) \cap \mathcal{Z}(R)$$

and corresponding equations for DG-R-right-modules.

(2.16) **Lemma.** *Let D be a dualizing DG-module for R. Then we have the following equalities,*

$$\mathcal{X}(R) = \mathcal{Z}(R) = \mathcal{Y}(R).$$

There are of course corresponding equalities for DG-R-right-modules.

Proof. The equalities for DG-R-right-modules follow by symmetry.

To prove the claim for DG-R-left-modules we proceed as follows. We first prove $\mathcal{X}(R) = \mathcal{Z}(R)$:

"$\subseteq$": Let $X \in \mathcal{X}(R)$. Then $D \overset{\mathrm{L}}{\otimes}_R X \in \mathsf{fin}(R)$ and by [D4] we have

$$\mathrm{RHom}_R(D \overset{\mathrm{L}}{\otimes}_R X, D) \in \mathsf{fin}(R^{\mathrm{opp}}).$$

But we have the following isomorphisms,

$$\begin{aligned} \mathrm{RHom}_R(X, R) &\overset{(a)}{\cong} \mathrm{RHom}_R(X, \mathrm{RHom}_R(D, D)) \\ &\overset{(b)}{\cong} \mathrm{RHom}_R(D \overset{\mathrm{L}}{\otimes}_R X, D), \end{aligned}$$

where (a) is by [D2], and (b) is due to adjointness. We now conclude that X is in $\mathcal{Z}(R)$.

"$\supseteq$": Let $Z \in \mathcal{Z}(R)$. Then $\operatorname{RHom}_R(Z,R) \in \mathsf{fin}(R^{\mathrm{opp}})$ and by [D4] we have

$$\operatorname{RHom}_{R^{\mathrm{opp}}}(\operatorname{RHom}_R(Z,R),D) \in \mathsf{fin}(R).$$

But we have the following isomorphisms,

$$\begin{aligned} D \overset{\mathrm{L}}{\otimes}_R Z &\cong \operatorname{RHom}_{R^{\mathrm{opp}}}(R,D) \overset{\mathrm{L}}{\otimes}_R Z \\ &\overset{(a)}{\cong} \operatorname{RHom}_{R^{\mathrm{opp}}}(\operatorname{RHom}_R(Z,R),D). \end{aligned}$$

Here (a) is due to [D3] since Z is in $\mathsf{fin}(R)$. We now conclude that Z is in $\mathcal{X}(R)$.

We next prove $\mathcal{Z}(R) = \mathcal{Y}(R)$:

"$\subseteq$": Let $Z \in \mathcal{Z}(R)$. Then $Z \in \mathsf{fin}(R)$ so by the duality (1.2) we have

$$Z \cong \operatorname{RHom}_{R^{\mathrm{opp}}}(\operatorname{RHom}_R(Z,D),D)$$

and $A = \operatorname{RHom}_R(Z,D) \in \mathsf{fin}(R^{\mathrm{opp}})$. We will have proved that Z is in $\mathcal{Y}(R)$ when we have proved that $\operatorname{RHom}_{R^{\mathrm{opp}}}(D,A)$ is in $\mathsf{fin}(R^{\mathrm{opp}})$. But

$$\begin{aligned} \operatorname{RHom}_{R^{\mathrm{opp}}}(D,A) &\cong \operatorname{RHom}_{R^{\mathrm{opp}}}(D,\operatorname{RHom}_R(Z,D)) \\ &\overset{(a)}{\cong} \operatorname{RHom}_R(Z,\operatorname{RHom}_{R^{\mathrm{opp}}}(D,D)) \\ &\overset{(b)}{\cong} \operatorname{RHom}_R(Z,R). \end{aligned}$$

Here (a) is due to the so-called swap isomorphism, and (b) is due to [D2]. Now, since $Z \in \mathcal{Z}(R)$, we have $\operatorname{RHom}_R(Z,R) \in \mathsf{fin}(R^{\mathrm{opp}})$ so $\operatorname{RHom}_{R^{\mathrm{opp}}}(D,A) \in \mathsf{fin}(R^{\mathrm{opp}})$.

"$\supseteq$": Let $Y \in \mathcal{Y}(R)$. Then $Y \cong \operatorname{RHom}_{R^{\mathrm{opp}}}(A,D)$ for an $A \in \mathsf{fin}(R^{\mathrm{opp}})$ so $Y \in \mathsf{fin}(R)$. Moreover, $\operatorname{RHom}_{R^{\mathrm{opp}}}(D,A) \in \mathsf{fin}(R^{\mathrm{opp}})$. Now, the duality (1.2) says

$$A \cong \operatorname{RHom}_R(\operatorname{RHom}_{R^{\mathrm{opp}}}(A,D),D) \cong \operatorname{RHom}_R(Y,D).$$

And so

$$\begin{aligned} \operatorname{RHom}_{R^{\mathrm{opp}}}(D,A) &\cong \operatorname{RHom}_{R^{\mathrm{opp}}}(D,\operatorname{RHom}_R(Y,D)) \\ &\overset{(a)}{\cong} \operatorname{RHom}_R(Y,\operatorname{RHom}_{R^{\mathrm{opp}}}(D,D)) \\ &\overset{(b)}{\cong} \operatorname{RHom}_R(Y,R), \end{aligned}$$

where (a) is due to the so-called swap isomorphism, and (b) is due to [D2]. We conclude $\operatorname{RHom}_R(Y,R) \in \mathsf{fin}(R^{\mathrm{opp}})$ so $Y \in \mathcal{Z}(R)$. $\square$

(2.17) **Proposition.** *Let D be a dualizing DG-module for R. Then*

$$\mathcal{A}^{\mathrm{f}}(R) = \mathcal{R}^{\mathrm{f}}(R).$$

There is of course a corresponding equality for DG-R-right-modules.

Proof. As everything is left/right-symmetric, it is enough to check the claim for DG-R-left-modules.

But remark (2.15) says $\mathcal{A}^{\mathrm{f}}(R) = \mathcal{A}(R) \cap \mathcal{X}(R)$ and $\mathcal{R}^{\mathrm{f}}(R) = \mathcal{R}(R) \cap \mathcal{Z}(R)$, and we have $\mathcal{A}(R) \cap \mathsf{fin}(R) = \mathcal{R}(R) \cap \mathsf{fin}(R)$ by proposition (2.11) and $\mathcal{X}(R) = \mathcal{Z}(R)$ by lemma (2.16). □

(2.18) **Maximality.** Note from definitions (2.3) and (2.12) that in the presence of a weak dualizing DG-module we have the following inclusions:

- $$\begin{cases} \mathcal{A}^{\mathrm{f}}(R) \subseteq \mathsf{fin}(R) \quad \text{and} \\ \mathcal{A}^{\mathrm{f}}(R^{\mathrm{opp}}) \subseteq \mathsf{fin}(R^{\mathrm{opp}}). \end{cases}$$
- $$\begin{cases} \mathcal{B}^{\mathrm{f}}(R) \subseteq \mathsf{fin}(R) \quad \text{and} \\ \mathcal{B}^{\mathrm{f}}(R^{\mathrm{opp}}) \subseteq \mathsf{fin}(R^{\mathrm{opp}}). \end{cases}$$
- $$\begin{cases} \mathcal{R}^{\mathrm{f}}(R) \subseteq \mathsf{fin}(R) \quad \text{and} \\ \mathcal{R}^{\mathrm{f}}(R^{\mathrm{opp}}) \subseteq \mathsf{fin}(R^{\mathrm{opp}}). \end{cases}$$

So the maximal possible size of either of the classes $\mathcal{R}^{\mathrm{f}}$, $\mathcal{A}^{\mathrm{f}}$, and $\mathcal{B}^{\mathrm{f}}$ is fin. The following main theorem now characterizes the DGAs for which this maximal size is attained.

(2.19) **Gorenstein Theorem.** *Let R be a DGA satisfying* [Grade] *and* [G1]. *Moreover, let D be a dualizing DG-module for R. The following conditions are equivalent:*

(1) *R is a Gorenstein DGA (i.e.,* [G2] *and* [G3] *hold).*

(2) $$\begin{cases} \mathcal{A}^{\mathrm{f}}(R) = \mathsf{fin}(R) \quad \textit{and} \\ \mathcal{A}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}}). \end{cases}$$

(3) $$\begin{cases} \mathcal{B}^{\mathrm{f}}(R) = \mathsf{fin}(R) \quad \textit{and} \\ \mathcal{B}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}}). \end{cases}$$

(4) $$\begin{cases} \mathcal{R}^{\mathrm{f}}(R) = \mathsf{fin}(R) \quad \textit{and} \\ \mathcal{R}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}}). \end{cases}$$

Proof. We shall prove (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

(1) $\Leftrightarrow$ (4): Holds by (2.13).

(4) $\Leftrightarrow$ (2): Follows by proposition (2.17).

(2) $\Rightarrow$ (3): When

$$\mathcal{A}^{\mathrm{f}}(R) = \mathsf{fin}(R) \qquad \text{and} \qquad \mathcal{A}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}}),$$

let us prove $\mathcal{B}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}})$. The inclusion "$\subseteq$" is clear. To prove the inclusion "$\supseteq$", consider $A \in \mathsf{fin}(R^{\mathrm{opp}})$. To see $A \in \mathcal{B}^{\mathrm{f}}(R^{\mathrm{opp}})$, it is enough to see

$$\mathrm{RHom}_{R^{\mathrm{opp}}}(D, A) \in \mathcal{A}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}}),$$

by theorem (2.8).

If we let

$$Y = \mathrm{RHom}_{R^{\mathrm{opp}}}(A, D)$$

then we have $Y \in \mathsf{fin}(R)$, and

$$\mathrm{RHom}_R(Y, D) \cong \mathrm{RHom}_R(\mathrm{RHom}_{R^{\mathrm{opp}}}(A, D), D) \cong A$$

by the duality (1.2). Clearly up to isomorphism, only this particular A can give Y under the duality. So the desired conclusion, $\mathrm{RHom}_{R^{\mathrm{opp}}}(D, A) \in \mathsf{fin}(R^{\mathrm{opp}})$, is equivalent to $Y \in \mathcal{Y}(R)$ where $\mathcal{Y}(R)$ is one of the sets from definition (2.14).

But lemma (2.16) shows that $\mathcal{Y}(R) = \mathcal{X}(R)$, and $\mathcal{X}(R) = \mathsf{fin}(R)$ since $\mathcal{A}^{\mathrm{f}}(R) = \mathsf{fin}(R)$ by assumption. So we conclude $Y \in \mathcal{Y}(R)$, as desired.

A symmetric argument shows $\mathcal{B}^{\mathrm{f}}(R) = \mathsf{fin}(R)$.

(3) $\Rightarrow$ (2): When

$$\mathcal{B}^{\mathrm{f}}(R) = \mathsf{fin}(R) \qquad \text{and} \qquad \mathcal{B}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}}),$$

let us prove $\mathcal{A}^{\mathrm{f}}(R) = \mathsf{fin}(R)$. The inclusion "$\subseteq$" is clear. To prove the inclusion "$\supseteq$", consider $X \in \mathsf{fin}(R)$. To see $X \in \mathcal{A}^{\mathrm{f}}(R)$, it is enough to see

$$D \overset{\mathrm{L}}{\otimes}_R X \in \mathcal{B}^{\mathrm{f}}(R) = \mathsf{fin}(R)$$

by theorem (2.8).

Now, since $\mathcal{B}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}})$ by assumption, any $A \in \mathsf{fin}(R^{\mathrm{opp}})$ has $\mathrm{RHom}_{R^{\mathrm{opp}}}(D, A) \in \mathsf{fin}(R^{\mathrm{opp}})$. This proves $\mathcal{Y}(R) = \mathsf{fin}(R)$. But lemma (2.16) shows that $\mathcal{X}(R) = \mathcal{Y}(R)$, so we get $\mathcal{X}(R) = \mathsf{fin}(R)$, whence $D \overset{\mathrm{L}}{\otimes}_R X \in \mathsf{fin}(R)$, as desired.

A symmetric argument shows $\mathcal{A}^{\mathrm{f}}(R^{\mathrm{opp}}) = \mathsf{fin}(R^{\mathrm{opp}})$. $\square$

3. Natural DGAs satisfying Grade

It is natural to ask if one can actually come up with DGAs which satisfy [Grade]. Let A be a commutative noetherian local ring and let B be a non-commutative noetherian local PI algebra over a field (see [20]). Paragraphs (3.1), (3.2), (3.5), and (3.6) show that

- The ring A viewed as a DGA concentrated in degree zero satifies [Grade].
- The algebra B viewed as a DGA concentrated in degree zero satisfies [Grade].
- The Koszul complex $\mathrm{K}(\boldsymbol{a})$ on a sequence $\boldsymbol{a} = (a_1, \ldots , a_n)$ of elements in the maximal ideal of A satisfies [Grade]. (Note that by [21, exer. 4.5.1] the Koszul complex is a commutative DGA.)

- The endomorphism DGA $\mathcal{E} = \mathrm{Hom}_A(L, L)$, where L is a bounded complex of finitely generated projective A-modules which is not exact, satisfies [Grade]. (For details on $\mathcal{E}$ we refer the reader to [9] and [12, rem. (1.2)].)

The two last statements are special cases of lemma (3.4) which says that if there is a so-called finite Gorenstein morphism of DGAs $A \xrightarrow{\varphi} R$, where A is a noetherian local commutative ring viewed as a DGA concentrated in degree zero and R is a DGA, then R satisfies [Grade].

(3.1) **Commutative rings satisfying [Grade].** *Let A be a noetherian local commutative ring. Then A viewed as a DGA concentrated in degree zero satisfies* [Grade].

Proof. First observe that since A is commutative, the two halves of condition [Grade] coincide.

Without loss of generality we may assume that A is complete. Then A admits a dualizing complex D. Let M be any non-trivial complex in $\mathsf{fin}(A) = \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$. We may compute as follows:

$$\begin{aligned}\mathrm{RHom}_A(M, A) &\cong \mathrm{RHom}_A(M, \mathrm{RHom}_A(D, D)) \\ &\overset{(a)}{\cong} \mathrm{RHom}_A(D \overset{\mathrm{L}}{\otimes}_A M, D),\end{aligned}$$

where (a) is by adjointness. Evoking that M is non-trivial and that D is a dualizing complex for A, we conclude using [10, prop. 2.2] that $D \overset{\mathrm{L}}{\otimes}_A M$ is non-trivial and bounded to the right; it has finitely generated homology by [4, proof of 4.7.I]. Since the functor $\mathrm{RHom}_A(-, D)$ gives a duality between $\mathsf{D}^{\mathrm{f}}_{+}(A)$ and a subcategory of $\mathsf{D}^{\mathrm{f}}_{-}(A)$ (see [18, prop. V.2.1] and [2, p. 27, thm. 1(4)]), this ensures that $\mathrm{RHom}_A(D \overset{\mathrm{L}}{\otimes}_A M, D)$ is non-trivial. ☐

(3.2) **Non-commutative rings satisfying [Grade].** *Let B be a noetherian local PI algebra over a field. Then B satifies* [Grade].

Proof. By [20, thm. B] and [23, lem. 4.1] we may assume that B is complete. Now, by the mirror version of [22, cor. 0.2] there exists a noetherian complete semi-local PI algebra T and a dualizing complex ${}_TD_B$ over (T, B) (see [22] for an explanation of this use of the terminology "dualizing complex" which generalizes paragraph (1.5)). Let M be any non-trivial complex in $\mathsf{fin}(B) = \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(B)$. As in the proof of (3.1) we may compute as follows:

$$\begin{aligned}\mathrm{RHom}_B(M, B) &\cong \mathrm{RHom}_B(M, \mathrm{RHom}_T({}_TD_B, {}_TD_B)) \\ &\overset{(a)}{\cong} \mathrm{RHom}_T({}_TD_B \overset{\mathrm{L}}{\otimes}_B M, {}_TD_B).\end{aligned}$$

Here (a) is by adjointness. Thus (again as in the proof of (3.1)) it clearly suffices to check that ${}_T D_B \overset{\mathrm{L}}{\otimes}_B M$ is non-trivial with finitely generated homology bounded to the right. This, however, follows from the non-commutative Nakayama's lemma.

The mirror version of the above argument applies to complexes in $\mathsf{fin}(B^{\mathrm{opp}}) = \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(B^{\mathrm{opp}})$. □

(3.3) **Gorenstein morphisms.** In the next result the notion of *finite Gorenstein morphisms* of DGAs plays an important role. For details on such morphisms we refer the reader to [15, sec. 2].

(3.4) **Lemma.** *Let A be a noetherian local commutative ring. Let $A \xrightarrow{\varphi} R$ be a finite Gorenstein morphism of DGAs (A is viewed as a DGA concentrated in degree zero). Then R satisfies* [Grade].

Proof. Since the morphism $A \xrightarrow{\varphi} R$ is a Gorenstein morphism we have the following two isomorphisms in $\mathsf{D}(A)$,

$$ {}_R R_A \cong \mathrm{RHom}_A(R_R, \Sigma^n A_A), \qquad (*)$$

$$ {}_A R_R \cong \mathrm{RHom}_A({}_R R, \Sigma^n {}_A A), \qquad (**)$$

for some $n \in \mathbb{Z}$, where Σ^n denotes the n'th suspension of a DG-module (see [15, def. (2.4)]). Note that since $A \xrightarrow{\varphi} R$ is a finite morphism (see [15, def. (2.1)]) any M in $\mathsf{fin}(R)$ or $\mathsf{fin}(R^{\mathrm{opp}})$ is in $\mathsf{fin}(A)$ when viewed over A.

Given $M \in \mathsf{fin}(R)$, we may now perform the following computations,

$$\begin{aligned} \mathrm{RHom}_R(M, R) &\overset{(a)}{\cong} \mathrm{RHom}_R(M, \Sigma^n \mathrm{RHom}_A(R, A)) \\ &\overset{(b)}{\cong} \Sigma^n \mathrm{RHom}_A(M, A). \end{aligned}$$

Here (a) is due to $(*)$, and (b) is due to adjointness.

Since M is in $\mathsf{fin}(A)$ when viewed over A, this computation and the corresponding computation for DG-R-right-modules using $(**)$ show that since A satisfies [Grade] by (3.1), so does R. □

(3.5) **The Koszul complex satisfies [Grade].** Let A be a noetherian local commutative ring, and let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a sequence of elements in the maximal ideal of A. Now, there is a canonical morphism of DGAs

$$A \xrightarrow{\theta} \mathrm{K}(\boldsymbol{a}).$$

Moreover, from [15, lem. (3.3)], we know that θ is a finite Gorenstein morphism whence $\mathrm{K}(\boldsymbol{a})$ satisfies [Grade] by lemma (3.4).

(3.6) **The endomorphism DGA satisfies [Grade].** Let A be a noetherian local commutative ring, let L be a bounded complex of finitely generated projective A-modules which is not exact, and let $\mathcal{E} = \mathrm{Hom}_A(L, L)$ be the endomorphism DGA of L. Now, there is a canonical morphism of DGAs

$$A \xrightarrow{\psi} \mathcal{E}, \qquad a \longmapsto (L \xrightarrow{a\cdot} L)$$

(see [15, (3.7)]). Moreover, from [15, lem. (3.8)], we know that ψ is a finite Gorenstein morphism whence $\mathcal{E}$ satisfies [Grade] by lemma (3.4).

References

[1] D. Apassov, *Dualizing modules over differential graded rings*, pp. 41–47 in "Complexes and Differential Graded Modules", ph.d. thesis, Lund University, 1999.

[2] D. Apassov, *Homological dimensions over differential graded rings*, pp. 25–39 in "Complexes and Differential Graded Modules", ph.d. thesis, Lund University, 1999.

[3] M. Auslander and D. Buchsbaum, *Homological dimensions in Noetherian rings. II*, Trans. Amer. Math. Soc. **88** (1958), 194–206.

[4] L. L. Avramov and H.-B. Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), 129–155.

[5] ——— and ———, *Ring homomorphisms and finite Gorenstein dimension,* Proc. London Math. Soc. (3) **75** (1997), 241–270.

[6] ———, ———, and S. Halperin, Differential graded homological algebra, in preparation.

[7] ——— and S. Halperin, *Through the looking glass: a dictionary between rational homotopy theory and local algebra*, pp. 1–27 in "Algebra, algebraic topology and their interactions" (proceedings of the conference in Stockholm, 1983), Lecture Notes in Math., Vol. 1183, Springer, Berlin, 1986.

[8] L. W. Christensen, "Gorenstein Dimensions", Lecture Notes in Math., Vol. 1747, Springer, Berlin, 2000.

[9] W. G. Dwyer and J. P. C. Greenlees, Complete modules and torsion modules, preprint (2000).

[10] H.-B. Foxby, *Isomorphisms between complexes with applications to the homological theory of modules*, Math. Scand. **40** (1977), 5–19.

[11] A. Frankild, S. Iyengar, and P. Jørgensen, Existence of dualizing DG-modules for Differential Graded Algebras, manuscript (2001).

[12] ——— and P. Jørgensen, Affine equivalence and Gorensteinness, preprint (2001).

[13] ——— and ———, Dualizing DG-modules for Differential Graded Algebras, preprint (2001).

[14] ——— and ———, Foxby equivalence, complete modules, and torsion modules, preprint (2001).

[15] ——— and ———, Gorenstein Differential Graded Algebras, preprint (2001).

[16] ——— and ———, Homological identities for Differential Graded Algebras, preprint (2001).

[17] ——— and ———, Homological identities for Differential Graded Algebras, preprint (2002).

[18] R. Hartshorne, *Residues and duality*, Lecture Notes in Math., Vol. 20, Springer, Berlin, 1966, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne.

[19] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 63–102.

[20] R. Vámos, *Semi-local noetherian PI rings*, Bull. London Math. Soc. **9** (1977) 251–256.

[21] C. A. Weibel, "An introduction to homological algebra", Cambridge Stud. Adv. Math., Vol. 38, Cambridge University Press, Cambridge, 1997, paperback edition.

[22] Q. S. Wu and J. J. Zhang, *Dualizing Complexes over Noncommutative Local Rings*, J. Algebra **239** (2001), 513–548.

[23] ——— and ———, *Some homological invariants of local PI algebras*, J. Algebra **225** (2000), 904–935.

[24] A. Yekutieli, *Dualizing complexes over noncommutative graded algebras*, J. Algebra **153** (1992), 41–84.

