



# Forward and backward methods in life insurance: Theoretical advances and applications

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# Preface

This thesis has been prepared in partial fulfillment of the requirements for the PhD degree at the Department of Mathematical Sciences, Faculty of Science, University of Copenhagen. The work was carried out between August 2023 and August 2026 in association with the project frame InterAct, an extensive agreement of collaboration within research and education between the Department of Mathematical Sciences at the University of Copenhagen and, as of August 2026, the seven Danish insurance and pension companies AP Pension, Danica Pension, Industriens Pension, PFA Pension, PensionDanmark, Tryg, and Velliv (listed in alphabetical order). The topics covered by this thesis were determined jointly by the industrial partners, the university, and myself. My supervisors were Professor Mogens Steffensen and PhD Christian Furrer (Associate Professor at the University of Copenhagen until 31st of January 2026; thereafter part-time lecturer).

This thesis consists of an introduction and four separate manuscripts. Each of the manuscripts constitutes a scientific contribution in its own right and they can be read independently. As a consequence, terminology and notation may vary significantly across the different chapters of this thesis.

## **Declaration on the use of generative AI**

Generative AI tools (specifically Gemini 3.6 Flash and older, free versions) were used during the preparation of this thesis solely for information acquisition and literature search, akin to a search engine. All analysis, synthesis and writing were performed by myself and I take full responsibility for the final content.

## **Acknowledgments**

This thesis is the result of three fulfilling years as a PhD student. I would like to thank my supervisors Mogens Steffensen and Christian Furrer for their supportive supervision and for giving me the opportunity to be one of the two inaugural PhD students within the project frame InterAct. Christian, you have taken a great interest in my personal and professional development and you have always been available when I needed guidance, advice, or technical sparring even though you

changed tack in your career. For that I am deeply grateful. I have greatly enjoyed learning from you (and occasionally with you) and I look forward to continue our collaboration. Mogens, your reliability and ability to bring out the essential has been a great help and inspiration, especially when scoping a research project or preparing a course, and I hope to continue our discussions in the future.

I would also like to thank the industrial partners of InterAct for providing me with fresh perspectives. Your input at bilateral meetings or at InterAct days has given me a deeper insight and understanding of the research problems discussed in this thesis. It has also been a great honour to return the favour to those of you who participated the CPD course ‘Design and Risk Management in Life Insurance’. I have thoroughly enjoyed teaching you and I have learned a lot in the process.

In the autumn of 2025 I had the great pleasure of visiting Sascha Desmettre at the Johannes Kepler Universität in Linz. I would like to thank Sascha and his colleagues at the Institute of Financial Mathematics and Applied Number Theory for a wonderful time in Linz and the opportunity to present parts of my research at the PhD Seminar in Mathematics. Sascha, I have enjoyed our collaboration and I hope that the future holds opportunities for further discussions or collaborations.

My time as a PhD student would have been a lot less entertaining if it had been without my fellow PhD students at the Department of Mathematical Sciences. I would like to thank Gnaverhytten and The Worming Office for their steady supply of good vibes and funny memes, Rasmus for sharing his inexhaustible supply of high-quality tea with me, and Tessa for being the other inaugural InterAct PhD student.

I thank my parents Annegret and Felix and my brother Frederick for their eternal support and for removing my head from the books from time to time regardless of what part of the world life brings them to. Last but not least I would like to thank my partner Freja for her love, support, patience, and for motivating me to achieve more by working less.

Philipp C. Hornung  
Copenhagen, August 2026

This final version of my thesis differs only from the previous version submitted to the PhD School on 6 August 2026 in that a few typographical errors have been corrected.

Philipp C. Hornung  
Copenhagen, October 2026

# List of papers

Besides Chapter 1 which has been prepared specifically for this thesis, each of the four remaining chapters presents a self-contained and submitted or already published manuscript according to the following scheme:

**Chapter 2:** Hornung, P.C. (2026). Mean-field approximations in insurance. *Scandinavian Actuarial Journal*. DOI: 10.1080/03461238.2026.2693555.

**Chapter 3:** Furrer, C. and P.C. Hornung (2026). Disability insurance with collective health claims: A mean-field approach. *Insurance: Mathematics and Economics* **129**. DOI: 10.1016/j.insmatheco.2026.103266.

**Chapter 4:** Desmettre, S. and P.C. Hornung (2026). *Robust control for marked point processes under transition rate uncertainty*. ARXIV: 2607.16935.

**Chapter 5:** Hornung, P.C. and M. Steffensen (2025). Optimal smooth consumption and its trade-offs. *Mathematical Methods of Operations Research* **102**(1), pp. 29–78. DOI: 10.1007/s00186-025-00902-6.

There might be minor differences between the contents of a chapter and the corresponding manuscript.



# Abstract

This thesis contains a series of separate investigations related to the use of forward and backward methods for the purpose of valuation and design of multi-state life insurance contracts. While Chapter 1 provides some background and an overview of the contributions of this thesis, the Chapters 2-5 each contain one investigation.

In Chapter 2 we provide a mathematical framework for the valuation of insurance contracts with collective effects using mean-field approximations. Due to the collective effects the individuals of a cohort are no longer independent and the valuation of an individual insurance contract requires the solution of a high-dimensional system of linear forward integro-differential equations. This is computationally infeasible and can be circumvented by using a mean-field model, since this replaces the high-dimensional system of linear forward equations with a low-dimensional, non-linear one. Under suitable regularity conditions we derive a law of large numbers and show that for large cohorts the value of an individual insurance contract converges towards its mean-field counterpart.

In Chapter 3 we expand the classical semi-Markov disability model with individual and collective health claims. This is motivated by its application to group experience rating where the inclusion of collective health claims can improve predictive power. The computational challenges introduced by the collective health claims are addressed by using a mean-field approximation, courtesy of Chapter 2. We adapt an existing numerical method to the mean-field model's system of non-linear forward equations and compare the mean-field approximation to the naïve Monte Carlo method in a practice-oriented simulation study. We conclude that the mean-field approximation is a necessary and rather efficient alternative.

In Chapter 4 we consider a novel robust utility maximisation problem under transition-rate uncertainty within the class of non-Markovian multi-state models, derive a martingale optimality principle, and prove a novel existence and uniqueness result for a class of worst-case backward stochastic differential equations (BSDEs). We use these results to solve the robust problem in two specific cases. First, we show that the worst- and best-case reserves of an insurance contract with reserve-dependent payments are characterised by a worst-case BSDE, thus combining

current backward methods for reserve-depedendent payments with those of worst-case reserves, while generalising them to non-Markovian models. Second, we solve a novel robust consumption-insurance problem with power utility preferences whose solution is characterised by a worst-case BSDE.

In Chapter 5 we discuss the preferences for a smooth consumption rate (with the purpose of informing the design of smooth pension products) in terms of the components (initial value, drift, volatility) of the consumption rate. We distinguish between pathwise smoothness (variation over time along one path) quantified by the expected quadratic variation and pointwise smoothness (variation at fixed time across paths) quantified by the variance and qualitatively identify the following trade-offs. First, increasing pathwise smoothness requires a reduction of the initial value, the drift or the value of the bequest. Second, increasing pathwise smoothness without reducing risky investments leads to an increase in trend risk and a decrease in diffusive risk. Finally we obtain explicit solutions to a novel consumption-investment problem, allowing us to discuss these trade-offs quantitatively.

# Resumé

Denne afhandling indeholder en række separate undersøgelser vedrørende anvendelsen af forlæns- og baglæns metoder med henblik på værdiansættelse og design af flertilstands livsforsikringskontrakter. Mens Kapitel 1 indeholder en introduktion og et overblik over afhandlingens bidrag, så indeholder Kapitlerne 2-5 hver især én undersøgelse.

I Kapitel 2 præsenterer vi en matematisk ramme for værdiansættelse af forsikringskontrakter med kollektive effekter ved hjælp af mean-field approksimationer. Introduktionen af de kollektive effekter medfører, at de enkelte individer i en kohorte ikke længere er uafhængige, og værdiansættelsen af en individuel forsikringskontrakt kræver derfor den numeriske løsning af et højdimensionelt system af lineære forlæns integro-differentialligninger. Dette er beregningsmæssigt krævende, og kan omgås ved at anvende en mean-field model, hvilket erstatter det højdimensionelle system af lineære forlæns ligninger med et lavdimensionelt, ikke-lineært system. Under passende regularitetsbetingelser udleder vi en store tals lov og viser, at værdien af en individuel forsikringskontrakt for store kohorter konvergerer mod dens mean-field ækvivalent.

I Kapitel 3 udvider vi den klassiske semi-Markov invalidemodell til også at inddrage individuelle og kollektive sundhedsskader. Dette er motiveret af modellens potentielle anvendelse inden for gruppebaseret erfaringstarifiering, hvor inddragelsen af kollektive sundhedsskader kan forbedre forudsigelsesevnen. De beregningsmæssige udfordringer, som inklusionen af de kollektive sundhedsskader fører til, løses ved hjælp af en mean-field approksimation, hentet fra Kapitel 2. Vi modificerer en eksisterende numerisk metode, så den kan bruges til mean-field modellens system af ikke-lineære forlæns ligninger, og vi sammenligner mean-field approksimationen med den naive Monte Carlo metode i et praksisorienteret simulationsstudie. Vi konkluderer, at mean-field approksimationen er et nødvendigt og effektivt alternativ.

I Kapitel 4 betragter vi et nyt robust nyttemaksimeringsproblem for ikke-Markovianske flertilstandsmodeller med ukendte overgangsrater, vi udleder et tilhørende martingaloptymalitetsprincip, og beviser et nyt eksistens- og entydighedsresultat for en klasse af worst-case baglæns stokastiske differentiaalligninger

(BSDE'er). Vi anvender disse resultater til at løse det robuste problem i to specifikke tilfælde. For det første viser vi, at worst-case og best-case reserve af en forsikringskontrakt med reserveafhængige betalinger karakteriseres af en worst-case BSDE. Dermed kombinerer vi de eksisterende baglæns metoder for reserveafhængige betalinger med de eksisterende baglæns metoder for worst-case reserver, samtidig med at vi generaliserer dem til ikke-Markovianske modeller. For det andet løser vi et nyt robust forbrugs- og forsikringsproblem med potensnyttepræferencer, hvis løsning ligeledes karakteriseres af en worst-case BSDE.

I Kapitel 5 diskuterer vi præferencer for udjævning af finansiell risiko i forbrugsraten (med henblik på at informere designet af udjævningsmekanismer for pensionsudbetalinger) baseret på forbrugsratens komponenter (startværdi, drift, volatilitet). Vi skelner mellem stivis variabilitet (variation over tid langs en bestemt sti), kvantificeret ved den forventede kvadratiske variation, og punktvis variabilitet (variation på et fast tidspunkt på tværs af stier), kvantificeret ved variansen, og identificerer kvalitativt følgende trade-offs. For at reducere stivis variabilitet kræves en reduktion af startværdien, driften eller værdien af arven. Hvis man derudover ønsker reduceret stivis variabilitet uden at reducere risikofyldte investeringer fører det til en stigning i trendrisikoen og et fald i diffusivrisikoen. Slutendeligt finder vi eksplicitte løsninger til et nyt forbrugs- og investeringsproblem, hvilket giver os mulighed for at diskutere disse trade-offs kvantitativt.

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# Chapter 1

## Introduction

This thesis contains a series of investigations concerned with the design and valuation of multi-state life insurance contracts (henceforth called insurance contracts) using forward and backward methods. This introductory chapter provides the background and context for the topics covered by this thesis and describes the contributions of the remaining Chapters 2-5. Section 1.1 starts with the background and consists of three subsections. Subsection 1.1.1 describes how insurance contracts carrying both actuarial and financial risk can be valued using either forward or backward methods, while discussing either method's advantages and disadvantages. Subsection 1.1.2 describes how optimal consumption-investment-insurance problems can be utilised to inspire different product designs in accordance with different policyholder preferences and discusses the insurance company's constraints for a viable product design. As the Chapters 2-4 focus on actuarial risk modelled by jump processes, Subsection 1.1.3 summarises how jump processes can be constructed from marked point processes or random counting measures. Finally, the contributions of each chapter are described Section 1.2.

### 1.1 Background

When an insurance company wants to offer an insurance contract, which pays benefits contingent on the biometric (alive, dead, disabled, etc.) or economic (unemployed, retired, etc.) state of the policyholder and possibly the returns of the financial market, then the insurance company has two main concerns. First they have to design the contract and then they have to determine the value of the contract for pricing and reserving purposes. In order to determine the value of the contract, the insurance company should have a thorough understanding of the different types of risk that the contract carries, they should be able to model those risks accurately, and they should be able to compute the value efficiently. When designing the contract, the company has to strike the right balance between the preferences of the target customer and their own constraints. This requires to

understand the interplay between different types of individual preferences (risk, flexibility) what product features they imply (guarantees, policyholder options), and what valuation methods can be used to value those features accurately and efficiently.

To formalise our discussion, we model the state of the policyholder as a jump process  $Z$  with finite state-space  $\mathcal{J}$  and the financial market by a  $d + 1$ -dimensional semi-martingale  $S = (S_0, \dots, S_d)$ , which can be interpreted as the asset prices of  $d + 1$  financial assets. Both  $S$  and  $Z$  are defined on an underlying probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and for simplicity we assume that they are independent. By  $\mathbb{F}^S = (\mathcal{F}_t^S)_{t \geq 0}$  we denote the financial information generated by  $S$  and by  $\mathbb{F}^Z = (\mathcal{F}_t^Z)_{t \geq 0}$  we denote the actuarial information generated by the state of the policyholder  $Z$ , while  $\mathbb{F} = \mathbb{F}^S \vee \mathbb{F}^Z$  is the total information. The asset  $S_0$  is the numeraire, which we assume to have the form

$$S_0(dt) = r(t)S_0(t)dt, \quad S_0(0) = 1,$$

for a sufficiently regular  $\mathbb{F}$ -optional interest rate process  $r$ .

The (cumulative) contractual payments  $B$  of an insurance contract equal the difference  $B^+ - B^-$  of two non-negative, non-decreasing processes accounting for (cumulative) benefits  $B^+$  and (cumulative) premiums  $B^-$ . Both (thus also  $B$ ) are typically of the form

$$B(dt) = \sum_{i \in \mathcal{J}} \mathbf{1}_{Z(t)=i} b_i(t)dt + \sum_{i,j:i \neq j} b_{ij}(t)N_{ij}(dt), \quad B(0) \in \mathbb{R},$$

where  $(b_i)_{i \in \mathcal{J}}$  are sufficiently regular,  $\mathbb{F}$ -optional sojourn payment rates that are paid continuously while the insured is in state  $i$ , and  $(b_{ij})_{i,j:i \neq j}$  are sufficiently regular,  $\mathbb{F}$ -predictable transition payments paid upon transition from  $i$  to  $j$ . The processes  $N_{ij}$  count the number of transitions of  $Z$  from  $i$  to  $j$ . The policyholder account associated with the insurance contract is given by

$$\begin{aligned} dX(t) &= h(t)dS(t) - \underbrace{(B^+(dt) - B^-(dt))}_{=B(dt)} - \tilde{B}(dt) \\ X(0) &= -(B(0) + \tilde{B}(0)), \end{aligned} \tag{1.1.1}$$

where the first term describes the investment returns of investing in the financial market  $S$  according to the investment strategy  $h$ , the process  $B$  describes the contractual payments, while the process  $\tilde{B}$  consists of (cumulative) payments to and from the equity of the insurance company. The process  $\tilde{B}$  thus determines the profit/loss of the insurance company associated with the contract. The problem of design is to determine how exactly the contractual elements  $(h, B, \tilde{B})$  and the state space  $\mathcal{J}$  of  $Z$  should be specified and the problem of valuation is to determine a probabilistic model for  $Z$  (and  $S$  if necessary) that allows for an accurate and

efficient evaluation of the risks carried by the resulting contract in order to determine what the value of the contract should be.

*Remark 1.1.1.* Throughout we will assume that there exists a deterministic maximal contract expiration time  $\mathcal{T} > 0$ , in the sense that  $b_i(t) = b_{ij}(t) = 0$  for all  $t > \mathcal{T}$  and  $i, j \in \mathcal{J}$  with  $i \neq j$ .

### 1.1.1 Valuation

The value of an insurance contract can be defined as the amount of money an interested party would have to be paid in order to take over the liability specified by the contractual payments  $B$ . An insurance contract is called an actuarial contract if  $B$  is  $\mathbb{F}^Z$ -adapted and a financial contract if  $B$  is  $\mathbb{F}^S$ -adapted. Traditionally actuarial and financial contracts are valued using fundamentally different principles.

The valuation of an actuarial contract with contractual payments  $B^A$  is based on the principle of diversification. Assuming that there is no systematic risk, the actuarial contract only depends on one individual, thus if the insurance company can pool a large number of sufficiently homogeneous and independent actuarial contracts into one portfolio, a law of large numbers can be applied. Assuming that the interest rate  $r$  is deterministic, the value of the liability (the reserve) would therefore be the expected present value of future payments (best estimate) given by

$$V^{BE}(t) = \mathbb{E} \left[ \int_{(t, \mathcal{T}]} e^{-\int_t^s r(u) du} B^A(ds) \middle| \mathcal{F}_t^Z \right]. \quad (1.1.2)$$

Practically the use of  $V^{BE}(t)$  as the value of the contract is challenged by three factors. First, real-world portfolios are not infinitely large, leaving a non-negligible amount of unsystematic risk. Second, the policyholders in a portfolio might not be independent due to systematic risk factors; for example policyholders infecting each other during an epidemic can cause mortality rates to spike in the short term, while technological advances can cause mortality rates to decrease in the long term. Third, as the measure  $\mathbb{P}$  is not known but has to be estimated, there is model risk. All these factors justify the addition of a safety margin  $\text{SM}(t)$  to (1.1.2), yielding the value  $V(t) = V^{BE}(t) + \text{SM}(t)$ . A popular way to determine the safety margin or to quantify the risk of the contract, is to determine a conservative or risk-adjusted measure  $\mathbb{P}^*$  on  $(\Omega, \mathcal{F}_{\mathcal{T}}^Z)$ , calculate the value of the liability as

$$V(t) = \mathbb{E}^{\mathbb{P}^*} \left[ \int_{(t, \mathcal{T}]} e^{-\int_t^s r(u) du} B^A(ds) \middle| \mathcal{F}_t^Z \right], \quad (1.1.3)$$

and then implicitly determine  $\text{SM}(t) = V(t) - V^{BE}(t)$ . Note that  $\mathbb{P}^*$  does not have to be equivalent to  $\mathbb{P}$ . Model risk and systematic risk in particular can be handled by specifying a confidence set  $\mathcal{M}$  of alternative measures, setting

$$V(t) = \sup_{P \in \mathcal{M}} \mathbb{E}^P \left[ \int_{(t, \mathcal{T}]} e^{-\int_t^s r(u) du} B^A(ds) \middle| \mathcal{F}_t^Z \right], \quad (1.1.4)$$

and choosing  $\mathbb{P}^*$  as the measure realising the supremum. One way to specify the set  $\mathcal{M}$  is to determine upper and lower bounds for the transition rates of  $Z$ . For Markovian  $Z$  and deterministic upper and lower bounds this has initially been explored by Christiansen (2010) and extended to include dependent transition and interest rates in Christiansen and Steffensen (2013) and Christiansen et al. (2016). For a thorough discussion of different methods for the calculation of implicit safety margins related to unsystematic, systematic and model risk see Christiansen (2012).

The diversification principle does apply to financial contracts, as they all are determined by the same  $d + 1$  assets, which cannot reasonably be assumed to be independent. Instead financial contracts are valued using the principle of arbitrage free pricing, which involves finding a replicating strategy under the assumption that the financial market is free of arbitrage. An arbitrage is an investment opportunity that can yield a profit with zero cost and no risk of loss. For a  $\mathcal{F}_T^S$ -adapted financial claim  $X$ , an investment strategy is a replicating strategy if it is  $X$ -admissible (the terminal value at time  $T$  of the corresponding portfolio equals  $X$ ) and self-financing (the value of the corresponding portfolio only changes due to fluctuations of the asset prices  $S$ ). For a financial contract  $B^S$  an investment strategy is a replicating strategy if it is 0-admissible and self-financing with respect to  $B^S$  (the value of the corresponding portfolio only changes due to fluctuations of the asset prices  $S$  and payments according to  $B$ ). Having found a replicating strategy for the financial claim or contract, its value is then equal to the cost of buying the replicating portfolio, while the assumption of no arbitrage ensures that this value must be unique. A financial claim or contract is called attainable, if there exists a replicating strategy, and if all financial claims are attainable then the financial market is complete and all financial claims and contracts have a unique value. By the two fundamental theorems of asset pricing, completeness and no arbitrage implies the existence of a unique probability measure  $\mathbb{Q}^S$  on  $(\Omega, \mathcal{F}_T^S)$  equivalent to  $\mathbb{P}|_{\mathcal{F}_T^S}$  such that the discounted price processes  $S^* := \frac{S}{S_0^S}$  are local  $(\mathbb{F}^S, \mathbb{Q}^S)$ -martingales. Under the assumption that  $S^*$  are true  $(\mathbb{F}^S, \mathbb{Q}^S)$ -martingales ( $\mathbb{Q}^S$  is an equivalent martingale measure), the unique value of any financial contract  $B^S$  at time  $t \in [0, T]$  can be obtained via the risk neutral valuation formula given by

$$V(t) = \mathbb{E}^{\mathbb{Q}^S} \left[ \int_{(t, T]} e^{-\int_t^s r(u) du} B^S(ds) \middle| \mathcal{F}_t^S \right]. \quad (1.1.5)$$

Going forward, we assume that  $\mathbb{Q}^S$  is unique.

If we want to value an insurance contract (carrying both actuarial and financial risk), then one can take an overall financial point of view by considering the actuarial risks and financial risks as part of the same financial market, as is done in Møller (1998), Steffensen (2000), and Møller (2001). Assuming absence of arbitrage and the existence of an equivalent martingale measure, the price of the insurance contract

should be

$$V(t) = \mathbb{E}^{\mathbb{Q}} \left[ \int_{(t,T]} e^{-\int_t^s r(u) du} B(ds) \middle| \mathcal{F}_t \right], \quad (1.1.6)$$

As the actuarial contracts cannot be replicated, the extension to insurance contracts results in an incomplete market, which implies that  $\mathbb{Q}$  (and the value of the insurance contract) is not unique. Thus we have to choose a reasonable  $\mathbb{Q}$ . Since  $S$  and  $Z$  are independent under  $\mathbb{P}$  it is natural to require the same of  $\mathbb{Q}$  and using the Girsanov theorem we can construct the measure  $\mathbb{Q}$  on  $(\Omega, \mathcal{F}_{\mathcal{T}})$  given by

$$d\mathbb{Q} = \frac{d\mathbb{Q}^S}{d\mathbb{P}|_{\mathcal{F}_{\mathcal{T}}^S}} L_{\mathcal{T}}^Z d\mathbb{P}, \quad (1.1.7)$$

for suitable  $\mathcal{F}_{\mathcal{T}}^Z$ -measurable  $L_{\mathcal{T}}^Z$ , see Dahl (2004) and Dahl and Møller (2006). This construction implies that  $S$  and  $Z$  are independent,  $S^*$  is a  $(\mathbb{F}, \mathbb{Q})$ -martingale,  $\mathbb{Q}|_{\mathcal{F}_{\mathcal{T}}^S} = \mathbb{Q}^S$ , and  $\mathbb{Q}|_{\mathcal{F}_{\mathcal{T}}^Z} = \mathbb{P}^*$  for  $\mathbb{P}^* = L_{\mathcal{T}}^Z d\mathbb{P}$ . Thus  $L_{\mathcal{T}}^Z$  implicitly determines the safety margin for actuarial risk. A special case of the class defined by (1.1.7) is the best estimate measure  $\mathbb{Q}^{BE}$  for which  $L_{\mathcal{T}}^Z = 1$  and  $\mathbb{P}^* = \mathbb{P}|_{\mathcal{F}_{\mathcal{T}}^Z}$ . We can see that (1.1.6) in the case of financial contracts reduces to (1.1.5) and in the case of actuarial contracts reduces to (1.1.3), or in the case of  $\mathbb{Q}^{BE}$  to (1.1.2). We can therefore conclude that this approach is consistent with both traditional valuation principles.

### The forward method

We start by assuming that  $r$  is  $\mathbb{F}^S$ -adapted and that the insurance payment process  $B$  is  $\mathbb{F}^Z$ -adapted. Without loss of generality we further assume that  $\mathbb{Q} = \mathbb{Q}^{BE}$ . Then the reserve is equal to

$$V(t) = \int_{(t,\mathcal{T}]} \mathbb{E}^{\mathbb{Q}^S} \left[ e^{-\int_t^s r(u) du} \middle| \mathcal{F}_t^S \right] A(t, ds),$$

where  $A$  is the so-called cash flow defined as

$$A(t, s) = \mathbb{E}[B(s) - B(t) | \mathcal{F}_t^Z], \quad 0 \leq t \leq s \leq \mathcal{T}.$$

Using the concept of forward interest rates, one obtains the valuation formula

$$V(t) = \int_{(t,\mathcal{T}]} e^{-\int_t^s f(t,u) du} A(t, ds). \quad (1.1.8)$$

This has the big advantage that it completely separates the actuarial and the financial components, both from a modelling perspective but also from a computational perspective. The forward interest rates can be determined using standard methods from mathematical finance, while the cash flows for Markov (Norberg, 1991), semi-Markov (Buchardt, Møller, and Schmidt, 2015) and doubly stochastic Markov (Buchardt, 2017) models can be calculated by solving the forward equations satisfied by the transition probabilities of the Markov process defining the model.

**Example 1.1.2.** Assume that the process  $Z$  is a Markov chain with deterministic transition intensities  $(\mu_{ij})_{i,j:i \neq j}$  and that the payments  $(b_i)_{i \in \mathcal{J}}$  and  $(b_{ij})_{i,j:i \neq j}$  are deterministic. Then the state-wise cash flows are given by

$$A_i(t, ds) = \sum_{j \in \mathcal{J}} p_{ij}(t, s) \left( b_j(s) + \sum_{k:k \neq j} \mu_{jk}(s) b_{jk}(s) \right) ds,$$

while the cash flow is given by  $A(t, ds) = A_{Z(t)}(t, ds)$ . The functions  $p_{ij}(t, s)$  are the transition probabilities  $\mathbb{P}(Z(s) = j | Z(t) = i)$ , which for  $0 \leq t \leq s$  satisfy the Kolmogorov forward equations

$$\frac{d}{ds} p_{ij}(t, s) = \sum_{k:k \neq j} p_{ik}(t, s) \mu_{kj}(s) - \sum_{k:k \neq j} p_{ij}(t, s) \mu_{jk}(s), \quad p_{ij}(t, t) = \mathbb{1}_{(i=j)}.$$

These forward equations can easily be solved numerically using standard methods for ordinary differential equations.

**Example 1.1.3.** Let  $Z$  be a jump process with finite state space  $\mathcal{J}$  and define the process  $T(t) := \sup\{s \in [0, t] | Z(s) \neq Z(s-)\}$  tracking the time of the last jump of  $Z$ . Thus  $U(t) = t - T(t)$  is the duration since the last jump. Assume that the bivariate process  $(Z, U)$  (or equivalently  $(Z, T)$ ) is a Markov process. In that case  $Z$  is a so-called semi-Markov process and it has the duration-dependent transition intensities  $(\mu_{ij}(t, U(t-)))_{i \neq j}$ , where  $\mu_{ij} : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ ,  $i \neq j$  are measurable functions bounded on compact sets. Similarly we assume that  $B$  is given by

$$B(dt) = \sum_{i \in \mathcal{J}} \mathbb{1}_{(Z(t)=i)} \left( b_i(t, U(t-)) dt + \sum_{j:j \neq i} b_{ij}(t, U(t-)) N_{ij}(dt) \right).$$

Then the state-wise cash flows are given by

$$A_i(t, u, ds) = \sum_{j \in \mathcal{J}} \int_0^{u+s-t} p_{ij}(t, u; s, dv) \left( b_j(s, v) + \sum_{k:k \neq j} \mu_{jk}(s, v) b_{jk}(s, v) \right) ds,$$

while the cash flow is given by  $A(t, ds) = A_{Z(t)}(t, U(t), ds)$ . The transition probabilities

$$p_{ij}(t, u; s, v) = \mathbb{P}(Z(s) = j, U(s) \leq v | Z(t) = i, U(t) = u), \quad (t, u) \in [0, \mathcal{T}] \times [0, \mathcal{T}]$$

satisfy the system of Kolmogorov forward integro-differential equations given by

$$\begin{aligned} \frac{d}{ds} p_{ij}(t, u; s, s-d) &= \sum_{k:k \neq j} \int_0^{u+s-t} \mu_{kj}(s, v) p_{ik}(t, u, s, dv) \\ &\quad - \sum_{k:k \neq j} \int_0^{s-d} \mu_{jk}(s, v) p_{ij}(t, u; s, dz), \end{aligned} \tag{1.1.9}$$

for  $d \in [t-u, \mathcal{T}]$ ,  $s \in [\max(t, d), \mathcal{T}]$ , and with boundary conditions  $p_{ij}(t, u; t, t-d) = \mathbb{1}_{(i=j)} \mathbb{1}_{(d=t-u)}$  and  $p_{ij}(t, u; s, 0) = 0$ . These forward equations can be solved numerically, see Buchardt, Møller, and Schmidt (2015), but at a significantly larger computational cost than the differential equations of Example 1.1.2. Note that while one usually specifies all quantities in terms of  $(Z, U)$ , it is the representation in terms of  $(Z, T)$  that is mathematically more convenient as  $(Z, T)$  is a jump process which eases the derivation of the forward equations (1.1.9). It is always possible to switch back and forth between these two representations by means of substituting  $U(t) = t - T(t)$ .

The forward method has further advantages. Once one has calculated the cash flows, it is computationally very efficient to calculate the reserve for different interest rate curves, which is very useful in practice when quantifying interest rate risk. Apart from the reserve, one also obtains the cash flows and the transition probabilities which can be used for further analysis. It is also possible to handle certain linearly reserve-dependent payments, which appear in the presence of policyholder options such as surrender or free-policy, by finding an alternative representation of the model where the payments are no longer reserve dependent, see Buchardt, Møller, and Schmidt (2015) for the semi-Markovian case and Furrer (2022) for the general non-Markovian case. If the reserve-dependence is more complicated, then the forward method is no longer applicable. When  $Z$  is a general non-Markovian process, the transition probabilities can no longer be obtained via forward equations, nevertheless by using forward transition rates a variation of the forward method can still be used to calculate the reserve for contractual payments with deterministic sojourn and transition payments (Christiansen, 2021) or stochastically scaled versions thereof (Bathke and Furrer, 2025). If the payment stream is  $\mathbb{F}$ -adapted and thus depends on financial risk as well the forward method cannot be applied directly, but can be combined with a simulation approach, see Ahmad, Buchardt, and Furrer (2022).

### The backward method

We start by assuming that the interest rate  $r$  is deterministic, that  $B$  is  $\mathbb{F}^Z$ -adapted and that the jump process  $Z$  has  $\mathbb{F}^Z$ -predictable transition intensities. For now, we have thus entirely removed financial risk and without loss of generality we assume that  $\mathbb{P}^* = \mathbb{P}|_{\mathcal{F}_T^Z}$ . Then the prospective reserve is uniquely determined by the backwards stochastic differential equation (BSDE)

$$V(dt) = r(t)V(t-)dt - B(dt) - \sum_{i,j:i \neq j} R_{ij}(t)M_{ij}(dt), \quad V(\mathcal{T}) = 0, \quad (1.1.10)$$

where  $M_{ij}(dt) = N_{ij}(dt) - \mathbb{1}_{(Z(t-)=i)}\mu_{ij}(t)dt$  and

$$\mathbb{1}_{(Z(t-)=i)}R_{ij}(t) = \mathbb{1}_{(Z(t-)=i)}(b_{ij}(t) + V_j(t) - V_i(t))$$

almost surely for all  $t \geq 0$  and  $i, j \in \mathcal{J}$  with  $i \neq j$ . The processes  $(V_i)_{i \in \mathcal{J}}$  are the state-wise reserves which point-wise almost surely satisfy

$$V_i(t) = \mathbb{E} \left[ \int_{(t, \mathcal{T}]} e^{-\int_t^s r(u) du} B(ds) \middle| \mathcal{F}_{t-}^Z, Z(t) = i \right]$$

and are (almost surely) uniquely characterised by the stochastic Thiele equations

$$\mathbb{1}_{(Z(t-)=i)} V_i(dt) = \mathbb{1}_{(Z(t-)=i)} \left( V_i(t-)r(t) - b_i(t) - \sum_{j:j \neq i} R_{ij}(t) \mu_{ij}(t) \right) dt \quad (1.1.11)$$

with terminal condition  $V_i(\mathcal{T}) = 0$ . The BSDE (1.1.10) and the stochastic Thiele equations (1.1.11) initially appeared in Norberg (1992, 1996) lacking a uniqueness result and a rigorous definition of the state-wise reserves. Uniqueness of (1.1.10) was shown in Christiansen and Djehiche (2020), the definition of the state-wise reserves was resolved in Christiansen and Furrer (2021), and the uniqueness of (1.1.11) was proven in Christiansen and Furrer (2024) using a specific canonical probability space, while also allowing  $Z$  to have cumulative transition rates (allowing point mass in the distribution of jump times). In that case (1.1.11) not only holds almost surely, but surely, see Christiansen and Furrer (2024).

If  $Z$  is a Markov or semi-Markov process, then the stochastic Thiele equations can be reduced to deterministic systems of Thiele differential equations, which can be solved numerically. In that case one obtains the state-wise reserves for all time-points simultaneously.

**Example 1.1.4.** Assume that the process  $Z$  is a Markov chain with deterministic transition intensities  $(\mu_{ij})_{i,j:i \neq j}$  and that the payments  $(b_i)_{i \in \mathcal{J}}$  and  $(b_{ij})_{i,j:i \neq j}$  are deterministic. Then the BSDE (1.1.10) has the unique solution  $V(t) = V_{Z(t)}(t)$ , where the functions  $V_i : [0, \mathcal{T}] \rightarrow \mathbb{R}$  are the solutions of the Thiele differential equations

$$\frac{d}{dt} V_i(t, t-d) = r(t) V_i(t, t-d) - b_i(t) - \sum_{j:j \neq i} (b_{ij}(t) + V_j(t) - V_i(t)) \mu_{ij}(t), \quad V_i(\mathcal{T}) = 0.$$

These differential equations can easily be solved using standard numerical methods.

**Example 1.1.5.** Let  $Z$  and  $B$  be the semi-Markov process and the corresponding payment process from Example 1.1.3. Then the BSDE (1.1.10) has the unique solution  $V(t) = V_{Z(t)}(t, U(t))$ , where the functions  $V_i : [0, \mathcal{T}] \times [0, \mathcal{T}] \rightarrow \mathbb{R}$  are the solutions of the Thiele differential equations

$$\begin{aligned} \frac{d}{dt} V_i(t, t-d) &= r(t) V_i(t, t-d) - b_i(t, t-d) - \sum_{j:j \neq i} R_{ij}(t, t-d) \mu_{ij}(t, t-d), \\ R_{ij}(t, t-d) &= b_{ij}(t, t-d) + V_j(t, t-d) - V_i(t, t-d), \end{aligned}$$

for  $d \in [0, \mathcal{T}]$  and  $t \in [d, \mathcal{T}]$  with terminal condition  $V_i(\mathcal{T}, \mathcal{T} - d) = 0$ . These differential equations can be solved numerically, see Adékambi and Christiansen (2017), at greater computational cost than the Thiele differential equations of Example 1.1.4, but using a simpler numerical scheme than the forward equations of Example 1.1.3.

Apart from providing a different avenue for numerical calculations, the BSDE (1.1.10) and the stochastic Thiele equations (1.1.11) also have theoretical applications as they can serve as a definition of the reserve and the state-wise reserves. First, this is particularly useful for resolving the circularity arising from reserve-dependent payments; it was first suggested for Markovian models with transition intensities and linear reserve-dependence (Norberg, 1991; Christiansen, Denuit, and Dhaene, 2014), subsequently generalised to Lipschitz continuous reserve-dependence (Djehiche and Löfdahl, 2016), and ultimately extended to non-Markovian models with transition intensities (Christiansen and Djehiche, 2020). Second, this can be used to find, characterise and calculate worst-case reserves in the spirit of (1.1.4); for Markovian models with cumulative transition rates and reserve-independent payments see Christiansen (2010). Third, this can be used to find alternative representations of the same model, which are computationally much more convenient, see Christiansen and Furrer (2024). For example they may be used to prove the modifications in the case of linear reserve-dependence, which then allow for the application of the forward method.

A final advantage of the backward method is that it can provide a way to calculate  $V$  in the case where  $B$  is  $\mathbb{F}$ -adapted. As long as the whole joint model  $(S, Z)$  is Markovian, a suitable version of the Feynman-Kac theorem can be used to obtain a backwards partial differential equation for  $V$ , see Steffensen (2000) and Asmussen and Steffensen (2020). The disadvantage of this is of course that one has to specify a full joint model for  $(S, Z)$ .

### 1.1.2 Design

An individual buys an insurance contract to protect themselves (or their dependents) from the economic consequences of adverse biometric events. That could be the loss of income and treatment costs due to long term illness, disability or death, or outliving one's savings after retirement. How much insurance an individual buys, and what features the insurance coverage should have depends on the individual's preferences for risk, which could depend both on the source of risk (financial or biometric) or how the risk manifests itself (fluctuations in the short term or varying trends in the long term). Thus in order to be able to design tailored insurance products for a wide spectrum of individuals with different risk preferences, the insurance company has to understand how different preferences can influence the components of an insurance contract. Mathematically this can be done by specifying

the individual's preferences as a real-valued function of the available options, which then can be maximised (or minimised) in order to determine what the *optimal* choice is for that particular individual. This approach is the foundation of a large body of literature studying the optimal choice of consumption, investment and insurance.

All of these problems start by modelling the wealth of the individual, given by

$$dX(t) = h(t)dS(t) + A(dt) - C(dt) + \bar{B}(dt), \quad X(0) = x_0 \geq 0. \quad (1.1.12)$$

The first term describes the investment returns of investing in the financial market  $S$  according to the investment strategy  $h$ , the processes  $A$ ,  $C$ , and  $\bar{B}$  are  $\mathbb{F}$ -adapted and describe the (cumulative) labour income, the (cumulative) consumption, and the (cumulative) insurance payments consisting of both premiums and benefits to and from the insurance company. The goal is to find  $(h, C, \bar{B})$  that is optimal given the individual's specified preferences.

**Example 1.1.6.** Consider an individual that has just retired at age 74 (this author's current statutory retirement age) and who during their long career has saved  $x_0 > 0$ . They now have to decide how to spend, invest and insure their savings in retirement. Let  $Z$  be a Markovian jump process on the state space  $\mathcal{J} = \{0 = \text{alive}, 1 = \text{dead}\}$  with deterministic mortality rate  $\mu$  and assume that  $S = (S_0, S_1)$  is a Black-Scholes market with constant parameters. Then the preferences of this individual could be formalised as

$$\max_{c, h, b} \mathbb{E} \left[ \int_0^T \mathbb{1}_{(Z(t)=0)} \frac{1}{1-\gamma} c^{1-\gamma}(t) ds + \int_{(0, T]} \frac{w_{01}^\gamma}{1-\gamma} X^{1-\gamma}(t) dN_{01}(t) \right] \quad (1.1.13)$$

for  $\gamma \in (0, 1) \cup (1, \infty)$ ,  $w_{01} \geq 0$  with wealth process

$$dX(t) = h(t)dS(t) - \underbrace{\mathbb{1}_{(Z(t-)=0)} c(t)dt}_{=C(dt)} + \underbrace{b(t)(N_{01}(dt) - \mathbb{1}_{(Z(t-)=0)}\mu(t)dt)}_{=\bar{B}(dt)}.$$

The first integral in (1.1.13) measures the utility accumulated by consuming the consumption rate  $c$  while being alive throughout the entire retirement period and the second integral measures the utility of the remaining wealth (the bequest) at the time of death. Thus by solving the maximisation problem, we obtain the optimal choice of consumption rate  $c$ , investment strategy  $h$  and life insurance  $b$  for an individual with these preferences. In the case where  $w_{01} = 0$  the optimal solution is given by

$$c(t) = \frac{X(t)}{f(t)}, \quad h_1(t) = \frac{\theta}{\sigma\gamma} \frac{X(t)}{S_1(t)}, \quad h_0(t) = \left(1 - \frac{\theta}{\sigma\gamma}\right) \frac{X(t)}{S_0(t)}, \quad b(t) = -X(t),$$

where the function  $f$  is a unit annuity with a constant interest rate  $\tilde{r}$  and with mortality rate  $\mu$ . As there is no utility from bequest, the optimal insurance strategy

is to give up all wealth to the insurance company upon death in exchange for receiving the mortality credits  $\mu(t)X(t)$  while alive. Thus the optimal solution for this individual is to consume according to a life annuity while investing a constant proportion of their wealth in stocks. How large this proportion should be depends on the market price of risk  $\theta$ , the volatility  $\sigma$  and the individual risk aversion  $\gamma$ .

Mathematically these optimisation problems are typically solved using dynamic programming, which is related to the backwards method as the optimal solution can be found by solving a non-linear backwards partial differential equation called the Hamilton-Jacobi-Bellman equation, see Fleming and Soner (2006) and Asmussen and Steffensen (2020). The specific problem (1.1.13) was first posed without mortality by Merton (1969, 1971) and with mortality by Richard (1975). Since then, it has been generalised in many different directions, notably to Markovian multi-state models by Kraft and Steffensen (2008), and the solutions of all these problems can, in principle, be directly translated into an insurance product design. The link can be established via the definition of the policyholder account. Comparing (1.1.12) with (1.1.1) we see that there are clear parallels. The premiums  $B^-$  can be interpreted as the income process  $A$  that the policyholder pays into their account, the benefits  $B^+$  correspond to the consumption process  $C$  which is paid out to the policyholder, while the process  $\tilde{B}$  corresponds to the payments  $\bar{B}$  between the policyholder and the insurance company. Since the money in a policyholder's account rightfully belongs to the policyholder, we can view the policyholder account as 'wealth in the custody of the insurance company' and by solving an optimisation problem akin to (1.1.13) we learn how an individual with given preferences prefers to spend this wealth. Thus the solution of (1.1.13) directly yields a product design  $(h, B, \tilde{B})$ .

We now turn towards the constraints of the insurance company, which often includes the following:

1. Explainability: It should be possible to describe the contractual components  $(h, B, \tilde{B})$  with explicit formulas that easily can be interpreted and explained.
2. Efficiency: It should be possible to numerically calculate the value of the contract in an efficient manner.
3. Manageable risk: The contract should not result in an unmanageable exposure of unhedgeable and undiversifiable risk.

Explainability is required as the insurance contract has to be understood by various stakeholders such as regulators, board members and policyholders, and efficiency is required as the insurance company only has limited resources available. Finally a large exposure of unhedgeable and undiversifiable risk would yield large capital requirements and an increased risk of bankruptcy.

A class of products which always satisfies the second point and mostly also the first and third point is the class of pure unit link contracts. An insurance contract is pure unit link if the payment process  $\tilde{B}$  to and from the insurance company's equity is a mean-zero  $(\mathbb{F}, \mathbb{Q})$ -martingale and if  $X(T) = 0$ . In that case, it was shown in Furrer, Hornung, and Lemvig (2026) that  $V(0) = -B(0)$  and

$$V(t) = \mathbb{E}^{\mathbb{Q}} \left[ \int_{(t, T]} e^{-\int_t^s r(u) du} B(ds) \middle| \mathcal{F}_t \right] = X(t), \quad \forall t \in [0, T].$$

The problem of valuation is thus reduced to looking up the balance of the corresponding policyholder account.

**Example 1.1.7.** Inspired by Example 1.1.6, we can design the following product. The contractual payments are given by

$$B(dt) = \mathbb{1}_{(Z(t)=0)} \frac{X(t)}{f(t)} dt, \quad B(0) = -x_0,$$

where  $f$  is a unit annuity with a constant interest rate  $\tilde{r}$  and a deterministic mortality rate  $\tilde{\mu}$  while the investment strategy is given by

$$h_0(t) = \frac{(1 - \pi)X(t)}{S_0(t)} \quad \text{and} \quad h_\ell(t) = \frac{\pi p_\ell(t)X(t-)}{S_\ell(t-)}, \quad \ell = 1, \dots, d,$$

where  $\pi \in [0, 1]$  and  $(p_\ell)_{\ell=1, \dots, d}$  are  $\mathbb{F}^S$ -predictable processes such that  $\sum_{\ell=1}^d p_\ell(t) = 1$ . This investment strategy corresponds to investing the proportion  $\pi$  in an index fund  $\bar{S}$  consisting of the  $d$  risky assets  $(S_1, \dots, S_d)$  weighted according to  $(p_\ell)_{\ell=1, \dots, d}$ , while investing the proportion  $1 - \pi$  in the locally risk free asset  $S_0$ . The process  $\tilde{B}$  is given by

$$\tilde{B}(dt) = X(t-)(N_{01}(dt) - \mathbb{1}_{(Z(t-)=0)}\mu^{\mathbb{Q}}(t)dt), \quad \tilde{B}(0) = 0,$$

where  $\mu^{\mathbb{Q}}$  is the mortality rate under  $\mathbb{Q}$ . Clearly  $\tilde{B}$  is a mean-zero  $(\mathbb{F}, \mathbb{Q})$ -martingale and since  $\frac{1}{f(t)} \uparrow \infty$  for  $t \rightarrow T$  it holds that  $X(t) \downarrow 0$  almost surely for  $t \rightarrow T$ . Thus we have created a pure unit link contract, which is explainable and efficient. The risk exposure of the company is determined by  $\tilde{B}$ , which can be rewritten into

$$\tilde{B}(dt) = \mathbb{1}_{(Z(t-)=0)}X(t-)(\mu(t) - \mu^{\mathbb{Q}}(t))dt + X(t-)(N_{01}(dt) - \mathbb{1}_{Z(t-)=0}\mu(t)dt).$$

The second term is a mean-zero  $(\mathbb{F}, \mathbb{P})$ -martingale and can therefore be controlled by diversification (assuming that individuals cannot die simultaneously). This leaves the first term, which can be controlled by choosing a proper  $\mathbb{Q}$ . In particular we see that if mortality credits are paid according to the true mortality rate  $\mu$  the term cancels out, leaving the insurance company only with unsystematic, diversifiable risk. Note that even though this design is strongly related to the optimal solution presented in Example 1.1.6, it is not optimal for the preferences (1.1.13) as we allow for a much more general probabilistic model.

The product design described in Example 1.1.7 is called a pure unit linked annuity and it has become the default annuity product in Denmark. One of the reasons for its popularity is that it completely falls within the company's constraints. It is easy to understand, easy to value and most importantly it carries only unsystematic, diversifiable risk for the insurance company. The financial risk and the systematic longevity risk is borne entirely by the policyholder. To see this, assume for simplicity that  $\mu^Q = \mu$ . Then the relative rate of change of the benefit rate  $b(t) = \frac{X(t)}{f(t)}$  (while the policyholder is alive) is given by

$$\frac{db(t)}{b(t-)} = \left( (1 - \pi) \frac{dS_0(t)}{S_0(t-)} + \pi \frac{d\bar{S}(t)}{\bar{S}(t-)} - \tilde{r}dt \right) + (\mu(t) - \tilde{\mu}(t))dt. \quad (1.1.14)$$

The first term is the difference between the investment returns and the design interest rate  $\tilde{r}$ . We can thus conclude that if the financial returns are worse than  $\tilde{r}$ , then the benefit is reduced and the other way around. The second term is the difference between the design mortality rate  $\tilde{\mu}$  and the true mortality rate  $\mu$ . Thus if  $\mu$  decreases relative to  $\tilde{\mu}$ , then the benefit is reduced.

Note that the effect of the financial risk on  $b$  is in conformance with the preferences (1.1.13), as the first term of (1.1.14) does not really change if  $(S_0, \bar{S})$  were a Black-Scholes market with constant parameters. That does not mean that everyone has those preferences, as evidenced by the 'consumption smoothing puzzle', which states that the relative change in wealth resulting from a shock is larger than the relative change in consumption (Hansen and Singleton, 1983; Campbell and Deaton, 1989). This is clearly not the case for (1.1.13) where the optimal consumption is proportional to the wealth. Consequently a variety of alternative preferences have been proposed, such as habit formation (Constantinides, 1990; Munk, 2008), cumulative prospect theory preferences (Bilsen and Laeven, 2020), and quadratic preferences (Bruhn and Steffensen, 2013), some resulting in suggestions for alternative pension product designs, such as 'tidspension' (Jørgensen and Linnemann, 2011; Linnemann, Bruhn, and Steffensen, 2014) conforming to the quadratic preferences of Bruhn and Steffensen (2013), or the design of Steffensen and Vikkelsøe (2024) based on habit formation preferences. As none of these suggestions constitutes a single 'correct' alternative to (1.1.13) it is important to be able to compare and understand the resulting optimal solutions or derivative product designs in a common framework.

Returning to (1.1.14) we note that effect of longevity risk would disappear if  $\mu = \tilde{\mu}$ . This would then be in line with the preferences (1.1.13); the individual fully annuitises their wealth. If  $\mu$  is stochastic, then we have moved outside of the setting in Example 1.1.6, since this implies the existence of systematic risk. One approach would be to specify a stochastic model for the mortality risk and then attempt to solve (1.1.13) again, but as longevity improvements are notoriously hard to predict, one can instead take a robust perspective and maximise the utility in (1.1.13) while simultaneously minimising over a set of alternative mortality rates, in

the spirit of (1.1.4). In the Markovian case, this has been done by Han and Hung (2021) and they conclude that aversion against deviations of the true mortality rate from a statistical best estimate can be an explanation of the so-called ‘annuitisation puzzle’, which states that, empirically, individuals annuitise less than they should if they had the preferences (1.1.13), see Modigliani (1986). It is thus also important to consider the effect that preferences for systematic risk (or model risk) can have.

### 1.1.3 The construction of jump processes

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $(E, \mathcal{E})$  be a Borel space and let  $X$  be an  $E$ -valued non-explosive jump process. We equip the probability space with the filtration  $\mathbb{F}^X = (\mathcal{F}_t^X)_{t \geq 0}$  given by  $\mathcal{F}_t^X := \sigma(X(s) : s \leq t)$ . The distribution of  $X$  can be characterised either via a marked point process or a random counting measure. For two excellent textbook references on point process theory see Last and Brandt (1999) and Jacobsen (2006).

#### Using marked point processes

Based on the jump times of  $X$  we can define the non-explosive marked point process  $(T_n, Y_n)_{n \in \mathbb{N}_0}$ , with  $T_0 := 0$  and  $Y_0 := X(0)$  and

$$\forall n \in \mathbb{N} : \quad T_n := \inf\{t > T_{n-1} : X(t) \neq X(T_{n-1})\} \quad \text{and} \quad Y_n := X(T_n),$$

where we use the convention that  $\inf \emptyset = \infty$  and  $Y_n = \nabla$  whenever  $T_n = \infty$ . Note that the marked point process  $(T_n, Y_n)_{n \in \mathbb{N}_0}$  does not allow for trivial jumps, that is

$$\mathbb{P}\left(\bigcup_{n \in \mathbb{N}_0} (T_n < \infty, Y_n = Y_{n+1})\right) = 0. \quad (1.1.15)$$

As long as there are no trivial jumps, the jump process  $X$  and the marked point process  $(T_n, Y_n)_{n \in \mathbb{N}_0}$  generate the same filtration, and are in a one-to-one correspondence, since  $X$  can be directly recovered from  $(T_n, Y_n)_{n \in \mathbb{N}_0}$  via

$$X(t) = \sum_{n \in \mathbb{N}_0} \mathbb{1}_{[T_n, T_{n+1})}(t) Y_n. \quad (1.1.16)$$

Thus due to (1.1.16), and as long as (1.1.15) is satisfied, the distributions of  $X$  and  $(T_n, Y_n)_{n \in \mathbb{N}_0}$  uniquely characterise each other and we can construct a non-explosive jump process by constructing a non-explosive marked point process without trivial jumps. By the Ionescu-Tulcea theorem the distribution of  $(T_n, Y_n)_{n \in \mathbb{N}_0}$  is uniquely characterised by the distribution of  $Y_0$  and regular versions of

$$\mathbb{P}(T_{n+1} \leq t | (T_i, Y_i)_{i \leq n}) \quad \text{and} \quad \mathbb{P}(Y_{n+1} \in \cdot | T_{n+1} = t, (T_i, Y_i)_{i \leq n}), \quad n \in \mathbb{N}_0.$$

In order to ensure that (1.1.15) is satisfied it is sufficient to require

$$\forall n \in \mathbb{N}_0 : \quad \mathbb{P}(Y_{n+1} = Y_n | T_{n+1} = t, (T_i, Y_i)_{i \leq n}) = 0, \quad \forall t \in [0, \infty).$$

**Example 1.1.8.** We now show how to construct the bivariate Markovian jump process  $X = (Z, T)$  with state space  $\mathcal{J} \times [0, \infty)$  defining the semi-Markov process  $Z$  considered in Example 1.1.3. Let  $T_0 = 0$  and let  $Y_0 = (Z(0), 0)$  for some  $\mathcal{J}$ -valued random variable  $Z(0)$ . Construct now  $(T_n, Y_n)_{n \in \mathbb{N}_0} = (T_n, (Z_n, T_n))_{n \in \mathbb{N}_0}$  using

$$\mathbb{P}(T_{n+1} > t | (T_\ell, Y_\ell)_{\ell \leq n}) = e^{-\int_{T_n}^t \sum_{j: j \neq Y_n} \mu_{Y_n j}(s, s - T_n) ds}, \quad t > T_n,$$

$$\mathbb{P}(Y_{n+1} = (j, \tau) | T_{n+1} = t, (T_\ell, Y_\ell)_{\ell \leq n}) = \frac{\mu_{Y_n j}(t, t - T_n)}{\sum_{k: k \neq Y_n} \mu_{Y_n k}(t, t - T_n)} \mathbb{1}_{(\tau=t)}, \quad t > T_n.$$

Then the process  $X = (Z, T)$  can be constructed from such  $(T_n, Y_n)_{n \in \mathbb{N}_0}$  via (1.1.16).

### Using random counting measures

Using the non-explosive marked point process  $(T_n, Y_n)_{n \in \mathbb{N}_0}$  we can furthermore define the non-explosive random counting measure  $Q$  on  $\mathcal{B}([0, \infty)) \otimes \mathcal{E}$  given by

$$Q(dt, dy) := \sum_{n \in \mathbb{N}: T_n < \infty} \delta_{(T_n, Y_n)}(dt, dy).$$

Note that as long as (1.1.15) holds, the jump process  $X$  and the pair of random counting measure and initial value,  $(Y_0, Q)$ , generate the same filtration and are in a one-to-one correspondence, since  $X$  can be directly recovered from  $(Y_0, Q)$  via

$$X(t) = Y_0 + \int_{(0, t] \times E} (y - X(s-)) Q(ds, dy). \quad (1.1.17)$$

Thus due to (1.1.17), and as long as (1.1.15) is satisfied, the distributions of  $X$  and  $(Y_0, Q)$  uniquely characterise each other and we can construct a non-explosive jump process by constructing a non-explosive random counting measure without trivial jumps and initial value  $Y_0$ . The distribution of  $Q$  is uniquely characterised by the compensating measure  $L$  on  $\mathcal{B}([0, \infty)) \otimes \mathcal{E}$  given by

$$L(dt, dy) = \sum_{n \in \mathbb{N}_0} \mathbb{1}_{(T_n, T_{n+1}]}(t) \Lambda^n(dt, dy, (T_i, Y_i)_{i \leq n})$$

for a kernel  $\Lambda^n$  of the form

$$\Lambda^n(dt, dy, (T_i, Y_i)_{i \leq n}) = \kappa_t^n(dy, (T_i, Y_i)_{i \leq n}) \nu^n(dt, (T_i, Y_i)_{i \leq n}),$$

where  $\nu^n$  for all  $n \in \mathbb{N}_0$  is a regular version of the hazard-measure

$$\nu^n(dt, (T_i, Y_i)_{i \leq n}) = \frac{\mathbb{P}(T_{n+1} \in dt | (T_i, Y_i)_{i \leq n})}{\mathbb{P}(T_{n+1} \geq t | (T_i, Y_i)_{i \leq n})},$$

and  $\kappa_t^n$  for all  $n \in \mathbb{N}_0$  is a regular version of

$$\mathbb{P}(Y_{n+1} \in \cdot | T_{n+1} = t, (T_i, Y_i)_{i \leq n}).$$

In particular (1.1.15) is satisfied if

$$\forall n \in \mathbb{N}_0 : \quad \Lambda^n(\{t\} \times \{Y_n\}, (T_i, Y_i)_{i \leq n}) = 0, \quad \forall t \in [0, \infty).$$

**Example 1.1.9.** The bivariate Markovian process  $X = (Z, T)$  from Example 1.1.8 can also be constructed via the pair  $(Y_0, Q)$ . For this we let  $Y_0$  be given and define the kernels  $(\Lambda^n)_{n \in \mathbb{N}_0}$  by

$$\Lambda^n(dt, \{j\} \times d\tau, (T_\ell, Y_\ell)_{\ell \leq n}) = \mu_{Z_{nj}}(t, t - T_n) \delta_{\{t\}}(d\tau) dt, \quad j \in \mathcal{J}.$$

The process  $X = (Z, T)$  can then be constructed via (1.1.17). Since it holds that  $Z(t-) = Z_n$  and  $T(t-) = T_n$  whenever  $t \in (T_n, T_{n+1}]$ , we obtain

$$L(dt, \{j\} \times d\tau) = \sum_{i: i \neq j} \mathbb{1}_{(Z(t-)=i)} \mu_{ij}(t, t - T(t-)) \delta_{\{t\}}(d\tau) dt, \quad j \in \mathcal{J}.$$

By substituting  $U(t) = t - T(t)$  and integrating out  $\tau$  we get

$$L(dt, \{j\} \times [0, \infty)) = \sum_{i: i \neq j} \mathbb{1}_{(Z(t-)=i)} \mu_{ij}(t, U(t-)) dt, \quad j \in \mathcal{J},$$

which can be recognised as the compensator of the counting process  $N_j = \sum_{i: i \neq j} N_{ij}$ .

## 1.2 Contributions

In addition to this introduction, this thesis consists of the four self-contained Chapters 2-5, each with their own scientific contributions. As a consequence, terminology and notation may vary significantly between them. The Chapters 2-4 only concern themselves with actuarial risk and disregard financial risk. Chapter 2 and Chapter 3 focus on the valuation of actuarial contracts with collective effects using mean-field approximations and the forward method. Chapter 4 considers both valuation and design under actuarial model uncertainty, generalising both existing backward methods for the determination of worst-case reserves and existing consumption-insurance problems. The final Chapter 5 differs from the previous ones by exclusively considering financial risk. Its main contribution is the specification of preferences for smooth consumption in terms of the components of the consumption rate and the solution of a novel, related consumption-investment problem.

### 1.2.1 Valuation of insurance contracts with collective effects using mean-field approximations

In Chapter 2 and Chapter 3 we consider the valuation of insurance contracts, where the distribution of the individual biometric state modelled by a jump process may depend on the collective (for example via cohort averages), which implies that the insured individuals are no longer independent. A prominent example of this are insurance contracts subject to contagion risk such as life insurance during an epidemic or cyber insurance. In both cases the individuals/computers can infect each other, and thus the model should allow for an individual infection rate dependent on the proportion of currently infectious individuals. This causes

two problems when valuing the contract. First, the diversification principle may break down and second, one has to model the high-dimensional joint distribution of the entire cohort which is computationally expensive. In the actuarial literature on epidemic and cyber insurance, the computational issues have been resolved using so-called mean-field approximations, see Fahrenwaldt, Weber, and Weske (2018), Hillairet and Lopez (2021), and Hillairet et al. (2022) for cyber insurance, and Feng and Garrido (2011), Feng et al. (2022), and Francis and Steffensen (2024) for life insurance during epidemics. So while the application of mean-field approximations within actuarial mathematics is nothing new, they have not yet been formally justified, and it has not been verified that the diversification principle still applies. There is a rich literature on mean-field approximations outside of actuarial mathematics (see Chaintron and Diez (2022a,b) for a thorough review) going back decades (McKean, 1966, 1969), but if one searches this literature for suitable convergence results one encounters two issues. First, existing results are mostly formulated in the setting of (jump)-diffusions requiring the intensity measure to be Lipschitz continuous in the state variable and the contractual payments to be continuous in the Skorokhod  $J_1$ -topology to achieve the desired convergence; both assumptions can be restrictive in actuarial applications. Second, there are no results on the convergence of conditional expectations, which is essential if one wants to approximate individual values. Therefore in Chapter 2 we prove such convergence results tailored to actuarial applications under less restrictive regularity conditions, while we in Chapter 3 use the results of Chapter 2 to extend the classic semi-Markov disability model by including collective health claims.

### The mathematical framework

In Chapter 2 we consider a cohort of  $n$  individuals, where the attributes of each individual  $\ell = 1, \dots, n$  are modelled by a jump process  $X^{\ell,n}$  on a Borel state space  $(E, \mathcal{E})$ , such that the joint process  $X^n = (X^{1,n}, \dots, X^{n,n})$  is Markovian and the family  $(X^{1,n}, \dots, X^{n,n})$  is exchangeable (symmetric and identically distributed). We assume that the distribution of  $X^n(0)$  is denoted by  $\pi^n$ , with identical 1-dimensional marginals  $\pi^{n,1}$ . The process  $X^n$  on  $(E^n, \mathcal{E}^n)$  is constructed via (1.1.17) using the compensating measure

$$L^n(dt, dy^{1:n}) = \sum_{\ell=1}^n \left( \mu_t(X_{t-}^{\ell,n}, \varepsilon_{t-}^n, dy_\ell) \prod_{j=1, j \neq \ell}^n \delta_{\{X_{t-}^{j,n}\}}(dy_j) \right) dt, \quad (1.2.1)$$

where  $\mu : [0, T] \times E \times \mathcal{P}(E) \times \mathcal{E} \rightarrow [0, \infty)$  is a suitable intensity kernel identical for all individuals, and  $\varepsilon_{t-}^n$  is the empirical distribution of  $(X_{t-}^{1,n}, \dots, X_{t-}^{n,n})$ , representing the influence of the collective on the individual. The structure of  $L^n$  ensures that there are no simultaneous jumps across individuals.

We consider an insurance contract with terminal time  $\mathcal{T} > 0$ , which has to be valued at time  $\tau \in [0, \mathcal{T}]$ . The remaining contractual payments are assumed to be

of the form  $g((X_t^{1,n})_{t \in [\tau, T]})$  for a measurable, real-valued function  $g$  defined on a suitable space of jump process paths. By the diversification principle the reserve of the contract at time  $\tau$  should be one of the following

$$\begin{aligned} V^{1,n}(\tau) &:= \mathbb{E}[g((X_t^{1,n})_{t \in [\tau, T]})], \\ V^{1,n}(\tau, A) &:= \mathbb{E}[g((X_t^{1,n})_{t \in [\tau, T]}) | X_\tau^{1,n} \in A], \quad A \in \mathcal{E}, \end{aligned}$$

depending on whether one wants to reserve on an individual, grouped or portfolio level. As  $X^n$  is Markovian we could, in principle, apply the forward method if the form of  $g$  allows it, but this would require the numerical solution of the system of (linear) Kolmogorov forward integro-differential equations satisfied by the occupation/transition probabilities of  $X^n$ . The dimension of this system grows exponentially in  $n$  rendering this approach computationally infeasible. Instead, we suggest a mean-field approximation based on the idea that for large  $n$  the individuals become independent and identically distributed, allowing us to replace the empirical distribution  $\varepsilon_t^n$  with its average. We thus obtain the mean-field process  $\bar{X}$  with state space  $(E, \mathcal{E})$ , initial distribution  $\pi$ , constructed via (1.1.17) using the compensating measure given by

$$\bar{L}(dt, dy) = \mu(t, \bar{X}_{t-}, \bar{X}_t(\mathbb{P}), dy)dt,$$

where  $\bar{X}_t(\mathbb{P})$  denotes the distribution of  $\bar{X}_t$ . We show that the process  $\bar{X}$  is a so-called distribution-dependent non-linear Markov process. As with a (linear) Markov process, conditioning on the past reduces to conditioning on the present state and one can obtain a system of Kolmogorov forward integro-differential equations satisfied by the occupation/transition probabilities, but since the intensity measure depends on the distribution of the process itself, this system is non-linear. Furthermore, and contrary to (linear) Markov processes, the transition probabilities depend on the initial distribution  $\pi$ . The intuition is, that the process  $\bar{X}$  represents a *typical* individual of a *specific* cohort whose (initial) composition is given by  $\pi$ . Changing  $\pi$  changes the composition of the cohort and consequently the transition probabilities of its typical individual.

Replacing  $X^{1,n}$  with  $\bar{X}$ , the mean-field reserves of the insurance contract becomes

$$\begin{aligned} \bar{V}(\tau) &:= \mathbb{E}[g((\bar{X}_t)_{t \in [\tau, T]})], \\ \bar{V}(\tau, A) &:= \mathbb{E}[g((\bar{X}_t)_{t \in [\tau, T]}) | \bar{X}_\tau \in A], \quad A \in \mathcal{E}. \end{aligned}$$

The system of non-linear forward equations we have obtained for the occupation and transition probabilities is low-dimensional and can be solved numerically using the same methods as one would for the linear case. The idea is therefore to use the mean-field approximation  $V^{1,n} \approx \bar{V}$  when  $n$  is large enough.

We show that within our class of models both the diversification principle and the mean-field approximation are valid. Under reasonable regularity conditions for

the intensity kernel  $\mu$  (bounded and measurable in the state variable), the function  $g$  (measurable and sufficiently integrable), and the set  $A \in \mathcal{E}$  we prove that

$$\lim_{n \rightarrow \infty} V^{1,n}(\tau) = \bar{V}(\tau) \quad \text{and} \quad \lim_{n \rightarrow \infty} V^{1,n}(\tau, A) = \bar{V}(\tau, A),$$

which means that we for large  $n$  can use the approximation  $V^{1,n} \approx \bar{V}$ . Furthermore, we show that

$$\frac{1}{n} \sum_{\ell=1}^n g((X_t^{1,n})_{t \in [\tau, T]}) \xrightarrow{L^2} \bar{V}(\tau),$$

which not only proves that the diversification principle still applies, but also that for an infinitely large cohort the mean-field reserve  $\bar{V}(\tau)$  is the exact reserve and not only an approximation. For the conditional versions  $\bar{V}(\tau, A)$  we similarly obtain the slightly weaker

$$\frac{\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_A(X_\tau^{\ell,n}) g((X_t^{1,n})_{t \in [\tau, T]})}{\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_A(X_\tau^{\ell,n})} \xrightarrow{P} \bar{V}(\tau, A), \quad \mathbb{P}(\bar{X}_\tau \in A > 0).$$

Chapter 2 thus provides a solid mathematical foundation for the application of mean-field approximations in actuarial mathematics.