[25] ——— and J. J. Zhang, *Rings with Auslander Dualizing Complexes*, J. Algebra **213** (1999), 1–51.

Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, 2100 København Ø, DK–Danmark

E-mail address: frankild@math.ku.dk, popjoerg@math.ku.dk

Journal of the London Math. Soc.
(to appear)

DUALIZING DG MODULES AND GORENSTEIN DG ALGEBRAS

ANDERS FRANKILD, SRIKANTH IYENGAR, AND PETER JØRGENSEN

Introduction

There is a growing body of research concerned with extending the rich theory of commutative Gorenstein rings to DG (=Differential Graded) algebras. This subject began with the work of Félix, Halperin, and Thomas [6] on a Gorenstein condition for (cochains complexes of) topological spaces, which extends classical Poincaré duality for manifolds. Their study was complemented by that of Avramov and Foxby [3], who adopted a similar definition of the Gorenstein property, but focused on DG algebras arising in commutative ring theory.

In a recent article, Frankild and Jørgensen [10] proposed a new notion of 'Gorenstein' DG algebras. The natural problem arises: *How does their approach relate to those of Félix-Halperin-Thomas and Avramov-Foxby?* The content of one of the main theorems in this paper is that the Avramov and Foxby definition of Gorenstein, for the class of DG algebras considered by them, is equivalent to the one of Frankild-Jørgensen.

In order to facilitate the ensuing discussion, we adopt the convention that the term 'Gorenstein DG algebra' is used in the sense of Frankild and Jørgensen. Their definition is recalled in Section 4; see also the discussion below.

Theorem I. *Let $R = \{R_i\}_{i \geqslant 0}$ be a commutative DG algebra where R_0 is a local ring, with residue field k, and the $\mathrm{H}_0(R)$-module $\mathrm{H}(R)$ is finitely generated.*

Then R is Gorenstein if and only if $\operatorname{rank}_k \operatorname{Ext}_R(k, R) = 1$.

The implication: when R is Gorenstein the k-vector space $\operatorname{Ext}_R(k, R)$ is one dimensional, was proved in [10]. The converse settles a conjecture in *loc. cit.*

We establish a similar result for cochain complexes of certain topological spaces. This amounts to the statement: Let k be a field and and let X be a finite simply connected space. If X is Gorenstein at k, in the sense of Félix, Halperin, and Thomas, then the cochain complex of X (with coefficients in k) is Gorenstein. The converse holds if the characteristic of k is 0.

2000 *Mathematics Subject Classification.* 13D99,13H10,16E45.

Key words and phrases. dualizing DG modules, Gorenstein DGA, Gorenstein space, ascent-descent of Gorenstein property for DGAs.

Part of this research was done when S. I. was at the University of Sheffield, supported by a grant from the E. P. S. R. C.

Theorem I is established in Section 4, as is a corollary related to ascent and descent of the Gorenstein property. The topological counterpart of Theorem I is established in Section 5.

By definition, a DG algebra is Gorenstein if it is *dualizing*, in the sense of [9], as a module over itself. Thus, it is no surprise that the protagonists in this work are dualizing DG modules over DG algebras. These objects, which are DG analogues of dualizing complexes over commutative rings, were introduced in *loc. cit.*, wherein one finds also DG versions of some central results in the theory of dualizing complexes for commutative rings.

Section 3 delves deeper into this subject, focusing on finite commutative local DG algebras: a class of DG algebras which is slightly larger than that considered in Theorem I. For instance, one finds there the following result that detects dualizing DG modules. The adjective 'balanced' explained in (1.1).

Theorem II. *Let R be a finite commutative local DG algebra and D a balanced DG R-module. Then D is dualizing if and only if the* $\mathrm{H}_0(R)$*-module* $\mathrm{H}(D)$ *is finitely generated and* $\operatorname{rank}_k \operatorname{Ext}_R(k, D) = 1$.

This is one of the principal results of this section, and indeed of this work. One piece of evidence for its efficacy is that one can immediately deduce Theorem I from it. Section 3 contains also the following result:

Theorem III. *Let R be a finite commutative local DG algebra. Then any pair of balanced dualizing DG R-modules are quasiisomorphic up to suspension.*

The results obtained demonstrate that the theory of dualizing DG modules over finite commutative local DG algebras is akin to that over commutative local rings.

There are numerous questions that have not been addressed as yet, especially when one considers dualizing DG modules over general, not necessarily commutative, DG algebras. Among these, perhaps the most important one is: *Do dualizing DG modules exist?*

Section 2 deals exclusively with this question. The following portmanteau result sums up the results arrived at there; it shows that many of the naturally arising DG algebras possess dualizing DG modules. The paper is so organized that statements (1)–(4) are to be found in Section 2, while the last one is in Section 5.

Theorem IV. *Let A be a commutative noetherian ring with a dualizing complex. Then the following statements hold:*

1. *The Koszul complex on each finite set of elements in A admits a dualizing DG module;*
2. *For each bounded complex of projective A-modules P with* $\mathrm{H}(P) \neq 0$*, the endomorphism DG algebra* $\operatorname{Hom}_A(P, P)$ *admits a dualizing DG module;*

(3) *Suppose that A is local and that $\varphi\colon A' \to A$ is a local homomorphism of finite flat dimension. The DG fibre of φ admits a dualizing DG module;*
(4) *The chain complex (with coefficients in A) of each finite topological monoid admits a dualizing complex.*
(5) *The cochain DG algebra (with coefficients in a field) of each finite simply connected CW complex admits a dualizing complex;*

Besides being of intrinsic interest, we anticipate that this existence result will prove useful in future applications of the theory of dualizing DG modules. Already, in Section 5, we make crucial use of (5) above.

As is to be expected, extensive use is made of techniques from homological algebra. For ease of reference, Section 1 recapitulates of some of the notions and constructions in this subject that arise frequently in this paper.

1. DG homological algebra

It is assumed that the reader is familiar with basic definitions concerning DG algebras and DG modules; if this is not the case, then they may consult, for instance, [7]. Moreover, in what follows, a few well known results in this subject are used; for these we quote from [4] whenever possible, in the interest of uniformity.

A few words about notation: Most graded objects that appear here are assumed to be graded homologically; for instance, a graded set X is a collection of sets $\{X_i\}_{i\in\mathbb{Z}}$. Also, in a complex M, the differential ∂ decreases degree: $\partial_i\colon M_i \to M_{i-1}$. All exceptions to this convention are to be found in Section 5 for it deals with examples from topology, and where the natural grading is the cohomological one.

1.1. **Modules.** Over a DG algebra R, it is convenient to refer to left DG R-modules (respectively, right DG R-modules) as *left R-modules* (respectively, *right R-modules*), thus omitting the 'DG'. One exception: When speaking of a *ring*, rather than a DG algebra, we distinguish between modules, in the classical sense of the word, and complexes, which are the DG modules.

We denote R° the opposite DG algebra. Note that a right R-module may be viewed as a left module over R°, and vice-versa.

An *R-bimodule* is an abelian group with compatible left and right R-module structures. One example of an R-bimodule is R itself, viewed as a left and right R-module via left and right multiplication respectively. In the sequel, this will be canonical bimodule structure on R.

Suppose that R is commutative, that is to say, $rs = (-1)^{|r||s|}sr$ for each $r, s \in R$. Then an R-bimodule M is said to be *balanced* if the left and right structures determine each other, in the sense that, for $r \in R$ and $m \in M$, one has $rm = (-1)^{|r||m|}mr$. For example, R is itself balanced. Moreover, any left (or right) R-module can be naturally enriched to a balanced R-bimodule.

Next we recall the basics of projective and injective resolutions.

1.2. **Resolutions.** Let R be a DG algebra and M a left R-module. A left R-module P is K-*projective* if $\mathrm{Hom}_R(P,-)$ preserves quasiisomorphisms. A K-*projective resolution* of M is a quasiisomorphism of left R-modules $P \to M$, where P is K-projective. Analogously, a left R-module I is K-*injective* if $\mathrm{Hom}_R(-,I)$ preserves quasiisomorphisms; a K-*injective resolution* of M is a quasiisomorphism of left R-modules $M \to I$ such that I is K-injective.

It is well known that every module has a K-projective and a K-injective resolution, and that they are unique up to quasiisomorphism; for example, see [4]. Thus one can construct the derived functor $-\otimes^{\mathbf{L}}_R -$ of the tensor product functor, as well as the derived functor $\mathbf{R}\mathrm{Hom}_R(-,-)$ of the homomorphism functor. Let L be a right R-module, and let M and N be left R-modules. Set

$$\mathrm{Tor}^R_i(L,M) = \mathrm{H}_i(L \otimes^{\mathbf{L}}_R M) \quad \text{and} \quad \mathrm{Ext}^i_R(M,N) = \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(M,N)).$$

When R is a ring, these correspond with the classically defined objects.

Suppose that X is an R-bimodule. We call a *biprojective resolution* of X a quasiisomorphism of R-bimodules $P \to X$ such that P is K-projective both as a left R-module and as a right R-module. One has also the analogous notion of a *biinjective resolution* of X.

It turns out that for the specific modules that interest us biinjective resolutions always exist, but biprojective resolutions may not. This problem explains our interest in the next result. First, some notation.

Henceforth, $\Bbbk$ denotes a fixed commutative noetherian ring. A *DG algebra over* $\Bbbk$ is a DG algebra R equipped with a morphism of DG algebras $\Bbbk \to R$ with the property that the image of $\Bbbk$ lies in the centre of R. Here $\Bbbk$ is viewed as a DG algebra (necessarily concentrated in degree 0) with trivial differential.

For each complex X of $\Bbbk$-modules, $X^\natural$ denotes the underlying graded $\Bbbk$-module.

1.3. **Proposition.** *Let R be a DG algebra over $\Bbbk$ and let X be an R-bimodule. Then X has a biprojective resolution under either one of the following conditions.*

(a) *R is commutative and X is balanced.*
(b) *$R_i = 0$ for $i \ll 0$ and the $\Bbbk$-module $R^\natural$ is projective.*
(c) *$\Bbbk$ is a field.*

Proof. (a) Let $P \to X$ be a K-projective resolution of the left R-module X. Since R is commutative, P can be endowed with a structure of a balanced R-module, and with this structure, the homomorphism $P \to X$ is in one of R-bimodules. Moreover, P is K-projective also when considered as a right R-module. Thus, P is a biprojective resolution of X.

(b) Since X is a bimodule, it can be viewed as a left module over the enveloping DG algebra $R^e = R \otimes_\Bbbk R^\circ$. Let P be a K-projective resolution of X over R^e; in particular, $P \to X$ is a quasiisomorphism of R-bimodules. Under our hypothesis, R is a K-projective (left) $\Bbbk$-module, so by base change we deduce that R^e is a K-projective left R-module. Thus, the adjunction isomorphism:

$$\operatorname{Hom}_R(P,-) \cong \operatorname{Hom}_R(R^e \otimes_{R^e} P,-) \cong \operatorname{Hom}_{R^e}(P,\operatorname{Hom}_R(R^e,-))$$

yields that P is K-projective on the left.

An analogous argument establishes that P is K-projective also on the right.

(c) This proof in this case is akin to the one in (b). □

The next few paragraphs concern finiteness issues for DG algebras and modules.

1.4. **Finiteness.** Let R be a DG algebra and M a (left or right) R-module.

We say that M is *bounded above* (respectively, *bounded below*) if $\mathrm{H}_i(M) = 0$ for $i \gg 0$ (respectively, for $i \ll 0$); it is *bounded* if it is bounded both above and below. We say that M is *degreewise finite* if for each integer i the $\mathrm{H}_0(R)$-module $\mathrm{H}_i(M)$ is finitely generated; M is *finite* if it is degreewise finite and bounded. In other words, the graded $\mathrm{H}_0(R)$-module $\mathrm{H}(M)$ is finitely generated.

One often encounters the following problem: One begins with a pair of (degreewise) finite modules and would like to know whether their derived tensor product has the same property. This turns out to be true in many situations of interest; this is the content of the next two results.

A DG algebra R is said to be *connective* if $\mathrm{H}_i(R) = 0$ for $i < 0$, and *coconnective* if $\mathrm{H}_i(R) = 0$ for $i > 0$. The following facts are well known; one way to prove them is to use the Eilenberg-Moore spectral sequence that converges, under either sets of hypotheses, from $\operatorname{Tor}^{\mathrm{H}(R)}(\mathrm{H}(L),\mathrm{H}(M))$ to $\operatorname{Tor}^R(L,M)$; see, for example, [5, (1.10)] or [6, (1.3.2)].

1.5. *Let R be DG $\Bbbk$-algebra that is connective, and degreewise finite over $\Bbbk$. Let L be a right R-module and M a left R-module. If L and M are degreewise finite and bounded below, then the same holds for the complex of $\Bbbk$-modules $L \otimes^{\mathbf{L}}_R M$.* □

1.6. *Let k be a field and R a degreewise finite coconnective DG k-algebra such that $\mathrm{H}_0(R) = k$ and $\mathrm{H}_{-1}(R) = 0$. Let L be a right R-module and M a left R-module. If L and M are degreewise finite and bounded above, then the same holds for the complex of k-vector spaces $L \otimes^{\mathbf{L}}_R M$.* □

Next we wish to introduce, following [9], dualizing DG modules. To this end, it is expedient to have on hand the following notion.

1.7. Let R be a DG algebra and X an R-bimodule. We say that a left R-module M is *X-reflexive* if the following biduality morphism is an isomorphism:

$$M \to \mathbf{R}\operatorname{Hom}_{R^\circ}(\mathbf{R}\operatorname{Hom}_R(M,X),X)\ .$$

A right R-module N is *X-reflexive* if it is X-reflexive over R°; in other words, if the biduality morphism $N \to \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_{R^\circ}(N,X),X)$ is an isomorphism.

1.8**. Dualizing modules.** Let R be a DG algebra. An R-bimodule D is *dualizing* (for R) if for any finite left R-module M and finite right R-module N, the following conditions are satisfied.

(1) D has a biprojective resolution and a biinjective resolution;
(2) The R-modules $\mathbf{R}\mathrm{Hom}_R(M,D)$ and $\mathbf{R}\mathrm{Hom}_{R^\circ}(N,D)$ are both finite;
(3) The left R-modules M and $D \otimes^{\mathbf{L}}_R M$ are D-reflexive, as are the right R-modules N and $N \otimes^{\mathbf{L}}_R D$;
(4) R is D-reflexive as left R-module and as a right R-module.

The properties asked of a dualizing module are inspired by the ones enjoyed by dualizing complexes over commutative noetherian rings; confer Hartshorne [11, Chapter V] or P. Roberts [12, Chapter 2]. Over such rings, the following strengthening of (1.8.3), contained in [11, V.2.1], holds; this is useful in the sequel.

1.9**.** *Let C be a balanced dualizing complex for $\Bbbk$. If a complex of $\Bbbk$-modules V is degreewise finite, then V is C-reflexive.* □

For the remainder of this section let R be a DG $\Bbbk$-algebra.

In Section 2 we prove that in many cases one can obtain a dualizing module over R by coinducing one from $\Bbbk$. This calls for a recap on this process.

1.1. **Coinduction.** Let V be a complex of $\Bbbk$-modules.

The complex of $\Bbbk$-modules $\mathbf{R}\mathrm{Hom}_{\Bbbk}(R,V)$ acquires a structure of an R-bimodule as follows: Let I be a K-injective resolution of V. Since (the image of) $\Bbbk$ is central in R, the complex of $\Bbbk$-modules $\mathrm{Hom}_{\Bbbk}(R,I)$ inherits a structure of an R-bimodule from the canonical one on R. This R-bimodule structure does not depend on the choice of the injective resolution: If J is another K-injective resolution of V, then there is a quasiisomorphism $I \to J$; this induces the quasiisomorphism $\mathrm{Hom}_{\Bbbk}(R,I) \to \mathrm{Hom}_{\Bbbk}(R,J)$, which is a morphism of R-bimodules.

There are a couple of simple observations that are worth recording:

(a) If R is commutative, then the R-bimodule $\mathbf{R}\mathrm{Hom}_{\Bbbk}(R,V)$ is balanced.
(b) If R is K-projective when viewed as a complex of $\Bbbk$-modules, then the R-bimodules $\mathbf{R}\mathrm{Hom}_{\Bbbk}(R,V)$ and $\mathrm{Hom}_{\Bbbk}(R,V)$ are quasiisomorphic.

Indeed, the R-bimodule structure on $\mathbf{R}\mathrm{Hom}_{\Bbbk}(R,V)$ is induced by the one on $\mathrm{Hom}_{\Bbbk}(R,I)$, where I is a K-injective resolution of V. Thus, (a) holds because the canonical R-bimodule structure on R is balanced. As to (b): Since R is K-projective over $\Bbbk$, the canonical map $\mathrm{Hom}_{\Bbbk}(R,V) \to \mathrm{Hom}_{\Bbbk}(R,I)$ is a quasiisomorphism. A straightforward calculation verifies that this homomorphism is one of R-bimodules.

1.10**.** Let V be a complex of $\Bbbk$-modules, and M a left R-module. The derived category avatar of the classical Hom-Tensor adjointness yield a quasiisomorphism

$$\mathbf{R}\mathrm{Hom}_R\,(M, \mathbf{R}\mathrm{Hom}_{\Bbbk}\,(R, V)\,) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_{\Bbbk}\,(M, V) \ .$$

Now, $\mathbf{R}\mathrm{Hom}_{\Bbbk}\,(R, V)$ is an R-bimodule, so its *right* R-structure induces a *left* R-structure on $\mathbf{R}\mathrm{Hom}_R\,(M, \mathbf{R}\mathrm{Hom}_{\Bbbk}\,(R, V)\,)$. On the other hand, since the image of $\Bbbk$ is central in R, the left R-module structure of M carries over to $\mathbf{R}\mathrm{Hom}_{\Bbbk}\,(M, V)$. The quasiisomorphism above is compatible with this additional data.

2. Dualizing DG modules

The gist of this section is that many of the naturally occurring DG algebras possess dualizing modules. We begin by describing the main results; their proofs are to be found at the end of the section.

2.1. **Theorem.** *Let A be a commutative noetherian ring and C a dualizing complex for A. Let K be the Koszul complex on a finite set of elements in A. Then the K-bimodule* $\mathrm{Hom}_A\,(K, C)$ *is dualizing.*

2.2. Let $\varphi\colon A' \to A$ be a local homomorphism, so that A' and A are local rings and φ maps the maximal ideal of A' to the maximal ideal of A. Let k denote the residue field of A'.

One can construct a DG A'-algebra resolution $A'[X] \xrightarrow{\simeq} k$ with the property that X is a positively graded set and $A'[X]$ is a free commutative A'-algebra on X. Such a *resolvent* is unique up to quasiisomorphism of DG A'-algebras. The reader may refer to, for instance, [2, (2.1)] for details on these matters.

The DG algebra $A \otimes_{A'} A'[X]$ is called the *DG fibre* of φ; it is immediate from the uniqueness of resolvents that the DG fibre is well defined up to quasiisomorphism of DG algebras. Since $A'[X]$ is a free resolution of k, one has an isomorphism of k-algebras: $\mathrm{H}(A \otimes_{A'} A'[X]) \cong \mathrm{Tor}^{A'}\,(A, k)$.

Theorem. *Let $\varphi\colon A' \to A$ be a local homomorphism of finite flat dimension, and let C be a dualizing complex for A. Let $A[X]$ be the DG fibre of φ. Then the $A[X]$-bimodule* $\mathrm{Hom}_A\,(A[X], C)$ *is dualizing.*

Remark. For commutative DG algebras, Apassov [1, §4] has considered a weaker notion of dualizing modules and has proved that DG fibres possess such dualizing modules; see [1, (4.3)]. That result is contained in the theorem above.

2.3. Let P be a complex over a commutative ring A and set $\mathcal{E} = \mathrm{Hom}_A\,(P, P)$. Composition defines a product structure on $\mathcal{E}$ making it a DG algebra; this is the *endomorphism DG algebra* of P. Since A is commutative, homothety induces a map $A \to \mathcal{E}$; this is a homomorphism of DG algebras with A lying in the centre of $\mathcal{E}$. In other words, $\mathcal{E}$ is an A-algebra.

For the next result, recall that P is said to be *perfect* if $P^\natural$ is bounded and, for each integer i, the A-module P_i is finitely generated and projective.

Theorem. *Let A be a commutative noetherian ring and C a dualizing complex for A. Let P be a perfect complex of A-modules with* $\mathrm{H}(P) \neq 0$ *and $\mathcal{E}$ the endomorphism DG algebra* $\mathrm{Hom}_A(P,P)$. *Then the $\mathcal{E}$-bimodule* $\mathrm{Hom}_A(\mathcal{E},C)$ *is dualizing.*

2.4. Recall our standing assumption that $\Bbbk$ denotes a commutative noetherian ring.

Let G be a topological monoid and let $\mathrm{C}_*(G;\Bbbk)$ denote the singular chain complex of G with coefficients in $\Bbbk$. It is well known that the product on G induces a structure of a DG $\Bbbk$-algebra on $\mathrm{C}_*(G;\Bbbk)$; for example, see [6, pp. 28,88].

Theorem. *Let C be a dualizing complex for $\Bbbk$. If the homology of G with coefficients in $\Bbbk$ is finitely generated, then the* $\mathrm{C}_*(G;\Bbbk)$*-bimodule* $\mathrm{Hom}_\Bbbk(\mathrm{C}_*(G;\Bbbk),C)$ *is dualizing.*

The preceding theorems are deduced from more general results described below.

2.5. **Proposition.** *Let R be a DG $\Bbbk$-algebra that is finite over $\Bbbk$, and C a dualizing complex for $\Bbbk$. Set* $D = \mathbf{R}\mathrm{Hom}_\Bbbk(R,C)$. *Suppose that the following conditions hold.*

(1) *D has a biprojective resolution;*
(2) *For any finite left R-module M (respectively, finite right R-module N), the $\Bbbk$-module $D \otimes^{\mathbf{L}}_R M$ (respectively, $N \otimes^{\mathbf{L}}_R D$) is C-reflexive.*

Then D is a dualizing module for R.

Proof. We verify that D has the properties required of it by (1.8).

Condition (1): The desired biprojective resolution is provided by our hypothesis. As to the biinjective one: Let J be an injective resolution of C over $\Bbbk$. Then, the R-bimodule $I = \mathrm{Hom}_\Bbbk(R,J)$ is K-injective both on the left and the right, as is easily verified using the adjunction isomorphism in (1.10).

A few remarks before we proceed with the verification of (1.8.2) and (1.8.3): By symmetry, it suffices to establish the assertions concerning left R-modules. Also,

(a) For any left R-module L and right R-module N, (1.10) yields that:

$$\mathbf{R}\mathrm{Hom}_R(L,D) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_\Bbbk(L,C) \quad \text{and} \quad \mathbf{R}\mathrm{Hom}_{R^\circ}(N,D) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_\Bbbk(N,C)\ .$$

(b) Since R is finite over $\Bbbk$, a module over R is finite if and only if it is finite when viewed as a complex of $\Bbbk$-modules.

Condition (2): As C is a dualizing complex for $\Bbbk$, condition (1.8) yields that for any finite left R-module M, the R-module $\mathbf{R}\mathrm{Hom}_\Bbbk(M,C)$ is finite as well. It remains to note that, by (a), the latter module is isomorphic to $\mathbf{R}\mathrm{Hom}_R(M,D)$.

Condition (3): We claim that if a left R-module L is C-reflexive when viewed as a complex of $\Bbbk$-modules, then it is D-reflexive. This is justified by the following commutative diagram where θ and γ are the respective biduality morphisms, and α and β are (induced by) the adjunction quasiisomorphisms; see (a) above.

$$\begin{tikzcd} L \arrow[rr, "\theta", "\simeq"'] \arrow[d, "\gamma"'] & & \mathbf{R}\mathrm{Hom}_{\Bbbk}(\mathbf{R}\mathrm{Hom}_{\Bbbk}(L,C),C) \arrow[d, "\simeq"', "\alpha"] \\ \mathbf{R}\mathrm{Hom}_{R^{\circ}}(\mathbf{R}\mathrm{Hom}_{R}(L,D),D) \arrow[rr, "\simeq", "\beta"'] & & \mathbf{R}\mathrm{Hom}_{\Bbbk}(\mathbf{R}\mathrm{Hom}_{R}(L,D),C) \end{tikzcd}$$

Now, if M is a finite left R-module, then, viewed as complexes of $\Bbbk$-modules, both M and $D \otimes^{\mathbf{L}}_{R} M$ are C-reflexive - the former by (1.9) and the latter by assumption - so it follows from the preceding discussion that they are D-reflexive as well.

Condition (4): Since R is finite over $\Bbbk$, this is the special case $M = R$ of the already verified Condition (3).

This completes our proof that D is dualizing for R. □

The preceding proposition allows us to establish the existence of dualizing modules for diverse classes of DG algebras.

2.6. **Proposition.** *Let R be a connective DG $\Bbbk$-algebra that is finite over $\Bbbk$, and C a dualizing complex for $\Bbbk$. If either $R_i = 0$ for $i \ll 0$ and the $\Bbbk$-module $R^{\natural}$ is projective, or R is commutative, then the R-bimodule $\mathbf{R}\mathrm{Hom}_{\Bbbk}(R,C)$ is dualizing.*

Proof. Set $D = \mathbf{R}\mathrm{Hom}_{\Bbbk}(R,C)$. Our hypothesis place us in the situation treated by (2.5). So it suffices to verify that D satisfies conditions (1) and (2) of *loc. cit.*

Condition (1): When R is commutative, D is balanced; see (1.1). Thus, under either sets of hypotheses, (1.3) provides us with a biprojective resolution.

Condition (2): The $\Bbbk$-module R is finite and C is a dualizing complex over $\Bbbk$, so D is finite when viewed as a complex of $\Bbbk$-modules. In particular, D is finite over R. Therefore, R being connective, for any finite left R-module M, the $\Bbbk$-module $\mathrm{H}(D \otimes^{\mathbf{L}}_{R} M)$ is degreewise finite and bounded below; see (1.5). Now (1.9) yields that $D \otimes^{\mathbf{L}}_{R} M$ is C-reflexive, as desired.

An analogous argument settles the assertion concerning right R-modules. □

This completes our preparation for proving the results announced at the beginning of this section.

Proof of Theorem 2.1. The DG A-algebra K is finite, connective, and commutative. Thus, by (2.6), the K-bimodule $\mathbf{R}\mathrm{Hom}_{A}(K,C)$ is dualizing. It remains to note that since $K^{\natural}$ is a bounded complex of projective A-modules, the K-bimodules $\operatorname{Hom}_A(K,C)$ and $\mathbf{R}\operatorname{Hom}_A(K,C)$ are quasiisomorphic; see (1.1). □

Proof of Theorem 2.2. Let $A'[X]$ be a resolvent of k; see the discussion in (2.2). Set $A[X] = A \otimes_{A'} A'[X]$; this is the DG fibre of φ. This is a commutative DG A-algebra with structure map the canonical inclusion $A \to A[X]$. The homology of $A[X]$ is $\operatorname{Tor}^{A'}(A,k)$, so $A[X]$ is connective; it is also finite because the flat dimension of A' over A is finite. Thus, $\mathbf{R}\operatorname{Hom}_A(A[X],C)$ is a dualizing module for $A[X]$, by (2.6). It remains to observe that $\mathbf{R}\operatorname{Hom}_A(A[X],C) \simeq \operatorname{Hom}_A(A[X],C)$ as $A[X]$-bimodules, since the A-module $A[X]$ is free. □

Proof of Theorem 2.3. Let $D = \mathbf{R}\operatorname{Hom}_A(\mathcal{E},C)$. Since $\mathcal{E}$ is a bounded complex of projective A-modules, the R-bimodule $\operatorname{Hom}_A(\mathcal{E},C)$ is quasiisomorphic to D, by (1.1). So it suffices to prove that D is dualizing for $\mathcal{E}$. To this end, we verify that the DG A-algebra $\mathcal{E}$ meets the criteria set out in (2.5).

To begin with, note that $\mathcal{E}$ is finite over A; this is by construction.

Condition (1): For each i, the A-module $\mathcal{E}_i$ is projective and $\mathcal{E}_i = 0$ when $i \ll 0$, so (1.3) yields that D has a biprojective resolution.

Condition (2): In the following sequence of isomorphisms of right $\mathcal{E}$-modules, the one on the left holds because $\mathcal{E}$ is a bounded complex of projective A-modules, the one in the middle is by [10], and the last one is the canonical one.

$$\mathbf{R}\operatorname{Hom}_A(\mathcal{E},C) \simeq C \otimes^{\mathbf{L}}_A \mathbf{R}\operatorname{Hom}_A(\mathcal{E},A) \simeq C \otimes^{\mathbf{L}}_A \Sigma^n \mathcal{E} = \Sigma^n C \otimes^{\mathbf{L}}_A \mathcal{E}\,.$$

Let M be a finite left $\mathcal{E}$-module. The quasiisomorphism above implies that the left module $D \otimes^{\mathbf{L}}_{\mathcal{E}} M$ is quasiisomorphic to $\left(\Sigma^n C \otimes^{\mathbf{L}}_A \mathcal{E}\right) \otimes^{\mathbf{L}}_{\mathcal{E}} M$, that is to say, to $\Sigma^n C \otimes^{\mathbf{L}}_A M$. By (1.5), this last module is degreewise finite and bounded below, hence (1.9) yields that it is C-reflexive, as desired.

An analogous argument may be used to verify that $N \otimes^{\mathbf{L}}_{\mathcal{E}} D$ is C-reflexive for any right R-module N. □

Proof of Theorem 2.4. Set $R = \mathrm{C}_*(G;\Bbbk)$. By construction, R is concentrated in non-negative degrees, so it is connective. Moreover, the homology of G is the homology of the chain complex of G so the hypothesis translates to: The $\Bbbk$-module $\mathrm{H}(R)$ is finitely generated, that is to say R is finite over $\Bbbk$. Finally, since the $\Bbbk$-module $R^\natural$ is projective, (2.6) yields that $\mathbf{R}\operatorname{Hom}_{\Bbbk}(R,C)$ is dualizing for R; it remains to invoke (1.1) to obtain that $\operatorname{Hom}_{\Bbbk}(R,C)$ is quasiisomorphic to $\mathbf{R}\operatorname{Hom}_{\Bbbk}(R,C)$, and hence, dualizing for R. □

3. Dualizing modules over local DG algebras

This section focuses on properties of dualizing modules over commutative local DG algebras. First, the relevant definition.

A commutative DG algebra R is said to be *local* if

(1) R is concentrated in non-negative degrees: $R = \{R_i\}_{i \geqslant 0}$;
(2) R_0 is noetherian;

(3) $\mathrm{H}_0(R)$ is local and $\mathrm{H}(R)$ is degreewise finite.

The *residue field* of such a DG algebra R is the residue field of the local ring $\mathrm{H}_0(R)$. The residue field k has the structure of an R-module via the canonical surjections: $R \to \mathrm{H}_0(R) \to k$.

Remark. Typically, when dealing with commutative *rings* it is tacitly assumed that dualizing complexes are balanced. And indeed all the result in this section are proved under such a hypothesis. However, we have opted not to incorporate this as part of our definition of dualizing modules (over commutative DG algebras), see 1.8. This is keeping with line with the point of view adopted here that commutative DG algebras are specializations of general DG algebras.

The following existence theorem for dualizing modules is a DG algebra version of [11, (V.3.4)]; it differs from Theorem II stated in the introduction only in the form. It can be used, for example, to give quick proofs of theorems 2.1 and 2.2.

3.1. **Theorem.** *Let R be a finite commutative local DG algebra with residue field k, and let D be a balanced R-module. The following conditions are equivalent.*

(1) *D is dualizing.*
(2) *D is finite and* $\operatorname{rank}_k \operatorname{Ext}_R(k, D) = 1$.

The theorem in proved in (3.6).

When R has a dualizing module, the preceding result is contained in the next one. It is a generalization of the well known result that (balanced) dualizing complexes over a commutative local ring are quasiisomorphic up to suspension [11, Chapter 5. §3]. It contains also Theorem III from the introduction.

3.2. **Theorem.** *Let R be a finite commutative local DG algebra, and let D, E be balanced R-modules. If D is dualizing, then the following conditions are equivalent.*

(1) *$E \simeq \Sigma^m D^r$ for some integers m and r.*
(2) *E is finite,* $\operatorname{Ext}_R(k, E)$ *is concentrated in a single degree, and it has rank r.*

In particular, if E is dualizing, then it is quasiisomorphic to D, up to suspension.

The proof is given in (3.9).

The following result is a first step towards theorems (3.1) and (3.2); note that it does not require that R be finite. Numerically, it can be interpreted as stating that, up to a shift, the Bass (respectively, Betti) numbers of D equal the Betti (respectively, Bass) numbers of R. In fact, one can extend this result such that it deals with any finite R-module M and its dual $\mathbf{R}\mathrm{Hom}_R(M, D)$. The purpose on hand does not call for that generality.

3.3. **Proposition.** *Let R be a commutative local DG algebra with residue field k, and let D be a balanced dualizing R-module. Then there is an integer n such that*

(a) $\mathbf{R}\mathrm{Hom}_R(k, D) \simeq \Sigma^n k.$
(b) $\mathbf{R}\mathrm{Hom}_R(D, k) \simeq \Sigma^{-n}\mathbf{R}\mathrm{Hom}_R(k, R)$.

Proof. Since R is commutative and D is balanced, the action of R on $\mathbf{R}\mathrm{Hom}_R(k, D)$ - which is apriori induced from D - coincides with the action induced from k. Thus, as an R-module, $\mathbf{R}\mathrm{Hom}_R(k, D)$ is quasiisomorphic to a complex of k-vector spaces with trivial differential:

$$\mathbf{R}\mathrm{Hom}_R(k, D) \simeq \bigoplus_{i\in\mathbb{Z}} \Sigma^i k^{b_i}.$$

Since k is a finite R-module, and D is dualizing, $\mathbf{R}\mathrm{Hom}_R(k, D)$ is finite. Therefore, $b_i = 0$ for all but a finitely many i, so that

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(k, D), D) &\simeq \bigoplus_{i\in\mathbb{Z}} \Sigma^{-i}\mathbf{R}\mathrm{Hom}_R(k, D)^{b_i}\\ &\simeq \bigoplus_{i,j\in\mathbb{Z}} \Sigma^{j-i} k^{b_i b_j}.\end{aligned}$$

The biduality morphism $k \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(k, D), D)$ is a quasiisomorphism since k is finite. Thus, the quasiisomorphism above entails: There exists an integer n such that $b_n = 1$, whilst $b_i = 0$ for $i \neq n$. This is the result we seek.

(b) In the following sequence of quasiisomorphisms, the first is given by (a), the third holds because R is commutative, while the fourth is by the definition of dualizing modules.

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(D, k) &\simeq \mathbf{R}\mathrm{Hom}_R\big(D, \Sigma^{-n}\mathbf{R}\mathrm{Hom}_R(k, D)\big)\\ &\simeq \Sigma^{-n}\mathbf{R}\mathrm{Hom}_R(D, \mathbf{R}\mathrm{Hom}_R(k, D))\\ &\simeq \Sigma^{-n}\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(D, D))\\ &\simeq \Sigma^{-n}\mathbf{R}\mathrm{Hom}_R(k, R).\end{aligned}$$

This is the desired quasiisomorphism. □

Our proof of Theorem 3.1 is built on the following special case. It is well known when, in addition, C is bounded above. Although this is all that is required in the sequel, it might be worthwhile to record the more general statement.

3.4. **Proposition.** *Let $(S, \mathfrak{m}, k)$ be a local ring and C a degreewise finite complex of balanced S-modules. If $\mathrm{rank}_k \mathrm{Ext}_S(k, C) = 1$, then C is dualizing.*

Proof. Since $\mathrm{rank}_k \mathrm{Ext}_S(k, C)$ is finite, $\sup\{i \mid \mathrm{Ext}^i_S(k, C) \neq 0\}$ is finite. This last number is, by definition, the depth of the complex C - see [8, (2.3)].

Thus, since C is degreewise finite, [*loc. cit.*, (2.5)] yields that C is bounded above. Now apply, for example, [11, (V.3.4)]. □

Here is the last piece of machinery required in the proof of Theorem 3.1.

3.5. **Truncations.** Let R be a commutative local DG algebra and M a finite R-module with $\mathrm{H}(M) \neq 0$. Set $s = \sup\{i \mid \mathrm{H}_i(M) \neq 0\}$, and

$$\tau(M)\colon\ \cdots \to M_{s+2} \xrightarrow{\partial_{s+2}} M_{s+1} \xrightarrow{\partial_{s+1}} \mathrm{Ker}(\partial_s) \to 0\,.$$

Apriori, the inclusion $\iota\colon \tau(M) \subseteq M$ is only one of complexes of abelian groups. However, a direct calculation establishes that $\tau(M)$ is stable under multiplication by elements in R. Therefore, it inherits an R-module structure from M, and ι is a homomorphism of R-modules. It extends to a triangle

$$\tau(M) \xrightarrow{\iota} M \to C(\iota) \to \Sigma\tau(M) \tag{†}$$

in the derived category of R-modules.

Since $\mathrm{H}_s(M) = \mathrm{Ker}(\partial_s)/\partial_{s+1}(M_{s+1})$, there is a surjection $\pi\colon \tau(M) \to \Sigma^s\mathrm{H}_s(M)$; it is a degree 0 homomorphism of R-modules, with R acting on $\mathrm{H}_s(M)$ via $\mathrm{H}_0(R)$.

We require the following observations. Recall that the *amplitude* of M, denoted $\mathrm{amp}(M)$, is the integer $\sup\{i \mid \mathrm{H}_i(M) \neq 0\} - \inf\{i \mid \mathrm{H}_i(M) \neq 0\}$.

(a) π is a quasiisomorphism. Moreover, if $\mathrm{amp}(M) = 0$, then $\iota\colon \tau(M) \to M$ is also a quasiisomorphism, so $M \simeq \Sigma^s\mathrm{H}_s(M)$ as R-modules.
(b) $\mathrm{amp}(\tau(M)) = 0$ and $\mathrm{amp}(C(\iota)) \leq \mathrm{amp}(M) - 1$.
(c) If M is (degreewise) finite, then so is $C(\iota)$.

Furthermore, for any R-modules D and X one has:

(d) If both $\mathbf{R}\mathrm{Hom}_R(\tau(M), D)$ and $\mathbf{R}\mathrm{Hom}_R(C(\iota), D)$ are (degreewise) finite, then so is $\mathbf{R}\mathrm{Hom}_R(M, D)$.
(e) If both $X \otimes^{\mathbf{L}}_R \tau(M)$ and $X \otimes^{\mathbf{L}}_R C(\iota)$ are D-reflexive, then so is $X \otimes^{\mathbf{L}}_R M$.

Indeed, by construction, π is a quasiisomorphism, $\mathrm{H}_s(\iota)\colon \mathrm{H}_s(\tau(M)) \to \mathrm{H}_s(M)$ is bijective and $\mathrm{H}_n(\tau(M)) = 0$ for $n \neq s$. This explains (a), and the equality $\mathrm{amp}(\tau(M)) = 0$ in (b). From (a) and the homology long exact sequence of $\mathrm{H}_0(R)$-modules engendered by (†) above

$$\cdots \to \mathrm{H}_n(\tau(M)) \to \mathrm{H}_n(M) \to \mathrm{H}_n(C(\iota)) \to \mathrm{H}_{n-1}(\tau(M)) \to \cdots\,,$$

one obtains that $\mathrm{H}_n(C(\iota)) \cong \mathrm{H}_n(M)$ for $n \leq s-1$ and 0 otherwise. This yields $\mathrm{amp}(C(\iota)) \leq \mathrm{amp}(M) - 1$, and also (c). As to (d), it is immediate from the homology exact sequence associated to the triangle

$$\mathbf{R}\mathrm{Hom}_R(C(\iota), D) \to \mathbf{R}\mathrm{Hom}_R(M, D) \to \mathbf{R}\mathrm{Hom}_R(\tau(M), D)$$

obtained by applying $\mathbf{R}\mathrm{Hom}_R(-, D)$ to (†). Finally, denote F the biduality functor $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(-, D), D)$ associated to D. The triangle (†) and the naturality of the biduality morphism yields the ladder:

$$\begin{tikzcd}
X \otimes^{\mathbf{L}}_R \tau(M) \arrow[r] \arrow[d, "\simeq"] & X \otimes^{\mathbf{L}}_R M \arrow[r] \arrow[d] & X \otimes^{\mathbf{L}}_R C(\iota) \arrow[d, "\simeq"] \\
F(X \otimes^{\mathbf{L}}_R \tau(M)) \arrow[r] & F(X \otimes^{\mathbf{L}}_R M) \arrow[r] & F(X \otimes^{\mathbf{L}}_R C(\iota))
\end{tikzcd}$$

where the quasiisomorphism are due to $X \otimes^{\mathbf{L}}_R \tau(M)$ and $X \otimes^{\mathbf{L}}_R C(\iota)$ being D-reflexive. The associated homology exact ladder and the five lemma yield that the middle vertical arrow is a quasiisomorphism as well; that is to say, $X \otimes^{\mathbf{L}}_R M$ is D-reflexive.

3.6. *Proof of* Theorem 3.1.

(1) $\implies$ (2). Since R is finite and D is dualizing, $\mathbf{R}\mathrm{Hom}_R(R, D)$ is finite; that is to say, D is finite. The remaining assertion is contained in Proposition 3.3.a.

(2) $\implies$ (1) We verify that D satisfies the conditions set out in (1.8). The basic strategy is to reduce everything to questions concerning modules over the ring $\mathrm{H}_0(R)$. To facilitate its implementation, let $S = \mathrm{H}_0(R)$ and $C = \mathbf{R}\mathrm{Hom}_R(S, D)$. Note that C is endowed with a natural structure of a balanced S-module; see (1.1).

Claim. C is dualizing for S.

Indeed, for any S-module M, viewed as an R-module via the surjection $R \to S$, adjointness yields the quasiisomorphism of R-modules

$$(*) \qquad \mathbf{R}\mathrm{Hom}_S(M, C) \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(M, D)\,.$$

In particular, $\mathbf{R}\mathrm{Hom}_S(k, C) \simeq \mathbf{R}\mathrm{Hom}_R(k, D)$, so that $\mathrm{rank}_k \mathrm{Ext}_S(k, C) = 1$. Moreover, since R is connective, and both S and D are finite, C is degreewise finite and bounded above. It remains to invoke Proposition 3.4.

Note that since D is balanced one need worry only about left R-modules. Also, for any R-module M the left and right structures on $M \otimes^{\mathbf{L}}_R D$ and $\mathbf{R}\mathrm{Hom}_R(M, D)$ coincide. These remarks will be used without further ado.

Condition (1): Since R is commutative and D is balanced, any K-projective, respectively, K-injective, resolution can be enriched to a biprojective, respectively, biinjective, one.

Condition (2): If M is a finite S-module, then so is $\mathbf{R}\mathrm{Hom}_S(M, C)$, since C is dualizing for S. Thus, $\mathbf{R}\mathrm{Hom}_R(M, D)$ is finite, by (*). This entails the finiteness of $\mathbf{R}\mathrm{Hom}_R(M, D)$ also when M is a finite R-module.

Indeed, the required finiteness is tautological if $\mathrm{H}(M) = 0$, so assume that $\mathrm{H}(M)$ is nontrivial. Now induce on the amplitude of M. Suppose that $\mathrm{amp}\, M = 1$. Then $M \simeq \Sigma^s \mathrm{H}_s(M)$ for some integer s, by (3.5.a), so by passing to $\mathrm{H}_s(M)$ one may assume that M is a finite S-module; this case has been dealt with.

Let n be a positive integer such that the desired finiteness holds for any finite R-module with amplitude at most n. Let M be a finite R-module with $\mathrm{amp}(M) = n+1$. Consider the R-modules $\tau(M)$ and $C(\iota)$ constructed as in (3.5). Both these have amplitude $\leq n$, by (3.5.b), and are finite, by (3.5.c). Thus, the induction hypothesis yields that $\mathbf{R}\mathrm{Hom}_R(\tau(M), D)$ and $\mathbf{R}\mathrm{Hom}_R(C(\iota), D)$ are both finite. Now (3.5.d) yields that $\mathbf{R}\mathrm{Hom}_R(M, D)$ is finite as well.

Thus, Condition (2) is verified.