### Disability insurance with collective health claims

In Chapter 3 we utilise the framework developed in Chapter 2 to extend the classic semi-Markov model used in disability insurance to include individual and collective health claims. The motivation for this extension can be found in its application to group experience rating. In some countries, including Denmark, disability (and health) insurance is sold as part of occupational pension plans. As the company-specific psychological and physical work environment likely has a significant impact on the occurrence of disability claims, it is important that insurance companies can distinguish the ‘good’ companies from the ‘bad’ when pricing disability insurance contracts, which suggests the application of group experience rating techniques. Current approaches, see Furrer (2019) and Furrer, Sørensen, and Yslas (2025), struggle to distinguish different companies as the low frequency of disability claims can lead to a lack of data. Since health claims occur with a much larger frequency and are partially driven by the same company-specific factors, we propose to mitigate this issue by including collective health claims into the model. As we are interested in pricing, we only consider the valuation time  $\tau = 0$ .

We consider a cohort of  $n$  exchangeable individuals, where each individual has an associated trivariate process  $X^{\ell,n} = (Z^{\ell,n}, T^{\ell,n}, H^{\ell,n})$ . The process  $Z^{\ell,n}$  equals the state of the insured and has state-space  $\mathcal{J} = \{0 = \text{active}, 1 = \text{disabled}, 2 = \text{dead}\}$ , while  $T^{\ell,n}$  is the time of the last jump of  $Z^{\ell,n}$ . The process  $H^{\ell,n}$  is a counting process counting individual health claims. The process  $X^{\ell,n}$  therefore has state space

$E = \mathcal{J} \times [0, \infty) \times \mathbb{N}_0$ . Note that we use the parametrisation  $(Z^{\ell,n}, T^{\ell,n}, H^{\ell,n})$  instead of  $(Z^{\ell,n}, U^{\ell,n}, H^{\ell,n})$ , see Example 1.1.9, because this allows for the application of the results of Chapter 2. Let  $g : E \rightarrow \mathbb{R}$  be measurable and bounded and define the cohort averages

$$\nu_t^n := \frac{1}{n} \sum_{\ell=1}^n g(Z_t^{\ell,n}, t - T_t^{\ell,n}, H_t^{\ell,n}).$$

For example, the choice  $g(z, t - \tau, h) = \min(h, K)$  for some  $K > 0$  would let  $\nu_t^n$  be the (truncated) cohort average number of health claims at time  $t$ . Furthermore for  $y \in \mathbb{R}$  let  $(t, \tau, h, y) \mapsto \mu_{ij}(t, t - \tau, h, y)$  for  $i \neq j$  and  $(t, t - \tau, h, y) \mapsto \lambda_i(t, t - \tau, h, y)$  for  $i \in \mathcal{J}$  be duration-, health- and collective-dependent transition rates and health claim hazards that are measurable and bounded on compact sets. The process  $X^n$  is constructed via (1.1.17). We set  $T_0^{\ell,n} = H_0^{\ell,n} = 0$  and let  $(Z_0^{\ell,n})_{1 \leq \ell \leq n}$  have initial distribution  $\pi^n$  with identical marginals  $\pi^{n,1}$ . The compensating measure  $L^n$  is assumed to be of the form (1.2.1), with intensity kernel  $\mu$  given by

$$\begin{aligned} \mu_t(X_{t-}^{\ell,n}, \nu_{t-}^n, \{j\} \times d(\tau, h)) &= \mu_{Z_{t-}^{\ell,n}j}(t, t - T_{t-}^{\ell,n}, H_{t-}^{\ell,n}, \nu_{t-}^n) \delta_{\{t, H_{t-}^{\ell,n}\}}(d(\tau, h)) dt \\ &\quad + \lambda_{Z_{t-}^{\ell,n}}(t, t - T_{t-}^{\ell,n}, H_{t-}^{\ell,n}, \nu_{t-}^n) \delta_{\{Z_{t-}^{\ell,n}, T_{t-}^{\ell,n}\}}(\{j\} \times d\tau) dt, \end{aligned}$$

for each  $j \in \mathcal{J}$  and  $\ell = 1, \dots, n$ . The first term of the right-hand side determines the jumps of  $(Z^{\ell,n}, T^{\ell,n})$  (compare with Example 1.1.9), while the second term determines the jumps of  $H^{\ell,n}$ . The associated mean-field model  $\bar{X} = (\bar{Z}, \bar{T}, \bar{H})$  is determined by the initial distribution  $\bar{T}_0 = \bar{H}_0 = 0$  and  $\bar{Z}_0 \sim \pi$ , and the compensating measure

$$\begin{aligned} \bar{L}(dt, \{j\} \times d(\tau, h)) &= \mu_{\bar{Z}_t j}(t, t - \bar{T}_t, \bar{H}_t, v(t)) \delta_{\{t, \bar{H}_t\}}(d(\tau, h)) dt \\ &\quad + \lambda_{\bar{Z}_t}(t, t - \bar{T}_t, \bar{H}_t, v(t)) \delta_{\{\bar{Z}_t, \bar{T}_t\}}(\{j\} \times d\tau) dt, \end{aligned}$$

where  $v(t) := \mathbb{E}[g(\bar{Z}_t, t - \bar{T}_t, \bar{H}_t)]$ . Using the results from Chapter 2 we verify that the mean-field model is well-defined, that the mean-field approximation is valid, and that the diversification principle holds. Then we show that for a payment process  $B$  of the form given in Example 1.1.3 the resulting mean-field reserves can be calculated using the forward method.

For this we define the occupation and transition probabilities by

$$\begin{aligned} \bar{p}_i(t, u, h) &:= \mathbb{P}(\bar{Z}_t = i, \bar{U}_t \leq u, \bar{H}_t = h) \\ \bar{p}_{ij}(t, u, h) &:= \mathbb{P}(\bar{Z}_t = j, \bar{U}_t \leq u, \bar{H}_t = h | \bar{Z}_0 = i). \end{aligned}$$

For  $i \in \mathcal{J}$ ,  $h \in \mathbb{N}_0$  and  $d \geq 0$  the occupation probabilities satisfy the following

system of non-linear forward equations

$$\begin{aligned}
\frac{d}{ds}\bar{p}_i(s, s-d, h) &= \sum_{j:j \neq i} \int_0^s \mu_{ji}(s, u, h, v(s)) \bar{p}_j(s, du, h) \\
&\quad - \sum_{j:j \neq i} \int_0^{s-d} \mu_{ij}(s, u, h, v(s)) \bar{p}_i(s, du, h) \\
&\quad + \int_0^{s-d} \lambda_i(s, u, h-1, v(s)) \bar{p}_i(s, du, h-1) \\
&\quad - \int_0^{s-d} \lambda_i(s, u, h, v(t)) \bar{p}_i(s, du, h),
\end{aligned} \tag{1.2.2}$$

with boundary conditions  $\bar{p}_i(s, 0, h) = \mathbb{1}_{\{s=0\}} \mathbb{1}_{\{h=0\}} \pi(i)$ . Compared to (1.1.9) there are two significant changes. First, the addition of health claims increases the dimension of the system to include  $h$  and adds the last two terms. Second, the intensities now directly depend on  $(\bar{p}_i)_{i \in \mathcal{J}}$  through  $v$ , as  $v$  can be written as

$$v(t) = \sum_{i \in \mathcal{J}} \int_0^t \sum_{h \in \mathbb{N}_0} g(i, u, h) \bar{p}_i(t, du, h).$$

This has big implications for the calculation of the transition probabilities. If there were no dependence on  $v$  the individual would be decoupled from the collective and we could simply set  $\pi(i) = 1$  for some  $i \in \mathcal{J}$  and directly obtain the individual transition probabilities by solving the resulting system of forward equations; changing  $\pi$  corresponds to changing the initial distribution of the individual. Since there is dependence on  $v$ , changing  $\pi$  will change  $v$  which is a collective quantity; changing  $\pi$  corresponds to changing the initial composition of the collective. Nevertheless we can calculate  $(\bar{p}_{ij})_{j:j \neq i}$  using a two-step procedure. First we calculate  $(\bar{p}_i)_{i \in \mathcal{J}}$  and  $v$  by solving the system (1.2.2). Then we consider (1.2.2) with  $v$  as given and fixed, change the initial distribution and solve the resulting system again to obtain  $(\bar{p}_{ij})_{j:j \neq i}$ . As  $v$  is given and fixed, this system is linear and changing the initial distribution does not change  $v$ . From there, we find explicit expressions for the cash flows and use (1.1.8) to obtain the mean-field reserves.

We conclude by comparing the mean-field reserve with the reserve of the  $n$ -individual model in a practice-oriented example. The mean-field reserve is calculated via the forward method, by adapting the semi-Markovian numerical schemes of Buchardt, Møller, and Schmidt (2015), while the reserve of the  $n$ -individual model is obtained by naive Monte Carlo methods. Our results show that the use of the mean-field reserve already yields a significant improvement in terms of runtime for  $n = 25$  compared to the Monte Carlo method with a sample size of 40.000, and we find that this sample size is not sufficient to ensure convergence of the Monte Carlo method. Thus already for moderate  $n$  the mean-field approximation is an efficient alternative. Finally we discuss how our methods can be adjusted to lapse

risk and suggest three approaches, depending on how lapse affects the individual and the collective.

### 1.2.2 Robust control under transition-rate uncertainty

In Chapter 4 we study a novel robust utility maximisation problem under transition rate uncertainty within the class of non-Markovian, non-explosive marked point processes on a finite state space with the purpose of advancing our understanding of the impact model uncertainty or systematic risk can have on both valuation and design of multi-state life insurance contracts. Throughout this subsection we let  $(\Omega, \mathcal{F})$  denote the canonical measurable space constructed by Christiansen and Furrer (2024) and we equip this space with the filtration  $\mathbb{F}^Z = (\mathcal{F}_t^Z)_{t \geq 0}$  generated by the canonical jump process  $Z$  on the finite state space  $\mathcal{J}$ .

The jump process  $Z$  is assumed to be constructed via (1.1.17), with an initial distribution  $\pi$  and compensating measure

$$L(dt, \{j\}) = \sum_{i: i \neq j} \mathbb{1}_{(Z(t-)=i)} \Lambda_{ij}(dt), \quad \forall j \in \mathcal{J},$$

where the processes  $\Lambda = (\Lambda_{ij})_{i \neq j}$  are unique, path-dependent cumulative transition rates which specify a unique probability measure  $\mathbb{P}^\Lambda$  on  $(\Omega, \mathcal{F})$ . Thus the process  $Z$  is fully non-Markovian. We assume that the true  $\Lambda$  is unknown and instead consider the set  $\mathcal{M}$  of probability measures  $\mathbb{P}^\Lambda$  for which

$$L_{ij}(s, \omega) - L_{ij}(t, \omega) \leq \Lambda_{ij}(s, \omega) - \Lambda_{ij}(t, \omega) \leq U_{ij}(s, \omega) - U_{ij}(t, \omega)$$

for all  $i, j \in \mathcal{J} : i \neq j$ ,  $0 \leq t \leq s$  and  $\omega \in \Omega$ , where  $L = (L_{ij})_{i \neq j}$  and  $U = (U_{ij})_{i \neq j}$  are known lower and upper bounds, which are path-dependent cumulative transition rates themselves. The problem we then consider is given by

$$\max_{\theta \in \mathcal{A}} \min_{\Lambda \in \mathcal{M}} \mathbb{E}^\Lambda \left[ \int_{(0, \mathcal{T})} \frac{1}{\mathcal{E}_\Phi(t)} U^{\theta, \Lambda}(dt) + \frac{1}{\mathcal{E}_\Phi(\mathcal{T})} Y^{\theta, \Lambda} \right]. \quad (1.2.3)$$

The process  $U^{\theta, \Lambda}$  is a cumulative utility process measuring utility gained over time while  $Y^{\theta, \Lambda}$  measures the terminal utility. Both depend on the control  $\theta$  and the biometric scenario  $\Lambda$ . The control  $\theta$  is an optional process taking values in some measurable space. For example,  $\theta$  could encompass sojourn consumption rates, consumption upon transition and insurance coverages. The process  $\mathcal{E}_\Phi(t)$  is the generalised exponential of the finite variation process  $\Phi$  which can be interpreted as a subjective (cumulative) discount rate. We prove a martingale optimality principle which serves as a sufficient condition of optimality and we use this to solve the problem in two special cases.

The first special case of (1.2.3) we consider is the problem of determining the worst- and best-case reserves given an  $\mathbb{F}^Z$ -adapted payment process  $B$  with terminal

time  $\mathcal{T}$ . The worst case prospective reserve is then, akin to (1.1.4), given by

$$V(t) := \sup_{\Lambda \in \mathcal{M}} \mathbb{E}^\Lambda \left[ \int_{(t, \mathcal{T})} \frac{\mathcal{E}_\Phi(t)}{\mathcal{E}_\Phi(s)} B(ds) \middle| \mathcal{F}_t^Z \right],$$

while the best-case reserve is obtained by replacing the supremum with an infimum. The  $\mathbb{F}^Z$ -predictable process  $\Phi$  is a cumulative interest rate and we allow reserve-dependent payments of the form

$$B(dt) = \sum_{i \in \mathcal{J}} \mathbb{1}_{(Z(t-) = i)} \left( b_i(t, V_i(t)) \gamma_i(dt) + \sum_{j: j \neq i} b_{ij}(t, V_j(t), V_i(t)) N_{ij}(dt) \right)$$

for suitable processes  $(\gamma_i, b_i, (b_{ij})_{i \neq j})_{i \in \mathcal{J}}$ . We show that  $V$  is characterised by the BSDE

$$\underline{V}(dt) = \underline{V}(t-) \Phi(dt) - B(dt) + \sum_{i, j: i \neq j} R_{ij}(t) \underline{M}_{ij}(dt), \quad \underline{V}(\mathcal{T}) = 0, \quad (1.2.4)$$

where  $\underline{M}_{ij}(dt) = N_{ij}(dt) - \mathbb{1}_{(Z(t-) = i)} \underline{\Delta}_{ij}(dt)$  and

$$\mathbb{1}_{(Z(t-) = i)} R_{ij}(t) = \mathbb{1}_{(Z(t-) = i)} (b_{ij}(t) + V_j(t) - V_i(t)).$$

The worst case biometric scenario  $\underline{\Delta}_{ij}(t)$  is given by

$$\underline{\Delta}_{ij}(dt) = \mathbb{1}_{(R_{ij}(t) \geq 0)} U_{ij}(dt) + \mathbb{1}_{(R_{ij}(t) < 0)} L_{ij}(dt). \quad (1.2.5)$$

Whenever the sum-at-risk  $R_{ij}(t)$  is positive the transition from  $i$  to  $j$  results in a financial loss for the insurance company and it is disadvantageous to have many such transitions, which results in  $U_{ij}$  as the choice of worst-case scenario. Whenever  $R_{ij}(t) < 0$ , the situation is reversed. The best-case biometric scenario is given by (1.2.5) with  $U_{ij}$  and  $L_{ij}$  switching places and the best-case reserve is characterised by (1.2.4) with  $\underline{\Delta}$  replaced by the corresponding best-case scenario. This result thus unifies the backward methods for reserve-dependent payments and worst-case reserves described in Subsection 1.1.1 by considering both simultaneously and generalises the existing results by additionally using a more general stochastic model.

The second special case of (1.2.3) we consider is a robust and non-Markovian extension of the consumption-insurance problem treated in Kraft and Steffensen (2008). The goal is to maximise the utility gained from sojourn consumption rates  $(c_i)_{i \in \mathcal{J}}$  and transition consumption  $(c_{ij})_{i \neq j}$  with access to different insurance coverages  $(b_{ij})_{i \neq j}$ . Thus  $\theta = (c_i, (c_{ij}, b_{ij})_{j: j \neq i})_{i \in \mathcal{J}}$  and the process  $U^{\theta, \Lambda}$  is given by

$$U(dt) = \sum_{i \in \mathcal{J}} \mathbb{1}_{(Z(t-) = i)} \left( u_i(t, c_i(t)) dt + \sum_{i: i \neq j} u_{ij}(t, c_{ij}(t)) N_{ij}(dt) \right),$$

while  $Y^{\theta, \Lambda}$  is given by  $u_{Z(T)}^T(X(T))$ , where  $X$  is the wealth of the individual. Using the martingale optimality principle we prove a verification theorem which we

then use to obtain explicit solutions in the case of power utility preferences with path-dependent utility weights corresponding to

$$u_i(t, c) = \frac{w_i^\gamma(t)}{1 - \gamma} c^{1-\gamma} \quad \text{and} \quad u_{ij}(t, c) = \frac{w_{ij}^\gamma(t)}{1 - \gamma} c^{1-\gamma} \quad \text{and} \quad u_i^T(x) = \frac{w_{i,T}^\gamma}{1 - \gamma} x^{1-\gamma}$$

for  $\gamma \in (0, 1) \cup (1, \infty)$ . The robust optimal value process is given by

$$V(t) = \frac{1}{1 - \gamma} Y^\gamma(t) X^{1-\gamma}(t),$$

where  $Y$  satisfies a BSDE of the form (1.2.4) and can be interpreted as a subjective present value of the future stream of utility determined by the weights  $(w_i, w_{i,T}, (w_{ij})_{j:j \neq i})_{i \in \mathcal{I}}$  under the worst-case measure. The optimal consumption processes and insurance coverages are of the same functional form as in Kraft and Steffensen (2008), with all deterministic and time-dependent quantities replaced by random, path-dependent quantities evaluated under the worst-case biometric scenario. The worst case biometric scenario itself is of the form (1.2.5) where the quantities  $\underline{R}_{ij}$  are replaced by  $\frac{\gamma}{1-\gamma} R_{ij}^Y$ , with  $R_{ij}^Y$  being utility-valued sums-at-risk. Thus the individual's conception of what the worst case is depends on the risk-aversion parameter  $\gamma$ . If  $\gamma > 1$  the individual views the future stream of utility as a liability, maximises it under the worst-case case measure, and consequently consumes more conservatively relative to other biometric scenarios. If  $\gamma < 1$  the individual views the future stream of utility as an asset, minimises it under the worst-case measure, and consequently consumes more aggressively relative to other biometric scenarios.

The BSDE (1.2.4) is a non-linear generalisation of (1.1.10) and therefore the existence of a solution is not guaranteed. In fact (1.2.4) violates both the assumption of inaccessible jump times (no point masses in the distribution of jump times) and the assumption of a Lipschitz generator, which are the two main assumptions that usually guarantee existence for this type of marked point process driven BSDE (Confortola and Fuhrman, 2013; Confortola, Fuhrman, and Jacod, 2016; Christiansen and Djehiche, 2020). Our main mathematical contribution is therefore that we under reasonable regularity conditions prove the existence of (1.2.4).

### 1.2.3 Design of smooth pension products

In Chapter 5 we focus specifically on the design of pension products in relation to an individual's preference for smoothing of the financial risk in the consumption rate. In order to focus our discussion on financial risk, we disregard biometric risk, and, as in Example 1.1.6, we assume that  $S = (S_0, S_1)$  is a Black-Scholes market with constant parameters  $(r, \lambda, \sigma)$ . As discussed in Subsection 1.1.2 a variety of alternatives to (1.1.13) have been proposed, but most of these contributions implicitly lead to smoothing, making it difficult to compare different preferences specified

by different utility parameters. Instead, we discuss preferences for smoothing in terms of the components of the consumption rate. This is done by viewing the consumption rate as a stochastic differential equation

$$dc_t = a_t dt + b_t dW_t, \quad C_0 = c_0 > 0. \quad (1.2.6)$$

with an initial value  $c_0$ , a drift term  $a$  and a volatility term  $b$ , as these are quantities one can compare across different utility parameters and preferences. We distinguish between two types of smoothness for the consumption rate: Pathwise smoothness related to the fluctuations (noise) of the consumption rate over time along a single path measured by the expected quadratic variation, and pointwise smoothness related to variability across paths at a fixed time point measured by the variance. We claim that a preference for smoothness relates to a preference for increased pathwise smoothness, but not necessarily increased pointwise smoothness. This can also be stated in terms of different types of risk. Due to (1.2.6), we can decompose the total risk (the variance):

$$\text{Var}(c_t) = \int_0^t \text{Cov}(a_s, c_s) ds + \underbrace{\int_0^t \mathbb{E}[b_s^2] ds}_{=:\mathbb{E}[[c_t]]}.$$

The first term measures trend risk (changes in the level of the consumption rate over long periods of time), while the second term measures the (accumulated) diffusive risk (fluctuations in the consumption rate over short periods of time with mean zero). This decomposition allows us to discuss the preferences for overall risk (pointwise smoothness) in terms of preferences for diffusive risk (pathwise smoothness) and preferences for trend risk. We thus claim that a preference for smoothness is a preference for trend risk over diffusive risk without necessarily being a preference for reduced overall risk.

By considering the components of the consumption rate we can also directly discuss what trade-offs a preference for smoothness induces amongst them. By requiring that the initial wealth equals the value of future consumption, we get that

$$\begin{aligned} X_0 = c_0 \int_0^T e^{-rs} ds + \mathbb{E}^Q[e^{-r(\cdot)} X_T] + \mathbb{E}^Q \left[ \int_0^T e^{-rs} \int_0^s a_u du ds \right] \\ - \mathbb{E}^Q \left[ \int_0^T e^{-rs} \int_0^s \theta b_u du ds \right], \end{aligned} \quad (1.2.7)$$

which indicates that if one wants to increase pathwise smoothness (reduce  $b$ ), then one has to reduce either the trend  $a$ , the initial value  $c_t$ , or the value of the bequest.

In order to study these trade-offs more closely, we consider the following extension of the optimal control problem studied in Bruhn and Steffensen (2013):

$$\min_{a,b,\pi} \mathbb{E} \left[ \int_0^T \frac{w_a}{2} (a_t - \bar{a}(t)c_t)^2 + \frac{w_b}{2} (b_t - \bar{b}(t)c_t)^2 + \frac{w}{2} b_t^2 dt + \frac{B}{2} (X_T - \xi c_T)^2 \right].$$

The consumption rate  $c$  is of the form (1.2.6), while the wealth process  $X$  satisfies

$$dX_t = (r + (\lambda - r)\pi_t)X_t dt - c_t dt + \sigma\pi_t X_t dW_t, \quad X_0 = x_0 > 0.$$

The controls are the drift  $a$  and volatility  $b$  of the consumption rate, and the proportion of wealth invested in risky assets  $\pi$ . There are two important sets of parameters that specify the preferences. First, the constant  $\xi$  determines the target bequest, while the functions  $\bar{a}$  and  $\bar{b}$ , determine the target drift and target volatility of the consumption rate. Inserting them into (1.2.7) and solving for  $c_0$  yields a target initial consumption  $\bar{c}_0$ . Second, the weights  $w_a$ ,  $w_b$ ,  $w$  and  $B$  determine what trade-offs one is willing to make. Note that it is  $w$  that determines the desire for pathwise smoothness as it penalises the volatility of consumption. We find an optimal solution of this problem and use this to study how smoothing can be achieved in two cases. First we reduce the initial consumption rate  $c_0$  below the target  $\bar{c}_0$ , while matching the target drift and without penalising volatility. Second, we reduce the drift while keeping  $c_0$  on target and penalising the volatility.

In the first case we find that by reducing  $c_0$  below  $\bar{c}_0$  we obtain surplus wealth, which can be used to reduce  $\pi$  and  $b$ , resulting in smoothing without penalising volatility. For this approach we have two comments. First, smoothing is achieved purely by reducing  $\pi$  and thus the overall risk. Second, we note that this smoothing approach has its limits, as indefinitely growing the surplus only yields smoothing until a certain point, and for each  $t \in [0, \mathcal{T}]$  we obtain an explicit expression for the initial value of the surplus which minimises  $\mathbb{E}[[c]_t]$ . Consequently this approach on its own cannot yield a perfectly pathwise smooth consumption rate ( $\mathbb{E}[[c]_{\mathcal{T}}] = 0$ ).

In the second case we find that by penalising  $b$  while allowing for a reduction of  $a$ , smoothing can be obtained in two ways. First, the penalty in itself leads to smoothing as it induces an adjusted target consumption rate with reduced drift and reduced volatility relative to  $\bar{a}$  and  $\bar{b}$ . Second, the introduction of the adjusted target consumption rate leads to surplus wealth. If this surplus is positive, then it can have an additional smoothing effect just as before. Contrary to only using the surplus for smoothing, the introduction of the penalty allows us to achieve a perfectly pathwise smooth consumption rate by setting  $w = \infty$ . In that case it is still possible to invest in the risky asset, but one can only use the surplus wealth to do so. Nevertheless it shows, that a preference for pathwise smoothness (diffusive risk) is somewhat decoupled from the preference for overall risk (pointwise smoothness).

## Chapter 2

# Mean-field approximations in insurance

This chapter contains the paper *Hornung (2026)*.

### ABSTRACT

The calculation of the insurance liabilities of a cohort of dependent individuals in general requires the solution of a high-dimensional system of coupled linear forward integro-differential equations, which is infeasible for a larger cohort. However, by using a mean-field model, the high dimensional system of linear forward equations can be replaced by a low-dimensional system of non-linear forward integro-differential equations. We show that, subject to certain regularity conditions, the insurance liability viewed as a (conditional) expectation of a functional of an underlying jump process converges to its mean-field counterpart, as the number of individuals in the cohort goes to infinity. Examples from both life- and non-life insurance illuminate the practical importance of mean-field approximations.

**Keywords:** Reserving; non-linear forward equations; total variation chaos; McKean–Vlasov jump process

## 2.1 Introduction

When modelling the insurance liabilities of a cohort, the individual liability can depend on the other individuals' liabilities, either because the insurance payments of one individual depend on the insurance payments of the other individuals, while the individuals themselves are independent, or because the individuals themselves are dependent. Because of this dependency, the exact calculation of the insurance liabilities requires the numerical solution of high-dimensional systems of equations which often is computationally infeasible. This necessitates the development of viable approximations.

A popular way of approximating the model for a cohort of  $n$  homogeneous but dependent individuals is to use a mean-field model, where instead of modelling the entire cohort jointly, one models only one typical individual. Since this mean-field model reduces the dimension to a single individual, the calculation of the corresponding mean-field insurance liabilities typically becomes computationally feasible. In order for the mean-field model to be a viable approximation of a single individual in the  $n$ -individual model, the individuals of any fixed group of size  $k$  should be asymptotically independent and the insurance liability of a single individual in the  $n$ -individual model should converge to the mean-field insurance liability whenever  $n$  approaches infinity.

In the case of independent individuals with dependent insurance payments it suffices to show convergence of the insurance liabilities as has been done in Djehiche and Löfdahl (2016) with applications to reserve dependent payments in life insurance. In this paper we will focus on the case of dependent individuals, where the asymptotic independence and convergence of insurance liabilities, to the best of our knowledge, has received little to no attention in the literature, even though mean-field models have been used in many instances due to their usefulness in the modelling of risks with contagion effects. In the case of epidemic health insurance, for example, Feng and Garrido (2011) and Feng et al. (2022) apply mean-field models to calculate portfolio-wide premiums and reserves, while Francis and Steffensen (2024) emphasises the importance of calculating individual premiums and reserves and develops an approach to do so using a mean-field model. This is followed by Tran (2024) which in a model similar to Francis and Steffensen (2024) analyses the individual's time to infection and performs parameter estimation. In the case of cyber insurance Fahrenwaldt, Weber, and Weske (2018) and Hillairet et al. (2022) apply different mean-field models to study the impact of network structures on cyber insurance losses, while Hillairet and Lopez (2021) uses a mean-field model to study the role and impact of countermeasures. None of these contributions formally verify that their proposed mean-field models are viable approximations of an appropriate  $n$ -individual model. The purpose and contribution of this paper is to provide a unifying characterisation of a general class of mean-field models and their corresponding  $n$ -individual models and within this class to prove asymptotic independence for a fixed group of  $k$  individuals and the convergence of the insurance liability in the  $n$ -individual model to the corresponding mean-field insurance liability. This not only provides a theoretical foundation for the use of mean-field models as approximations within epidemic health and cyber insurance, but it also enables novel applications such as the modelling of disability insurance using collective information about health insurance claims proposed in Furrer and Horning (2026).

We consider a cohort of  $n$  homogeneous individuals jointly modelled by an  $n$ -dimensional Markov jump process, with each coordinate representing one individual. The compensating measure of the jump process is assumed to be absolutely

continuous with respect to the Lebesgue measure and we allow the intensity kernel to depend on collective quantities, such as cohort averages or functions thereof. Thus the individuals are dependent. The insurance payments of each individual are given by a functional of the corresponding individual's jump process path and the insurance liability of interest is one of three types: The cohort-wide liability defined as the unconditional expectation of the individual insurance payments, the grouped insurance liability defined as the conditional expectation of individual insurance payments given the individual's specific group in a grouping of the state or covariate space and the individual insurance liability defined as the conditional expectation of individual insurance payments given the individual's specific state or covariates. When using the forward method, the calculation of the insurance liability of a single individual thus requires one to solve the system of linear forward integro-differential equations satisfied by the occupation or transition probabilities of the joint  $n$ -dimensional Markov jump process. This is computationally infeasible when  $n$  is large.

The corresponding mean-field model is obtained by replacing all collective quantities by their expectations. In this case the forward integro-differential equations become non-linear, but the dimension of the system remains the same as for a single individual. These non-linear equations are solved by the occupation- or transition probabilities of a distribution dependent non-linear Markov jump process, which is distribution dependent in the sense that the intensity kernel depends on the distribution of the process itself and non-linear Markov in the sense of Rehmeier and Röckner (2025). Thus by switching to the mean-field model, one changes the probabilistic model from a high-dimensional Markov jump process modelling  $n$  dependent individuals to a low-dimensional non-linear Markov jump process modelling one typical individual. The mean-field liability can therefore be obtained as the (conditional) expectation of a functional of a non-linear Markov jump process path.

We show that under suitable regularity conditions the mean-field model exists, is unique and that any fixed group of  $k$  individuals in the  $n$ -individual model becomes asymptotically independent. Furthermore we show that if the individual insurance payments are measurable and uniformly integrable, then the corresponding insurance liability converges to the corresponding mean-field insurance liability, when  $n$  approaches infinity, in both the cohort-wide and grouped case and under some additional assumptions in the individual case as well. Additionally we prove that the cohort average of insurance payments converges in  $L^2$  to the cohort-wide mean-field insurance liability and that the average of insurance payments of a group with certain characteristics converges in probability to the grouped mean-field insurance liability when  $n$  approaches infinity. This shows that the diversification effect of large cohorts persists both on the level of the cohort and on the level of groupings, even though the individuals are dependent.

The key to these results is to show that for large  $n$ , the joint distribution of the jump processes for a fixed group of  $k$  individuals in a cohort of  $n$  individuals converges to the joint distribution of  $k$  independent individuals with a distribution-dependent non-linear Markov jump process. This type of convergence is known as chaos and was first introduced by Kac (1956), while the concept of distribution dependent processes for diffusions stems from McKean (1966, 1969). Ever since these concepts have been further developed in many directions and have found numerous applications outside of insurance (for a very comprehensive review, see Chaintron and Diez (2022a,b)). The convergence of distributions required by chaos can be metrised using different metrics, see Hauray and Mischler (2014). While the original notion of chaos uses weak convergence, we choose the stronger notion of chaos in total variation allowing us to obtain stronger results.

While the papers Shiga and Tanaka (1985) and Djehiche and Kaj (1995) provide different chaosticity results specifically for jump processes, the assumptions on the distribution dependence are too strict for many actuarial applications, as they do not allow for distribution dependent jump sizes. We therefore adapt methods from the jump-diffusion literature. In particular, we modify a coupling construction introduced by Graham (1992b) combined with an approach used by Andreis and Pra (2018) to innovatively prove chaos in total variation for a class of Markov jump processes with potentially unbounded jump sizes. This is also sufficient to obtain convergence of insurance liabilities in the cohort-wide and grouped case and given that the state space is countable, it is also sufficient for the individual case.

If the state space is uncountable, this result is insufficient for the individual case and more work is required. For any fixed  $k$  we can condition on the initial state or covariates for the first  $k$  individuals. Under the assumption that the joint conditional distribution of the initial state or covariates of the remaining  $n - k$  individuals given the initial state or covariates of the first  $k$  individuals is chaotic, we show that the joint conditional distribution of the first  $k$  individuals, given their initial state or covariates, converges in total variation to the joint distribution of  $k$  independent individuals, each following the conditional distribution of a distribution dependent non-linear Markov jump process given the respective initial value or covariates. While this result is not surprising, it has (to the best of our knowledge) not previously received attention in the literature.

Finally we note that the mean-field liabilities considered in this paper can naturally be calculated via the forward method by solving the non-linear forward integro-differential equations for the occupation or transition probabilities of the distribution dependent non-linear Markov jump process. Since only the initial distribution is known and the intensity kernel depends on the occupation probabilities themselves, a backwards approach appears cumbersome. If the individuals are independent but the insurance payments are dependent, then Djehiche and Löfdahl

(2016) shows that a backwards approach is possible. In that case the liability can be calculated by solving a non-linear version of Thiele's backward differential equation, which has been generalised to the non-Markovian case in Christiansen and Djehiche (2020) and the as-if-Markov case in Christiansen and Djehiche (2025).

The structure of the paper is now as follows: Section 2.2 illustrates the programme of the paper using examples from epidemic health insurance and cyber insurance. Section 2.3 introduces distribution dependent non-linear Markov jump processes, and characterises their conditional distributions as the distributions of a certain class of linearised Markov jump processes. Section 2.4 proves total variation chaos in the unconditional case, while Section 2.5 proves the conditional case. Section 2.6 shows the convergence of the three types of insurance liabilities, verifies the two laws of large numbers and provides a central limit theorem followed by a return to the examples from Section 2.2. Finally Section 2.7 concludes with a discussion of the impact different types of chaos have on our results.

## 2.2 Spread of infections in a network

Before developing the general results, we use two examples from epidemic health insurance and cyber insurance to showcase how mean-field models are used in actuarial applications, to pinpoint where the theoretical challenges lie and to indicate how we will solve them. In both cases insurance claims are caused by the spread of an infection in a network of individuals (either people or computers) and since the infection spreads throughout the network by individuals infecting each other, the probability of infection for a particular individual depends not only on their location in the network, but also on the health status of the other individuals. Thus any realistic stochastic model requires all individuals to be dependent. We consider a variation of the network-based Susceptible-Infected-Susceptible (SIS) model of Fahrenwaldt, Weber, and Weske (2018) as a driver for both epidemic health insurance payments inspired by Francis and Steffensen (2024) and cyber insurance payments inspired by Fahrenwaldt, Weber, and Weske (2018).

For this let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and consider a graph with nodes  $\{1, \dots, J\}$ . The edges between the nodes can be represented by the  $J \times J$  adjacency matrix  $A = (a_{ij})_{i,j=1,\dots,J}$ , where the entries  $a_{ij}$  take the values zero or one and we assume that the diagonal entries  $a_{ii}$  are always equal to one for  $i \in \{1, \dots, J\}$ . We can think of the graph as a network, the nodes as locations, and the edges as connections between locations, where  $a_{ij} = 1$  means that individuals in location  $i$  can be infected by individuals in location  $j$ . We now consider  $n$  individuals whose location in the network at time zero is given by the  $\{1, \dots, J\}^n$ -valued random vector  $I_0^n = (I_0^{1,n}, \dots, I_0^{n,n})$  and for simplicity we assume that the individuals stay in the same location and cannot move as time passes. Additionally, each individual  $\ell = 1, \dots, n$  has associated to it a jump process  $Z^{\ell,n}$  which takes values in the state

space  $\{0, 1\}$ , where 0 means susceptible and 1 means infected. The collective model of the entire cohort of individuals can be seen as an  $n$ -dimensional jump process  $X^n = ((Z^{1,n}, I^{1,n}), \dots, (Z^{n,n}, I^{n,n}))$  on state space  $E^n$  with  $E = \{0, 1\} \times \{1, \dots, J\}$ . The processes  $Z^{\ell,n}$  are governed by transition rates  $(\mu_{01}(t, i, X_{t-}^n))_{i=1, \dots, J}$  and  $(\mu_{10}(t, i, X_{t-}^n))_{i=1, \dots, J}$  given by

$$\mu_{01}(t, i, X_{t-}^n) = \beta_i \sum_{j=1}^J a_{ij} \left( \frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_{(Z_{t-}^{\ell,n}=1, I_{t-}^{\ell,n}=j)} \right) \quad \text{and} \quad \mu_{10}(t, i, X_{t-}^n) = \gamma_i,$$

for positive constants  $(\beta_i)_{i=1, \dots, J}$  and  $(\gamma_i)_{i=1, \dots, J}$ . The recovery rates  $\mu_{10}(t, i, X_{t-}^n)$  are location dependent but constant in time, while the infection rates  $\mu_{01}(t, i, X_{t-}^n)$  are a little more complicated. The term

$$\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_{(Z_{t-}^{\ell,n}=1, I_{t-}^{\ell,n}=j)}$$

is equal to the proportion of individuals in location  $j$  that are infected, meaning that the infection rate of an individual in location  $i$  depends on the proportion of infected individuals in all connected locations  $j$ . Thus there is a contagion effect that depends on the collective: The higher the proportion of infected individuals in connected locations, the larger the rate of infection for a susceptible individual in location  $i$ .

If we want to describe the whole joint process  $X^n$  precisely we can write

$$X_t^n = Y_0^n + \int_{(0,t] \times E^n} (y^{1:n} - X_{s-}^n) Q^n(dt, dy^{1:n}), \quad (2.2.1)$$

where  $Y_0^n$  has some exchangeable initial distribution  $\zeta^n$  and  $Q^n$  is a random counting measure with state space  $E^n$  and compensating measure

$$L^n(dt, dy^{1:n}) = \sum_{\ell=1}^n \left( \mu_t(X_{t-}^{\ell,n}, X_{t-}^n, dy_{\ell}) \prod_{j=1, j \neq \ell}^n \delta_{\{X_{t-}^{j,n}\}}(dy_j) \right) dt.$$

Here  $\mu_t(X_{t-}^{\ell,n}, X_{t-}^n, d(z, i))$  is the individual intensity kernel given by

$$\begin{aligned} \mu_t(X_{t-}^{\ell,n}, X_{t-}^n, d(z, i)) &= \mathbb{1}_{(Z_{t-}^{\ell,n}=0)} \mu_{01}(t, I_{t-}^{\ell,n}, X_{t-}^n) \delta_{\{1, I_{t-}^{\ell,n}\}}(d(z, i)) \\ &\quad + \mathbb{1}_{(Z_{t-}^{\ell,n}=1)} \gamma_{I_{t-}^{\ell,n}} \delta_{\{1, I_{t-}^{\ell,n}\}}(d(z, i)). \end{aligned}$$

It follows that while the process  $X^n$  describing the entire collective is a Markov process, the individual processes  $X^{\ell,n}$  generally are not Markov by themselves.

**Example 2.2.1** (Epidemic health insurance). In the case of epidemic health insurance, the process  $X^n$  describes the spread of an infectious disease amongst a population whose individuals are located at different geographic locations. Thus

each node in the graph  $\{1, \dots, J\}$  can be seen as a geographic location, such as a country or city, while  $a_{ij} = 1$  means that location  $i$  is susceptible to infection from location  $j$ . Each individual has contractual payments given by

$$B^{\ell,n}(dt) = -\mathbb{1}_{(Z_{t-}^{\ell,n}=0)}\pi(I_{t-}^{\ell,n})dt + \mathbb{1}_{(Z_{t-}^{\ell,n}=1)}b(I_{t-}^{\ell,n})dt,$$

where  $\pi : \{1, \dots, J\} \rightarrow (0, \infty)$  is a positive, location dependent premium rate paid continuously while susceptible and  $b : \{1, \dots, J\} \rightarrow (0, \infty)$  is a positive, location dependent benefit rate received continuously while infected.

Assume that the expiration of the insurance contract is at time  $T > 0$  and that  $r \in \mathbb{R}$ . Throughout the duration of the contract, the individual insurance liability at time  $\tau \in [0, T]$ , which the insurance company should be able to calculate for reserving purposes, is given by

$$\begin{aligned} V_z^{\ell,n}(\tau, i) &= \mathbb{E} \left[ \int_{\tau}^T e^{-r(t-\tau)} B^{\ell,n}(dt) \middle| Z_{\tau}^{\ell,n} = z, I_{\tau}^{\ell,n} = i \right] \\ &= -\pi(i) \int_{\tau}^T e^{-r(t-\tau)} p_{z0}^i(\tau, t) dt + b(i) \int_{\tau}^T e^{-r(t-\tau)} p_{z1}^i(\tau, t) dt, \end{aligned}$$

where  $p_{z0}^i(\tau, t) = \mathbb{P}(Z_t^{\ell,n} = 0 | Z_{\tau}^{\ell,n} = z, I_{\tau}^{\ell,n} = i)$ . If the process  $(Z^{\ell,n}, I^{\ell,n})$  in itself was a Markov process, then  $p_{z0}^i(\tau, t)$  would correspond to the transition probabilities and could be calculated by solving the classical Kolmogorov forward differential equations. But since the individuals are not independent, the process  $(Z^{\ell,n}, I^{\ell,n})$  is not Markov. Instead it is only the process  $X^n$  describing the state of the entire cohort that is Markov. While this means that we in principle could obtain  $p_{z0}^i(\tau, t)$  from the transition probabilities of the process  $X^n$ , which can be calculated by solving the classical Kolmogorov forward differential equations, this system grows exponentially in dimension with the number of individuals, rendering this approach computationally infeasible.

**Example 2.2.2** (Cyber insurance). In the case of cyber insurance, the process  $X^n$  describes the spread of malware amongst individual computers who are part of different companies. Thus each node in the graph  $\{1, \dots, J\}$  can be seen as a company, while  $a_{ij} = 1$  means that company  $i$  is susceptible to malware from company  $j$ . Assume that the time of expiration of the insurance contract is  $T > 0$ . Then the insured losses caused by a malware infection of computer  $\ell$  throughout the duration of the contract are given by

$$B^{\ell,n}(T) = \sum_{m=1}^{N_{01}^{\ell,n}(T)} Y_m^{\ell,n}(I_0^{\ell,n}),$$

where  $N_{01}^{\ell,n}(T) := \#\{t \in (0, T] : Z_{t-}^{\ell,n} = 0, Z_t^{\ell,n} = 1\}$  is the number of times computer  $\ell$  has been infected with malware throughout the period  $(0, T]$ . Thus

every time computer  $\ell$  is infected with malware, it causes a loss  $Y_m^{\ell,n}(I_0^{\ell,n})$ . For each  $i = 1, \dots, J$  and  $\ell = 1, \dots, n$  the sequence  $(Y_m^{\ell,n}(i))_{m \in \mathbb{N}}$  is an i.i.d. sequence of  $[0, \infty)$ -valued random variables with distribution  $r(i, dy) = h_i(y)dy$ . The sequences are independent of each other and of  $X^n$  and identically distributed across different individuals. Since all individuals and claim sizes are identically distributed, so are the losses. Thus when pricing this type of contract the insurance company should at least be able to calculate the cohort-wide expected loss of a computer in the network given by

$$V^n(0) := \mathbb{E}[B^{1,n}(T)].$$

Since the different companies might have different levels of risk, charging the same premium  $V^n(T)$  to all the companies might not be fair. Therefore the insurance company should ideally be able to calculate the individual expected loss of a computer in the network, given that it is in a specific company and that it is free of malware at contract inception:

$$V_0^{1,n}(0, i) := \mathbb{E}[B^{1,n}(T) | Z_0^{1,n} = 0, I_0^{1,n} = i].$$

With our assumptions, we can obtain the expressions:

$$\begin{aligned} V^{1,n}(0) &= \int_0^T \mathbb{E} \left[ \mathbb{1}_{(Z_{t-}^{\ell,n}=0)} \mu_{01}(t, i, X_{t-}^n) \int_{[0,\infty)} y r(I_0^{\ell,n}, dy) \right] dt \\ V_0^{1,n}(0, i) &= \int_0^T \mathbb{E} \left[ \mathbb{1}_{(Z_{t-}^{\ell,n}=0)} \mu_{01}(t, i, X_{t-}^n) | Z_0^{\ell,n} = z, I_0^{\ell,n} = i \right] \int_{[0,\infty)} y r(i, dy) dt. \end{aligned}$$

As in the previous example, the calculation of the expectations in the integrals requires the calculation of the transition probabilities of  $X^n$ , yielding the same computational issues.

One way around this computational issue for large  $n$  is to approximate the  $n$ -individual model (2.2.1) by a mean-field model. If one can argue that for large  $n$  we have that

$$\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_{(Z_{t-}^{\ell,n}=1, I_{t-}^{\ell,n}=j)} \approx \mathbb{P}(Z_{t-}^{\ell,n} = 1, I_{t-}^{\ell,n} = j), \quad (2.2.2)$$

then the former can be replaced by the latter, an approach which is often used without formal verification. Since the initial distribution  $\zeta^n$  is exchangeable all individuals have the same marginal initial distribution  $\zeta^{n,1}(dx) := \zeta^n(dx \times E^{n-1})$  and since the intensities are the same for all individuals, they are all identically distributed. Thus we obtain an approximating model  $\bar{X} := (\bar{Z}, \bar{I})$  for a typical individual, where  $(\bar{Z}_0, \bar{I}_0)$  has distribution  $\zeta$  (for now think  $\zeta(dx) = \zeta^n(dx \times E^{n-1})$ ) and  $\bar{Z}$  is governed by the intensities

$$\mu_{01}(t, i, \bar{\mathbf{p}}_t^\zeta) = \beta_i \sum_{j=1}^J a_{ij} \bar{p}_1^\zeta(t, j) \quad \text{and} \quad \mu_{10}(t, i, \bar{\mathbf{p}}_t^\zeta) = \gamma_i,$$

where  $\bar{\mathbf{p}}_t^\zeta := (\bar{p}_1^\zeta(t, 1), \dots, \bar{p}_1^\zeta(t, J))$  and  $\bar{p}_1^\zeta(t, i) := \mathbb{P}(\bar{Z}_t = 1, \bar{I}_t = i)$  are the occupation probabilities. The process  $\bar{X} = (\bar{Z}, \bar{I})$  with state space  $E$  can thus be described as

$$\bar{X}_t = \bar{Y}_0 + \int_{(0,t] \times E} (y - \bar{X}_{s-}) \bar{Q}(dt, dy), \quad (2.2.3)$$

where  $\bar{Y}_0 = (\bar{Z}_0, \bar{I}_0)$  has initial distribution  $\zeta$  and  $\bar{Q}$  is a random counting measure with state space  $E$  and compensating measure

$$\bar{L}(dt, dx) = \mu_t(\bar{X}_{t-}, \bar{\mathbf{p}}_t^\zeta, dx) dt.$$

Here  $\mu_t(\bar{X}_{t-}, \bar{\mathbf{p}}_t^\zeta, dx)$  is the individual intensity kernel given by

$$\begin{aligned} \mu_t(\bar{X}_{t-}, \bar{\mathbf{p}}_t^\zeta, d(z, i)) = & \mathbb{1}_{(\bar{Z}_{t-}=0)} \mu_{01}(t, \bar{I}_{t-}, \bar{\mathbf{p}}_t^\zeta) \delta_{\{1, \bar{I}_{t-}\}}(d(z, i)) \\ & + \mathbb{1}_{(\bar{Z}_{t-}=1)} \gamma_{\bar{I}_{t-}} \delta_{\{1, \bar{I}_{t-}\}}(d(z, i)). \end{aligned} \quad (2.2.4)$$

Note that we again have assumed that the individuals all remain in their initial locations. Assuming that  $\bar{X}$  is well-defined, we see that the intensities of  $\bar{Z}$  no longer depend on the random state of the entire cohort, but instead depend on the deterministic probabilities  $\bar{\mathbf{p}}_t^\zeta$ . As a consequence, the process  $\bar{X}$  only depends on its own state, thus achieving a dimension reduction, but it also depends on its own distribution and is therefore no Markov process in the classical sense. Apart from the future only depending on the past through the present state, the transition probabilities of a Markov process are independent of the initial distribution. The latter property breaks down in the case of  $\bar{X}$ , because the transition intensities at time  $t > 0$  depend on the occupation probabilities  $\bar{\mathbf{p}}_t^\zeta$ , which of course vary with the initial distribution  $\zeta$ . This becomes crystal clear from a description of how to calculate the occupation and transition probabilities. First one finds  $\bar{\mathbf{p}}_t^\zeta$  by solving the following system of non-linear Kolmogorov forward differential equations

$$\begin{aligned} \frac{d}{dt} \bar{p}_0^\zeta(t, i) &= \gamma_i \bar{p}_1^\zeta(t, i) - \mu_{01}(t, i, \bar{\mathbf{p}}_t^\zeta) \bar{p}_0^\zeta(t, i), \quad \bar{p}_0^\zeta(0, i) = \zeta(0, i), \\ \frac{d}{dt} \bar{p}_1^\zeta(t, i) &= \mu_{01}(t, i, \bar{\mathbf{p}}_t^\zeta) \bar{p}_0^\zeta(t, i) - \gamma_i \bar{p}_1^\zeta(t, i), \quad \bar{p}_1^\zeta(0, i) = \zeta(1, i), \end{aligned} \quad (2.2.5)$$

for  $i = 1, \dots, J$  and  $t \in [0, T]$ . Note that this system is coupled across different values of  $i$  through the vector  $\bar{\mathbf{p}}_t^\zeta$ . The system can be solved numerically using standard methods for non-linear ordinary differential equations. After this, one can then find the transition probabilities  $\bar{p}_{zy}^\zeta(i, \tau, t) := \mathbb{P}(\bar{Z}_t = y | \bar{Z}_\tau = z, \bar{I}_\tau = i)$  by

solving the system of linear Kolmogorov forward differential equations

$$\begin{aligned}
\frac{d}{dt}\tilde{p}_{00}^\zeta(\tau, t, i) &= \gamma_i\tilde{p}_{01}^\zeta(\tau, t, i) - \mu_{01}(t, i, \bar{\mathbf{p}}_t^\zeta)\tilde{p}_{00}^\zeta(\tau, t, i), & \tilde{p}_{00}^\zeta(\tau, \tau, i) &= 1, \\
\frac{d}{dt}\tilde{p}_{01}^\zeta(\tau, t, i) &= \mu_{01}(t, i, \bar{\mathbf{p}}_t^\zeta)\tilde{p}_{00}^\zeta(\tau, t, i) - \gamma_i\tilde{p}_{01}^\zeta(\tau, t, i), & \tilde{p}_{01}^\zeta(\tau, \tau, i) &= 0, \\
\frac{d}{dt}\tilde{p}_{11}^\zeta(\tau, t, i) &= \mu_{01}(t, i, \bar{\mathbf{p}}_t^\zeta)\tilde{p}_{10}^\zeta(\tau, t, i) - \gamma_i\tilde{p}_{11}^\zeta(\tau, t, i), & \tilde{p}_{11}^\zeta(\tau, \tau, i) &= 1, \\
\frac{d}{dt}\tilde{p}_{10}^\zeta(\tau, t, i) &= \gamma_i\tilde{p}_{11}^\zeta(\tau, t, i) - \mu_{01}(t, i, \bar{\mathbf{p}}_t^\zeta)\tilde{p}_{10}^\zeta(\tau, t, i), & \tilde{p}_{10}^\zeta(\tau, \tau, i) &= 0,
\end{aligned} \tag{2.2.6}$$

for  $i = 1, \dots, J$  and  $0 \leq \tau \leq t \leq T$ . Note that this system is not coupled across  $i$ , since  $(\bar{\mathbf{p}}_t^\zeta)_{t \in [0, T]}$  is known. Again this system can be solved using standard methods.

The intuition behind the dependence of the transition probabilities on the initial distribution can be explained as follows. The initial distribution determines the composition of the cohort for which the process  $\bar{X}$  represents the typical individual. Changing the initial distribution thus corresponds to changing the composition of the cohort, which the typical individual is a representative of. This illustrates why the mean-field approximation is useful. It reduces the dimension of the problem to one typical individual without completely ignoring the characteristics of the cohort that this individual represents.

**Example 2.2.3** (Epidemic health insurance). When using the mean-field model in the case of epidemic health insurance the contractual payments of the typical individual are given by

$$\bar{B}(dt) = -\mathbb{1}_{(\bar{Z}_{t-}=0)}\pi(\bar{I}_{t-})dt + \mathbb{1}_{(\bar{Z}_{t-}=1)}b(\bar{I}_{t-})dt,$$

while the insurance liability of the typical individual is given by

$$\bar{V}_z(\tau, i) = -\pi(i) \int_\tau^T e^{-r(t-\tau)} \tilde{p}_{z0}(\tau, t, i) dt + b(i) \int_\tau^T e^{-r(t-\tau)} \tilde{p}_{z1}(\tau, t, i) dt.$$

Thus to calculate the insurance liability we need to solve the two systems (2.2.5) and (2.2.6) and then approximate the integrals. As this is computationally feasible, the mean-field approximation  $V_z^{1,n}(\tau, i) \approx \bar{V}_z(\tau, i)$  would be a useful and viable alternative.

**Example 2.2.4** (Cyber insurance). When using the mean-field model in the case of cyber insurance, the insured loss of a typical computer now takes the form

$$\bar{B}(T) = \sum_{m=1}^{\bar{N}_{01}(T)} \bar{Y}_m(\bar{I}_0),$$

where  $(\bar{Y}_m(i))_{m \in \mathbb{N}}$  for each  $i = 1, \dots, J$  is an i.i.d. sequence of  $[0, \infty)$ -valued random variables, independent of  $\bar{X}$ , with distribution  $r(i, dy) = h_i(y)dy$ . The cohort-wide

and individual expected losses are now given by

$$\bar{V}(0) := \mathbb{E}[\bar{B}(T)] \quad \text{and} \quad \bar{V}_0^n(0, i) := \mathbb{E}[\bar{B}(T) | \bar{Z}_0 = 0, \bar{I}_0 = i],$$

which are equal to

$$\begin{aligned} \bar{V}(0) &= \int_0^T \sum_{i=1}^J \bar{p}_0^\zeta(t, i) \mu_{01}(t, i, \bar{\mathbf{p}}_t^\zeta) \int_{[0, \infty)} y r(i, dy) dt, \\ \bar{V}_0(0, i) &= \int_0^T \bar{p}_{00}^\zeta(0, t, i) \mu_{01}(t, i, \bar{\mathbf{p}}_t^\zeta) \int_{[0, \infty)} y r(i, dy) dt. \end{aligned}$$

Again the calculation of these two quantities requires the solution of the two systems (2.2.5) and (2.2.6) and an approximation of the integrals. As this is computationally feasible, the mean-field approximations  $V^{1,n}(0) \approx \bar{V}(0)$  and  $V_0^{1,n}(0, i) \approx \bar{V}_0(0, i)$  would be useful and viable alternatives.

Thus it is clear that the mean-field approximation can be very useful in practice when dealing with situations where the insured individuals can no longer be assumed to be independent, but in order to actually apply them, we have to formally justify their use. This requires us to show three things.

1. The distribution of  $\bar{X}$  denoted by  $\bar{\mathbb{Q}}_{0,\zeta} := \bar{X}(\mathbb{P})$  exists and is unique.
2. The approximation (2.2.2) can be justified in an appropriate sense.
3. It holds that  $V_z^{1,n}(\tau, i) \rightarrow \bar{V}_z(\tau, i)$  and

$$V^{1,n}(0) \rightarrow \bar{V}(0) \quad \text{and} \quad V_0^{1,n}(0, i) \rightarrow \bar{V}_0(0, i).$$

for  $n \rightarrow \infty$ .

Point (1) can be achieved by an appropriate fixed point argument. Point (2) can be achieved by showing that the distribution of  $(X^{1,n}, \dots, X^{k,n})$ , given by  $\mathbb{Q}_{0,\zeta^n}^{n,k} := (X^{1,n}, \dots, X^{k,n})(\mathbb{P})$ , for any fixed  $k \in \mathbb{N}$  converges to the product measure  $\mathbb{Q}_{0,\zeta}^{\otimes k}$ , where  $\bar{\mathbb{Q}}_{0,\zeta} := \bar{X}(\mathbb{P})$ , either weakly or in total variation. Point (3) then requires us to show convergence of both unconditional and conditional expectations. In point (2) we have in particular already shown  $\mathbb{Q}_{0,\zeta^n}^{n,1} \rightarrow \bar{\mathbb{Q}}_{0,\zeta}$  weakly or in total variation, so in order to conclude convergence of unconditional expectations we have to show that the insurance payments are sufficiently regular to strengthen convergence of probability measures to convergence of moments. In order to conclude convergence of conditional moments, we have to show that

$$\mathbb{Q}_{0,\zeta^n}^{n,1}(df | X_\tau^{1,n} = x) \rightarrow \bar{\mathbb{Q}}_{0,\zeta}(df | \bar{X}_\tau = x)$$

either weakly or in total variation for any  $\tau \in [0, T]$  for some regular versions of these conditional probabilities and then similarly be able to strengthen that convergence to convergence of moments.

On a more fundamental level we are also interested in verifying that the diversification effect of large portfolios persists, even though the individuals are no longer independent and can infect each other. In particular we would like to obtain a result like

$$\frac{1}{n} \sum_{\ell=1}^n B^{\ell,n}(T) \rightarrow \bar{V}(0),$$

where the convergence is either almost surely, in  $L^p$  or in probability. This would confirm that the risks remain insurable and that the cohort-wide mean-field liability actually equals the limit of the average insurance payments.

This wish list now provides the outline for the rest of the paper. In Section 2.3 we prove an existence and uniqueness result for a class of distribution dependent non-linear Markov jump processes on a standard Borel state space, which can be interpreted as a class of mean-field models. In Section 2.4 we prove total variation convergence of the marginal distributions of  $k$  individuals in an appropriate class of  $n$ -individual models to the  $k$ -fold product distribution of a distribution dependent non-linear Markov jump process, which enables us to prove the convergence of conditional distributions in Section 2.5. In Section 2.6 we show the desired convergence of cohort-wide, individual, and grouped insurance liabilities, followed by a law of large numbers for insurance payments of quite general form. We conclude Section 2.6 by specialising these results to typical non-life and life insurance settings, while revisiting the examples of this section. Finally in Section 2.7 we discuss the implications for our results when switching to weak convergence instead.

## 2.3 Jump processes

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $T > 0$  be a terminal time and let  $(E, \mathcal{B}(E))$  be a standard Borel space with  $\mathcal{P}(E)$  denoting the set of probability measures on  $(E, \mathcal{B}(E))$ . The set of non-explosive jump process paths on  $[\tau, T]$  for  $0 \leq \tau < T$  with state space  $E$  is given by:

**Definition 2.3.1.** *Let  $0 \leq \tau < T$  and let  $f : [\tau, T] \rightarrow E$ . Then the function  $f$  is a jump process path if and only if*

- (i)  $t \mapsto f(t)$  is càdlàg and piecewise constant.
- (ii) The jump times  $\tau_0 := \tau$  and  $\tau_n := \inf\{t > \tau_{n-1} : f(t) \neq f(\tau_{n-1})\}$  with the convention  $\inf \emptyset := \infty$  satisfy  $\lim_{n \rightarrow \infty} \tau_n = \infty$ .

The set of non-explosive jump process paths is denoted by  $\mathbb{H}([\tau, T], E)$ .

We equip the set  $\mathbb{H}([\tau, T], E)$  with the sigma-algebra  $\mathcal{B}(\mathbb{H}([\tau, T], E))$  generated by the coordinate projections  $\pi_t(f) = f(t)$ . Note that  $\mathbb{H}([\tau, T], E)$  is a Borel

subset of the Skorokhod space  $\mathbb{D}([\tau, T], E)$  of càdlàg paths equipped with the Borel sigma-algebra generated by the  $J_1$ -topology and that  $\mathcal{B}(\mathbb{H}([\tau, T], E))$  is equal to the Borel sigma-algebra generated by the corresponding  $J_1$ -subspace topology. The set  $\mathcal{P}(\mathbb{H}([\tau, T], E))$  is the set of probability measures on  $(\mathbb{H}([\tau, T], E), \mathcal{B}(\mathbb{H}([\tau, T], E)))$ . The canonical jump process  $X^\circ : \mathbb{H}([\tau, T], E) \rightarrow \mathbb{H}([\tau, T], E)$  is given by the identity mapping  $X^\circ(f) = f$  (we will also use the notation  $X_t^\circ := \pi_t(X^\circ)$ ) and any probability measure  $\mathbb{Q} \in \mathcal{P}(\mathbb{H}([\tau, T], E))$  can be seen as a jump process distribution.

Any random variable  $X : \Omega \rightarrow \mathbb{H}([\tau, T], E)$  can be viewed as a non-explosive jump process and for the purposes of this paper, we construct jump processes and their distributions via stochastic differential equations driven by a random counting measure (and its associated marked point process).

### 2.3.1 Markov jump processes

For an initial time  $\tau \in [0, T]$  and an initial distribution  $\zeta \in \mathcal{P}(E)$  we consider the jump process  $X^{\tau, \zeta}$  given by

$$X_t^{\tau, \zeta} = Y_0^{\tau, \zeta} + \int_{(\tau, t] \times A} (y - X_{s-}^{\tau, \zeta}) Q(ds, dy), \quad t \in [\tau, T], \quad (2.3.1)$$

where  $Y_0^{\tau, \zeta}$  has distribution  $\zeta$  and  $Q$  is a random counting measure with state space  $E$  and compensating measure

$$L(dt, dy) = \mu_t(X_{t-}^{\tau, \zeta}, dy)dt.$$

Here  $\mu : [0, T] \times E \times \mathcal{B}(E) \rightarrow [0, \infty)$  is the intensity kernel and it is assumed that  $\mu$  satisfies the following:

- (i)  $(t, x) \mapsto \mu_t(x, B)$  is measurable for all  $B \in \mathcal{B}(E)$
- (ii)  $B \mapsto \mu_t(x, B)$  is a positive and bounded measure for all  $(t, x) \in [\tau, T] \times E$
- (iii)  $\mu_t(x, \{x\}) = 0$

The first two assumptions ensure that  $\mu$  is a kernel, while the third assumption ensures that there are no trivial jumps. The intensity measure can be decomposed into the Borel-measurable jump intensity  $\lambda_t(x) := \mu_t(x, E)$  and the probability kernel  $r_t(x, dy) := \frac{\mu_t(x, dy)}{\lambda_t(x)}$ , which is a regular version of the conditional distribution of the next jump destination given that there is a jump at time  $t$  and that  $X_{t-}^{\tau, \zeta} = x$ .

The random counting measure  $Q$  is given by

$$Q(dt, dy) = \sum_{i=1}^{\infty} \delta_{\{(T_i^{\tau, \zeta}, Y_i^{\tau, \zeta})\}}(dt, dy),$$

for a marked point process  $(T_i^{\tau, \zeta}, Y_i^{\tau, \zeta})_{i \in \mathbb{N}} \subset ([\tau, T] \cup \{\infty\}) \times (E \cup \{\nabla\})$ . Since  $Q$  has no trivial jumps, the sequence of jump times  $(T_i^{\tau, \zeta})_{i \in \mathbb{N}} \subset [\tau, T] \cup \{\infty\}$  coincides

with the jump times of  $X^{\tau, \zeta}$ , while the marks  $(Y_i^{\tau, \zeta})_{i \in \mathbb{N}} \subset E \cup \{\nabla\}$  coincide with the value of  $X_t^{\tau, \zeta}$  whenever  $T_i^{\tau, \zeta} \leq t < T_{i+1}^{\tau, \zeta}$ . We can thus write

$$X_t^{\tau, \zeta} = \sum_{i=0}^{\infty} Y_i^{\tau, \zeta} \mathbb{1}_{(T_i^{\tau, \zeta} \leq t < T_{i+1}^{\tau, \zeta})}, \quad (2.3.2)$$

where  $T_0^{\tau, \zeta} := \tau$  and  $Y_0^{\tau, \zeta}$  is the initial value. The mark  $\nabla$  can be interpreted as the event that never happens and  $Y_i^{\tau, \zeta} = \nabla$  if and only if  $T_i^{\tau, \zeta} = \infty$ . The jump process distribution of (2.3.1) will be denoted by  $\mathbb{Q}_{\tau, \zeta} := X^{\tau, \zeta}(\mathbb{P})$  and it is an element of  $\mathcal{P}(\mathbb{H}([\tau, T], E))$ . We now have the following result:

**Theorem 2.3.2.** *Assume that there exists a constant  $C_\lambda > 0$  such that  $\lambda_t(x) \leq C_\lambda$  for all  $t \in [0, T]$  and  $x \in E$ . Then the jump process (2.3.1) has a unique non-explosive jump process distribution  $\mathbb{Q}_{\tau, \zeta} \in \mathcal{P}(\mathbb{H}([\tau, T], E))$ .*

*Proof.* The random counting measure  $Q$  can be directly identified with its associated marked point process  $(T_i^{\tau, \zeta}, Y_i^{\tau, \zeta})_{i \in \mathbb{N}_0}$ , see Section 2.2 of Last and Brandt (1999) and by Theorem 8.1.4 in Last and Brandt (1999) the distribution of  $(T_i^{\tau, \zeta}, Y_i^{\tau, \zeta})_{i \in \mathbb{N}_0}$  is uniquely determined by the compensating measure  $L$  of  $Q$  and the initial distribution  $\zeta \in \mathcal{P}(E)$ . Theorem 8.2.2 of Last and Brandt (1999) yields existence of this distribution which by Lemma 2.B.2 is equivalent to existence and uniqueness of the jump process distribution of (2.3.1). The non-explosiveness follows, as

$$\mathbb{E}[Q((\tau, T] \times E)] = \mathbb{E}\left[\int_{\tau}^T \mu_t(X_{t-}^{\tau, \zeta}, E) dt\right] \leq C_\lambda(T - \tau) < \infty.$$

□

*Remark 2.3.3.* Note that the assumption  $\mu_t(x, \{x\}) = 0$  is not required for Theorem 2.3.2 to apply. In that case the same jump process distribution might be constructed using different random counting measures, but the same random counting measure will not yield different jump process distributions, see Lemma 2.B.2.

A special case of (2.3.1) that is of particular interest is the jump process

$$X_t^{\tau, x} = x + \int_{(\tau, t] \times E} (y - X_{s-}^{\tau, x}) Q(ds, dy), \quad t \in [\tau, T], \quad (2.3.3)$$

which has a deterministic initial value, corresponding to  $\zeta = \delta_{\{x\}}$ . Let  $\mathbb{Q}_{\tau, x} := X^{\tau, x}(\mathbb{P})$  denote the jump process distribution of (2.3.6) and  $(\mathcal{F}_t^{\tau, \zeta})_{\tau \leq t \leq T}$  given by  $\mathcal{F}_t^{\tau, \zeta} := \sigma(X_s^{\tau, \zeta} : \tau \leq s \leq T)$  the natural filtration of  $X^{\tau, \zeta}$ . Then we have that:

**Theorem 2.3.4.** *Let  $B \in \mathcal{B}(\mathbb{H}([s, T], E))$ . For fixed  $\tau \in [0, T]$  the family  $(\mathbb{Q}_{\tau, x})_{x \in E}$  constitutes a regular conditional probability of  $\mathbb{P}((X_t^{\tau, \zeta})_{t \in [\tau, T]} \in B | X_\tau^{\tau, \zeta} = x)$ . Furthermore for any  $0 \leq \tau \leq s \leq t \leq T$  it holds that*

$$\mathbb{P}((X_t^{\tau, \zeta})_{t \in [s, T]} \in B | \mathcal{F}_s^{\tau, \zeta}) = \mathbb{Q}_{s, X_s^{\tau, \zeta}}(B), \quad \mathbb{P} - a.s.$$

*Proof.* For a proof see Appendix 2.B.  $\square$

Theorem 2.3.4 shows that  $X^{\tau,\zeta}$  is a Markov process. Note that the conditional distribution of  $(X_t^{\tau,\zeta})_{t \in [s,T]}$  given  $X_s^{\tau,\zeta} = x$  for any  $s \in [\tau, T]$  has a regular version which does not depend on the initial distribution  $\zeta$ . This means that  $(\mathbb{Q}_{s,x})_{x \in E}$  is a regular conditional distribution of  $(X_t^{\tau,\zeta})_{s \leq t \leq T}$  given  $X_s^{\tau,\zeta} = x$  for any  $\zeta \in \mathcal{P}(E)$ .