Condition (3): We check: If an R-module M is finite, then $D \otimes^{\mathbf{L}}_R M$ is D-reflexive.

To begin with, suppose that M is an S*-module*, with R on it via S. Then the action of R on $D \otimes^{\mathbf{L}}_R M$ induced by D coincides with the one induced by M, so $D \otimes^{\mathbf{L}}_R M$ is quasiisomorphic, to a (DG) S-module U, as an R-module. Although U may not be finite, it is degreewise finite and bounded below; see (1.5).

The biduality morphism $\theta\colon U \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(U, D), D)$ fits in the following commutative square. In it γ is the biduality map associated to U when viewed as an S-module; by (1.9), it is a quasiisomorphism since U is degreewise finite and C is dualizing. Adjunction - see (*) - induces the remaining quasiisomorphisms.

$$\begin{tikzcd}
U \arrow[r, "\theta"] \arrow[d, "\gamma"', "\simeq"] & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(U, D), D) \arrow[d, "\simeq"] \\
\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_S(U, C), C) \arrow[r, "\simeq"] & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(U, C), D)
\end{tikzcd}$$

So θ is a quasiisomorphism; that is to say, U, and hence $D \otimes^{\mathbf{L}}_R M$, is D-reflexive.

Now an induction argument akin to the one used in verifying Condition (2) settles the issue for any finite R-module M. Here is a sketch: If $\mathrm{amp}(M) = 0$, then, by passing to a quasiisomorphic R-module, one may assume that M is an S-module; this case has already been settled. If $\mathrm{amp}(M) \geq 2$, then the R-modules $\tau(M)$ and $C(\iota)$ from (3.5.c) are both finite, and have amplitudes less than $\mathrm{amp}(M)$. So the induction hypothesis ensures that both $D \otimes^{\mathbf{L}}_R \tau(M)$ and $D \otimes^{\mathbf{L}}_R C(\iota)$ are D-reflexive. Therefore, (3.5.e) - with $X = D$ - yields that $D \otimes^{\mathbf{L}}_R M$ is also D-reflexive.

A similar argument establishes that any finite R-module is D-reflexive.

Condition (4): This is covered by Condition (3) as R is finite.

Thus D is dualizing. This completes the proof of the theorem. □

Now to prepare for the proof of Theorem 3.2.

3.7. Let R be a commutative local DG algebra and M an R-module. If M is degreewise finite and bounded below, then one can construct a resolution F of M such that there is an isomorphism of $R^\natural$-modules

$$F^\natural \cong \bigoplus_{i\in\mathbb{Z}} \Sigma^i \left(R^\natural\right)^{b_i} \qquad \text{with} \quad b_i = 0 \quad \text{for} \quad i \ll 0\,,$$

and the complex $F \otimes_R k$ has trivial differential. Such an F is said to be a *minimal semifree resolution* of M; see [4, §10], [6, (A.3)].

Here is a simple application of the preceding construction.

3.8. **Lemma.** *Let R be a commutative local DG algebra with residue field k, and let M be an R-module that is degreewise finite and bounded below. If there integers n and b such that* $\operatorname{Ext}_R(M,k) \cong \Sigma^n k^b$*, then* $M \simeq \Sigma^{-n} R^b$.

Proof. Let F be a minimal semifree resolution of M given by (3.7). Then the complex of k-vector spaces $\operatorname{Hom}_R(F,k)$, being isomorphic to $\operatorname{Hom}_k(F \otimes_R k, k)$, has trivial differential. The isomorphism $\operatorname{Ext}_R(M,k) \cong \Sigma^n k^b$ translates to the isomorphism: $\operatorname{Hom}_R(F,k) \cong \Sigma^n k^b$. Therefore, $b_i = 0$ for $i \neq -n$ whilst $b_{-n} = b$. In other words, $F \cong \Sigma^{-n} R^b$; thus $M \simeq \Sigma^{-n} R^b$. □

3.9. *Proof of* Theorem 3.2. (1) $\implies$ (2): Since R is finite and D is dualizing, $\mathbf{R}\operatorname{Hom}_R(R,D)$ is finite; thus D is finite. Therefore, E is also finite. Moreover

$$\mathbf{R}\operatorname{Hom}_R(k,E) \simeq \mathbf{R}\operatorname{Hom}_R(k,\Sigma^m D^r) \simeq \Sigma^m \mathbf{R}\operatorname{Hom}_R(k,D)^r \simeq \Sigma^{m+n} k^r ,$$

while the last quasiisomorphism is due to Proposition 3.3.a. Taking homology yields the desired conclusion.

(2) $\implies$ (1): Since E is finite and D is dualizing, $\mathbf{R}\operatorname{Hom}_R(E,D)$ is finite. Moreover, the following biduality morphism is a quasiisomorphism.

$$(\dagger) \qquad E \xrightarrow{\cong} \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(E,D),D) .$$

This engenders the first of the following sequence of quasiisomorphisms; the second is due to the commutativity of R, whilst the third is by (3.3).

$$\begin{aligned}
\mathbf{R}\operatorname{Hom}_R(k,E) &\simeq \mathbf{R}\operatorname{Hom}_R(k,\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(E,D),D)) \\
&\simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(E,D),\mathbf{R}\operatorname{Hom}_R(k,D)) \\
&\simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(E,D),\Sigma^n k) \\
&\simeq \Sigma^n \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(E,D),k) .
\end{aligned}$$

The hypothesis is that $\mathrm{H}_l(\mathbf{R}\operatorname{Hom}_R(k,E)) \cong k^r$ for some integers l,r. This is tantamount to: $\mathbf{R}\operatorname{Hom}_R(k,E) \simeq \Sigma^l k^r$; see (3.5.a). Thus, the quasiisomorphism above yields that $\mathbf{R}\operatorname{Hom}_R(\mathbf{R}\operatorname{Hom}_R(E,D),k) \simeq \Sigma^{l-n} k^r$. Now Lemma (3.8) implies that $\mathbf{R}\operatorname{Hom}_R(E,D) \simeq \Sigma^{n-l} R^r$. Feeding this into equation (†) yields:

$$E \simeq \mathbf{R}\operatorname{Hom}_R\left(\Sigma^{n-l} R^r, D\right) \simeq \Sigma^{l-n} D^r .$$

This completes the proof that (2) $\implies$ (1).

Finally, if E is dualizing, then $\operatorname{rank}_k \operatorname{Ext}_R(k,E) = 1$, by Proposition 3.3.a, so the already established implication (2) $\implies$ (1) yields that $E \simeq \Sigma^m D$, as desired. □

In attempting to extend Theorem 3.2 to dualizing modules that are not balanced, one has to contend with the following phenomenon.

Example. Let R be a commutative local ring and $\sigma\colon R \to R$ a nontrivial ring automorphism. Let R^σ denote the R-bimodule on the set underlying R with multiplication defined by $r \cdot x = rx$ and $x \cdot r = x\sigma(r)$, for $r \in R$ and $x \in R^\sigma$. Then R^σ and R are not quasiisomorphic as bimodules.

Indeed, suppose there is such a quasiisomorphism. This induces a bimodule isomorphism $\varphi\colon R^\sigma \to R$. Hence, for each $r \in R$ one has

$$\varphi(r - \sigma(r)) = \varphi(r \cdot 1 - 1 \cdot r) = r\varphi(1) - \varphi(1)r = 0\,.$$

In particular, $r = \sigma(r)$; this contradicts the non-triviality of σ.

On the other hand, if R is Gorenstein, then both R and R^σ are dualizing; the first by definition, and the second by a direct verification.

We hope to investigate this matter elsewhere.

4. Gorenstein DG algebras

In this short section we apply the results in Section 3 to the study of Gorenstein DG algebras, as introduced in [10]. To begin with, a definition.

4.1. A DG algebra R is *Gorenstein* if R is dualizing, in sense of (1.8). Since R is biprojective and R-reflexive, it follows from the definition that R is Gorenstein if and only if for each finite left R-module M and finite right R-module N:

(1) R has a biinjective resolution;
(2) The R-modules $\mathbf{R}\mathrm{Hom}_R(M,R)$ and $\mathbf{R}\mathrm{Hom}_{R^\circ}(N,R)$ are both finite;
(3) M and N are R-reflexive.

4.2. *Remark.* Let R be a commutative local DG algebra with residue field k, as defined in (3.3). If R is Gorenstein, then $\mathrm{Ext}_R(k,R) \cong \Sigma^n k$ for some integer n; this is by (3.3.a).

In [3] Avramov and Foxby define a commutative local DG algebra R to be 'Gorenstein' if it is *finite* and the k-vector space $\mathrm{Ext}_R(k,R)$ has rank 1. Thus, among *finite* commutative local DG algebras, the Gorenstein class as identified by [10] is potentially smaller than that identified by [3]. In fact, these classes coincide; this is immediate from Theorem 3.1. We record this fact in the following result; it contains Theorem I from the introduction.

4.3. **Theorem.** *Let R be a finite commutative local DG algebra with residue field k. The following conditions are equivalent.*

(1) *R is Gorenstein;*
(2) $\mathrm{rank}_k\,\mathrm{Ext}_R(k,R) = 1$. □

As a corollary, we obtain the following result which completes [10, (2.11)].

4.4. Let Q be a commutative local DG algebra with residue field k, and let $\varphi\colon Q \to T$ be a homomorphism of DG algebras such that $\varphi(Q)$ lies in the centre of T and $T \otimes_Q k \neq 0$. Assume that the following conditions hold:

(1) T has a biinjective resolution over itself;
(2) Viewed as an Q-module, T admits a K-projective resolution $P \xrightarrow{\simeq} T$ such that $P \otimes_Q k$ has trivial differential.

Theorem. *Suppose φ is finite and Gorenstein. Then, Q is Gorenstein if and only if T is Gorenstein.*

Proof. Suppose Q is Gorenstein. Since Q is commutative, it has a biinjective resolution; this follows from an argument analogous to that of 1.3.a., and T has a biinjective resolution, by hypothesis. Thus [10, (2.6)] yields that T is Gorenstein.

If T is Gorenstein, then according to [10, (2.11)], the k-vector space $\operatorname{Ext}_Q(k, Q)$ is one dimensional. It remains to invoke Theorem 4.3. □

5. Cochain complexes

The results in this section are geared towards application in topology. For this reason, all graded objects that appear will be graded *cohomologically.* For instance, any complex M is assumed to be of the form $\{M^i\}_{i\in\mathbb{Z}}$ with differential that increases degrees: $\partial^i\colon M^i \to M^{i+1}$.

For the rest of this section, let k be a field. The following definition is motivated by the example of cochain complexes of topological spaces; cf. (5.5).

5.1. A DG k-algebra R is said to be *cochain* if

(1) R is concentrated in non-negative degrees: $R = \{R^i\}_{i\geqslant 0}$;
(2) $R^0 = k$ and $\mathrm{H}^1(R) = 0$;
(3) the graded k-vector space $\mathrm{H}(R)$ is degreewise finite.

Given such a DG algebra R, we call k its *residue field.* It has the structure of an R-module via the canonical surjection $R \to k$.

Cochain DG algebras are cohomological versions of local DG algebras introduced in Section 3. With one major difference: they are not assumed to be commutative.

The following result ensures that cochain DG algebras posses dualizing modules. Its proof is akin to that of Proposition 2.6, except that, in verifying Condition 2 of (2.5) one invokes (1.6) instead of (1.5).

5.2. **Proposition.** *Let R be a finite cochain DG k-algebra. Then the DG R-bimodule* $\operatorname{Hom}_k(R, k)$ *is dualizing.* □

The remaining results concern cochain DG k-algebras that are Gorenstein, that is to say, those that are dualizing; see (4.1). To begin with, one has the following counterpart of implication (2) $\implies$ (1) of Theorem 4.3.

5.3. **Theorem.** *Let R be a finite cochain DG k-algebra such that* $\mathrm{H}(R)$ *is commutative. If* $\operatorname{rank}_k \operatorname{Ext}_R(k, R) = 1$*, then R is Gorenstein.*

Proof. Set $D = \operatorname{Hom}_k(R,k)$; by (5.2), this R-module is dualizing.

Claim. It suffices to prove that D is quasiisomorphic to a suspension of R as a left R-module and also as a right R-module.

Indeed, assume that D has this property. We verify conditions set out in (4.1).

Condition (1): This follows from arguments analogous to the proof of (1.3.c).

By symmetry, it is enough to check the remaining conditions for left R-modules.

Condition (2): Say $D \simeq \Sigma^n R$ as left R-modules, for some integer n. Then, for any finite left R-module M, the k-modules $\mathbf{R}\operatorname{Hom}_R(M,D)$ and $\Sigma^n \mathbf{R}\operatorname{Hom}_R(M,R)$ are quasiisomorphic. Since D is dualizing, this yields that the latter module is finite over k, and hence over R, because $\mathrm{H}^0(R) = R^0 = k$.

Condition (3): Let M be a finite left R-module. The quasiisomorphism in the diagram below is by adjunction, while evaluation yields the first homomorphism.

$$\mathbf{R}\operatorname{Hom}_R(M,R) \xrightarrow{\varepsilon} \mathbf{R}\operatorname{Hom}_R\left(D \otimes^{\mathbf{L}}_R M, D\right) \xrightarrow{\simeq} \mathbf{R}\operatorname{Hom}_R(M, \mathbf{R}\operatorname{Hom}_R(D,D))\ .$$

The composed homomorphism coincides with the one induced by the homothety $R \to \mathbf{R}\operatorname{Hom}_R(D,D)$. Since this last map is a quasiisomorphism one obtains that ε is also one such. Note that ε is compatible with the right R-module structures. This results in the second of the following quasiisomorphisms; the first one is the biduality morphism, and hence also a quasiisomorphism.

$$\begin{aligned} D \otimes^{\mathbf{L}}_R M &\xrightarrow{\simeq} \mathbf{R}\operatorname{Hom}_{R^\circ}\left(\mathbf{R}\operatorname{Hom}_R\left(D \otimes^{\mathbf{L}}_R M, D\right), D\right) \\ &\xrightarrow{\simeq} \mathbf{R}\operatorname{Hom}_{R^\circ}\left(\mathbf{R}\operatorname{Hom}_R(M,R), D\right)\ . \end{aligned}$$

Therefore, the composed map $\gamma\colon D\otimes^{\mathbf{L}}_R M \to \mathbf{R}\operatorname{Hom}_{R^\circ}(\mathbf{R}\operatorname{Hom}_R(M,R), D)$ is a quasiisomorphism. Note that γ is the canonical evaluation homomorphism. Since D and R are quasiisomorphic as right R-modules and the left R-module structure of D is not involved in the definition of γ, we deduce that the corresponding evaluation morphism $M = R \otimes_R M \to \mathbf{R}\operatorname{Hom}_{R^\circ}(\mathbf{R}\operatorname{Hom}_R(M,R), R)$, which is the biduality morphism, is a quasiisomorphism. Thus, Condition (3) is satisfied.

This completes the justification of our claim.

It remains to prove that D and R are quasiisomorphic, up to suspension, as left R-modules and also as right R-modules. To this end, note that since k is a field, $\mathrm{H}(D) = \operatorname{Hom}_k(\mathrm{H}(R), k)$. By [6, (3.6)], our hypothesis entails that $\mathrm{H}(R)$ satisfies Poincaré duality, that is to say, the $\mathrm{H}(R)$-module $\mathrm{H}(D)$ is isomorphic to $\Sigma^n \mathrm{H}(R)$, for some integer n. Observe that, since $\mathrm{H}(R)$ is commutative, $\mathrm{H}(D)$ is balanced and the last isomorphism respects this structure.

The Eilenberg-Moore spectral sequence that converges from $\mathrm{Tor}^{\mathrm{H}(R)}(k, \mathrm{H}(D))$ to $\mathrm{Tor}^{R}(k, D)$ - see [6, (1.3.2)] - collapses and the k-vector space $\mathrm{Tor}^{R}(k, D)$ is one dimensional. This entails $D \simeq \Sigma^n R$-module as left R-modules; this follows by an argument akin to that in the proof of Proposition 3.8.

Similarly, one deduces that $D \simeq \Sigma^n R$ also as right R-modules.

This completes the proof of the theorem. □

Here is a partial converse to the preceding theorem; it is the cohomological version of (the special case $D = R$) of (3.3.a), and can be proved in the same way.

5.4. **Theorem.** *Let R be a cochain DG k-algebra. Suppose that R is commutative. If R is Gorenstein, then* $\mathrm{rank}_k \mathrm{Ext}_R(k, R) = 1$. □

Given Theorem 5.3, one is tempted to raise the following

Question. In the result above, can one weaken the hypothesis 'R is commutative' to '$\mathrm{H}(R)$ is commutative'?

The last result is a topological avatars of the ones above.

5.5. Let X be a topological space. The complex of cochains on X with coefficients in k, which is denoted $\mathrm{C}^*(X; k)$, is a DG k-algebra, with product defined by the Alexander-Whitney map; see [7, §5]. Although this product is not itself commutative, it is *homotopy commutative*, so that the cohomology of $\mathrm{C}^*(X; k)$ is a commutative k-algebra. It is denoted $\mathrm{H}^*(X; k)$.

According to Félix, Halperin, and Thomas a topological space X is Gorenstein at k if the k-vector space $\mathrm{Ext}_{\mathrm{C}^*(X;k)}(k, \mathrm{C}^*(X; k))$ is one dimensional.

5.6. **Theorem.** *Let X be a simply connected topological space such that the k-vector space $\mathrm{H}^*(X; k)$ is finite dimensional. If X is Gorenstein at k, then the DG algebra $\mathrm{C}^*(X; k)$ is Gorenstein. The converse holds if the characteristic of k is 0.*

Proof. Since $\mathrm{H}(\mathrm{C}^*(X; k))$ is the cohomology of X, our hypotheses translate to: $\mathrm{C}^*(X; k)$ is a cochain DG k-algebra. Thus, by Theorem 5.3, if X is Gorenstein at k, then $\mathrm{C}^*(X; k)$ is Gorenstein.

Suppose that the characteristic of k is 0. Then $\mathrm{C}^*(X; k)$ is quasiisomorphic to a *commutative* DG k-algebra R; see [6, §10]. It immediate from the definition that $\mathrm{C}^*(X; k)$ and R are Gorenstein simultaneously; moreover, $\mathrm{Ext}_{\mathrm{C}^*(X;k)}(k, \mathrm{C}^*(X; k))$ and $\mathrm{Ext}_R(k, R)$ are isomorphic. The desired result is thus contained in (5.4). □

References

[1] D. Apassov, *Dualizing modules over differential graded rings*, pp. 41–47 in "Complexes and Differential Graded Modules", ph.d. thesis, Lund University, 1999.

[2] L. L. Avramov, *Infinite free resolutions*, in: Six lectures in commutative algebra (Bellaterra, 1996), Progress in Math. **166**, Birkhäuser, Boston, 1998; pp. 1–118.

[3] L. L. Avramov, H.-B. Foxby, *Locally Gorenstein homomorphisms*, Amer. J. Math. **114** (1992), 1007–1047.

[4] L. L. Avramov, H.-B. Foxby, S. Halperin, Differential graded homological algebra, in preparation.

[5] L. L. Avramov, S. Halperin, *Through the looking glass: a dictionary between rational homotopy theory and local algebra*, pp. 1–27 in "Algebra, algebraic topology and their interactions" (proceedings of the conference in Stockholm, 1983), Lecture Notes in Math., Vol. 1183, Springer, Berlin, 1986.

[6] Y. Félix, S. Halperin, J.-C. Thomas, *Gorenstein spaces*, Adv. Math. **71** (1988), 92–112.

[7] Y. Félix, S. Halperin, J.-C. Thomas, "Rational Homotopy Theory", Grad. Texts in Math., Vol. 205, Springer, Berlin, 2001.

[8] H.-B. Foxby, S. Iyengar, *Depth and amplitude of unbounded complexes*, preprint (2001).

[9] A. Frankild, P. Jørgensen, *Dualizing DG-modules for Differential Graded Algebras*, preprint (2001).

[10] A. Frankild, P. Jørgensen, *Gorenstein Differential Graded Algebras*, preprint (2001).

[11] R. Hartshorne, "Residues and duality", Lecture Notes in Math., Vol. 20, Springer, Berlin, 1966, notes of a seminar given by A. Grothendieck.

[12] P. Roberts, *Homological invariants of modules over commutative rings*, Sèm. Math. Sup., Univ. Montréal, 1980.

Matematisk Afdeling, Københavns Universitet, København Ø, DK–Danmark

E-mail address: `frankild@math.ku.dk`

Mathematics Department, University of Missouri, Columbia, MO 65211, USA

E-mail address: `iyengar@math.missouri.edu`

Danish National Library of Science and Medicine, Nørre Allé 49, 2200 København N, DK–Danmark

E-mail address: `pej@dnlb.dk`

Journal of Algebra
(to appear)

HOMOLOGICAL IDENTITIES FOR DIFFERENTIAL GRADED ALGEBRAS

ANDERS FRANKILD AND PETER JØRGENSEN

0. Introduction

Our original motivation for this paper was to answer [4, question (3.10)] on gaps in the sequence of Bass numbers of a Differential Graded Algebra (DGA). We do this for some important classes of DGAs in paragraph (3.1).

[4, question (3.10)] asks for a sort of No Holes Theorem for Bass numbers of DGAs. More precisely, it asks for a certain bound on the length of gaps in the sequence of Bass numbers; namely, that if one has $\mu^{\ell} \neq 0$ and $\mu^{\ell+1} = \cdots = \mu^{\ell+g} = 0$ and $\mu^{\ell+g+1} \neq 0$, then g is at most equal to the degree of the highest non-vanishing homology of the DGA. This is the best possible bound one can hope for, as shown in [4, exam. (3.9)]).

As mentioned, we provide this bound in paragraph (3.1) and thereby answer the question. The bound arises as corollary to a more general Gap Theorem, theorem (2.5), which is the natural generalization to the world of DGAs of the classical No Holes Theorem from homological ring theory (see [11, thm. (1.1)], [13], [18, thm. 2]), and [20, thm. 0.3]).

Since the classical No Holes Theorem lives in the world of so-called homological identities, such as the Auslander-Buchsbaum and Bass Formulae (see [2, thm. 3.7], [8, lem. (3.3)], [20, thm. 0.3], and [21, thm. 1.1]), it seemed natural also to generalize these to DGAs. This is done in theorems (2.3) and (2.4). The results work for some important classes of DGAs, among them DG-fibres, Koszul complexes, and DGAs of the form $\mathrm{C}_*(G;k)$ where k is a field and G a path connected topological monoid with $\dim_k \mathrm{H}_*(G;k) < \infty$ (see remark (2.2)).

We prove theorems (2.3), (2.4), and (2.5) by means of what we call dualizing DG-modules (DG-module being our abbreviation of Differential Graded module). These are the natural generalization of dualizing complexes from homological ring theory, and were made available in [14] and [15]. As any reader of the ring theoretic literature will know, dualizing complexes can be used to give nice proofs of homological identities; it is hence not surprising that dualizing DG-modules enable us to prove homological identities for DGAs.

1991 *Mathematics Subject Classification.* 16E45, 16E10, 55P62.

Key words and phrases. Differential Graded Algebra, dualizing DG-module, Auslander-Buchsbaum Formula, Bass Formula, Gap Theorem, DG-fibre of a ring homomorphism, chain DGA of a topological monoid, No Holes Theorem.

Indeed, this is a very simple paper. Our proofs are close in spirit to homological ring theory (see [12], [18], [20], and [21]) and use dualizing DG-modules much as ring theory uses dualizing complexes. If anything, our proofs are slightly simpler than the corresponding ones from ring theory, because they have the benefit of so-called semi-free resolutions, the device from DGA theory which replaces free resolutions.

Our results have other consequences than the answer to [4, question (3.10)]. For instance, applying the Auslander-Buchsbaum Formula to $\mathrm{C}_*(G;k)$ proves additivity of homological dimension on G-Serre-fibrations (see paragraph (3.2)). Also, the classical Auslander-Buchsbaum and Bass Formulae and No Holes Theorem for noetherian local commutative rings are special cases (see paragraph (3.3)).

The paper is organized thus: This section ends with a few blanket items. Section 1 introduces some homological invariants for DG-modules and proves some elementary properties. Section 2 proves our main results. Section 3 gives the applications we have mentioned.

(0.1) **Some notation.** Most of our notation for DGAs and DG-modules is standard, in particular concerning derived categories and functors and the various resolutions used to compute them, cf. [6], [10, chaps. 3 and 6], and [17]. There are a few items we want to mention explicitly:

Let R be a DGA. By $\mathsf{D}(R)$ we denote the derived category of DG-R-left-modules. By R^{opp} we denote the *opposite DGA* of R, whose product is defined as $s \cdot r = (-1)^{|r||s|} rs$ for graded elements r and s. The idea of R^{opp} is that we can identify DG-R-right-modules with DG-R^{opp}-left-modules. So for instance, we will identify $\mathsf{D}(R^{\mathrm{opp}})$ with the derived category of DG-R-right-modules. This approach enables us to state many of our definitions and results for DG-R-left-modules only; applying them to DG-R^{opp}-left-modules then takes care of DG-R-right-modules.

Let M be a DG-R-module. The *amplitude* of M is defined by

$$\operatorname{amp} M = \sup\{\, i \mid \mathrm{H}_i(M) \neq 0\,\} - \inf\{\, i \mid \mathrm{H}_i(M) \neq 0\,\}.$$

We operate with the convention $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

Finally, $R^\natural$ denotes the graded algebra obtained by forgetting the differential of R, and $M^\natural$ denotes the graded $R^\natural$-module obtained by forgetting the differential of M.

(0.2) **The category fin.** Let R be a DGA for which $\mathrm{H}_0(R)$ is a noetherian ring. Then $\mathsf{fin}(R)$ denotes the full trianguated subcategory of $\mathsf{D}(R)$ which consists of M's so that the homology $\mathrm{H}(M)$ is bounded, and so that each $\mathrm{H}_i(M)$ is finitely generated as an $\mathrm{H}_0(R)$-left-module.

(0.3) **Dagger Duality.** In [15], the theory of dualizing DG-modules and the duality they define ("dagger duality") is developed (see [15, (1.2)] in particular). Here is a short summary:

Let R be a DGA for which $\mathrm{H}_0(R)$ is a noetherian ring, and suppose that R has the dualizing DG-module D. For any DG-R-left-module M and any DG-R-right-module N we define the *dagger duals* by

$$M^\dagger = \mathrm{RHom}_R(M, D) \qquad \text{and} \qquad N^\dagger = \mathrm{RHom}_{R^{\mathrm{opp}}}(N, D).$$

Strictly speaking, these should be called the dagger duals with respect to D, but we always only have a single D around, so there is no risk of confusion.

Dagger duality is now the pair of quasi-inverse contravariant equivalences of categories between $\mathsf{fin}(R)$ and $\mathsf{fin}(R^{\mathrm{opp}})$,

$$\begin{tikzcd} \mathsf{fin}(R) \arrow[r, shift left, "(-)^\dagger"] & \mathsf{fin}(R^{\mathrm{opp}}). \arrow[l, shift left, "(-)^\dagger"] \end{tikzcd}$$

Note the slight abuse of notation in that $(-)^\dagger$ denotes two different functors.

An alternative way of expressing the duality is to say that

$$\begin{array}{l} \text{the biduality morphism } M \longrightarrow M^{\dagger\dagger} \text{ is an} \\ \text{isomorphism for any } M \text{ in } \mathsf{fin}(R) \text{ or } \mathsf{fin}(R^{\mathrm{opp}}). \end{array} \tag{0.3.1}$$

For $M, N \in \mathsf{fin}(R)$ we even have

$$\begin{aligned} \mathrm{RHom}_{R^{\mathrm{opp}}}(N^\dagger, M^\dagger) &= \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(N, D), \mathrm{RHom}_R(M, D)) \\ &\overset{\text{(a)}}{\cong} \mathrm{RHom}_R(M, \mathrm{RHom}_{R^{\mathrm{opp}}}(\mathrm{RHom}_R(N, D), D)) \\ &= \mathrm{RHom}_R(M, N^{\dagger\dagger}) \\ &\cong \mathrm{RHom}_R(M, N), \end{aligned} \tag{0.3.2}$$

where (a) is by the so-called swap isomorphism.

The name "dagger duality" is due to Foxby.

(0.4) **Blanket Setup.** For the rest of this paper, R denotes a DGA satisfying:

- $R_i = 0$ for $i < 0$ (that is, R is a chain DGA).
- $\mathrm{H}_0(R)$ is a noetherian ring which is local in the sense that it has a unique maximal two sided ideal J such that $\mathrm{H}_0(R)/J$ is a skew field.
- ${}_R R \in \mathsf{fin}(R)$ and $R_R \in \mathsf{fin}(R^{\mathrm{opp}})$.

We denote the skew field $\mathrm{H}_0(R)/J$ by k.

Note that k can be viewed in a canonical way as a DG-R-left-R-right-module concentrated in degree zero.

(0.5) **Semi-free resolutions.** Let M be a DG-R-left-module with $\mathrm{H}(M)$ bounded to the right and each $\mathrm{H}_i(M)$ finitely generated as an $\mathrm{H}_0(R)$-module.

Since R satisfies the conditions of setup (0.4), there is a minimal semi-free resolution $F \xrightarrow{\simeq} M$ with

$$F^\natural \cong \coprod_{v \le j} \Sigma^j (R^\natural)^{\beta_j},$$

where $v = \inf\{\, i \mid \mathrm{H}_i(M) \neq 0 \,\}$ and where each β_j is finite. Here Σ^j denotes the j'th suspension. Minimality of F means that the differential ∂^F maps into $\mathfrak{m}F$, where $\mathfrak{m}$ is the DG-ideal

$$\cdots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow J \longrightarrow 0 \longrightarrow \cdots .$$

As consequence, $\mathrm{Hom}_R(F, k)$ and $k \otimes_R F$ have vanishing differentials. See [1, prop. 2], [6], and [9, lem. (A.3)(iii)].

1. Invariants

(1.1) **Definition.** For a DG-R-left-module M, we define the k-projective dimension, the k-injective dimension, and the depth as

$$\begin{aligned}
k.\mathrm{pd}_R\, M &= -\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(M,k)) \neq 0 \,\},\\
k.\mathrm{id}_R\, M &= -\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(k,M)) \neq 0 \,\},\\
\mathrm{depth}_R\, M &= -\sup\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(k,M)) \neq 0 \,\}.
\end{aligned}$$

(1.2) **Remark.** By the existence of minimal semi-free resolutions (see paragraph (0.5)), it is easy to prove for $M \in \mathsf{fin}(R)$ that

$$k.\mathrm{pd}_R\, M = \sup\{\, i \mid \mathrm{H}_i(k \overset{\mathrm{L}}{\otimes}_R M) \neq 0 \,\}.$$

(1.3) **Definition.** For a DG-R-left-module M, we define the j'th Bass number and the j'th Betti number as

$$\begin{aligned}
\mu_R^j(M) &= \dim_k \mathrm{H}_{-j}(\mathrm{RHom}_R(k,M)),\\
\beta_j^R(M) &= \dim_{k^{\mathrm{opp}}} \mathrm{H}_{-j}(\mathrm{RHom}_R(M,k)).
\end{aligned}$$

(Note that $\mu_R^j(M)$ and $\beta_j^R(M)$ may well equal $+\infty$.)

(1.4) **Remark.** Let M be a DG-R-left-module. It is clear from the definitions that

$$\begin{aligned}
k.\mathrm{pd}_R\, M &= \sup\{\, j \mid \beta_j^R(M) \neq 0 \,\},\\
k.\mathrm{id}_R\, M &= \sup\{\, j \mid \mu_R^j(M) \neq 0 \,\},\\
\mathrm{depth}_R\, M &= \inf\{\, j \mid \mu_R^j(M) \neq 0 \,\}.
\end{aligned}$$

See also lemma (1.7).

(1.5) **Proposition.** *Let M and N be DG-R-left-modules with $\mathrm{H}(M)$ bounded to the right and $\mathrm{H}(N)$ bounded to the left, and each $\mathrm{H}_i(M)$*

finitely generated as an $\mathrm{H}_0(R)$*-left-module. Then*

$$\begin{aligned}\sup\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(M,N)) \neq 0\} \\ \leq \sup\{\, i \mid \mathrm{H}_i(N) \neq 0\} - \inf\{\, i \mid \mathrm{H}_i(M) \neq 0\}.\end{aligned}$$

Proof. Paragraph (0.5) gives that M admits a semi-free resolution $F \xrightarrow{\simeq} M$ with

$$F^\natural \cong \coprod_{v \leq j} \Sigma^j (R^\natural)^{\beta_j},$$

where $v = \inf\{\, i \mid \mathrm{H}_i(M) \neq 0\,\}$ and where each β_j is finite. Let T be a truncation of N which is quasi-isomorphic to N and is concentrated in degrees smaller than or equal to $\sup\{\, i \mid \mathrm{H}_i(N) \neq 0\,\}$. We then have $\mathrm{RHom}_R(M,N) \cong \mathrm{Hom}_R(F,T)$.

Now,

$$\begin{aligned}\mathrm{Hom}_R(F,T)^\natural &= \mathrm{Hom}_{R^\natural}(F^\natural, T^\natural) \\ &\cong \mathrm{Hom}_{R^\natural}(\coprod_{v \leq j} \Sigma^j (R^\natural)^{\beta_j}, T^\natural) \\ &\cong \prod_{v \leq j} \Sigma^{-j} (T^\natural)^{\beta_j}\end{aligned}$$

is concentrated in degrees smaller than or equal to $\sup\{\, i \mid \mathrm{H}_i(N) \neq 0\,\} - v$. This gives the last $\leq$ in

$$\begin{aligned}\sup\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(M,N)) \neq 0\} \\ &= \sup\{\, i \mid \mathrm{H}_i(\mathrm{Hom}_R(F,T)) \neq 0\,\} \\ &\leq \sup\{\, i \mid (\mathrm{Hom}_R(F,T)^\natural)_i \neq 0\,\} \\ &\leq \sup\{\, i \mid \mathrm{H}_i(N) \neq 0\,\} - v \\ &= \sup\{\, i \mid \mathrm{H}_i(N) \neq 0\} - \inf\{\, i \mid \mathrm{H}_i(M) \neq 0\}.\end{aligned}$$

□

(1.6) **Lemma.** *Let F be a K-projective DG-R-left-module with*

$$F^\natural \cong \coprod_{j \leq p} \Sigma^j (R^\natural)^{\beta_j},$$

and let N be a DG-R-left-module with $\mathrm{H}(N)$ *bounded to the right. Then*

$$\inf\{\, i \mid \mathrm{H}_i(\mathrm{Hom}_R(F,N)) \neq 0\,\} \geq -p + \inf\{\, i \mid \mathrm{H}_i(N) \neq 0\,\}.$$

Proof. This is just like the proof of proposition (1.5): Let T be a truncation of N so that T is quasi-isomorphic to N and so that T is concentrated in degrees larger than or equal to $\inf\{\, i \mid \mathrm{H}_i(N) \neq 0\,\}$. We then have a quasi-isomorphism $\operatorname{Hom}_R(F,N) \simeq \operatorname{Hom}_R(F,T)$, and

$$\begin{aligned}\operatorname{Hom}_R(F,T)^\natural &\cong \operatorname{Hom}_{R^\natural}(F^\natural,T^\natural)\\ &\cong \operatorname{Hom}_{R^\natural}(\coprod_{j\leq p}\Sigma^j(R^\natural)^{\beta_j},T^\natural)\\ &\cong \prod_{j\leq p}\Sigma^{-j}(T^\natural)^{\beta_j}\end{aligned}$$

is concentrated in degrees larger than or equal to $-p+\inf\{\,i \mid \mathrm{H}_i(N)\neq 0\,\}$. This gives the last "$\geq$" in

$$\begin{aligned}\inf\{\,i \mid \mathrm{H}_i(\operatorname{Hom}_R(F,N))\neq 0\,\} &= \inf\{\,i \mid \mathrm{H}_i(\operatorname{Hom}_R(F,T))\neq 0\,\}\\ &\geq \inf\{\,i \mid (\operatorname{Hom}_R(F,T)^\natural)_i\neq 0\,\}\\ &\geq -p+\inf\{\,i \mid \mathrm{H}_i(N)\neq 0\,\}.\end{aligned}$$

□

(1.7) **Lemma.** *Let M be a DG-R-left-module and suppose that $F \xrightarrow{\simeq} M$ is a minimal K-projective resolution with*

$$F^\natural \cong \coprod_{v\leq j}\Sigma^j(R^\natural)^{\beta_j},$$

where each β_j is finite. Then

$$\beta_j^R(M) = \begin{cases} 0 & \text{for } j<v,\\ \beta_j & \text{for } j\geq v,\end{cases} \tag{1}$$

and we have

$$k.\mathrm{pd}_R\, M = \sup\{\,j \mid \beta_j^R(M)\neq 0\,\} = \sup\{\,j \mid \beta_j\neq 0\,\} \tag{2}$$

and

$$\inf\{\,i \mid \mathrm{H}_i(M)\neq 0\,\} = \inf\{\,j \mid \beta_j^R(M)\neq 0\,\} = \inf\{\,j \mid \beta_j\neq 0\,\}. \tag{3}$$

Proof. To see (1), we use $\operatorname{RHom}_R(M,k)\cong \operatorname{Hom}_R(F,k)$ to compute,

$$\begin{aligned}\beta_j^R(M) &= \dim_{k^{\mathrm{opp}}}\mathrm{H}_{-j}(\operatorname{RHom}_R(M,k))\\ &= \dim_{k^{\mathrm{opp}}}\mathrm{H}_{-j}(\operatorname{Hom}_R(F,k))\\ &\overset{(a)}{=} \dim_{k^{\mathrm{opp}}}(\operatorname{Hom}_R(F,k)^\natural)_{-j}\\ &= \dim_{k^{\mathrm{opp}}}(\operatorname{Hom}_{R^\natural}(F^\natural,k^\natural))_{-j}\\ &= \dim_{k^{\mathrm{opp}}}(\operatorname{Hom}_{R^\natural}(\coprod_{v\leq \ell}\Sigma^\ell(R^\natural)^{\beta_\ell},k^\natural))_{-j}\\ &= \dim_{k^{\mathrm{opp}}}(\prod_{v\leq \ell}\Sigma^{-\ell}(k^\natural)^{\beta_\ell})_{-j}\\ &= \begin{cases} 0 & \text{for } j<v,\\ \beta_j & \text{for } j\geq v,\end{cases}\end{aligned}$$

where (a) is because F is minimal.

In (2), the first $=$ is known from remark (1.4), and the second $=$ is clear from (1).

As for (3), the second $=$ is again clear from (1). We will therefore be done if we can prove $\inf\{\, i \mid \mathrm{H}_i(M) \neq 0 \,\} = \inf\{\, j \mid \beta_j \neq 0 \,\}$, and this is equivalent to

$$\inf\{\, i \mid \mathrm{H}_i(F) \neq 0 \,\} = \inf\{\, j \mid \beta_j \neq 0 \,\}.$$

So let $u = \inf\{\, j \mid \beta_j \neq 0 \,\}$. Then we have

$$F^\natural \cong \coprod_{u \leq j} \Sigma^j (R^\natural)^{\beta_j}, \tag{b}$$

so F is concentrated in degrees larger than or equal to u, proving

$$\inf\{\, i \mid \mathrm{H}_i(F) \neq 0 \,\} \geq u.$$

It hence only remains to prove $\mathrm{H}_u(F) \neq 0$, and this is easy, using that F is minimal, and that by (b), the right-most summand in $F^\natural$ is $\Sigma^u (R^\natural)^{\beta_u}$. □

(1.8) **Proposition.** *Let M and N be DG-R-left-modules with $\mathrm{H}(M)$ and $\mathrm{H}(N)$ bounded to the right, and each $\mathrm{H}_i(M)$ and each $\mathrm{H}_i(N)$ finitely generated as an $\mathrm{H}_0(R)$-module. Suppose that $k.\mathrm{pd}_R\, M$ is finite. Then*

$$\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(M,N)) \neq 0 \,\} = -\, k.\mathrm{pd}_R\, M + \inf\{\, i \mid \mathrm{H}_i(N) \neq 0 \,\}.$$

Proof. If $N \cong 0$ then both sides of the equation are $+\infty$, so we can assume $N \not\cong 0$.

First, we let $F \xrightarrow{\simeq} M$ be a semi-free resolution. By paragraph (0.5) we can pick F minimal with $F^\natural \cong \coprod_{v \leq j} \Sigma^j (R^\natural)^{\beta_j}$ and all β_j finite, and by lemma (1.7)(2) we can even write

$$F^\natural \cong \coprod_{v \leq j \leq p} \Sigma^j (R^\natural)^{\beta_j}$$

with $p = k.\mathrm{pd}_R\, M$ and $\beta_p \neq 0$.

Secondly, we write $u = \inf\{\, i \mid \mathrm{H}_i(N) \neq 0 \,\}$ and let T be a truncation of N so that T is quasi-isomorphic to N and so that T is concentrated in degrees larger than or equal to u.

We now have $\mathrm{RHom}_R(M,N) \cong \mathrm{Hom}_R(F,T)$, and the lemma's equation amounts to

$$\inf\{\, i \mid \mathrm{H}_i(\mathrm{Hom}_R(F,T)) \neq 0 \,\} = -p + u.$$

Now, lemma (1.6) gives

$$\inf\{\, i \mid \mathrm{H}_i(\mathrm{Hom}_R(F,T)) \neq 0 \,\} \geq -p + u,$$

so we will be done when we have proved

$$\mathrm{H}_{-p+u}(\mathrm{Hom}_R(F,T)) \neq 0.$$

As T_u is the right-most non-zero component of T, there is a surjection of R_0-left-modules $T_u \longrightarrow H_u(T)$. Moreover, since $\mathrm{H}_u(T) \cong \mathrm{H}_u(N)$ is finitely generated as an $\mathrm{H}_0(R)$-left-module, Nakayama's lemma gives that there is a surjection of $\mathrm{H}_0(R)$-left-modules $\mathrm{H}_u(T) \longrightarrow k$. Altogether, there is a surjection of R_0-left-modules,

$$T_u \longrightarrow k.$$

It is clear how this gives rise to a surjection of DG-R-left-modules $T \longrightarrow \Sigma^u k$, and denoting the kernel by T', there is a short exact sequence of DG-R-left-modules,

$$0 \to T' \longrightarrow T \longrightarrow \Sigma^u k \to 0. \tag{a}$$

Note that

$$\begin{aligned} \inf\{\, i \mid \mathrm{H}_i(\mathrm{Hom}_R(F, T')) \neq 0 \,\} &\geq -p + \inf\{\, i \mid \mathrm{H}_i(T') \neq 0 \,\} \\ &\geq -p + u, \end{aligned} \tag{b}$$

where the first $\geq$ is by lemma (1.6), and the second $\geq$ is because T' is a DG-submodule of T, so is concentrated in degrees larger than or equal to u.