When it comes to practical calculations, we are mostly interested in the transition probabilities  $p_t^{\tau,x} := \pi_t(\mathbb{Q}_{\tau,x})$  and occupation probabilities  $p_t^{\tau,\zeta} := \pi_t(\mathbb{Q}_{\tau,\zeta})$ . The former satisfy the well-known (see Feller (1940) and Feinberg, Mandava, and Shiryaev (2014)) forward integro-differential equations given by:

**Proposition 2.3.5.** *Let  $B \in \mathcal{B}(E)$ . The transition probability  $p_t^{\tau,x}(B)$  satisfies the forward integro-differential equation*

$$\frac{d}{dt} p_t^{\tau,x}(B) = \int_{E \setminus B} \mu_t(y, B) p_t^{\tau,x}(dy) - \int_B \mu_t(y, E \setminus B) p_t^{\tau,x}(dy),$$

with  $p_\tau^{\tau,x}(B) = \delta_{\{x\}}(B)$  for  $t \in [\tau, T]$ ,  $x \in E$  and  $\tau \in [0, T]$ .

Since Theorem 2.3.4 directly implies

$$p_t^{\tau,\zeta}(B) = \int_E p_t^{\tau,x}(B) \zeta(dx) \quad (2.3.4)$$

and the transition probabilities  $(p_t^{\tau,x})_{x \in E}$  do not depend on  $\zeta$ , one can easily calculate the occupation probabilities  $p_t^{\tau,\zeta}$  for any  $\zeta \in \mathcal{P}(E)$ , once  $(p_t^{\tau,x})_{x \in E}$  is obtained, but by using (2.3.4) we can also prove that  $p_t^{\tau,\zeta}$  can be calculated by directly solving the following integro-differential equations:

**Proposition 2.3.6.** *Let  $B \in \mathcal{B}(E)$ . The occupation probability  $p_t^{\tau,\zeta}(B)$  satisfies the forward integro-differential equation*

$$\frac{d}{dt} p_t^{\tau,\zeta}(B) = \int_{E \setminus B} \mu_t(x, B) p_t^{\tau,\zeta}(dx) - \int_B \mu_t(x, E \setminus B) p_t^{\tau,\zeta}(dx),$$

with  $p_\tau^{\tau,\zeta}(B) = \zeta(B)$  for  $t \in [\tau, T]$  and  $(\tau, \zeta) \in [0, T] \times \mathcal{P}(E)$ .

*Proof.* By (2.3.4) and Proposition 2.3.5 we have that:

$$\begin{aligned} p_t^{\tau,\zeta}(B) &= \int_E p_\tau^{\tau,x}(B) \zeta(dx) + \int_{(\tau,t]} \int_{E \setminus B} \mu_s(y, B) \int_E p_s^{\tau,x}(dy) \zeta(dx) ds \\ &\quad - \int_{(\tau,t]} \int_B \mu_s(y, E \setminus B) \int_E p_s^{\tau,x}(dy) \zeta(dx) ds \\ &= \zeta(B) + \int_{(\tau,t]} \int_{E \setminus B} \mu_s(y, B) p_s^{\tau,\zeta}(dy) ds \\ &\quad - \int_{(\tau,t]} \int_B \mu_s(y, E \setminus B) p_s^{\tau,\zeta}(dy) ds. \end{aligned}$$

Differentiating with respect to  $t$  finishes the proof.  $\square$

### 2.3.2 Distribution dependent non-linear Markov jump processes

For an initial time  $\tau \in [0, T]$  and an initial distribution  $\zeta \in \mathcal{P}(E)$  we now consider the distribution dependent non-linear Markov jump process  $\bar{X}^{\tau, \zeta}$  given by

$$\bar{X}_t^{\tau, \zeta} = \bar{Y}_0^{\tau, \zeta} + \int_{(\tau, t] \times E} (y - \bar{X}_{s-}^{\tau, \zeta}) \bar{Q}(ds, dy), \quad t \in [\tau, T], \quad (2.3.5)$$

where  $\bar{Y}_0^{\tau, \zeta}$  has distribution  $\zeta$  and  $\bar{Q}$  is a random counting measure with compensating measure

$$\bar{L}(dt, dy) = \mu_t(\bar{X}_t^{\tau, \zeta}, \bar{p}_t^{\tau, \zeta}, dy) dt.$$

The jump process distribution of (2.3.5) will be denoted by  $\bar{\mathbb{Q}}_{\tau, \zeta} = \bar{X}^{\tau, \zeta}(\mathbb{P})$  and it is an element of  $\mathcal{P}(\mathbb{H}([\tau, T], E))$ . Contrary to (2.3.1), the process  $\bar{X}^{\tau, \zeta}$  is distribution dependent, because  $\bar{p}_t^{\tau, \zeta} := \pi_t(\bar{\mathbb{Q}}_{\tau, \zeta}) = \bar{X}_t^{\tau, \zeta}(\mathbb{P})$  denotes the occupation probabilities of  $\bar{X}^{\tau, \zeta}$  itself, which means that the process depends on its own distribution via the intensity measure  $\mu$ . As before  $\mu : [0, T] \times E \times \mathcal{P}(E) \times \mathcal{B}(E) \rightarrow [0, \infty)$  is the intensity kernel satisfying

- (i)  $(t, x, \rho) \mapsto \mu_t(x, \rho, B)$  is measurable for all  $B \in \mathcal{B}(E)$
- (ii)  $B \mapsto \mu_t(x, \rho, B)$  is a positive and bounded measure for all  $(t, x, \rho) \in [\tau, T] \times E \times \mathcal{P}(E)$
- (iii)  $\mu_t(x, \rho, \{x\}) = 0$ .

It can still be decomposed into  $\mu_t(dy, x, \rho) = \lambda_t(x, \rho) r_t(x, \rho, dy)$ . The only difference to (2.3.1) is that  $\mu$  is measure dependent. Similarly to (2.3.2),  $\bar{Q}$  and  $\bar{X}^{\tau, \zeta}$  can be written as

$$\bar{Q}(dt, dy) = \sum_{i=1}^{\infty} \delta_{\{(\bar{T}_i^{\tau, \zeta}, \bar{Y}_i^{\tau, \zeta})\}}(dt, dy) \quad \text{and} \quad \bar{X}_t^{\tau, \zeta} = \sum_{i=0}^{\infty} \bar{Y}_i^{\tau, \zeta} \mathbf{1}_{(\bar{T}_i^{\tau, \zeta} \leq t < \bar{T}_{i+1}^{\tau, \zeta})},$$

where  $(\bar{T}_i^{\tau, \zeta}, \bar{Y}_i^{\tau, \zeta})_{i \in \mathbb{N}_0}$  is the associated marked point process including the initial time  $\bar{T}_0^{\tau, \zeta} := \tau$  and initial value  $\bar{Y}_0^{\tau, \zeta}$ .

As the process now depends on its own distribution, Theorem 2.3.2 cannot be used to establish existence and uniqueness of the jump process distribution  $\bar{\mathbb{Q}}_{\tau, \zeta}$ . Instead we have the following result:

**Theorem 2.3.7.** *Assume that there exists constants  $C_\lambda, C_\mu > 0$  such that  $\lambda_t(x, \rho) \leq C_\lambda$  for all  $t \in [0, T]$ ,  $x \in E$ ,  $\rho \in \mathcal{P}(E)$  and such that*

$$d_{TV}(\mu_t(x_1, \rho_1, dy), \mu_t(x_2, \rho_2, dy)) \leq C_\mu(\mathbf{1}_{(x_1 \neq x_2)} + d_{BL}(\rho_1, \rho_2))$$

*for all  $x_1, x_2 \in E$  and  $\rho_1, \rho_2 \in \mathcal{P}(E)$ . Then the jump process  $\bar{X}^{\tau, \zeta}$  given by (2.3.5) has a unique non-explosive jump process distribution  $\bar{\mathbb{Q}}_{\tau, \zeta} \in \mathcal{P}(\mathbb{H}([\tau, T], E))$ .*

*Proof.* For the proof, see Section 2.3.4.  $\square$

*Remark 2.3.8.* Here  $d_{TV}$  denotes the total variation distance on the space of positive bounded measures  $\mathcal{M}_b(E)$  while  $d_{BL}$  denotes the bounded-Lipschitz distance on the space of probability measures  $\mathcal{P}(E)$ . Both distances are defined as follows:

$$d_{TV}(\mu, \nu) := \frac{1}{2} \sup_{f \in \text{BM}} \left\{ \left| \int_S f(x) \mu(dx) - \int_S f(x) \nu(dx) \right| \right\},$$

$$d_{BL}(\mu, \nu) := \frac{1}{2} \sup_{f \in \text{BL}} \left\{ \left| \int_S f(x) \mu(dx) - \int_S f(x) \nu(dx) \right| \right\},$$

where BM is the class of measurable functions  $f : E \rightarrow [-1, 1]$  and BL is the class of Lipschitz functions  $f : E \rightarrow [-1, 1]$  with a Lipschitz constant of at most 1. While the bounded-Lipschitz distance typically is defined without the factor of  $\frac{1}{2}$ , this normalisation is convenient. For additional information about both metrics, see Appendix 2.A.

The occupation probability  $\bar{p}_t^{\tau, \zeta}$  satisfies a forward integro-differential equation, but contrary to Proposition 2.3.6 the equations are now non-linear.

**Proposition 2.3.9.** *Let  $B \in \mathcal{B}(E)$ . The transition probability  $\bar{p}_t^{\tau, \zeta}(B)$  satisfies the forward integro-differential equation*

$$\frac{d}{dt} \bar{p}_t^{\tau, \zeta}(B) = \int_{E \setminus B} \mu_t(y, \bar{p}_t^{\tau, \zeta}, B) \bar{p}_t^{\tau, \zeta}(dy) - \int_B \mu_t(y, \bar{p}_t^{\tau, \zeta}, E \setminus B) \bar{p}_t^{\tau, \zeta}(dy),$$

with  $\bar{p}_\tau^{\tau, \zeta}(B) = \zeta(B)$  for  $t \in [\tau, T]$  and  $(\tau, \zeta) \in [0, T] \times \mathcal{P}(E)$ .

*Proof.* Let  $B \in \mathcal{B}(E)$ . By an argument made on p.150 in Jacobsen (2006) it holds that

$$\mathbb{1}_B(\bar{X}_t^{\tau, \zeta}) = \mathbb{1}_B(\bar{X}_\tau^{\tau, \zeta}) + \int_{(0, t]} \int_E (\mathbb{1}_B(y) - \mathbb{1}_B(X_{s-}^{\tau, \zeta})) \mu_s(X_{s-}^{\tau, \zeta}, \bar{p}_s^{\tau, \zeta}, dy) ds + M_t^B,$$

where  $M_t^B$  is given by

$$M_t^B = \int_{(\tau, t] \times E} (\mathbb{1}_B(y) - \mathbb{1}_B(X_{s-}^{\tau, \zeta})) (Q(dt, dy) - L(dt, dy)).$$

Since  $\mathbb{1}_B$  is bounded and  $\mathbb{E}[\bar{Q}((\tau, T] \times E)] < \infty$  we have that  $M_t^B$  is a mean-zero martingale with respect to the natural filtration of  $\bar{X}_t^{\tau, \zeta}$  and thus taking the expectation on both sides yields

$$\mathbb{E}[\phi(\bar{X}_t)] = \mathbb{E}[\mathbb{1}_B(\bar{X}_\tau^{\tau, \zeta})] + \mathbb{E} \left[ \int_{(0, t]} \int_E (\mathbb{1}_B(y) - \mathbb{1}_B(X_{s-}^{\tau, \zeta})) \mu_s(X_{s-}^{\tau, \zeta}, \bar{p}_s^{\tau, \zeta}, dy) ds \right].$$

Applying Fubini, this can also be written as

$$\bar{p}_t^{\tau, \zeta}(B) = \bar{p}_\tau^{\tau, \zeta}(B) + \int_{(\tau, t]} \int_E \mu_s(x, \bar{p}_s^{\tau, \zeta}, B) \bar{p}_s^{\tau, \zeta}(dx) - \int_B \mu_s(x, \bar{p}_s^{\tau, \zeta}, E) \bar{p}_s^{\tau, \zeta}(dx) ds.$$

Using that  $\bar{p}_\tau^{\tau,\zeta} = \zeta$  and that each of the inner integrals can be decomposed into two terms, of which two cancel out with each other, we arrive at

$$\bar{p}_t^{\tau,\zeta}(B) = \zeta(B) + \int_{(\tau,t]} \int_{E \setminus B} \mu_s(x, \bar{p}_s^{\tau,\zeta}, B) \bar{p}_s^{\tau,\zeta}(dx) - \int_B \mu_s(x, \bar{p}_s^{\tau,\zeta}, E \setminus B) \bar{p}_s^{\tau,\zeta}(dx) ds.$$

The result follows by differentiating with respect to  $t$ .  $\square$

Similarly to (2.3.3), we can for each  $x \in E$  consider the linearised jump process

$$\tilde{X}_t^{\tau,x} = x + \int_{(\tau,t] \times E} (y - \tilde{X}_t^{\tau,x}) \tilde{Q}(dt, dy), \quad (2.3.6)$$

where  $\tilde{Q}$  is a random counting measure with compensating measure

$$\tilde{L}(dt, dy) = \mu_t(\tilde{X}_{t-}^{\tau,x}, \bar{p}_t^{\tau,\zeta}, dy) dt,$$

and where  $\bar{p}_t^{\tau,\zeta} = \pi_t(\tilde{\mathbb{Q}}_{\tau,\zeta})$  is considered known and fixed. The process  $\tilde{X}^{\tau,x}$  thus is a Markov jump process, since it does not depend on its own distribution. We can therefore invoke Theorem 2.3.2 to obtain existence and uniqueness of the jump process distribution  $\tilde{\mathbb{Q}}_{\tau,\zeta,x} := \tilde{X}^{\tau,x}(\mathbb{P})$  of (2.3.6) for all  $x \in E$ . We can now obtain the following analogue to Theorem 2.3.4, where  $\bar{\mathcal{F}}_t^{\tau,\zeta} := \sigma(\tilde{X}_t^{\tau,\zeta} : \tau \leq t \leq T)$ .

**Theorem 2.3.10.** *Let  $B \in \mathcal{B}(\mathbb{H}([s, T], E))$ . For fixed  $\tau \in [0, T]$  the family  $(\tilde{\mathbb{Q}}_{\tau,\zeta,x})_{x \in E}$  is a regular conditional probability of  $\mathbb{P}((\tilde{X}_t^{\tau,\zeta})_{t \in [\tau, T]} \in B | \bar{X}_\tau^{\tau,\zeta} = x)$ . Furthermore for any  $0 \leq \tau \leq s \leq t \leq T$  it holds that*

$$\mathbb{P}((\bar{X}_t^{\tau,\zeta})_{t \in [s, T]} \in B | \bar{\mathcal{F}}_s^{\tau,\zeta}) = \tilde{\mathbb{Q}}_{s, \bar{p}_s^{\tau,\zeta}, \bar{X}_s^{\tau,\zeta}}(B), \quad \mathbb{P} - a.s.$$

*Proof.* See Appendix 2.B.  $\square$

Theorem 2.3.10 proves two central facts. First it shows that the conditional distribution of  $(\bar{X}_t^{\tau,\zeta})_{s \leq t \leq T}$  given  $\bar{X}_s^{\tau,\zeta} = x$  for any  $s \geq \tau$  has a regular version equal to  $\tilde{\mathbb{Q}}_{s, \bar{p}_s^{\tau,\zeta}, x}$ , which is the jump process distribution of the linearised jump process (2.3.6) started at time  $s$  and based on  $\bar{\mathbb{Q}}_{s, \bar{p}_s^{\tau,\zeta}}$ . Secondly it shows that  $\bar{X}^{\tau,\zeta}$  is a non-linear Markov process in the sense of Rehmeier and Röckner (2025). Contrary to the classical Markov property of Theorem 2.3.4 the conditional distributions of  $\bar{X}^{\tau,\zeta}$  now depend on  $\zeta$ , which means that the future  $(\bar{X}_t^{\tau,\zeta})_{s \leq t \leq T}$  not only depends on the past through the present state  $\bar{X}_s^{\tau,\zeta}$ , but also through the initial distribution  $\zeta$ . Thus changing the initial distribution at time  $\tau$  changes all subsequent conditional jump process distributions.

As a consequence of the non-linear Markov property, the transition probabilities of  $\bar{X}^{\tau,\zeta}$  are given by  $\tilde{p}_t^{\tau,\zeta}(x, \cdot) := \tilde{X}_t^{\tau,x}(\mathbb{P})$ . Since  $\tilde{X}^{\tau,x}$  is a Markov jump process we can invoke Proposition 2.3.5 to conclude that, given  $(\bar{p}_t^{\zeta,\tau})_{t \in [\tau, T]}$ , the transition probabilities  $\tilde{p}_t^{\tau,\zeta}(x, B)$  satisfy the linear forward integro-differential equations:

**Proposition 2.3.11.** *Let  $B \in \mathcal{B}(E)$ . Given  $(\bar{p}_t^{\tau, \zeta})_{t \in [\tau, T]}$  the transition probabilities  $\bar{p}_x^{\tau, \zeta}(t, B)$  satisfy the forward integro-differential equation*

$$\frac{d}{dt} \bar{p}_t^{\tau, \zeta}(x, B) = \int_{E \setminus B} \mu_t(y, \bar{p}_t^{\tau, \zeta}, B) \bar{p}_t^{\tau, \zeta}(x, dy) - \int_B \mu_t(y, \bar{p}_t^{\tau, \zeta}, E \setminus B) \bar{p}_t^{\tau, \zeta}(x, dy),$$

with  $\bar{p}_\tau^{\tau, \zeta}(x, \cdot) = \delta_{\{x\}}$  for  $t \in [\tau, T]$ ,  $x \in E$  and  $\tau \in [0, T]$ .

If one wishes to solve the equations of Proposition 2.3.11 numerically, then there are two possibilities. The first is to solve the non-linear forward equations of Proposition 2.3.9 first, in order to obtain  $(\bar{p}_t^{\tau, \zeta})_{t \in [\tau, T]}$ , and then to solve the linear forward equations. Both can often be done by employing the same numerical schemes which one would use for the linear versions of Proposition 2.3.5 and Proposition 2.3.6. Alternatively, we can utilise Theorem 2.3.10 to obtain the following analogue of (2.3.4):

$$\bar{p}_t^{\tau, \zeta}(B) = \int_E \bar{p}_t^{\tau, \zeta}(x, B) \zeta(dx). \quad (2.3.7)$$

We can thus substitute (2.3.7) into the equations of Proposition 2.3.11 to obtain a new non-linear equation which could be solved directly.

*Remark 2.3.12.* Note that Propositions 2.3.9 and 2.3.11 make no statement about uniqueness of the non-linear equations. Thus numerical solutions should always be treated with care.

*Remark 2.3.13.* The forward equations of Propositions 2.3.9 and 2.3.11 are the pure jump equivalent of the non-linear and linearised Fokker-Planck-Kolmogorov equations known from McKean-Vlasov diffusion processes, see Rehmeier and Röckner (2025).

In practical applications the distribution dependence of the intensity kernel often comes through quantities such as

$$\int_E h(t, x, y) \rho(dy)$$

for some function  $h : [\tau, T] \times E^2 \rightarrow \mathbb{R}^d$ , while the intensity kernel also has a density with respect to some measure  $\nu$  on  $E$ . In particular we are interested in intensity kernels of the form

$$\mu_t(x, \rho, dy) = q_t^y \left( x, \int_E h(t, x, z) \rho(dz) \right) \nu(dy),$$

where  $q_t^y : E \times \mathbb{R}^d \rightarrow [0, \infty)$  for each  $(t, y) \in [\tau, T] \times E$  and when they satisfy the assumptions of Theorem 2.3.7. The following result provides sufficient conditions in terms of  $q_t^y$  and  $h$ .

**Proposition 2.3.14.** *Let  $d_E$  be a metric on  $E$  whose induced topology generates the Borel sigma-algebra  $\mathcal{B}(E)$  and let  $\|u\| = \sum_{i=1}^d |u_i|$  for  $u \in \mathbb{R}^d$ . Assume that*

(i) *There exists  $C_1$  such that*

$$\|h(t, x_1, y_1) - h(t, x_2, y_2)\| \leq C_1(\mathbb{1}_{(x_1 \neq x_2)} + \min(1, d_E(y_1, y_2)))$$

*for all  $(x_1, y_1), (x_2, y_2) \in E^2$ .*

(ii) *There exists  $C_2 : E \rightarrow [0, \infty)$  with  $\int_E C_2(y) \nu(dy) < \infty$  such that*

$$|q_t^y(x_1, u_1) - q_t^y(x_2, u_2)| \leq C_2(y)(\mathbb{1}_{(x_1 \neq x_2)} + \|u_1 - u_2\|)$$

*for all  $x_1, x_2 \in E$  and  $u_1, u_2 \in \mathbb{R}^d$ .*

(iii) *There exists  $C_\lambda > 0$  such that*

$$\int_E q_t^y(x, u) \kappa(dy) \leq C_\lambda$$

*for all  $(x, u) \in E \times \mathbb{R}^d$ .*

*Then the intensity kernel*

$$\mu_t(x, \rho, dy) = q_t^y\left(x, \int_E h(t, x, z) \rho(dz)\right) \nu(dy),$$

*satisfies the assumptions of Theorem 2.3.7.*

*Proof.* We start with the Lipschitz assumption of Theorem 2.3.7. Let  $f : E \rightarrow [-1, 1]$  be measurable. Then by (ii) we obtain

$$\begin{aligned} \left| \int_E f(y) q_t^y(x_1, u_1) \nu(dy) - \int_E f(y) q_t^y(x_2, u_2) \nu(dy) \right| \\ \leq \int_E |q_t^y(x_1, u_1) - q_t^y(x_2, u_2)| \nu(dy) \\ \leq \left( \int_E C_2(y) \nu(dy) \right) (\mathbb{1}_{(x_1 \neq x_2)} + \|u_1 - u_2\|). \end{aligned}$$

As this inequality holds for all such  $f$  the inequality also holds when taking the supremum and multiplying with  $\frac{1}{2}$ . Now define  $F : [\tau, T] \times E \times \mathcal{P}(E) \rightarrow \mathbb{R}^d$  by  $F(t, x, \rho) := \int_E h(t, x, y) \rho(dy)$  and let  $\rho_1, \rho_2 \in \mathcal{P}(E)$  be arbitrary. Then we have that

$$\begin{aligned} \|F(t, x_1, \rho_1) - F(t, x_2, \rho_2)\| &\leq \sum_{j=1}^d \left| \int_E h_j(t, x_2, y) \rho_1(dy) - \int_E h_j(t, x_2, y) \rho_2(dy) \right| \\ &\quad + \int_E \|h(t, x_1, y) - h(t, x_2, y)\| \rho_1(dy). \end{aligned}$$

For the first term we note that (i) implies that  $y \mapsto |h_j(t, x, y)|$  is bounded by some  $K > 0$  for all  $j = 1, \dots, d$  and  $(t, x) \in [\tau, T] \times E$ . Thus we have that  $y \mapsto \frac{h_j(t, x_2, y)}{\max(K, C_1)}$  is in the class BL for all  $j = 1, \dots, d$  and  $(t, x) \in [\tau, T] \times E$ . By using the definition of  $d_{BL}$  we see that the sum is bounded by  $2d \max(K, C_1) d_{BL}(\rho_1, \rho_2)$ . The second term is by (i) bounded by  $C_1 \mathbb{1}_{(x_1 \neq x_2)}$  and we thus arrive at

$$\|F(x_1, \rho_1) - F(x_2, \rho_2)\| \leq 2d \max(K, C_1) (\mathbb{1}_{(x_1 \neq x_2)} + d_{BL}(\rho_1, \rho_2)).$$

Combining this with the inequality for  $q_t^y$  yields the desired Lipschitz property of Theorem 2.3.7. The boundedness assumption of Theorem 2.3.7 follows directly from (iii).  $\square$

**Example 2.3.15** (SIS-model). Returning to the SIS-model on a network from Section 2.2, we can see that the intensity measure  $\mu$  given by (2.2.4) is absolutely continuous with respect to the counting measure  $\nu$  on  $E = \{0, 1\} \times \{1, \dots, J\}$  with density

$$q_t^{z,i}(\bar{z}, \bar{i}, \rho) = \mathbb{1}_{\{1, \bar{i}\}}(z, i) \mathbb{1}_{(\bar{z}=0)} \beta_{\bar{i}} \sum_{j=1}^J a_{\bar{i}j} \rho(\{1, j\}) + \mathbb{1}_{\{0, \bar{i}\}}(z, i) \mathbb{1}_{(\bar{z}=1)} \gamma_{\bar{i}}.$$

This can be seen to be a composition of  $q_t^{z,i} : E \times \mathbb{R} \rightarrow [0, \infty)$  given by

$$q_t^{z,i}(\bar{z}, \bar{i}, y) = \mathbb{1}_{\{1, \bar{i}\}}(z, i) \mathbb{1}_{(\bar{z}=0)} \beta_{\bar{i}} y + \mathbb{1}_{\{0, \bar{i}\}}(z, i) \mathbb{1}_{(\bar{z}=1)} \gamma_{\bar{i}}.$$

and of the mapping

$$\rho \mapsto \int_E h(\bar{i}, z, i) \rho(d(z, i)), \quad \text{where} \quad h(\bar{i}, z, i) = \sum_{j=1}^J a_{\bar{i}j} \mathbb{1}_{\{1, j\}}(z, i).$$

Using the discrete distance  $d_E(x, y) = \mathbb{1}_{(x \neq y)}$  on  $E$  and the fact that the  $a_{ij}$  are equal to 0 or 1, we get

$$|h(\bar{i}_1, z_1, i_1) - h(\bar{i}_2, z_2, i_2)| \leq \mathbb{1}_{(\bar{i}_1 \neq \bar{i}_2)} + \min(1, \mathbb{1}_{((z_1, i_1) \neq (z_2, i_2))}).$$

Similarly we obtain

$$|q_t^{z,i}(\bar{z}_1, \bar{i}_1, y_1) - q_t^{z,i}(\bar{z}_2, \bar{i}_2, y_2)| \leq K (\mathbb{1}_{((\bar{z}_1, \bar{i}_1) \neq (\bar{z}_2, \bar{i}_2))} + |y_1 - y_2|),$$

where  $K := \max_i \beta_i + \max_i \gamma_i$ . Finally since for any  $\rho \in \mathcal{P}(E)$  we have

$$\sum_{j=1}^J a_{ij} \rho(\{1, j\}) \leq \rho(\{1\} \times \{1, \dots, J\}) \leq 1,$$

we arrive at

$$\int_E q_t^{z,i}(\bar{z}, \bar{i}, \rho) \nu(d(z, i)) \leq K, \quad \forall (\bar{z}, \bar{i}) \in E, \rho \in \mathcal{P}(E).$$

Thus Proposition 2.3.14 in conjunction with Theorem 2.3.7 yields existence and uniqueness of the mean-field model (2.2.3). Furthermore by Proposition (2.3.9) the occupation probabilities satisfy the non-linear forward differential equations given by (2.2.5) and by Proposition 2.3.11 the transition probabilities satisfy the linear forward equations given by (2.2.6).

### 2.3.3 Coupling of jump process distributions

For all major proofs in this paper it is essential to obtain sensible bounds of the total variation distance between two jump process distributions. Let  $\bar{\mathbb{Q}}_{\tau, \zeta_1}$  and  $\bar{\mathbb{Q}}_{\tau, \zeta_2}$  be two (potentially distribution dependent) jump process distributions with the same intensity kernel  $\mu$  but different initial distributions  $\zeta_1, \zeta_2 \in \mathcal{P}(E)$ . Then by (2.A.5)

$$d_{TV}(\bar{\mathbb{Q}}_{\tau, \zeta_1}, \bar{\mathbb{Q}}_{\tau, \zeta_2}) \leq \mathbb{P}(\bar{X}^{\tau, \zeta_1} \neq \bar{X}^{\tau, \zeta_2}) = \mathbb{P}\left(\bigcup_{t \in [\tau, T]} (\bar{X}_t^{\tau, \zeta_1} \neq \bar{X}_t^{\tau, \zeta_2})\right)$$

for any joint jump process  $(\bar{X}^{\tau, \zeta_1}, \bar{X}^{\tau, \zeta_2})$  with state space  $E^2$  whose joint jump process distribution  $(\bar{X}^{\tau, \zeta_1}, \bar{X}^{\tau, \zeta_2})(\mathbb{P})$  is a coupling of  $\bar{\mathbb{Q}}_{\tau, \zeta_1}$  and  $\bar{\mathbb{Q}}_{\tau, \zeta_2}$ , meaning that the marginals  $\bar{X}^{\tau, \zeta_1}(\mathbb{P})$  and  $\bar{X}^{\tau, \zeta_2}(\mathbb{P})$  are equal to  $\bar{\mathbb{Q}}_{\tau, \zeta_1}$  and  $\bar{\mathbb{Q}}_{\tau, \zeta_2}$ . The goal is to find a coupling with small  $\mathbb{P}(\bar{X}^{\tau, \zeta_1} \neq \bar{X}^{\tau, \zeta_2})$  or equivalently large  $\mathbb{P}(\bar{X}^{\tau, \zeta_1} = \bar{X}^{\tau, \zeta_2})$ .

The coupling that we intend to use can be constructed by marking and thinning a Poisson process using the following procedure inspired by Graham (1992b):

1. Obtain the potential jump times  $(T_i)_{i \in \mathbb{N}}$  from a homogeneous Poisson process with intensity  $C_\lambda$ .
2. Obtain the initial values  $\bar{Y}_0^{\zeta_1}$  and  $\bar{Y}_0^{\zeta_2}$  from a coupling of  $\zeta_1$  and  $\zeta_2$ .
3. Starting at zero and iterating upwards, for each  $i$  choose the jump destinations from the maximal coupling of the respective jump destination distributions, that is the coupling which maximises the probability of both processes jumping to the same destination.

Thus by fixing the potential common jump times  $(T_i)_{i \in \mathbb{N}}$  and then choosing the jump destinations from the maximal coupling, we ensure that the paths of  $\bar{X}^{\tau, \zeta_1}$  and  $\bar{X}^{\tau, \zeta_2}$  are close. Mathematically this construction is specified by

$$\begin{pmatrix} \bar{X}_t^{\tau, \zeta_1} \\ \bar{X}_t^{\tau, \zeta_2} \end{pmatrix} = \begin{pmatrix} Y_0^{\tau, \zeta_1} \\ Y_0^{\tau, \zeta_2} \end{pmatrix} + \int_{(\tau, t] \times E^2} \begin{pmatrix} y_1 - \bar{X}_{s-}^{\tau, \zeta_1} \\ y_2 - \bar{X}_{s-}^{\tau, \zeta_2} \end{pmatrix} \mathcal{N}(dt, d(y_1, y_2)) \quad (2.3.8)$$

for  $t \in [\tau, T]$ , where the joint distribution of  $Y_0^{\tau, \zeta_1}$  and  $Y_0^{\tau, \zeta_2}$  is a coupling of  $\zeta_1$  and  $\zeta_2$ , while  $\mathcal{N}$  is a random counting measure with compensating measure

$$L(dt, d(y_1, y_2)) = C_\lambda \gamma_t(X_{t-}^{\tau, \zeta_1}, X_{t-}^{\tau, \zeta_2}, \bar{p}_t^{\tau, \zeta_1}, \bar{p}_t^{\tau, \zeta_2}, d(y_1, y_2)) dt.$$

Here  $\gamma : [\tau, T] \times E^2 \times \mathcal{P}(E)^2 \times \mathcal{B}(E^2) \rightarrow [0, 1]$  is a probability kernel, which for each fixed  $t, (x_1, x_2), (\rho_1, \rho_2)$  is a maximal coupling of  $\kappa_t(x_1, \rho_1, dy)$  and  $\kappa_t(x_2, \rho_2, dy)$ , where  $\kappa : [\tau, T] \times E \times \mathcal{P}(E) \times \mathcal{B}(E) \rightarrow [0, 1]$  is a probability kernel given by

$$\kappa_t(x, \rho, dy) = \frac{1}{C_\lambda} \mu_t(x, \rho, dy) + \left(1 - \frac{\lambda_t(x, \rho)}{C_\lambda}\right) \delta_{\{x\}}(dy). \quad (2.3.9)$$

The next result shows that (2.3.8) indeed constitutes a coupling of  $\bar{\mathbb{Q}}_{\tau, \zeta_1}$  and  $\bar{\mathbb{Q}}_{\tau, \zeta_2}$ .

**Proposition 2.3.16.** *Assume that  $\bar{\mathbb{Q}}_{\tau, \zeta_1}$  and  $\bar{\mathbb{Q}}_{\tau, \zeta_2}$  exist and are unique. The jump process distribution of (2.3.8) exists, is unique and has marginals  $X^{\tau, \zeta_1}(\mathbb{P}) = \bar{\mathbb{Q}}_{\tau, \zeta_1}$  and  $X^{\tau, \zeta_2}(\mathbb{P}) = \bar{\mathbb{Q}}_{\tau, \zeta_2}$ . Furthermore the process  $N_t := \mathcal{N}_t((\tau, t] \times E^2)$  is a Poisson process with intensity  $C_\lambda$ .*

*Proof.* Note that since  $\bar{\mathbb{Q}}_{\tau, \zeta_1}$  and  $\bar{\mathbb{Q}}_{\tau, \zeta_2}$  exist and are unique, we can treat  $\bar{p}^{\tau, \zeta_1}$  and  $\bar{p}^{\tau, \zeta_2}$  as given and fixed, which makes (2.3.8) a linear jump process. Lemma 2.C.1 ensures that  $\gamma$  exists and since  $C_\lambda < \infty$  Theorem 2.3.2 yields existence and uniqueness of the jump process distribution of (2.3.8). We can now set

$$Q^1(dt, dy_1) := \mathbb{1}_{(y_1 \neq \bar{X}_{t-}^{\tau, \zeta_1})} \mathcal{N}(dt, dy_1 \times E)$$

and note that we can write

$$\bar{X}_t^{\tau, \zeta_1} = Y_0^{\tau, \zeta_1} + \int_{(\tau, t] \times E} (y_1 - \bar{X}_{s-}^{\tau, \zeta_1}) Q^1(ds, dy_1).$$

Since  $\gamma$  is a coupling of  $\kappa_t(x_1, \rho_1, dy)$  and  $\kappa_t(x_2, \rho_2, dy)$  the compensating measure of  $Q^1$  is given by

$$\begin{aligned} L^1(dt, dy_1) &= \mathbb{1}_{(y_1 \neq \bar{X}_{t-}^{\tau, \zeta_1})} L(dt, dy_1 \times E) = \mathbb{1}_{(y_1 \neq \bar{X}_{t-}^{\tau, \zeta_1})} C_\lambda \kappa_t(\bar{X}_{t-}^{\tau, \zeta_1}, \bar{p}_t^{\tau, \zeta_2}, dy_1) dt \\ &= \mu_t(\bar{X}_{t-}^{\tau, \zeta_1}, \bar{p}_t^{\tau, \zeta_1}) dt, \end{aligned}$$

where the last equality follows from (2.3.9). Thus we can conclude that  $\bar{X}^{\tau, \zeta_1}(\mathbb{P}) = \bar{\mathbb{Q}}_{\tau, \zeta_1}$ . Similar calculations yield  $\bar{X}^{\tau, \zeta_2}(\mathbb{P}) = \bar{\mathbb{Q}}_{\tau, \zeta_2}$ . Finally the compensating measure for  $N_t = \mathcal{N}((\tau, t] \times E^2)$  is given by  $C_\lambda dt$  proving that  $N_t$  is a Poisson process with intensity  $C_\lambda$ .  $\square$

The Lipschitz assumption of  $\mu$  from Theorem 2.3.7 carries over to  $\kappa_t(x, \rho, dy)$ .

**Lemma 2.3.17.** *Assume that  $\mu$  satisfies the assumptions of Theorem 2.3.7. Then it holds that*

$$d_{TV}(\kappa_t(dy, x_1, \rho_1), \kappa_t(dy, x_2, \rho_2)) \leq \left(1 + \frac{C_\mu}{2C_\lambda}\right) (\mathbb{1}_{(x_1 \neq x_2)} + d_{BL}(\rho_1, \rho_2)).$$

*Proof.* Let  $f : E \rightarrow [-1, 1]$  be given. Then

$$\begin{aligned} & \left| \int_E f(y) \kappa_t(dy, x_1, \rho_1) - \int_E f(y) \kappa_t(dy, x_2, \rho_2) \right| \\ & \leq \frac{1}{C_\lambda} \left| \int_E f(y) \mu_t(dy, x_1, \rho_1) - \int_E f(y) \mu_t(dy, x_2, \rho_2) \right| \\ & \quad + \left| \left(1 - \frac{\lambda_t(x_1, \rho_1)}{C_\lambda}\right) f(x_1) - \left(1 - \frac{\lambda_t(x_2, \rho_2)}{C_\lambda}\right) f(x_2) \right|. \end{aligned}$$

The term in the third line is bounded by  $2\mathbb{1}_{(x_1 \neq x_2)}$ , since  $f(x_i) \in [-1, 1]$  and  $1 - \frac{\lambda_t(x_i, \rho_i)}{C_\lambda} \in [0, 1]$  for  $i = 1, 2$ . Using that bound and taking the supremum over measurable  $f : E \rightarrow [-1, 1]$  and multiplying with  $\frac{1}{2}$  yields

$$\begin{aligned} & d_{TV}(\kappa_t(dy, x_1, \rho_1), \kappa_t(dy, x_2, \rho_2)) \\ & \leq \frac{1}{2C_\lambda} d_{TV}(\mu_t(dy, x_1, \rho_1), \mu_t(dy, x_2, \rho_2)) + \mathbb{1}_{(x_1 \neq x_2)}. \end{aligned}$$

Inserting the Lipschitz assumption from Theorem 2.3.7 yields the desired result.  $\square$

Finally we use this coupling construction to prove a bound of  $\mathbb{P}(\bar{X}^{\tau, \zeta_1} \neq \bar{X}^{\tau, \zeta_2})$  which is essential for proving the main results of this paper. For each  $t \in [\tau, T]$  let

$$A_t := \bigcup_{s \in [\tau, t]} (\bar{X}_s^{\tau, \zeta_1} \neq \bar{X}_s^{\tau, \zeta_2})$$

be the set where  $\bar{X}_s^{\tau, \zeta_1}$  differs from  $\bar{X}_s^{\tau, \zeta_2}$  for some  $s \in [\tau, t]$ . We then obtain the following bound:

**Lemma 2.3.18.** *Assume that  $\bar{\mathbb{Q}}_{\tau, \zeta_1}$  and  $\bar{\mathbb{Q}}_{\tau, \zeta_2}$  exist and are unique and let  $\bar{X}^{\tau, \zeta_1}$  and  $\bar{X}^{\tau, \zeta_2}$  be given by (2.3.8). Then it holds that*

$$\mathbb{P}(A_t) \leq \mathbb{P}(\bar{Y}_0^{\tau, \zeta_1} \neq \bar{Y}_0^{\tau, \zeta_2}) + C_1 \int_\tau^t \mathbb{P}(A_s) + \mathbb{E}[d_{BL}(\bar{p}_s^{\tau, \zeta_1}, \bar{p}_s^{\tau, \zeta_2})] ds,$$

for all  $t \in [\tau, T]$ , where  $C_1 := C_\lambda + \frac{C_\mu}{2}$ .

*Proof.* By construction we have that

$$\bar{X}_t^{\tau, \zeta_1} = \sum_{i=0}^{\infty} \bar{Y}_i^{\tau, \zeta_1} \mathbb{1}_{(T_i \leq t < T_{i+1})} \quad \text{and} \quad \bar{X}_t^{\tau, \zeta_2} = \sum_{i=0}^{\infty} \bar{Y}_i^{\tau, \zeta_2} \mathbb{1}_{(T_i \leq t < T_{i+1})},$$

for the same jump times  $T_0 := \tau$  and  $(T_i)_{i \in \mathbb{N}}$ . Since the jump times align, we have for any  $t \in [\tau, T]$  that  $\bar{X}_s^{\tau, \zeta_1} \neq \bar{X}_s^{\tau, \zeta_2}$  for some  $s \in [\tau, t]$  if and only if there exists  $i \in \{0, \dots, N_t\}$ , such that  $\bar{Y}_i^{\tau, \zeta_1} \neq \bar{Y}_i^{\tau, \zeta_2}$ . We can thus conclude that

$$A_t := \bigcup_{s \in [\tau, t]} (\bar{X}_s^{\tau, \zeta_1} \neq \bar{X}_s^{\tau, \zeta_2}) = \bigcup_{m=0}^{\infty} \left( (N_t = m) \cap \bigcup_{i=0}^m (\bar{Y}_i^{\tau, \zeta_1} \neq \bar{Y}_i^{\tau, \zeta_2}) \right),$$

is measurable with respect to the natural filtration of  $(\bar{Y}_0^{\tau, \zeta_1}, \bar{Y}_0^{\tau, \zeta_2}, \mathcal{N})$ . Furthermore  $A_s \subseteq A_t$  for  $s \leq t$ , which means that  $t \mapsto \mathbb{P}(A_t)$  is increasing and therefore measurable.

Next we note that

$$\mathbb{1}_{A_t} \leq \mathbb{1}_{(\bar{Y}_0^{\tau, \zeta_1} \neq \bar{Y}_0^{\tau, \zeta_2})} + \mathcal{N}((\tau, t] \times E_{x \neq y}^2),$$

where  $E_{x \neq y}^2 := \{(x, y) \in E^2 : x \neq y\}$ . While the left hand side simply indicates whether  $\bar{X}_s^{\tau, \zeta_1} \neq \bar{X}_s^{\tau, \zeta_2}$  for some  $s \in [\tau, t]$ , the right hand side counts how many marks are different. Taking expectation on both sides yields

$$\mathbb{P}(A_t) \leq \mathbb{P}(\bar{Y}_0^{\tau, \zeta_1} \neq \bar{Y}_0^{\tau, \zeta_2}) + \int_{\tau}^t C_{\lambda} \mathbb{E}[\gamma_s(\bar{X}_{s-}^{\tau, \zeta_1}, \bar{X}_{s-}^{\tau, \zeta_2}, \bar{p}_s^{\tau, \zeta_1}, \bar{p}_s^{\tau, \zeta_2}, E_{x \neq y}^2)] ds.$$

Since  $\gamma$  is a maximal coupling, we get by Lemma 2.3.17 that

$$\mathbb{P}(A_t) \leq \mathbb{P}(\bar{Y}_0^{\tau, \zeta_1} \neq \bar{Y}_0^{\tau, \zeta_2}) + C_1 \int_{\tau}^t \mathbb{P}(\bar{X}_{s-}^{\tau, \zeta_1} \neq \bar{X}_{s-}^{\tau, \zeta_2}) + \mathbb{E}[d_{BL}(\bar{p}_s^{\tau, \zeta_1}, \bar{p}_s^{\tau, \zeta_2})] ds.$$

Using that  $(\bar{X}_{s-}^{\tau, \zeta_1} \neq \bar{X}_{s-}^{\tau, \zeta_2}) \subset A_s$  for any  $s \leq t$  we get the desired result.  $\square$

### 2.3.4 Proof of Theorem 2.3.7

The proof is essentially a fixed point argument in the complete metric space  $(\mathcal{P}(\mathbb{H}([\tau, T], E)), d_{TV})$ . We fix  $(\tau, \zeta) \in [0, T] \times \mathcal{P}(E)$ , let  $\mathbb{Q} \in \mathcal{P}(\mathbb{H}([\tau, T], E))$  and define

$$Z_t^{\mathbb{Q}} = Y_0 + \int_{(\tau, t] \times E} (y - Z_{s-}^{\mathbb{Q}}) Q(dt, dy), \quad t \in [\tau, T],$$

where  $Y_0$  has distribution  $\zeta$  and  $Q$  has compensating measure

$$L(dt, dy) = \mu_t(Z_{t-}^{\mathbb{Q}}, \pi_t(\mathbb{Q}), dy) dt.$$

As this is a Markov jump process, Theorem 2.3.2 yields existence and uniqueness of the jump process distribution of  $Z^{\mathbb{Q}}$ . Thus  $F_{\zeta}(\mathbb{Q}) = Z^{\mathbb{Q}}(\mathbb{Q})$  defines a mapping from  $\mathcal{P}(\mathbb{H}([\tau, T], E))$  to itself. We now let  $F_{\zeta}^n$  denote the  $n$ -fold composition of  $F_{\zeta}$  with itself.

**Lemma 2.3.19.** *Let  $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{P}(\mathbb{H}([\tau, T], E))$ . It holds that*

$$d_{TV}(F_{\zeta}^n(\mathbb{Q}_1), F_{\zeta}^n(\mathbb{Q}_2)) \leq K^n \frac{(T - \tau)^n}{n!} d_{TV}(\mathbb{Q}_1, \mathbb{Q}_2), \quad \forall n \in \mathbb{N},$$

where  $K = C_1 e^{C_1(T - \tau)}$ .

*Proof.* We prove by induction that the statement not only holds for  $T$ , but any  $t \in [\tau, T]$ . That is

$$d_{TV}(F_{\zeta}^n(\mathbb{Q}_1)^t, F_{\zeta}^n(\mathbb{Q}_2)^t) \leq K^n \frac{(t - \tau)^n}{n!} d_{TV}(\mathbb{Q}_1^t, \mathbb{Q}_2^t), \quad \forall n \in \mathbb{N},$$

where the notation  $\mathbb{Q}^t$  denotes the restriction of the measure  $\mathbb{Q} \in \mathcal{P}(\mathbb{H}([\tau, T], E))$  to  $\mathbb{H}([\tau, t], E)$ .

For  $n = 1$  we utilise the coupling construction of Subsection 2.3.3 and apply Lemma 2.3.18 in conjunction with Grönwall's inequality to get

$$\mathbb{P}(A_t) \leq C_1 e^{C_1(t-\tau)} \int_{\tau}^t d_{BL}(\pi_s(\mathbb{Q}_1), \pi_s(\mathbb{Q}_2)) ds,$$

where  $A_t := \bigcup_{s \in [\tau, t]} (Z_s^{\mathbb{Q}_1} \neq Z_s^{\mathbb{Q}_2})$ . By (2.A.3) we have

$$d_{BL}(\pi_s(\mathbb{Q}_1), \pi_s(\mathbb{Q}_2)) \leq d_{TV}(\pi_s(\mathbb{Q}_1), \pi_s(\mathbb{Q}_2)) \leq d_{TV}(\mathbb{Q}_1^t, \mathbb{Q}_2^t),$$

for  $s \in [\tau, t]$  and by (2.A.5) we obtain  $d_{TV}(F_{\zeta}(\mathbb{Q}_1)^t, F_{\zeta}(\mathbb{Q}_2)^t) \leq \mathbb{P}(A_t)$ . Thus we arrive at

$$d_{TV}(F_{\zeta}(\mathbb{Q}_1)^t, F_{\zeta}(\mathbb{Q}_2)^t) \leq K(t - \tau) d_{TV}(\mathbb{Q}_1^t, \mathbb{Q}_2^t).$$

Assume now that the claim holds for some  $n > 1$ . Then by the coupling construction of Subsection 2.3.3, Lemma 2.3.18 and Grönwall's inequality we obtain

$$d_{TV}(F_{\zeta}^{n+1}(\mathbb{Q}_1)^t, F_{\zeta}^{n+1}(\mathbb{Q}_2)^t) \leq K \int_{\tau}^t d_{TV}(F_{\zeta}^n(\mathbb{Q}_1)^s, F_{\zeta}^n(\mathbb{Q}_2)^s) ds.$$

Inserting the induction hypothesis for  $t = s$  and applying (2.A.3) yields

$$d_{TV}(F_{\zeta}^{n+1}(\mathbb{Q}_1)^t, F_{\zeta}^{n+1}(\mathbb{Q}_2)^t) \leq K^{n+1} d_{TV}(\mathbb{Q}_1^t, \mathbb{Q}_2^t) \int_{\tau}^t \frac{(s - \tau)^n}{n!} ds.$$

Calculating the integral and setting  $t = T$  yields the desired result.  $\square$

As  $\mathcal{P}(\mathbb{H}([\tau, T], E))$  endowed with  $d_{TV}$  is a complete metric space and since

$$\sum_{n=0}^{\infty} \frac{(K(T - \tau))^n}{n!} = e^{K(T - \tau)} < \infty,$$

Weissinger's Fixed Point Theorem yields the existence of a unique fixed point  $\bar{\mathbb{Q}}_{\tau, \zeta} \in \mathcal{P}(\mathbb{H}([\tau, T], E))$  such that  $F_{\zeta}(\bar{\mathbb{Q}}_{\tau, \zeta}) = \bar{\mathbb{Q}}_{\tau, \zeta}$ .

## 2.4 Mean-field approximation

We now specify a class  $n$ -individual models and subsequently prove that the sequence of associated jump process distributions is chaotic in total variation with respect to the distribution of a distribution dependent non-linear Markov jump process. Note that we from now on and for the rest of the paper in the name of readability will refrain from decorating the jump processes with their initial conditions  $(\tau, \zeta)$ , while we for the sake of mathematical accuracy keep the decorations in place for jump process distributions and occupation and transition probabilities.

Throughout the section fix  $\tau \in [0, T]$  and a sequence  $(\zeta^n)_{n \in \mathbb{N}}$  of probability measures  $\zeta^n \in \mathcal{P}(E^n)$  such that each  $\zeta^n$  is exchangeable. For any  $n \in \mathbb{N}$  we now model a cohort of  $n$  individuals by a jump process  $X^n = (X^{1,n}, \dots, X^{n,n})$  with state space  $E^n$ , given by

$$X_t^n = Y_0^n + \int_{(\tau, t] \times E^n} (y^{1:n} - X_{s-}^n) Q^n(dt, dy^{1:n}), \quad t \in [\tau, T], \quad (2.4.1)$$

where the vector  $Y_0^n = (Y_0^{1,n}, \dots, Y_0^{n,n})$  has distribution  $\zeta^n \in \mathcal{P}(E^n)$  and the random counting measure  $Q^n$  has compensating measure

$$L^n(dt, dy^{1:n}) = \sum_{\ell=1}^n \left( \mu_t(X_{t-}^{\ell,n}, \varepsilon_{t-}^n, dy_\ell) \prod_{j=1, j \neq \ell}^n \delta_{\{X_{t-}^{j,n}\}}(dy_j) \right) dt.$$

Here  $\varepsilon_t^n := \frac{1}{n} \sum_{\ell=1}^n \delta_{\{X_t^{\ell,n}\}}$  is the empirical measure of  $(X_t^{1,n}, \dots, X_t^{n,n})$  and  $\mu : [\tau, T] \times E \times \mathcal{P}(E) \times \mathcal{B}(E) \rightarrow [0, 1]$  is an intensity kernel satisfying the assumptions of Theorem 2.3.7. Since the empirical measure is a function of  $(X^{1,n}, \dots, X^{n,n})$ , the process  $X^n$  is a Markov jump process and thus by Theorem 2.3.2 there exists a unique jump process distribution  $X^n(\mathbb{P}) := \mathbb{Q}_{\tau, \zeta^n}^n \in \mathcal{P}(\mathbb{H}([\tau, T], E^n))$ .

The jump process of individual  $\ell$  is given by

$$X_t^{\ell,n} = Y_t^{\ell,n} + \int_{(\tau, t] \times E} (y - X_{s-}^{\ell,n}) Q^{\ell,n}(ds, dy), \quad t \in [\tau, T],$$

where  $Y^{\ell,n}$  has distribution  $\zeta^{n,1}(dy_1) := \zeta^n(dy_1 \times E^{n-1})$  and the random counting measure  $Q^{\ell,n}(dt, dy_\ell) := Q^n(dt, E^{\ell-1} \times dy_\ell \times E^{n-\ell})$  has compensating measure

$$L^{\ell,n}(dt, dy) = \mu_t(X_{t-}^{\ell,n}, \varepsilon_{t-}^n, dy) dt.$$

The definition of  $L^n$  implies that the individuals have no common jumps. We can now clearly see, that each single individual  $X^{\ell,n}$  only depends on the other individuals through their collective empirical distribution  $\varepsilon_t^n$ . This can be interpreted as the dependence of the individual on collective quantities, such as cohort averages or frequencies. Furthermore, since  $\mu$  is the same for any individual and because the single individual depends on the collective in a symmetric manner, all individuals are identically distributed albeit not independent.

Since  $X^n$  is a Markov jump process, all results in Section 2.3.1 are valid. In particular we could write down and solve the linear forward integro-differential equations of Propositions 2.3.5 and 2.3.6, but as already discussed in Section 2.2, dependence of the individual on the entire cohort renders this computationally infeasible in most practical applications.

Instead we can hope to use the corresponding mean-field model instead, which is obtained by replacing the empirical distribution  $\varepsilon_{t-}^n$  of the collective with the

distribution of the individual processes themselves. This mean-field model is given by the distribution dependent non-linear Markov jump process (2.3.5):

$$\bar{X}_t = \bar{Y}_0 + \int_{(\tau, t] \times E} (y - \bar{X}_{s-}) \bar{Q}(ds, dy),$$

where  $\bar{Y}_0$  has distribution  $\zeta \in \mathcal{P}(E)$  and  $\bar{Q}$  has compensating measure

$$\bar{L}(dt, dy) = \mu_t(\bar{X}_{t-}, \bar{p}_t^{\tau, \zeta}, dy) dt.$$

Thus  $\bar{X}(\mathbb{P}) = \bar{\mathbb{Q}}_{\tau, \zeta}$  and existence and uniqueness follows directly from Theorem 2.3.7. Using this mean-field model, The entire collective of  $n$  individuals can then be modelled by  $n$  independent and identically distributed individuals, each with distribution  $\bar{\mathbb{Q}}_{\tau, \zeta}$ . Mathematically this is written as

$$\bar{X}_t^n = \bar{Y}_0^n + \int_{(\tau, t] \times E^n} (y^{1:n} - \bar{X}_{s-}^n) \bar{Q}^n(dt, dy^{1:n}), \quad t \in [\tau, T], \quad (2.4.2)$$

where the vector  $\bar{Y}^n = (\bar{Y}^{1,n}, \dots, \bar{Y}^{n,n})$  is a collection of independent and identically distributed random variables with distribution  $\zeta$  while the random counting measure  $\bar{Q}^n$  has compensating measure

$$\bar{L}^n(dt, dy^{1:n}) = \sum_{\ell=1}^n \left( \mu_t(\bar{X}_{t-}^{\ell, n}, \bar{p}_t^{\tau, \zeta}, dy_\ell) \prod_{j=1, j \neq \ell}^n \delta_{\{\bar{X}_{t-}^{j, n}\}}(dy_j) \right) dt.$$

The jump process distribution of (2.4.2) is denoted by  $\bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes n} = \bar{X}^n(\mathbb{P})$ .

In order to use the mean-field approximation we have to show that  $\mathbb{Q}_{\tau, \zeta}^{n, k} := (X^{1,n}, \dots, X^{k,n})(\mathbb{P})$  converges to  $\mathbb{Q}_{\tau, \zeta}^{\otimes k}$  in an appropriate sense. This is made precise by the notion of chaoticity.

**Definition 2.4.1.** Let  $(S, d_S)$  be a separable metric space,  $\mathbb{Q}$  a probability measure on  $S$  and let  $(\mathbb{Q}^n)_{n \in \mathbb{N}}$  be a sequence of exchangeable probability measures, each defined on the product space  $S^n$ . Then the sequence  $(\mathbb{Q}^n)_{n \in \mathbb{N}}$  is  $\mathbb{Q}$ -chaotic if and only if for any fixed  $k \in \mathbb{N}$  it holds that

$$\mathbb{Q}^{n, k} \xrightarrow{w^k} \mathbb{Q}^{\otimes k} \quad \text{for } n \rightarrow \infty,$$

where  $\mathbb{Q}^{n, k}$  denotes the joint marginal distribution of the first  $k$  individuals.

*Remark 2.4.2.* Assuming that each  $\mathbb{Q}^n$  is the distribution of the random variables  $(X^{1,n}, \dots, X^{n,n})$ , we have that  $\mathbb{Q}^n$  is exchangeable if

$$(X^{1,n}, \dots, X^{n,n}) \stackrel{d}{=} (X^{\sigma(1),n}, \dots, X^{\sigma(n),n})$$

for each permutation  $\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ . This means that the joint distribution of the individuals does not change when reordering them, implying that all individuals have the same marginal distribution. A sufficient, but not necessary condition for this to hold is that all individuals are independent and identically distributed.

The intuitive interpretation of chaosticity is that any fixed group of  $k$  individuals becomes independent and identically distributed with distribution  $\mathbb{Q}$ , when the size of the cohort tends to infinity.

Definition 2.4.1 goes back to Kac (1956), but as chaosticity is equivalent to weak convergence of the marginals  $\mathbb{Q}^{n,k}$  to the product measure  $\mathbb{Q}^{\otimes k}$ , it is possible to relate the notion of chaos to convergence in a metric space (see Hauray and Mischler (2014) and Chaintron and Diez (2022a) for chaosticity in terms of different metrics). Definition 2.4.1 for example can be stated in terms of the bounded-Lipschitz distance, which metrises weak convergence. Other choices leading to slightly different notions of chaos are the Wasserstein(1) distance or the total variation distance. For a discussion of the different notions and the consequences for the results of this paper, see Section 2.7. We will use the stronger notion of chaos in total variation, which always implies Definition 2.4.1.

**Definition 2.4.3.** *Let  $(S, \mathcal{S})$  be a measurable space,  $\mathbb{Q}$  a probability measure on  $(S, \mathcal{S})$  and let  $(\mathbb{Q}^n)_{n \in \mathbb{N}}$  be a sequence of exchangeable probability measures, each defined on  $(S^n, \mathcal{S}^n)$ . Then the sequence  $(\mathbb{Q}^n)_{n \in \mathbb{N}}$  is  $\mathbb{Q}$ -chaotic in total variation if and only if for any fixed  $k \in \mathbb{N}$  it holds that*

$$\lim_{n \rightarrow \infty} d_{TV}(\mathbb{Q}^{n,k}, \mathbb{Q}^{\otimes k}) = 0.$$

In our case the measurable space is  $\mathbb{H}([\tau, T], E)$  equipped with  $\mathcal{B}(\mathbb{H}([\tau, T], E))$ , the sequence of probability measures for which we want to prove chaosticity are the jump process distributions  $(\mathbb{Q}_{\tau, \zeta^n}^n)_{n \in \mathbb{N}}$  of (2.4.1) and the measure for which we would like the sequence to be chaotic for is the jump process distribution  $\bar{\mathbb{Q}}_{\tau, \zeta}$  of (2.3.5). Note that each  $\mathbb{Q}_{\tau, \zeta^n}^n$  should be exchangeable in the sense of Remark 2.4.2, which is the case, since the initial distribution  $\zeta^n$  is exchangeable, the intensity kernels of the random counting measures  $Q^{\ell, n}$  are the same for all  $1 \leq \ell \leq n$  and the individuals depend on each other in a symmetric way through  $\varepsilon_{i-}^n$  only. We now have the following result.

**Theorem 2.4.4.** *Assume that the assumptions of Theorem 2.3.7 are satisfied and that the sequence  $(\zeta^n)_{n \in \mathbb{N}}$  is  $\zeta$ -chaotic in total variation. Then for any fixed  $k \in \mathbb{N}$ , it holds that*

$$\lim_{n \rightarrow \infty} d_{TV}(\mathbb{Q}_{\tau, \zeta^n}^{n,k}, \bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes k}) = 0.$$

### 2.4.1 Proof of Theorem 2.4.4

The proof of Theorem 2.4.4 requires a bound for the distance  $d_{TV}(\mathbb{Q}_{\tau, \zeta^n}^{n,k}, \bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes k})$  for any fixed  $k \in \mathbb{N}$ . Constructing a direct coupling between  $\mathbb{Q}_{\tau, \zeta}^{n,k}$  and  $\bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes k}$  is impossible, since this would require a coupling  $\nu^n$  of  $\zeta^n$  and  $\zeta^{\otimes n}$  on  $(E^{2n}, \mathcal{B}(E^{2n}))$

such that each marginal

$$\nu^n((E^{\ell-1} \times dx_\ell \times E^{n-\ell}) \times (E^{\ell-1} \times dy_\ell \times E^{n-\ell})), \quad \ell = 1, \dots, n$$

is equal to the same maximal coupling  $\nu_1^n$  of  $\zeta^{n,1}$  and  $\zeta$ . This would only be possible if  $\zeta^n$  were a product measure. Instead by Lemma 2.C.3 we can construct a measure  $\nu^n$  on  $(E^{2n}, \mathcal{B}(E^{2n}))$  with the following properties:

- (i) The marginal  $\nu^n(dx^{1:n} \times E^n)$  is equal to  $\zeta^n(dx^{1:n})$ .
- (ii) Let  $\xi^n(dy^{1:n}) := \nu^n(E^n \times dy^{1:n})$ . Then  $\xi^n$  has identical 1-dimensional marginals  $\xi^{n,1} = \zeta$ .
- (iii) The marginal  $\nu^n((E^{\ell-1} \times dx_\ell \times E^{n-\ell}) \times (E^{\ell-1} \times dy_\ell \times E^{n-\ell}))$  equals the same maximal coupling  $\nu_1^n$  of  $\zeta^{n,1}$  and  $\zeta$  for all  $\ell \in \{1, \dots, n\}$ .
- (iv) The sequence  $(\xi^n)_{n \in \mathbb{N}}$  is  $\zeta$ -chaotic in total variation.

Using this we can construct the intermediate jump process  $\hat{X}^n = (\hat{X}^{1,n}, \dots, \hat{X}^{n,n})$  given by

$$\hat{X}_t^n = \hat{Y}_0^n + \int_{(\tau, t] \times E^n} (y^{1:n} - \hat{X}_{s-}^n) \hat{Q}^n(ds, dy^{1:n}), \quad t \in [\tau, T], \quad (2.4.3)$$

where  $(\hat{Y}_0^{1,n}, \dots, \hat{Y}_0^{n,n})$  has distribution  $\xi^n$  and  $\hat{Q}^n$  has compensating measure

$$\hat{L}^n(dt, dy^{1:n}) = \sum_{\ell=1}^n \left( \mu_t(\hat{X}_{t-}^{\ell,n}, \bar{p}_t^{\tau, \zeta}, dy_\ell) \prod_{j=1, j \neq \ell}^n \delta_{\{\hat{X}_{t-}^{j,n}\}}(dy_j) \right) dt,$$

where  $\bar{p}_t^{\tau, \zeta} = \pi_t(\bar{\mathbb{Q}}_{\tau, \zeta})$ . Thus  $\hat{X}^n$  is a Markov jump process with state space  $E^n$  and existence and uniqueness of  $\hat{\mathbb{Q}}_{\tau, \xi^n}^n := \hat{X}^n(\mathbb{P})$  follows from Theorem 2.3.2.

The triangle inequality yields

$$d_{TV}(\mathbb{Q}_{\tau, \zeta^n}^{n,k}, \bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes k}) \leq d_{TV}(\mathbb{Q}_{\tau, \zeta^n}^{n,k}, \hat{\mathbb{Q}}_{\tau, \xi^n}^{n,k}) + d_{TV}(\hat{\mathbb{Q}}_{\tau, \xi^n}^{n,k}, \bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes k}).$$

Thus the two steps of the proof are to show that each of the two distances on the right hand side will vanish for  $n \rightarrow \infty$ . This is done by employing the coupling construction from Section 2.3.3 coordinate-wise to create a coupling between  $\mathbb{Q}_{\tau, \zeta^n}^n$  and  $\hat{\mathbb{Q}}_{\tau, \xi^n}^n$  and between  $\hat{\mathbb{Q}}_{\tau, \xi^n}^n$  and  $\bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes n}$ . We start with the latter pair.

For each  $n \in \mathbb{N}$  we set

$$\begin{pmatrix} \hat{X}_t^n \\ \bar{X}_t^n \end{pmatrix} = \begin{pmatrix} \hat{Y}_0^n \\ \bar{Y}_0^n \end{pmatrix} + \int_{(\tau, t] \times E^{2n}} \begin{pmatrix} \hat{y}^{1:n} - \hat{X}_{s-}^{\ell,n} \\ \bar{y}^{1:n} - \bar{X}_{s-}^{\ell,n} \end{pmatrix} \mathcal{N}^n(ds, d(\hat{y}^{1:n}, \bar{y}^{1:n})), \quad (2.4.4)$$

for  $t \in [\tau, T]$ , where the joint distribution of  $\hat{Y}_0^n = (\hat{Y}_0^{1,n}, \dots, \hat{Y}_0^{n,n})$  and  $\bar{Y}_0^n = (\bar{Y}_0^{1,n}, \dots, \bar{Y}_0^{n,n})$  is a coupling of  $\xi^n$  and  $\zeta^{\otimes n}$  which is maximal for the marginals

$\zeta^{n,k}$  and  $\zeta^{\otimes k}$  (see Lemma 2.C.2 for existence), while  $\mathcal{N}^n$  is a random counting measure with compensating measure

$$\begin{aligned} L^n(dt, d(\hat{y}^{1:n}, \bar{y}^{1:n})) \\ = C_\lambda \sum_{\ell=1}^n \left( \gamma_t(\hat{X}_{t-}^{\ell,n}, \bar{X}_{t-}^{\ell,n}, \bar{p}_t^{\tau,\zeta}, \bar{p}_t^{\tau,\zeta}, d(\hat{y}_\ell, \bar{y}_\ell)) \prod_{j=1, j \neq \ell}^n \delta_{\{\hat{X}_{t-}^{j,n}, \bar{X}_{t-}^{j,n}\}}(d(\hat{y}_j, \bar{y}_j)) \right) dt. \end{aligned}$$

Here  $\gamma : [\tau, T] \times E^2 \times \mathcal{P}(E)^2 \times \mathcal{B}(E^2) \rightarrow [0, 1]$  is the probability kernel from Subsection 2.3.3. By construction there occur common jumps within each pair  $(\hat{X}^{\ell,n}, \bar{X}^{\ell,n})$ , but there are no common jumps across different pairs. Thus each pair  $(\hat{X}^{\ell,n}, \bar{X}^{\ell,n})$  corresponds to the coupling construction of Section 2.3.3 and by an argument similar to the proof of Proposition 2.3.16, we can conclude that the distribution of  $(\hat{X}^{1,n}, \dots, \hat{X}^{k,n}, \bar{X}^{1,n}, \dots, \bar{X}^{k,n})$  is a coupling of  $\hat{\mathbb{Q}}_{\tau, \xi^n}^{n,k}$  and  $\bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes k}$  for any  $k \leq n$ .