As F is semi-free, acting with the functor $\mathrm{Hom}_R(F, -)$ on (a) gives a new short exact sequence

$$0 \to \mathrm{Hom}_R(F, T') \longrightarrow \mathrm{Hom}_R(F, T) \longrightarrow \mathrm{Hom}_R(F, \Sigma^u k) \to 0$$

whose homology long exact sequence contains

$$\mathrm{H}_{-p+u}(\mathrm{Hom}_R(F, T)) \longrightarrow \mathrm{H}_{-p+u}(\mathrm{Hom}_R(F, \Sigma^u k)) \longrightarrow \mathrm{H}_{-p+u-1}(\mathrm{Hom}_R(F, T')).$$

The last term is zero because of (b), so we will be done if we can prove that the middle term is non-zero. But by minimality of F we have the first $\cong$ in

$$\begin{aligned} \mathrm{H}(\mathrm{Hom}_R(F, \Sigma^u k)) &\cong \mathrm{Hom}_R(F, \Sigma^u k)^\natural \\ &\cong \mathrm{Hom}_{R^\natural}(F^\natural, (\Sigma^u k)^\natural) \\ &\cong \mathrm{Hom}_{R^\natural}(\coprod_{v \leq j \leq p} \Sigma^j (R^\natural)^{\beta_j}, (\Sigma^u k)^\natural) \\ &\cong \prod_{v \leq j \leq p} \Sigma^{-j+u} (k^\natural)^{\beta_j}, \end{aligned}$$

and as we have $\beta_p \neq 0$, this is non-zero in degree $-p + u$, as required. $\square$

(1.9) **Lemma.** *Suppose that R has a dualizing DG-module D satisfying the extra conditions*

$$\mathrm{RHom}_R({}_R k, {}_R D_R) \cong k_R \quad \textit{and} \quad \mathrm{RHom}_{R^{\mathrm{opp}}}(k_R, {}_R D_R) \cong {}_R k.$$

Let M be in $\mathsf{fin}(R)$. Then

$$\mu_R^j(M) \textit{ and } \beta_j^{R^{\mathrm{opp}}}(M^\dagger) \textit{ are zero simultaneously,} \tag{1}$$

and we have

$$k.\mathrm{id}_R\, M = k.\mathrm{pd}_{R^{\mathrm{opp}}}\, M^{\dagger} \tag{2}$$

and

$$\mathrm{depth}_R\, M = \inf\{\, i \mid \mathrm{H}_i(M^{\dagger}) \neq 0 \,\}. \tag{3}$$

Proof. The lemma's extra conditions on D can also be expressed

$$({}_Rk)^{\dagger} \cong k_R \quad \text{and} \quad (k_R)^{\dagger} \cong {}_Rk. \tag{a}$$

Thus,

$$\mathrm{RHom}_R({}_Rk, {}_RM) \cong \mathrm{RHom}_R((k_R)^{\dagger}, ({}_RM)^{\dagger\,\dagger}) \cong \mathrm{RHom}_{R^{\mathrm{opp}}}(({}_RM)^{\dagger}, k_R),$$

where the first $\cong$ follows from equations (a) and (0.3.1), and the second $\cong$ follows from equation (0.3.2). Hence we get isomorphisms of abelian groups,

$$\mathrm{H}_{-j}(\mathrm{RHom}_R(k, M)) \cong \mathrm{H}_{-j}(\mathrm{RHom}_{R^{\mathrm{opp}}}(M^{\dagger}, k)),$$

and (1) follows.

As for (2), it follows immediately from (1) and remark (1.4).

To see (3), we can compute,

$$\begin{aligned}\mathrm{depth}_R\, M &\overset{\text{(b)}}{=} \inf\{\, j \mid \mu_R^j(M) \neq 0 \,\} \\ &\overset{\text{(c)}}{=} \inf\{\, j \mid \beta_j^{R^{\mathrm{opp}}}(M^{\dagger}) \neq 0 \,\} \\ &\overset{\text{(d)}}{=} \inf\{\, i \mid \mathrm{H}_i(M^{\dagger}) \neq 0 \,\},\end{aligned}$$

where (b) is by remark (1.4) and (c) is by (1), while (d) is by lemma (1.7)(3) because $M^{\dagger}$ is in $\mathsf{fin}(R^{\mathrm{opp}})$ and hence by paragraph (0.5) admits a resolution as required in (1.7)(3). $\square$

(1.10) **Corollary.** *Suppose that R has a dualizing DG-module D satisfying the extra conditions*

$$\mathrm{RHom}_R({}_Rk, {}_RD_R) \cong k_R \quad \textit{and} \quad \mathrm{RHom}_{R^{\mathrm{opp}}}(k_R, {}_RD_R) \cong {}_Rk.$$

Then

$$\mathrm{depth}_R\, R = \inf\{\, i \mid \mathrm{H}_i(D) \neq 0 \,\} = \mathrm{depth}_{R^{\mathrm{opp}}}\, R.$$

Proof. The corollary's first $=$ can be proved as follows,

$$\begin{aligned}\mathrm{depth}_R\, R &= \inf\{\, i \mid \mathrm{H}_i(({}_RR)^{\dagger}) \neq 0 \,\} \\ &= \inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(R, D)) \neq 0 \,\} \\ &= \inf\{\, i \mid \mathrm{H}_i(D) \neq 0 \,\},\end{aligned}$$

where the first $=$ is by lemma (1.9)(3). The corollary's second $=$ follows by an analogous computation. $\square$

2. Identities

(2.1) **Setup.** Recall that R denotes a DGA satisfying the conditions of setup (0.4). In this section, R will also satisfy:

- R admits a dualizing DG-module D satisfying

$$\mathrm{RHom}_R({}_Rk, {}_RD_R) \cong k_R \quad \text{and} \quad \mathrm{RHom}_{R^{\mathrm{opp}}}(k_R, {}_RD_R) \cong {}_Rk.$$

(2.2) **Remark.** From [14] we know that, in suitable circumstances, one can get a dualizing DG-module for the DGA R by coinducing a dualizing complex from a commutative central base ring A. That is, if A has the dualizing complex C, then

$$D = \mathrm{RHom}_A(R, C)$$

is a dualizing DG-module for R.

A small computation with the generic pattern

$$\begin{aligned}\mathrm{RHom}_R(k, D) &= \mathrm{RHom}_R(k, \mathrm{RHom}_A(R, C)) \\ &\cong \mathrm{RHom}_A(R \overset{\mathrm{L}}{\otimes}_R k, C) \\ &\cong \mathrm{RHom}_A(k, C) \\ &\cong k\end{aligned}$$

proves frequently that such a dualizing DG-module D also satisfies the extra conditions of setup (2.1). (Some care is needed when making this concrete; for instance, we have made no assumptions on the behaviour of k when viewed as an A-module, so the last $\cong$ does not necessarily apply.)

Summing up, when this method works, the conditionss of setup (2.1) hold for R, and hence the results of this section apply to R.

In particular, the DGAs in the following list satisfy the standing conditions of setup (0.4), and the method we have sketched shows that they also satisfy the conditions of setup (2.1). Hence the results of this section apply to them:

- The DG-fibre $F(\alpha')$, where $A' \xrightarrow{\alpha'} A$ is a local ring homomorphism of finite flat dimension between noetherian local commutative rings A' and A, and where A has a dualizing complex. See [7, (3.7)].
- The Koszul complex $\mathrm{K}(\boldsymbol{a})$, where $\boldsymbol{a} = (a_1, \ldots, a_n)$ is a sequence of elements in the maximal ideal of the noetherian local commutative ring A, and where A again has a dualizing complex. See [19, exer. 4.5.1].
- The chain DGA $\mathrm{C}_*(G; k)$ where k is a field and G is a path connected topological monoid with $\dim_k \mathrm{H}_*(G; k) < \infty$. See [10, chap. 8].

Finally, let us mention a "degenerate" case: Let A be a noetherian ring which is local in the sense that A has a unique maximal two sided ideal J such that A/J is a skew field. We can then consider A as a DGA concentrated in degree zero, and A falls under setup (0.4). So if A satisfies the conditions of setup (2.1), then the results of this section apply to A.

A special case of this is that A is even a noetherian local commutative ring. Then "dualizing DG-module" just means "dualizing complex" by [15, thm. (1.7)], and if D is a dualizing complex for A then the extra conditions of setup (2.1) hold automatically by [16, prop. V.3.4] (we might need to replace D by some $\Sigma^i D$). So we can extend the list above: The results of this section also apply to

- The noetherian local commutative ring A, where A has a dualizing complex.

(2.3) **Theorem (Auslander-Buchsbaum Formula).** *Let M be in $\mathsf{fin}(R)$ and suppose that $k.\mathrm{pd}_R\, M$ is finite. Then*

$$k.\mathrm{pd}_R\, M + \mathrm{depth}_R\, M = \mathrm{depth}_R\, R.$$

Proof. Proposition (1.8) applies to $\mathrm{RHom}_R(M, D)$: We have $M \in \mathsf{fin}(R)$ by assumption, so M satisfies the lemma's finiteness conditions. And R_R is in $\mathsf{fin}(R^{\mathrm{opp}})$ by setup (0.4), so

$$(R_R)^\dagger = \mathrm{RHom}_{R^{\mathrm{opp}}}(R_R, {}_R D_R) \cong {}_R D$$

is in $\mathsf{fin}(R)$, so ${}_R D$ also satisfies the lemma's finiteness conditions. Finally, we have $k.\mathrm{pd}_R\, M < \infty$ by assumption.

We can now compute,

$$\begin{aligned}
\mathrm{depth}_R\, M &\overset{\text{(a)}}{=} \inf\{\, i \mid \mathrm{H}_i(M^\dagger) \neq 0 \,\} \\
&= \inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_R(M, D)) \neq 0 \,\} \\
&\overset{\text{(b)}}{=} -\, k.\mathrm{pd}_R\, M + \inf\{\, i \mid \mathrm{H}_i(D) \neq 0 \,\} \\
&\overset{\text{(c)}}{=} -\, k.\mathrm{pd}_R\, M + \mathrm{depth}_R\, R,
\end{aligned}$$

where (a) is by lemma (1.9)(3) and (b) is by proposition (1.8), while (c) is by corollary (1.10). □

(2.4) **Theorem (Bass Formula).** *Let N be in $\mathsf{fin}(R)$ and suppose that $k.\mathrm{id}_R\, N$ is finite. Then*

$$k.\mathrm{id}_R\, N + \inf\{\, i \mid \mathrm{H}_i(N) \neq 0 \,\} = \mathrm{depth}_R\, R.$$

Proof. From the duality (0.3) we know that $N^\dagger$ is in $\mathsf{fin}(R^{\mathrm{opp}})$, and from lemma (1.9)(2) we have $k.\mathrm{pd}_{R^{\mathrm{opp}}}\, N^\dagger = k.\mathrm{id}_R\, N$, so $k.\mathrm{pd}_{R^{\mathrm{opp}}}\, N^\dagger$ is finite.

Now

$$\begin{aligned}
k.\mathrm{id}_R N &= k.\mathrm{pd}_{R^{\mathrm{opp}}} N^\dagger \\
&\overset{\text{(a)}}{=} \mathrm{depth}_{R^{\mathrm{opp}}} R - \mathrm{depth}_{R^{\mathrm{opp}}} N^\dagger \\
&\overset{\text{(b)}}{=} \mathrm{depth}_{R^{\mathrm{opp}}} R - \inf\{\, i \mid \mathrm{H}_i(N^{\dagger\dagger}) \neq 0 \,\} \\
&\overset{\text{(c)}}{=} \mathrm{depth}_{R^{\mathrm{opp}}} R - \inf\{\, i \mid \mathrm{H}_i(N) \neq 0 \,\} \\
&\overset{\text{(d)}}{=} \mathrm{depth}_R R - \inf\{\, i \mid \mathrm{H}_i(N) \neq 0 \,\},
\end{aligned}$$

where (a) is by the Auslander-Buchsbaum Formula, theorem (2.3), and (b) is by lemma (1.9)(3), while (c) is by equation (0.3.1) and (d) is by corollary (1.10). □

(2.5) **Gap Theorem.** *Let M be in $\mathsf{fin}(R)$ and let $g \in \mathbb{Z}$ satisfy $g > \operatorname{amp} R$. Assume that the sequence of Bass numbers of M has a gap of length g, in the sense that there exists $\ell \in \mathbb{Z}$ such that*

- $\mu_R^\ell(M) \neq 0$.
- $\mu_R^{\ell+1}(M) = \cdots = \mu_R^{\ell+g}(M) = 0$.
- $\mu_R^{\ell+g+1}(M) \neq 0$.

Then we have

$$\operatorname{amp} M \geq g + 1.$$

Proof. We can replace R by a quasi-isomorphic quotient DGA concentrated between degrees 0 and $\sup\{\, i \mid \mathrm{H}_i(R) \neq 0 \,\} = \operatorname{amp} R$.

Observe from the duality (0.3) that $M^\dagger$ is in $\mathsf{fin}(R^{\mathrm{opp}})$. So paragraph (0.5) gives that $M^\dagger$ admits a minimal semi-free resolution $F \xrightarrow{\simeq} M^\dagger$ with

$$F^\natural \cong \coprod_{v \leq j} \Sigma^j (R^\natural)^{\beta_j},$$

where $v = \inf\{\, i \mid \mathrm{H}_i(M^\dagger) \neq 0 \,\}$ and where each β_j is finite. Lemma (1.7)(1) yields

$$\beta_j^{R^{\mathrm{opp}}}(M^\dagger) = \begin{cases} 0 & \text{for } j < v, \\ \beta_j & \text{for } j \geq v. \end{cases} \tag{a}$$

Note that we have $F \cong M^\dagger$ in $\mathsf{D}(R^{\mathrm{opp}})$ and $F^\dagger \cong M$ in $\mathsf{D}(R)$.

By assumption we have

$$\mu_R^\ell(M) \neq 0,\ \mu_R^{\ell+1}(M) = \cdots = \mu_R^{\ell+g}(M) = 0,\ \mu_R^{\ell+g+1}(M) \neq 0.$$

Applying lemma (1.9)(1) this translates to

$$\beta_\ell^{R^{\mathrm{opp}}}(M^\dagger) \neq 0,\ \beta_{\ell+1}^{R^{\mathrm{opp}}}(M^\dagger) = \cdots = \beta_{\ell+g}^{R^{\mathrm{opp}}}(M^\dagger) = 0,\ \beta_{\ell+g+1}^{R^{\mathrm{opp}}}(M^\dagger) \neq 0.$$

And by equation (a) this says

$$\beta_\ell \neq 0,\ \beta_{\ell+1} = \cdots = \beta_{\ell+g} = 0,\ \beta_{\ell+g+1} \neq 0. \tag{b}$$

But then the graded $R^\natural$-right-module $F^\natural$ splits as

$$F^\natural \cong F_1^\natural \amalg F_2^\natural,$$

where the summands have the form

$$F_1^\natural = \coprod_{v \leq j \leq \ell} \Sigma^j (R^\natural)^{\beta_j}, \tag{c}$$

$$F_2^\natural = \coprod_{\ell+g+1 \leq j} \Sigma^j (R^\natural)^{\beta_j}. \tag{d}$$

Now observe that

- The leftmost summand in $F_1^\natural$ has index $j = \ell$, so is concentrated between degrees ℓ and $\ell + \operatorname{amp} R < \ell + g$ because $R^\natural$ itself is concentrated between degrees 0 and $\operatorname{amp} R$.
- The rightmost summand in $F_2^\natural$ has index $j = \ell + g + 1$, so has its rightmost component in degree $\ell + g + 1$ (and continues to the left).

In other words, the summands $F_1^\natural$ and $F_2^\natural$ are separated by at least one zero in degree $\ell + g$ so the differential of F does not map between $F_1^\natural$ and $F_2^\natural$. Hence the splitting of $F^\natural$ is induced by a splitting of the DG-R-right-module F,

$$F \cong F_1 \amalg F_2.$$

Clearly, both F_1 and F_2 are minimal K-projective, as F itself is. Also, we have $F_1 \not\cong 0$ and $F_2 \not\cong 0$ in $\mathsf{D}(R^{\mathrm{opp}})$, as one sees easily from $\beta_\ell \neq 0$ and $\beta_{\ell+g+1} \neq 0$ (see equation (b)).

The rest of the proof consists of computations with $\operatorname{RHom}_{R^{\mathrm{opp}}}(F_1, F_2)$. Let us first check that we have

$$\operatorname{RHom}_{R^{\mathrm{opp}}}(F_1, F_2) \not\cong 0. \tag{e}$$

From $F \cong M^\dagger$ we get

$$\mathrm{H}(F_1) \amalg \mathrm{H}(F_2) \cong \mathrm{H}(F) \cong \mathrm{H}(M^\dagger),$$

so it is clear that F_1 and F_2 are in $\mathsf{fin}(R^{\mathrm{opp}})$. Moreover, we know $\beta_\ell \neq 0$ from equation (b), so equation (c) and lemma (1.7)(2) give

$$k.\mathrm{pd}_{R^{\mathrm{opp}}} F_1 = \ell,$$

in particular $k.\mathrm{pd}_{R^{\mathrm{opp}}} F_1 < \infty$. Finally, F_2 is bounded to the right and has $F_2 \not\cong 0$, so $\inf\{\, i \mid \mathrm{H}_i(F_2) \neq 0 \,\}$ is finite. Proposition (1.8) can now be applied and proves

$$\begin{aligned} \inf\{\, i \mid \mathrm{H}_i(\operatorname{RHom}_{R^{\mathrm{opp}}}(F_1, F_2)) \neq 0\} \\ &= -\, k.\mathrm{pd}_{R^{\mathrm{opp}}} F_1 + \inf\{\, i \mid \mathrm{H}_i(F_2) \neq 0\} \\ &= -\ell + \inf\{\, i \mid \mathrm{H}_i(F_2) \neq 0\}, \end{aligned} \tag{f}$$

and this is a finite number, from which (e) follows.

Let us now compute a lower bound for

$$\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_{R^{\mathrm{opp}}}(F_1, F_2)) \neq 0\}.$$

Equation (d) gives

$$\inf\{\, i \mid \mathrm{H}_i(F_2) \neq 0\,\} \geq \ell + g + 1.$$

So starting with equation (f) we get

$$\begin{aligned}\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_{R^{\mathrm{opp}}}(F_1, F_2)) \neq 0\} &= -\ell + \inf\{\, i \mid \mathrm{H}_i(F_2) \neq 0\} \\ &\geq -\ell + \ell + g + 1 \\ &= g + 1. \end{aligned} \tag{g}$$

Next we want to compute an upper bound for

$$\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_{R^{\mathrm{opp}}}(F_1, F_2)) \neq 0\}.$$

From $F^\dagger \cong M$ we get

$$\mathrm{H}(F_1^\dagger) \amalg \mathrm{H}(F_2^\dagger) \cong \mathrm{H}(F^\dagger) \cong \mathrm{H}(M), \tag{h}$$

so it is clear that $F_1^\dagger$ and $F_2^\dagger$ are in $\mathsf{fin}(R)$. We also get the estimate

$$\sup\{\, i \mid \mathrm{H}_i(F_1^\dagger) \neq 0\} \leq \sup\{\, i \mid \mathrm{H}_i(M) \neq 0\}.$$

Hence

$$\begin{aligned}&\inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_{R^{\mathrm{opp}}}(F_1, F_2)) \neq 0\} \\ &\qquad \overset{\text{(i)}}{\leq} \sup\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_{R^{\mathrm{opp}}}(F_1, F_2)) \neq 0\} \\ &\qquad \overset{\text{(j)}}{=} \sup\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_{R}(F_2^\dagger, F_1^\dagger)) \neq 0\} \\ &\qquad \overset{\text{(k)}}{\leq} \sup\{\, i \mid \mathrm{H}_i(F_1^\dagger) \neq 0\} - \inf\{\, i \mid \mathrm{H}_i(F_2^\dagger) \neq 0\} \\ &\qquad \leq \sup\{\, i \mid \mathrm{H}_i(M) \neq 0\} - \inf\{\, i \mid \mathrm{H}_i(F_2^\dagger) \neq 0\},\end{aligned} \tag{l}$$

where (i) holds because of (e) and (j) is by equation (0.3.2), while (k) is by proposition (1.5).

Combining (g) and (l) we may write

$$\begin{aligned} g + 1 &\leq \inf\{\, i \mid \mathrm{H}_i(\mathrm{RHom}_{R^{\mathrm{opp}}}(F_1, F_2)) \neq 0\} \\ &\leq \sup\{\, i \mid \mathrm{H}_i(M) \neq 0\} - \inf\{\, i \mid \mathrm{H}_i(F_2^\dagger) \neq 0\},\end{aligned}$$

hence

$$\sup\{\, i \mid \mathrm{H}_i(M) \neq 0\} \geq \inf\{\, i \mid \mathrm{H}_i(F_2^\dagger) \neq 0\} + (g + 1). \tag{m}$$

Finally, from equation (h) we also get the estimate

$$\inf\{\, i \mid \mathrm{H}_i(M) \neq 0\} \leq \inf\{\, i \mid \mathrm{H}_i(F_2^\dagger) \neq 0\}. \tag{n}$$

Subtracting (n) from (m) we get

$$\begin{aligned}\operatorname{amp} M &= \sup\{\, i \mid \mathrm{H}_i(M) \neq 0\} - \inf\{\, i \mid \mathrm{H}_i(M) \neq 0\} \\ &\geq \inf\{\, i \mid \mathrm{H}_i(F_2^{\dagger}) \neq 0\} + (g+1) - \inf\{\, i \mid \mathrm{H}_i(F_2^{\dagger}) \neq 0\} \\ &= g + 1.\end{aligned}$$

□

3. Applications

(3.1) **Gaps in Bass series.** We can use theorem (2.5) to answer [4, question (3.10)], on gaps in Bass series of DGAs, for a DGA R which satisfies the conditions of setup (2.1) (and as always, the standing conditions of setup (0.4)). By remark (2.2), this includes DG-fibres, Koszul complexes, and DGAs of the form $\mathrm{C}_*(G;k)$ where k is a field and G a topological monoid with $\dim_k \mathrm{H}_*(G;k) < \infty$.

First a short recap on [4]. Following theorem (2.5), we say that the sequence of Bass numbers of a DG-R-left-module M has a *gap of length* g if there exists an ℓ with

- $\mu_R^{\ell}(M) \neq 0$,
- $\mu_R^{\ell+1}(M) = \cdots = \mu_R^{\ell+g}(M) = 0$,
- $\mu_R^{\ell+g+1}(M) \neq 0$.

Now, [4] defines the *Bass series* of M by

$$I_M(t) = \sum_n \mu_R^n(M) t^n,$$

and defines the *gap of* $I_M(t)$ by

$$\operatorname{gap} I_M(t) = \sup\left\{ g \;\middle|\; \begin{array}{l}\text{the sequence of Bass numbers} \\ \text{of } M \text{ has a gap of length } g\end{array} \right\}.$$

[4, question (3.10)] asks whether $\operatorname{gap} I_R(t) \leq \operatorname{amp} R$ holds. (The number $\operatorname{fd} R$ from [4] is just $\operatorname{amp} R$ in our notation.) And indeed, using theorem (2.5), we can prove even more: Let M be any DG-R-left-module in $\mathsf{fin}(R)$ with $\operatorname{amp} M \leq \operatorname{amp} R + 1$. If we had $\operatorname{gap} I_M(t) = g > \operatorname{amp} R$ then theorem (2.5) would give $\operatorname{amp} M \geq g + 1 > \operatorname{amp} R + 1 > \operatorname{amp} R$, hence $\operatorname{amp} M \geq \operatorname{amp} R + 2$, a contradiction. So we must have

$$\operatorname{gap} I_M(t) \leq \operatorname{amp} R.$$

Note that in the case of R being the DG-fibre of a local ring homomorphism of finite flat dimension, one can prove the stronger result that there are no gaps in $I_R(t)$ by using [5, (7.2) and thm. (7.4)].