**Lemma 2.4.5.** *It holds for each fixed  $k \in \mathbb{N}$  that*

$$\lim_{n \rightarrow \infty} d_{TV}(\hat{\mathbb{Q}}_{\tau, \xi^n}^{n,k}, \bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes k}) = 0.$$

*Proof.* Set  $A_T^{\ell,n} := \bigcup_{t \in [\tau, T]} (\hat{X}_t^{\ell,n} \neq \bar{X}_t^{\ell,n})$ . We then have that

$$d_{TV}(\hat{\mathbb{Q}}_{\tau, \xi^n, k}, \bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes k}) \leq \mathbb{P} \left( \bigcup_{\ell=1}^k A_T^{\ell,n} \right) \leq \sum_{\ell=1}^k \mathbb{P}(A_T^{\ell,n}).$$

where the last inequality is due to the subadditivity of measures. Since  $(\hat{Y}_0^{1,n}, \dots, \hat{Y}_0^{k,n}, \bar{Y}_0^{1,n}, \dots, \bar{Y}_0^{k,n})$  are chosen according to a maximal coupling we have

$$\mathbb{P}(\hat{Y}_0^{\ell,n} \neq \bar{Y}_0^{\ell,n}) \leq \mathbb{P} \left( \bigcup_{\ell=1}^k \hat{Y}_0^{\ell,n} \neq \bar{Y}_0^{\ell,n} \right) = d_{TV}(\zeta^{n,k}, \zeta^{\otimes k}),$$

for  $\ell = 1, \dots, k$ . Lemma 2.3.18 and an application of Grönwall's inequality yields

$$\sum_{\ell=1}^k \mathbb{P}(A_T^{\ell,n}) \leq k e^{C_1(T-\tau)} d_{TV}(\zeta^{n,k}, \zeta^{\otimes k}).$$

As  $(\zeta^n)_{n \in \mathbb{N}}$  is  $\zeta$ -chaotic in total variation, the result follows.  $\square$

We continue with the pair  $\mathbb{Q}_{\tau, \zeta^n}^n$  and  $\hat{\mathbb{Q}}_{\tau, \xi^n}^n$ . For each  $n \in \mathbb{N}$  we set

$$\begin{pmatrix} X_t^n \\ \hat{X}_t^n \end{pmatrix} = \begin{pmatrix} Y_0^n \\ \hat{Y}_0^n \end{pmatrix} + \int_{(\tau, t] \times E^{2n}} \begin{pmatrix} y^{1:n} - X_{s-}^{\ell,n} \\ \hat{y}^{1:n} - \hat{X}_{s-}^{\ell,n} \end{pmatrix} \mathcal{N}^n(ds, d(y^{1:n}, \hat{y}^{1:n})), \quad (2.4.5)$$

for  $t \in [\tau, T]$ , where  $Y_0^n = (Y_0^{1,n}, \dots, Y_0^{n,n})$  and  $\hat{Y}_0^n = (\hat{Y}_0^{1,n}, \dots, \hat{Y}_0^{n,n})$  are from the joint distribution  $\nu^n$  satisfying (i)-(iv) above, while  $\mathcal{N}^n$  is a random counting measure with compensating measure

$$L^n(dt, d(y^{1:n}, \hat{y}^{1:n})) \\ = C_\lambda \sum_{\ell=1}^n \left( \gamma_t(X_{t-}^{\ell,n}, \hat{X}_{t-}^{\ell,n}, \varepsilon_{t-}^n, \bar{p}_t^{\tau,\zeta}, d(y_\ell, \hat{y}_\ell)) \prod_{j=1, j \neq \ell}^n \delta_{\{X_{t-}^{\ell,n}, \hat{X}_{t-}^{\ell,n}\}}(d(y_j, \hat{y}_j)) \right) dt,$$

with  $\gamma : [\tau, T] \times E^2 \times \mathcal{P}(E)^2 \times \mathcal{B}(E^2) \rightarrow [0, 1]$  from Subsection 2.3.3. Again each pair  $(X^{\ell,n}, \hat{X}^{\ell,n})$  corresponds to the coupling construction of Subsection 2.3.3 and by an argument similar to the proof of Proposition 2.3.16 we can conclude that the distribution of  $(X^{1,n}, \dots, X^{k,n}, \hat{X}^{1,n}, \dots, \hat{X}^{k,n})$  is a coupling of  $\mathbb{Q}_{\tau, \zeta^n}^{n,k}$  and  $\hat{\mathbb{Q}}_{\tau, \xi^n}^{n,k}$  for any  $k \leq n$ . Note that by property (iii) of  $\nu^n$  the initial pairs  $(Y_0^{\ell,n}, \hat{Y}_0^{\ell,n})$  and consequently the pairs  $(X^{\ell,n}, \hat{X}^{\ell,n})$  are identically distributed for all  $\ell = \{1, \dots, n\}$ .

**Lemma 2.4.6.** *It holds for each fixed  $k \in \mathbb{N}$  that*

$$\lim_{n \rightarrow \infty} d_{TV}(\mathbb{Q}_{\tau, \zeta^n}^{n,k}, \hat{\mathbb{Q}}_{\tau, \xi^n}^{n,k}) = 0.$$

*Proof.* Set  $A_T^{\ell,n} := \bigcup_{t \in [\tau, T]} (X_t^{\ell,n} \neq \hat{X}_t^{\ell,n})$ . We then have that

$$d_{TV}(\mathbb{Q}_{\tau, \zeta^n}^{n,k}, \hat{\mathbb{Q}}_{\tau, \xi^n}^{n,k}) \leq \mathbb{P}\left(\bigcup_{\ell=1}^k A_T^{\ell,n}\right) \leq k \mathbb{P}(A_T^{\ell,n}).$$

where the last inequality is due to the subadditivity of measures and the fact that the pairs  $(X^{\ell,n}, \hat{X}^{\ell,n})$  are identically distributed. Since by property (iii) of  $\nu^n$

$$\mathbb{P}(Y_0^{\ell,n} \neq \hat{Y}_0^{\ell,n}) = \nu_1^n(E_{x \neq y}^2) = d_{TV}(\zeta^{n,1}, \zeta),$$

Lemma 2.3.18 yields

$$\mathbb{P}(A_T^{\ell,n}) \leq d_{TV}(\zeta^{n,1}, \zeta) + C_1 \int_{\tau}^T \mathbb{P}(A_t^{\ell,n}) + \mathbb{E}[d_{BL}(\varepsilon_{t-}^n, \bar{p}_t^{\tau,\zeta})] dt.$$

By the triangle inequality we have

$$\mathbb{E}[d_{BL}(\varepsilon_{t-}^n, \bar{p}_t^{\tau,\zeta})] \leq \mathbb{E}[d_{BL}(\varepsilon_{t-}^n, \hat{\varepsilon}_{t-}^n)] + \mathbb{E}[d_{BL}(\hat{\varepsilon}_{t-}^n, \bar{p}_t^{\tau,\zeta})],$$

where  $\hat{\varepsilon}_t^n := \frac{1}{n} \sum_{\ell=1}^n \delta_{\{\hat{X}_t^{\ell,n}\}}$ . For the first term, we note that

$$\mathbb{E}[d_{BL}(\varepsilon_{t-}^n, \hat{\varepsilon}_{t-}^n)] \leq \mathbb{E}[d_{TV}(\varepsilon_{t-}^n, \hat{\varepsilon}_{t-}^n)] \leq \frac{1}{n} \sum_{\ell=1}^n \mathbb{P}(A_t^{\ell,n}) = \mathbb{P}(A_t^{\ell,n}).$$

Combining this and applying Grönwall's inequality yields

$$\mathbb{P}(A_T^{\ell,n}) \leq e^{2C_1(T-\tau)} \left( d_{TV}(\zeta^{n,1}, \zeta) + \int_{\tau}^T \mathbb{E}[d_{BL}(\hat{\varepsilon}_{t-}^n, \bar{p}_t^{\tau,\zeta})] dt \right).$$

Since  $\varepsilon_{t-}^n \neq \varepsilon_t^n$  for at most countably many  $t \in [\tau, T]$ , we can replace  $\varepsilon_{t-}^n$  with  $\varepsilon_t^n$  in the integral. By Lemma 2.4.5 we have that  $(\hat{\mathbb{Q}}_{\tau, \xi^n}^n)_{n \in \mathbb{N}}$  is  $\bar{\mathbb{Q}}_{\tau, \zeta}$ -chaotic in total variation, which by Proposition 2.2 in Sznitman (1991) implies that  $\lim_{n \rightarrow \infty} \mathbb{E}[d_{BL}(\hat{\varepsilon}_t^n, \bar{p}_t)] = 0$  for each  $t \in [\tau, T]$ . Applying the Dominated Convergence Theorem yields convergence to zero of the second term. The first term goes to zero since  $(\zeta^n)_{n \in \mathbb{N}}$  is  $\zeta$ -chaotic in total variation. The desired result follows.  $\square$

## 2.5 Conditional mean-field approximation

We now turn to the convergence of the conditional distributions of the  $n$ -individual model (2.4.1). In particular we are interested in the convergence of the following two kinds

$$\begin{aligned} \mathbb{Q}_{\tau, \zeta}^{n,1}(df|X_\tau^\circ \in B) &\xrightarrow{TV} \bar{\mathbb{Q}}_{\tau, \zeta}(df|X_\tau^\circ \in B), \\ \mathbb{Q}_{\tau, \zeta}^{n,1}(df|X_\tau^\circ = x) &\xrightarrow{TV} \bar{\mathbb{Q}}_{\tau, \zeta}(df|X_\tau^\circ = x), \end{aligned}$$

where  $B \in \mathcal{B}(E)$  and  $x \in E$ . We will start with the first type of convergence which can be shown to be a direct consequence of Theorem 2.4.4, followed by the second which requires extra assumptions if  $\bar{\mathbb{Q}}_{\tau, \zeta}(X_\tau^\circ = x) = 0$ .

### 2.5.1 Conditioning on a set

Fix  $\tau \in [0, T]$ ,  $m \in \mathbb{N}$ , let  $n > m$ , set  $\mathbf{B}_m \in \mathcal{B}(E^m)$  and let  $\beta_m^n$  and  $\bar{\beta}_m^n$  denote a regular versions of  $\zeta^n(dy^{1:n}|y^{1:m} \in \mathbf{B}_m)$  and  $\zeta^{\otimes n}(dy^{1:n}|y^{1:m} \in \mathbf{B}_m)$ . We now consider the jump process

$$X_t^n = Y_0^n + \int_{(\tau, t] \times E^n} (y^{1:n} - X_{s-}^n) Q^n(ds, dy^{1:n}), \quad t \in [\tau, T], \quad (2.5.1)$$

where  $(Y_0^{1,n}, \dots, Y_0^{n,n})$  has distribution  $\beta_m^n$  and the random counting measure  $Q^n$  has compensating measure

$$L^n(dt, dy) = \sum_{\ell=1}^n \left( \mu_t(X_{t-}^{\ell,n}, \varepsilon_{t-}^n, dy_\ell) \prod_{j=1, j \neq \ell}^n \delta_{\{X_{t-}^{j,n}\}}(dy_j) \right) dt.$$

This closely resembles the jump process (2.4.1), but the difference is that the initial distribution now is  $\beta_m^n$ , which is a conditional distribution of  $\zeta^n$ . Thus we can interpret (2.5.1) as the model for a cohort of  $n$  individuals, where we already know that the first  $m$  individuals have their initial values in the set  $\mathbf{B}_m$ . This interpretation is correct, since the distribution of (2.5.1) denoted by  $\mathbb{Q}_{\tau, \zeta^n, \mathbf{B}_m}^n$  is the conditional distribution of  $\mathbb{Q}_{\tau, \zeta^n}^n$  given that the initial values of first  $m$  individuals are in  $\mathbf{B}_m$ .

**Proposition 2.5.1.** *Assume that  $\zeta^{n,m}(\mathbf{B}_m) > 0$ . Then it holds that*

$$\mathbb{Q}_{\tau, \zeta^n, \mathbf{B}_m}^n(df) = \mathbb{Q}_{\tau, \zeta^n}^n(df|(X_\tau^{\circ,1}, \dots, X_\tau^{\circ,m}) \in \mathbf{B}_m).$$

*Proof.* Let  $E_{\mathbf{B}_m}^n := \mathbf{B}_m \times E^{n-m}$ . Since  $\mathbb{Q}_{\tau, x^{1:n}}^n$  by Theorem 2.3.4 is a regular conditional distribution of  $\mathbb{Q}_{\tau, \zeta^n}^n$  given all initial values we have that

$$\begin{aligned} \mathbb{Q}_{\tau, \zeta}^n(df \cap (X_\tau^\circ \in E_{\mathbf{B}_m}^n)) &= \int_{E_{\mathbf{B}_m}^n} \mathbb{Q}_{\tau, x^{1:n}}^n(df) \zeta^n(dx^{1:n}) \\ &= \int_{E^n} \mathbb{1}_{E_{\mathbf{B}_m}^n}(x^{1:n}) \mathbb{Q}_{\tau, x^{1:n}}^n(df) \zeta^n(dx^{1:n}). \end{aligned}$$

Since  $\zeta$  is the initial distribution and  $\beta_m^n$  is a conditional distribution we have

$$\mathbb{Q}_{\tau, \zeta}^n(X_\tau^\circ \in E_{\mathbf{B}_m}^n) = \zeta^n(E_{\mathbf{B}_m}^n) \quad \text{and} \quad \beta_m^n(dx^{1:n}) = \mathbb{1}_{E_{\mathbf{B}_m}^n}(x^{1:n}) \frac{\zeta^n(dx^{1:n})}{\zeta^n(E_{\mathbf{B}_m}^n)},$$

which we can use to arrive at

$$\frac{\mathbb{Q}_{\tau, \zeta}^n(df \cap (X_\tau^\circ \in E_{\mathbf{B}_m}^n))}{\mathbb{Q}_{\tau, \zeta}^n(X_\tau^\circ \in E_{\mathbf{B}_m}^n)} = \int_{E^n} \mathbb{Q}_{\tau, x^{1:n}}^n(df) \beta_m^n(dx^{1:n}) = \mathbb{Q}_{\tau, \zeta^n, \mathbf{B}_m}^n(df),$$

where the last equality follows from Lemma 2.B.3.  $\square$

Similarly we can consider the jump process

$$\tilde{X}_t^n = \tilde{Y}_0^n + \int_{(\tau, t] \times E^n} (y^{1:n} - \tilde{X}_{s-}^n) \tilde{Q}^n(ds, dy^{1:n}), \quad t \in [\tau, T], \quad (2.5.2)$$

where the initial values  $\tilde{Y}_0^n = (\tilde{Y}_0^{1,n}, \dots, \tilde{Y}_0^{n,n})$  have joint distribution  $\bar{\beta}_m^n$  and the random counting measure  $\tilde{Q}^n$  has compensating measure

$$\tilde{L}^n(dt, dy^{1:n}) = \sum_{\ell=1}^n \left( \mu_t(\tilde{X}_{t-}^{\ell,n}, \bar{p}_t^{\tau, \zeta}, dy_\ell) \prod_{j=1, j \neq \ell}^n \delta_{\{\tilde{X}_{t-}^{j,n}\}}(dy_j) \right) dt.$$

This closely resembles the jump process (2.4.2), but since  $\bar{p}_t^{\tau, \zeta} = \pi_t(\bar{\mathbb{Q}}_{\tau, \zeta})$ , the process (2.5.2) is a linearised Markov jump process in the spirit of (2.3.6). The process (2.5.2) with distribution  $\tilde{\mathbb{Q}}_{\tau, \zeta, \mathbf{B}_m}^{\otimes n} := \tilde{X}^n(\mathbb{P})$  can thus be interpreted as the conditional distribution of  $\bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes n}$  given that the initial values of the first  $m$  individuals are in  $\mathbf{B}_m$ .

**Proposition 2.5.2.** *Assume that  $\zeta^{\otimes m}(\mathbf{B}_m) > 0$ . Then it holds that*

$$\tilde{\mathbb{Q}}_{\tau, \zeta, \mathbf{B}_m}^{\otimes n}(df) = \bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes n}(df | (X_\tau^{\circ, 1}, \dots, X_\tau^{\circ, m}) \in \mathbf{B}_m).$$

*Proof.* The proof is the same as the proof of Proposition 2.5.1 with Lemma 2.B.3 replaced by Lemma 2.B.4.  $\square$

We are now ready to prove the following conditional version of Theorem 2.4.4.

**Theorem 2.5.3.** *Assume that  $\zeta^{\otimes m}(\mathbf{B}_m) > 0$ . Under the assumptions of Theorem 2.4.4 it holds for any fixed  $k \geq m$  that*

$$\lim_{n \rightarrow \infty} d_{TV}(\mathbb{Q}_{\tau, \zeta^n, \mathbf{B}_m}^{n,k}, \tilde{\mathbb{Q}}_{\tau, \zeta, \mathbf{B}_m}^{\otimes k}) = 0.$$

*Proof.* Let  $A \in \mathcal{B}(\mathbb{H}([\tau, T], E^k))$  be arbitrary and let  $E_{\mathbf{B}_m}^k := \mathbf{B}_m \times E^{k-m}$ . By Proposition 2.5.1 it holds that

$$\mathbb{Q}_{\tau, \zeta^n, \mathbf{B}_m}^{n,k}(A) = \frac{\mathbb{Q}_{\tau, \zeta^n}(A \cap (X_\tau^\circ \in E_{\mathbf{B}_m}^k))}{\zeta^{n,m}(\mathbf{B}_m)}$$

and by Proposition 2.5.2 it holds that

$$\tilde{\mathbb{Q}}_{\tau, \zeta, \mathbf{B}_m}^{\otimes k}(A) = \frac{\bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes k}(A \cap (X_\tau^\circ \in E_{\mathbf{B}_m}^k))}{\zeta^{\otimes m}(\mathbf{B}_m)}.$$

Utilising these two formulas in conjunction with (2.A.2) we arrive at

$$|\mathbb{Q}_{\tau, \zeta^n, \mathbf{B}_m}^{n,k}(A) - \tilde{\mathbb{Q}}_{\tau, \zeta, \mathbf{B}_m}^{\otimes k}(A)| \leq \frac{d_{TV}(\mathbb{Q}_{\tau, \zeta^n}, \bar{\mathbb{Q}}_{\tau, \zeta})}{\zeta^{n,m}(\mathbf{B}_m)} + \left| \frac{1}{\zeta^{n,m}(\mathbf{B}_m)} - \frac{1}{\zeta^{\otimes m}(\mathbf{B}_m)} \right|.$$

For  $n \rightarrow \infty$ , the first term on the right hand side goes to zero due to Theorem 2.4.4 and the second term goes to zero since  $(\zeta^n)_{n \in \mathbb{N}}$  is  $\zeta$ -chaotic in total variation, which implies set-wise convergence. Since this upper bound does not depend on  $A$  it survives taking the supremum over  $A \in \mathcal{B}(\mathbb{H}([\tau, T], E^k))$  and the desired result follows.  $\square$

Theorem 2.5.3 shows, that we have

$$\mathbb{Q}_{\tau, \zeta^n}^{n,k}(\mathrm{d}f|(X_\tau^{\circ,1}, \dots, X_\tau^{\circ,m}) \in \mathbf{B}_m) \xrightarrow{TV} \bar{\mathbb{Q}}_{\tau, \zeta}^{\otimes k}(\mathrm{d}f|(X_\tau^{\circ,1}, \dots, X_\tau^{\circ,m}) \in \mathbf{B}_m)$$

whenever  $\zeta^{\otimes m}(\mathbf{B}_m) > 0$  for any  $k \geq m$ . Choosing  $k = m = 1$ , we thus in particular arrive at

$$\mathbb{Q}_{\tau, \zeta}^{n,1}(\mathrm{d}f|X_\tau^\circ \in B) \xrightarrow{TV} \bar{\mathbb{Q}}_{\tau, \zeta}(\mathrm{d}f|X_\tau^\circ \in B)$$

for any  $B \in \mathcal{B}(E)$  for which  $\zeta(B) > 0$ .

## 2.5.2 Conditioning on a point

Fix  $\tau \in [0, T]$  and  $m \in \mathbb{N}$ , let  $n \geq m$  and set  $\mathbf{x}^m = (x_1, \dots, x_m) \in E^m$ . Let  $(\zeta_{\mathbf{x}^m}^n)_{\mathbf{x}^m \in E^m} \subset \mathcal{P}(E^n)$  denote a regular version of

$$\zeta^n(\mathrm{d}(y_1, \dots, y_n)|(y_1, \dots, y_m) = \mathbf{x}^m)$$

for  $n \geq m$ . Clearly  $\zeta_{\mathbf{x}^m}^n = \delta_{\{\mathbf{x}^m\}} \otimes \beta_{\mathbf{x}^m}^{n-m}$ , where  $(\beta_{\mathbf{x}^m}^{n-m})_{\mathbf{x}^m \in E^m} \subset \mathcal{P}(E^{n-m})$  is a regular conditional probability of

$$\zeta^{n, n-m}(\mathrm{d}(y_{m+1}, \dots, y_n)|(y_1, \dots, y_m) = \mathbf{x}^m).$$

Unfortunately  $\zeta_{\mathbf{x}^m}^n$  is no longer exchangeable, but luckily  $\beta_{\mathbf{x}^m}^{n-m}$  remains so. Using  $\zeta_{\mathbf{x}^m}^n$  as initial distribution, we can define the jump process

$$X_t^n = Y_0^n + \int_{(\tau, t] \times E^n} (y^{1:n} - X_{s-}^n) Q^n(ds, dy^{1:n}), \quad t \in [\tau, T], \quad (2.5.3)$$

where the random counting measure  $Q^n$  has compensating measure

$$L^n(dt, dy^{1:n}) = \sum_{\ell=1}^n \left( \mu_t(X_{t-}^{\ell, n}, \varepsilon_{t-}^n, dy_\ell) \prod_{j=1, j \neq \ell}^n \delta_{\{X_{t-}^{j, n}\}}(dy_j) \right) dt.$$

This closely resembles (2.5.1), the only difference being that the first  $m$  individuals now have known and deterministic starting values  $(Y_0^{1, n}, \dots, Y_0^{m, n}) = \mathbf{x}_m$ , while the rest have random starting values  $(Y_0^{m+1, n}, \dots, Y_0^{n, n})$  from distribution  $\beta_{\mathbf{x}^m}^{n-m}$ . Thus it is not suprising, that the distribution of (2.5.3) denoted by  $\mathbb{Q}_{\tau, \zeta^n, \mathbf{x}^m}^n$  turns out to be a regular conditional distribution of  $\mathbb{Q}_{\tau, \zeta^n}^n$ .

**Proposition 2.5.4.** *The family  $(\mathbb{Q}_{\tau, \zeta^n, \mathbf{x}^m}^n)_{\mathbf{x}^m \in E^m}$  constitutes a regular conditional distribution of  $\mathbb{Q}_{\tau, \zeta^n}^n(df | (X_\tau^{\circ, 1}, \dots, X_\tau^{\circ, m}) = \mathbf{x}^m)$ . Thus it holds that*

$$\mathbb{Q}_{\tau, \zeta}^n(df) = \int_{E^m} \mathbb{Q}_{\tau, \zeta^n, \mathbf{x}^m}^n(df) \zeta^{n, m}(d\mathbf{x}^m).$$

*Proof.* Let  $B_m \in \mathcal{B}(E^m)$  and set  $E_{B_m}^n = B_m \times E^{n-m}$ . Since  $\zeta_{\mathbf{x}^m}^n$  is a regular conditional distribution of  $\zeta^n$  given the first  $m$  coordinates, we have

$$\int_{B_m} \zeta_{\mathbf{x}^m}^n(dy^{1:n}) \zeta^{n, m}(d\mathbf{x}_m) = \zeta^n(dy^{1:n} \cap E_{B_m}^n) = \mathbb{1}_{E_{B_m}^n}(y^{1:m}) \zeta^n(dy^{1:n})$$

and similarly since  $\mathbb{Q}_{\tau, y^{1:n}}^n$  by Theorem 2.3.4 is a regular conditional distribution of  $\mathbb{Q}_{\tau, \zeta^n}^n$  given all initial values, we get

$$\mathbb{Q}_{\tau, \zeta^n}^n(df \cap (X^\circ \in E_{B_m}^n)) = \int_{E^n} \mathbb{1}_{E_{B_m}^n}(y^{1:n}) \mathbb{Q}_{\tau, y^{1:n}}^n(df) \zeta^n(dy^{1:n}).$$

Using these two identities we arrive at

$$\begin{aligned} \int_{B_m} \mathbb{Q}_{\tau, \zeta^n, \mathbf{x}^m}^n(df) \zeta^{n, m}(d\mathbf{x}_m) &= \int_{E^n} \mathbb{Q}_{\tau, y^{1:n}}^n(df) \int_{B_m} \zeta_{\mathbf{x}^m}^n(dy^{1:n}) \zeta^{n, m}(d\mathbf{x}_m) \\ &= \int_{E^n} \mathbb{1}_{E_{B_m}^n}(y^{1:n}) \mathbb{Q}_{\tau, y^{1:n}}^n(df) \zeta^n(dy^{1:n}) \\ &= \mathbb{Q}_{\tau, \zeta^n}^n(df \cap (X^\circ \in E_{B_m}^n)). \end{aligned}$$

□

Similarly to (2.5.2) the mean-field model we consider now is given by

$$\tilde{X}_t^n = \tilde{Y}_0^n + \int_{(\tau, t] \times E^n} (y^{1:n} - \tilde{X}_{s-}^n) \tilde{Q}^n(ds, dy^{1:n}), \quad t \in [\tau, T], \quad (2.5.4)$$

where the initial values  $\tilde{Y}_0^n = (\tilde{Y}_0^{1,n}, \dots, \tilde{Y}_0^{n,n})$  have joint distribution

$$\delta_{\mathbf{x}^m}(\mathrm{d}y^{1:m}) \otimes \zeta^{\otimes n-m}(\mathrm{d}y^{m+1:n})$$

and the random counting measure  $\tilde{Q}^n$  has compensating measure

$$\tilde{L}^n(\mathrm{d}t, \mathrm{d}y^{1:n}) = \sum_{\ell=1}^n \left( \mu_t(\tilde{X}_{t-}^{\ell,n}, \bar{p}_t^{\tau,\zeta}, \mathrm{d}y_\ell) \prod_{j=1, j \neq \ell}^n \delta_{\{\tilde{X}_{t-}^{j,n}\}}(\mathrm{d}y_j) \right) \mathrm{d}t.$$

The individuals are all independent and the jump process distribution of (2.5.4) satisfies

$$\tilde{X}^n(\mathbb{P}) = \bigotimes_{\ell=1}^m \tilde{\mathbb{Q}}_{\tau,\zeta,x_\ell} \otimes \bar{\mathbb{Q}}_{\tau,\zeta}^{\otimes n-m} := \tilde{\mathbb{Q}}_{\tau,\zeta,\mathbf{x}^m}^{\otimes n}.$$

Thus all individuals are independent and while the first  $m$  individuals each have distribution  $\tilde{\mathbb{Q}}_{\tau,\zeta,x_\ell}$ , which is the distribution of the linearised Markov jump process (2.3.6), the remaining  $n - m$  individuals have distribution  $\bar{\mathbb{Q}}_{\tau,\zeta}$ . Thus by Theorem 2.3.10, we can conclude that  $(\tilde{\mathbb{Q}}_{\tau,\zeta,\mathbf{x}^m}^{\otimes n})_{\mathbf{x}^m \in E^m}$  is a regular conditional distribution of

$$\tilde{\mathbb{Q}}_{\tau,\zeta,\mathbf{x}^m}^{\otimes n}(\mathrm{d}f) = \bar{\mathbb{Q}}_{\tau,\zeta}(\mathrm{d}f | (X_\tau^{\circ,1}, \dots, X_\tau^{\circ,m}) = \mathbf{x}^m).$$

If  $\zeta$  has point masses, then we can obtain the following corollary to Theorem 2.5.3:

**Corollary 2.5.5.** *Let  $\mathbf{x}^m \in E^m$ . If  $\zeta^{\otimes m}(\{\mathbf{x}^m\}) > 0$ , then under the assumptions of Theorem 2.4.4 it holds for any fixed  $k \geq m$  that*

$$\lim_{n \rightarrow \infty} d_{TV} \left( \mathbb{Q}_{\tau,\zeta^n,\mathbf{x}^m}^{n,k}, \tilde{\mathbb{Q}}_{\tau,\zeta,\mathbf{x}^m}^{\otimes k} \right) = 0.$$

This result can be interpreted as follows: If we have a group of  $m$  individuals with known initial values that are embedded into a large cohort of individuals whose initial values are random according to a chaotic distribution, then these  $m$  individuals become asymptotically independent, each with distribution  $\tilde{\mathbb{Q}}_{\tau,\zeta,x_\ell}$ . Simultaneously for any fixed  $k > m$  the remaining  $k - m$  individuals become asymptotically independent of the first  $m$  individuals and of each other, each with distribution  $\bar{\mathbb{Q}}_{\tau,\zeta}$ , even though they also depend on the individuals  $1 \leq \ell \leq m$ . Thus even though the collective as a whole is no longer exchangeable, we still recover a conditional version of chaos in total variation. The intuition behind this result is, that changing the initial distribution of a finite number of individuals has no effect on the empirical distribution of the collective, when the total number of individuals tends to infinity.

Sadly Corollary 2.5.5 only applies whenever  $\zeta^{\otimes m}(\{\mathbf{x}^m\}) > 0$ . This is mostly the case if the state space  $E$  is at most countably infinite, but if  $E$  is uncountable, then it most likely happens that  $\zeta^{\otimes m}(\{\mathbf{x}^m\}) = 0$ . In that case, we have to make an extra assumption.

**Theorem 2.5.6.** *Let  $\mathbf{x}^m \in E^m$  and assume that the sequence  $(\beta_{\mathbf{x}^m}^{n-m})_{n>m}$  is  $\zeta$ -chaotic in total variation. Then for any fixed  $k \geq m$  it holds that*

$$\lim_{n \rightarrow \infty} d_{TV}(\mathbb{Q}_{\tau, \zeta^n, \mathbf{x}^m}^{n,k}, \tilde{\mathbb{Q}}_{\tau, \zeta, \mathbf{x}^m}^{\otimes k}) = 0.$$

*Remark 2.5.7.* Note that the assumption that  $(\beta_{\mathbf{x}^m}^{n-m})_{n \geq m}$  is  $\zeta$ -chaotic in total variation is not automatically implied by the fact that  $(\zeta^n)_{n \in \mathbb{N}}$  is  $\zeta$ -chaotic in total variation. Thus it is a stronger assumption. What always is implied is the following. By Theorem 3 of Gänsler and Pfanzagl (1971) it holds for any  $k \in \mathbb{N}$  that

$$\lim_{n \rightarrow \infty} \int_{E^m} d_{TV}(\beta_{\mathbf{x}^m}^{n-m,k}, \zeta^{\otimes k}) \zeta^{\otimes m}(d\mathbf{x}^m) = 0.$$

So in that sense  $(\beta_{\mathbf{x}^m}^{n-m})_{n \geq m}$  is  $\zeta$ -chaotic in total variation in  $L^1$ .

### 2.5.3 Proof of Theorem 2.5.6

The strategy of the proof is similar to the proof of Theorem 2.4.4, with the added complication that the first  $m$  individuals are no longer identically distributed. Similar to the proof of Theorem 2.4.4 the key is to construct an appropriate intermediate jump process distribution  $\hat{\mathbb{Q}}_{\tau, \xi^n, \mathbf{x}^m}^n$  and then use the triangle inequality to conclude that is sufficient to show

$$\lim_{n \rightarrow \infty} d_{TV}(\mathbb{Q}_{\tau, \zeta^n, \mathbf{x}^m}^{n,k}, \hat{\mathbb{Q}}_{\tau, \xi^n, \mathbf{x}^m}^{n,k}) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} d_{TV}(\hat{\mathbb{Q}}_{\tau, \xi^n, \mathbf{x}^m}^{n,k}, \tilde{\mathbb{Q}}_{\tau, \zeta, \mathbf{x}^m}^{\otimes k}) = 0.$$

For the direct approach we would need a coupling of  $\zeta_{\mathbf{x}^m}^n$  and  $\zeta_{\mathbf{x}^m}^{\otimes n}$  which is maximal for each marginal and this is again impossible. Instead we again construct a measure  $\nu_{\mathbf{x}^m}^n$  on  $(E^{2n}, \mathcal{B}(E^{2n}))$  now with the properties:

- (i) The marginal  $\nu_{\mathbf{x}^m}^n(dx^{1:n} \times E^n)$  is equal to  $\zeta_{\mathbf{x}^m}^n$ .
- (ii) Let  $\xi_{\mathbf{x}^m}^n(dy^{1:n}) := \nu_{\mathbf{x}^m}^n(E^n \times dy^{1:n})$ . Then  $\xi_{\mathbf{x}^m}^{n,m}(dy^{1:m}) = \delta_{\mathbf{x}^m}(dy^{1:m})$  and  $\xi_{\mathbf{x}^m}^n(E^{\ell-1} \times dy_\ell \times E^{n-\ell}) = \zeta(dy_\ell)$  for all  $\ell = m+1, \dots, n$ .
- (iii) The marginal  $\nu_{\mathbf{x}^m}^n((E^{\ell-1} \times dx_\ell \times E^{n-\ell}) \times (E^{\ell-1} \times dy_\ell \times E^{n-\ell}))$  equals  $\delta_{\{\mathbf{x}_\ell^m, \mathbf{x}_\ell^m\}}(dx_\ell, dy_\ell)$  for  $\ell = 1, \dots, m$  and the same maximal coupling of  $\beta_{\mathbf{x}^m}^{n-m,1}$  and  $\zeta$  for all  $\ell = m+1, \dots, n$ .
- (iv) Let  $\alpha_{\mathbf{x}^m}^{n-m}(dy^{m+1:n}) := \xi_{\mathbf{x}^m}^n(E^m \times dy^{m+1:n})$ . Then  $(\alpha_{\mathbf{x}^m}^{n-m})_{n>m}$  is  $\zeta$ -chaotic in total variation.

The existence of such a  $\nu_{\mathbf{x}^m}^n$  for all  $n \geq m$  can be verified by using Lemma 2.C.3 and Lemma 2.C.4. The intermediate jump process  $\hat{X}^n = (\hat{X}^{1,n}, \dots, \hat{X}^{n,n})$  is given by (2.4.3) with initial distribution  $\xi_{\mathbf{x}^m}^n$  instead of  $\xi^n$ . The distribution of  $\hat{X}^n$  is denoted by  $\hat{\mathbb{Q}}_{\tau, \xi^n, \mathbf{x}^m}^n$ . We start by considering the pair  $\hat{\mathbb{Q}}_{\tau, \xi^n, \mathbf{x}^m}^n$  and  $\tilde{\mathbb{Q}}_{\tau, \zeta, \mathbf{x}^m}^{\otimes n}$ .

**Lemma 2.5.8.** *It holds for each fixed  $k \geq m$  that*

$$\lim_{n \rightarrow \infty} d_{TV}(\hat{\mathbb{Q}}_{\tau, \xi^n, \mathbf{x}^m}^{n,k}, \tilde{\mathbb{Q}}_{\tau, \zeta, \mathbf{x}^m}^{\otimes k}) = 0.$$

*Proof.* The coupling  $(\widehat{X}^n, \widetilde{X}^n)(\mathbb{P})$  of  $\widehat{\mathbb{Q}}_{\tau, \xi^n, \mathbf{x}^m}^n$  and  $\widetilde{\mathbb{Q}}_{\tau, \zeta, \mathbf{x}^m}^{\otimes n}$  is given by (2.4.4), where the joint distribution of the initial values  $(\widehat{Y}_0^n, \widetilde{Y}_0^n)$  is a coupling of  $\zeta_{\mathbf{x}^m}^n$  and  $\xi_{\mathbf{x}^m}^n$  with the following properties

- (i)  $\mathbb{P}\left(\bigcup_{\ell=1}^m (\widehat{Y}_0^{\ell, n} \neq \widetilde{Y}_0^{\ell, n})\right) = d_{TV}(\delta_{\{\mathbf{x}^m\}}, \delta_{\{\mathbf{x}^m\}}) = 0,$
- (ii)  $\mathbb{P}\left(\bigcup_{\ell=m+1}^k (\widehat{Y}_0^{\ell, n} \neq \widetilde{Y}_0^{\ell, n})\right) = d_{TV}(\alpha_{\mathbf{x}^m}^{n-m, k-m}, \zeta^{\otimes k-m}).$

The existence of such a coupling can be verified using Lemma 2.C.2 and Lemma 2.C.4. Set  $A_T^{\ell, n} := \bigcup_{t \in [\tau, T]} (\widehat{X}_t^{\ell, n} \neq \widetilde{X}_t^{\ell, n})$ . We then have that

$$d_{TV}(\widehat{\mathbb{Q}}_{\tau, \xi^n, k}, \widetilde{\mathbb{Q}}_{\tau, \zeta}^{\otimes k}) \leq \mathbb{P}\left(\bigcup_{\ell=1}^k A_T^{\ell, n}\right) \leq \sum_{\ell=1}^m \mathbb{P}(A_T^{\ell, n}) + \sum_{\ell=m+1}^k \mathbb{P}(A_T^{\ell, n}),$$

where the last inequality is due to the subadditivity of measures. Due to property (i) and (ii) of the joint distribution of  $(\widehat{Y}_0^n, \widetilde{Y}_0^n)$  we have

$$\mathbb{P}(\widehat{Y}_0^{\ell, n} \neq \widetilde{Y}_0^{\ell, n}) \leq \mathbb{P}\left(\bigcup_{\ell=1}^m (\widehat{Y}_0^{\ell, n} \neq \widetilde{Y}_0^{\ell, n})\right) = d_{TV}(\delta_{\{\mathbf{x}^m\}}, \delta_{\{\mathbf{x}^m\}}) = 0$$

for  $\ell = 1, \dots, m$  and

$$\mathbb{P}(\widehat{Y}_0^{\ell, n} \neq \widetilde{Y}_0^{\ell, n}) \leq \mathbb{P}\left(\bigcup_{\ell=m+1}^k (\widehat{Y}_0^{\ell, n} \neq \widetilde{Y}_0^{\ell, n})\right) = d_{TV}(\alpha_{\mathbf{x}^m}^{n-m, k-m}, \zeta^{\otimes k-m})$$

for  $\ell = m+1, \dots, k$ . Lemma 2.3.18 and an application of Grönwall's inequality yields

$$\begin{aligned} \sum_{\ell=1}^m \mathbb{P}(A_T^{\ell, n}) &\leq m e^{C_1(T-\tau)} d_{TV}(\delta_{\{\mathbf{x}^m\}}, \delta_{\{\mathbf{x}^m\}}) = 0, \\ \sum_{\ell=m+1}^k \mathbb{P}(A_T^{\ell, n}) &\leq (k-m) e^{C_1(T-\tau)} d_{TV}(\alpha_{\mathbf{x}^m}^{n-m, k-m}, \zeta^{\otimes k-m}). \end{aligned}$$

As  $(\alpha_{\mathbf{x}^m}^{n-m})_{n>m}$  is  $\zeta$ -chaotic in total variation, the result follows.  $\square$

We now turn towards proving

$$\lim_{n \rightarrow \infty} d_{TV}(\mathbb{Q}_{\tau, \zeta^n, \mathbf{x}^m}^{n, k}, \widehat{\mathbb{Q}}_{\tau, \xi^n, \mathbf{x}^m}^{n, k}) = 0.$$

The coupling  $(X^n, \widehat{X}^n)(\mathbb{P})$  of  $\mathbb{Q}_{\tau, \zeta^n, \mathbf{x}^m}^n$  and  $\widehat{\mathbb{Q}}_{\tau, \xi^n, \mathbf{x}^m}^n$  is given by (2.4.5), where the joint distribution of the initial values  $(Y_0^n, \widehat{Y}_0^n)$  is given by the measure  $\nu_{\mathbf{x}^m}^n$  satisfying points (i)-(iv) above. Note that by property (iii) of  $\nu_{\mathbf{x}^m}^n$  it is now only the pairs  $(Y_0^{\ell, n}, \widehat{Y}_0^{\ell, n})_{\ell=m+1, \dots, n}$  and consequently  $(X^{\ell, n}, \widehat{X}^{\ell, n})_{\ell=m+1, \dots, n}$  that are identically distributed. We start with the following result.

**Lemma 2.5.9.** *Let  $\widehat{\varepsilon}_t^n := \frac{1}{n} \sum_{\ell=1}^n \delta_{\{\widehat{X}_t^{\ell,n}\}}$ . Then it holds that*

$$\lim_{n \rightarrow \infty} \mathbb{E}[d_{BL}(\widehat{\varepsilon}_t^n, \bar{p}_t^{\tau, \zeta})] = 0.$$

*Proof.* For any function  $f : E \rightarrow [-1, 1]$  let  $\bar{m}_t(f) := \int_E f(x) \bar{p}_t^{\tau, \zeta}(\mathrm{d}x)$ . By definition of  $d_{BL}$ , the triangle inequality and subadditivity of the supremum we get

$$\begin{aligned} d_{BL}(\widehat{\varepsilon}_t^n, \bar{p}_t^{\tau, \zeta}) &= \frac{1}{2} \sup_{f \in \text{BL}} \left\{ \left| \frac{1}{n} \sum_{\ell=1}^n f(\widehat{X}_t^{\ell,n}) - \bar{m}_t(f) \right| \right\} \\ &\leq \frac{n-m}{n} \frac{1}{2} \sup_{f \in \text{BL}} \left\{ \left| \frac{1}{n-m} \sum_{\ell=m+1}^n (f(\widehat{X}_t^{\ell,n}) - \bar{m}_t(f)) \right| \right\} \\ &\quad + \frac{1}{2} \sup_{f \in \text{BL}} \left\{ \left| \frac{1}{n} \sum_{\ell=1}^m (f(\widehat{X}_t^{\ell,n}) - \bar{m}_t(f)) \right| \right\} \\ &\leq d_{BL}(\widehat{\rho}_t^{n-m}, \bar{p}_t) + \frac{m}{n}. \end{aligned}$$

where  $\widehat{\rho}_t^{n-m} := \frac{1}{n-m} \sum_{\ell=m+1}^n \delta_{\{\widehat{X}_t^{\ell,n}\}}$ . Substituting  $i = n - m$  we get

$$\widehat{\rho}_t^{n-m} := \frac{1}{n-m} \sum_{\ell=m+1}^n \delta_{\{\widehat{X}_t^{\ell,n}\}} = \frac{1}{i} \sum_{\ell=1}^i \delta_{\{\widehat{X}_t^{m+\ell, m+i}\}}.$$

We thus arrive at

$$\mathbb{E}[d_{BL}(\widehat{\varepsilon}_t^{m+i}, \bar{p}_t^{\tau, \zeta})] \leq 2 \frac{m}{m+i} + \mathbb{E}[d_{BL}(\widehat{\rho}_t^i, \bar{p}_t^{\tau, \zeta})].$$

The first term obviously goes to zero for  $i \rightarrow \infty$ . The second term requires a little more work. For  $i \in \mathbb{N}$  and  $t \in [\tau, T]$  let  $\theta_t^i$  denote the distribution of  $(\widehat{X}_t^{m+1, m+i}, \dots, \widehat{X}_t^{m+i, m+i})$ . Then for any fixed  $j \in \mathbb{N}$  and  $t \in [\tau, T]$  we have

$$d_{TV}(\theta_t^{i,j}, (\bar{p}_t^{\tau, \zeta})^{\otimes j}) \leq d_{TV}(\widehat{\mathbb{Q}}_{\tau, \zeta^{m+i}, \mathbf{x}^m}^{m+i, m+j}, \bar{\mathbb{Q}}_{\tau, \zeta, \mathbf{x}^m}^{\otimes m+j}).$$

By Lemma 2.5.8 we thus have that

$$\lim_{n \rightarrow \infty} d_{TV}(\theta_t^{i,j}, \bar{p}_t^{\otimes j}) = 0$$

for any fixed  $t \in [\tau, T]$  and  $j \in \mathbb{N}$ . Thus  $(\theta_t^i)_{i \in \mathbb{N}}$  is  $\bar{p}_t^{\tau, \zeta}$ -chaotic in total variation, which by Proposition 2.2 in Sznitman (1991) implies that  $\lim_{i \rightarrow \infty} \mathbb{E}[d_{BL}(\widehat{\rho}_t^i, \bar{p}_t^{\tau, \zeta})] = 0$  for each  $t \in [\tau, T]$ .  $\square$

We finish the proof of Theorem 2.5.6 with the following final result.

**Lemma 2.5.10.** *It holds for each fixed  $k \geq m$  that*

$$\lim_{n \rightarrow \infty} d_{TV}(\mathbb{Q}_{\tau, \zeta^n, \mathbf{x}^m}^{n,k}, \widehat{\mathbb{Q}}_{\tau, \xi^n, \mathbf{x}^m}^{n,k}) = 0.$$

*Proof.* Set  $A_T^{\ell,n} := \bigcup_{t \in [\tau, T]} (X_t^{\ell,n} \neq \widehat{X}_t^{\ell,n})$ . We then have that

$$d_{TV}(\mathbb{Q}_{\tau, \zeta^n}^{n,k}, \widehat{\mathbb{Q}}_{\tau, \zeta^n}^{n,k}) \leq \mathbb{P}\left(\bigcup_{\ell=1}^k A_T^{\ell,n}\right) \leq \sum_{\ell=1}^m \mathbb{P}(A_T^{\ell,n}) + (k-m)\mathbb{P}(A_T^{k,n}),$$

where the last inequality is due to the subadditivity of measures and the fact that  $(X^{\ell,n}, \widehat{X}^{\ell,n})$  are identically distributed for  $\ell = m+1, \dots, n$ . By property (iii) of  $\nu_{\mathbf{x}^m}^n$  we have

$$\begin{aligned} \mathbb{P}(Y_0^{\ell,n} \neq \widehat{Y}_0^{\ell,n}) &\leq d_{TV}(\delta_{\{\mathbf{x}^m\}}, \delta_{\{\widehat{\mathbf{x}}^m\}}) = 0, \quad \ell = 1, \dots, m, \\ \mathbb{P}(Y_0^{\ell,n} \neq \widehat{Y}_0^{\ell,n}) &= d_{TV}(\beta_{\mathbf{x}^m}^{n-m,1}, \zeta), \quad \ell = m+1, \dots, n. \end{aligned}$$

Lemma 2.3.18 thus yields

$$\mathbb{P}(A_T^{\ell,n}) \leq C_1 \int_{\tau}^T \mathbb{P}(A_t^{\ell,n}) + \mathbb{E}[d_{BL}(\varepsilon_{t-}^n, \bar{p}_t^{\tau, \zeta})] dt, \quad (2.5.5)$$

for  $\ell = 1, \dots, m$  and

$$\mathbb{P}(A_T^{k,n}) \leq d_{TV}(\beta_{\mathbf{x}^m}^{n-m,1}, \zeta) + C_1 \int_{\tau}^T \mathbb{P}(A_t^{k,n}) + \mathbb{E}[d_{BL}(\varepsilon_{t-}^n, \bar{p}_t^{\tau, \zeta})] dt. \quad (2.5.6)$$

By the triangle inequality we have

$$\mathbb{E}[d_{BL}(\varepsilon_{t-}^n, p_t^{\tau, \zeta})] \leq \mathbb{E}[d_{BL}(\varepsilon_{t-}^n, \widehat{\varepsilon}_{t-}^n)] + \mathbb{E}[d_{BL}(\widehat{\varepsilon}_{t-}^n, p_t^{\tau, \zeta})],$$

where  $\widehat{\varepsilon}_t^n := \frac{1}{n} \sum_{\ell=1}^n \delta_{\{\widehat{X}_t^{\ell,n}\}}$ . For the first term, we note that

$$\begin{aligned} \mathbb{E}[d_{BL}(\varepsilon_{t-}^n, \widehat{\varepsilon}_{t-}^n)] &\leq \mathbb{E}[d_{TV}(\varepsilon_{t-}^n, \widehat{\varepsilon}_{t-}^n)] \leq \frac{1}{n} \sum_{\ell=1}^m \mathbb{P}(A_t^{\ell,n}) + \frac{1}{n} \sum_{\ell=m+1}^n \mathbb{P}(A_t^{\ell,n}) \\ &\leq \frac{m}{n} + \mathbb{P}(A_t^{k,n}). \end{aligned}$$

Inserting this back into (2.5.5) and applying Grönwall's inequality we arrive at

$$\sum_{\ell=1}^m \mathbb{P}(A_T^{\ell,n}) \leq C_1 e^{C_1(T-\tau)} \left( (T-\tau) \left( \frac{m}{n} + \mathbb{P}(A_T^{k,n}) \right) + \int_{\tau}^T \mathbb{E}[d_{BL}(\widehat{\varepsilon}_t^n, \bar{p}_t^{\tau, \zeta})] dt \right).$$

The fraction  $\frac{m}{n}$  obviously goes to zero for  $n \rightarrow \infty$  and Lemma 2.5.9 in conjunction with the Dominated Convergence Theorem implies that the integral goes to zero for  $n \rightarrow \infty$ . It remains to show that  $\mathbb{P}(A_T^{k,n})$  goes to zero.

Inserting the bound for the expected bounded-Lipschitz distance into (2.5.6) and applying Grönwall's inequality we get

$$\mathbb{P}(A_T^{k,n}) \leq e^{2C_1(T-\tau)} \left( d_{TV}(\beta_{\mathbf{x}^m}^{n-m,1}, \zeta) + C_1 \int_{\tau}^T \frac{m}{n} + \mathbb{E}[d_{BL}(\widehat{\varepsilon}_t^n, \bar{p}_t^{\tau, \zeta})] dt \right).$$

The fact that  $(\beta_{\mathbf{x}^m}^{n-m})_{n>m}$  is  $\zeta$ -chaotic in total variation and Lemma 2.5.9 in conjunction with the Dominated Convergence Theorem yields

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_T^{k,n}) = 0.$$

The desired result follows.  $\square$

## 2.6 Convergence of insurance liabilities

Consider a cohort of  $n$  individuals modelled by (2.4.1). The future insurance payments at time  $\tau \in [0, T]$  of each individual are given by  $g((X_t^{\ell,n})_{t \in [\tau, T]})$ , where  $g : \mathbb{H}([\tau, T], E) \rightarrow \mathbb{R}$  is a functional of the individual's jump process path. For pricing and reserving purposes, we are interested in three different types of insurance liabilities:

**Definition 2.6.1.** *Let  $g : \mathbb{H}([\tau, T], E) \rightarrow \mathbb{R}$  be measurable. Then the cohort-wide insurance liability is given by*

$$V^{1,n}(\tau) := \mathbb{E}[g((X_t^{1,n})_{t \in [\tau, T]})].$$

For  $A \in \mathcal{B}(E)$  with  $\zeta^{n,1}(A) > 0$  the grouped insurance liability is given by

$$V^{1,n}(\tau, A) := \mathbb{E}[g((X_t^{1,n})_{t \in [\tau, T]}) | X_\tau^{1,n} \in A].$$

For  $x \in E$  the individual insurance liability is given by

$$V^{1,n}(\tau, x) := \mathbb{E}[g((X_t^{1,n})_{t \in [\tau, T]}) | X_\tau^{1,n} = x].$$

Since the individuals are identically distributed, the cohort-wide insurance liability satisfies

$$V^{1,n}(\tau) = \frac{1}{n} \mathbb{E} \left[ \sum_{\ell=1}^n g((X_t^{\ell,n})_{t \in [\tau, T]}) \right]$$

and is thus the individual's share of the cohort's total insurance liability under an equally weighted distribution. If the state space  $E$  is partitioned into  $m$  disjoint subsets  $(A_i)_{i=1,\dots,m} \subset \mathcal{B}(E)$  such that  $\zeta^{n,1}(A_i) > 0$  for all  $i = 1, \dots, m$ , and  $\bigcup_{i=1}^m A_i = E$ , and we know that the individual is part of group  $A_i$  at time  $\tau$ , then  $V^{1,n}(\tau, A_i)$  can be understood as the insurance liability of an individual, given that that they are part of group  $i$ . Finally the individual insurance liability is the insurance liability of an individual given that particular individual's exact characteristics at time  $\tau$ , represented by  $X_\tau^{1,n} = x$ , and not just their membership of a certain group. The corresponding insurance liabilities of the mean-field model (2.4.2) are given by:

**Definition 2.6.2.** *Let  $g : \mathbb{H}([\tau, T], E) \rightarrow \mathbb{R}$  be measurable. Then the cohort-wide mean-field insurance liability is given by*

$$\bar{V}(\tau) := \mathbb{E}[g((\bar{X}_t)_{t \in [\tau, T]})].$$

For  $A \in \mathcal{B}(E)$  with  $\zeta^{n,1}(A) > 0$  the grouped mean-field insurance liability is given by

$$\bar{V}(\tau, A) := \mathbb{E}[g((\bar{X}_t)_{t \in [\tau, T]}) | \bar{X}_\tau \in A].$$

For  $x \in E$  the individual mean-field insurance liability is given by

$$\bar{V}(\tau, x) := \mathbb{E}[g((\bar{X}_t)_{t \in [\tau, T]}) | \bar{X}_\tau = x].$$

The following three Propositions now show when the mean-field approximations

$$V^{1,n}(\tau) \approx \bar{V}(\tau) \quad \text{and} \quad V^{1,n}(\tau, A) \approx \bar{V}(\tau, A) \quad \text{and} \quad V^{1,n}(\tau, x) \approx \bar{V}(\tau, x)$$

can be formally justified.

**Proposition 2.6.3.** *Assume that the assumptions of Theorem 2.3.7 and Theorem 2.4.4 are satisfied. If there exists an  $\varepsilon > 0$  such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|g((X_t^{1,n})_{t \in [\tau, T]})|^{1+\varepsilon}] < \infty,$$

*then we have  $\lim_{n \rightarrow \infty} V^{1,n}(\tau) = \bar{V}(\tau)$ .*

*Proof.* By Theorem 2.3.7 the mean-field model exists and is unique. By Theorem 2.4.4 the sequence  $(\mathbb{Q}_{\tau, \zeta^n}^n)_{n \in \mathbb{N}}$  is  $\bar{\mathbb{Q}}_{\tau, \zeta}$ -chaotic in total variation. We can therefore apply Proposition 2.D.1 to get the desired result.  $\square$

**Proposition 2.6.4.** *Assume that the assumptions of Theorem 2.3.7 and Theorem 2.5.3 are satisfied. If there exists an  $\varepsilon > 0$  such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|g((X_t^{1,n})_{t \in [\tau, T]})|^{1+\varepsilon} | X_\tau^{1,n} \in A] < \infty,$$

*then we have  $\lim_{n \rightarrow \infty} V^{1,n}(\tau, A) = \bar{V}(\tau, A)$ .*

*Proof.* By Theorem 2.3.7 the mean-field model exists and is unique. By Theorem 2.5.3 we have that  $\mathbb{Q}_{\tau, \zeta^n, A}^{n,1} \xrightarrow{TV} \tilde{\mathbb{Q}}_{\tau, \zeta, A}$ . The result follows directly from Corollary 2.9 in Feinberg, Kasyanov, and Zgurovsky (2016).  $\square$

**Proposition 2.6.5.** *Assume that the assumptions of Theorem 2.3.7 and additionally of Corollary 2.5.5 or Theorem 2.5.6 are satisfied. If there exists an  $\varepsilon > 0$  such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|g((X_t^{1,n})_{t \in [\tau, T]})|^{1+\varepsilon} | X_\tau^{1,n} = x] < \infty,$$

*then we have  $\lim_{n \rightarrow \infty} V^{1,n}(\tau, x) = \bar{V}(\tau, x)$ .*

*Proof.* By Theorem 2.3.7 the mean-field model exists and is unique. By Corollary 2.5.5 or Theorem 2.5.6 we have that  $\mathbb{Q}_{\tau, \zeta^n, x}^{n,1} \xrightarrow{TV} \tilde{\mathbb{Q}}_{\tau, \zeta, x}$ . The result follows directly from Corollary 2.9 in Feinberg, Kasyanov, and Zgurovsky (2016).  $\square$

Assuming a little more integrability, we can derive the following law of large numbers, which shows that the diversification effect of large portfolios persists, even though all individuals are dependent. The diversification effect persists both on a cohort or cohort-wide level, but also on a group level.

**Proposition 2.6.6.** *Assume that the assumptions of Theorem 2.3.7 and Theorem 2.4.4 are satisfied. If there exists an  $\varepsilon > 0$  such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|g(X^{1,n})|^{2+\varepsilon}] < \infty,$$

*then it holds that*

$$\frac{1}{n} \sum_{\ell=1}^n g((X_t^{\ell,n})_{t \in [\tau, T]}) \xrightarrow{L^2} \bar{V}(\tau).$$

*If  $A \in \mathcal{B}(E)$  with  $\zeta(A) > 0$ , then*

$$\frac{\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_{(X_\tau^{\ell,n} \in A)} g((X_t^{\ell,n})_{t \in [\tau, T]})}{\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_{(X_\tau^{\ell,n} \in A)}} \xrightarrow{P} \bar{V}(\tau, A).$$

*Proof.* By Theorem 2.3.7 the mean-field model exists and is unique. By Theorem 2.4.4 the sequence  $(\mathbb{Q}_{\tau, \zeta^n}^n)_{n \in \mathbb{N}}$  is  $\mathbb{Q}_{\tau, \zeta}$ -chaotic in total variation. We can therefore apply Proposition 2.D.2 to directly get the first result. For the second result we note that

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|\mathbb{1}_{(X_\tau^{\ell,n} \in A)} g((X_t^{\ell,n})_{t \in [\tau, T]})|^{2+\varepsilon}] \leq \sup_{n \in \mathbb{N}} \mathbb{E}[|g((X_t^{\ell,n})_{t \in [\tau, T]})|^{2+\varepsilon}],$$

and that  $\mathbb{1}_{(X_\tau^{\ell,n} \in A)} \leq 1$  for all  $n \in \mathbb{N}$ . We can thus apply Proposition 2.D.2 to get  $L^2$ -convergence and thus convergence in probability of both nominator and denominator. An application of the continuous mapping theorem yields the second result.  $\square$

Let  $\sigma_\tau^2 := \text{Var}(g((\bar{X}_t)_{t \in [\tau, T]}))$ . If we assume further integrability and specific convergence speeds, then we can obtain the following central limit theorem.

**Proposition 2.6.7.** *Assume that the assumptions of Theorem 2.3.7 and Theorem 2.4.4 are satisfied, that*

$$\lim_{n \rightarrow \infty} n \text{Cov}(g((X_t^{1,n})_{t \in [\tau, T]}), g((X_t^{2,n})_{t \in [\tau, T]})) = 0,$$

*and that  $\lim_{n \rightarrow \infty} \sqrt{n}(V^{1,n}(\tau) - \bar{V}(\tau)) = 0$ . If there exists an  $\varepsilon > 0$  such that*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|g((X_t^{1,n})_{t \in [\tau, T]})|^{4+\varepsilon}] < \infty,$$

*then it holds that*

$$\frac{1}{\sqrt{n}} \sum_{\ell=1}^n \frac{V^{1,n}(\tau) - \bar{V}(\tau)}{\sigma_\tau} \xrightarrow{D} N(0, 1).$$

Note that since total variation chaos implies that the individuals become asymptotically independent it always holds that

$$\lim_{n \rightarrow \infty} \text{Cov}(g((X_t^{1,n})_{t \in [\tau, T]}), g((X_t^{2,n})_{t \in [\tau, T]})) = 0,$$

and in light of Proposition 2.6.3 it always holds that  $\lim_{n \rightarrow \infty} (V^{1,n}(\tau) - \bar{V}(\tau)) = 0$ . Unfortunately since total variation chaos in itself is not enough to automatically guarantee the required convergence speeds these would have to be verified on a case by case basis. This appears to be very difficult to do theoretically.

### 2.6.1 Non-life insurance liabilities

In many non-life insurance applications the main quantity of interest is the expected loss throughout the contract period  $[0, T]$ . If we have a cohort of  $n$  individuals, the loss of each individual is typically modelled by

$$W_T^{\ell, n} = \sum_{i=1}^{N_T^{\ell, n}} Y_i^{\ell, n},$$

where the counting process  $N_T^{\ell, n}$  can be interpreted as the individual claim count, while the non-negative random variables  $(Y_i^{\ell, n})_{i \in \mathbb{N}}$  are the individual claim sizes. Often the individual claim counts and claim sizes depend on individual characteristics, such as covariates (constant in time) or status (exposed/not exposed variable over time) of the individual, which we here will model by a process  $U^{\ell, n}$  taking values in the space  $\mathcal{U} \subseteq \mathbb{R}^d$ . Usually the individuals are assumed to be independent given their individual characteristics, but as we have seen in the case of cyber insurance, this is not always the case.

The total loss and the individual characteristics can be modelled as one joint jump process starting at  $\tau = 0$ . We therefore now consider a cohort of  $n$  individuals of the type (2.4.1), where each  $X^{\ell, n} = (W^{\ell, n}, U^{\ell, n})$  has state space  $E = [0, \infty) \times \mathcal{U} \subset \mathbb{R}^{d+1}$  and is defined by

$$X_t^{\ell, n} = \begin{pmatrix} W_t^{\ell, n} \\ U_t^{\ell, n} \end{pmatrix} = \begin{pmatrix} 0 \\ U_0^{\ell, n} \end{pmatrix} + \int_{(0, t] \times E} \begin{pmatrix} (w - W_{s-}^{\ell, n}) \\ (u - U_{s-}^{\ell, n}) \end{pmatrix} Q^{\ell, n}(ds, d(w, u)),$$

where  $Q^{\ell, n}$  has compensating measure

$$L^{\ell, n}(dt, d(w, u)) = \mu_t(W_{t-}^{\ell, n}, U_{t-}^{\ell, n}, \varepsilon_{t-}^n, d(w, u))dt.$$

The initial distribution of the process  $X = (X^{1,n}, \dots, X^{n,n})$  is given by  $\zeta^n = \delta_{\{0\}}^{\otimes n} \otimes \nu^n$ , where  $\nu^n \in \mathcal{P}(\mathcal{U}^n)$  is the initial distribution of  $(U^{\ell, n})_{\ell=1, \dots, n}$ . We assume that  $\nu^n$  is  $\nu$ -chaotic in total variation for some  $\nu \in \mathcal{P}(\mathcal{U})$  which implies that  $\zeta^n$  is  $\zeta$ -chaotic in total variation, where  $\zeta := \delta_{\{0\}} \otimes \nu$ .

The typical non-life insurance liabilities of interest are the cohort-wide and individual expected loss

$$V^{1,n}(0) := \mathbb{E}[W_T^{\ell,n}] \quad \text{and} \quad V^{1,n}(0, u) := \mathbb{E}[W_T^{1,n} | U_0^{1,n} = u].$$

Since it is customary to partition the space of individual characteristics  $\mathcal{U}$  into a finite number of groups, we are also interested in the expected loss for a particular group  $A \in \mathcal{B}(E)$  given by

$$V^{1,n}(0, A) := \mathbb{E}[W_T^{1,n} | U_0^{1,n} \in A].$$

The corresponding mean-field model is given by

$$\bar{X}_t = \begin{pmatrix} \bar{W}_t \\ \bar{U}_t \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{U}_0 \end{pmatrix} + \int_{(0,t] \times E} \begin{pmatrix} (w - \bar{W}_{s-}) \\ (u - \bar{U}_{s-}) \end{pmatrix} \bar{Q}(ds, d(w, u)),$$

where  $\bar{U}_0$  has distribution  $\nu$  and  $\bar{Q}$  has compensating measure

$$\bar{L}(dt, d(w, u)) = \mu_t(\bar{W}_{t-}, \bar{U}_{t-}, \bar{p}_t^{\tau, \zeta}, d(w, u))dt.$$

The mean-field insurance liabilities are given by

$$\bar{V}(0) := \mathbb{E}[\bar{W}_T] \quad \text{and} \quad \bar{V}(0, u) := \mathbb{E}[\bar{W}_T | \bar{U}_0 = u],$$

and in the case of a grouping of the space  $\mathcal{U}$  we have

$$\bar{V}(0, A) := \mathbb{E}[\bar{W}_T | \bar{U}_0 \in A].$$

The next result shows that the integrability conditions required for the convergence results of Propositions 2.6.3-2.6.7 to hold are automatically satisfied if the intensity measure and the initial distribution satisfy simple moment conditions.

**Proposition 2.6.8.** *Let  $\|x\| := \sum_{i=1}^{d+1} |x_i|$  for  $x \in \mathbb{R}^{d+1}$  and let  $q > 1$ . If there exists a constant  $C > 0$  such that*

$$\int_E \|y\|^q \mu_t(x, \rho, dy) \leq C, \quad \forall t \in [0, T], x \in E, \rho \in \mathcal{P}(E).$$

*and if  $\sup_{n \in \mathbb{N}} \int_E \|y\|^q \zeta^{n,1}(dy) < \infty$  then there exists  $g : \mathbb{H}([\tau, T], E) \rightarrow \mathbb{R}$  such that*

- (i)  $W_T^{\ell,n} = g((X_t^{\ell,n})_{t \in [\tau, T]})$  and  $\bar{W}_T = g((\bar{X}_t)_{t \in [\tau, T]})$ ,
- (ii)  $\sup_{n \in \mathbb{N}} \mathbb{E}[|g((X_t^{1,n})_{t \in [\tau, T]})|^q] < \infty$ ,
- (iii)  $\sup_{n \in \mathbb{N}} \mathbb{E}[|g((X_t^{1,n})_{t \in [\tau, T]})|^q | X_\tau^{1,n} = x] < \infty$ ,
- (iv)  $\sup_{n \in \mathbb{N}} \mathbb{E}[|g((X_t^{1,n})_{t \in [\tau, T]})|^q | X_\tau^{1,n} \in A] < \infty$ , whenever  $\zeta^{n,1}(A) > 0$  for all  $n \in \mathbb{N}$  and  $\sup_{n \in \mathbb{N}} \int_E \|y\|^q \zeta^{n,1}(dy) < \infty$ .

*Proof.* (i): Note that since  $E \subset \mathbb{R}^{d+1}$ , we have that any element  $f \in \mathbb{H}([\tau, T], E)$  has  $d + 1$  coordinates, that is  $f = (f_1, \dots, f_{d+1})$ . Let now  $g : \mathbb{H}([\tau, T], E) \rightarrow \mathbb{R}$  be given by  $g(f) = \pi_T^1(f)$ , where  $\pi_T^1(f) = f_1(T)$ . Then we have that

$$g((X_t^{\ell,n})_{t \in [\tau, T]}) = W_T^{\ell,n} \quad \text{and} \quad g((\bar{X}_t)_{t \in [\tau, T]}) = \bar{W}_T.$$

(ii): Clearly  $|g(f)| \leq \sup_{t \in [\tau, T]} \|\pi_t(f)\|$  and we have arrive at

$$\mathbb{E}[|g((X_t^{1,n})_{t \in [\tau, T]})|^q] \leq \mathbb{E}\left[\sup_{t \in [\tau, T]} \|X_t^{1,n}\|^q\right].$$

Due to to the representation (2.3.2) it holds that

$$\sup_{t \in [\tau, T]} \|X_t^{1,n}\|^q \leq \|Y_0^{1,n}\|^q + \int_{(\tau, T] \times E} \|y\|^q Q^{1,n}(dt, dy).$$

Thus by assumption we have

$$\begin{aligned} \mathbb{E}\left[\sup_{t \in [\tau, T]} \|X_t^{1,n}\|^q\right] &\leq \mathbb{E}[\|Y_0^{1,n}\|^q] + \mathbb{E}\left[\int_{\tau}^T \int_E \|y\|^q \mu_t(X_{t-}^{1,n}, dy) dt\right] \\ &\leq \sup_{n \in \mathbb{N}} \int_E \|y\|^q \zeta^{n,1}(dy) + C(T - \tau) < \infty. \end{aligned}$$

This bound holds for all  $n \in \mathbb{N}$  and the desired result follows.

(iii)+(iv): By Proposition 2.5.4 and Proposition 2.5.1 we can prove the second and third result by repeating the argument with  $X^n$  given by (2.5.3) and (2.5.1) instead.  $\square$

**Example 2.6.9** (Cyber insurance). One case that fits into this framework is the cyber insurance example from Section 2.2. We can view the total loss  $W_T$  as the terminal value of a jump process that has jump sizes with distribution  $r_i(dy) = h_i(y)\kappa(dy)$  whenever  $Z$  jumps from the susceptible state to the infected state and has no jump when  $Z$  jumps back to the susceptible state. We can thus view  $(Z, I)$  as the process  $U$ , which provides the individual characteristics: The status susceptible/infected and the location in the network. If we modify the intensity kernel (2.2.4) appropriately, then we can set  $X = (W, Z, I)$ . This modification is given as follows:

$$\begin{aligned} \mu_t((\bar{w}, \bar{z}, \bar{i}), \rho, d(w, z, i)) &= \mathbf{1}_{(\bar{z}=0)} \mu_{01}(t, \bar{i}, \rho) \delta_{\{1, \bar{i}\}}(d(z, i)) \mathbf{1}_{[\bar{w}, \infty)}(w) h_{\bar{i}}(w - \bar{w}) dw \\ &\quad + \mathbf{1}_{(\bar{z}=1)} \gamma_{\bar{i}} \delta_{\{\bar{w}, 0, \bar{i}\}}(d(w, z, i)). \end{aligned}$$

Note that  $W$  only has a jump whenever  $Z$  jumps from 0 to 1. Even though this modified intensity kernel does not have a density anymore, we can use similar arguments as in the proof of Proposition 2.3.14 and Example 2.3.15 to show that the assumptions of Theorem 2.3.7 are satisfied. Assume that the initial distribution

of  $(Z^{\ell,n}, I^{\ell,n})_{\ell=1,\dots,n}$  given by  $\nu_n$  is  $\nu$ -chaotic in total variation for some  $\nu$ . Then the initial distribution of  $X^n$  given by  $\zeta^n = \delta_{\{0\}}^{\otimes n} \otimes \nu^n$  is  $\zeta$ -chaotic in total variation, where  $\zeta = \delta_{\{0\}} \otimes \nu$ .

If  $\int_{[0,\infty)} |y|^q h_i(y) dy < \infty$  for some  $q > 1$  and all  $i \in \{1, \dots, J\}$ , Proposition 2.6.3 yields

$$\lim_{n \rightarrow \infty} V^{1,n}(0) = \bar{V}(0),$$

since the initial distribution  $\zeta^{n,1}$  has the same finite support for all  $n \in \mathbb{N}$ . As  $(Z^{1,n}, I^{1,n})$  and  $(\bar{Z}, \bar{I})$  are defined on a discrete space, we can by Proposition 2.6.5 furthermore conclude that

$$\lim_{n \rightarrow \infty} V_0^{1,n}(0, i) = \bar{V}_0(0, i)$$

whenever  $\nu(\{z, i\}) > 0$ . Finally if  $\int_{[0,\infty)} |y|^q h_i(y) dy < \infty$  for some  $q > 2$  and all  $i \in \{1, \dots, J\}$  we have by Proposition 2.6.6

$$\frac{1}{n} \sum_{\ell=1}^n W_T^{\ell,n} \xrightarrow{L^2} \bar{V}(0) \quad \text{and} \quad \frac{\frac{1}{n} \sum_{\ell=1}^n \mathbf{1}_{(Z_0^{\ell,n}=0, I_0^{\ell,n}=i)} W_T^{\ell,n}}{\frac{1}{n} \sum_{\ell=1}^n \mathbf{1}_{(Z_0^{\ell,n}=0, I_0^{\ell,n}=i)}} \xrightarrow{P} \bar{V}_0(0, i).$$

We conclude that the insurance liabilities in the  $n$ -individual model converge to their mean-field counterparts and that the dependency between the individuals does not remove the diversification effect of large portfolios. Thus the mean-field liabilities  $\bar{V}(0)$  and  $\bar{V}_0(0, i)$  are indeed viable approximations of  $V^{1,n}(0)$  and  $V_0^{1,n}(0, i)$ .

## 2.6.2 Life insurance liabilities

In life insurance applications the contractual payments are deterministic functions of the jump process describing the biometric state of each individual and other quantities of interest. Therefore there is no need to include the payments in the jump process itself and we directly consider the  $n$ -individual model (2.4.1) with a state space  $E \in \mathcal{B}(\mathbb{R}^d)$ . Within this model, each individual receives the contractual payments given by

$$B^{\ell,n}(dt) = b(t, X_{t-}^{\ell,n})dt + \int_E b^y(t, X_{t-}^{\ell,n}) Q^{\ell,n}(dt, dy), \quad \ell = 1, \dots, n,$$

where  $Q^{\ell,n}$  is the same random counting measure which also drives  $X^{\ell,n}$  and  $b$  and  $(b^y)_{y \in E}$  are measurable payment functions. The function  $b$  describes the sojourn payment rate, while the functions  $(b^y)_{y \in E}$  describe the transition payments. Let  $r : [0, T] \rightarrow \mathbb{R}$  be a bounded and measurable function. The present value of future payments at time  $\tau \in [0, T]$  are given by

$$PV^{\ell,n}(\tau) := \int_{\tau}^T e^{-\int_{\tau}^t r(u) du} B^{\ell,n}(dt).$$

The insurance liabilities of interest are then the cohort-wide and state-wise (individual) reserves

$$V^{1,n}(\tau) = \mathbb{E}[PV^{1,n}(\tau)] \quad \text{and} \quad V^{1,n}(\tau, x) = \mathbb{E}[PV^{1,n}(\tau) | X_\tau^{1,n} = x].$$

In the corresponding mean-field model (2.4.2) the individual payments are given by

$$\bar{B}(dt) = b(t, \bar{X}_{t-})dt + \int_E b^y(t, \bar{X}_{t-})\bar{Q}(dt, dy),$$

the present value of future payments is given by

$$\bar{P}V(\tau) := \int_\tau^T e^{-\int_\tau^t r(u)du} \bar{B}(dt),$$

and the mean-field liabilities are given by

$$\bar{V}(\tau) = \mathbb{E}[\bar{P}V(\tau)] \quad \text{and} \quad \bar{V}(\tau, x) = \mathbb{E}[\bar{P}V(\tau) | \bar{X}_\tau = x].$$

The following result shows that in order for the convergence results of Propositions 2.6.3–2.6.7 to apply it suffices to assume that  $b(t, x)$  and  $b^y(t, x)$  are uniformly bounded.

**Proposition 2.6.10.** *Assume that there exists  $C > 0$  such that  $|b(t, x)| < C$  for all  $(t, x) \in [\tau, T] \times E$  and  $|b^y(t, x)| < C$  for all  $(t, x, y) \in [\tau, T] \times E^2$ . Then there exists  $g : \mathbb{H}([\tau, T], E) \rightarrow \mathbb{R}$  such that for all  $q > 1$*

- (i)  $PV^{\ell,n}(\tau) = g((X_t^{\ell,n})_{t \in [\tau, T]})$  and  $\bar{P}V(\tau) = g((\bar{X}_t)_{t \in [\tau, T]})$ ,
- (ii)  $\sup_{n \in \mathbb{N}} \mathbb{E}[|g((X_t^{1,n})_{t \in [\tau, T]})|^q] < \infty$ ,
- (iii)  $\sup_{n \in \mathbb{N}} \mathbb{E}[|g((X_t^{1,n})_{t \in [\tau, T]})|^q | X_\tau^{1,n} = x] < \infty$ ,
- (iv)  $\sup_{n \in \mathbb{N}} \mathbb{E}[|g((X_t^{1,n})_{t \in [\tau, T]})|^q | X_\tau^{1,n} \in A] < \infty$ , whenever  $\zeta^{n,1}(A) > 0$  for all  $n \in \mathbb{N}$ .

*Proof.* (i): Let  $g : \mathbb{H}([\tau, T], E) \rightarrow \mathbb{R}$  be given by

$$g(f) := \int_\tau^T b(t, \pi_{t-}(f))dt + \sum_{\tau < t \leq T} \mathbf{1}_{(\pi_t(f) \neq \pi_{t-}(f))} b^{\pi_t(f)}(t, \pi_{t-}(f)),$$

for  $f \in \mathbb{H}([\tau, T], E)$ . Then clearly  $PV^{\ell,n}(\tau) = g((X_t^{\ell,n})_{t \in [\tau, T]})$  and  $\bar{P}V(\tau) = g((\bar{X}_t)_{t \in [\tau, T]})$ .

(ii): Due to the boundedness of  $b$  and  $(b^y)_{y \in E}$  we have that

$$|g(f)| \leq K \left( 1 + \sum_{\tau < t \leq T} \mathbf{1}_{(\pi_t(f) \neq \pi_{t-}(f))} \right),$$

for all  $f \in \mathbb{H}([\tau, T], E)$ . Here  $K = C \max(1, (T - \tau))$ . Since

$$\sum_{\tau < t \leq T} \mathbb{1}_{(X_t^{1,n} \neq X_{t-}^{1,n})} = Q^{1,n}((\tau, T] \times E),$$

Lemma 2.E.2 and the bound on  $g$  yields

$$\mathbb{E}[|g(X^{1,n})|^q] \leq K^q 2^{q-1} \left( 1 + \mathbb{E}[Q^{1,n}((\tau, T] \times E)^q] \right).$$

The counting process  $t \mapsto Q^{1,n}([\tau, t] \times E)$  has intensity process

$$t \mapsto \int_{\tau}^t \int_E \mu_s(X_{s-}^{1,n}, \varepsilon_{s-}^n, dy) ds \leq C_{\lambda}(t - \tau) \leq C_{\lambda}(t - \tau),$$

which is bounded, without the bound depending on  $n$ . Thus it follows that all  $Q^{1,n}([\tau, t] \times E)$  are dominated by the same time-homogeneous Poisson process  $M$  with rate  $C_{\lambda}(t - \tau)$  in the sense of first order stochastic dominance. This implies that  $\mathbb{E}[Q^{1,n}([\tau, T] \times E)^q] \leq \mathbb{E}M_T^q < \infty$  for all  $n \in \mathbb{N}$ . Since a Poisson random variable has all moments, the desired result follows.

(iii)+(iv): By Proposition 2.5.4 and Proposition 2.5.1 we can prove the second and third result by repeating the argument with  $X^n$  given by (2.5.3) and (2.5.1) instead.  $\square$

**Example 2.6.11** (Epidemic health insurance). In the case of the epidemic health insurance example of Section 2.2, we have shown in Example 2.3.15 that the assumptions of Theorem 2.3.7 are satisfied, yielding existence and uniqueness of the mean-field model. We now assume that the initial distribution  $\nu^n$  of  $(Z^n, I^n)$  is  $\nu$ -chaotic in total variation. The individual contractual payments in the  $n$ -individual model are given by

$$B^{\ell,n}(dt) = -\mathbb{1}_{(Z_{t-}^{\ell,n}=0)}\pi(I^{\ell,n})dt + \mathbb{1}_{(Z_{t-}^{\ell,n}=1)}b(I^{\ell,n})dt,$$

while the contractual payments in the mean-field model are given by

$$\bar{B}(dt) = -\mathbb{1}_{(\bar{Z}_{t-}=0)}\pi(\bar{I})dt + \mathbb{1}_{(\bar{Z}_{t-}=1)}b(\bar{I})dt.$$

Since  $\pi : \{1, \dots, J\} \rightarrow [0, \infty)$  and  $b : \{1, \dots, J\} \rightarrow [0, \infty)$  only take a finite number of values they are bounded and we can thus by Propositions 2.6.10+2.6.3 conclude that

$$\lim_{n \rightarrow \infty} V^{1,n}(\tau) = \bar{V}(\tau).$$

Furthermore since  $(Z^{1,n}, I^{1,n})$  and  $(\bar{Z}, \bar{I})$  are defined on a discrete state space, Propositions 2.6.10+2.6.5 yield

$$\lim_{n \rightarrow \infty} V_z^{1,n}(\tau, i) = \bar{V}_z(\tau, i)$$

whenever  $\zeta(\{z, i\}) > 0$ . Finally Propositions 2.6.10+2.6.6 yield

$$\frac{1}{n} \sum_{\ell=1}^n PV^{\ell,n}(\tau) \xrightarrow{L^2} \bar{V}(\tau) \quad \text{and} \quad \frac{\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_{(Z_{\tau}^{\ell,n}=0, I_{\tau}^{\ell,n}=i)} PV^{\ell,n}(\tau)}{\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_{(Z_{\tau}^{\ell,n}=0, I_{\tau}^{\ell,n}=i)}} \xrightarrow{P} \bar{V}_z(\tau, i).$$

We conclude that the insurance liabilities in the  $n$ -individual model converge to their mean-field counterparts and that the dependency between the individuals does not remove the diversification effect of large portfolios. Thus the mean-field liabilities  $\bar{V}(\tau)$  and  $\bar{V}_z(\tau, i)$  are indeed viable approximations of  $V^{1,n}(\tau)$  and  $V_z^{1,n}(\tau, i)$ .

## 2.7 Discussion of different notions of chaos

As we have already noted in Section 2.4 it is possible to choose different kinds of chaos metrised by different metrics. Apart from the chaos in Definition 2.4.1 metrised by the bounded-Lipschitz distance and the total variation chaos of Definition 2.4.3, another popular option often used in the literature is Wasserstein(1)-chaos (see Hauray and Mischler (2014) and Chaintron and Diez (2022a)). This requires  $\lim_{n \rightarrow \infty} d_W(\mathbb{Q}^{n,k}, \mathbb{Q}) = 0$  for all fixed  $k \in \mathbb{N}$ , where  $d_W$  is the Wasserstein(1) distance. While Definition 2.4.3 merely requires a measurable space  $(S, \mathcal{S})$ , Definition 2.4.1 and Wasserstein(1)-chaos require that there is a metric  $d_S$  such that  $(S, d_S)$  is a separable metric space. If the metric space  $(S, d_S)$  is bounded in the sense of the metric  $d_S$ , then Wasserstein(1) chaos is equivalent to Definition 2.4.1, and thus weaker than total variation chaos. If the metric space  $(S, d_S)$  is unbounded in the sense of the metric  $d_S$ , the situation is more complicated. Since convergence in the Wasserstein(1) distance is equivalent to weak convergence on the space of probability measures with first moment and convergence of first moments (see Theorem 6.9 in Villani (2009)) it is stronger than the bounded-Lipschitz chaos. At the same time convergence in the total variation distance is a stronger mode of convergence than convergence in the Wasserstein(1) distance, but the latter distance is not dominated by the former. Thus the interesting case to compare is the case of unbounded  $(S, d_S)$ .

In our case  $(S, d_S) = (\mathbb{H}([\tau, T], E), d_{J_1})$ , where  $d_{J_1}$  is the  $J_1$ -metric which generates the  $J_1$ -topology on the Skorokhod space of càdlàg paths  $\mathbb{D}([\tau, T], E)$ . Since  $\mathbb{D}([\tau, T], E)$  equipped with the  $J_1$ -metric is a separable metric space and  $\mathbb{H}([\tau, T], E)$  is a subset of  $\mathbb{D}([\tau, T], E)$ , using the  $J_1$ -metric on  $\mathbb{H}([\tau, T], E)$  renders this a separable metric space as well. By the definition of  $d_{J_1}$  (see e.g. (12.13) on p.124 of Billingsley (1999) and Section 4.3 of Kern (2024)) the space  $(\mathbb{H}([\tau, T], E), d_{J_1})$  is bounded whenever the state space  $(E, d_E)$  is bounded. We will therefore focus on the case of an unbounded state space  $(E, d_E)$ , where total variation chaos and Wasserstein(1) chaos imply different types of convergence and turn out to require different regularity conditions for the intensity kernel and initial distributions in

order for the respective convergence to hold. We start by discussing the implications of the difference in regularity conditions followed a discussion of the difference in convergence types.

Let  $E = \mathbb{R}^d$  and let  $\|x\| := \sum_{i=1}^d |x_i|$  for  $x \in \mathbb{R}^d$ . For simplicity we restrict the intensity kernels to the time homogeneous form

$$\mu(x, \rho, dy) = q^y\left(x, \int_{\mathbb{R}^d} h(x, z)\rho(dz)\right)\nu(dy)$$

for  $q^y : \mathbb{R}^d \times \mathbb{R}^d \rightarrow [0, \infty)$  and  $h : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and for some measure  $\nu$  on  $\mathbb{R}^d$ . In the case of total variation chaos, Theorem 2.4.4 requires the initial distributions  $\zeta^n$  to be elements of  $\mathcal{P}(E^n)$  and  $(\zeta^n)_{n \in \mathbb{N}}$  to be  $\zeta$ -chaotic in total variation for some  $\zeta \in \mathcal{P}(E)$ , while we by Proposition 2.3.14 must assume the Lipschitz conditions

$$\begin{aligned} |q^y(x_1, u_1) - q^y(x_1, u_2)| &\leq C(\mathbf{1}_{(x_1 \neq x_2)} + \|u_1 - u_2\|), \\ \|h(x_1, z_1) - h(x_2, z_2)\| &\leq C(\mathbf{1}_{(x_1 \neq x_2)} + \min(1, \|z_1 - z_2\|)). \end{aligned}$$

In the case of Wasserstein(1) chaos the results of Andreis and Pra (2018) and Graham (1992a,b) show that  $\zeta^n$  must be in  $\mathcal{P}^1(E)$  and  $(\zeta^n)_{n \in \mathbb{N}}$  must be  $\zeta$ -chaotic in Wasserstein(1) for  $\zeta \in \mathcal{P}^1(E)$ , while we must assume the Lipschitz conditions

$$\begin{aligned} |q^y(x_1, u_1) - q^y(x_1, u_2)| &\leq C(\|x_1 - x_2\| + \|u_1 - u_2\|), \\ \|h(x_1, z_1) - h(x_2, z_2)\| &\leq C(\|x_1 - x_2\| + \|z_1 - z_2\|), \end{aligned}$$

and the linear growth condition  $h(x, z) \leq C(1 + \|y\|)$ .

The first difference is that we have to restrict ourselves to initial distributions with first moments, as the Wasserstein(1)-distance would be undefined otherwise. The more significant difference lies in the Lipschitz assumptions though. In the total variation case, the mappings  $x \mapsto q^y(x, u)$  and  $x \mapsto h(x, z)$  must be bounded without having to be continuous, while in the case of Wasserstein(1), they must Lipschitz continuous without having to be to be bounded. Similary the mapping  $z \mapsto h(x, z)$  must be bounded and Lipschitz continuous in the case of total variation, while it suffices to be Lipschitz continuous and of linear growth in the case of Wasserstein(1). This shows that there is a trade-off to be made in terms of regularity conditions when working with an unbounded state space.

We continue with a discussion of the difference in convergence types. Wasserstein(1) chaos implies weak convergence which is weaker than convergence in total variation. As a consequence the convergence of conditional distributions proved by Theorem 2.5.3 and Corollary 2.5.5 will no longer be valid for uncountable state spaces as its proof relies on setwise convergence of measures, which is stronger than weak convergence, but implied by total variation convergence. As a consequence this invalidates the convergence of grouped insurance liabilities obtained

by Proposition 2.6.4. The convergence of the cohort-wide insurance liability of Proposition 2.6.3 and the unconditional law of large numbers obtained by Proposition 2.6.6 can only be preserved if the mapping  $g : \mathbb{H}([\tau, T], E) \rightarrow \mathbb{R}$  describing the insurance payments is  $\bar{\mathbb{Q}}_{\tau, \zeta}$ -almost surely continuous.

In conclusion using Wasserstein(1) chaos instead of total variation chaos in the case of an unbounded state space can lead to slightly less restrictive regularity conditions in the measure dependence of the intensity kernel and a trade-off in regularity conditions for the state dependence of the intensity kernel, while leading to more restrictive convergence results for the insurance liabilities. Bridging this gap could be an interesting topic for future research.

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## 2.A Distances on the spaces of measures

Let  $(S, \mathcal{S})$  be a measurable space, let  $\mathcal{M}_b(S)$  denote the set of bounded measures on  $(S, \mathcal{S})$  and let BM be the class of measurable functions  $f : S \rightarrow [-1, 1]$ . We can then define the total variation distance on the set  $\mathcal{M}_b(S)$  as

$$d_{TV}(\mu, \nu) := \frac{1}{2} \sup_{f \in \text{BM}} \left\{ \left| \int_S f(x) \mu(dx) - \int_S f(x) \nu(dx) \right| \right\}. \quad (2.A.1)$$

If  $\mu$  and  $\nu$  are probability measures, then we have the following identity

$$d_{TV}(\mu, \nu) = \sup_{A \in \mathcal{S}} |\mu(A) - \nu(A)|. \quad (2.A.2)$$

If  $\mathcal{A}$  is a sigma-algebra such that  $\mathcal{A} \subseteq \mathcal{S}$  then (2.A.2) yields

$$d_{TV}(\mu_{\mathcal{A}}, \nu_{\mathcal{A}}) = d_{TV}(\mu, \nu), \quad (2.A.3)$$

where  $\mu_{\mathcal{A}}$  and  $\nu_{\mathcal{A}}$  denote the respective restriction to  $\mathcal{A}$ . Furthermore if  $\mu$  and  $\nu$  are probability measures, then the total variation distance can be written in terms of an infimum over couplings. Let

$$C(\mu, \nu) := \{\gamma \in \mathcal{P}(S^2) : \gamma(dx \times S) = \mu(dx) \text{ and } \gamma(S \times dy) = \nu(dy)\}$$

denote the set of couplings between  $\mu$  and  $\nu$  and define  $S^2_{x \neq y} := \{(x, y) \in S^2 : x \neq y\}$ . If  $S^2_{x \neq y} \in \mathcal{S} \otimes \mathcal{S}$ , then (see Theorems 7.2, 7.3 in Chapter 3 of Thorisson (2000))

$$d_{TV}(\mu, \nu) = \inf_{\gamma \in C(\mu, \nu)} \gamma(S^2_{x \neq y}). \quad (2.A.4)$$

This yields the following very useful inequality. If  $X$  and  $Y$  are  $S$ -valued random variables on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with joint distribution  $\gamma \in C(\mu, \nu)$ , then

$$d_{TV}(\mu, \nu) \leq \mathbb{P}(X \neq Y). \quad (2.A.5)$$

A coupling  $\gamma \in C(\mu, \nu)$  is called maximal, if equality is achieved. The space  $(\mathcal{P}(S), d_{TV})$  is a complete metric space and if the sequence  $(\mu_n)_{n \in \mathbb{N}} \subset \mathcal{P}(S)$  converges to  $\mu \in \mathcal{P}(S)$  in the metric  $d_{TV}$ , then (2.A.2) implies the setwise convergence  $\mu_n(A) \rightarrow \mu(A)$  for all  $A \in \mathcal{B}(S)$ .

If  $S$  is metrisable with a metric  $d_S$  such that  $(S, d_S)$  is a separable metric space, we will also consider the so-called bounded-Lipschitz distance on  $\mathcal{P}(S)$ . For this let BL denote the class of Lipschitz continuous functions  $f : S \rightarrow [-1, 1]$  whose smallest Lipschitz constant is less than or equal to one. Then

$$d_{BL} := \frac{1}{2} \sup_{f \in \text{BL}} \left\{ \left| \int_S f(x) \mu(dx) - \int_S f(x) \nu(dx) \right| \right\}.$$

By (2.A.1) it holds that  $d_{BL}(\mu, \nu) \leq d_{TV}(\mu, \nu)$ . The metric  $d_{BL}$  metrises weak convergence. That is  $\mu_n \xrightarrow{wk} \mu$  if and only if  $\lim_{n \rightarrow \infty} d_{BL}(\mu_n, \mu) = 0$ .

## 2.B Characterisation of jump process distributions

There is a close correspondance between non-explosive marked point process distributions and non-explosive jump process distributions, see Section 2.5 of Last and Brandt (1999). Let  $(E, \mathcal{B}(E))$  be a standard Borel space and define the space of non-explosive marked point process paths on the time interval  $[\tau, T]$  as follows:

**Definition 2.B.1.** *The sequence  $(t_i, y_i)_{i \in \mathbb{N}_0} \subset ([\tau, T] \cup \{\infty\} \times E \cup \{\nabla\})^{\mathbb{N}_0}$  is a marked point process realisation started at  $\tau$  and ending at  $T$  if and only if*

- (i)  $\tau = t_0 < t_1 \leq t_2 \leq \dots$  with  $t_i < t_{i+1}$  whenever  $t_i < \infty$  and  $t_i = t_{i+1}$  whenever  $t_i = \infty$ ,
- (ii)  $y_i \in E$  whenever  $t_i < \infty$  and  $y_i = \nabla$  whenever  $t_i = \infty$ ,
- (iii)  $\lim_{i \rightarrow \infty} t_i = \infty$ .

The set of marked point process realisations started at  $\tau$  is denoted by  $\bar{\mathbb{M}}([\tau, T], E)$ .

Similarly we can define the space of non-explosive marked point process paths without trivial jumps as

$$\mathbb{M}([\tau, T], E) := \{(t_i, y_i)_{i \in \mathbb{N}_0} \in \bar{\mathbb{M}}([\tau, T], E) : y_i \neq y_{i+1} \Leftrightarrow t_i < \infty\}.$$

The sets  $\bar{\mathbb{M}}([\tau, T], E)$  and  $\mathbb{M}([\tau, T], E)$  are measurable subsets of  $([\tau, T] \cup \{\infty\} \times E \cup \{\nabla\})^{\mathbb{N}_0}$  and can be endowed with sigma-algebra generated by the coordinate projections  $t_i^o((t_i, y_i)_{i \in \mathbb{N}_0}) = t_i$  and  $y_i^o((t_i, y_i)_{i \in \mathbb{N}_0}) = y_i$ .

As in Section 2.5 in Last and Brandt (1999), define now the mapping  $\Gamma$  from either  $\mathbb{M}([\tau, T], E)$  or  $\bar{\mathbb{M}}([\tau, T], E)$  into  $\mathbb{H}([\tau, T], E)$  given by

$$\Gamma((t_i, y_i)_{i \in \mathbb{N}_0}) := \left( \sum_{i=0}^{\infty} y_i \mathbb{1}_{(t_i \leq t < t_{i+1})} \right)_{t \in [\tau, T]},$$

and the mapping  $\Gamma^* : \mathbb{H}([\tau, T], E) \rightarrow \mathbb{M}([\tau, T], E)$  given by

$$\Gamma^*(f) := (t_i(f), y_i(f))_{i \in \mathbb{N}_0}, \quad \text{where} \quad y_i(f) := \begin{cases} f(t_i(f)) & \text{if } t_i(f) < \infty \\ \nabla & \text{if } t_i(f) = \infty, \end{cases}$$

and  $t_i(f) := \inf\{t \geq t_{i-1}(f) : f(t) \neq f(t_{i-1}(f))\}$ , with  $t_0(f) := \tau$ . Theorems 2.5.10 and 2.5.11 in Last and Brandt (1999) yield that if  $\Gamma$  is defined on  $\mathbb{M}([\tau, T], E)$  it is a bimeasurable bijection with  $\Gamma^*$  as its inverse, while if  $\Gamma$  is defined on  $\bar{\mathbb{M}}([\tau, T], E)$  it is only surjective. We thus get the following result:

**Lemma 2.B.2.** *It holds that:*

- (i) *For any  $\mathbb{Q} \in \mathcal{P}(\mathbb{H}([\tau, T], E))$  there exists a unique  $\mathbb{P} \in \mathcal{P}(\mathbb{M}([\tau, T], E))$  such that  $\mathbb{Q} = \Gamma(\mathbb{P})$ .*
- (ii) *For any  $\mathbb{Q} \in \mathcal{P}(\mathbb{H}([\tau, T], E))$  there exists at least one  $\mathbb{P} \in \mathcal{P}(\bar{\mathbb{M}}([\tau, T], E))$  such that  $\mathbb{Q} = \Gamma(\mathbb{P})$ .*
- (iii) *For any  $\mathbb{P} \in \mathcal{P}(\mathbb{M}([\tau, T], E))$  there exists a unique  $\mathbb{Q} \in \mathcal{P}(\mathbb{H}([\tau, T], E))$  such that  $\mathbb{P} = \Gamma^*(\mathbb{Q})$ .*

*Proof.* Result (i) and (iii) follow if we show that  $\mathbb{P} \mapsto \Gamma(\mathbb{P})$  is bijection with inverse  $\mathbb{Q} \mapsto \Gamma^*(\mathbb{Q})$ . Using that  $\Gamma = (\Gamma^*)^{-1}$  and that  $\Gamma$  is a bijection, we have that

$$\Gamma(\Gamma^*(\mathbb{Q}))(B) = \mathbb{Q}(\Gamma^{-1}(\Gamma(B))) = \mathbb{Q}(B), \quad \forall B \in \mathcal{B}(\mathbb{H}([\tau, T], E))$$

and

$$\Gamma^*(\Gamma(\mathbb{P}))(B) = \mathbb{P}(\Gamma^{-1}(B)) = \mathbb{P}(B), \quad \forall B \in \mathcal{B}(\mathbb{M}([\tau, T], E)).$$

For result (ii) let  $\mathbb{P} \in \mathcal{P}(\mathbb{M}([\tau, T], E))$  be the unique measure from (i) and let  $\bar{\mathbb{P}} := \mathbb{P}(\cdot \cap \mathbb{M}([\tau, T], E))$  be the extension to  $\bar{\mathbb{M}}([\tau, T], E)$ . Thus there exists at least one measure with the desired property. Since  $\Gamma$  is surjective when defined on  $\bar{\mathbb{M}}([\tau, T], E)$  we cannot be sure that it is the only one.  $\square$

Thus existence and uniqueness of jump process distributions is equivalent to existence and uniqueness of marked point process distributions with initial distribution and without trivial jumps. If trivial jumps are allowed, then several marked point process distributions might give rise to the same jump process distribution, but for each marked point process distribution and initial value there will only be exactly one jump process distribution.

We now turn to the characterisation of the jump process distributions  $\mathbb{Q}_{\tau,\zeta}$  of (2.3.1) and  $\mathbb{Q}_{\tau,x}$  of (2.3.3) in terms of the associated marked point process distribution.

**Lemma 2.B.3.** *Let  $x \in E$  and  $B \in \mathcal{B}(\mathbb{H}([\tau, T], E))$ . Then the kernel  $(x, B) \mapsto \mathbb{Q}_{\tau,x}(B)$  is a regular conditional probability of  $\mathbb{Q}_{\tau,\zeta}(B|X_\tau^\circ = x)$  and thus*

$$\mathbb{Q}_{\tau,\zeta}(B|X_\tau^\circ = x) = \mathbb{Q}_{\tau,x}(B), \quad \text{for } \zeta - a.e. x.$$

Furthermore there exists a sequence of sets  $C_n \in \mathcal{B}([\tau, T] \times E)^n$  such that

$$\mathbb{Q}_{\tau,x}(B) = \sum_{n=1}^{\infty} R_{\tau,x}^n((t_i^\circ, y_i^\circ)_{i=1,\dots,n-1} \in C_{n-1}, t_n^\circ > T),$$

where  $R_{\tau,x}^n : E \times \mathcal{B}([\tau, T] \times E)^{n+1} \rightarrow [0, 1]$  is the probability kernel given by

$$\begin{aligned} R_{\tau,x}^n(d((t_i, y_i)_{i \leq n})) &:= \mathbb{1}_{(\tau < t_1 < \dots < t_n)} r_{t_n}(y_{n-1}, dy_n) U_{t_{n-1}, y_{n-1}}(dt_n) \\ &\quad \dots r_{t_1}(y_0, dy_1) U_{t_0, y_0}(dt_1) \delta_{\{\tau, x\}}(d(t_0, y_0)) \end{aligned}$$

and  $U_{t,y}(ds) = \lambda_s(y) e^{-\int_t^s \lambda_u(y) du} ds$  for  $s \geq t$ .

*Proof.* Let  $B \in \mathcal{B}(\mathbb{H}([\tau, T], E))$ . By construction there exists a unique  $\mathbb{P}_{\tau,x} \in \mathcal{P}(\mathbb{M}([\tau, T], E))$  determined by  $x$  and the compensating measure  $L$  such that

$$\mathbb{Q}_{\tau,x}(B) = \mathbb{P}_{\tau,x}(\Gamma^*(B)).$$

Since  $\Gamma^*(B) \in \mathcal{B}(\mathbb{M}([\tau, T], E))$ , we can apply Lemma 2.2.21 and Exercise 2.2.16 in Last and Brandt (1999) to conclude that there exists a sequence of sets  $C_n \in \mathcal{B}([\tau, T] \times E)^{n+1}$  such that

$$\Gamma^*(B) \cap (t_{n-1}^\circ \leq T < t_n^\circ) = ((t_i^\circ, y_i^\circ)_{i=0,\dots,n-1} \in C_{n-1}) \cap (t_n^\circ > T)$$

for all  $n \in \mathbb{N}$ . Since this partition is disjoint, we get by Theorem 8.1.2 of Last and Brandt (1999) that

$$\begin{aligned} \mathbb{P}_{\tau,x}(\Gamma^*(B)) &= \sum_{n=1}^{\infty} \mathbb{P}_{\tau,x}(((t_i^\circ, y_i^\circ)_{i=0,\dots,n-1} \in C_{n-1}) \cap (t_n^\circ > T)) \\ &= \sum_{n=1}^{\infty} R_{\tau,x}^n((t_i^\circ, y_i^\circ)_{i=1,\dots,n-1} \in C_{n-1}, t_n^\circ > T). \end{aligned}$$

This proves the second statement. By construction there exists a unique  $\mathbb{P}_{\tau,\zeta} \in \mathcal{P}(\mathbb{M}([\tau, T], E))$  determined by  $\zeta$  and the compensating measure  $L$  such that

$$\mathbb{Q}_{\tau,\zeta}(B) = \mathbb{P}_{\tau,\zeta}(\Gamma^*(B)).$$

By Theorem 8.2.2 of Last and Brandt (1999)

$$\mathbb{P}_{\tau,\zeta}(\Gamma^*(B)|y_0^\circ = x) = \mathbb{P}_{\tau,x}(\Gamma^*(B)), \quad \text{for } \zeta - a.e. x.$$

The first statement follows. □

We can obtain a similar characterisation of the jump process distributions  $\bar{\mathbb{Q}}_{\tau,\zeta}$  of (2.3.5) and  $\tilde{\mathbb{Q}}_{\tau,\zeta,x}$  of (2.3.6). Since existence and uniqueness of  $\bar{\mathbb{Q}}_{\tau,\zeta}$  implies existence and uniqueness of  $\bar{p}_t^{\tau,\zeta}$  we can view  $(\bar{p}_t^{\tau,\zeta})_{t \in [\tau, T]}$  as given and fixed and treat (2.3.5) as a standard jump process. Thus by directly invoking Lemma 2.B.3 we obtain the following result:

**Lemma 2.B.4.** *Let  $x \in E$  and  $B \in \mathcal{B}(\mathbb{H}([\tau, T], E))$  and assume that  $\bar{\mathbb{Q}}_{\tau,\zeta}$  exists and is unique. Then the kernel  $(x, B) \mapsto \tilde{\mathbb{Q}}_{\tau,\zeta,x}(B)$  is a regular conditional probability of  $\bar{\mathbb{Q}}_{\tau,\zeta}(B|X_\tau^\circ = x)$  and thus*

$$\bar{\mathbb{Q}}_{\tau,\zeta}(B|X_\tau^\circ = x) = \tilde{\mathbb{Q}}_{\tau,\zeta,x}(B), \quad \text{for } \zeta - \text{a.e. } x.$$

Furthermore there exists a sequence of sets  $C_n \in \mathcal{B}([\tau, T] \times E)^n$  such that

$$\tilde{\mathbb{Q}}_{\tau,\zeta,x}(B) = \sum_{n=1}^{\infty} \tilde{R}_{\tau,\zeta,x}^n((t_i^\circ, y_i^\circ)_{i=1,\dots,n-1} \in C_{n-1}, t_n^\circ > T),$$

where  $\tilde{R}_{\tau,\zeta,x}^n : E \times \mathcal{B}([\tau, T] \times E)^{n+1} \rightarrow [0, 1]$  is the probability kernel given by

$$\begin{aligned} \tilde{R}_{\tau,\zeta,x}^n(d((t_i, y_i)_{i \leq n})) &:= \mathbb{1}_{(\tau < t_1 < \dots < t_n)} r_{t_n}(y_{n-1}, \bar{p}_{t_n}^{\tau,\zeta}, dy_n) \tilde{U}_{t_{n-1}, y_{n-1}}(dt_n) \\ &\quad \dots r_{t_1}(y_0, \bar{p}_{t_1}^{\tau,\zeta}, dy_1) \tilde{U}_{t_0, y_0}(dt_1) \delta_{\{\tau, x\}}(d(t_0, y_0)) \end{aligned}$$

and  $\tilde{U}_{t,y}(ds) = \lambda_s(y, \bar{p}_s^{\tau,\zeta}) e^{-\int_t^s \lambda_u(y, \bar{p}_u^{\tau,\zeta}) du} ds$  for  $s \geq t$ .

We are now ready to prove Theorem 2.3.10. Note that Theorem 2.3.4 can be proven by a similar argument using Lemma 2.B.3 instead of Lemma 2.B.4 in the proof below.

*Proof of Theorem 2.3.10.* Let  $B \in \mathcal{B}(\mathbb{H}([\tau, T], E))$ . As  $(\bar{\mathbb{Q}}_{\tau,\zeta,x})_{x \in E}$  by Lemma 2.B.4 is a regular conditional distribution of  $\bar{\mathbb{Q}}_{\tau,\zeta}(B|X_\tau^\circ = x)$  and since  $\bar{X}^{\tau,\zeta}(\mathbb{P}) = \bar{\mathbb{Q}}_{\tau,\zeta}$  we have that  $(\bar{\mathbb{Q}}_{\tau,\zeta,x})_{x \in E}$  is a regular conditional distribution of  $\mathbb{P}((\bar{X}_t^{\tau,\zeta})_{t \in [\tau, T]} \in B | \bar{X}_\tau^{\tau,\zeta} = x)$  as well. This proves the first assertion.

Fix  $0 \leq \tau \leq s \leq t \leq T$ , let  $(\bar{\mathcal{F}}_u^{\tau,\zeta})_{u \in [\tau, T]}$  be the natural filtration of the jump process  $\bar{X}^{\tau,\zeta}$  and let  $(\bar{\mathcal{G}}_u^{\tau,\zeta})_{u \in [\tau, T]}$  with  $\bar{\mathcal{G}}_u^{\tau,\zeta} := \sigma(\bar{Y}_0, ((\bar{T}_n, \bar{Y}_n) \mathbb{1}_{(\bar{T}_n \leq u)})_{n \in \mathbb{N}_0})$  be the filtration generated by the initial value and marked point process. By Theorem 2.5.10 in Last and Brandt (1999), we have that  $\bar{\mathcal{F}}_u^{\tau,\zeta} = \bar{\mathcal{G}}_u^{\tau,\zeta}$  for all  $u \in [\tau, T]$ . Furthermore we can write

$$\bar{X}_t^{\tau,\zeta} = \sum_{n=0}^{\infty} \bar{Y}_{s,n} \mathbb{1}_{(\bar{T}_{s,n} \leq t < \bar{T}_{s,n+1})},$$

where  $\bar{T}_{s,n} := \bar{T}_{N(s)+n}$  and  $\bar{Y}_{s,n} := \bar{Y}_{N(s)+n}$  are the  $n$ 'th jump time and mark after time  $s$ , with  $\bar{T}_{s,0} := s$  and  $\bar{Y}_{s,0} := \bar{Y}_{N(s)}$ . Thus for any  $B \in \mathcal{B}(E)$  we obtain

$$\begin{aligned} \mathbb{P}(\bar{X}_t^{\tau,\zeta} \in B | \bar{\mathcal{F}}_s^{\tau,\zeta}) &= \mathbb{P}\left(\bigcup_{n=1}^{\infty} (\bar{Y}_{s,n-1} \in B) \cap (\bar{T}_{s,n-1} \leq t < \bar{T}_{s,n}) \middle| \bar{\mathcal{G}}_s^{\tau,\zeta}\right) \\ &= \sum_{n=1}^{\infty} \mathbb{P}(\bar{Y}_{s,n-1} \in B, \bar{T}_{s,n-1} \leq t < \bar{T}_{s,n} | \bar{\mathcal{G}}_s^{\tau,\zeta}). \end{aligned}$$

By Theorem 8.1.2 of Last and Brandt (1999), it follows that

$$\mathbb{P}(\bar{X}_t^{\tau,\zeta} \in B | \bar{\mathcal{F}}_s^{\tau,\zeta}) = \sum_{n=1}^{\infty} \tilde{R}_{s,\zeta,\bar{X}_s^{\tau,\zeta}}^n(t_{n-1}^\circ \leq t, y_{n-1}^\circ \in B, t_n^\circ > t), \quad \mathbb{P} - \text{a.s.},$$

where  $\tilde{R}_{s,\zeta,x}^n$  for each  $x \in E$  is given by

$$\begin{aligned} \bar{R}_{s,\zeta,x}^n(\text{d}((t_i, y_i)_{i \leq n})) &:= \mathbf{1}_{(\tau < t_1 < \dots < t_n)} r_{t_n}(y_{n-1}, \bar{p}_{t_n}^{\tau,\zeta}, \text{d}y_n) \bar{U}_{t_{n-1}, y_{n-1}}(\text{d}t_n) \\ &\quad \dots r_{t_1}(y_0, \bar{p}_{t_1}^{\tau,\zeta}, \text{d}y_1) \bar{U}_{t_0, y_0}(\text{d}t_1) \delta_{\{\tau, x\}}(\text{d}(t_0, y_0)) \end{aligned}$$

and  $\bar{U}_{t,y}(\text{d}s) = \lambda_s(y, \bar{p}_s^{\tau,\zeta}) e^{-\int_t^s \lambda_u(y, \bar{p}_u^{\tau,\zeta}) \text{d}u} \text{d}s$  for  $s \geq t$ . By Proposition 2.3.9 it holds that  $\bar{p}_t^{\tau,\zeta} = \bar{p}_t^{s, \bar{p}_s^{\tau,\zeta}}$  for any  $t \geq s$  and thus we can replace  $\bar{p}_t^{\tau,\zeta}$  with  $\bar{p}_t^{s, \bar{p}_s^{\tau,\zeta}}$ . By Lemma 2.B.4 we can thus conclude that

$$\mathbb{P}(\bar{X}_t^{\tau,\zeta} \in B | \bar{\mathcal{F}}_s^{\tau,\zeta}) = \tilde{\mathbb{Q}}_{s, \bar{p}_s^{\tau,\zeta}, X_s^{\tau,\zeta}}(X_t^\circ \in B), \quad \mathbb{P} - \text{a.s.}$$

for any  $B \in \mathcal{B}(E)$ . An extension argument as in Remark 2.4 of Rehmeier and Röckner (2025) via the finite dimensional distributions and the Monotone Class Theorem then yields the desired result.  $\square$

## 2.C Coupling results

The first result we need is the following extension of the well-known (see Theorem 7.3 in Chapter 4 of Thorisson (2000)) fact that it is always possible to construct a coupling between two probability measures defined on the same space that is maximal with respect to the total variation distance.

**Lemma 2.C.1.** *Let  $(S, \mathcal{B}(S))$  be a standard Borel space and let  $p : S \times \mathcal{B}(S) \rightarrow [0, 1]$  be a probability kernel. Then there always exists a probability kernel  $\gamma : S^2 \times \mathcal{B}(S^2) \rightarrow [0, 1]$  such that*

- (i) *For any  $x_1, x_2 \in S$  and  $A \in \mathcal{B}(S)$  it holds that  $\pi(x_1, x_2, A \times S) = p(x_1, S)$  and  $\pi(x_1, x_2, S \times A) = p(x_2, S)$ .*
- (ii) *For any  $x_1, x_2 \in S$  we have that*

$$\gamma(x_1, x_2, S_{x \neq y}^2) = d_{TV}(p(x_1, \text{d}y), p(x_2, \text{d}y)),$$

where  $S_{x \neq y}^2 := \{(x, y) \in S^2 | x \neq y\}$ .

*Proof.* Set  $q(x_1, x_2, dy) := \frac{1}{2}(p(x_1, dy) + p(x_2, dy))$ . Then by Theorem 58 in Dellacherie and Meyer (1982), which is a variant of the Radon-Nikodym Theorem, we have that there exists measurable functions  $f_1, f_2 : S^3 \rightarrow [0, \infty)$  such that

$$p(x_1, dy) = f_1(x_1, x_2, y)q(x_1, x_2, dy) \quad \text{and} \quad p(x_2, dy) = f_2(x_1, x_2, y)q(x_1, x_2, dy).$$

Since  $q(x_1, x_2, dy) = q(x_2, x_1, dy)$ , we also get

$$p(x_2, dy) = f_1(x_2, x_1, y)q(x_1, x_2, dy) \quad \text{and} \quad p(x_1, dy) = f_2(x_2, x_1, y)q(x_1, x_2, dy).$$

We thus get that  $f_2(x_1, x_2, y) = f_1(x_2, x_1, y)$  for  $q(x_1, x_2, \cdot)$ -a.a.  $y \in S$ . Hence we can conclude that there exists a measurable function  $f : S^3 \rightarrow [0, \infty)$  such that

$$p(x_1, dy) = f(x_1, x_2, y)q(x_1, x_2, dy) \quad \text{and} \quad p(x_2, dy) = f(x_2, x_1, y)q(x_1, x_2, dy).$$

Define now the measures

$$\begin{aligned} \nu(x_1, x_2, dy) &:= \min(f(x_1, x_2, y), f(x_2, x_1, y))q(x_1, x_2, dy), \\ \nu^+(x_1, x_2, dy) &:= (f(x_1, x_2, y) - f(x_2, x_1, y))^+ q(x_1, x_2, dy), \\ \nu^-(x_1, x_2, dy) &:= (f(x_1, x_2, y) - f(x_2, x_1, y))^- q(x_1, x_2, dy), \end{aligned}$$

which all are measurable functions of  $(x_1, x_2)$ . Thus  $\nu, \nu^+, \nu^- : S^2 \times \mathcal{B}(S) \rightarrow [0, \infty)$  are kernels. As a consequence, we also have that

$$(x_1, x_2) \mapsto \nu(x_1, x_2, S) = \int_S \min(f(x_1, x_2, y), f(x_2, x_1, y))q(x_1, x_2, dy)$$

is a measurable mapping. We can now define the probability measure  $\gamma$  on  $S^2$  as

$$\gamma(x_1, x_2, A \times B) := \nu(x_1, x_2, A \cap B) + \frac{\nu^+(x_1, x_2, A)\nu^-(x_1, x_2, B)}{1 - \nu(x_1, x_2, S)}$$

for any  $A, B \in \mathcal{B}(S)$  whenever  $\nu(x_1, x_2, S) < 1$  and

$$\gamma(x_1, x_2, A \times B) := \nu(x_1, x_2, A \cap B),$$

for any  $A, B \in \mathcal{B}(S)$  whenever  $\nu(x_1, x_2, S) = 1$ . As  $(x_1, x_2) \mapsto \gamma(\cdot, x_1, x_2)$  can be seen as a composition of measurable mappings, it is measurable as well, and thus it is a probability kernel. Using the identities of Theorems 8.1 and 8.2 in Chapter 3 of Thorisson (2000) we can for each fixed  $(x_1, x_2)$  identify  $\gamma(x_1, x_2, \cdot)$  as the maximal coupling from Theorem 7.3 in Chapter 3 of Thorisson (2000) and thus property (i) and (ii) follow.  $\square$

The next result shows that it is possible to construct a coupling between two multivariate distributions which is maximal for the marginal distribution of a fixed subset of the coordinates.

**Lemma 2.C.2.** *Let  $(S, \mathcal{B}(S))$  be standard Borel space. Let  $(X^1, \dots, X^n)$  have distribution  $\zeta_1 \in \mathcal{P}(S^n)$  and  $(Y_1, \dots, Y^n)$  have distribution  $\zeta_2 \in \mathcal{P}(S^n)$ . Fix a  $k \in \{1, \dots, n\}$ . Then there exists a coupling  $\gamma \in \mathcal{P}(S^{2n})$  of  $\zeta^1$  and  $\zeta^2$  such that the marginal*

$$\gamma((dx_{1:k} \times S^{n-k}) \times (dy_{1:k} \times S^{n-k}))$$

*is a maximal coupling of the marginals  $\zeta_1^k := \zeta_1(dx_{1:k} \times S^{n-k})$  and  $\zeta_2^k := \zeta_2(dy_{1:k} \times S^{n-k})$ .*

*Proof.* By Theorem 7.3 in Chapter 3 of Thorisson (2000) there exists a measure  $\nu_k \in \mathcal{P}(S^{2k})$  such that  $\nu_k$  is the maximal coupling of  $\zeta_1^k$  and  $\zeta_2^k$ . Let

$$\zeta_1(dx_{k+1:n}|x_{1:k}) \quad \text{and} \quad \zeta_2(dy_{k+1:n}|y_{1:k})$$

denote regular versions of the conditional probability of  $\zeta_1$  and  $\zeta_2$  given the value of the first  $k$  coordinates. Define now

$$\gamma(d(x_{1:n}, y_{1:n})) := \zeta_1(dx_{k+1:n}|x_{1:k})\zeta_2(dy_{k+1:n}|y_{1:k})\nu_k(d(x_{1:k}, y_{1:k})).$$

Then we have that

$$\begin{aligned} \gamma(S^n \times dy_{1:n}) &= \zeta_2(dy_{k+1:n}|y_{1:k})\zeta_2^k(dy_{1:k}) = \zeta_2(dy_{1:n}), \\ \gamma(dx_{1:n} \times S^n) &= \zeta_1(dx_{k+1:n}|x_{1:k})\zeta_1^k(dx_{1:k}) = \zeta_1(dx_{1:n}), \end{aligned}$$

which confirms that  $\gamma$  indeed is a coupling of  $\zeta_1$  and  $\zeta_2$ . Furthermore

$$\gamma((dx_{1:k} \times S^{n-k}) \times (dy_{1:k} \times S^{n-k})) = \nu_k(d(x_{1:k}, y_{1:k})),$$

which confirms the last property.  $\square$

**Lemma 2.C.3.** *Let  $(S, \mathcal{B}(S))$  be a standard Borel space. Let  $(\zeta^n)_{n \in \mathbb{N}}$  with  $\zeta^n \in \mathcal{P}(S^n)$  be  $\zeta$ -chaotic in total variation for  $\zeta \in \mathcal{P}(S)$ . Then there exists a sequence of probability measures  $\gamma^n \in \mathcal{P}(S^{2n})$  such that:*

- (i) *The marginal  $\gamma^n(dx_{1:n} \times S^n)$  is equal to  $\zeta^n$ .*
- (ii) *Let  $\tilde{\zeta}^n := \gamma^n(S^n \times dy_{1:n})$ . The marginals  $\tilde{\zeta}^{n,i} := \gamma^n(S^n \times (S^{i-1} \times dy_i \times S^{n-i}))$  are equal to  $\zeta$  for all  $i \in \{1, \dots, n\}$ .*
- (iii) *The marginals  $\gamma^n((S^{i-1} \times dx_i \times S^{n-i}) \times (S^{i-1} \times dy_i \times S^{n-i}))$  are equal to a maximal coupling  $\gamma_1^n(dx_1, dy_1)$  of  $\zeta_1^{n,1}$  and  $\zeta$  for all  $i \in \{1, \dots, n\}$ .*
- (iv) *The sequence of marginals  $(\tilde{\zeta}^n)_{n \in \mathbb{N}} = (\gamma^n(S^n \times dy_{1:n}))_{n \in \mathbb{N}}$  is  $\zeta$ -chaotic in total variation.*

*Proof.* By Theorem 7.3 in Chapter 3 of Thorisson (2000) there always exists a maximal coupling  $\gamma_1^n(dx, dy)$  of  $\zeta^{n,1}$  and  $\zeta$ . Define now

$$\gamma(dx_{1:n}, dy_{1:n}) := \prod_{i=1}^n \gamma_1^n(dy_i | x_i) \zeta^n(dx_{1:n}),$$

where  $\gamma_1^n(dy_i | x_i)$  is a regular conditional distribution of  $\gamma_1^n$  given  $x_1$ . Next we check the properties.

(i) is satisfied since

$$\gamma^n(dx_{1:n} \times S^n) = \prod_{i=1}^n \gamma_1^n(S | x_i) \zeta^n(dx_{1:n}) = \zeta^n(dx_{1:n}).$$

(ii) is satisfied since

$$\begin{aligned} \gamma^n(S^n \times (S^{i-1} \times dy_i \times S^{n-i})) &= \int_{S^n} \gamma_1^n(dy_i | x_i) \prod_{j=1, j \neq i}^n \gamma_1^n(S | x_j) \zeta^n(dx_{1:n}) \\ &= \int_S \gamma_1^n(dy_i | x_i) \zeta^{n,1}(dx_i) = \gamma_1^1(S, dy_i) = \zeta(dy_i). \end{aligned}$$

(iii) is satisfied since for any  $B_1, B_2 \in \mathcal{B}(S)$

$$\begin{aligned} \gamma^n((S^{i-1} \times B_1 \times S^{n-i}) \times (S^{i-1} \times B_2 \times S^{n-i})) \\ &= \int_{S^{i-1} \times B_1 \times S^{n-i}} \gamma_1^n(B_2 | x_i) \prod_{j=1, j \neq i}^n \gamma_1^n(S | x_j) \zeta^n(dx_{1:n}) \\ &= \int_{B_1} \gamma_1^n(B_2 | x_i) \zeta^{n,1}(dx_i) = \gamma(B_1 \times B_2). \end{aligned}$$

(iv) Set  $\tilde{\zeta}^{n,k} := \gamma_1^n(S^n \times dy_{1:k} \times S^{n-k})$ . Let  $((X_i)_{i=1,\dots,n}, (Y_i)_{i=1,\dots,n})$  a family with distribution  $\gamma$ . Here  $(X_i)_{i=1,\dots,n}$  has distribution  $\zeta^n$ , while  $(Y_i)_{i=1,\dots,n}$  has distribution  $\tilde{\zeta}^n$ . By the coupling representation of the total variation distance, we now have

$$\begin{aligned} d_{TV}(\zeta^{n,k}, \tilde{\zeta}^{n,k}) &\leq \mathbb{P}\left(\bigcup_{i=1}^k (X_i \neq Y_i)\right) \leq \sum_{i=1}^k \mathbb{P}(X_i \neq Y_i) \\ &= k \gamma_1^n(S_{x \neq y}^2) = k d_{TV}(\zeta^{n,1}, \zeta), \end{aligned}$$

which converges to zero for  $n \rightarrow \infty$  as  $(\zeta^n)_{n \in \mathbb{N}}$  is  $\zeta$ -chaotic in total variation. Next we have by the triangle inequality that

$$d_{TV}(\tilde{\zeta}^{n,k}, \zeta^{\otimes k}) \leq d_{TV}(\tilde{\zeta}^{n,k}, \zeta^{n,k}) + d_{TV}(\zeta^{n,k}, \zeta^{\otimes k}).$$

As we have just shown, the first term goes to zero for  $n \rightarrow \infty$ , while the second term goes to zero since  $(\zeta^n)_{n \in \mathbb{N}}$  is  $\zeta$ -chaotic in total variation.  $\square$

Finally we show that it is possible to extend a given coupling.

**Lemma 2.C.4.** *Let  $(S, \mathcal{B}(S))$  be standard Borel space and fix  $m \in \mathbb{N}$ . For  $i \in \mathbb{N}$ , let  $\zeta_{\mathbf{x}}^i : S^m \times \mathcal{B}(S^i) \rightarrow [0, 1]$  and  $\xi_{\mathbf{x}}^i : S^m \times \mathcal{B}(S^i) \rightarrow [0, 1]$  be two probability kernels and let  $\nu_{\mathbf{x}}^i \in \mathcal{P}(S^{2i})$  be a coupling of  $\zeta_{\mathbf{x}}^i$  and  $\xi_{\mathbf{x}}^i$ . Then for any  $\mathbf{x} \in S^m$  there exists a probability measure  $\gamma_{\mathbf{x}} \in \mathcal{P}(S^{m+i})$  such that*

- (i)  $\gamma_{\mathbf{x}}(dx^{1:m+i} \times S^{m+i}) = \delta_{\mathbf{x}}(dx^{1:m}) \otimes \zeta_{\mathbf{x}}^i(dx^{m+1:m+i})$ ,
- (ii)  $\gamma_{\mathbf{x}}^i(S^{m+i} \times dy^{1:m+i}) = \delta_{\mathbf{x}}(dy^{1:m}) \otimes \xi_{\mathbf{x}}^i(dy^{m+1:m+i})$ ,
- (iii)  $\gamma_{\mathbf{x}}^{1:m}(S_{x \neq y}^m) = d_{TV}(\delta_{\{\mathbf{x}\}}, \delta_{\{\mathbf{x}\}}) = 0$ ,  
where  $\gamma_{\mathbf{x}}^{1:m}(d(x^{1:m}, y^{1:m})) := \gamma_{\mathbf{x}}((dx^{1:m} \times S^i) \times (dy^{1:m} \times S^i))$ ,
- (iv)  $\gamma_{\mathbf{x}}((S^m \times dx^{1:i}) \times (S^m \times dy^{1:i})) = \nu_{\mathbf{x}}^i(d(x^{1:i}, y^{1:i}))$ .

*Proof.* Define the measure  $\gamma_{\mathbf{x}}$  as

$$\gamma_{\mathbf{x}}(d(x^{1:m+i}, y^{1:m+i})) := \nu_{\mathbf{x}}^i(d(x^{m+1:m+i}, y^{m+1:m+i}))\delta_{\{\mathbf{x}, \mathbf{x}\}}(d(x^{1:m}, y^{1:m})).$$

(i): Since  $\nu_{\mathbf{x}}^i$  is a coupling, we have that

$$\begin{aligned} \gamma_{\mathbf{x}}(dx^{1:m+i} \times S^{m+i}) &= \nu_{\mathbf{x}}^i(dx^{m+1:m+i} \times S^i)\delta_{\{\mathbf{x}, \mathbf{x}\}}(dx^{1:m} \times S^m) \\ &= \delta_{\mathbf{x}}(dx^{1:m})\zeta_{\mathbf{x}}^i(dx^{m+1:m+i}). \end{aligned}$$

(ii): Proven by a similar argument as (i).

(iii): Note that  $\gamma_{\mathbf{x}}^{1:m}(d(x^{1:m}, y^{1:m})) = \delta_{\{\mathbf{x}, \mathbf{x}\}}(d(x^{1:m}, y^{1:m}))$ , which can be recognised as the maximal coupling of  $\delta_{\{\mathbf{x}\}}$  with itself. The result follows.

(iv): It holds that

$$\begin{aligned} \gamma_{\mathbf{x}}((S^m \times dx^{1:i}) \times (S^m \times dy^{1:i})) &= \nu_{\mathbf{x}}^i(d(x^{1:i}, y^{1:i}))\delta_{\{\mathbf{x}, \mathbf{x}\}}(S^m \times S^m) \\ &= \nu_{\mathbf{x}}^i(d(x^{1:i}, y^{1:i})). \end{aligned}$$

□

## 2.D LLN and CLT for chaotic random variables

Let  $(S, \mathcal{S})$  be a measurable space and let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Consider the triangular array  $((X^{1,n}, \dots, X^{n,n}))_{n \in \mathbb{N}}$  of random variables  $X^{\ell,n} : \Omega \rightarrow S$ , where each row  $X^n = (X^{1,n}, \dots, X^{n,n})$  has distribution  $X^n(\mathbb{P}) = \mathbb{Q}^n \in \mathcal{P}(S)$ . Furthermore let  $X : \Omega \rightarrow S$  be a random variable with  $X(\mathbb{P}) = \mathbb{Q} \in \mathcal{P}(S)$ .

**Proposition 2.D.1.** *Assume that  $(\mathbb{Q}^n)_{n \in \mathbb{N}}$  is  $\mathbb{Q}$ -chaotic in total variation and that  $f : S \rightarrow \mathbb{R}$  is measurable. If the sequence  $(f(X^{1,n}))_{n \in \mathbb{N}}$  is uniformly integrable, then it holds that*

$$\lim_{n \rightarrow \infty} \mathbb{E}[f(X^{\ell,n})] = \mathbb{E}[f(X)].$$

*Proof.* Due to chaoticity we have that  $X^{\ell,n}(\mathbb{P}) = \mathbb{Q}^{n,1} \xrightarrow{TV} \mathbb{Q} = X(\mathbb{P})$ . The result follows directly from Corollary 2.9 in Feinberg, Kasyanov, and Zgurovsky (2016).  $\square$

The next result is a law of large numbers:

**Proposition 2.D.2.** *Assume that  $(\mathbb{Q}^n)_{n \in \mathbb{N}}$  is  $\mathbb{Q}$ -chaotic in total variation and let  $f : S \rightarrow \mathbb{R}$  be measurable with*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|f(X^{1,n})|^{2+\varepsilon}] < \infty, \quad \text{for some } \varepsilon > 0.$$

*Then it holds that*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{\ell=1}^n f(X^{\ell,n}) - \mathbb{E}[f(X)] \right)^2 \right] = 0.$$

*Proof.* The proof is based on part of the proof of Theorem 3.2 in Gottlieb (1998). Set  $\mu := \mathbb{E}[f(X)]$ .

$$\begin{aligned} \mathbb{E} \left[ \left( \frac{1}{n} \sum_{\ell=1}^n f(X^{\ell,n}) - \mu \right)^2 \right] &= \frac{1}{n^2} \sum_{i,j=1}^n \mathbb{E}[(f(X^{i,n}) - \mu)(f(X^{j,n}) - \mu)] \\ &= \frac{1}{n} \mathbb{E}[(f(X^{1,n}) - \mu)^2] \\ &\quad + \frac{n-1}{n} \mathbb{E}[(f(X^{1,n}) - \mu)(f(X^{2,n}) - \mu)], \end{aligned}$$

The last equality is due the fact that all individuals are identically distributed. Our assumptions, Lemma 2.E.2 and (3.18) on p. 31 of Billingsley (1999) imply that  $(f(X^{1,n}) - \mu)^2$  and  $(f(X^{1,n}) - \mu)(f(X^{2,n}) - \mu)$  are uniformly integrable sequences. Thus by Definition 2.4.1 and Proposition 2.D.1, it holds that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}[(f(X^{1,n}) - \mu)^2] &= \left( \lim_{n \rightarrow \infty} \frac{1}{n} \right) \left( \lim_{n \rightarrow \infty} \mathbb{E}[(f(X^{1,n}) - \mu)^2] \right) \\ &= 0 \cdot \mathbb{E}[(f(X) - \mu)^2] = 0 \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \mathbb{E}[(f(X^{1,n}) - \mu)(f(X^{2,n}) - \mu)] = 2(\mathbb{E}[f(X)] - \mu) = 0.$$

The result follows.  $\square$

Now set  $\mu_n := \mathbb{E}[f(X^{1,n})]$  and  $\sigma_n^2 := \mathbb{E}[(f(X^{1,n}) - \mu)^2]$  and similarly set  $\mu := \mathbb{E}[f(X)]$  and  $\sigma^2 := \mathbb{E}[(f(X) - \mu)^2]$ . It is also possible (under additional assumptions) to derive a central limit theorem.

**Proposition 2.D.3.** *Assume that  $(\mathbb{Q}^n)_{n \in \mathbb{N}}$  is  $\mathbb{Q}$ -chaotic in total variation and that*

$$\lim_{n \rightarrow \infty} n \mathbb{E}[(f(X^{1,n}) - \mu_n)(f(X^{2,n}) - \mu_n)] = 0.$$

*Let  $f : S \rightarrow \mathbb{R}$  be measurable with*

$$\sup_{n \in \mathbb{N}} \mathbb{E}[|f(X^{1,n})|^{4+\varepsilon}] < \infty, \quad \text{for some } \varepsilon > 0.$$

*Then*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{f(X^{i,n}) - \mu_n}{\sigma_n} \xrightarrow{D} N(0, 1).$$

*If furthermore  $\lim_{n \rightarrow \infty} \sqrt{n}(\mu_n - \mu) = 0$ , then*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{f(X^{i,n}) - \mu}{\sigma} \xrightarrow{D} N(0, 1).$$

*Proof.* Since  $\{X^{\ell,n}; \ell = 1, \dots, n\}_{n \in \mathbb{N}}$  are exchangeable and so are  $\{f(X^{\ell,n}); \ell = 1, \dots, n\}_{n \in \mathbb{N}}$ . Therefore we would like to apply a CLT for exchangeable processes (see Theorem 2 in Blum et al. (1958)).

The first step is to prove the following three equalities:

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu_n &= \lim_{n \rightarrow \infty} \mathbb{E}[f(X^{1,n})] = \mathbb{E}[f(X)] =: \mu, \\ \lim_{n \rightarrow \infty} \sigma_n &= \lim_{n \rightarrow \infty} \mathbb{E}[(f(X^{1,n}) - \mu_n)^2] = \mathbb{E}[(f(X) - \mu)^2] =: \sigma, \\ \lim_{n \rightarrow \infty} \rho_n &= \lim_{n \rightarrow \infty} \mathbb{E}[f(X^{1,n})f(X^{2,n})] = \mathbb{E}[f(X)]^2 = \mu^2. \end{aligned}$$

Due to our Assumptions, Lemma 2.E.2 and (3.18) on p.31 of Billingsley (1999) all involved sequences are uniformly integrable. Thus all three identities are implied by chaosticity and Proposition 2.D.1. Using the same argument and the three identities it follows that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \left( \frac{f(X^{1,n}) - \mu_n}{\sigma_n} \right)^2 \left( \frac{f(X^{2,n}) - \mu_n}{\sigma_n} \right)^2 \right] = \frac{1}{\sigma^4} \mathbb{E}[(f(X) - \mu)^2]^2 = 1.$$

So condition (2) of Theorem 2 in Blum et al. (1958) is satisfied.