(3.2) ***G*-Serre-fibrations.** The Auslander-Buchsbaum Formula (theorem (2.3)) can be applied to G-Serre-fibrations: Let k be a field, G a path connected topological monoid, and

$$G \longrightarrow P \xrightarrow{p} X$$

a G-Serre-fibration (see [10, chap. 2]). Assume that $\mathrm{H}_*(G;k)$, $\mathrm{H}_*(P;k)$ and $\mathrm{H}_*(X;k)$ are finite dimensional over k. Note that it is the composition in G that turns $\mathrm{C}_*(G;k)$ into a DGA which is potentially highly non-commutative (see [10, chap. 8]).

By remark (2.2), the standing conditions of setup (0.4) and the conditions of setup (2.1) hold for $\mathrm{C}_*(G;k)$. Hence the results of section 2 hold for $\mathrm{C}_*(G;k)$, in particular the Auslander-Buchsbaum Formula.

In fact, note that by remark (2.2), the conditions of setup (2.1) are satisfied with the dualizing DG-module

$$_{\mathrm{C}_*(G;k)}D_{\mathrm{C}_*(G;k)} = \mathrm{RHom}_k({}_{k,\mathrm{C}_*(G;k)}\mathrm{C}_*(G;k)_{\mathrm{C}_*(G;k)}, k).$$

Dagger dualization with respect to this D is particularly simple: For a DG-R-left-module M we have

$$\begin{aligned} M^\dagger &= \mathrm{RHom}_{\mathrm{C}_*(G;k)}(M, D) \\ &= \mathrm{RHom}_{\mathrm{C}_*(G;k)}(M, \mathrm{RHom}_k(\mathrm{C}_*(G;k), k)) \\ &\cong \mathrm{RHom}_k(\mathrm{C}_*(G;k) \overset{\mathrm{L}}{\otimes}_{\mathrm{C}_*(G;k)} M, k) \\ &\cong \mathrm{RHom}_k(M, k), \end{aligned} \tag{a}$$

that is, dagger dualization is just dualization with respect to k.

We now want to use the Auslander-Buchsbaum Formula on the DG-$\mathrm{C}_*(G;k)$-right-module $\mathrm{C}_*(P;k)$. Clearly $\mathrm{C}_*(P;k)$ is in $\mathsf{fin}(\mathrm{C}_*(G;k)^{\mathrm{opp}})$. Next note that

$$\mathrm{C}_*(P;k) \overset{\mathrm{L}}{\otimes}_{\mathrm{C}_*(G;k)} k \cong \mathrm{C}_*(X;k)$$

by [10, thm. 8.3], so using remark (1.2) we may compute,

$$\begin{aligned} k.\mathrm{pd}_{\mathrm{C}_*(G;k)^{\mathrm{opp}}}\, \mathrm{C}_*(P;k) &= \sup\{\, i \mid \mathrm{H}_i(\mathrm{C}_*(P;k) \overset{\mathrm{L}}{\otimes}_{\mathrm{C}_*(G;k)} k) \neq 0 \,\} \\ &= \sup\{\, i \mid \mathrm{H}_i(\mathrm{C}_*(X;k)) \neq 0 \,\} \\ &= \sup\{\, i \mid \mathrm{H}_i(X;k) \neq 0 \,\}. \end{aligned}$$

This is finite by assumption.

Thus we may apply the Auslander-Buchsbaum Formula, and get

$$\begin{aligned} k.\mathrm{pd}_{\mathrm{C}_*(G;k)^{\mathrm{opp}}}\, \mathrm{C}_*(P;k) &+ \mathrm{depth}_{\mathrm{C}_*(G;k)^{\mathrm{opp}}}\, \mathrm{C}_*(P;k) \\ &= \mathrm{depth}_{\mathrm{C}_*(G;k)^{\mathrm{opp}}}\, \mathrm{C}_*(G;k). \end{aligned}$$

Substituting the above expression for $k.\mathrm{pd}_{\mathrm{C}_*(G;k)^{\mathrm{opp}}}\, \mathrm{C}_*(P;k)$, this becomes

$$\begin{aligned} \sup\{\, i \mid \mathrm{H}_i(X;k) \neq 0 \,\} &+ \mathrm{depth}_{\mathrm{C}_*(G;k)^{\mathrm{opp}}}\, \mathrm{C}_*(P;k) \\ &= \mathrm{depth}_{\mathrm{C}_*(G;k)^{\mathrm{opp}}}\, \mathrm{C}_*(G;k). \end{aligned} \tag{b}$$

Finally, for any $M \in \mathsf{fin}(\mathrm{C}_*(G;k)^{\mathrm{opp}})$ we have

$$\begin{aligned}\operatorname{depth}_{\mathrm{C}_*(G;k)^{\mathrm{opp}}} M &\overset{\text{(c)}}{=} \inf\{\, i \mid \mathrm{H}_i(M^\dagger) \neq 0 \,\} \\ &\overset{\text{(d)}}{=} -\sup\{\, i \mid \mathrm{H}_i(M) \neq 0 \,\},\end{aligned}$$

where (c) is by lemma (1.9)(3) and (d) follows from (a).

Using this in equation (b), we finally get

$$\sup\{\, i \mid \mathrm{H}_i(P;k) \neq 0 \,\} = \sup\{\, i \mid \mathrm{H}_i(G;k) \neq 0 \,\} + \sup\{\, i \mid \mathrm{H}_i(X;k) \neq 0 \,\},$$

stating that homological dimension is additive on G-Serre-fibrations.

(3.3) **Commutative rings.** We noted already in remark (2.2) that the results of section 2 apply to noetherian local commutative rings with dualizing complexes.

Indeed, let us show that for any noetherian local commutative ring A, the classical Auslander-Buchsbaum and Bass Formulae and the No Holes Theorem (see [2, thm. 3.7], [8, lem. (3.3)], [13], [11, thm. (1.1)], and [18, thm. 2]) follow from theorems (2.3), (2.4), and (2.5):

First, to prove the three classical results for A, it suffices to prove them for the completion $\widehat{A}$, so we can assume that A is complete. Hence A has a dualizing complex D by [16, p. 299], and by remark (2.2), the results of section 2 apply to A.

The classical Auslander-Buchsbaum Formula now follows from theorem (2.3) since our notions of $k.\mathrm{pd}$ and depth coincide with the classical notions of projective dimension and depth for complexes in $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$, by [3, prop. 5.5].

The classical Bass Formula likewise follows from theorem (2.4) since our notion of $k.\mathrm{id}$ coincides with the classical notion of injective dimension for complexes in $\mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$, again by [3, prop. 5.5].

The classical No Holes Theorem can be obtained as follows from theorem (2.5): Consider $M \in \mathsf{D}^{\mathrm{f}}_{\mathrm{b}}(A)$ and suppose that there is a "hole" in the sequence of Bass numbers of M, that is, we have $\mu^j_A(M) = 0$, but there are non-zero Bass numbers both below and above $\mu^j_A(M)$. In the terminology of theorem (2.5), this says that the sequence of Bass numbers of M has a gap. If we let g be the length of the gap, then we have $g > 0$ whence $g > \operatorname{amp} A$ since $\operatorname{amp} A = 0$, so theorem (2.5) states

$$\operatorname{amp} M \geq g + 1 > 1,$$

so M is certainly not an ordinary A-module, since it is not concentrated in one degree. So if M *is* an ordinary A-module, then there are no holes in the sequence of Bass numbers of M.

(3.4) **Non-commutative rings.** The method of paragraph (3.3) could also be used on a suitable non-commutative noetherian ring, and, when successful, would recover the Auslander-Buchsbaum and Bass Formulae and the No Holes Theorem (see [20, thm. 0.3] and [21, thm. 1.1]).

However, the question of existence of a suitable dualizing DG-module satisfying the conditions of setup (2.1) is much more delicate in this case (see [20] and [15, thm. (1.9)]), so we prefer to leave the matter with this remark.

References

[1] D. Apassov, *Homological dimensions over differential graded rings*, pp. 25–39 in "Complexes and Differential Graded Modules", ph.d. thesis, Lund University, 1999.

[2] M. Auslander and D. Buchsbaum, *Homological dimension in Noetherian rings*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 36–38.

[3] L. L. Avramov and H.-B. Foxby, *Homological dimensions of unbounded complexes*, J. Pure Appl. Algebra **71** (1991), 129–155.

[4] ——— and ———, *Locally Gorenstein homomorphisms*, Amer. J. Math. **114** (1992), 1007–1047.

[5] ——— and ———, *Ring homomorphisms and finite Gorenstein dimension,* Proc. London Math. Soc. (3) **75** (1997), 241–270.

[6] ———, ———, and S. Halperin, Differential graded homological algebra, in preparation.

[7] ——— and S. Halperin, *Through the looking glass: a dictionary between rational homotopy theory and local algebra*, pp. 1–27 in "Algebra, algebraic topology and their interactions" (proceedings of the conference in Stockholm, 1983), Lecture Notes in Math., Vol. 1183, Springer, Berlin, 1986.

[8] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.

[9] Y. Félix, S. Halperin, and J.-C. Thomas, *Gorenstein spaces*, Adv. Math. **71** (1988), 92–112.

[10] ———, ———, and ———, "Rational Homotopy Theory", Grad. Texts in Math., Vol 205, Springer, Berlin, 2000.

[11] R. Fossum, H.-B. Foxby, P. Griffith, and I. Reiten, *Minimal injective resolutions with applications to dualizing modules and Gorenstein modules*, Inst. Hautes Études Sci. Publ. Math. **45** (1975), 193–215.

[12] H.-B. Foxby, *Homological dimension of complexes of modules*, pp. 360–368 in "Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin" (proceedings, Paris, 1979), Lecture Notes in Math., Vol. 795, Springer, Berlin, 1980.

[13] ———, *On the μ^i in a minimal injective resolution*, Math. Scand. **29** (1971), 175–186.

[14] A. Frankild, S. Iyengar, and P. Jørgensen, Existence of dualizing DG-modules for Differential Graded Algebras, manuscript (2001).

[15] ——— and P. Jørgensen, Dualizing DG-modules for Differential Graded Algebras, preprint (2001).

[16] A. Grothendieck, "Residues and duality", Lecture Notes in Math., Vol. 20, Springer, Berlin, 1966. Notes by R. Hartshorne.

[17] B. Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), 63–102.

[18] P. Roberts, *Two applications of dualizing complexes over local rings*, Ann. Sci. École Norm. Sup. (4) **9** (1976), 103–106.

[19] C. A. Weibel, "An introduction to homological algebra", Cambridge Stud. Adv. Math., Vol. 38, Cambridge University Press, Cambridge, 1997, paperback edition.

[20] Q. S. Wu and J. J. Zhang, *Dualizing Complexes over Noncommutative Local Rings*, J. Algebra **239** (2001), 513–548.
[21] ——— and ———, *Some Homological Invariants of Local PI Algebras*, J. Algebra **225** (2000), 904–935.

Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, 2100 København Ø, DK–Danmark
E-mail address: frankild@math.ku.dk, popjoerg@math.ku.dk

HOMOLOGICAL IDENTITIES FOR DIFFERENTIAL GRADED ALGEBRAS, II

ANDERS FRANKILD AND PETER JØRGENSEN

0. Introduction

This is a direct sequel to [8]. That paper was very simple: It took a few so-called homological identities from ring theory and generalized them to chain Differential Graded Algebras.

This paper is even simpler: It takes note of the so-called *looking glass principle* of [2], which propounds symmetry between chain and cochain Differential Graded Algebras (abbreviated DGAs from now on). This principle indicates that each result in [8] on chain DGAs ought to have a mirror version for cochain DGAs.

Indeed, this is exactly what we shall prove. The mirror results are closely parallel to the original results, and can be obtained from them by applying an extension of the dictionary between chain and cochain DGAs contained in [2]; see below.

However, proving the new results is not a matter of simply translating the old proofs. Just as in [2], we shall see that although the results themselves translate perfectly, some of the proofs do not.

The extension of the dictionary from [2] results from the following considerations: The looking glass principle basically tells us to interchange left and right. A way of doing this is to interchange homology and cohomology, so if M and N denote DG-modules, then we clearly must have in the dictionary

<u>Chain DGAs</u>	<u>Cochain DGAs</u>
$\sup\{\, i \mid \mathrm{H}_i M \neq 0 \,\}$	$\sup\{\, i \mid \mathrm{H}^i N \neq 0 \,\}$
$\inf\{\, i \mid \mathrm{H}_i M \neq 0 \,\}$	$\inf\{\, i \mid \mathrm{H}^i N \neq 0 \,\}$.

To continue the dictionary with some more subtle entries, let us look at the behaviour of semi-free resolutions:

Over (sufficiently nice) chain DGAs, it is well-known that semi-free resolutions can be constructed very much like free resolutions over a ring: One starts at the right end of a DG-module and "kills" homology, and

---

1991 *Mathematics Subject Classification.* 16E45, 16E10, 55P62.

Key words and phrases. Differential Graded Algebra, dualizing DG-module, Auslander-Buchsbaum Formula, Bass Formula, Gap Theorem, cochain DGA of a topological space, semi-free resolution.

proceeds to the left. Thus, to measure the "size" of a semi-free resolution over a chain DGA, one should measure how far it extends to the left. This is exactly what is done by the invariant *k-projective dimension* which was introduced in [8] and is abbreviated $k.\mathrm{pd}$.

It is less well-known that over (sufficiently nice) cochain DGAs, semi-free resolutions can be constructed in the opposite way: One starts at the *left* end of a DG-module and "kills" cohomology, and proceeds to the *right*. This is implicit in [4, appendix], and is made explicit in the appendix of the present paper. Thus, to measure the size of a semi-free resolution over a cochain DGA, one should measure how far it extends to the *right*. This is exactly what is done by the invariant *width* to be defined below in definition (1.1). So we can continue the dictionary

$$k.\mathrm{pd}(M) \qquad -\mathrm{width}(N)$$

(the sign on width is made necessary by our sign conventions).

By entirely similar considerations we can go on, adding to the dictionary

$$k.\mathrm{id}(M) \qquad -\mathrm{depth}(N)$$

$$\mathrm{depth}(M) \qquad -\sup\{\, i \mid \mathrm{H}^i N \neq 0 \,\}.$$

For definitions of the chain DGA invariants in the left-hand column, see [8]; for definitions of the cochain DGA invariants in the right-hand column, see definition (1.1).

The paper is hence organized as follows: Sections 1 and 2 contain the results obtained when applying the dictionary above to the results from [8] (but as we said, some of the proofs are new and cannot be obtained from the proofs in [8]). We have named the results in section 2 in parallel to the results in [8, sec. 2], as the *Auslander-Buchsbaum* and *Bass Formulae* and the *Gap Theorem*. Section 3 gives an application: It applies the Auslander-Buchsbaum Formula to the singular cochain DGA of a topological space, and recovers additivity of cohomological dimension on fibrations. Finally, the appendix does semi-free resolutions over cochain DGAs.

Let us finish the introduction with some blanket items.

(0.1) **Notation.** The notation of this paper is the same as in [8].

(0.2) **Blanket Setup.** For the rest of this paper, k is a field and R is a DGA over k satisfying:

- $R^i = 0$ for $i < 0$ (that is, R is a cochain DGA).
- $R^0 = k$ and $R^1 = 0$.
- $\dim_k \mathrm{H}R < \infty$.

Note that since we have $R/R^{\geq 1} \cong k$, we can view k as a DG-R-left-R-right-module concentrated in degree zero.

(0.3) **Duality.** As described in [6] and [7], there is a theory of dualizing DG-modules over DGAs, and a dualizing DG-module induces so-called dagger duality between DG-left-modules and DG-right-modules.

However, in the present simple circumstances, dagger duality turns out to degenerate into simple duality over the ground field k. Let us briefly state what we need about this duality:

Defining

$$M' = \operatorname{Hom}_k(M, k),$$

we have that $(-)'$ can be viewed as a functor from DG-R-left-modules to DG-R-right-modules, or vice versa. These functors are well-defined on the relevant derived categories, and when M and N are DG-R-left-modules with $\mathrm{H}M$ and $\mathrm{H}N$ finite dimensional over k, then we have

$$M'' \cong M \tag{1}$$

and

$$\operatorname{RHom}_{R^{\mathrm{opp}}}(N', M') \cong \operatorname{RHom}_R(M, N). \tag{2}$$

Also,

$$\inf\{\, i \mid \mathrm{H}^i(M') \neq 0 \,\} = -\sup\{\, i \mid \mathrm{H}^i M \neq 0 \,\}. \tag{3}$$

(0.4) **Minimal resolutions.** A DG-R-left-module F is called minimal if the differential ∂_F takes values inside $R^{\geq 1} \cdot F$.

A minimal semi-free resolution of a DG-module is a semi-free resolution F so that F is minimal; the notion of minimal K-projective resolution is defined similarly. Any minimal semi-free resolution is in particular a minimal K-projective resolution. For an existence theorem on minimal semi-free resolutions, see theorem (A.2).

Note that if F is minimal then $\operatorname{Hom}_R(F, k)$ and $k \otimes_R F$ have zero differentials. This is handy for computations.

1. Invariants

(1.1) **Definition.** For a DG-R-left-module M, we define width and depth by

$$\begin{aligned} \operatorname{width} M &= -\sup\{\, i \mid \mathrm{H}^i(k \overset{\mathrm{L}}{\otimes}_R M) \neq 0 \,\}, \\ \operatorname{depth} M &= \inf\{\, i \mid \mathrm{H}^i(\operatorname{RHom}_R(k, M)) \neq 0 \,\}. \end{aligned}$$

(1.2) **Definition.** For a DG-R-left-module M, we define the j'th Bass number and the j'th Betti number as

$$\begin{aligned} \mu^j(M) &= \dim_k \mathrm{H}^j(\operatorname{RHom}_R(k, M)), \\ \beta_j(M) &= \dim_k \mathrm{H}^j(\operatorname{RHom}_R(M, k)). \end{aligned}$$

(Note that $\mu^j(M)$ and $\beta_j(M)$ may well equal $+\infty$.)

Of the following two lemmas, the first is trivial, and the second only uses existence of minimal semi-free resolutions (theorem (A.2)).

(1.3) **Lemma.** *Let M be a DG-R-left-module which has a minimal K-projective resolution $F \xrightarrow{\simeq} M$ with*

$$F^\natural \cong \coprod_j \Sigma^j (R^\natural)^{(\beta_j)},$$

where $\natural$ indicates the graded modules obtained by forgetting the differentials. Then

$$\operatorname{width} M = \inf\{\, j \mid \beta_j \neq 0 \,\}.$$

(1.4) **Lemma.** *Let M be a DG-R-left-module with* $\mathrm{H}M$ *finite dimensional over k. Then*

$$\operatorname{width}_{R^{\mathrm{opp}}} M' = \operatorname{depth}_R M.$$

The proof of the following lemma is completely analogous to the proof of [8, lem. (1.6)], so we omit it.

(1.5) **Lemma.** *Let F be a K-projective DG-R-left-module with*

$$F^\natural \cong \coprod_{j \geq w} \Sigma^j (R^\natural)^{(\beta_j)},$$

and let N be a DG-R-left-module with $\mathrm{H}N$ *bounded to the left. Then*

$$\inf\{\, i \mid \mathrm{H}^i(\operatorname{Hom}_R(F,N)) \neq 0 \,\} \geq w + \inf\{\, i \mid \mathrm{H}^i N \neq 0 \,\}.$$

The formula in the following proposition can be obtained by applying the dictionary of the introduction to [8, prop. (1.8)], but the proof cannot.

(1.6) **Proposition.** *Let M and N be DG-R-left-modules with* $\mathrm{H}M$ *and* $\mathrm{H}N$ *bounded to the left. Suppose that each* $\mathrm{H}^i M$ *is finite dimensional over k, and that* $\operatorname{width} M$ *is finite. Then*

$$\inf\{\, i \mid \mathrm{H}^i(\operatorname{RHom}_R(M,N)) \neq 0 \,\} = \operatorname{width} M + \inf\{\, i \mid \mathrm{H}^i N \neq 0 \,\}.$$

Proof. We use theorem (A.2)(1) to construct a minimal semi-free resolution $F \xrightarrow{\simeq} M$, along with a semi-free filtration of F. If the semi-free filtration continued indefinitely, then $F^\natural$ would contain as summands arbitrarily high negative suspensions of $R^\natural$. But then lemma (1.3) would give $\operatorname{width} M = -\infty$, contradicting that $\operatorname{width} M$ is finite.