A similar argument yields

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \left| \frac{f(X^{1,n}) - \mu_n}{\sigma_n} \right|^3 \right] = \frac{1}{\sigma^3} \mathbb{E}[(f(X) - \mu)^3] < \infty,$$

and therefore condition (3) of Theorem 2 in Blum et al. (1958) is satisfied. Finally we have that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[ \left( \frac{f(X^{1,n}) - \mu_n}{\sigma_n} \right) \left( \frac{f(X^{2,n}) - \mu_n}{\sigma_n} \right) \right] = \frac{1}{\sigma^2} (\mathbb{E}[(f(X) - \mu)])^2 = 0.$$

This shows that we have the right convergence, which is required for condition (1) of Theorem 2 in Blum et al. (1958), but not necessarily the required convergence speed. Therefore we had to assume this in addition to the chaoticity assumption. Now all conditions of Theorem 2 in Blum et al. (1958) are satisfied and thus we obtain

$$\frac{1}{\sqrt{n}} \sum_{\ell=1}^n \frac{f(X^{\ell,n}) - \mu_n}{\sigma_n} \xrightarrow{D} N(0, 1).$$

Furthermore as  $\lim_{n \rightarrow \infty} \mu_n = \mu$  and  $\lim_{n \rightarrow \infty} \sigma_n = \sigma$  and since we have assumed that  $\lim_{n \rightarrow \infty} \sqrt{n}(\mu_n - \mu) = 0$ , an application of Slutsky's Lemma yields

$$\frac{1}{\sqrt{n}} \sum_{\ell=1}^n \frac{f(X^{\ell,n}) - \mu}{\sigma} = \frac{\sigma_n}{\sigma} \left( \sum_{\ell=1}^n \frac{f(X^{\ell,n}) - \mu_n}{\sigma_n} \right) + \frac{\sqrt{n}(\mu_n - \mu)}{\sigma} \xrightarrow{D} N(0, 1).$$

□

## 2.E Auxiliary results

**Lemma 2.E.1** (Gronwall's inequality). *Let  $u, v$  be non-negative real functions on the interval  $[\tau, T]$ , with  $v$  integrable and  $u$  bounded and measurable. Let  $c \geq 0$  and assume that*

$$u(t) \leq c + \int_{\tau}^t v(s)u(s)ds.$$

*Then it holds that*

$$u(t) \leq ce^{\int_{\tau}^t v(s)ds}.$$

**Lemma 2.E.2.** *Let  $(x_i)_{i=1, \dots, n} \subset \mathbb{R}$  and let  $p > 1$ . Then*

$$\left| \sum_{i=1}^n x_i \right|^p \leq n^{p-1} \sum_{i=1}^n |x_i|^p.$$



## Chapter 3

# Disability insurance with collective health claims: A mean-field approach

This chapter contains the paper *Furrer and Hornung (2026)*.

### ABSTRACT

The classic semi-Markov disability model is expanded with individual and collective health claims to improve its explanatory and predictive power – in particular in the context of group experience rating. The inclusion of collective health claims leads to a computationally challenging many-body problem. By adopting a mean-field approach, this many-body problem can be approximated by a non-linear one-body problem, which in turn leads to a transparent pricing method for disability coverages based on a lower-dimensional system of non-linear forward integro-differential equations. In a practice-oriented simulation study, the mean-field approximation clearly stands its ground in comparison to naïve Monte Carlo methods.

**Keywords:** Group experience rating; non-linear forward equations; semi-Markov model.

## 3.1 Introduction

Disability insurance plays a vital role in ensuring income stability and supporting part-time employment during periods with reduced earning capacity. In many countries, disability coverages are sold not only directly to individuals, but also as part of a company pension scheme. It is therefore essential for insurers to be able to accurately price such coverages – taking into account the fact that the physical and psychological work environment of each company likely has a substantial effect on the frequency, but also the severity, of disability claims. This suggests the application of group experience rating to disability insurance, which

has been explored via an empirical Bayes approach for Markov chains in Furrer (2019) and Furrer, Sørensen, and Yslas (2025). However, since the frequency of disability claims is rather low and single claims with long durations tend to have a substantial impact on the total loss, such a direct approach to experience rating is greatly challenged. Simply put: It can be near impossible to distinguish between ‘good’ and ‘bad’ companies, since there is just too limited data available.

In this paper, we address the ‘small data’ issue of infrequent disability claims by drawing on three observations. First, disability insurance and health insurance are increasingly, at least in certain countries such as Denmark, sold together. Second, it is reasonable to expect – at least when adjusting for covariates – that there is a relationship between an employees disability frequency and the extent of health claims across all employees; this observation is indirectly based on the assumption that both factors are largely attributable to the physical and psychological working environment. For example, a toxic work culture would lead to increased mental health risk, which would show itself in the form of health claims (consulting a psychologist, etc.) and, later, in actual disability (loss of working capacity due to severe stress). Third and finally, the scope of health insurance data is much more extensive, as this type of insurance is often used on a regular basis. Based on these observations, we formulate a multi-state model for disability insurance with integrated information about health claims.

To be specific, we expand the classic semi-Markov disability model (see Hoem (1972), Helwich (2008), Christiansen (2012), and Buchardt, Møller, and Schmidt (2015)) with collective and individual health claims. In the classic model, transition probabilities, expected cash flows, and prospective reserves may efficiently be calculated using Kolmogorov’s integro-differential equations. However, if we denote the number of individuals by  $n$ , the inclusion of collective health claims entails that the computational complexity of the relevant forward equations grows exponentially in  $n$ . Consequently, we adopt the mean-field approach outlined in Hornung (2026) to obtain, as an approximation in the limit  $n \rightarrow \infty$ , a lower-dimensional system of non-linear forward integro-differential equations. Furthermore, we briefly address statistical aspects as well as practical implementation.

In general, the insurance liabilities of an individual may depend on the other individuals either through the payments or, as is the case here, because the individuals themselves are dependent. In the area of actuarial multi-state modeling, mean-field approximations have hitherto received the most interest in the former case, see in particular Djehiche and Löfdahl (2016). In light hereof, we believe this paper offers a fresh perspective on the application of mean-field theory to the actuarial field.

The remainder of the paper is organized as follows. In Section 3.2, we expand the classic semi-Markov disability model with individual and collective health

claims. Section 3.3 contains a description of the corresponding mean-field model, including the associated system of non-linear forward integro-differential equations, and proofs of the required convergences. The next two sections are more oriented towards practice. In Section 3.4, we introduce and briefly discuss a meta-algorithm for solving the relevant system of differential equations, while Section 3.5 is devoted to a simulation study and the comparison of the mean-field approximation to naïve Monte Carlo methods. The paper concludes with Section 3.6 in which we offer some extensions and avenues for future work.

## 3.2 Disability model with health claims

This section is devoted to the expansion of the classic semi-Markov disability model with initially individual health claims and subsequently also collective health claims.

### 3.2.1 Semi-Markov model

Let  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  be a filtered probability space and write  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ . The multi-state approach to classic disability insurance considers a non-explosive jump process  $Z$  on the finite state space  $\mathcal{J} = \{1, 2, 3\}$  according to Figure 3.1. The initial distribution of  $Z$  is denoted  $\pi$ . We associate to  $Z$  a multivariate counting process  $N$  with components  $N_{jk}$ ,  $j \neq k$ , given by

$$N_{jk}(t) = \#\{s \in (0, t] : Z_{s-} = j, Z_s = k\}.$$

In this paper, we consider contractual payments prescribed by a payment process  $B$  given by

$$B(dt) = \sum_j \mathbf{1}_{\{Z_{t-}=j\}} b_j(t, U_{t-}) dt + \sum_{j \neq k} b_{jk}(t, U_{t-}) N_{jk}(dt), \quad (3.2.1)$$

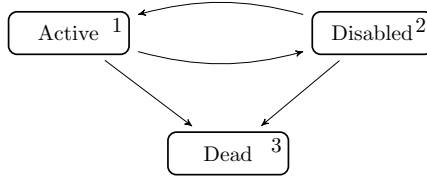
where  $U$  is the duration process associated with  $Z$  given by

$$U_t = t - \sup\{s \in [0, t] : Z_s \neq Z_t\} \text{ for } t > 0 \text{ and } U_0 = 0,$$

while  $(t, u) \mapsto b_j(t, u)$  and  $(t, u) \mapsto b_{jk}(t, u)$  are measurable functions that are bounded on compacts and which describe sojourn payment rates and transition payments, respectively.

Semi-Markov modeling entails the assumption that  $(Z, U)$  is Markov, and smooth semi-Markov modeling adds the assumption that the distributions of the jump times should be absolutely continuous (with respect to the Lebesgue measure). An alternative, but equivalent, formulation is in terms of the compensators of the multivariate counting process, which then should read

$$\mathbb{E}[N_{jk}(dt) | \mathcal{F}_{t-}] = \mathbf{1}_{\{Z_{t-}=j\}} \mu_{jk}(t, U_{t-}) dt.$$



**Figure 3.1:** State space  $\mathcal{J} = \{1, 2, 3\}$  for classic disability insurance. The arrows represent the possible transitions.

The functions  $(t, u) \mapsto \mu_{jk}(t, u)$ ,  $j \neq k$ , are the so-called (duration-dependent) transition rates, which we assume to be measurable and bounded on compacts.

Throughout, we assume implicitly that there exists a maximal contract time  $T \in (0, \infty)$  after which all payments are zero. We may then define the state-wise prospective reserves at contract inception,  $(V_i)_i$ , according to

$$V_i = \mathbb{E} \left[ \int_0^T e^{-\int_0^t r(s) ds} B(dt) \mid Z_0 = i \right],$$

where the deterministic interest rate  $t \mapsto r(t)$  is a measurable function that is bounded on compacts. If we denote by  $(A_i)_i$  the state-wise expected accumulated cash flows given by

$$A_i(dt) = \mathbb{E}[B(dt) \mid Z_0 = i],$$

then it holds that

$$V_i = \int_0^T e^{-\int_0^t r(s) ds} A_i(dt).$$

In similar fashion, the portfolio-wide prospective reserve at contract inception  $V$  reads

$$V = \mathbb{E} \left[ \int_0^T e^{-\int_0^t r(s) ds} B(dt) \right] = \int_0^T e^{-\int_0^t r(s) ds} A(dt),$$

where the portfolio-wide expected accumulated cash flow  $A$  is given by

$$A(dt) = \mathbb{E}[B(dt)].$$

In the following, let

$$p_{ij}(t, u) := \mathbb{P}(Z_t = j, U_t \leq u \mid Z_0 = i) \text{ and } p_j(t, u) := \mathbb{P}(Z_t = j, U_t \leq u).$$

be the transition probabilities and occupation probabilities, respectively. Note that

$$p_j = \sum_i \pi(i) p_{ij}, \quad A = \sum_i \pi(i) A_i, \quad V = \sum_i \pi(i) V_i.$$

**Example 3.2.1.** Consider again a disability annuity with a waiting period of  $\varepsilon \geq 0$ , corresponding to  $b_2(t, u) = b\mathbb{1}_{\{u \geq \varepsilon\}}$  with  $b > 0$ , while all other payments are zero.

$$\begin{aligned} A_i(dt) &= \int_0^t b\mathbb{1}_{\{u \geq \varepsilon\}} p_{i2}(t, du) dt = b\mathbb{1}_{\{t \geq \varepsilon\}} (p_{i2}(t, t) - p_{i2}(t, \varepsilon)) dt, \\ A(dt) &= \int_0^t b\mathbb{1}_{\{u \geq \varepsilon\}} p_2(t, du) dt = b\mathbb{1}_{\{t \geq \varepsilon\}} (p_2(t, t) - p_2(t, \varepsilon)) dt. \end{aligned}$$

Consequently,

$$\begin{aligned} V_i &= b \int_{\varepsilon}^T e^{-\int_0^t r(s) ds} (p_{i2}(t, t) - p_{i2}(t, \varepsilon)) dt, \\ V &= b \int_{\varepsilon}^T e^{-\int_0^t r(s) ds} (p_2(t, t) - p_2(t, \varepsilon)) dt. \end{aligned}$$

The following results are standard – see for instance Buchardt, Møller, and Schmidt (2015).

**Proposition 3.2.2.** *It holds that*

$$\begin{aligned} A_i(dt) &= \sum_j \int_0^t \left( b_j(t, u) + \sum_{k:k \neq j} b_{jk}(t, u) \mu_{jk}(t, u) \right) p_{ij}(t, du) dt, \\ A(dt) &= \sum_j \int_0^t \left( b_j(t, u) + \sum_{k:k \neq j} b_{jk}(t, u) \mu_{jk}(t, u) \right) p_j(t, du) dt. \end{aligned}$$

**Proposition 3.2.3.** *Let  $d \geq 0$ . It holds almost everywhere on  $[d, \infty)$  that*

$$\frac{d}{dt} p_{ij}(t, t-d) = \sum_{k:k \neq j} \int_0^t \mu_{kj}(t, u) p_{ik}(t, du) - \int_0^{t-d} \mu_{j\bullet}(t, u) p_{ij}(t, du)$$

*with boundary conditions  $p_{ij}(t, 0) = \mathbb{1}_{\{t=0\}} \mathbb{1}_{\{i=j\}}$ . It further holds almost everywhere on  $[d, \infty)$  that*

$$\frac{d}{dt} p_j(t, t-d) = \sum_{k:k \neq j} \int_0^t \mu_{kj}(t, u) p_k(t, du) - \int_0^{t-d} \mu_{j\bullet}(t, u) p_j(t, du)$$

*with boundary conditions  $p_j(t, 0) = \mathbb{1}_{\{t=0\}} \pi(j)$ .*

**Remark 3.2.4.** In light of the reparameterization of Remark 3.2.6 below, Proposition 3.2.3 states that the transition probabilities solve a version of Kolmogorov's forward equations. In fact, they *uniquely* solve these equations, confer with Feinberg, Mandava, and Shiryaev (2014) and Feinberg, Mandava, and Shiryaev (2022).

**Remark 3.2.5.** Note that the occupation probabilities may be calculated directly or by first calculating the transition probabilities and then taking a weighted average with respect to the initial distribution; the latter would of course be more

time-consuming. In particular, the occupation probability  $p_j$  corresponds to the transition probability  $p_{ij}$  if  $\pi(i) = 1$ . This is due to the fact that they have the same evolution forward through time and this evolution does not depend on the initial distribution. We emphasize this at the present time because precisely this property is lost as we turn to mean-field approximations in Section 3.3.

The terms of the integro-differential equations for the transition probabilities admit an intuitive interpretation. The first term is the aggregate probability mass stemming from paths starting in  $i$  at time zero, transitioning into state  $k$  at some time  $t - u \in [0, t)$ , and staying in state  $k$  until transitioning into state  $j$  at exactly time  $t$  (inflow). Similarly, the second term subtracts the probability mass stemming from paths starting in state  $i$  at time zero, transitioning into  $j$  no earlier than time  $d$  (as the duration at time  $t$  should be smaller than  $t - d$ ), and staying there until transitioning out of state  $j$  at exactly time  $t$  (outflow).

The systems of integro-differential equations of Proposition 3.2.3 can be solved using, for instance, the algorithm presented in Section 3 of Buchardt, Møller, and Schmidt (2015). Numerical schemes and practical implementation is discussed in greater detail in the latter Section 3.4.

*Remark 3.2.6.* The semi-Markov model can be reparameterized, using the process  $Y$  given by  $Y_t = t - U_t$ ,  $t \geq 0$ , instead of  $U$ . Since  $U$  is the duration *since* the last jump,  $Y$  is the time *of* the last jump. Therefore, while  $(Z, U)$  is a piecewise deterministic process,  $(Z, Y)$  is actually a jump process. This proves mathematically convenient.

In fact, the model  $(Z, U)$  with transition rates  $\mu_{jk}$ ,  $j \neq k$ , and transition probabilities  $p_{ij}$  is equivalent to the model  $(Z, Y)$  with transition rates  $(t, y) \mapsto \tilde{\mu}_{jk}(t, y) := \mu_{jk}(t, t - y)$ ,  $j \neq k$ , and transition probabilities  $\tilde{p}_{ij}$  given by

$$\tilde{p}_{ij}(t, d) = \mathbb{P}(Z_t = j, Y_t \geq d \mid Z_0 = i, Y_0 = 0), \quad 0 \leq d \leq t.$$

To see this, we may use the definition of  $Y$  to obtain

$$\begin{aligned} \tilde{p}_{ij}(t, d) &= \mathbb{P}(Z_t = j, Y_t \geq d \mid Z_0 = i, Y_0 = 0) \\ &= \mathbb{P}(Z_t = j, U_t \leq t - d \mid Z_0 = i, U_0 = 0) \\ &= p_{ij}(t, t - d) \end{aligned}$$

for  $0 \leq d \leq t$ . Applying a change of variables, we arrive at a version of Kolmogorov's forward equations for  $(Z, Y)$ :

$$\begin{aligned} \frac{d}{dt} \tilde{p}_{ij}(t, d) &= \sum_{k: k \neq j} \int_0^t \tilde{\mu}_{kj}(t, y) \tilde{p}_{ik}(t, dy) - \int_d^t \tilde{\mu}_{j\bullet}(t, y) \tilde{p}_{ij}(t, dy), \quad 0 \leq d \leq t, \\ \tilde{p}_{ij}(0, 0) &= \mathbf{1}_{\{i=j\}}. \end{aligned}$$

### 3.2.2 Extension: Individual health claims

It is reasonable to assume that the past number of health insurance claims of an individual is informative about the likelihood of a disability claim. Frequent use of the health insurance policy might predate a disability claim and, similarly, limited or no use might indicate good overall health. While disability claims are rare, health claims are frequent; this makes it particularly attractive to utilize the latter in risk profiling the individual. Therefore, we add to the model a non-explosive counting process  $H$  describing the number of health insurance claims of the individual. Foremost for notational convenience, we make the simplifying assumption that  $H$  and  $N$  admit no simultaneous jumps. Further, we assume that

$$\begin{aligned}\mathbb{E}[N_{jk}(dt) | \mathcal{F}_{t-}] &= \mathbb{1}_{\{Z_{t-}=j\}} \mu_{jk}(t, U_{t-}, H_{t-}) dt, \\ \mathbb{E}[H(dt) | \mathcal{F}_{t-}] &= \lambda_{Z_{t-}}(t, U_{t-}, H_{t-}) dt.\end{aligned}$$

The functions  $(t, u, h) \mapsto \mu_{jk}(t, u, h)$ ,  $j \neq k$ , are duration- and health-dependent transition rates, which we assume to be measurable and bounded on compacts. The functions  $(t, u, h) \mapsto \lambda_j(t, u, h)$  are health claim hazards of similar nature, which we also assume to be measurable and bounded on compacts. It follows that  $(Z, U, H)$  is a Markov process, confer for instance with Theorem 7.3.1(a) in Jacobsen (2006).

**Example 3.2.7.** In modeling the health claims, a simple choice would be

$$\mathbb{E}[H(dt) | \mathcal{F}_{t-}] = \lambda_{Z_{t-}} dt,$$

for non-negative constants  $\lambda_j$ . In this case,  $H$  is a doubly stochastic Poisson process with hazards depending only on the state (active, disabled, and dead) of the individual.

In the following, let

$$\begin{aligned}p_{ij}(t, u, h) &:= \mathbb{P}(Z_t = j, U_t \leq u, H_t = h | Z_0 = i), \\ p_j(t, u, h) &:= \mathbb{P}(Z_t = j, U_t \leq u, H_t = h)\end{aligned}$$

be the transition probabilities and occupation probabilities, respectively. Note that  $p_j = \sum_i \pi(i) p_{ij}$ .

**Example 3.2.8.** Consider again a disability annuity with a waiting period of  $\varepsilon \geq 0$ , corresponding to  $b_2(t, u) = b \mathbb{1}_{\{u \geq \varepsilon\}}$  with  $b > 0$ , while all other payments are zero. In that case, we now have that

$$\begin{aligned}A_i(dt) &= \sum_{h=0}^{\infty} \int_0^t b \mathbb{1}_{\{u \geq \varepsilon\}} p_{i2}(t, du, h) dt = b \mathbb{1}_{\{t \geq \varepsilon\}} \sum_{h=0}^{\infty} (p_{i2}(t, t, h) - p_{i2}(t, \varepsilon, h)) dt, \\ A(dt) &= \sum_{h=0}^{\infty} \int_0^t b \mathbb{1}_{\{u \geq \varepsilon\}} p_2(t, du, h) dt = b \mathbb{1}_{\{t \geq \varepsilon\}} \sum_{h=0}^{\infty} (p_2(t, t, h) - p_2(t, \varepsilon, h)) dt.\end{aligned}$$

Consequently,

$$\begin{aligned} V_i &= b \int_{\varepsilon}^T e^{-\int_0^t r(s) ds} \sum_{h=0}^{\infty} (p_{i2}(t, t, h) - p_{i2}(t, \varepsilon, h)) dt, \\ V &= b \int_{\varepsilon}^T e^{-\int_0^t r(s) ds} \sum_{h=0}^{\infty} (p_2(t, t, h) - p_2(t, \varepsilon, h)) dt. \end{aligned}$$

Since the payments themselves do not depend on  $H$ , this variable is simply summed out.

The following two propositions expand Propositions 3.2.2 and 3.2.3 to also include individual health claims.

**Proposition 3.2.9.** *It holds that*

$$\begin{aligned} A_i(dt) &= \sum_j \sum_{h=0}^{\infty} \int_0^t \left( b_j(t, u) + \sum_{k:k \neq j} b_{jk}(t, u) \mu_{jk}(t, u, h) \right) p_{ij}(t, du, h) dt, \\ A(dt) &= \sum_j \sum_{h=0}^{\infty} \int_0^t \left( b_j(t, u) + \sum_{k:k \neq j} b_{jk}(t, u) \mu_{jk}(t, u, h) \right) p_j(t, du, h) dt. \end{aligned}$$

*Proof.* Referring to the martingale property of differences between the counting processes and their (absolutely continuous) compensators, we find that

$$A_i(dt) = \mathbb{E} \left[ b_{Z_t}(t, U_t) + \sum_{k:k \neq Z_t} b_{Z_t k}(t, U_t) \mu_{Z_t k}(t, U_t, H_t) \mid Z_0 = i \right] dt.$$

Invoking the relevant push-forward measure, the first statement of the proposition is immediate. The second statement is proveable in an identical fashion.  $\square$

In the following, we adopt the convention that  $(t, u) \mapsto \lambda_j(t, u, -1)$  is constantly zero.

**Proposition 3.2.10.** *Let  $d \geq 0$ . It holds almost everywhere on  $[d, \infty)$  that*

$$\begin{aligned} & \frac{d}{dt} p_{ij}(t, t-d, h) \\ &= \sum_{k:k \neq j} \int_0^t \mu_{kj}(t, u, h) p_{ik}(t, du, h) - \int_0^{t-d} \mu_{j\bullet}(t, u, h) p_{ij}(t, du, h) \\ & \quad + \int_0^{t-d} \lambda_j(t, u, h-1) p_{ij}(t, du, h-1) - \int_0^{t-d} \lambda_j(t, u, h) p_{ij}(t, du, h) \end{aligned}$$

with boundary conditions  $p_{ij}(t, 0, h) = \mathbb{1}_{\{t=0\}} \mathbb{1}_{\{h=0\}} \mathbb{1}_{\{i=j\}}$ . It further holds almost everywhere on  $[d, \infty)$  that

$$\begin{aligned} & \frac{d}{dt} p_j(t, t-d, h) \\ &= \sum_{k: k \neq j} \int_0^t \mu_{kj}(t, u, h) p_k(t, du, h) - \int_0^{t-d} \mu_{j\bullet}(t, u, h) p_j(t, du, h) \\ & \quad + \int_0^{t-d} \lambda_j(t, u, h-1) p_j(t, du, h-1) - \int_0^{t-d} \lambda_j(t, u, h) p_j(t, du, h) \end{aligned}$$

with boundary conditions  $p_j(t, 0, h) = \mathbb{1}_{\{t=0\}} \mathbb{1}_{\{h=0\}} \pi(j)$ .

*Proof.* The second statement of the proposition follows, for instance, from the first statement and the identity  $p_j = \sum_i \pi(i) p_{ij}$ . To prove the first statement, we consider the trivariate process  $(Z, Y, H)$  with  $Y$  defined according to  $Y_t = t - U_t$ ,  $t \geq 0$ ; this is a Markov jump process. By definition, the compensators of the counting processes associated with  $(Z, Y, H)$  are predictable w.r.t. the information generated by themselves, and thus by Theorem 4.8.1 of Jacobsen (2006), the distribution of  $(Z, Y, H)$  is fully characterized by these compensators. Consequently, we may invoke (7.17) on p. 151 in Jacobsen (2006) to obtain a system of integro-differential equations which, after a change of variable similar to Remark 3.2.6, yields the desired result.  $\square$

*Remark 3.2.11.* In light of the reparameterization utilized in the proof, Proposition 3.2.10 states that the transition probabilities solve a version of Kolmogorov's forward equations. In fact, they *uniquely* solve these equations, confer with Feinberg, Mandava, and Shiryaev (2014) and Feinberg, Mandava, and Shiryaev (2022).

*Remark 3.2.12.* Note that the occupation probabilities may be calculated directly or by first calculating the transition probabilities and then taking a weighted average with respect to the initial distribution. In particular, the occupation probability  $p_j$  corresponds to the transition probability  $p_{ij}$  if  $\pi(i) = 1$ . This is due to the fact that they have the same evolution forward through time and this evolution does not depend on the initial distribution. We emphasize this once again because precisely this property is lost as we turn to mean-field approximations in Section 3.3.

The addition of individual health claims has not hurt the intuitive interpretation of the terms of the integro-differential equations. The first line is concerned with jumps of  $Z$ , while the second line is concerned with jumps of  $H$ ; this split is possible since simultaneous jumps were disallowed – otherwise, two additional terms would be necessary. The two terms of the first line are similar to before, but with an added requirement of  $H$  having reached state  $h$  strictly before time  $t$ . The two terms of the second line are new. The first term adds the aggregate probability mass stemming from paths for which the  $h$ 'th health claim occurs at time  $t$  (inflow),

while the second term subtracts the aggregate probability mass stemming from path for which the  $(h + 1)$ 'th health claim occurs at time  $t$  (outflow).

The systems of integro-differential equations of Proposition 3.2.3 can, for instance, be solved by adapting the algorithm presented in Section 3 of Buchardt, Møller, and Schmidt (2015). Compared to the situation without individual health claims, the procedure would have to be used for every triplet  $(i, j, h)$  rather than only every pair  $(i, j)$ . In practice, one must choose a cut-off  $K_H$ , calculate the transition probabilities for  $h = 0, \dots, K_H$ , and perhaps extrapolate for  $h > K_H$ . Numerical schemes and practical implementation is discussed in greater detail in the latter Section 3.4.

### 3.2.3 Extension: Collective health claims

So far, we have only considered a single individual – implicitly assuming individuals to be independent. However, one could easily imagine this to not be the case. There is of course the obvious example of events and trends that impact society as a whole, such as pandemics and climate change, but also technological and medical advancements. However, as explained in the introduction, we rather have the example of company level insurance plans and collective health claims in mind. Dependence between individuals here stems from the fact that each employee is affected by the same physical and psychological work environment. If the insurance plan contains both disability and health coverage, aggregate information about health claims, which are much more extensive than disability claims, could serve as a reasonable predictor of, for example, the disability rate.

To formalize a model able to capture the stylized facts, we consider  $n \in \mathbb{N}$  individuals, each with an associated trivariate process  $X^{\ell,n} = (Z^{\ell,n}, U^{\ell,n}, H^{\ell,n})$ . Foremost for notational convenience, we make the additional simplifying assumption that  $H^{1,n}, N^{1,n}, \dots, H^{n,n}, N^{n,n}$  admit no simultaneous jumps. We denote by  $E$  the state space of each  $X^{\ell,n}$ , that is  $E = \mathcal{J} \times [0, \infty) \times \mathbb{N}_0$ , and consider a measurable and bounded function  $g : E \mapsto \mathbb{R}^d$ ,  $d \in \mathbb{N}$ . Based hereon, we define the averaged process  $\nu^n$  according to

$$\nu_t^n = \frac{1}{n} \sum_{\ell=1}^n g(X_t^{\ell,n}),$$

and we assume that

$$\begin{aligned} \mathbb{E}[N_{jk}^{\ell,n}(dt) | \mathcal{F}_{t-}] &= \mathbf{1}_{\{Z_{t-}^{\ell,n}=j\}} \mu_{jk}(t, U_{t-}^{\ell,n}, H_{t-}^{\ell,n}, \nu_{t-}^n) dt, \\ \mathbb{E}[H^{\ell,n}(dt) | \mathcal{F}_{t-}] &= \lambda_{Z_{t-}^{\ell,n}}(t, U_{t-}^{\ell,n}, H_{t-}^{\ell,n}, \nu_{t-}^n) dt. \end{aligned} \tag{3.2.2}$$

The functions  $(t, u, h, y) \mapsto \mu_{jk}(t, u, h, y)$ ,  $j \neq k$ , are duration-, health- as well as collective-dependent transition rates, which we assume to be measurable and

bounded on compacts. The functions  $(t, u, h, y) \mapsto \lambda_j(t, u, h, y)$  are health claim hazards of similar nature, which we also assume to be measurable and bounded on compacts.

**Example 3.2.13.** As a concrete example of how the function  $g$  might be chosen in practice, consider  $g(z, u, h) = \max\{h, \kappa_H\}$  for some  $\kappa_H > 0$ . In this case,

$$\nu_t^n = \frac{1}{n} \sum_{\ell=1}^n \max\{H_t^{\ell,n}, \kappa_H\}.$$

Consequently, the transition rate from the active to the disabled state may now depend on the average truncated number of health insurance claims. In practice, the impact of truncation can be removed by selecting  $\kappa_H$  sufficiently large.

If the individuals have the same initial distribution, meaning  $\pi^{\ell,n}$  does not depend on  $\ell$ , then (3.2.2) entails that the individuals are actually identically distributed. However, they are in general not independent and the Markov property holds neither for  $X^{\ell,n}$  or  $\nu^n$  nor  $(X^{\ell,n}, \nu^n)$ ; only the high-dimensional process  $X^n := (X^{1,n}, \dots, X^{n,n})$  is always Markov. If we may think of  $\nu^n$  as an external process driving  $Z^{1,n}$ , which in the context of Example 3.2.13 would be the case if  $H^{\ell,n}$  is *locally independent* of  $Z^{\ell,n}$ , confer with Aalen (1987), computational simplifications could surface. It is the (causal) interaction between disabilities and health claims on a collective level that especially complicate the computational aspects.

The fact that  $X^n$  is a Markov process signifies, at least in principle, that transition probabilities and other quantities of interest may be computed by solving (high-dimensional) systems of integro-differential equations. However, even if we disregard health claims and duration effects, the computational complexity grows exponentially in  $n$ . Thus, already for moderate  $n$  such a direct approach likely has to be abandoned. Monte Carlo methods constitute an alternative, but are rather slow and should, therefore, only be used as a kind of last resort.

In the next section, we explore an attractive alternative: mean-field approximations, where the process  $\nu^n$  is replaced by its mean, restoring the independence between individuals as well as the computational tractability.

### 3.3 Mean-field approximation

The idea behind mean-field approximations is to replace averages by mean values – noting that as  $n \rightarrow \infty$ , the average of  $n$  exchangeable random variables converges to their mean. Specifically, we want to replace the averaged process  $\nu^n$  in (3.2.2) by its mean. The advantage of such an approximation would be the replacement of the high-dimensional  $n$ -individual model by a one-individual limiting model, the so-called mean-field model.

### 3.3.1 Setup and mean-field convergence

The mean-field model corresponding to the  $n$ -individual model of Subsection 3.2.3 is given by the trivariate process  $\bar{X} = (\bar{Z}, \bar{U}, \bar{H})$ , where  $\bar{U}$  is the duration process associated with  $\bar{Z}$  and  $\bar{H}$  is a counting process, with the following characteristics. First, and foremost for notational convenience, we assume that  $\bar{H}$  and  $\bar{N}$ , where  $\bar{N}$  is the multivariate counting process associated with  $\bar{Z}$ , admit no simultaneous jumps. Further, we assume that

$$\begin{aligned}\mathbb{E}[\bar{N}_{jk}(dt) | \mathcal{F}_{t-}] &= \mathbb{1}_{\{Z_{t-}=j\}} \mu_{jk}(t, \bar{U}_{t-}, \bar{H}_{t-}, v(t-)) dt, \\ \mathbb{E}[\bar{H}(dt) | \mathcal{F}_{t-}] &= \lambda_{\bar{Z}_{t-}}(t, \bar{U}_{t-}, \bar{H}_{t-}, v(t-)) dt,\end{aligned}\tag{3.3.1}$$

where  $t \mapsto v(t) := \mathbb{E}[g(\bar{X}_t)]$ . It is apparent that the empirical average  $\nu^n$  has been replaced by its expectation  $v$ . Instead of depending on other individuals, the compensators of an individual now depend on the distribution of the process  $\bar{X}$  through the mean  $v$ . In other words,  $\bar{X} = (\bar{Z}, \bar{U}, \bar{H})$  is a distribution-dependent trivariate process. This has certain mathematical implications, some of which are beneficial and some of which are not, as we shall soon demonstrate. The type of convergence that lies behind a mean-field approximation relates to the notion of chaosticity. In the mean-field literature, one usually employs a notion of chaosticity based on weak convergence. However, as discussed in Section 7 of Horning (2026), it can be preferable to use a stronger notion based on the total variation distance. In this manuscript, we exclusively use this stronger notion of chaosticity.

**Definition 3.3.1.** *Let  $(S, \mathcal{S})$  be a measurable space, and let  $\mathbb{Q}$  be a probability measure on  $(S, \mathcal{S})$ . Then a sequence of exchangeable probability measures  $(\mathbb{Q}_n)_{n \in \mathbb{N}}$ , with each  $\mathbb{Q}_n$  defined on the corresponding product space  $(S^n, \otimes_{\ell=1}^n \mathcal{S})$ , is said to be  $\mathbb{Q}$ -chaotic if*

$$\forall k \in \mathbb{N}: \quad \mathbb{Q}_n^k \xrightarrow{TV} \otimes_{\ell=1}^k \mathbb{Q},$$

where  $\mathbb{Q}_n^k(\cdot) := \mathbb{Q}_n(\cdot \times S^{n-k})$  for  $k < n$ .

*Remark 3.3.2.* By using the superscript TV, we signify convergence in the total variation distance. As a gentle reminder to the reader, we may define the total variation distance  $d_{TV}$  on the set of probability measure  $\mathcal{P}(S)$  of a measurable space  $(S, \mathcal{S})$  according to

$$d_{TV}(\mathbb{Q}_1, \mathbb{Q}_2) := \sup_{A \in \mathcal{S}} |\mathbb{Q}_1(A) - \mathbb{Q}_2(A)|, \quad \mathbb{Q}_1, \mathbb{Q}_2 \in \mathcal{P}(S).$$

An important technical observation is that  $(\mathcal{P}(S), d_{TV})$  is complete.

*Remark 3.3.3.* As another gentle reminder to the reader, if  $\mathbb{Q}_n$  describes the distribution of random variables  $(Y^{1,n}, \dots, Y^{n,n})$ , then  $\mathbb{Q}_n$  (or the random variables themselves) is said to be *exchangeable* if

$$(Y^{1,n}, \dots, Y^{1,n}) \stackrel{d}{=} (Y^{\sigma(1),n}, \dots, Y^{\sigma(n),n})$$

for all permutations  $\sigma : \{1, \dots, n\} \mapsto \{1, \dots, n\}$ . Intuitively, the joint distribution of the individuals is unchanged when the individuals are reordered and, consequently, all individuals must share the same marginal distribution. Note that independent and identically distributed random variables are automatically exchangeable.

Chaoticity, as Definition 3.3.1 reveals, entails that the individuals become asymptotically independent in the following manner: any fixed number of individuals become independent as the overall number of individuals goes to infinity.

In the following, we consider the processes  $\bar{X}, X^{1,n}, \dots, X^{n,n}$  as random variables taking values in the path space of non-explosive jump processes. This space can be equipped with the  $\sigma$ -algebra generated by the coordinate projections, and this  $\sigma$ -algebra corresponds to the Borel  $\sigma$ -algebra generated by the  $J_1$ -subspace topology. We want to show that the sequence  $(Q_n)_{n \in \mathbb{N}}$  is  $\bar{Q}$ -chaotic; this is what is understood by ‘mean-field convergence’.

So far, we have refrained from restricting the initial distribution of  $X^n$  and  $\bar{X}$ . By definition,  $U_0^{\ell,n} = H_0^{\ell,n} = \bar{U}_0 = \bar{H}_0 = 0$ . The minimal condition would thus be to assume  $\pi$ -chaoticity of  $(Z_0^{1,n}, \dots, Z_0^{n,n})(\mathbb{P})$  for some  $\pi$ , which is the assumption we adopt in the remainder of this section. Note that if the individuals are independent and identically distributed at time zero, then this minimal condition is automatically satisfied with  $\pi = \pi^{1,1}$ .

The following conditions on the transition rates, the health claim hazards, and the averaging function  $g$  are sufficient to ensure the desired mean-field convergence. For just the existence and uniqueness of the mean-field model, less may do.

**Condition 1.**

- a) *The transition rates  $(t, u, h, y) \mapsto \mu_{jk}(t, u, h, y)$ ,  $j \neq k$ , and the hazards  $(t, u, h, y) \mapsto \lambda_j(t, u, h, y)$ , in addition to being bounded on compacts, are bounded in  $h$  and satisfy the following Lipschitz conditions: There exist non-negative constants  $C_{jk}$  and  $C_j$  such that, for all  $y_1, y_2 \in \mathbb{R}^d$ , all  $u \in [0, T]$ , and all  $h \in \mathbb{N}_0$ , it holds that*

$$\begin{aligned} |\mu_{jk}(t, u, h, y_1) - \mu_{jk}(t, u, h, y_2)| &\leq C_{jk} \|y_1 - y_2\|, \\ |\lambda_j(t, u, h, y_1) - \lambda_j(t, u, h, y_2)| &\leq C_j \|y_1 - y_2\|. \end{aligned}$$

- b) *The averaging function  $(z, u, h) \mapsto g(z, u, h)$ , in addition to being bounded, satisfies the following Lipschitz condition: There exists a non-negative constant  $C_g$  such that, for all  $u_1, u_2 \in [0, T]$ , all  $z \in \mathcal{J}$ , and all  $h \in \mathbb{N}_0$ , it holds that*

$$\|g(z, u_1, h) - g(z, u_2, h)\| \leq C_g |u_1 - u_2|.$$

*Remark 3.3.4.* Let  $\mathcal{P}(E)$  denote the set of all probability measures on  $E$  equipped with the Borel  $\sigma$ -algebra, and let  $d_{\text{BL}}$  denote the so-called bounded-Lipschitz distance on  $\mathcal{P}(E)$ . Condition 1 is sufficient to ensure that  $(t, u, h, \rho) \mapsto \mu_{jk}(t, u, h, \int_E g \, d\rho)$ ,

$j \neq k$ , and  $(t, u, h, \rho) \mapsto \lambda_j(t, u, h, \int_E g \, d\rho)$  are bounded and satisfy the following Lipschitz conditions: There exist non-negative constants  $C_{jk}$  and  $C_j$  such that, for all  $\rho_1, \rho_2 \in \mathcal{P}(E)$ , all  $u \in [0, T]$ , and all  $h \in \mathbb{N}_0$ , it holds that

$$\begin{aligned} \left| \mu_{jk} \left( t, u, h, \int_E g \, d\rho_1 \right) - \mu_{jk} \left( t, u, h, \int_E g \, d\rho_2 \right) \right| &\leq C_{jk} d_{\text{BL}}(\rho_1, \rho_2), \\ \left| \lambda_j \left( t, u, h, \int_E g \, d\rho_1 \right) - \lambda_j \left( t, u, h, \int_E g \, d\rho_2 \right) \right| &\leq C_j d_{\text{BL}}(\rho_1, \rho_2). \end{aligned}$$

These are the Lipschitz conditions that are actually exploited to verify the existence and uniqueness of the mean-field model and to establish the desired mean-field convergence.

Due to Condition 1, we have conveniently and firmly positioned ourselves within the general mean-field theory, see for instance Hornung (2026). Consequently, we immediately obtain the following result, which confirms the desired mean-field convergence and establishes the mean-field model as an approximation to the  $n$ -individual model.

**Theorem 3.3.5.** *Suppose that Condition 1 is met. Then the mean-field model exists and is unique. Further, with  $\mathbb{Q}_n := X^n(\mathbb{P})$  and  $\bar{\mathbb{Q}} := \bar{X}(\mathbb{P})$ , it holds that  $(\mathbb{Q}_n)_{n \in \mathbb{N}}$  is  $\bar{\mathbb{Q}}$ -chaotic.*

*Proof.* We adopt the reparameterization of Remark 3.2.6 for both  $X^{\ell, n}$  and  $\bar{X}$ . This yields the following systems of integral equations:

$$\begin{aligned} dZ_t^{\ell, n} &= \sum_k (k - Z_{t-}^{\ell, n}) N_k^{\ell, n}(dt), \quad dY_t^{\ell, n} = (t - Y_{t-}^{\ell, n}) \sum_k N_k^{\ell, n}(dt), \quad dH_t^{\ell, n} = d\bar{H}_t^{\ell, n}, \\ d\bar{Z}_t &= \sum_k (k - \bar{Z}_{t-}) \bar{N}_k(dt), \quad d\bar{Y}_t = (t - \bar{Y}_{t-}) \sum_k \bar{N}_k(dt), \quad d\bar{H}_t = d\bar{H}_t, \end{aligned}$$

where  $N_k^{\ell, n} := \sum_j N_{jk}^{\ell, n}$  and  $\bar{N}_k := \sum_j \bar{N}_{jk}$ . Proposition 3.14 of Hornung (2026) shows that Condition 1 is sufficient for the validity of Theorem 3.7 and Theorem 4.4 in Hornung (2026), with the former ensuring the existence and uniqueness of the mean-field model and the latter confirming the postulated chaoticity.  $\square$

We conclude this subsection with a brief discussion on regular conditional distributions in mean-field models. If  $Y$  is an ordinary Markov process, then it is completely described by its initial distribution and its transition probabilities. In particular, we may determine its occupation probabilities by integrating the transition probabilities with respect to the initial distribution or, vice versa, we may determine the initial distribution via disintegration of the occupation probabilities with respect to the transition probabilities. In other words, changing the initial distribution only affects the occupation probabilities – not the transition probabilities; confer also with the discussions in Section 3.2.1.

In a mean-field model, the transition probabilities depend on the initial distribution, so changing the initial distribution affects not only the occupation probabilities, but also the transition probabilities. This means that caution should be exercised when interpreting conditional expectations such as  $\mathbb{E}[h(\bar{X}) \mid \bar{Z}_0 = i]$  for some suitably regular function  $h$ . This expectation represents integration with respect to a regular conditional distribution of  $\bar{X}$  given  $\bar{Z}_0$ , that is it involves the transition probabilities of  $\bar{X}$ , which depend on the limiting initial distribution  $\pi$ .

### 3.3.2 Actuarial applications

The contractual payments remain on the form (3.2.1), meaning we consider  $B^{\ell,n}$  and  $\bar{B}$  given by

$$\begin{aligned} B^{\ell,n}(dt) &= \sum_j \mathbb{1}_{\{Z_{t-}^{\ell,n}=j\}} b_j(t, U_{t-}^{\ell,n}) dt + \sum_{j \neq k} b_{jk}(t, U_{t-}^{\ell,n}) N_{jk}^{\ell,n}(dt), \\ \bar{B}(dt) &= \sum_j \mathbb{1}_{\{\bar{Z}_{t-}=j\}} b_j(t, \bar{U}_{t-}) dt + \sum_{j \neq k} b_{jk}(t, \bar{U}_{t-}) \bar{N}_{jk}(dt). \end{aligned}$$

Recall that  $(t, u) \mapsto b_j(t, u)$  and  $(t, u) \mapsto b_{jk}(t, u)$  are measurable functions that are bounded on compacts.

The following results provide an actuarial perspective on Theorem 3.3.5, illuminating how the total variation convergence of state processes gives rise to laws of large numbers that substantiate the use of mean-field approximations for, among other things, pricing and reserving purposes.

**Proposition 3.3.6.** *Suppose that Condition 1 is met. It then holds that*

$$\frac{1}{n} \sum_{\ell=1}^n \int_0^T e^{-\int_0^t r(s) ds} B^{\ell,n}(dt) \xrightarrow{L^2} \mathbb{E} \left[ \int_0^T e^{-\int_0^t r(s) ds} \bar{B}(dt) \right].$$

Furthermore, if  $\pi(i) > 0$ , then

$$\frac{\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_{\{Z_0^{\ell,n}=i\}} \int_0^T e^{-\int_0^t r(s) ds} B^{\ell,n}(dt)}{\frac{1}{n} \sum_{\ell=1}^n \mathbb{1}_{\{Z_0^{\ell,n}=i\}}} \xrightarrow{P} \mathbb{E} \left[ \int_0^T e^{-\int_0^t r(s) ds} \bar{B}(dt) \mid \bar{Z}_0 = i \right].$$

*Proof.* We can view  $\bar{B}$  as a measurable mapping from the path space of non-explosive jump processes into  $\mathbb{R}$ . Theorem 3.3.5 ensures chaoticity and Proposition 6.10 of Hornung (2026), Condition 1, and the boundedness on compacts of  $(t, u) \mapsto b_j(t, u)$  and  $(t, u) \mapsto b_{jk}(t, u)$  ensure sufficient integrability. The desired results then follow from Proposition 6.6 of Hornung (2026).  $\square$

The limits appearing in Proposition 3.3.6 are so-called mean-field prospective reserves. The state-wise prospective mean-field prospective reserves at contract

inception,  $(\bar{V}_i)_i$ , are defined according to

$$\bar{V}_i = \mathbb{E} \left[ \int_0^T e^{-\int_0^t r(s) ds} \bar{B}(dt) \mid \bar{Z}_0 = i \right],$$

while the portfolio-wide prospective mean-field reserve at contract inception  $\bar{V}$  reads

$$\bar{V} = \mathbb{E} \left[ \int_0^T e^{-\int_0^t r(s) ds} \bar{B}(dt) \right].$$

If we denote by  $(\bar{A}_i)_i$  the so-called state-wise mean-field accumulated cash flows, which are given by

$$\bar{A}_i(dt) = \mathbb{E}[\bar{B}(dt) \mid \bar{Z}_0 = i],$$

then it holds that

$$\bar{V}_i = \int_0^T e^{-\int_0^t r(s) ds} \bar{A}_i(dt).$$

In similar fashion, with  $\bar{A}$  the mean-field accumulated cash flow given by

$$\bar{A}(dt) = \mathbb{E}[\bar{B}(dt)],$$

it holds that

$$\bar{V} = \int_0^T e^{-\int_0^t r(s) ds} \bar{A}(dt).$$

These prospective reserves and expected cash flows should be seen in comparison to those of the  $n$ -individual model, namely  $(V_i^{1,n})_i$ ,  $V^{1,n}$ ,  $(A_i^{1,n})_i$ , and  $A^{1,n}$  given by

$$V_i^{1,n} = \mathbb{E} \left[ \int_0^T e^{-\int_0^t r(s) ds} B^{1,n}(dt) \mid Z_0^{1,n} = i \right], \quad V^{1,n} = \mathbb{E} \left[ \int_0^T e^{-\int_0^t r(s) ds} B^{1,n}(dt) \right],$$

$$A_i^{1,n}(dt) = \mathbb{E}[B^{1,n}(dt) \mid Z_0^{1,n} = i], \quad A^{1,n}(dt) = \mathbb{E}[B^{1,n}(dt)],$$

and satisfying

$$V_i^{1,n} = \int_0^T e^{-\int_0^t r(s) ds} A_i^{1,n}(dt), \quad V^{1,n} = \int_0^T e^{-\int_0^t r(s) ds} A^{1,n}(dt).$$

The following proposition establishes the mean-field cash flows and reserves as viable approximations of their  $n$ -individual counterparts.

**Proposition 3.3.7.** *Suppose that Condition 1 is met. It then holds that*

$$V^{1,n} \rightarrow \bar{V} \quad \text{and} \quad \forall t \geq 0 : A^{1,n}(t) - A^{1,n}(0) \rightarrow \bar{A}(t) - \bar{A}(0).$$

Furthermore, if  $\pi(i) > 0$ , then

$$V_i^{1,n} \rightarrow \bar{V}_i \quad \text{and} \quad \forall t \geq 0 : A_i^{1,n}(t) - A_i^{1,n}(0) \rightarrow \bar{A}_i(t) - \bar{A}_i(0).$$

*Proof.* The argument for the expected cash flows is similar to that of the reserves, so we focus on the latter. Theorem 3.3.5 ensures chaoticty and Proposition 6.10 of Hornung (2026), Condition 1, and the boundedness on compacts of  $(t, u) \mapsto b_j(t, u)$  and  $(t, u) \mapsto b_{jk}(t, u)$  ensure sufficient integrability. The desired result then follows from Proposition 6.3 and Proposition 6.4 of Hornung (2026).  $\square$

Having formally verified the usefulness of the mean-field model, we now discuss how to calculate mean-field cash flows and reserves. To this end, let

$$\begin{aligned}\bar{p}_{ij}(t, u, h) &:= \mathbb{P}(\bar{Z}_t = j, \bar{U}_t \leq u, \bar{H}_t = h \mid \bar{Z}_0 = i), \\ \bar{p}_j(t, u, h) &:= \mathbb{P}(\bar{Z}_t = j, \bar{U}_t \leq u, \bar{H}_t = h)\end{aligned}$$

be the mean-field transition probabilities and occupation probabilities, respectively. Note that, with  $\pi$  the limiting initial distribution,

$$\bar{p}_j = \sum_i \pi(i) \bar{p}_{ij}, \quad \bar{A} = \sum_i \pi(i) \bar{A}_i, \quad \bar{V} = \sum_i \pi(i) \bar{V}_i. \quad (3.3.2)$$

The following proposition expands Proposition 3.2.2 to not only include individual health claims, but also mean-field effects. It should be compared to Proposition 3.2.9.

**Proposition 3.3.8.** *It holds that*

$$\begin{aligned}\bar{A}_i(dt) &= \sum_j \sum_{h=0}^{\infty} \int_0^t \left( b_j(t, u) + \sum_{k:k \neq j} b_{jk}(t, u) \mu_{jk}(t, u, h, v(t)) \right) \bar{p}_{ij}(t, du, h) dt, \\ \bar{A}(dt) &= \sum_j \sum_{h=0}^{\infty} \int_0^t \left( b_j(t, u) + \sum_{k:k \neq j} b_{jk}(t, u) \mu_{jk}(t, u, h, v(t)) \right) \bar{p}_j(t, du, h) dt, \\ v(t) &= \sum_j \sum_{h=0}^{\infty} \int_0^t g(j, u, h) \bar{p}_j(t, du, h) = \sum_i \pi(i) \sum_j \sum_{h=0}^{\infty} \int_0^t g(j, u, h) \bar{p}_{ij}(t, du, h).\end{aligned}$$

*Proof.* The core argument is exactly the same as in the proof of Proposition 3.2.9.  $\square$

It is worth noting that the calculation of mean-field accumulated cash flows, and therefore also mean-field reserves, is thus no more involved than the calculation of expected accumulated cash flows and reserves in the one-individual model – if occupation and transition probabilities are readily available and intermediaries such as  $v$  are easy to calculate.

**Example 3.3.9.** Continuing Example 3.2.13 with  $g(z, u, h) = \max\{h, \kappa_H\}$ , it follows that

$$v(t) = \sum_j \sum_{h=0}^{\infty} \max\{h, \kappa_H\} \bar{p}_j(t, t, h) = \sum_i \pi(i) \sum_j \sum_{h=0}^{\infty} \max\{h, \kappa_H\} \bar{p}_{ij}(t, t, h),$$

which is a rather straightforward to calculate based on the mean-field occupation or transition probabilities.

Let us begin by exploring the calculation of the mean-field occupation probabilities. In the following, we adopt the convention that  $(t, u, v) \mapsto \lambda_j(t, u, -1, v)$  is constantly zero.

**Proposition 3.3.10.** *Let  $d \geq 0$ . It holds almost everywhere on  $[d, \infty)$  that*

$$\begin{aligned} & \frac{d}{dt} \bar{p}_j(t, t-d, h) \\ &= \sum_{k: k \neq j} \int_0^t \mu_{kj}(t, u, h, v(t)) \bar{p}_k(t, du, h) - \int_0^{t-d} \mu_{j\bullet}(t, u, h, v(t)) \bar{p}_j(t, du, h) \\ & \quad + \int_0^{t-d} \lambda_j(t, u, h-1, v(t)) \bar{p}_j(t, du, h-1) - \int_0^{t-d} \lambda_j(t, u, h, v(t)) \bar{p}_j(t, du, h) \end{aligned}$$

with boundary conditions  $\bar{p}_j(t, 0, h) = \mathbb{1}_{\{t=0\}} \mathbb{1}_{\{h=0\}} \pi(j)$ .

*Proof.* Adopting the reparameterization and change of variables from Remark 3.2.6, the result follows from Proposition 3.9 of Horning (2026), which yields a forward equation for  $(\bar{Z}, \bar{Y}, \bar{H})$ .  $\square$

Contrary to the second part of Proposition 3.2.10, the forward equation for the mean-field occupation probabilities is non-linear since the transition rates and health claim hazards depend on  $v$  (in a possibly non-linear fashion) and  $v$  is a function of the occupation probabilities themselves, confer with Proposition 3.3.8. As a consequence, we can no longer be completely assured that the equations admit a unique solution. The practical consequence is that the actuary must be particularly vigilant regarding a solution's mathematical characteristics and concrete impact.

Furthermore, the non-linearity of the equations gives reason to handle the mean-field transition probabilities with additional care. In the one-individual model, confer with Remark 3.2.5, the evolution in time of the occupation and transition probabilities was identical and, hence, we could calculate the transition probabilities using the same equations as for the occupation probabilities, but with changed initial conditions (from  $\pi(j)$  to  $\mathbb{1}_{\{i=j\}}$ ). However, as we discussed briefly at the end of Subsection 3.3.1, in the mean-field model the evolution in time depends on the initial distribution through  $v$ . The intuition is as follows. The mean-field occupation probabilities  $(\bar{p}_j)_j$  represent the occupation probabilities for a typical individual in a very large group of identically distributed individuals, who *all* have initial distribution  $\pi$  and all depend on each other through the group average. Thus changing the initial distribution  $\pi$  not only corresponds to changing the initial distribution of one individual, but it also corresponds to changing the initial

distribution of the *entire* group and thus the group average. Or, in other words, trying to equate occupation probabilities from equations with different initial conditions corresponds to equating the occupation probabilities of two individuals with different initial conditions, but also from two different groups!

In summary, we *cannot* calculate the mean-field transition probabilities by simply changing the initial (or boundary) conditions. However, if we treat  $v$ , which depends on the occupation probabilities, as fixed, we may actually calculate the mean-field transition probabilities in the usual manner.

**Proposition 3.3.11.** *Let  $d \geq 0$ . It holds almost everywhere on  $[d, \infty)$  that*

$$\begin{aligned} & \frac{d}{dt} \bar{p}_{ij}(t, t-d, h) \\ &= \sum_{k:k \neq j} \int_0^t \mu_{kj}(t, u, h, v(t)) \bar{p}_{ij}(t, du, h) - \int_0^{t-d} \mu_{j\bullet}(t, u, h, v(t)) \bar{p}_{ij}(t, du, h) \\ & \quad + \int_0^{t-d} \lambda_j(t, u, h-1, v(t)) \bar{p}_{ij}(t, du, h-1) - \int_0^{t-d} \lambda_j(t, u, h, v(t)) \bar{p}_{ij}(t, du, h) \end{aligned}$$

with boundary conditions  $\bar{p}_{ij}(t, 0, h) = \mathbb{1}_{\{t=0\}} \mathbb{1}_{\{h=0\}} \mathbb{1}_{\{i=j\}}$ .

*Proof.* Adopting the reparameterization and change of variables from Remark 3.2.6, the result follows from Proposition 3.11 of Hornung (2026), which yields a forward equation for  $(\bar{Z}, \bar{Y}, \bar{H})$ .  $\square$

The intuition behind these linearized forward equations is as follows. Imagine a very large group of  $M$  individuals of which all but one have initial distribution  $\pi$ , while the remaining individual has a degenerate initial distribution, say this individual's initial state is almost surely  $i$ . Since the individuals solely depend on each other through their group average, and the contribution of one individual to this average is negligible as  $M \rightarrow \infty$ , we may still replace the average by  $v$  – also when calculating the occupation probabilities of the remaining individual. However, for this individual the occupation probabilities now correspond to the mean-field transition probabilities  $(\bar{p}_{ij})_j$ .

Collecting results leaves us with two ways of calculating the mean-field transition probabilities. Either we first determine the mean-field occupation probabilities by solving the non-linear forward equations from Proposition 3.3.10, use these to calculate  $v$ , and then finally find the mean-field transition probabilities by solving the linearized forward equations from Proposition 3.3.11. Alternatively, we recall that

$$v(t) = \sum_i \pi(i) \sum_j \sum_{h=0}^{\infty} \int_0^t g(j, u, h) \bar{p}_{ij}(t, du, h),$$

consider the forward equations from Proposition 3.3.11 as non-linear, and solve these directly. If one is only interested in the mean-field transition probabilities  $(\bar{p}_{ij})_j$  for a specific  $i$ , the first method is to be preferred. Otherwise, neither method is inherently superior to the other and, in any case, for both methods the non-linearity entails that we may no longer be absolutely certain that the resulting solution is unique.

### 3.3.3 Statistical aspects

For most practical purposes, estimates of the (collective-dependent) health claims hazards  $(t, u, h, y) \mapsto \lambda_j(t, u, h, y)$  and transition rates  $(t, u, h, y) \mapsto \mu_{jk}(t, u, h, y)$ ,  $j \neq k$ , are required. If only a single collective is observed, identifiability of the collective effect may become particularly challenging. However, as briefly described in the introduction, we have the example of company level insurance plans in mind – with the insurer signing contracts with several (relatively independent) companies. In the following, we therefore outline how estimates may be obtained in the presence of multiple, mutually independent, groups; for notational convenience, we omit the inclusion of individual- and company-level covariates.

We begin by considering a single company consisting of  $n$  employees observed in the interval  $[0, R^n]$ , with  $R^n$  a common random time describing right-censoring of the company. Subject to classic assumptions, including *independent right-censoring*, the partial log-likelihoods

$$\begin{aligned} \log\{\mathcal{L}\} &= \sum_{\ell=1}^n \int_0^{R^n} \log\{\lambda_{Z_{t-}^{\ell,n}}(t, U_{t-}^{\ell,n}, H_{t-}^{\ell,n}, \nu_{t-}^n)\} H^{\ell,n}(dt) \\ &\quad - \sum_{\ell=1}^n \int_0^{R^n} \lambda_{Z_{t-}^{\ell,n}}(t, U_{t-}^{\ell,n}, H_{t-}^{\ell,n}, \nu_{t-}^n) dt, \\ \log\{\mathcal{L}_{jk}\} &= \sum_{\ell=1}^n \int_0^{R^n} \log\{\mu_{jk}(t, U_{t-}^{\ell,n}, H_{t-}^{\ell,n}, \nu_{t-}^n)\} N_{jk}^{\ell,n}(dt) \\ &\quad - \sum_{\ell=1}^n \int_0^{R^n} \mu_{jk}(t, U_{t-}^{\ell,n}, H_{t-}^{\ell,n}, \nu_{t-}^n) \mathbb{1}_{\{Z_{t-}^{\ell,n}=j\}} dt, \end{aligned}$$

offer a reasonable starting point for inference, confer with Section III.4 in Andersen et al. (1993).

In the presence of multiple, mutually independent, companies, the relevant partial log-likelihoods are simply sums of the each company's contribution. Temporarily discretizing the transition rates and health claims hazards using a grid with time, duration, etc., usually produces a good approximation, and the resulting expressions correspond to Poisson likelihoods with occurrences and exposures as one might

expect. Therefore, estimates of health claims hazards and transition rates may be obtained non-parametrically, semi-parametrically, or parametrically using standard techniques for occurrence and exposure data.

### 3.4 Practical implementation

The expected accumulated cash flows and prospective reserves may be computed from the transition and occupation probabilities via numerical integration and, for instance, using the trapezoidal rule. It is the computation of probabilities based on forward integro-differential equations that requires special attention. In the following, we briefly describe how the meta-algorithm of Buchardt, Møller, and Schmidt (2015) can be adapted to be fit for purpose for the task at hand. We focus on the system of Proposition 3.3.10; the other systems of integro-differential equations are, ultimately, special cases or of significantly less sophistication.

Obviously,  $\bar{p}_j(t, t-d, h) = 0$  for  $d > t$  and  $\bar{p}_j(t, t-d, h) = \bar{p}_j(t, t, h)$  for  $d < 0$ . We therefore for  $\eta > 0$  with  $T/\eta \in \mathbb{N}$  consider the discretization  $\mathcal{D}$  of  $\{(t, d, h) \in [0, T]^2 \times \mathbb{N}_0 : d \leq t\}$  consisting of points  $(\eta m, \eta n, h)$  for  $n, m, h \in \mathbb{N}_0$  with  $n \leq m \leq T/\eta$ .

The goal is to calculate  $(\bar{p}_j)_j$  on  $\mathcal{D}$ . This first involves selecting a cut-off  $K_H$  and for all  $j$  equating  $\bar{p}_j(\cdot, \cdot, h)$  with zero for  $h > K_H$ . To select the cut-off, one may look for a deterministic constant  $\tilde{\lambda}$  which uniformly bounds the health claims hazards on  $[0, T]$ , and then for an error threshold  $\text{err} > 0$  select  $K_H = \inf\{K \in \mathbb{N}_0 : \mathbb{P}(\tilde{H} > K) < \text{err}\}$ , where  $\tilde{H} \sim \text{Poisson}(\tilde{\lambda}T)$ . Next, one can apply the following meta-algorithm:

**Initial stage (0).** The boundary conditions yield the values  $\bar{p}_j(0, 0, h) = \mathbb{1}_{\{h=0\}}\pi(j)$  for all  $j$  and all  $h$ .

**Subsequent stages ( $m+1$ ).** The non-linear integro-differential equations of Proposition 3.3.10 together with the formula for  $v$  of Proposition 3.3.8 may be used to calculate

$$\begin{aligned} &\bar{p}_j(\eta(m+1), \eta(m+1) - d, h) \\ &\text{for all } j \text{ and all } h \text{ and } d \in \{0, \eta, \eta 2, \dots, \eta(m+1)\} \end{aligned}$$

based on

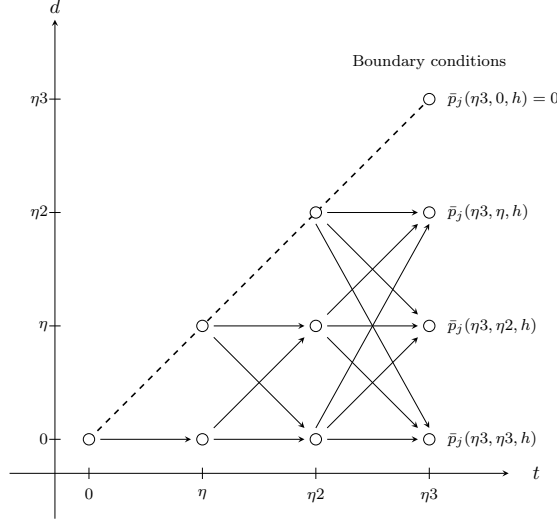
$$\bar{p}_j(\eta m, \eta m - d, h) \text{ for all } j \text{ and all } h \text{ and } d \in \{0, \eta, \eta 2, \dots, \eta m\}$$

and the boundary conditions

$$\bar{p}_j(\eta(m+1), 0, h) = 0 \text{ for all } j \text{ and all } h.$$

This can be done using for instance Euler steps, taking care of the integrals via the trapezoidal rule. Efficiency may be gained by storing and reusing computations related to the integrals.

The sequence of stages for  $m = 0, 1, 2, 3$  is illustrated in Figure 3.2, which mirrors Figure 2 in Buchardt, Møller, and Schmidt (2015). The time complexity given a cut-off  $K_H$  is of the same order as in the classic semi-Markov disability model, but differs by about a factor of  $K_H$ .



**Figure 3.2:** Illustration of a sequence of stages in the meta-algorithm.

### 3.5 Simulation study

The purpose of this section is to assess the quality of the mean-field approximation through a practice-oriented simulation study. In Subsection 3.5.1, we outline the specific model and its connection to practice. Subsection 3.5.2 concerns the quality of mean-field reserves compared to a naïve Monte Carlo approach, while Subsection 3.5.3 presents some additional insights on the nature and the convergence of the mean-field approximation. Finally, in Subsection 3.5.4 we discuss how to adapt the model to also handle lapse risk – from a practical perspective.

#### 3.5.1 Setup

In order to specify a concrete model, we must specify the initial distribution, transition rates  $(t, u, h, y) \mapsto \mu_{jk}(t, u, h, y)$ ,  $j \neq k$ , health claims hazards  $(t, u, h, y) \mapsto \lambda_j(t, u, h, y)$ , and the function  $g$ . For the initial distribution, we assume for simplicity that individuals are independent and active at time zero. For the health claims hazards, we follow Example 3.2.7 and choose  $\lambda_j(t, u, h, y) \equiv \lambda_j$  with parameter values

$$\lambda_1 = 0.2, \quad \lambda_2 = 0.3, \quad \lambda_3 = 0,$$

implying in particular that health claims are 50% more likely while disabled than active. For the transition rates, we choose

$$\begin{aligned}\mu_{12}(t, y) &= e^{-9.55+0.24(t+45)-0.0046(t+45)^2+0.000036(t+45)^3} e^{\beta \min\left\{\frac{1}{1+t}(y+\zeta_1)-\zeta_1, \zeta_0\right\}}, \\ \mu_{13}(t) &= 0.0005 + 10^{5.52+0.038(t+45)-10}, \\ \mu_{21}(t, u) &= e^{2.11-0.039(t+45)-1.44u}, \\ \mu_{23}(t, u) &= 0.0005 + 10^{5.52+0.038(t+45)-10} + e^{-2.79-0.23u}.\end{aligned}$$

In particular, solely the disability rate depends on the collective and solely the recovery rate and the disability mortality depend on duration. The collective effect on the disability rate is included via the term

$$\beta \min\left\{\frac{1}{1+t}(y + \zeta_1) - \zeta_1, \zeta_0\right\}$$

with parameter values

$$\beta = 2, \quad \zeta_1 = 0.1, \quad \zeta_0 = 0.4.$$

Finally, we follow Examples 3.2.13 and 3.3.9 and choose  $g(z, u, h) \equiv \max\{h, \kappa_H\}$  with  $\kappa_H = 100$ , meaning that the dependence on the collective stems practically speaking only from the average of health claims. This corresponds to the average and expectation

$$\nu_t^n = \frac{1}{n} \sum_{\ell=1}^n \max\{H_t^{\ell,n}, \kappa_H\}, \quad v(t) = \sum_j \sum_{h=0}^{\infty} \max\{h, \kappa_H\} \bar{p}_{1j}(t, t, h)$$

for the  $n$ -individual and mean-field model, respectively.