Hence the semi-free filtration must terminate, so we are in the situation of theorem (A.2)(3), so have a semi-split exact sequence

$$0 \to P \longrightarrow F \longrightarrow \Sigma^w R^{(\alpha)} \to 0 \tag{4}$$

with $\alpha \neq 0$, with P being K-projective, and with

$$P^{\natural} \cong \coprod_{j \geq w} \Sigma^j (R^{\natural})^{(\beta_j)}. \tag{5}$$

As the sequence (4) is semi-split, it gives

$$F^{\natural} \cong P^{\natural} \amalg \Sigma^w (R^{\natural})^{(\alpha)} \cong (\coprod_{j \geq w} \Sigma^j (R^{\natural})^{(\beta_j)}) \amalg \Sigma^w R^{(\alpha)}, \tag{6}$$

and lemma (1.3) hence gives

$$\operatorname{width} M = w. \tag{7}$$

If we introduce the notation

$$u = \inf\{\, i \mid \mathrm{H}^i N \neq 0 \,\}$$

and keep $\mathrm{RHom}_R(M, N) \cong \mathrm{Hom}_R(F, N)$ in mind, then the proposition's formula amounts to

$$\inf\{\, i \mid \mathrm{H}^i(\mathrm{Hom}_R(F, N)) \neq 0 \,\} = w + u.$$

Now, $\geq$ holds by lemma (1.5) and equation (6). So we must prove that $\leq$ holds. In other words, we must prove

$$\mathrm{H}^{w+u}(\mathrm{Hom}_R(F, N)) \neq 0.$$

For this, note that as the sequence (4) is semi-split, it remains semi-split when we apply $\mathrm{Hom}_R(-, N)$. This gives a semi-split exact sequence

$$0 \to \Sigma^{-w} N^{(\alpha)} \longrightarrow \mathrm{Hom}_R(F, N) \longrightarrow \mathrm{Hom}_R(P, N) \to 0. \tag{8}$$

From lemma (1.5) and equation (5) we get

$$\inf\{\, i \mid \mathrm{H}^i(\mathrm{Hom}_R(P, N)) \neq 0 \,\} \geq w + u.$$

Also,

$$\mathrm{H}^{w+u}(\Sigma^{-w} N^{(\alpha)}) = \mathrm{H}^u(N^{(\alpha)}) \neq 0.$$

But the long exact sequence induced by (8) contains

$$\mathrm{H}^{w+u-1}(\mathrm{Hom}_R(P, N)) \longrightarrow \mathrm{H}^{w+u}(\Sigma^{-w} N^{(\alpha)}) \longrightarrow \mathrm{H}^{w+u}(\mathrm{Hom}_R(F, N)),$$

and we have just proved that the first term is zero and the second non-zero, so it follows that the third term is non-zero, which is what we wanted. $\square$

2. Identities

(2.1) **Theorem (Cochain Auslander-Buchsbaum Formula).** *Let M be a DG-R-left-module with* $\mathrm{H}M$ *finite dimensional over k, and suppose that* $\operatorname{width} M$ *is finite. Then*

$$\operatorname{width} M + \sup\{\, i \mid \mathrm{H}^i M \neq 0 \,\} = \sup\{\, i \mid \mathrm{H}^i R \neq 0 \,\}.$$

Proof. We can compute,

$$\begin{aligned}
\sup\{\, i \mid \mathrm{H}^i M \neq 0 \,\} &\overset{\text{(a)}}{=} -\inf\{\, i \mid \mathrm{H}^i(M') \neq 0 \,\} \\
&= -\inf\{\, i \mid \mathrm{H}^i(\mathrm{RHom}_{R^{\mathrm{opp}}}(R, M')) \neq 0 \,\} \\
&\overset{\text{(b)}}{=} -\inf\{\, i \mid \mathrm{H}^i(\mathrm{RHom}_R(M'', R')) \neq 0 \,\} \\
&\overset{\text{(c)}}{=} -\inf\{\, i \mid \mathrm{H}^i(\mathrm{RHom}_R(M, R')) \neq 0 \,\} \\
&\overset{\text{(d)}}{=} -(\operatorname{width} M + \inf\{\, i \mid \mathrm{H}^i(R') \neq 0 \,\}) \\
&\overset{\text{(e)}}{=} -(\operatorname{width} M - \sup\{\, i \mid \mathrm{H}^i R \neq 0 \,\}),
\end{aligned}$$

where (a) and (e) are by equation (3), (b) is by equation (2), (c) is by equation (1), and (d) is by proposition (1.6). ☐

(2.2) **Theorem (Cochain Bass Formula).** *Let N be a DG-R-left-module with* $\mathrm{H}N$ *finite dimensional over k, and suppose that* $\operatorname{depth} N$ *is finite. Then*

$$\operatorname{depth} N - \inf\{\, i \mid \mathrm{H}^i N \neq 0 \,\} = \sup\{\, i \mid \mathrm{H}^i R \neq 0 \,\}.$$

Proof. From lemma (1.4) we have

$$\operatorname{width}_{R^{\mathrm{opp}}} N' = \operatorname{depth}_R N,$$

forcing $\operatorname{width}_{R^{\mathrm{opp}}} N'$ to be finite. So

$$\begin{aligned}
\operatorname{depth}_R N &= \operatorname{width}_{R^{\mathrm{opp}}} N' \\
&\overset{\text{(a)}}{=} \sup\{\, i \mid \mathrm{H}^i R \neq 0 \,\} - \sup\{\, i \mid \mathrm{H}^i(N') \neq 0 \,\} \\
&\overset{\text{(b)}}{=} \sup\{\, i \mid \mathrm{H}^i R \neq 0 \,\} + \inf\{\, i \mid \mathrm{H}^i N \neq 0 \,\},
\end{aligned}$$

where (a) is by the Cochain Auslander-Buchsbaum Formula (theorem (2.1)), and (b) is by equation (3). ☐

(2.3) **Theorem (Cochain Gap Theorem).** *Let M be a DG-R-left-module with* $\mathrm{H}M$ *finite dimensional over k, and let g be an integer satisfying* $g > \sup\{\, i \mid \mathrm{H}^i R \neq 0 \,\}$. *Assume that the sequence of Bass numbers of M has a gap of length g, in the sense that there exists an integer ℓ such that*

- $\mu^\ell(M) \neq 0$.
- $\mu^{\ell+1}(M) = \cdots = \mu^{\ell+g}(M) = 0$.
- $\mu^{\ell+g+1}(M) \neq 0$.

Then we have

$$\sup\{\, i \mid \mathrm{H}^i M \neq 0 \,\} - \inf\{\, i \mid \mathrm{H}^i M \neq 0 \,\} \geq g + 1.$$

Proof. The proof is almost verbatim to the proof of the Chain Gap Theorem, [8, thm. (2.5)], the main difference being to use proposition (1.6) in place of [8, prop. (1.8)]. ☐

3. A topological application

(3.1) **Fibrations of topological spaces.** Let k be a field and let

$$F \longrightarrow X \xrightarrow{p} Y$$

be a fibration with Y simply connected (see [5, chp. 2]). Assume that $\mathrm{H}^*(F;k)$, $\mathrm{H}^*(X;k)$, and $\mathrm{H}^*(Y;k)$ are finite dimensional over k.

Since Y is simply connected with $\dim_k \mathrm{H}^*(Y;k)$ finite, it follows from [4, proof of thm. 3.6] that $\mathrm{C}^*(Y;k)$ can be replaced with an equivalent cochain DGA, R, which falls under setup (0.2). Hence the Cochain Auslander-Buchsbaum Formula (theorem (2.1)) holds over $\mathrm{C}^*(Y;k)$. We will use it on the DG-$\mathrm{C}^*(Y;k)$-left-module $\mathrm{C}^*(X;k)$.

To see that this is possible, note that by assumption

$$\mathrm{H}(\mathrm{C}^*(X;k)) = \mathrm{H}^*(X;k)$$

is finite dimensional over k. Next note that

$$k \overset{\mathrm{L}}{\otimes}_{\mathrm{C}^*(Y;k)} \mathrm{C}^*(X;k) \cong \mathrm{C}^*(F;k)$$

by [5, thm. 7.5], so we may compute $\mathrm{width}_{\mathrm{C}^*(Y;k)} \mathrm{C}^*(X;k)$ as follows:

$$\begin{aligned}
\mathrm{width}_{\mathrm{C}^*(Y;k)} \mathrm{C}^*(X;k) &= -\sup\{\, i \mid \mathrm{H}^i(k \overset{\mathrm{L}}{\otimes}_{\mathrm{C}^*(Y;k)} \mathrm{C}^*(X;k)) \neq 0 \,\} \\
&= -\sup\{\, i \mid \mathrm{H}^i(\mathrm{C}^*(F;k)) \neq 0 \,\} \\
&= -\sup\{\, i \mid \mathrm{H}^i(F;k) \neq 0 \,\}.
\end{aligned}$$

This is finite by the assumptions, so the Cochain Auslander-Buchsbaum Formula can be applied to the DG-$\mathrm{C}^*(Y;k)$-left-module $\mathrm{C}^*(X;k)$.

Doing so, we get

$$\begin{aligned}
\mathrm{width}_{\mathrm{C}^*(Y;k)} \mathrm{C}^*(X;k) + \sup\{\, i \mid \mathrm{H}^i(\mathrm{C}^*(X;k)) \neq 0 \,\} = \\
\sup\{\, i \mid \mathrm{H}^i(\mathrm{C}^*(Y;k)) \neq 0 \,\}.
\end{aligned}$$

Inserting the above expression for $\mathrm{width}_{\mathrm{C}^*(Y;k)} \mathrm{C}^*(X;k)$, this becomes

$$-\sup\{\, i \mid \mathrm{H}^i(F;k) \neq 0 \,\} + \sup\{\, i \mid \mathrm{H}^i(X;k) \neq 0 \,\} = \sup\{\, i \mid \mathrm{H}^i(Y;k) \neq 0 \,\},$$

that is,

$$\begin{aligned}
\sup\{\, i \mid \mathrm{H}^i(X;k) \neq 0 \,\} = \sup\{\, i \mid \mathrm{H}^i(F;k) \neq 0 \,\} \\
+ \sup\{\, i \mid \mathrm{H}^i(Y;k) \neq 0 \,\}. \qquad (9)
\end{aligned}$$

This can also be written

$$\mathrm{cd}_k X = \mathrm{cd}_k F + \mathrm{cd}_k Y$$

if we follow [3] in defining the *cohomological dimension* $\mathrm{cd}_k X$ of a topological space X with respect to k by

$$\mathrm{cd}_k X = \sup\{\, i \mid \mathrm{H}^i(X;k) \neq 0 \,\}.$$

In other words, we have recovered the result that cd_k is additive on fibrations. See also [3, prop. 6.14].

Appendix A. Semi-free resolutions

We shall prove a result on minimal semi-free resolutions over R, our standing DGA. The first results of this type were given without proof in [4, appendix], but we need some details which are not in that source.

The following remark was made by Apassov [1, lemma], in a slightly weaker form.

(A.1) **Remark.** Let Q be a DGA with $\mathrm{H}^1 Q = 0$, let $L \xrightarrow{\alpha} M$ be a morphism of DG-Q-left-modules, and let n be in $\mathbb{N}$.

There is a "canonical" way of doctoring $\mathrm{H}^n\alpha$ to become a bijection: Let Y be a set of n-cocycles in L so that the cohomology classes of the elements of Y generate $\operatorname{Ker}\mathrm{H}^n\alpha$, and pick a system $\{m_y\}_{y \in Y}$ in M^{n-1} so that $\alpha(y) = \partial^M(m_y)$ for each $y \in Y$. Define $\Sigma^{-n}Q^{(Y)} \xrightarrow{\Delta} L$ by $\Sigma^{-n}1_y \longmapsto y$, where $\Sigma^{-n}1_y$ is the generator of the y'th copy of $\Sigma^{-n}Q$.

Now let F be the mapping cone of Δ. There is a mapping cone short exact sequence which is semi-split,

$$0 \to L \longrightarrow F \longrightarrow \Sigma^{-n+1}Q^{(Y)} \to 0,$$

in particular

$$F^\natural = (\Sigma^{-n+1}(Q^\natural)^{(Y)}) \amalg L^\natural.$$

Define a morphism of DG-Q-left-modules $F \xrightarrow{\widetilde{\alpha}} M$ by

$$\widetilde{\alpha}(\sum_y r_y(\Sigma^{-n+1}1_y), \ell) = \sum_y r_y m_y + \alpha(\ell).$$

Now $\widetilde{\alpha}$ extends α and by a diagram chase using the condition $\mathrm{H}^1 Q = 0$ one checks that $\mathrm{H}^n\widetilde{\alpha}$ is injective, and that if $\mathrm{H}^n\alpha$ is surjective then $\mathrm{H}^n\widetilde{\alpha}$ is even bijective.

(A.2) **Theorem.** *Let M be a DG-R-left-module with $\mathrm{H}M$ non-zero and bounded to the left, and each $\mathrm{H}^i M$ finite dimensional over k. Set $u = \inf\{\, i \mid \mathrm{H}^i M \neq 0 \,\}$.*

(1) *We can construct a minimal semi-free resolution $F \xrightarrow{\simeq} M$ which has a semi-free filtration with quotients as indicated,*

$$\begin{tikzcd}[column sep=small]
& \Sigma^{-u}R^{(\gamma_0)} \arrow[dl, dash] \arrow[dr, dash] & & & & \Sigma^{-u}R^{(\gamma_1)} \arrow[dl, dash] \arrow[dr, dash] & & & & \Sigma^{-u-1}R^{(\gamma_2)} \arrow[dl, dash] \arrow[dr, dash] & & & \cdots & & \\
0 & \subseteq & F(0) & \subseteq & L(1) & \subseteq & F(1) & \subseteq & L(2) & \subseteq & F(2) & \subseteq & \cdots & \subseteq & F, \\
& & & \Sigma^{-u-1}R^{(\delta_1)} \arrow[ul, dash] \arrow[ur, dash] & & & & \Sigma^{-u-2}R^{(\delta_2)} \arrow[ul, dash] \arrow[ur, dash] & & & & & \cdots & &
\end{tikzcd}$$

where each γ_j and each δ_j is finite.

(2) *In the construction from* (1), *we can write* $F^\natural$ *as*

$$F^\natural \cong \coprod_{j \leq -u} \Sigma^j (R^\natural)^{(\beta_j)},$$

where each β_j *is finite.*

(3) *In the construction from* (1), *if the filtration terminates, then there exists a semi-split exact sequence of DG-R-left-modules*

$$0 \to P \longrightarrow F \longrightarrow \Sigma^w R^{(\alpha)} \to 0 \tag{10}$$

with $\alpha \neq 0$, *with* P *being* K*-projective, and with*

$$P^\natural \cong \coprod_{j \geq w} \Sigma^j (R^\natural)^{(\epsilon_j)}. \tag{11}$$

Proof. All direct sums in the proof will be finite because each $\mathrm{H}^i M$ is finite dimensional over k. Without loss of generality, we can assume $u = 0$ throughout.

(1) We construct the semi-free filtration of F by induction, and define F as the union.

To start, we construct a morphism $R^{(\gamma_0)} \longrightarrow M$ so that $\mathrm{H}^0(R^{(\gamma_0)}) \longrightarrow \mathrm{H}^0 M$ is an isomorphism, and set $F(0) = R^{(\gamma_0)}$. This is possible because we work over the field $R^0 = k$.

Suppose now that $F(n-1)$ has been constructed along with a morphism $F(n-1) \longrightarrow M$ so that $\mathrm{H}^i(F(n-1)) \longrightarrow \mathrm{H}^i M$ is an isomorphism for $i = 0, \ldots, n-1$. Suppose also that $F(n-1)$ is minimal.

First, we construct $L(n)$ by adding some $\Sigma^{-n} R^{(\delta_n)}$ to $F(n-1)$ so that there is a morphism

$$L(n) = \Sigma^{-n} R^{(\delta_n)} \amalg F(n-1) \xrightarrow{\alpha} M$$

which extends the previously constructed morphism $F(n-1) \longrightarrow M$ and which induces a surjection $\mathrm{H}^n(L(n)) \xrightarrow{\mathrm{H}^n \alpha} \mathrm{H}^n M$. We do this in a way so that the image under $\mathrm{H}^n \alpha$ of $\mathrm{H}^n(\Sigma^{-n} R^{(\delta_n)})$ is a complement to the image of $\mathrm{H}^n(F(n-1))$, whence

$$\operatorname{Ker} \mathrm{H}^n \alpha \text{ is contained in the summand } \mathrm{H}^n(F(n-1)) \text{ of } \mathrm{H}^n(L(n)).$$

Again, this is possible because we work over the field $R^0 = k$. It is clear that $L(n)$ is minimal and that $\mathrm{H}^i \alpha$ is still an isomorphism for $i = 0, \ldots, n-1$.

Secondly, we construct $F(n)$ by using remark (A.1) on α. We pick the set of cocycles Y in a way so that

$$\Sigma^{-n} R^{(Y)} \xrightarrow{\Delta} L(n) \text{ has } \mathrm{H}^n \Delta \text{ injective},$$

which is possible because we work over the field $R^0 = k$. Also, since the cocycles in Y have their cohomology classes in $\operatorname{Ker} \mathrm{H}^n \alpha$ which is contained in the summand $\mathrm{H}^n(F(n-1))$ of $\mathrm{H}^n(L(n))$, we can choose the set Y in the summand $F(n-1)$ of $L(n)$. Hence, as Δ maps the generators of $\Sigma^{-n}R^{(Y)}$ to the cocycles in Y, we can assume that

$$\Delta \text{ maps into the summand } F(n-1) \text{ of } L(n).$$

Denoting $\Sigma^{-n}R^{(Y)}$ by $\Sigma^{-n}R^{(\gamma_n)}$, remark (A.1) now gives a mapping cone short exact sequence which is semi-split,

$$0 \rightarrow L(n) \longrightarrow F(n) \longrightarrow \Sigma^{-n+1}R^{(\gamma_n)} \rightarrow 0, \tag{12}$$

and a morphism $F(n) \xrightarrow{\widetilde{\alpha}} M$ extending α so that $\mathrm{H}^n\widetilde{\alpha}$ is an isomorphism. Using that $\mathrm{H}^n\Delta$ is injective, it is easy to see from the long exact sequence that $\mathrm{H}^i\widetilde{\alpha}$ is still an isomorphism for $i = 0, \ldots, n-1$.

So the only question is whether $F(n)$ is minimal. Now, $F(n)$ is constructed as the mapping cone of the morphism $\Sigma^{-n}R^{(\gamma_n)} \xrightarrow{\Delta} L(n)$ between two minimal DG-modules, so the only potential problem is the cross term in the differential. But the cross term equals Δ which maps into the summand $F(n-1)$ of $L(n)$. In particular, the generators of $\Sigma^{-n}R^{(\gamma_n)}$ map to the part of $F(n-1)$ which is in cohomological degree n. And the filtration leading up to $F(n-1)$ has quotients of the form $\Sigma^{\ell}R^{(\alpha)}$ with $\ell \geq -n+1$, so the image of Δ is in $R^{\geq 1} \cdot F(n-1)$.

(2) This is immediate from the semi-free filtration of F given in (1).

(3) Recall that we are assuming $u = 0$. Consider the iterative construction given in the proof of (1). For the filtration of F to terminate means that from a certain step, the iterations yield $F(n-1) = F(n)$. Let us consider the last iteration with $F(n-1) \neq F(n)$,

$$\begin{tikzcd}
 & & & & & \Sigma^{-n+1}R^{(\gamma_n)} \arrow[dl, dash] \arrow[dr, dash] & & & \\
\cdots & \subseteq & F(n-1) & \subseteq & L(n) & \subseteq & F(n) & = & F; \\
 & & & \Sigma^{-n}R^{(\delta_n)} \arrow[ul, dash] \arrow[ur, dash] & & & & &
\end{tikzcd}$$

here γ_n and δ_n are not both zero.

Now there are two possibilities: Either $\delta_n = 0$ or $\delta_n \neq 0$.

The case $\delta_n = 0$: Here we have $\gamma_n \neq 0$. From the construction in the proof of (1) we have the mapping cone short exact sequence (12). Using $\delta_n = 0$ and $\gamma_n \neq 0$, it is easy to check that (12) can be used as the sequence (10).

The case $\delta_n \neq 0$: We still have the sequence (12), but it can no longer be used as (10) because $\delta_n \neq 0$ introduces a summand $\Sigma^{-n}(R^{\natural})^{(\delta_n)}$ in $L(n)^{\natural}$, preventing $L(n)$ from being used as P in (10) because of the condition in equation (11).

We hence come up with a different idea: By construction we have

$$L(n) = \Sigma^{-n}R^{(\delta_n)} \amalg F(n-1);$$

that is, $\Sigma^{-n}R^{(\delta_n)}$ is a direct summand in $L(n)$. And in fact, $\Sigma^{-n}R^{(\delta_n)}$ remains a direct summand in $F(n)$:

From the construction in the proof of (1), we have that $F(n)$ is constructed as the mapping cone of the map $\Sigma^{-n}R^{(\gamma_n)} \xrightarrow{\Delta} L(n)$. Hence

$$F(n)^\natural \cong \Sigma^{-n+1}(R^\natural)^{(\gamma_n)} \amalg L(n)^\natural.$$

The differential of $F(n)$ is constructed from the differentials of $\Sigma^{-n+1}R^{(\gamma_n)}$ and $L(n)$, and from a cross term equal to Δ. But now recall that by construction, Δ maps into the summand $F(n-1)$ of $L(n)$. Writing $F(n)^\natural$ as

$$\begin{aligned} F(n)^\natural &= \Sigma^{-n+1}(R^\natural)^{(\gamma_n)} \amalg L(n)^\natural \\ &= \Sigma^{-n+1}(R^\natural)^{(\gamma_n)} \amalg \Sigma^{-n}(R^\natural)^{(\delta_n)} \amalg F(n-1)^\natural, \end{aligned}$$

it follows that the differential of $F(n)$, viewed as a map on $F(n)^\natural$, cannot map between $\Sigma^{-n}(R^\natural)^{(\delta_n)}$ and the rest of $F(n)^\natural$. Hence $\Sigma^{-n}R^{(\delta_n)}$ remains a direct summand in $F(n)$.

But then there exists a split exact sequence

$$0 \to P \longrightarrow F(n) \longrightarrow \Sigma^{-n}R^{(\delta_n)} \to 0,$$

where $P^\natural \cong \Sigma^{-n+1}(R^\natural)^{(\gamma_n)} \amalg F(n-1)^\natural$. It is easy to check that this can be used as the sequence (10). $\square$

References

[1] D. Apassov, *Homological dimensions over differential graded rings*, pp. 25–39 in "Complexes and Differential Graded Modules", ph.d. thesis, Lund University, 1999.

[2] L. L. Avramov and S. Halperin, *Through the looking glass: a dictionary between rational homotopy theory and local algebra*, pp. 1–27 in "Algebra, algebraic topology and their interactions" (proceedings of the conference in Stockholm, 1983), Lecture Notes in Math., Vol. 1183, Springer, Berlin, 1986.

[3] W. G. Dwyer and W. G. Wilkerson, *Homotopy fixed-point methods for Lie groups and finite loop spaces*, Ann. of Math. (2) **139** (1994), no. 2, 395–442.

[4] Y. Félix, S. Halperin, and J.-C. Thomas, *Gorenstein spaces*, Adv. Math. **71** (1988), 92–112.

[5] ———, ———, and ———, "Rational Homotopy Theory", Grad. Text in Math., Vol 205, Springer, Berlin, 2000.

[6] A. Frankild, S. Iyengar, and P. Jørgensen, Dualizing DG modules and Gorenstein DG algebras, preprint (2002).

[7] ——— and P. Jørgensen, Dualizing DG-modules for Differential Graded Algebras, preprint (2001).

[8] ——— and ———, Homological identitites for Differential Graded Algebras, preprint (2001).

Frankild: Matematisk Afdeling, Universitetsparken 5, 2100 København Ø, DK–Denmark
E-mail address: frankild@math.ku.dk

Jørgensen: Danish National Library of Science and Medicine, Nørre Allé 49, 2200 København N, DK–Denmark
E-mail address: pej@dnlb.dk