The collective health claims influence the disability rate  $\mu_{12}$  by means of a credibility factor, taking into account time passed. Inserting  $y = \nu^n$ , we identify the term

$$\frac{1}{1+t}(\nu_t^n + \zeta_1) = \frac{t}{t+1} \frac{\nu_t^n}{t} + \frac{1}{t+1} \zeta_1$$

as a credibility formula between the collective rate  $\nu_t^n/t$  and a baseline  $\zeta_1$ . Consequently,

$$\frac{1}{1+t}(\nu_t^n + \zeta_1) - \zeta_1$$

yields the deviation from the baseline. At time zero, where no collective information is available, all weight is placed on the baseline  $\zeta_1$ . As time passes – and more and more information becomes available – more and more weight is based on the collective rate  $\nu_t^n/t$ . However, by introducing a maximum positive deviation given by the parameter  $\zeta_0$ , we ensure that in no case can the deviation exceed  $\zeta_0$ . Finally, the parameter  $\beta$  controls the influence of health claims on the disability rate.

The parametrizations mirror forms seen in practice, and the parameter values are chosen to obtain rates which are reasonable for an individual of age 45 years. Select parameter values are collected in Table 3.1. It is worth noting that the baseline  $\zeta_1$  is chosen quite low compared to the health claims rates  $\lambda_1, \lambda_2, \lambda_3$ .

| Parameter   | Value | Description                                |
|-------------|-------|--------------------------------------------|
| $\lambda_1$ | 0.2   | Health claims rate, active                 |
| $\lambda_2$ | 0.3   | Health claims rate, disabled               |
| $\lambda_3$ | 0     | Health claims rate, dead                   |
| $\zeta_1$   | 0.1   | Collective health claims effect, baseline  |
| $\zeta_0$   | 0.5   | Collective health claims effect, maximum   |
| $\beta$     | 2     | Collective health claims effect, influence |

**Table 3.1:** *Select parameter values for the simulation study.*

Following Example 3.2.8, the contractual payments correspond to a disability annuity with a waiting period. That is, we consider

$$dB_t^{\ell,n} = \mathbb{1}_{(Z_{t-}^{\ell,n}=2)} b_2(t, U_{t-}^{\ell,n}) dt$$

with  $b_2(t, u) = \mathbb{1}_{(u \geq \varepsilon)} b$ . We choose  $b = 1$  and  $\varepsilon = 0.25$ , the latter corresponding to the rather common waiting period of three months. Finally, we choose  $r = 0.01$  and  $T = 25$ .

### 3.5.2 Main results

Since solving the forward integro-differential equations for the  $n$ -individual model is not computationally feasible for  $n \gg 1$ , we mainly consider two ways of calculating the reserve  $V^{1,n}$ :

1. By means of a naïve Monte Carlo method, repeatedly simulating the  $n$ -individual model
2. Employing the mean-field approximation  $\bar{V} \approx V^{1,n}$  and solving the resulting non-linear integro-differential equations.

In the naïve Monte Carlo method, we repeatedly sample paths of the process  $X^n = (X^{1,n}, \dots, X^{n,n})$  via inhomogeneous Poisson processes using the by now classic acceptance-rejection method described in Lewis and Shedler (1979). Denoting the samples by  $m = 1, \dots, M$ , this yields the estimate

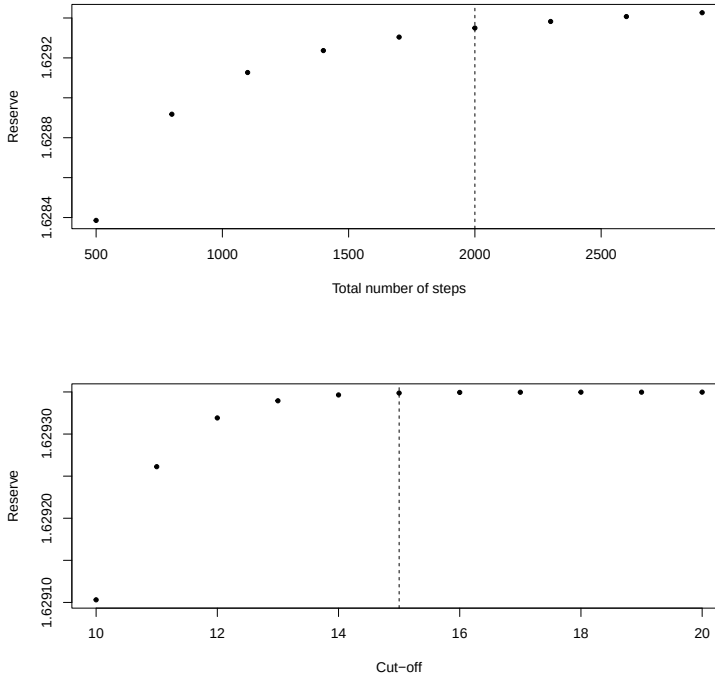
$$\frac{1}{M} \sum_{m=1}^M \left( \frac{1}{n} \sum_{\ell=1}^n \int_0^T e^{-\int_0^t r(s) ds} B^{\ell,n,m}(dt) \right),$$

which for  $n \gg 1$  should have substantially lower variance than the neonatal estimate

$$\frac{1}{M} \sum_{m=1}^M \int_0^T e^{-\int_0^t r(s) ds} B^{1,n,m}(dt)$$

due to chaosticity.

To solve the integro-differential equations, compute the expected accumulated cash flows, and finally calculate the reserves, we adopt the implementation outlined in Section 3.4. In particular, we employ the Euler method and, for numerical integration, the trapezoidal rule. The numerical methods are implemented in the programming language R. For the choice of cut-off,  $K_H = 20$  is more than sufficient since the health claims hazards are bounded by  $\tilde{\lambda} = 0.3$  and with  $\tilde{H} \sim \text{Poisson}(\tilde{\lambda}T)$  the excess probability  $\mathbb{P}(\tilde{H} > 20)$  is negligible. In Figure 3.3, we illustrate the convergence of the numerical method in terms of the total number of steps and the cut-off  $K_H$ . The top plot shows the computed reserve for a fixed cut-off  $K_H = 20$  and various total number of steps. Based hereon, we select a step length of 0.0125 corresponding to a total number of steps of 2,000. The bottom plot shows the computed reserve for this step length and various cut-offs. Based hereon, we select a cut-off of  $K_H = 15$ .



**Figure 3.3:** Convergence of the numerical method as a function of the total number of steps with the fixed cut-off  $K_H = 20$  (top) and as a function of the cut-off with 2,000 total number of steps (bottom); the dotted lines indicate the values selected for the subsequent analyses.

For  $n > 1$  we only consider the mean-field approximation. For  $n = 1$  we also consider the one-individual model, yielding the ‘true’ value for the reserve. We present our main results in Table 3.2; for the Monte Carlo method, a large sample size of  $M = 40,000$  is chosen. The first observation is that both the mean-field approximation and the Monte Carlo estimate deviate substantially from the true value for the one-individual model. While this is to be expected from the mean-field approximation, and indicates that the collective effect on the disability rate is non-negligible, it also signifies that the Monte Carlo estimate has not converged for  $n = 1$ . For  $n = 2, 5$  we continue to see substantial differences between the mean-field approximation and the Monte Carlo estimates, while for  $n = 25, 50, 100$  the differences are less pronounced. The seemingly increased stability of the Monte Carlo estimates for larger  $n$  are due to the aforementioned chaoticity-induced variance reduction.

| $n$ | Mean-field | Monte Carlo | True  |
|-----|------------|-------------|-------|
| 1   | 1.629      | 1.647       | 1.668 |
| 2   | 1.629      | 1.651       | —     |
| 5   | 1.629      | 1.661       | —     |
| 25  | 1.629      | 1.633       | —     |
| 50  | 1.629      | 1.630       | —     |
| 100 | 1.629      | 1.629       | —     |

**Table 3.2:** Reserves  $V^{1,n}$  computed using the mean-field approximation and a naïve Monte Carlo method (with sample size  $M = 40,000$ ), respectively, for  $n = 1, 2, 5, 25, 50, 100$ .

| $n$ | Mean-field | Monte Carlo |
|-----|------------|-------------|
| 1   | 1m34s      | 30s         |
| 2   | 1m34s      | 45s         |
| 5   | 1m34s      | 1m49s       |
| 25  | 1m34s      | 10m10s      |
| 50  | 1m34s      | 22m58s      |
| 100 | 1m34s      | 52m30s      |

**Table 3.3:** Runtimes using the mean-field approximation and a naïve Monte Carlo method (with sample size  $M = 40,000$ ), respectively, for  $n = 1, 2, 5, 25, 50, 100$ .

To further compare the performance of the mean-field approximation to the naïve Monte Carlo method, we include in Table 3.3 runtimes corresponding to the computations in Table 3.2. The runtime measurements are derived from non-optimized code executed on a hardware configuration with an Intel(R) Core(TM) i7-8700 CPU (using one of six cores) as well as 2x16 GB DDR4 2400 MHz RAM. Proper code optimization should lead to significant improvements for both the mean-field approximation and the Monte Carlo method, but perhaps to a lesser degree for the latter due to its inherent simplicity. In terms of runtimes, we see that the Monte Carlo method (using the large sample size of  $M = 40,000$ ) remains

competitive for  $n = 5$  but already for  $n = 25$  is much slower than the mean-field approximation.

| $n$ | Second lowest | Average | Second highest | Standard deviation |
|-----|---------------|---------|----------------|--------------------|
| 2   | 1.632         | 1.651   | 1.673          | 0.0122             |
| 5   | 1.627         | 1.643   | 1.661          | 0.0087             |
| 25  | 1.626         | 1.633   | 1.639          | 0.0035             |

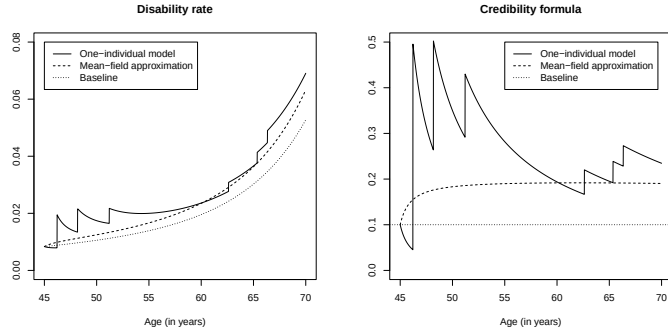
**Table 3.4:** *Statistics for 50 repeated applications of the naïve Monte Carlo method to estimate  $V^{1,n}$  with  $n = 2, 5, 25$ .*

The deviations between the mean-field approximation and the Monte Carlo estimates for moderate  $n$  could indicate that the mean-field approximation is somewhat poor, that the Monte Carlo estimate has yet to fully converge, or both. Table 3.4 contains statistics for 50 repeated applications of the naïve Monte Carlo method with  $n = 2, 5, 25$ . The quantiles and the standard deviations both indicate substantial variance in the Monte Carlo estimates, and for  $n = 5, 25$  the mean-field approximation is contained in the empirical 90 % confidence intervals. This confirms that  $M = 40,000$  does not suffice to ensure the convergence of the Monte Carlo estimates. Therefore, we must conclude that the mean-field approximation constitutes a necessary, and rather efficient, alternative already for moderate  $n$ .

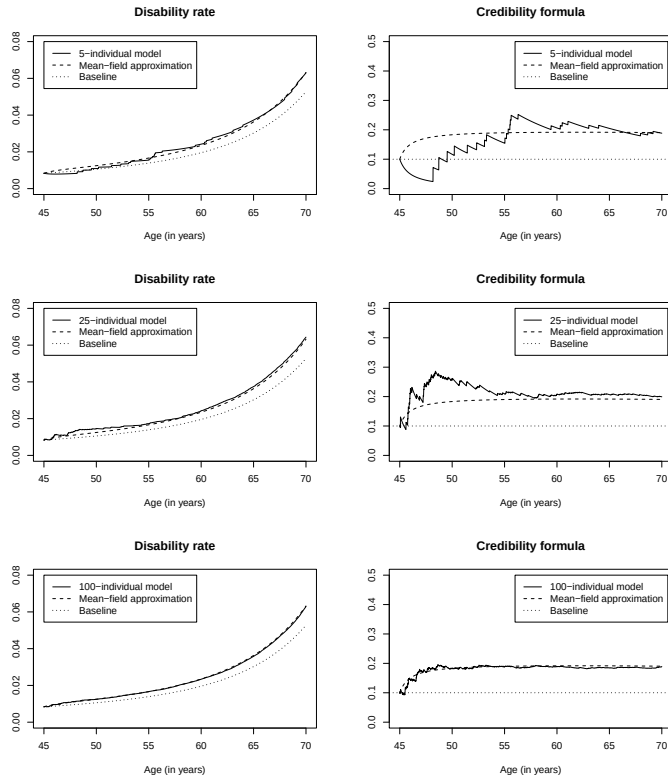
### 3.5.3 Further findings

The reserve in the one-individual model is about 2.38 % larger than the mean-field reserve. This is because in the one-individual model, the effect of health claims on the disability rate is calculated based only on the health claims history of a single individual, meaning that the (likely) occurrence of just one health claim causes the disability rate to spike quite violently upwards, confer also with Figure 3.4. Since this is not similarly counteracted by downwards movements, there is a larger probability of disability in the one-individual model compared to the mean-field approximation and consequently also a larger reserve.

The mean-field convergence implies that  $\nu^n \rightarrow v$  as  $n \rightarrow \infty$ . This convergence is neatly illustrated in Figure 3.5, which mirrors Figure 3.4, but has  $n = 5, 25, 100$  rather than  $n = 1$ . Substantial deviations in disability rate appear even for  $n = 25$ .



**Figure 3.4:** Disability rate (left) and credibility formula (right) for a single realization of the one-individual model ( $y = \nu^1$ ) with the mean-field approximation ( $y = v$ ) and the baseline ( $y = \zeta_1$ ).

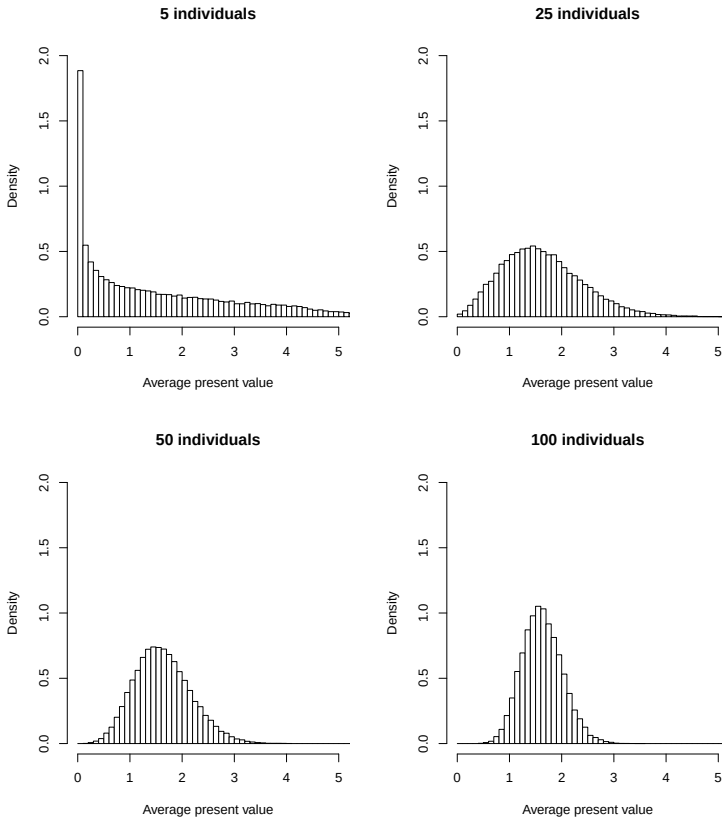


**Figure 3.5:** For  $n = 5, 25, 100$  disability rates (left) and credibility formulas (right) for a single realization of the  $n$ -individual model ( $y = \nu^n$ ) with the mean-field approximation ( $y = v$ ) and the baseline ( $y = \zeta_1$ ).

To assess the quality of the convergence, we may additionally study histograms of the average present values

$$\frac{1}{n} \sum_{\ell=1}^n \int_0^T e^{-\int_0^t r(s) ds} B^{\ell,n,m}(dt)$$

for  $m = 1, \dots, M$ , where  $M = 40,000$ . The resulting histograms for  $n = 5, 25, 50, 100$  can be found in Figure 3.6. For  $n \geq 50$  the clear shape of a bell curve emerges, which would seem to indicate that besides the laws of large numbers already covered in Section 3.3, a central limit theorem might also hold. Proposition 6.7 in Hornung (2026) confirms exactly such a central limit theorem, but subject to a covariance condition that is difficult to verify theoretically. Figure 3.6 offers empirical support for the conjecture that, in this specific model, the condition is met.



**Figure 3.6:** Histograms of average present values for  $n = 5, 25, 50, 100$ ; as  $n$  increases, a clear bell curve emerges.

### 3.5.4 Inclusion of lapse risk

Throughout, we have made the assumption that the group of insured is closed – that is, members do not leave and new members do not join. This is not realistic for the application we have in mind, where group members constitute employees and the group a specific employer. In this subsection, we study the impact of lapse risk on the calculation of the reserve; besides lapse risk affecting the individual, lapses may change the group composition and hereby the collective effect in the disability rate.

We suppose that lapses occur solely from state 1, that is when the individual is active, and denote the rate of lapse by  $t \mapsto \alpha(t)$ . We consider three different ways of including lapse risk:

**Approach A – individually.** Lapse risk affects only the individual and not the collective: We first calculate  $\bar{p}$  and  $v$  excluding lapse risk, that is with  $\mu_{1\bullet}$  unchanged, and then we repeat the calculation of  $\bar{p}$  using this  $v$  and including lapse risk, that is adding  $\alpha$  to  $\mu_{1\bullet}$ .

**Approach B – collectively.** Lapse risk affects both the individual and the collective: We calculate  $\bar{p}$  and  $v$  including lapse risk, that is adding  $\alpha$  to  $\mu_{1\bullet}$ .

**Approach C – collectively with adjustment.** Lapse risk affects both the individual and the collective, but with an adjustment to account for the effect it has on the group composition: We calculate  $\bar{p}$  and  $v$  including lapse risk, that is adding  $\alpha$  to  $\mu_{1\bullet}$ , but we adjust the formula of  $v$  according to

$$v(t) = \frac{\sum_j \sum_{h=0}^{\infty} \max\{h, \kappa_H\} \bar{p}_{1j}(t, t, h)}{\sum_j \sum_{h=0}^{\infty} \bar{p}_{1j}(t, t, h)}.$$

Note that – since  $\bar{p}$  has been calculated with  $\alpha$  added solely to the diagonal element  $\mu_{1\bullet}$  – the denominator is less than one.

Recall that  $v$  is, practically speaking, describing the expected number of health claims of a typical individual in the group. Introducing lapse risk, one might ask: What do we understand by ‘group’ and does its composition change over time? If we do not adjust the formula of  $v$ , the one understands the group as composed of both lapsed individuals (with future health claims unrecorded) and individuals yet to lapse. If we adjust the formula of  $v$  as in Approach C above, one understands the group as solely composed of the individuals yet to lapse.

In Table 3.5, we present mean-field approximations of the reserve for constant lapse rates of 0.02, 0.05, and 0.1. We see, as one would expect, that the reserve decreases as the lapse rate increases. The inclusion of lapse risk produces similar results in Approach A and C, while Approach B admits even lower reserves. This

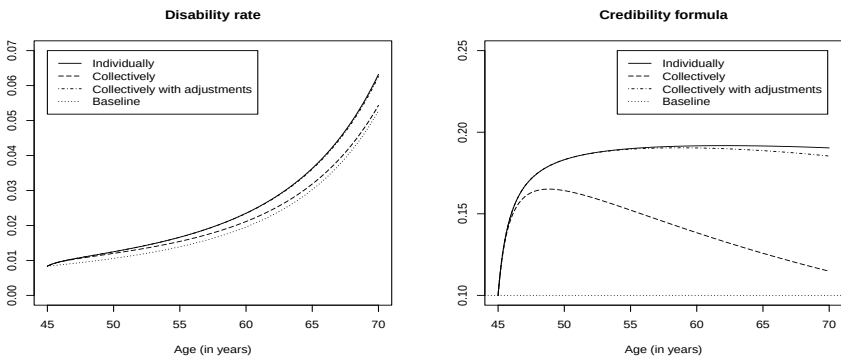
is because increasing the lapse rate not only decreases the likelihood that the individual utilizes their disability insurance, but also decreases the collective effect on the disability rate through the unadjusted  $v$ .

| $\alpha$ | Individually | Collectively | Collectively with adjustment |
|----------|--------------|--------------|------------------------------|
| 0 %      | 1.629        | 1.629        | 1.629                        |
| 2 %      | 1.311        | 1.271        | 1.310                        |
| 5 %      | 0.974        | 0.916        | 0.973                        |
| 10 %     | 0.639        | 0.586        | 0.637                        |

**Table 3.5:** Reserves  $V^{1,n}$  computed using the mean-field approximation and including lapse risk across Approach A, B, and C.

To further illuminate the different approaches, we plot in Figure 3.7 the disability rates and credibility formulas for a lapse rate of 0.05. The observed effects are consistent with the results for the reserve; however, we see that although the reserves in Approach A and C are virtually indistinguishable, the underlying credibility formulas are indeed different. This is because the implicitly assumed group compositions are different, with there being a higher proportion of disabled and dead individuals in Approach C compared to Approach A; this is due to the fact that only active individuals may lapse.

The most important conclusion to draw from the findings presented in this subsection is the following. There are indeed discrepancies in the reserves calculated across Approach A, B, and C, and these discrepancies stem from differences in implicit assumptions regarding the composition of the group. Which approach is appropriate depends on how changes to the composition of the group have been taken into account in the estimation of the disability intensity. This is about properly adjusting  $\nu^n$  in the presence of samling effects so it remains observable yet still related to the estimand of interest.



**Figure 3.7:** Disability rate (left) and credibility formula (right) for a lapse rate of 0.05 and across Approach A, B, and C together with the baseline.

### 3.6 Outlook

In Subsection 3.5.4, we gave a practical perspective on how to account for lapse risk. The idea was to make plausible that the methods developed in this paper for closed groups may also be adapted to the situation where members leave and new members join. From a more theoretical perspective, to capture such policyholder behavior and sampling effects, one can expand the state-space to encompass group entries and exits. In principle, this is not at odds with the mean-field theory or the likelihood-based estimation, except that one – as illustrated in Subsection 3.5.4 – may have to adjust  $\nu^n$  and  $v$  to the situation at hand. The detailed mathematical and statistical analysis of such adjustments, including alternative choices of  $\nu^n$ , constitute an interesting avenue for future work.

In this paper, we focus on valuation at contract inception by firmly placing ourselves at initial time  $t_0 = 0$  in the calculation of reserves, and we make the simplifying assumption that  $U_0^{\ell,n} = H_0^{\ell,n} = 0$  and thus  $\bar{U}_0 = \bar{H}_0 = 0$ . To establish mean-field convergence, we only needed the assumption of  $\pi$ -chaoticity for  $(Z_0^{1,n}, \dots, Z_0^{n,n})(\mathbb{P})$  for some  $\pi$ . If we instead place ourselves at initial time  $t_0 > 0$ , the values  $U_{t_0}^{\ell,n}$ ,  $H_{t_0}^{\ell,n}$ ,  $\bar{U}_{t_0}$ , and  $\bar{H}_{t_0}$  are random. If everything develops according to the model, one might like to initialize with the distribution of  $(X_{t_0}^{1,n}, \dots, X_{t_0}^{n,n})$ . It can be shown from Theorem 3.3.5 and properties of the total variance distance that the distribution of  $(X_{t_0}^{1,n}, \dots, X_{t_0}^{n,n})$  is  $\bar{X}_{t_0}(\mathbb{P})$ -chaotic. Consequently, the distribution of  $(X_{t_0}^{1,n}, \dots, X_{t_0}^{n,n})$  constitutes a theoretically sound choice of initial distribution, and most of the results for  $t_0 = 0$  thus also hold for  $t_0 > 0$ . However, it should be noted that it may be technically problematic to condition on null-sets – such as a specific duration – which poses a challenge for the state-wise quantities, confer also with the discussion between Corollary 5.5 and Theorem 5.6 in Horning (2026). In practice, however, this can be resolved by instead conditioning on a sufficiently small duration interval.

### Acknowledgments and declarations of interest

Both authors have carried out this research in association with the project frame InterAct. The expressed opinions are attributable solely to us and do not necessarily reflect the views of any of our past, current, and future employers.

## Chapter 4

# Robust control for marked point processes under transition rate uncertainty

This chapter contains the manuscript *Desmettre and Hornung (2026)*.

### ABSTRACT

We consider a novel robust utility maximisation problem under bounded cumulative transition rate uncertainty within the class of non-Markovian marked point processes on a finite state-space. Utility is maximised over the class of admissible controls, while Nature chooses a worst-case biometric scenario from the class of admissible, path-dependent cumulative transition rates restricted by path-dependent upper and lower bounds. We prove a martingale optimality principle and a novel existence and uniqueness result for a non-standard worst-case backwards stochastic differential equation, which allows us to establish existence and uniqueness of worst-case and best-case prospective reserves of life and health insurance contracts with reserve-dependent payments. Finally, we find an explicit solution of a novel robust consumption-insurance problem with power utility preferences.

**Keywords:** Robust reserving; Non-Markovian multi-state models; worst-case prospective reserves; robust consumption-insurance; biometric risk; model uncertainty

### 4.1 Introduction

Multi-state models are a classical and widely used framework for the stochastic modelling of an individual's evolution through a finite set of biometric, contractual, or economic states. This makes multi-state models particularly suitable for the valuation of life and health insurance contracts and the study of an individual's optimal choice of consumption and insurance when exposed to biometric risk,

within a unified continuous-time framework. Originally such multi-state models were developed for the purpose of valuation of life and health insurance contracts within the class of continuous-time Markov chains with finite state-spaces, see Hoem (1969), Norberg (1991), and Milbrodt and Stracke (1997), but in parallel the class of continuous-time Markov chains also came to play a central role in the literature on optimal consumption-insurance problems under biometric risk, see Richard (1975), Pliska and Ye (2007), and Kraft and Steffensen (2008).

Defining for the class of Markovian multi-state models is that the biometric scenario is modelled by deterministic, time-dependent cumulative transition rates, or, in the smooth case, deterministic, time-dependent transition intensities. However, as the cumulative transition rates for health-related risks such as disability, long-term care, and unemployment may depend on the duration already spent in the current state, or on other features of the observed history, the Markov assumption is often too restrictive for accurate modelling. Thus recent literature on the valuation of life and health insurance contracts has moved towards more general multi-state models, such as semi-Markov and duration-dependent models, Christiansen (2012), Buchardt, Møller, and Schmidt (2015), and Adékambi and Christiansen (2017), or so-called non-Markov models, Christiansen and Djehiche (2020) and Christiansen and Furrer (2021), where the biometric scenario may be modelled by path-dependent cumulative transition rates dependent on the entire observed history, see Christiansen and Furrer (2024). This move has been mirrored by the statistics literature for multi-state models, see Putter and Spitoni (2018), Nießl et al. (2023), and Bladt and Furrer (2024).

Another difficulty, which is common for all multi-state models, and the main focus of this paper, is model uncertainty. In actuarial practice, regardless of what class of multi-state models is used, the cumulative transition rates have to be estimated from finite, heterogeneous and often non-stationary data sets. This motivates robust or safe-side approaches in which one considers a whole class of admissible models whose cumulative transition rates are allowed to vary within a set of plausible alternatives. Within the class of Markovian models Christiansen (2010) studies the worst-case valuation of multi-state life and health insurance contracts, given deterministic upper and lower bounds for the cumulative transition rates, while Christiansen and Steffensen (2013) and Christiansen et al. (2016) extend this approach to joint financial and biometric worst-case scenarios. Within the literature on optimal consumption and insurance problems Han and Hung (2021) consider the robust optimal choice of consumption and annuity demand under mortality rate uncertainty within a Markovian survival model. In all of these contributions, the class of alternative models is restricted to Markovian models, and extensions allowing for non-Markovian alternative models are, to the best of our knowledge, absent from the literature. Thus, motivated by the recent shift towards non-Markov models by not only the actuarial and statistical literature, but

also actuarial practice, we aim to provide such a non-Markovian extension.

In this paper, we formulate a general max-min objective which aims at maximising a path-dependent utility process under the least favourable admissible biometric scenario, where the admissible biometric scenarios are described by path-dependent lower and upper bounds on the cumulative transition rates of a finite state non-Markovian marked point process. To formulate our problem, we use the canonical measurable space of non-explosive marked point process paths without trivial jumps on a finite state space, as constructed by Christiansen and Furrer (2024). This canonical construction has the crucial advantage that it separates the measurable path space from the choice of probability measure, since, as shown by Christiansen and Furrer (2024), each unique set of cumulative transition rates induces a unique probability measure on the same canonical measurable space. In particular, this setup does not require that all admissible probability measures have to be equivalent.

After formulating our max-min objective, we prove a martingale optimality principle which provides a sufficient condition of optimality and which we subsequently use to obtain verification theorems in two specialised cases. The first application of the martingale optimality principle shows that the worst- and best- case prospective reserves of an insurance contract with reserve-dependent payments are characterised by a non-standard backwards stochastic differential equation (BSDE). The second application of the martingale optimality principle yields a verification theorem for a robust and path-dependent generalisation of the consumption-insurance problem of Kraft and Steffensen (2008), where the individual aims to maximise utility of consumption and terminal wealth by choosing a consumption stream and transition-contingent insurance coverage, while Nature chooses, within the admissible upper and lower bounds, the transition intensities that minimise the individual's expected utility. In the case of power utility preferences, the classical separation Ansatz still applies, with the deterministic time-dependent function replaced by a stochastic process satisfying a BSDE of the same non-standard type as the worst- and best-case prospective reserves. This yields explicit path-dependent feedback forms for the robust optimal consumption stream and insurance coverage of the same structural form as in the non-robust Markov-chain model of Kraft and Steffensen (2008), but evaluated under worst-case preference-dependent transition intensities.

The BSDEs characterising the best- and worst-case prospective reserves and the solution of the power utility problem are non-standard in two ways. Firstly, the distribution of the jump times may have point masses, violating the assumption of inaccessible jump times. Secondly, due to the reserve-dependence of the worst/best-case cumulative transition rates, the generator of our BSDE no longer satisfies standard Lipschitz assumptions. Both assumptions are central for the existence and uniqueness results in the papers of Confortola and Fuhrman (2013), Confortola, Fuhrman, and Jacod (2016), and Christiansen and Djehiche (2020) and while Chris-

tiansen and Furrer (2024) allow for accessible jump times, they require a linear generator. Nevertheless, a key feature of the approach of Confortola, Fuhrman, and Jacod (2016) is the reduction of the BSDE to a finite or infinite system of ordinary differential equations between jumps. We similarly prove existence and uniqueness of our non-standard BSDE by first establishing equivalence to the existence and uniqueness of a solution of an infinite system of Lebesgue-Stieltjes integral equations and then by proving that this system indeed has a unique solution. This is aided by our chosen canonical setting, which allows for path-wise sure constructions.

The worst-case biometric scenario we obtain in the case of the worst-case prospective reserve is determined by the sums-at-risk. As long as a particular transition increases the prospective reserve, the upper bound is selected as the worst-case cumulative transition rate; if it decreases the prospective reserve, the lower bound is selected. Thus, our worst-case scenario can be viewed as a direct path-wise extension of Christiansen (2010), who, in the Markovian case, obtained a worst-scenario of the same structural form. In the power utility case, a similar worst-case scenario is obtained. The difference is that the worst-case cumulative transition rates are determined by the sums-at-risk of a utility-valued process and the relative risk aversion parameter.

Our robust approach is related to the broader literature on ambiguity and robust decision making. The max-min expected utility criterion originates from Gilboa and Schmeidler (1989), while robust control under model misspecification has been developed systematically by Hansen and Sargent (2008). Variational and multiple-prior approaches to ambiguity are discussed in Maccheroni, Marinacci, and Rustichini (2006). In continuous-time finance, robust portfolio and consumption problems have been studied, for example, by Maenhout (2004) and by Rieder and Woppperer (2012). In contrast to much of this literature, the model uncertainty considered here concerns the cumulative transition rates of a biometric finite-state marked point process rather than the drift or volatility of a financial diffusion.

The main contribution of the paper is twofold. First, we formulate a novel robust utility maximisation problem under bounded cumulative transition rate uncertainty and provide a sufficient condition for optimality. This allows us to characterise best- and worst-case prospective reserves under bounded cumulative transition rate uncertainty and identify the corresponding extremal biometric scenarios. Further, it allows us to characterise the solution of a robust consumption-insurance problem under transition intensity uncertainty, showing how the classical continuous-time Markov-chain solution is modified by the endogenous worst-case scenario. Second, within the framework of marked point processes on a finite state-space, we provide a novel existence and uniqueness result for a class of highly non-linear worst-case BSDEs, which allows us to provide sufficient conditions for the existence of the best- and worst-case prospective reserves and the solution of the robust

consumption-insurance problem.

The remainder of the paper is organised as follows. Section 2 introduces the canonical marked point process framework and develops necessary jump-count and state-wise decompositions of optional and predictable processes. Section 3 studies optional projections and proves the existence and uniqueness result of the worst-case BSDE. Section 4 formulates the general robust utility maximisation problem under bounded cumulative transition rate uncertainty and provides the martingale optimality principle. Section 5 characterises best- and worst-case prospective reserves. Section 6 studies the robust consumption-insurance problem, proves the verification theorem, and discusses explicit optimal controls in the power-utility case. Known technical results on finite-variation functions and Stieltjes integration are collected in the appendix for the reader's convenience.

## 4.2 The canonical probability space

In this section, we start with a description of the canonical filtered measurable space of non-explosive marked point process paths without trivial jumps on a finite state space, as constructed in Christiansen and Furrer (2024), which will be the filtered measurable space we use throughout the rest of the paper. This is followed by a discussion, which links the jump-count decomposition of optional and predictable processes known from the marked point process literature (see Jacobsen (2006) and Last and Brandt (1999)) with the state-wise decomposition of optional processes introduced in Christiansen and Furrer (2024). In particular, we introduce a unique jump-count decomposition in Proposition 4.2.3, which in Proposition 4.2.6 allows us to generalise and extend the state-wise decomposition result obtained by Christiansen and Furrer (2024). The combination of these two results links the two decompositions and enables us to seamlessly switch between them. This is crucial for our purposes because both decompositions play central (but different) roles throughout the paper. The state-wise representation is central for linking optional projections to BSDEs, and it is less unwieldy when solving the robust optimisation problems we are interested in. The jump-count representation is the key to proving the existence and uniqueness of the worst-case BSDE that characterises the worst- and best-case prospective reserves and the solution of the robust consumption-insurance problem with power utility preferences. Finally, we end this section by outlining how to endow the canonical filtered measurable space with a probability measure given by specified cumulative transition rates, as constructed in Christiansen and Furrer (2024).

### 4.2.1 The canonical filtered measurable space

Let  $\mathcal{J} \subset \mathbb{N}_0$  be a finite set. A marked point process path without trivial jumps consists of jump times  $t_0 = 0 < t_1 < t_2 < \dots$  and corresponding states  $z_0, z_1, \dots \in \mathcal{J}$  with  $z_0 \neq z_1, z_1 \neq z_2, z_2 \neq z_3, \dots$  which together form a sequence of events  $(0, z_0), (z_1, t_1), \dots$ . For our purposes, we can think of these events as the initial value and the jump times and values of a piecewise constant jump process describing, for example, the biometric state of an individual. Since we are only interested in non-explosive sequences, the total number of events may be countably infinite on the full time line  $[0, \infty)$ , but must be finite on any bounded time interval. In the case that the total number of events occurring in  $[0, \infty)$  is finite, there exists  $n \in \mathbb{N}_0$  such that the corresponding marked point process path  $(t_\ell, z_\ell)_{\ell \leq n}$  is an element of

$$\Omega^n := \{(t_\ell, z_\ell)_{\ell \leq n} \in [0, \infty)^{n+1} \times \mathcal{J}^{n+1} : 0 = t_0 < t_1 < \dots < t_n, z_\ell \neq z_{\ell+1}\}.$$

If there is an infinite number of events in  $[0, \infty)$ , then the corresponding marked point process path  $(t_\ell, z_\ell)_{\ell \in \mathbb{N}_0}$  is an element of

$$\Omega^\infty := \{(t_n, z_n)_{n \in \mathbb{N}_0} \in [0, \infty)^{\mathbb{N}_0} \times \mathcal{J}^{\mathbb{N}_0} : t_0 = 0, t_n < t_{n+1} < \infty, \\ z_n \neq z_{n+1}, \sup_n t_n = \infty\}.$$

All possible non-explosive marked point process paths without trivial jumps could thus be written as

$$\Omega^\infty \cup \bigcup_{n \in \mathbb{N}_0} \Omega^n.$$

As this is mathematically unwieldy to work with, we will extend any path with finite events to a path with countably infinite events, by extending the finite sequence with the artificial event that never happens  $(\infty, \nabla)$ . The state  $\nabla$  is unreachable, which is why its jump time is  $\infty$ . By defining the extended state space  $\bar{\mathcal{J}} := \mathcal{J} \cup \{\nabla\}$ , the canonical space of non-explosive marked point process paths without trivial jumps may be defined as

$$\Omega := \{(t_n, z_n)_{n \in \mathbb{N}_0} \in [0, \infty]^{\mathbb{N}_0} \times \bar{\mathcal{J}}^{\mathbb{N}_0} : t_n < t_{n+1} \Leftrightarrow t_n < \infty, t_n = t_{n+1} \Leftrightarrow t_n = \infty, \\ z_n \neq z_{n+1} \Leftrightarrow t_n < \infty, z_n = \nabla \Leftrightarrow t_n = \infty, \\ t_0 = 0, \sup_n t_n = \infty\},$$

which can be viewed as a subset of  $[0, \infty]^{\mathbb{N}_0} \times \bar{\mathcal{J}}^{\mathbb{N}_0}$ . Using the projection mappings

$$\tau_n : \Omega \rightarrow [0, \infty], \quad \tau_n(\omega) = t_n, \\ \zeta_n : \Omega \rightarrow \bar{\mathcal{J}}, \quad \zeta_n(\omega) = z_n,$$

we can define the canonical marked point process  $\omega \mapsto (\tau_n(\omega), \zeta_n(\omega))_{n \in \mathbb{N}_0}$  and the  $n$ -event projection mappings

$$\forall n \in \mathbb{N}_0 : \quad \xi^n : \Omega \rightarrow [0, \infty]^{n+1} \times \bar{\mathcal{J}}^{n+1}, \quad \xi^n(\omega) = (\tau_\ell(\omega), \zeta_\ell(\omega))_{0 \leq \ell \leq n},$$

which extracts the initial value and the first  $n$  jump times and states of a path  $\omega \in \Omega$ . The mapping  $\xi^n$  has the image  $\bar{\Omega}^n := \xi^n(\Omega) \subset [0, \infty]^{n+1} \times \bar{\mathcal{J}}^{n+1}$  equal to

$$\begin{aligned} \bar{\Omega}^n = \{ & (t_\ell, z_\ell)_{0 \leq \ell \leq n} \in [0, \infty]^{n+1} \times \bar{\mathcal{J}}^{n+1} : t_0 = 0, t_\ell < t_{\ell+1} \Leftrightarrow t_\ell < \infty, \\ & t_\ell = t_{\ell+1} \Leftrightarrow t_\ell = \infty, \\ & z_\ell \neq z_{\ell+1} \Leftrightarrow t_\ell < \infty, z_\ell = \nabla \Leftrightarrow t_\ell = \infty \}. \end{aligned}$$

Contrary to the set  $\Omega^n$ , which contains all possible sequences of exactly  $n$  events, the set  $\bar{\Omega}^n$  contains all possible sequences with at most  $n$  events and thus  $\Omega^n \subset \bar{\Omega}^n$ , the only exception being the case  $n = 0$ , where  $\bar{\Omega}^0 = \Omega^0 = \{0\} \times \mathcal{J}$ . A generic element of  $\Omega^n$  or  $\bar{\Omega}^n$  will be denoted by  $\omega^n = ((0, z_0), \dots, (t_n, z_n))$ .

The set  $\Omega$  can be equipped with the  $\sigma$ -algebra generated by the canonical marked point process

$$\mathcal{F} := \sigma((\tau_n, \zeta_n)_{n \in \mathbb{N}_0})$$

and the two filtrations  $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$  and  $\mathbb{F}^- := (\mathcal{F}_{t-})_{t \geq 0}$  generated by the canonical marked point process given by

$$\begin{aligned} \mathcal{F}_t &:= \sigma(\mathbb{1}_{(\tau_n \leq t)}(\tau_n, \zeta_n) : n \in \mathbb{N}_0) \\ \mathcal{F}_{t-} &:= \sigma(\mathbb{1}_{(\tau_n < t)}(\tau_n, \zeta_n) : n \in \mathbb{N}_0), \end{aligned}$$

where the former contains past and present information, while the latter only concerns itself with past information. Note that the filtration  $\mathbb{F}$  is right-continuous.

The canonical marked point process can be used to define three other types of important fundamental processes, which all generate the same above filtrations  $\mathbb{F}$  and  $\mathbb{F}^-$ . First, we consider the counting processes

$$N_{ij} : [0, \infty) \times \Omega \rightarrow \mathbb{N}_0, \quad N_{ij}(t) := \sum_{n \in \mathbb{N}} \mathbb{1}_{(\tau_n \leq t)} \mathbb{1}_{(\zeta_{n-1} = i, \zeta_n = j)}, \quad i, j \in \mathcal{J} : i \neq j,$$

which count the number of jumps from state  $i$  to state  $j$ , while

$$N : [0, \infty) \times \Omega \rightarrow \mathbb{N}_0, \quad N(t) := \sum_{n \in \mathbb{N}} \mathbb{1}_{(\tau_n \leq t)} = \sum_{i, j : i \neq j} N_{ij}(t)$$

counts the total number of jumps. Then, we consider the indicator processes

$$\begin{aligned} I^n &: [0, \infty) \times \Omega \rightarrow \{0, 1\}, \quad I^n(t) := \mathbb{1}_{(\tau_n \leq t < \tau_{n+1})}, \quad \text{for } n \in \mathbb{N}_0, \\ I_i &: [0, \infty) \times \Omega \rightarrow \{0, 1\}, \quad I_i(t) := \sum_{n \in \mathbb{N}_0} \mathbb{1}_{(\tau_n \leq t < \tau_{n+1})} \mathbb{1}_{(\zeta_n = i)}, \quad \text{for } i \in \mathcal{J}, \end{aligned}$$

which can be used to construct the jump process

$$Z : [0, \infty) \times \Omega \rightarrow \mathcal{J}, \quad Z(t) := \sum_{n \in \mathbb{N}_0} \zeta_n I^n(t) = \sum_{i \in \mathcal{J}} i I_i(t),$$

with  $Z(0-) := Z(0) = \zeta_0$ . By construction,  $Z$  has càdlàg paths of finite variation, with a finite number of jumps in finite time. It holds that

$$\begin{aligned}\mathcal{F}_t &= \sigma(\zeta_0, N_{ij}(s) : s \leq t, i, j \in \mathcal{J}, i \neq j) \\ &= \sigma(I_i(s) : s \leq t, i \in \mathcal{J}) = \sigma(Z(s) : s \leq t), \\ \mathcal{F}_{t-} &= \sigma(\zeta_0, N_{ij}(s) : s < t, i, j \in \mathcal{J}, i \neq j) \\ &= \sigma(I_i(s) : s < t, i \in \mathcal{J}) = \sigma(Z(s) : s < t).\end{aligned}$$

**Definition 4.2.1.** *The set  $\Omega$  equipped with the sigma-algebra  $\mathcal{F}$  and the filtration  $\mathbb{F}$  is the canonical filtered measurable space.*

### 4.2.2 Decompositions of optional and predictable processes

Let  $(\Omega, \mathcal{F}, \mathbb{F})$  be the canonical filtered measurable space. Each  $\mathbb{F}$ -optional or  $\mathbb{F}$ -predictable (henceforth just "optional" and "predictable") process  $X$  can be decomposed in two different ways. The first is the jump-count decomposition, which decomposes  $X$  into the family of functions  $(f^n)_{n \in \mathbb{N}_0}$ , each being a measurable function of the first  $n$  jump times and marks. The second is the state-wise decomposition, which decomposes  $X$  into a family of predictable processes  $(\tilde{X}_i)_{i \in \mathcal{J}}$ , which can be interpreted as counterfactuals in a certain sense. We start by discussing the former.

By Theorem 2.2.22 of Last and Brandt (1999), it holds that a process  $X$  is optional if and only if there exists a family of functions  $f^n : [0, \infty) \times \bar{\Omega}^n \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}_0$  such that

$$X(t, \omega) = \sum_{n \in \mathbb{N}_0} I^n(t, \omega) f^n(t, \xi^n(\omega)), \quad \forall (t, \omega) \in [0, \infty) \times \Omega, \quad (4.2.1)$$

and a process  $X$  is predictable if and only if

$$X(t, \omega) = \sum_{n \in \mathbb{N}_0} I^n(t-, \omega) f^n(t, \xi^n(\omega)), \quad \forall (t, \omega) \in [0, \infty) \times \Omega. \quad (4.2.2)$$

Thus, not only can any optional or predictable process be decomposed into a family of functions  $(f^n)_{n \in \mathbb{N}_0}$ , but this decomposition property actually characterises being optional or predictable. From (4.2.1) and (4.2.4), it can be seen that each  $f^n$  does not have to be uniquely defined on the entire domain  $[0, \infty) \times \bar{\Omega}^n$  in order for  $X$  to be unique. Since we later want to construct adapted processes by constructing the family  $(f^n)_{n \in \mathbb{N}_0}$ , it is important to know on what domain the  $f^n$  uniquely determine  $X$  and on what domain they are uniquely determined by  $X$ .

**Proposition 4.2.2.** *Let  $X$  and  $Y$  be two optional or predictable processes satisfying (4.2.1) or (4.2.2) with  $(f^n)_{n \in \mathbb{N}_0}$  and  $(g^n)_{n \in \mathbb{N}_0}$ . Then the following two statements hold:*

(i) If  $X$  and  $Y$  are optional, then  $X(t, \omega) = Y(t, \omega)$  for all  $(t, \omega) \in [0, \infty) \times \Omega$  if and only if for all  $n \in \mathbb{N}_0$ :

$$f^n(t, \omega^n) = g^n(t, \omega^n), \quad \forall \omega^n = ((t_0, z_0), \dots, (t_n, z_n)) \in \Omega^n, \forall t \geq t_n.$$

(ii) If  $X$  and  $Y$  are predictable, then  $X(t, \omega) = Y(t, \omega)$  for all  $(t, \omega) \in [0, \infty) \times \Omega$  if and only if for all  $n \in \mathbb{N}_0$ :

$$f^n(t, \omega^n) = g^n(t, \omega^n), \quad \forall \omega^n = ((t_0, z_0), \dots, (t_n, z_n)) \in \Omega^n, \forall t > t_n.$$

*Proof.* We only prove (i) as the proof of (ii) is analogous, and we start with the direction " $\Rightarrow$ ". By (4.2.1) it holds for all  $n \in \mathbb{N}_0$  and  $\omega \in \Omega$  that

$$f^n(t, \xi^n(\omega)) = X(t, \omega) = Y(t, \omega) = g^n(t, \xi^n(\omega)), \quad \tau_n(\omega) \leq t < \tau_{n+1}(\omega).$$

In particular for any  $n \in \mathbb{N}_0$  this holds for all paths of the form  $\omega = (\omega_n, (\infty, \nabla), \dots)$  with  $\omega^n \in \Omega^n$ . Thus we obtain

$$\forall n \in \mathbb{N}_0, \omega^n \in \Omega^n : \quad f^n(t, \omega^n) = g^n(t, \omega^n), \quad t_n \leq t < \infty.$$

For the direction " $\Leftarrow$ " let  $\omega \in \Omega$  be arbitrary. Note that since  $t \in [0, \infty)$ , it is only the  $f^n$  and  $g^n$  with  $n \in \{m \in \mathbb{N}_0 : \tau_m(\omega) < \infty\}$  that determine  $X(t, \omega)$  and  $Y(t, \omega)$ . By definition if  $\tau_n(\omega) < \infty$ , then  $\xi^n(\omega) \in \Omega^n$  and since for all  $n \in \mathbb{N}_0$  and  $\omega^n \in \Omega^n$  it holds that  $f^n(t, \omega^n) = g^n(t, \omega^n)$  for  $t \geq t_n$ , we have that

$$X(t, \omega) = f^n(t, \xi^n(\omega)) = g^n(t, \xi^n(\omega)) = Y(t, \omega), \quad \tau_n(\omega) \leq t < \tau_{n+1}(\omega),$$

whenever  $\tau_n(\omega) < \infty$ . Thus we can conclude  $X(t, \omega) = Y(t, \omega)$  for all  $(t, \omega) \in [0, \infty) \times \Omega$ .  $\square$

Proposition 4.2.2 shows that there are two cases in which the functions  $(f^n)_{n \in \mathbb{N}_0}$  are not uniquely determined. The first case is  $\omega^n \in \Omega^n$  and  $t < t_n$ , meaning that we would like to evaluate  $f^n$  before the  $n$ 'th jump has happened, while the second case is  $\omega^n \in \bar{\Omega}^n \setminus \Omega^n$  (implying  $t_n = \infty$ ), meaning that we would like to evaluate  $f^n$  when there never will be an  $n$ 'th jump. In order to obtain a unique decomposition  $(f^n)_{n \in \mathbb{N}_0}$  on the entire domain  $[0, \infty) \times \bar{\Omega}^n$ , we have to choose a convention for the two previously described cases. For optional processes we choose the convention,

$$f^n(t, \omega^n) = \begin{cases} f^n(t_n, \omega^n) & \text{when } \omega^n \in \Omega^n \text{ and } t < t_n \\ f^{n-1}(t, \xi^{n-1}(\omega^n)) & \text{when } \omega^n \in \bar{\Omega}^n \setminus \Omega^n \end{cases}, \quad (4.2.3)$$

and for predictable processes which have limits from the right, we choose

$$f^n(t, \omega^n) = \begin{cases} f^n(t_n+, \omega^n) & \text{when } \omega^n \in \Omega^n \text{ and } t < t_n \\ f^{n-1}(t, \xi^{n-1}(\omega^n)) & \text{when } \omega^n \in \bar{\Omega}^n \setminus \Omega^n \end{cases}. \quad (4.2.4)$$

The rationale behind this choice is very natural. In the first case, the  $n$ 'th jump has not happened yet, but will happen, so  $f^n$  equals the value that  $X$  will take, as soon as the  $n$ 'th jump happens. In the second case, the  $n$ 'th jump will never happen, which is why  $f^n$  equals the value  $X$  takes at time  $t$  after the  $n - 1$ 'st jump. Using these two conventions, we can obtain a unique jump-count decomposition.

**Proposition 4.2.3.** *Let  $X = (X_t)_{t \geq 0}$  be a real-valued stochastic process.*

(i)  *$X$  is  $\mathbb{F}$ -optional if and only if for every  $n \in \mathbb{N}_0$  there exists a unique measurable function  $f^n : [0, \infty) \times \bar{\Omega}^n \rightarrow \mathbb{R}$  satisfying (4.2.3) such that:*

$$X(t, \omega) = \sum_{n \in \mathbb{N}_0} I^n(t, \omega) f^n(t, \xi^n(\omega)), \quad \forall (t, \omega) \in [0, \infty) \times \Omega.$$

(ii) *Let  $X$  have limits from the right. Then  $X$  is  $\mathbb{F}$ -predictable if and only if for every  $n \in \mathbb{N}_0$  there exists a unique measurable function  $f^n : [0, \infty) \times \bar{\Omega}^n \rightarrow \mathbb{R}$  satisfying (4.2.4) such that:*

$$X(t, \omega) = \sum_{n \in \mathbb{N}_0} I^n(t-, \omega) f^n(t, \xi^n(\omega)), \quad \forall (t, \omega) \in [0, \infty) \times \Omega.$$

*Proof.* We prove only (i), since (ii) is proven analogously. The direction " $\Leftarrow$ " follows directly from Theorem 2.2.22 of Last and Brandt (1999) and Proposition 4.2.2. For the other direction Theorem 2.2.22 of Last and Brandt (1999) yields the existence of measurable functions  $\tilde{f}^n : [0, \infty) \times \bar{\Omega}^n \rightarrow \mathbb{R}$  such that

$$X(t, \omega) = \sum_{n \in \mathbb{N}_0} I^n(t, \omega) \tilde{f}^n(t, \xi^n(\omega)), \quad \forall (t, \omega) \in [0, \infty) \times \Omega.$$

From this we define  $f^n : [0, \infty) \times \bar{\Omega}^n \rightarrow \mathbb{R}$  given by

$$f^n(t, \omega^n) := \begin{cases} \tilde{f}^n(t, \omega^n) & \text{when } \omega^n \in \Omega \text{ and } t \geq t_n \\ \tilde{f}^n(t_n, \omega^n) & \text{when } \omega^n \in \Omega \text{ and } t < t_n \\ \tilde{f}^{n-1}(t, \xi^{n-1}(\omega^n)) & \text{when } \omega^n \in \bar{\Omega}^n \setminus \Omega \end{cases}$$

Rewriting this into the form

$$\begin{aligned} f^n(t, \omega^n) &= \mathbf{1}_{\Omega}(\omega^n) \mathbf{1}_{[t_n, \infty)}(t) \tilde{f}^n(t, \omega^n) + \mathbf{1}_{\Omega}(\omega^n) \mathbf{1}_{[0, t_n)}(t) \tilde{f}^n(t_n, \omega^n) \\ &\quad + \mathbf{1}_{\bar{\Omega}^n \setminus \Omega}(\omega^n) \tilde{f}^{n-1}(t, \xi^{n-1}(\omega^n)) \end{aligned}$$

shows that  $(t, \omega^n) \mapsto f^n(t, \omega^n)$  is measurable. By Proposition 4.2.2  $\tilde{f}^n(t, \omega^n)$  is unique whenever  $\omega^n \in \Omega^n$  and  $t \geq t_n$  and thus it holds that  $f^n$  is unique in the first two cases. In the third case, there are two options. Either  $\xi^{n-1}(\omega^n) \in \Omega^{n-1}$ , in which case the value is unique, or  $\xi^{n-1}(\omega^n) \in \bar{\Omega}^{n-1} \setminus \Omega^{n-1}$ . In that case, one repeats this argument over and over again, until one reaches some  $0 \leq m < n$  such

that  $\xi^m(\omega^n) \in \Omega^m$ . If this fails for all  $1 \leq m \leq n$ , the procedure terminates at  $m = 0$  where it holds that  $\bar{\Omega}^0 = \Omega^0 = \{0\} \times \mathcal{J}$  and thus Proposition 4.2.2 yields that  $\tilde{f}^0(t, \omega^0)$  is unique for all  $(t, \omega^0) \in [0, \infty) \times \bar{\Omega}^0$ . Thus  $\tilde{f}^0$  is always uniquely determined by  $X$  on all of  $[0, \infty) \times \bar{\Omega}^0$ .  $\square$

*Remark 4.2.4.* Note that as a consequence of Proposition 4.2.2 and the convention (4.2.3) or (4.2.4),  $t \mapsto X(t, \omega)$  has càdlàg paths of finite variation for all  $\omega \in \Omega$  if and only if  $t \mapsto f^n(t, \omega^n)$  is càdlàg and of finite variation for all  $\omega^n \in \Omega^n$  and  $n \in \mathbb{N}_0$ . Similarly, any boundedness conditions uniform in  $(t, \omega)$  that  $X$  satisfies carry over to  $(f^n)_{n \in \mathbb{N}_0}$  and vice-versa.

Proposition 4.2.3 provides a unique jump-count decomposition of optional and predictable processes, and throughout the rest of the paper, when referring to the jump-count decomposition of a process, we refer to the one given by Proposition 4.2.3.

We now turn towards the state-wise decomposition of optional and predictable processes given by

$$X(t) = \sum_{i \in \mathcal{J}} I_i(t) \tilde{X}_i(t) \quad \text{or} \quad X(t) = \sum_{i \in \mathcal{J}} I_i(t-) \tilde{X}_i(t)$$

for predictable processes  $(\tilde{X}_i)_{i \in \mathcal{J}}$ . The idea is that  $\tilde{X}_i(t, \omega)$  always equals the value that  $X(t, \omega)$  would have if  $Z(t, \omega)$  were equal to  $i$ , even though  $Z(t, \omega)$  for that particular  $(t, \omega)$  might be different from  $i$ . In that sense,  $\tilde{X}_i$  can be understood as a counterfactual. Following Christiansen and Furrer (2024), the construction of such processes  $(\tilde{X}_i)_{i \in \mathcal{J}}$  requires the construction of counterfactual paths. For this define for each  $i \in \mathcal{J}$  the  $(t, i)$ -stopped path given by

$$\omega_i^t((t_\ell, z_\ell)_{\ell \in \mathbb{N}_0}) := \begin{cases} ((t_\ell, z_\ell)_{0 \leq \ell \leq n}, (\infty, \nabla), \dots), & \text{for } t_n < t \leq t_{n+1}, z_n = i \\ ((t_\ell, z_\ell)_{0 \leq \ell \leq n}, (t, i), (\infty, \nabla), \dots), & \text{for } t_n < t \leq t_{n+1}, z_n \neq i \\ ((0, i), (\infty, \nabla), \dots), & \text{for } t = 0 \\ ((t_\ell, z_\ell)_{\ell \in \mathbb{N}_0}), & \text{for } t = \infty. \end{cases}$$

If the current state at time  $t$  is equal to  $i$ , then all future jumps are removed, and the path remains in state  $i$  forever. If the current state at time  $t$  is different from  $i$ , then a jump to state  $i$  at time  $t$  is added, and all future jumps are removed. Let  $\omega_{i,j}^{t_1, t_2}$  for  $t_1 \leq t_2$  denote the composition  $\omega_j^{t_2}(\omega_i^{t_1})$ , which first adds a jump to  $i$  at time  $t_1$  (if the state at  $t_1$  is different from  $i$ ) followed by a jump to  $j$  at time  $t_2$ . Using this, we construct the two counterfactuals

$$X_i(t, \omega) := \sum_{n \in \mathbb{N}} I^n(t-) X(t, \omega_i^{\tau_n}(\omega)), \quad (4.2.5)$$

$$X_{ij}(t, \omega) = \sum_{n \in \mathbb{N}_0} I^n(t-) X(t, \omega_{i,j}^{\tau_n, t}(\omega)). \quad (4.2.6)$$

For any  $t \geq 0$  the process  $X_i(t)$  equals the value  $X(t)$  would have whenever  $Z(t) = i$  for  $t$  in between jump times, while  $X_{ij}(t)$  equals the value  $X(t)$  would have if  $Z$  were to jump from state  $i$  to state  $j$  at time  $t$ . Invoking Proposition 4.2.3 we can also write

$$\begin{aligned} X_i(t, \omega) &= \sum_{n \in \mathbb{N}_0} I^n(t-) f^n(t, \xi^n(\omega_i^{\tau_n})) \\ X_{ij}(t, \omega) &= \sum_{n \in \mathbb{N}_0} I^n(t-) f^{n+1}(t, \xi^{n+1}(\omega_{i,j}^{\tau_n, t})), \end{aligned} \quad (4.2.7)$$

which shows that both (4.2.5) and (4.2.6) define predictable processes. In fact, they are not only predictable, but  $i$ -predictable:

**Definition 4.2.5.** *A process  $X$  is  $i$ -predictable if  $X(t)$  is deterministic for  $0 \leq t \leq \tau_1$  and  $\sigma((\tau_\ell, \zeta_\ell)_{\ell \leq n-1}, \mathbb{1}_{(\zeta_{n-1} \neq i)} \tau_n)$ -measurable for  $\tau_n < t \leq \tau_{n+1}$  for all  $n \in \mathbb{N}$ .*

This is slightly stricter than just being predictable, since by Proposition 4.2.3(ii) any predictable process is  $\sigma((\tau_\ell, \zeta_\ell)_{\ell \leq n})$ -measurable when  $\tau_n < t \leq \tau_{n+1}$ . In the case of (4.2.5) and (4.2.6), we have set  $\zeta_n = i$  whenever  $\zeta_n \neq i$  or removed the  $n$ 'th jump whenever  $\zeta_n = i$ , thus in both cases we have removed the dependence on  $\zeta_n$  and in the last case we have removed the dependence on  $\tau_n$ . By p.7 of Christiansen and Furrer (2024) and Proposition 4.2.3(ii) a process  $X$  with jump-count decomposition  $(f^n)_{n \in \mathbb{N}_0}$  being  $i$ -predictable is equivalent to

$$X(t, \omega) = X(t, \omega_i^{\tau_n}) \quad \text{and} \quad f^n(t, \xi^n(\omega)) = f^n(t, \xi^n(\omega_i^{\tau_n})), \quad (4.2.8)$$

for all  $n \in \mathbb{N}_0$ ,  $\omega \in \Omega$  and  $\tau_n(\omega) < t \leq \tau_{n+1}(\omega)$ . Using the concept of  $i$ -predictability, we now get the following state-wise decomposition result.

**Proposition 4.2.6.** *Let  $X$  be a real-valued stochastic process.*

- (i)  *$X$  is  $\mathbb{F}$ -optional if and only if there for all  $i \in \mathcal{J}$  exist unique  $i$ -predictable processes  $X_i$  and  $(X_{ij})_{j:j \neq i}$  satisfying (4.2.5) and (4.2.6) such that*

$$X(t) = \sum_{i \in \mathcal{J}} I_i(t) \tilde{X}_i(t) \quad \text{with} \quad \tilde{X}_i(t) = I_i(t-) X_i(t) + \sum_{j:j \neq i} I_j(t-) X_{ji}(t).$$

- (ii)  *$X$  is  $\mathbb{F}$ -predictable if and only if there for all  $i \in \mathcal{J}$  exist a unique  $i$ -predictable process  $X_i$  satisfying (4.2.5) such that*

$$X(t) = \sum_{i \in \mathcal{J}} I_i(t-) X_i(t)$$

*Proof.* (i): We start with the direction " $\Rightarrow$ ". For this let  $t \in [0, \infty)$  be arbitrary and let  $(X_i, (X_{ij})_{j:j \neq i})_{i \in \mathcal{J}}$  be the  $i$ -predictable processes uniquely defined by (4.2.5) and (4.2.6). It remains to check whether the decomposition holds. We begin with

the case where  $Z$  does not have a jump at time  $t$ . Then for all  $n \in \mathbb{N}_0$  the definition of  $\tilde{X}_i$ , Equation (4.2.7) and Proposition 4.2.3(i) yields

$$\tilde{X}_i(t, \omega) = X_i(t, \omega) = f^n(t, \xi^n(\omega_i^{\tau_n})) = f^n(t, \xi^n(\omega)) = X(t, \omega),$$

for all  $\omega \in (\tau_n < t < \tau_{n+1}, \zeta_n = i)$ , since  $\xi^n(\omega_i^{\tau_n}) = \xi^n(\omega)$  when  $\zeta_n = i$ . It remains to check the case where  $Z$  has a jump at time  $t$ . Then for all  $n \in \mathbb{N}_0$  and  $j \neq i$  we have by the definition of  $\tilde{X}_i$ , Equation (4.2.7) and Proposition 4.2.3(i) that

$$\tilde{X}_i(t, \omega) = X_{ji}(t, \omega) = f^n(t, \xi^n(\omega_{j,i}^{\tau_{n-1}, t})) = f^n(t, \xi^n(\omega)) = X(t, \omega),$$

for all  $\omega \in (\tau_n = t, \zeta_{n-1} = j, \zeta_n = i)$ , as  $\xi^n(\omega_{j,i}^{\tau_{n-1}, t}) = \xi^n(\omega)$  when  $\zeta_{n-1} = j$ ,  $\zeta_n = i$  and  $\tau_n = t$ . This concludes the proof of the direction " $\Rightarrow$ ". For the direction " $\Leftarrow$ ", note that  $X$  defined by the decomposition consists of sums and products of optional processes yielding optionality of  $X$  itself. Finally the definition of  $X$  and  $\tilde{X}_i$  in conjunction with (4.2.8) yields  $X(t, \omega_i^{\tau_n}) = X_i(t, \omega_i^{\tau_n}) = X_i(t, \omega)$  and

$$X(t, \omega_{i,j}^{\tau_n, t}) = X_{ij}(t, \omega_{i,j}^{\tau_n, t}) = X_{ij}(t, \omega_i^{\tau_n}(\omega_{i,j}^{\tau_n, t})) = X_{ij}(t, \omega)$$

for all  $\tau_n < t \leq \tau_{n+1}$  and all  $n \in \mathbb{N}_0$ . This concludes the proof of (i).

(ii): We start with the direction " $\Rightarrow$ ". Let  $t \in [0, \infty)$  be arbitrary and the  $i$ -predictable  $X_i$  be given by (4.2.5). Then for all  $n \in \mathbb{N}_0$  Equation (4.2.7) and Proposition 4.2.3(ii) yields

$$X_i(t, \omega) = f^n(t, \xi^n(\omega_i^{\tau_n})) = f^n(t, \xi^n(\omega)) = X(t, \omega),$$

for all  $\omega \in (\tau_n < t \leq \tau_{n+1}, \zeta_n = i)$ , as we again have  $\xi^n(\omega_i^{\tau_n}) = \xi^n(\omega)$ , concluding the proof of the direction " $\Rightarrow$ ". The direction " $\Leftarrow$ " is proven as in (i).  $\square$

*Remark 4.2.7.* Note that  $\tilde{X}_i(t)$  as defined in Proposition 4.2.6(i) is equal to  $X(t, \omega_i^t)$ , which is how the state-wise decomposition is defined in Proposition 6.3 of Christiansen and Furrer (2024). They do not explicitly discuss the decomposition of predictable processes, but since any predictable process  $X$  is optional, and since  $I_i(t-) \tilde{X}_i(t) = I_i(t-) X_i(t)$  for all  $t \geq 0$ , the decomposition  $X(t) = \sum_{i \in \mathcal{J}} I_i(t-) \tilde{X}_i(t)$  would be just as valid for predictable  $X$ . But unlike  $X_i(t)$ , the process  $\tilde{X}_i(t)$  is only predictable and not  $i$ -predictable, thus using superfluous information. In that sense using the decomposition  $(X_i)_{i \in \mathcal{J}}$  is minimal.

*Remark 4.2.8.* Note that since  $t \mapsto I_i(t-, \omega)$  and  $t \mapsto I^n(t-, \omega)$  for  $i \in \mathcal{J}$  and  $n \in \mathbb{N}_0$  are not right-continuous in the points  $t \in \{\tau_n : n \in \mathbb{N}_0\}$ , the paths  $t \mapsto X_i(t, \omega)$ ,  $t \mapsto X_{ij}(t, \omega)$  and  $t \mapsto \tilde{X}_i(t, \omega)$  might not be right-continuous in those same points, even though  $t \mapsto X(t, \omega)$  is. But due to (4.2.7) and Remark 4.2.4 it always holds that  $(\tilde{X}_i)_{i \in \mathcal{J}}$ ,  $(X_i, (X_{ij})_{j: j \neq i})_{i \in \mathcal{J}}$  are of finite variation, whenever  $X$  is of finite variation.

Proposition 4.2.6 provides a unique state-wise decomposition of optional and predictable processes, and throughout the rest of the paper, when referring to the state-wise decomposition of a process, we refer to the one given by Proposition 4.2.6.

Having introduced both the jump-count representation of Proposition 4.2.3 and the state-wise representation of Proposition 4.2.6, we now restrict ourselves to càdlàg processes of finite variation. In this case, one can obtain dynamic (in the sense of Lebesgue-Stieltjes integration) versions of the decomposition result. Throughout the paper we use for any subintervals  $I, J \subseteq [0, \infty)$  the shorthand notation

$$X(dt) = H(t)Y(dt) \quad \forall t \in I \quad \Leftrightarrow \quad \int_J F(dt) = \int_J H(t)Y(dt), \quad \forall J \subseteq I.$$

**Proposition 4.2.9.** *Let  $X$  be a real-valued càdlàg, finite variation process.*

(i) *If  $X$  is adapted with jump-count decomposition  $(f^n)_{n \in \mathbb{N}_0}$ , then*

$$X(dt) = \sum_{n \in \mathbb{N}_0} I^n(t-) \left( f^n(dt, \xi^n) + \sum_{j: j \neq \zeta_n} \tilde{\Delta} f_{\zeta_n j}^{n+1}(t, \xi^n) N_{\zeta_n j}(dt) \right),$$

*for all  $t \in (0, \infty)$ , where  $\tilde{\Delta} f_{\zeta_n j}^{n+1}(t, \xi^n) := f^{n+1}(t, \xi^{n+1}(\omega_{\zeta_n j}^{\tau_n, t})) - f^n(t, \xi^n)$ .*

(ii) *If  $X$  is predictable with jump-count decomposition  $(f^n)_{n \in \mathbb{N}_0}$ , then*

$$X(dt) = \sum_{n \in \mathbb{N}_0} I^n(t-) f^n(dt, \xi_n), \quad \forall t \in (0, \infty).$$

*Proof.* (i): Applying integration by parts to the decomposition of Proposition 4.2.3(i) we arrive at

$$X(dt) = \sum_{n \in \mathbb{N}_0} I^n(t-) f^n(dt, \xi^n) + \sum_{n \in \mathbb{N}_0} f^n(t, \xi^n) I^n(dt), \quad \forall t \in (0, \infty).$$

Utilising that  $I^n(dt) = (I^{n-1}(t-) - I^n(t-))N(dt)$  we obtain

$$X(dt) = \sum_{n \in \mathbb{N}_0} I^n(t-) \left( f^n(dt, \xi^n) + (f^{n+1}(t, \xi^{n+1}) - f^n(t, \xi^n)) N(dt) \right),$$

for all  $t \in (0, \infty)$ . The desired result then follows from the fact that  $I^n(t-)N(dt) = I^n(t-) \sum_{j: j \neq \zeta_n} N_{\zeta_n j}(dt)$  and the fact that  $\xi^{n+1}(\omega_{\zeta_n j}^{\tau_n, t}) = \xi^{n+1}$  when  $\Delta N_{\zeta_n j}(t) = 1$ .

(ii): Since any predictable process is optional, (i) holds true, and we obtain

$$\begin{aligned} X(t) - X(0) - \int_{(0, t]} \sum_{n \in \mathbb{N}_0} I^n(s-) f^n(ds, \xi^n) \\ = \sum_{0 < s \leq t} \sum_{n \in \mathbb{N}_0} I^n(s-) \sum_{j: j \neq \zeta_n} \tilde{\Delta} f_{\zeta_n j}^{n+1}(s, \xi^{n+1}) \Delta N_{\zeta_n j}(s), \end{aligned}$$

for  $t \geq 0$ . The left-hand side is predictable, and consequently, the right-hand side must be as well. However as the processes  $\Delta N_{ij}$  are optional but not predictable, the quantities  $\tilde{\Delta} f_{ij}^{n+1}(s, \xi^{n+1})$  must be zero whenever  $\Delta N_{ij}(s) \neq 0$ , implying that the entire right-hand side must be zero.  $\square$

**Proposition 4.2.10.** *Let  $X$  be a real-valued, càdlàg finite variation process.*

(i) *If  $X$  is adapted with state-wise decomposition  $(\tilde{X}_i)_{i \in \mathcal{J}}$  then*

$$\begin{aligned} X(dt) &= \sum_{i \in \mathcal{J}} I_i(t-) \left( \tilde{X}_i(dt) + \sum_{j: j \neq i} \tilde{X}_j(t) - \tilde{X}_i(t) N_{ij}(dt) \right) \\ &= \sum_{i \in \mathcal{J}} I_i(t-) \left( X_i(dt) + \sum_{j: j \neq i} \tilde{\Delta} X_{ij}(t) N_{ij}(dt) \right), \quad \forall t \in (0, \infty), \end{aligned}$$

where  $\tilde{\Delta} X_{ij}(t) := X_{ij}(t) - X_i(t)$ .

(ii) *If  $X$  is predictable with state-wise decomposition  $(X_i)_{i \in \mathcal{J}}$  then*

$$X(dt) = \sum_{i \in \mathcal{J}} I_i(t-) \tilde{X}_i(dt) = \sum_{i \in \mathcal{J}} I_i(t-) X_i(dt), \quad \forall t \in (0, \infty).$$

*Proof.* Note that by (4.2.7), Proposition 4.2.9(ii), and the fact that  $\sum_{i \in \mathcal{J}} I_i(t-) = 1$  we obtain

$$\begin{aligned} I_i(t-) I^n(t-) f^n(dt, \xi^n) &= I_i(t-) I^n(t-) f^n(dt, \xi^n(\omega_i^{\tau_n})) \\ &= I_i(t-) I^n(t-) \tilde{X}_i(dt) = I_i(t-) I^n(t-) X_i(dt) \end{aligned}$$

By (4.2.7) and the definition of  $\tilde{X}_i$  we obtain that

$$\begin{aligned} I_i(t-) I^n(t-) (f^{n+1}(t, \xi^{n+1}(\omega_{i,j}^{\tau_n, t})) - f^n(t, \xi^n)) &= I_i(t-) I^n(t-) (X_{ij}(t) - X_i(t)) \\ &= I_i(t-) I^n(t-) (\tilde{X}_j(t) - \tilde{X}_i(t)). \end{aligned}$$

The desired results follow now directly from Proposition 4.2.9.  $\square$

### 4.2.3 The probability measure

In order to specify a probability measure on the filtered canonical measurable space  $(\Omega, \mathcal{F}, \mathbb{F})$ , we will use the approach developed in Christiansen and Furrer (2024), which constructs a probability measure by specifying the cumulative transition rates and an initial distribution.

**Definition 4.2.11.** *Given a probability measure  $\mathbb{P}$  the predictable processes  $(\Lambda_{ij})_{i \neq j}$  with  $\Lambda_{ij} : [0, \infty) \times \Omega \rightarrow [0, \infty)$  are cumulative transition rates if for all  $n \in \mathbb{N}_0$*

$$I_i(t-) \Lambda_{ij}(dt) = I_i(t-) \frac{\mathbb{E}[\mathbf{1}_{(\tau_{n+1} \geq t)} N_{ij}(dt) | \mathcal{F}_{\tau_n}]}{\mathbb{E}[\mathbf{1}_{(\tau_{n+1} \geq t)} | \mathcal{F}_{\tau_n}]}, \quad \tau_n < t \leq \tau_{n+1}. \quad (4.2.9)$$

Note that (4.2.9) does not uniquely define a set of processes, which then are the cumulative transition rates, as it does not define the values of  $\Lambda_{ij}$  whenever  $Z(t-) \neq i$ . As an example, let  $C^j$  be the unique predictable compensator of  $N_j(t) = \sum_{i:i \neq j} N_{ij}(t)$  given by

$$C^j(dt) = \sum_{i:i \neq j} I_i(t-) \Lambda_{ij}(dt), \quad C^j(0) = 0.$$

By Proposition 4.2.10(ii) and Remark 4.2.7 it holds that

$$C^j(dt) = \sum_{i:i \neq j} I_i(t-) \tilde{C}_i^j(dt) = \sum_{i:i \neq j} I_i(t-) C_i^j(dt), \quad C^j(0) = 0,$$

meaning that both  $(\tilde{C}_i^j)_{i:i \neq j}$  and  $(C_i^j)_{i:i \neq j}$  for  $j \in \mathcal{J}$  satisfy Definition 4.2.11. Therefore, we have to choose a unique set of processes  $(\Lambda_{ij})_{i \neq j}$ , which then should end up satisfying Definition 4.2.11. Thus following Christiansen and Furrer (2024) we make the following assumptions about  $(\Lambda_{ij})_{i \neq j}$ :

**Assumption 4.2.12.** *Assume that*

- (i) *For each  $i \in \mathcal{J}$  the processes  $\Lambda_i$  and  $(\Lambda_{ij})_{j:j \neq i}$  are  $i$ -predictable.*
- (ii)  *$t \mapsto \Lambda_{ij}(t, \omega)$  is non-negative, non-decreasing and càdlàg with*

$$\sum_{j \in \mathcal{J}} \Delta \Lambda_{ij}(t, \omega) \leq 1, \quad \forall t \in [0, \infty), \omega \in \Omega.$$

- (iii) *For each  $i, j \in \mathcal{J}$  with  $i \neq j$  there exists a non-decreasing, non-negative càdlàg function  $\alpha_{ij} : [0, \infty) \rightarrow [0, \infty)$ , with  $\alpha_{ij}(0) = 0$  and bounded on compact sets, such that*

$$\Lambda_{ij}(s, \omega) - \Lambda_{ij}(t, \omega) \leq \alpha_{ij}(s) - \alpha_{ij}(t), \quad \forall 0 \leq t \leq s, \forall \omega \in \Omega.$$

*Remark 4.2.13.* Note that compared to Christiansen and Furrer (2024), we make more restrictive assumptions, as we do not allow the cumulative transition rates to have poles followed by reset points. The exclusion of poles and reset points brings our setup closer to the marked point process literature Jacobsen (2006) and Last and Brandt (1999), which does not include poles and reset points.

The  $i$ -predictability of Assumption 4.2.12(i) in conjunction with (4.2.8) ensures that

$$\sum_{n \in \mathbb{N}_0} I_n(t-) C^j(dt, \omega_i^{\tau_n}) = \sum_{n \in \mathbb{N}_0} I_n(t-) \Lambda_{ij}(dt, \omega_i^{\tau_n}) = \Lambda_{ij}(dt, \omega),$$

for all  $(t, \omega) \in [0, \infty) \times \Omega$ , which therefore guarantees that our chosen cumulative transition rates  $(\Lambda_{ij})_{i:i \neq j}$  for each  $j \in \mathcal{J}$  equal the state-wise decomposition of the compensator  $C_j$ , while using the minimal amount of information, see also 4.2.7.

Assumption 4.2.12(ii) ensures that the transition probabilities are always in the interval  $[0, 1]$ , while Assumption 4.2.12(iii) ensures non-explosiveness. Theorem 4.1 of Christiansen and Furrer (2024) now immediately gives the following result:

**Theorem 4.2.14.** *Let  $\pi : \mathcal{J} \rightarrow [0, 1]$  with  $\sum_{i \in \mathcal{J}} \pi(i) = 1$  and assume that  $(\Lambda_{ij})_{i \neq j}$  satisfy Assumption 4.2.12. Then there exists a unique probability measure  $\mathbb{P}^\Lambda$  and unique  $\mathcal{F}_{t-}$ -measurable probability kernels  $(\mathbb{P}_{t,i}^\Lambda)_{t \in [0, \infty), i \in \mathcal{J}}$  on  $(\Omega, \mathcal{F})$  such that*

(i) *For all  $i \in \mathcal{J}$  it holds that  $\mathbb{P}^\Lambda(Z(0) = i) = \pi(i)$ .*

(ii) *For all  $A \in \mathcal{F}$  and  $i \in \mathcal{J}$  it holds almost surely that*

$$I_i(t) \mathbb{P}_{t,i}^\Lambda(A) = I^i(t) \mathbb{P}^\Lambda(A | \mathcal{F}_t).$$

(iii) *Let  $\mathbb{E}_{t,i}^\Lambda$  denote the expectation under  $\mathbb{P}_{t,i}^\Lambda$ . Then for all  $i \neq j$  the processes  $(\Lambda_{ij})_{i \neq j}$  satisfy*

$$\Lambda_{ij}(dt) = \frac{\mathbb{E}_{\tau_n, i}^\Lambda[\mathbb{1}_{(\tau_{n+1} \geq t)} N_{ij}(dt)]}{\mathbb{E}_{\tau_n, i}^\Lambda[\mathbb{1}_{(\tau_{n+1} \geq t)}]}, \quad \tau_n < t \leq \tau_{n+1}, \quad \forall n \in \mathbb{N}_0.$$

Using the kernels  $\mathbb{P}_{t,i}^\Lambda$  we can for any  $\mathcal{F}$ -measurable random variable  $Y$ , which for all  $i \in \mathcal{J}$  is  $\mathbb{P}_{t,i}$ -integrable, define

$$\mathbb{E}_{t, Z(t)}^\Lambda[Y] := \sum_{i \in \mathcal{J}} I_i(t) \mathbb{E}_{t,i}^\Lambda[Y],$$

which has the following properties:

**Proposition 4.2.15.** *Let  $\pi$  be any initial distribution, let  $\Lambda = (\Lambda_{ij})_{i \neq j}$  satisfy Assumption 4.2.12 and let  $Y$  be an  $\mathcal{F}$ -measurable random variable.*

(i) *If  $Y$  is  $\mathbb{P}_{t,i}$ -integrable for all  $i \in \mathcal{J}$  then*

$$\mathbb{E}_{t, Z(t)}^\Lambda[Y] = \mathbb{E}^\Lambda[Y | \mathcal{F}_t], \quad \mathbb{P}^\Lambda - a.s.$$

(ii) *If  $Y$  is  $\mathcal{F}_{t-}$ -measurable and  $\mathbb{P}_{t,i}$ -integrable for all  $i \in \mathcal{J}$  then*

$$\forall i \in \mathcal{J} : \mathbb{E}_{t,i}^\Lambda[Y](\omega) = Y(\omega), \quad \forall \omega \in \Omega.$$

(iii) *If  $Y$  is  $\mathcal{F}_t$ -measurable and  $\mathbb{P}_{t,i}$ -integrable for all  $i \in \mathcal{J}$  then*

$$\begin{aligned} \forall i \in \mathcal{J} : \mathbb{E}_{t,i}^\Lambda[Y](\omega) &= \tilde{Y}_i(\omega), \quad \forall \omega \in \Omega, \\ \mathbb{E}_{t, Z(t)}^\Lambda[Y](\omega) &= Y(\omega), \quad \forall \omega \in \Omega. \end{aligned}$$

*Proof.* (i): This follows from Theorem 4.2.14(ii) by splitting  $Y$  in a positive and negative part, approximating each from below by an increasing sequence of simple

functions, and applying the dominated convergence theorem (for both conditional and unconditional expectations).

(ii): This follows from Proposition 4.4(i) of Christiansen and Furrer (2024) by splitting  $Y$  in a positive and negative part, approximating each from below by an increasing sequence of simple functions, and applying the dominated convergence theorem.

(iii): By Proposition 4.2.6(i), the just proven (ii) and Proposition 4.4(i) of Christiansen and Furrer (2024) it holds that

$$\mathbb{E}_{t,i}^\Lambda[Y] = \sum_{j \in \mathcal{J}} \mathbb{E}_{t,i}^\Lambda[I_j(t)\tilde{Y}_j] = \sum_{j \in \mathcal{J}} \tilde{Y}_j \mathbb{E}_{t,i}^\Lambda[I_j(t)] = \sum_{j \in \mathcal{J}} \tilde{Y}_j \mathbf{1}_{j=i} = \tilde{Y}_i, \quad \forall i \in \mathcal{J},$$

which is the first statement. Using this, we thus obtain

$$\mathbb{E}_{t,Z(t)}^\Lambda[Y] = \sum_{i \in \mathcal{J}} I_i(t) \mathbb{E}_{t,i}^\Lambda[Y] = \sum_{i \in \mathcal{J}} I_i(t) \tilde{Y}_i = Y.$$

□

The non-explosiveness assumption, Assumption 4.2.12(iii), implies the following integrability condition.

**Proposition 4.2.16.** *Let  $c > 0$ . Then for any  $\Lambda = (\Lambda_{ij})_{i \neq j}$  satisfying Assumption 4.2.12 it holds that*

$$\sup_{0 \leq t \leq s \leq T} \sup_{\omega \in \Omega} \mathbb{E}_{t,i}^\Lambda[e^{c(N(s)-N(t))}](\omega) < \infty, \quad \forall i \in \mathcal{J}.$$

*Proof.* By the change of variables formula for càdlàg finite variation functions, we obtain

$$e^{c(N(s)-N(t))} = 1 + \int_{(t,s]} (e^c - 1) e^{c(N(u-)-N(t))} N(du).$$

Taking the expectation and applying Proposition 4.2 of Christiansen and Furrer (2024) in conjunction with the monotone convergence theorem yields

$$\mathbb{E}_{t,i}^\Lambda[e^{c(N(s)-N(t))}] = 1 + \mathbb{E}_{t,i}^\Lambda \left[ \int_{(t,s]} (e^c - 1) e^{c(N(u-)-N(t))} \sum_{i,j:i \neq j} \Lambda_{ij}(du) \right]$$

Using Assumption 4.2.12(iii) and Tonelli's theorem we get

$$\mathbb{E}_{t,i}^\Lambda[e^{c(N(s)-N(t))}] \leq 1 + \int_{(t,s]} \mathbb{E}_{t,i}^\Lambda[e^{c(N(u-)-N(t))}] (e^c - 1) \sum_{i,j:i \neq j} \alpha_{ij}(du).$$

Set  $\alpha(t) := (e^c - 1) \sum_{i,j:i \neq j} \alpha_{ij}(t)$ . An application of Gronwall's inequality 4.A.7 yields

$$\mathbb{E}_{t,i}^\Lambda[e^{c(N_{ij}(s)-N_{ij}(t))}] \leq \frac{\mathcal{E}_\alpha(s)}{\mathcal{E}_\alpha(t)} \leq \mathcal{E}_\alpha(T).$$

As this bound does not depend on  $s, t$  or  $\omega$ , taking the suprema yields the desired result.  $\square$

In many practical applications, one assumes the cumulative transition rates to be absolutely continuous with respect to the Lebesgue measure, that is

$$\Lambda_{ij}(dt) = \mu_{ij}(t)dt, \quad t \in [0, \infty), i, j \in \mathcal{J}, i \neq j \quad (4.2.10)$$

for non-negative,  $i$ -predictable transition intensities  $(\mu_{ij})_{i \neq j}$ . In order for  $(\Lambda_{ij})_{i \neq j}$  defined as in (4.2.10) to satisfy Assumption 4.2.12 and Theorem 4.2.14 to remain valid, the following set of assumptions for  $(\mu_{ij})_{i \neq j}$  is sufficient:

**Assumption 4.2.17.** *Assume that*

- (i) *For any  $i \in \mathcal{J}$  the processes  $(\mu_{ij})_{j:j \neq i}$  are  $i$ -predictable.*
- (ii) *For each  $i, j$  with  $i \neq j$  there exists a measurable, non-negative function  $\alpha_{ij} : [0, \infty) \rightarrow [0, \infty)$  bounded on compact sets such that  $\mu_{ij}(t, \omega) \leq \alpha_{ij}(t)$  for all  $t \geq 0$  and  $\omega \in \Omega$ .*

Assumption 4.2.17(i)+(ii) guarantee the validity of Assumption 4.2.12(i)+(iii), while the non-negativity of the  $(\mu_{ij})_{i \neq j}$  and (4.2.10) implies the automatic satisfaction of Assumption 4.2.12(ii).

**Example 4.2.18** (Smooth semi-Markov process). In the case of the smooth semi-Markov process, the cumulative transition rates are of the form

$$\Lambda_{ij}(dt) = \mu_{ij}(t)dt \quad \text{with} \quad \mu_{ij}(t) = \tilde{\mu}_{ij}(t, t - \tau(t-)),$$

where  $\tilde{\mu}_{ij} : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  is a non-negative measurable function bounded on compact sets and

$$\tau(t) := \sum_{n \in \mathbb{N}_0} I_n(t) \tau_n$$

is the most recent jump time. Thus, the transition intensities are allowed to depend on the duration  $U(t) := t - \tau(t)$  since the last jump. The issue here is that  $\mu_{ij}$  defined in this way is not  $i$ -predictable. This is best seen by finding a violation of (4.2.8). Evaluating  $\tau(t-, \omega)$  in  $\omega_i^{\tau_n}$  yields

$$\forall n \in \mathbb{N}_0 : \quad \tau(t-, \omega_i^{\tau_n}) = \mathbb{1}_{(\zeta_{n-1}=i)} \tau_{n-1} + \mathbb{1}_{(\zeta_{n-1} \neq i)} \tau_n, \quad \tau_n < t \leq \tau_{n+1},$$

as setting  $\zeta_n$  to be equal to  $i$  when  $\zeta_{n-1}$  already equals  $i$  removes the  $n$ 'th jump. We therefore obtain

$$\mu_{ij}(t) = \tilde{\mu}_{ij}(t, t - \tau_n) \neq \tilde{\mu}_{ij}(t, t - \tau_{n-1}) = \mu_{ij}(t, \omega_i^{\tau_n}),$$

whenever  $\tau_n < t \leq \tau_{n+1}$  and  $\zeta_{n-1} = i$ , which violates (4.2.8). The process  $\mu_{ij}$  is not its own counterfactual. Instead, we can via (4.2.7) define

$$\tau_i(t) := I^0(t-) \tau_0 + \sum_{n \in \mathbb{N}} I^n(t-) (\mathbb{1}_{(\zeta_{n-1}=i)} \tau_{n-1} + \mathbb{1}_{(\zeta_{n-1} \neq i)} \tau_n), \quad i \in \mathcal{J},$$

and set  $\mu_{ij}(t) := \tilde{\mu}_{ij}(t, t - \tau_i(t))$ , which is  $i$ -predictable.

### 4.3 Optional projections

The central object of the robust optimisation problems we are interested in, are quantities such as the expected present value of future insurance payments or the expected present value of future utility. In both cases, the mathematical object of interest is thus the optional projection of either the (discounted) insurance payment process or the (discounted) utility process. As we are using the canonical filtered probability space, the filtration and the probability space are not complete, and thus the standard results guaranteeing the existence of the optional projection do not apply. Therefore, in Subsection 4.3.1 we start by showing in Proposition 4.3.2 that the optional projection exists under minimal regularity conditions. This is followed by Theorem 4.3.5 and Theorem 4.3.6, which for two different types of processes show that the optional projection is characterised by a BSDE. Note that Theorem 4.3.5 and Theorem 4.3.6 are closely related to Theorem 6.4 and Theorem 7.2 of Christiansen and Furrer (2024). While they formulate their result in terms of state-wise conditional expectation processes and stochastic Thiele equations, we use the optional projection and BSDEs and generalise the type of process that the optional projection can be taken of. We end Subsection 4.3.1 by using the link between the jump-count and state-wise decomposition established in Section 4.2 to show in Proposition 4.3.7 and Proposition 4.3.8 that the existence and uniqueness of the BDSs obtained in Theorem 4.3.5 and Theorem 4.3.6 is equivalent to the existence and uniqueness of an infinite-dimensional system of Lebesgue-Stieltjes integral equations. In Subsection 4.3.2 we wish to specify the optional projection of a process, where both the cumulative transition rates determining the probability measure under which the optional projection is taken and the process which the optional projection is taken of can depend on the optional projection itself. This circular case brings us to the two main results of this section, Theorem 4.3.10 and Theorem 4.3.11. They show existence and uniqueness of two worst-case BSDEs that characterise this type of circular optional projection, which will turn out to characterise the worst- and best-case prospective reserves of life insurance contracts.

#### 4.3.1 Optional projections and BSDEs

Let  $(\Omega, \mathcal{F}, \mathbb{F})$  be the canonical filtered measurable space, let  $\pi$  be any initial distribution and let  $\Lambda = (\Lambda_{ij})_{i \neq j}$  be any set of cumulative transition rates satisfying

Assumption 4.2.12. We endow the space  $(\Omega, \mathcal{F}, \mathbb{F})$  with the probability measure  $\mathbb{P}^\Lambda$  generated by the initial distribution  $\pi$  and the cumulative transition rates  $\Lambda = (\Lambda_{ij})_{i \neq j}$  as given by Theorem 4.2.14 and let  $(\mathbb{P}_{t,i}^\Lambda)_{t \in [0, \infty), i \in \mathcal{J}}$  be the associated kernels.

Let  $T > 0$  be a deterministic terminal time and for a càdlàg process of finite variation  $C$  and an  $\mathcal{F}_T$ -measurable random variable  $Y$  we define the process  $L(t, T)$  by

$$L(t, T) := C(T) - C(t) + Y, \quad t \in [0, T].$$

We are now interested in the optional projection of  $L(t, T)$ , that is a càdlàg process  $X$  which satisfies

$$X(t) = \mathbb{E}^\Lambda[L(t, T) | \mathcal{F}_t] \quad \mathbb{P}^\Lambda - \text{a.s.} \quad \forall t \in [0, T].$$

As the filtered probability space  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\Lambda)$  is not complete, it does not satisfy the usual conditions, and therefore we cannot rely on the standard results which guarantee the existence of such an optional projection. We therefore aim to show that the process

$$X(t) := \mathbb{E}_{t, Z(t)}^\Lambda[L(t, T)] = \sum_{i \in \mathcal{J}} I_i(t) \mathbb{E}_{t,i}^\Lambda[L(t, T)]$$

is an optional projection of  $L(t, T)$ . We start with the simple case  $L(t, T) = Y$  and prove the following extension of the martingale representation result of Remark 6.5 in Christiansen and Furrer (2024).

**Proposition 4.3.1.** *Let  $Y$  be an  $\mathcal{F}_T$ -measurable, real-valued random variable such that*

$$\forall i \in \mathcal{J} : \quad \sup_{0 \leq t \leq T} \mathbb{E}_{t,i}^\Lambda[|Y|](\omega) < \infty, \quad \forall \omega \in \Omega.$$

*Then the process  $X(t) = \mathbb{E}_{t, Z(t)}^\Lambda[Y]$  is an optional projection with paths of finite variation and*

$$X(t) = Y - \int_{(t, T]} \sum_{i, j: i \neq j} (\tilde{X}_j(s) - \tilde{X}_i(s)) (N_{ij}(ds) - I_i(s-) \Lambda_{ij}(dt)), \quad t \in [0, T].$$

*Proof.* The fact that  $X(t) = \mathbb{E}^\Lambda[Y | \mathcal{F}_t]$   $\mathbb{P}^\Lambda$ -a.s. for all  $t \in [0, T]$  follows directly from Proposition 4.2.15(i). It remains to show the path properties. We start with the simple case  $Y = \mathbb{1}_A$  for some  $A \in \mathcal{F}_T$ . Set  $\tilde{X}_i(t) = \mathbb{E}_{t,i}^\Lambda[\mathbb{1}_A]$  for  $i \in \mathcal{J}$ . Then by Theorem 5.2 of Christiansen and Furrer (2024) the processes

$$(t, \omega) \mapsto \int_{(0, t]} I_i(s-) \tilde{X}_i(ds), \quad i \in \mathcal{J}$$

are predictable, càdlàg and of finite variation, which implies that

$$\begin{aligned} X(t) &= \sum_{i \in \mathcal{J}} I_i(T) \tilde{X}_i(T) - \sum_{i \in \mathcal{J}} \int_{(t, T]} I_i(s-) \tilde{X}_i(ds) \\ &\quad - \int_{(t, T]} \sum_{i, j: i \neq j} (\tilde{X}_j(s) - \tilde{X}_i(s)) N_{ij}(ds). \end{aligned}$$

Since  $t \mapsto N_{ij}(t, \omega)$  is càdlàg and of finite variation for all  $i, j \in \mathcal{J} : i \neq j$ , and  $t \mapsto \tilde{X}_i(t, \omega)$  is bounded for all  $i \in \mathcal{J}$  it follows that  $X$  is càdlàg and of finite variation. The integration by parts formula now yields  $X(t) = \sum_{i \in \mathcal{J}} I_i(t) \tilde{X}_i(t) = \mathbb{E}_{t, Z(t)}^\Lambda[\mathbb{1}_A]$ . By Theorem 5.2 of Christiansen and Furrer (2024) it also holds for all  $i \in \mathcal{J}$  that

$$I_i(t-) \tilde{X}_i(dt) = - \sum_{j: j \neq i} (\tilde{X}_j(t) - \tilde{X}_i(t)) I_i(t-) \Lambda_{ij}(dt), \quad \tilde{X}_i(T) = \mathbb{E}_{T, i}^\Lambda[\mathbb{1}_A],$$

which, when inserted into the above expression for  $X$ , yields the desired

$$X(t) = \mathbb{1}_A - \sum_{i, j: i \neq j} \int_{(t, T]} (\tilde{X}_j(s) - \tilde{X}_i(s)) (N_{ij}(ds) - I_i(s-) \Lambda_{ij}(ds)), \quad \forall t \in [0, T].$$

Next, we let  $Y$  be non-negative such that the above assumed integrability condition is satisfied. Set  $X(t) := \mathbb{E}_{t, Z(t)}^\Lambda[Y]$  and  $\tilde{X}_i(t) = \mathbb{E}_{t, i}^\Lambda[Y]$  for  $i \in \mathcal{J}$ . Then there exists  $(Y_m)_{m \in \mathbb{N}}$  given by

$$Y_m(\omega) = \sum_{\ell=1}^m a_{\ell, m} \mathbb{1}_{A_{\ell, m}}(\omega) \quad \text{with} \quad \sup_{m \in \mathbb{N}} Y_m(\omega) = Y(\omega), \quad \forall \omega \in \Omega,$$

where  $(a_{\ell, m})_{\ell=1, \dots, n} \subset [0, \infty)$  and  $A_{\ell, m} \subset \mathcal{F}_T$  for all  $m \in \mathbb{N}$ . The dominated convergence theorem then yields

$$\tilde{X}_i(t) = \mathbb{E}_{t, i}^\Lambda \left[ \lim_{m \rightarrow \infty} Y_m \right] = \lim_{m \rightarrow \infty} \mathbb{E}_{t, i}^\Lambda[Y_m] = \lim_{m \rightarrow \infty} \sum_{\ell=1}^m a_{\ell, m} \mathbb{E}_{t, i}^\Lambda[\mathbb{1}_{A_{\ell, m}}],$$

for all  $t \in [0, T]$  and  $i \in \mathcal{J}$ . Combining this with the fact that our desired result holds for indicator random variables, we obtain

$$\begin{aligned} X(s) - X(t) &= \lim_{m \rightarrow \infty} \sum_{\ell=1}^m a_{\ell, m} (\mathbb{E}_{s, Z(s)}^\Lambda[\mathbb{1}_{A_{\ell, m}}] - \mathbb{E}_{t, Z(t)}^\Lambda[\mathbb{1}_{A_{\ell, m}}]) \\ &= - \lim_{m \rightarrow \infty} \sum_{\ell=1}^m a_{\ell, m} \left( \sum_{i, j: i \neq j} \int_{(t, s]} (\mathbb{E}_{u, j}^\Lambda[\mathbb{1}_{A_{\ell, m}}] - \mathbb{E}_{u, i}^\Lambda[\mathbb{1}_{A_{\ell, m}}]) M_{ij}(du) \right) \\ &= - \lim_{m \rightarrow \infty} \sum_{i, j: i \neq j} \int_{(t, s]} (\mathbb{E}_{u, j}^\Lambda[Y_m] - \mathbb{E}_{u, i}^\Lambda[Y_m]) M_{ij}(du), \quad 0 \leq t \leq s \leq T. \end{aligned}$$

The fact that all  $Y_m$  are dominated by  $Y$  in combination with the above assumed integrability condition allows another application of the dominated convergence

theorem, which yields

$$X(s) - X(t) = - \sum_{i,j:i \neq j} \int_{(t,s]} (\tilde{X}_j(u) - \tilde{X}_i(u)) M_{ij}(du), \quad \forall 0 \leq t \leq s \leq T,$$

which together with  $X(T) = \mathbb{E}_{T,Z(T)}^\Lambda[Y] = Y$  yields the desired result. That  $t \mapsto X(t, \omega)$  is càdlàg and of finite variation follows again from the fact that  $t \mapsto N_{ij}(t, \omega)$  and  $t \mapsto \Lambda_{ij}(t, \omega)$  are càdlàg and of finite variation for all  $i, j \in \mathcal{J} : i \neq j$ , and that  $t \mapsto \tilde{X}_i(t, \omega)$  and  $t \mapsto I_i(t-, \omega)$  are bounded for all  $i \in \mathcal{J}$ . The result for the case of real-valued random variables  $Y$  now follows from the result for non-negative random variables and from the fact that  $Y = Y^+ - Y^-$ , where  $Y^+ = \max(0, Y)$  and  $Y^- = -\min(0, Y)$  both are both non-negative.  $\square$

**Proposition 4.3.2.** *Assume that  $C$  is càdlàg and of finite variation and that*

$$\forall i \in \mathcal{J} : \sup_{t \in [0, T]} \mathbb{E}_{t,i}^\Lambda[|L(t, T)|](\omega) < \infty, \quad \forall \omega \in \Omega.$$

*Then  $X(t) := \mathbb{E}_{t,Z(t)}^\Lambda[L(t, T)]$  is an optional projection of  $L(t, T)$  with paths of finite variation.*

*Proof.* Proposition 4.2.6(i) and Proposition 4.2.15(iii) yield

$$\mathbb{E}_{t,i}^\Lambda[|L(0, T)|](\omega) \leq |C(0, \omega)| + |\tilde{C}_i(t, \omega)| + \mathbb{E}_{t,i}^\Lambda[|L(t, T)|](\omega) < \infty,$$

for all  $(t, \omega) \in [0, T] \times \Omega$ . Since  $C$  has paths of finite variation, so has  $\tilde{C}_i$ , see Remark 4.2.8, and thus we obtain

$$\begin{aligned} & \sup_{t \in [0, T]} \mathbb{E}_{t,i}^\Lambda[|L(0, T)|](\omega) \\ & \leq |C(0, \omega)| + |\tilde{C}_i(0, \omega)| + |\tilde{C}_i|(T, \omega) + \sup_{t \in [0, T]} \mathbb{E}_{t,i}^\Lambda[|L(t, T)|](\omega) < \infty, \end{aligned}$$

for all  $\omega \in \Omega$ , where  $|\tilde{C}_i|(T, \omega)$  denotes the variation of  $t \mapsto \tilde{C}_i(t, \omega)$  on the interval  $[0, T]$ . By Proposition 4.3.1 we thus obtain that  $W(t) = \mathbb{E}_{t,Z(t)}^\Lambda[L(0, T)] = C(t) - C(0) + \mathbb{E}_{t,Z(t)}^\Lambda[L(t, T)]$  is càdlàg and of finite variation, which implies the same for  $X(t) = \mathbb{E}_{t,Z(t)}^\Lambda[L(t, T)]$ .  $\square$

Now that we have proven the existence of the optional projection, we turn towards establishing that the optional projection constructed in Proposition 4.3.2 can be characterised by a BSDE. This requires additional integrability conditions, as one has to establish a priori that the integral on the right-hand side of the martingale representation result of Proposition 4.3.1 is a martingale. For this, we follow Christiansen and Furrer (2024) and define the following class of processes, which guarantees sufficient integrability, regardless of which specific pair  $(\pi, \Lambda)$  is used to generate  $\mathbb{P}^\Lambda$  and  $(\mathbb{P}_{t,i}^\Lambda)_{t \in [0, \infty), i \in \mathcal{J}}$ .

**Definition 4.3.3.** Let  $m \in \mathbb{N}_0$  and let  $\mathbb{Y}_m^+$  denote the set of all  $\mathbb{F}$ -adapted, non-negative, non-decreasing, univariate càdlàg processes  $X$  such that

$$X(s, \omega) - X(t, \omega) \leq g(t, s)(1 + N(s, \omega) - N(t, \omega))^m, \quad \forall 0 \leq t \leq s, \forall \omega \in \Omega$$

for some non-negative measurable function  $g : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$  which is bounded on compact sets. Set  $\mathbb{Y}^+ := \cup_{m \in \mathbb{N}_0} \mathbb{Y}_m^+$  and denote by  $\mathbb{Y}$  the class of processes that can be written as a difference of two processes from  $\mathbb{Y}^+$ .

**Proposition 4.3.4.** For any  $X \in \mathbb{Y}$  and any  $\Lambda = (\Lambda_{ij})_{i \neq j}$  satisfying Assumption 4.2.12 it holds that

$$\sup_{0 \leq t \leq s \leq T} \sup_{\omega \in \Omega} \mathbb{E}_{t,i}^\Lambda[|X(s) - X(t)|] < \infty.$$

*Proof.* This follows directly from the fact that  $X$  is the difference of two processes from  $\mathbb{Y}^+$ , the two inequalities  $(1+x)^m \leq e^{mx}$  and  $(x+y)^m \leq 2^{m-1}(x^m + y^m)$  and Proposition 4.2.16.  $\square$

We say that a process  $X$  is bounded on finite intervals if the mapping  $X : I \times \Omega \rightarrow \mathbb{R}$  is bounded for all finite intervals  $I \subset [0, \infty)$ .

**Theorem 4.3.5.** Assume that  $C \in \mathbb{Y}$  and that  $Y$  is bounded. The following two statements are equivalent:

- (i) The process  $X(t) = \mathbb{E}_{t,Z(t)}^\Lambda[L(t, T)]$  has state-wise decomposition  $\tilde{X}_i(t) = \mathbb{E}_{t,i}^\Lambda[L(t, T)]$  and satisfies

$$\forall t \in [0, T] : X(t) = \mathbb{E}^\Lambda[L(t, T) | \mathcal{F}_t] \quad \mathbb{P}^\Lambda - a.s.$$

- (ii) The process  $X$  is càdlàg, of finite variation, bounded on finite intervals and a solution of the BSDE

$$X(dt) = -C(dt) + \sum_{i,j:i \neq j} R_{ij}(t)(N_{ij}(dt) - I_i(t-)\Lambda_{ij}(dt)), \quad (4.3.1)$$

with terminal condition  $X(T) = Y$ , and where  $R_{ij}(t) := \tilde{\Delta}C_{ij}(t) + X_{ij}(t) - X_i(t)$ .

*Proof.* (i)  $\Rightarrow$  (ii): Let  $X(t) = \mathbb{E}_{t,Z(t)}^\Lambda[L(t, T)]$ . By definition  $\tilde{X}_i(t) = \mathbb{E}_{t,i}^\Lambda[L(t, T)]$  is the state-wise decomposition of  $X$  and since  $C \in \mathbb{Y}$  and  $Y$  is bounded, Proposition 4.3.4 yields that  $\tilde{X}_i(t)$  (and thus  $X$ ) is bounded on finite intervals. As a consequence, Proposition 4.3.2 yields that  $X$  is càdlàg and of finite variation and we can apply Proposition 4.3.1 to conclude that  $W(t) := \mathbb{E}_{t,Z(t)}^\Lambda[L(0, T)]$  satisfies

$$W(t) = L(0, T) - \sum_{i,j:i \neq j} \int_{(t,T]} \tilde{W}_j(s) - \tilde{W}_i(s)(N_{ij}(ds) - I_i(s-)\Lambda_{ij}(ds)),$$

for  $t \in [0, T]$  where  $\widetilde{W}_i(t) = \mathbb{E}_{t,i}^\Lambda[L(0, T)]$ . By applying Proposition 4.2.15(iii) we obtain

$$\begin{aligned} W(t) &= \mathbb{E}_{t,Z(t)}^\Lambda[Y + C(T) - C(0)] = C(t) - C(0) + \mathbb{E}_{t,Z(t)}^\Lambda[Y + C(T) - C(t)] \\ &= C(t) - C(0) + X(t). \end{aligned}$$

Equating this expression for  $W$  with the previous one, we get

$$X(t) = Y + C(T) - C(t) - \sum_{i,j:i \neq j} \int_{(t,T]} \widetilde{W}_j(s) - \widetilde{W}_i(s) M_{ij}(\mathrm{d}s), \quad (4.3.2)$$

where  $M_{ij}(\mathrm{d}s) = N_{ij}(\mathrm{d}s) - I_i(s-) \Lambda_{ij}(\mathrm{d}s)$ . By Proposition 4.2.10(i) and Proposition 4.2.15(ii) we have for all  $i \in \mathcal{J}$  that

$$\begin{aligned} \widetilde{W}_i(t) &= \mathbb{E}_{t,i}^\Lambda[L(0, T)] = \int_{(0,t]} \sum_{j \in \mathcal{J}} I_j(s-) C_j(\mathrm{d}t) + \int_{(0,t]} \sum_{j,k:j \neq k} \widetilde{\Delta} C_{jk}(s) N_{jk}(\mathrm{d}s) \\ &\quad + \mathbb{E}_{t,i}^\Lambda \left[ \sum_{j,k:j \neq k} \widetilde{\Delta} C_{jk}(t) \Delta N_{jk}(t) \right] + \widetilde{X}_i(t), \quad t \in [0, T], \end{aligned}$$

and utilising that  $\Delta N_{jk}(t) = I_j(t-) I_k(t)$  Proposition 4.4(i) of Christiansen and Furrer (2024) yields

$$\begin{aligned} \mathbb{E}_{t,i}^\Lambda \left[ \sum_{j,k:j \neq k} \widetilde{\Delta} C_{jk}(t) \Delta N_{jk}(t) \right] &= \sum_{j,k:j \neq k} \widetilde{\Delta} C_{jk}(t) \mathbb{E}_{t,i}[I_j(t-) I_k(t)] \\ &= \sum_{j:j \neq i} I_j(t-) \widetilde{\Delta} C_{ji}(t). \end{aligned}$$

These two identities after (4.3.2) yield

$$\begin{aligned} I_i(s-) (\widetilde{W}_j(s) - \widetilde{W}_i(s)) &= I_i(s-) (\widetilde{\Delta} C_{ij}(s) + \widetilde{X}_j(s) - \widetilde{X}_i(s)) \\ &= I_i(s-) (\widetilde{\Delta} C_{ij}(s) + X_{ij}(s) - X_i(s)), \end{aligned}$$

for all  $s \in (0, T]$ , which in conjunction with (4.3.2) allows us to conclude that  $X$  is a solution of (4.3.1).

(ii)  $\Rightarrow$  (i): Using (4.3.1) we get

$$X(t) = L(t, T) - \sum_{i,j:i \neq j} \int_{(t,T]} R_{ij}(s) (N_{ij}(\mathrm{d}s) - I_i(s-) \Lambda_{ij}(\mathrm{d}s)), \quad t \in [0, T].$$

Since  $C \in \mathbb{Y}$  and  $X$  is bounded on finite intervals,  $R_{ij}(t)$  is sufficiently integrable in order for Proposition 4.2 of Christiansen and Furrer (2024) to hold. Thus taking the expectations with respect to  $\mathbb{P}_{t,i}^\Lambda$  on both sides and applying Proposition 4.2.15(iii) yields

$$\forall i \in \mathcal{J} : \widetilde{X}_i(t) = \mathbb{E}_{t,i}^\Lambda[X(t)] = \mathbb{E}_{t,i}^\Lambda[L(t, T)] \quad \Rightarrow \quad X(t) = \mathbb{E}_{t,Z(t)}^\Lambda[L(t, T)],$$

for all  $t \in [0, T]$ , which in conjunction with Proposition 4.2.15(i) yields

$$X(t) = \mathbb{E}[L(t, T) | \mathcal{F}_t], \quad \mathbb{P}^\Lambda - \text{a.s.} \quad \forall t \in [0, T].$$

□

Often we are also interested in the optional projection of a discounted version of the process  $L(t, T)$  given by:

$$L^*(t, T) := \int_{(t, T]} \frac{\mathcal{E}_\Phi(t)}{\mathcal{E}_\Phi(s)} C(ds) + \frac{\mathcal{E}_\Phi(t)}{\mathcal{E}_\Phi(T)} Y, \quad t \in (0, T],$$

where the process  $\mathcal{E}_\Phi$  is defined as the generalised exponential

$$\mathcal{E}_\Phi(dt) = \mathcal{E}_\Phi(t-) \Phi(dt), \quad \mathcal{E}_\Phi(0) = 1 \quad \Leftrightarrow \quad \mathcal{E}_\Phi(t) = e^{\Phi^c(t) - \Phi^c(0)} \prod_{0 < s \leq t} (1 + \Delta\Phi(s)).$$

We assume that  $\Phi$  is a predictable, càdlàg, finite variation process that is bounded on finite intervals and satisfies  $\inf_{(t, \omega) \in [0, T] \times \Omega} \Delta\Phi(t, \omega) > -1$ . Since this implies that  $\mathcal{E}_\Phi$  and  $\mathcal{E}_\Phi^{-1}$  are bounded on finite intervals, Proposition 4.3.2 can be utilised to conclude that the process

$$X(t) = \mathbb{E}_{t, Z(t)}^\Lambda [L^*(t, T)]$$

indeed is an optional projection of  $L^*(t, T)$  with paths of finite variation. Furthermore, we obtain the following analogue to Theorem 4.3.5:

**Theorem 4.3.6.** *Assume that  $C \in \mathbb{Y}$ , that  $Y$  is bounded, and that  $\Phi$  is a predictable, càdlàg finite variation process bounded on finite intervals such that  $\inf_{(t, \omega) \in [0, T] \times \Omega} \Delta\Phi(t, \omega) > -1$ . The following two statements are equivalent:*

- (i) *The process  $X(t) = \mathbb{E}_{t, Z(t)}^\Lambda [L(t, T)]$  has state-wise decomposition  $\tilde{X}_i(t) = \mathbb{E}_{t, i}^\Lambda [L^*(t, T)]$  and satisfies*

$$\forall t \in [0, T] : X(t) = \mathbb{E}^\Lambda [L^*(t, T) | \mathcal{F}_t] \quad \mathbb{P}^\Lambda - \text{a.s.}$$

- (ii) *The process  $X$  is càdlàg, of finite variation, bounded on finite intervals and a solution of the BSDE*

$$X(dt) = X(t-) \Phi(dt) - C(dt) + \sum_{i, j: i \neq j} R_{ij}(t) (N_{ij}(dt) - I_i(t-) \Lambda_{ij}(dt)), \quad (4.3.3)$$

*with terminal condition  $X(T) = Y$ , and where  $R_{ij}(t) := \tilde{\Delta}C_{ij}(t) + X_{ij}(t) - X_i(t)$ .*

*Proof.* Let  $Y^* := \frac{Y}{\mathcal{E}_\Phi(t)}$  and  $C^*(t) := C(0) + \int_{[0,T]} \frac{1}{\mathcal{E}_\Phi(s)} C(ds)$ . Due to our assumptions on  $\Phi$ , we have that  $\frac{1}{\mathcal{E}_\Phi(t)}$  is strictly positive and bounded on finite intervals, which implies that  $Y^*$  and  $C^*$  satisfy the assumptions of Theorem 4.3.5.

(ii)  $\Rightarrow$  (i): Let  $W(t) := \frac{X(t)}{\mathcal{E}_\Phi(t)}$ . By Ito's formula, we get

$$\begin{aligned} W(dt) &= -\frac{X(t-)}{\mathcal{E}_\Phi(t)} \Phi(dt) + \frac{1}{\mathcal{E}_\Phi(t)} X(dt) \\ &= -C^*(dt) + \sum_{i,j:i \neq j} \frac{1}{\mathcal{E}_\Phi(t)} R_{ij}(t) (N_{ij}(dt) - I_i(t-) \Lambda_{ij}(dt)), \quad t \in (0, T]. \end{aligned}$$

As  $\mathcal{E}_\Phi$  is predictable and  $W$  is optional, Proposition 4.2.6 yields that  $W_i(t) = \frac{X_i(t)}{\mathcal{E}_{\Phi,i}(t)}$  and  $W_{ij}(t) = \frac{X_{ij}(t)}{\mathcal{E}_{\Phi,i}(t)}$  and thus we obtain

$$I_i(t-) \frac{1}{\mathcal{E}_\Phi(t)} R_{ij}(t) = I_i(t-) \left( \tilde{\Delta} C_{ij}^*(t) + W_{ij}(t) - W_i(t) \right) = I_i(t-) R_{ij}^W(t),$$

and we can conclude that  $W$  satisfies the BSDE

$$W(dt) = -C^*(dt) - \sum_{i,j:i \neq j} R_{ij}^W(t) (N_{ij}(dt) - I_i(t-) \Lambda_{ij}(dt)), \quad W(T) = Y^*.$$

Theorem 4.3.5 in conjunction with the relationship  $X(t) = \mathcal{E}_\Phi(t)W(t)$  and Proposition 4.2.15(iii) now yields the desired result.

(i)  $\Rightarrow$  (ii): By Proposition 4.2.15(iii) we have that

$$X(t) = \mathbb{E}_{t,Z(t)}^\Lambda [L(t, T)] = \mathcal{E}_\Phi(t) \mathbb{E}_{t,Z(t)} [Y^* + C^*(T) - C^*(t)].$$

By Theorem 4.3.5 we obtain the above BSDE for  $W(t) := \mathbb{E}_{t,Z(t)} [Y^* + C^*(T) - C^*(t)]$ . Since  $X(t) = \mathcal{E}_\Phi(t)W(t)$ , integration by parts yields the desired result.  $\square$

Theorem 4.3.5 and Theorem 4.3.6 provide a one-to-one correspondence between a BSDE and an optional projection. This turns out to be very useful for two reasons: The first reason is that one can establish existence and uniqueness of the optional projection by establishing existence and uniqueness of the BSDE, which is key when considering the case where the process  $C$  or the cumulative transition rates  $\Lambda$  depend on the optional projection itself. The second reason is that by using the jump-count representation, it is possible to show that the BSDE is equivalent to an infinite-dimensional system of Lebesgue-Stieltjes integral equations. This allows, in principle, to calculate the optional projection numerically and reduces the question of existence and uniqueness of the BSDE to the question of existence and uniqueness of the infinite-dimensional system of Lebesgue-Stieltjes integral equations.

By Proposition 4.2.3 the processes  $C$  and  $(\Lambda_{ij})_{i \neq j}$  each have a unique jump-count representation  $(C^n)_{n \in \mathbb{N}_0}$  and  $(\Lambda_{ij}^n)_{n \in \mathbb{N}_0}$ . Before we state the system of integral

equations properly, we have to define counterfactual mappings in the spirit of  $\omega_i^t$ , which have  $\Omega^n$  as their domain. For  $i \in \mathcal{J}$  and  $n \in \mathbb{N}_0$  define the mapping

$$\omega_i^n : \Omega^n \rightarrow \bar{\Omega}^n, \quad \omega_i^n(\omega^n) := \begin{cases} ((t_\ell, z_\ell)_{\ell \leq n-1}, (t_n, i)) & \text{for } z_{n-1} \neq i \\ ((t_\ell, z_\ell)_{\ell \leq n-1}, (\infty, \nabla)) & \text{for } z_{n-1} = i \end{cases},$$

which for any given sequence of  $n$  jumps sets  $z_n = i$  whenever  $z_{n-1} \neq i$ , while it removes the  $n$ 'th jump if  $z_{n-1} = i$ . Thus for any  $\omega \in \Omega$  with  $\omega = (\omega^n, \dots)$  for some  $\omega^n \in \Omega^n$  it holds that  $\omega_i^n = \xi^n(\omega_i^{t_n})$ . Furthermore for  $i, j \in \mathcal{J}$  with  $i \neq j$  and  $n \in \mathbb{N}_0$  we define the mapping  $\omega_{t,ij}^{n+1} : [0, \infty) \times \Omega^n \rightarrow \bar{\Omega}^{n+1}$  given by

$$\omega_{t,ij}^{n+1}(\omega^n) := \begin{cases} ((t_\ell, z_\ell)_{\ell \leq n-1}, (t_n, i), (t, j)) & \text{for } z_{n-1} \neq i, t > t_n \\ ((t_\ell, z_\ell)_{\ell \leq n-1}, (t, j), (\infty, \nabla)) & \text{for } z_{n-1} = i, t > t_n \end{cases},$$

which for any given sequence of  $n$  jumps sets  $z_n = i$  whenever  $z_{n-1} \neq i$  and then adds a jump to  $j$  at time  $t$ , while it removes the  $n$ 'th jump if  $z_{n-1} = i$  and then adds a jump to  $j$  at time  $t$ . Thus for any  $\omega \in \Omega$  with  $\omega = (\omega^n, \dots)$  for some  $\omega^n \in \Omega^n$  it holds that  $\omega_{t,ij}^{n+1} = \xi^{n+1}(\omega_{i,j}^{t_n,t})$ .

**Proposition 4.3.7.** *Let  $X$  be an adapted càdlàg process of finite variation with jump-count representation  $(f^n)_{n \in \mathbb{N}_0}$ . Then  $X$  is the unique solution of (4.3.5) if and only if  $(f^n)_{n \in \mathbb{N}_0}$  is the unique solution of the system*

$$\begin{aligned} f^n(dt, \omega^n) &= -C^n(dt, \omega^n) - \sum_{j:j \neq z_n} R_{z_n j}^{n+1}(t, \omega^n) \Lambda_{z_n j}^n(dt, \omega^n), \quad t \in (0, T], \\ f^n(T, \omega^n) &= Y^n(\omega^n), \quad \omega^n \in \Omega^n, n \in \mathbb{N}_0 \end{aligned} \tag{4.3.4}$$

and where  $R_{ij}^{n+1}(t, \omega^n) := \tilde{\Delta} C_{ij}^{n+1}(t) + f^{n+1}(t, \omega_{t,ij}^{n+1}) - f^n(t, \omega_i^n)$ .

*Proof.* Since  $X$  is càdlàg and of finite variation Proposition 4.2.9(i) yields

$$X(dt) = \sum_{n \in \mathbb{N}_0} I^n(t-) \left( f^n(dt, \xi^n) + \sum_{j:j \neq \zeta_n} f^{n+1}(t, \xi^n(\omega_{\zeta_n, j}^{t_n, t})) - f^n(t, \xi^n) N_{\zeta_n j}(dt) \right).$$

Using Proposition 4.2.3, Proposition 4.2.9 and (4.2.7), the BSDE (4.3.5) can equivalently be written as

$$\begin{aligned} X(dt) &= \sum_{n \in \mathbb{N}_0} I^n(t-) \left( -C^n(dt, \xi^n) - \sum_{j:j \neq \zeta_n} \tilde{\Delta} C_{\zeta_n j}^n(t, \xi^n) N_{ij}(dt) \right. \\ &\quad \left. + \sum_{j:j \neq \zeta_n} R_{\zeta_n j}^{n+1}(t, \xi^n) (N_{\zeta_n j}(dt) - I_{\zeta_n}(t-) \Lambda_{\zeta_n j}^n(dt, \xi^n)) \right). \end{aligned}$$

Equating the first and second equation for  $X(dt)$  and rearranging yields

$$0 = I^n(t-) \left( f^n(dt, \xi^n) + C^n(dt, \xi^n) + \sum_{j:j \neq \zeta_n} R_{\zeta_n j}^{n+1}(t, \xi^n) \Lambda_{\zeta_n j}^n(dt, \xi^n) \right).$$

If  $X$  is an adapted, càdlàg finite variation solution of (4.3.5) with jump-count decomposition  $(f^n)_{n \in \mathbb{N}_0}$ , then by Remark 4.2.4 each  $f^n$  is càdlàg and of finite variation and both equations for  $X(dt)$  are valid implying the above third equation. Since this equation holds for all  $\omega = (\omega^n, (\infty, \nabla), \dots)$  for  $\omega^n \in \Omega^n$ , it holds for all  $t > t_n$ . Due to the conventions (4.2.3) and (4.2.4) used to define the jump-count representation, it holds that  $f^n(dt) = C^n(dt) = \Lambda_{z_n j}^n(dt) = 0$  for  $t \in (0, t_n]$ . Thus  $(f^n)_{n \in \mathbb{N}_0}$  is a solution of the system (4.3.4) for all  $t \in (0, T]$ . If on the other hand  $(f^n)_{n \in \mathbb{N}_0}$  is a solution of the system (4.3.4), each  $f^n$  is càdlàg and of finite variation which implies the same for  $X(t) = \sum_{n \in \mathbb{N}_0} I^n(t) f^n(t, \xi^n)$ . Thus, the first equation of  $X(dt)$  is valid and inserting the above third equation and rearranging yields the second, which proves that  $X$  is a solution of (4.3.5).  $\square$

Similar to the undiscounted case, the existence and uniqueness of the BSDE (4.3.3) is equivalent to the existence and uniqueness of a system of Lebesgue-Stieltjes integral equations, which is the statement of the following analogue of Proposition 4.3.7.

**Proposition 4.3.8.** *Let  $X$  be an adapted càdlàg process of finite variation with jump-count representation  $(f^n)_{n \in \mathbb{N}_0}$ . Then  $X$  is the unique solution of (4.3.6) if and only if  $(f^n)_{n \in \mathbb{N}_0}$  is the unique solution of the system*

$$f^n(dt, \omega^n) = f^n(t-, \omega^n) \Phi^n(dt, \omega^n) - C^n(dt, \omega^n) - \sum_{j: j \neq z_n} R_{z_n j}^{n+1}(t, \omega^n) \Lambda_{z_n j}^n(dt, \omega^n),$$

$$f^n(T, \omega^n) = Y^n(\omega^n), \quad t \in (0, T], \omega^n \in \Omega^n, n \in \mathbb{N}_0$$

and where  $R_{ij}^{n+1}(t, \omega^n) := \tilde{\Delta} C_{ij}^{n+1}(t) + f^{n+1}(t, \omega_{t, ij}^{n+1}) - f^n(t, \omega_i^n)$ .

### 4.3.2 The worst-case BSDE

We now consider a case where both the process  $C$  and the cumulative transition rates may depend on the optional projection and provide an existence and uniqueness result for this type of circular problem. For this, let  $(U_{ij})_{i \neq j}$  and  $(L_{ij})_{i \neq j}$  be cumulative transition rates which in addition to Assumption 4.2.12 satisfy

$$L_{ij}(s, \omega) - L_{ij}(t, \omega) \leq U_{ij}(s, \omega) - U_{ij}(t, \omega), \quad \forall 0 \leq t \leq s, \forall \omega \in \Omega, \forall i, j \in \mathcal{J} : i \neq j$$

and consider the BSDE given by

$$X(dt) = -C(dt) + \sum_{i, j: i \neq j} R_{ij}(t) (N_{ij}(dt) - I_i(t-) \Delta_{ij}(dt)), \quad X(T) = Y, \quad (4.3.5)$$

where  $R_{ij}(t) := \Delta C_{ij}(t) + X_{ij}(t) - X_i(t)$  and  $Y$  is  $\mathcal{F}_T$ -measurable. We assume that the cumulative transition rates  $\underline{\Lambda} = (\underline{\Lambda}_{ij})_{i \neq j}$  have one of the following two forms

$$\underline{\Lambda}_{ij}(dt) = \mathbb{1}_{(R_{ij}(t) \geq 0)} U_{ij}(dt) + \mathbb{1}_{(R_{ij}(t) < 0)} L_{ij}(dt), \quad \forall i, j \in \mathcal{J} : i \neq j,$$

$$\underline{\Lambda}_{ij}(dt) = \mathbb{1}_{(R_{ij}(t) < 0)} U_{ij}(dt) + \mathbb{1}_{(R_{ij}(t) \geq 0)} L_{ij}(dt), \quad \forall i, j \in \mathcal{J} : i \neq j.$$

The  $(\underline{\Lambda}_{ij})_{i \neq j}$  are cumulative transition rates, as the processes  $(L_{ij})_{i \neq j}$  and  $(U_{ij})_{i \neq j}$  being cumulative transition rates ensures that  $(\underline{\Lambda}_{ij})_{i \neq j}$  satisfies condition (ii) and (iii) of Assumption 4.2.12, and the fact that the processes  $(R_{ij})_{i \neq j}$  are  $i$ -predictable ensures that the processes  $\underline{\Lambda}_{ij}$  are  $i$ -predictable (Assumption 4.2.12(i)). The process  $C$  is an optional process with càdlàg paths of finite variation given by

$$C(dt) = \sum_{i \in \mathcal{J}} I_i(t-) \left( C_i(dt) + \sum_{j: j \neq i} \Delta C_{ij}(t) N_{ij}(dt) \right),$$

for  $i$ -predictable  $(C_i, (\Delta C_{ij})_{j: j \neq i})_{i \in \mathcal{J}}$ , which are assumed to be of the form

$$\begin{aligned} C_i(dt, \omega) &= b_i(t, \omega, X_i(t, \omega)) \gamma_i(dt, \omega), \\ \Delta C_{ij}(dt, \omega) &= b_{ij}(t, \omega, X_i(t, \omega), X_{ij}(t, \omega)). \end{aligned}$$

Clearly if there exists a unique solution  $X$  of the BSDE (4.3.5) that is càdlàg, of finite variation and bounded on finite intervals, we can by Theorem 4.3.5 conclude that  $X(t) = \mathbb{E}_{t, Z(t)}^\Delta[Y + C(T) - C(t)]$ , but since both the process  $C$  and the cumulative transition rates  $\underline{\Lambda}$  determining the measure  $\mathbb{P}^\Delta$  depend on  $X$ , it is not clear, whether such a solution exists. In order to prove existence and uniqueness of the BSDE (4.3.5), we assume that the processes  $(\gamma_i)_{i \in \mathcal{J}}$ ,  $(b_i)_{i \in \mathcal{J}}$  and  $(b_{ij})_{i \neq j}$  satisfy the following regularity conditions.

**Assumption 4.3.9.** *Assume that*

- (i) *The processes  $(\gamma_i)_{i \in \mathcal{J}}$  are  $i$ -predictable càdlàg processes of finite variation, and there exists a non-decreasing càdlàg function  $\bar{\gamma} : [0, \infty) \rightarrow [0, \infty)$  bounded on compacts such that for all  $\omega \in \Omega$ :*

$$|\gamma_i(t, \omega)| \leq \bar{\gamma}(t), \quad t \geq 0 \quad \text{and} \quad |\gamma_i(t, \omega) - \gamma_i(s, \omega)| \leq \bar{\gamma}(t) - \bar{\gamma}(s), \quad 0 \leq s \leq t$$

- (ii) *The functions  $(b_i, (b_{ij})_{j: j \neq i})_{i \in \mathcal{J}}$  with  $b_i : [0, \infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  and  $b_{ij} : [0, \infty) \times \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  are measurable, and  $(t, \omega) \mapsto b_i(t, \omega, x)$  and  $(t, \omega) \mapsto b_{ij}(t, \omega, x, y)$  are  $i$ -predictable for all fixed  $x, y \in \mathbb{R}$ .*

- (iii) *There exists a positive function  $\beta : [0, \infty) \rightarrow (0, \infty)$  which is bounded on compacts such that for all  $i, j \in \mathcal{J} : i \neq j$ :*

$$|b_i(t, \omega, 0)| \leq \beta(t) \quad \text{and} \quad |b_{ij}(t, \omega, 0, 0)| \leq \beta(t)$$

*for all  $(t, \omega) \in [0, \infty) \times \Omega$ .*

- (iv) *There exists a positive function  $\beta : [0, \infty) \rightarrow (0, \infty)$  which is bounded on compacts such that for all  $i, j \in \mathcal{J} : i \neq j$ :*

$$\begin{aligned} |b_i(t, \omega, x_1) - b_i(t, \omega, x_2)| &\leq \beta(t) |x_1 - x_2|, \\ |b_{ij}(t, \omega, x_1, y_1) - b_{ij}(t, \omega, x_2, y_2)| &\leq \beta(t) (|x_1 - x_2| + |y_1 - y_2|), \end{aligned}$$

*for all  $(t, \omega) \in [0, \infty) \times \Omega$  and  $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ .*

Assuming those regularity conditions brings us to the first of two main results of this section.

**Theorem 4.3.10.** *Assume that Assumption 4.3.9 is satisfied and that  $Y$  is bounded. Then there exists a unique solution  $X$  of the BSDE (4.3.5) that is càdlàg, of finite variation and bounded on finite intervals.*

*Proof.* See Subsection 4.3.3. □

Returning to the discounted case, we can consider the BSDE

$$X(dt) = X(t-)\Phi(dt) - C(dt) + \sum_{i,j:i \neq j} R_{ij}(t)(N_{ij}(dt) - I_i(t-)\underline{\Lambda}_{ij}(dt)), \quad (4.3.6)$$

with terminal condition  $X(T) = Y$ , and where  $R_{ij}(t) := \Delta C_{ij}(t) + X_{ij}(t) - X_i(t)$ . The process  $\Phi$  is predictable, càdlàg, and of finite variation, while the terminal condition  $Y$ , the cumulative transition rates  $\underline{\Lambda}$ , and the process  $C$  are assumed to be of the same form as in the undiscounted case. Assuming that Assumption 4.3.9 is satisfied, we obtain the second main result of this section and analogue of Theorem 4.3.10.

**Theorem 4.3.11.** *Assume that Assumption 4.3.9 is satisfied, that  $Y$  is bounded and that  $\Phi$  is predictable, càdlàg, of finite variation and bounded on finite intervals with  $\inf_{(t,\omega) \in [0,T] \times \Omega} \Phi(t,\omega) > -1$ . Then there exists a unique solution  $X$  of the BSDE (4.3.6) that is càdlàg, of finite variation and bounded on finite intervals.*

*Proof.* We start by considering the process  $W$  solving the BSDE

$$W(dt) = -C^*(dt) + \sum_{i,j:i \neq j} R_{ij}^W(t)(N_{ij}(dt) - I_i(t-)\underline{\Lambda}_{ij}^W(dt)), \quad W(T) = Y^*,$$

$$R_{ij}^W(t) := \tilde{\Delta} C_{ij}^*(t) + W_{ij}(t) - W_i(t),$$

and where  $Y^* := \frac{Y}{\varepsilon_\Phi(T)}$ . The cumulative transition rates  $\underline{\Lambda}_{ij}^W$  all have one of the following forms:

$$\underline{\Lambda}_{ij}^W(dt) = \mathbf{1}_{(R_{ij}^W \geq 0)} U_{ij}(dt) + \mathbf{1}_{(R_{ij}^W < 0)} L_{ij}(dt)$$

$$\underline{\Lambda}_{ij}^W(dt) = \mathbf{1}_{(R_{ij}^W < 0)} U_{ij}(dt) + \mathbf{1}_{(R_{ij}^W \geq 0)} L_{ij}(dt),$$

while the process  $C^*$  is given by

$$C^*(dt) = \sum_{i \in \mathcal{I}} I_i(t-) \left( C_i^*(dt) + \sum_{j:j \neq i} \tilde{\Delta} C_{ij}^*(t) N_{ij}(dt) \right),$$

where  $C_i^*$  and  $\tilde{\Delta}C_{ij}^*$  have the form

$$C_i^*(dt, \omega) = \gamma_i(dt, \omega) = \underbrace{\frac{1}{\mathcal{E}_{\Phi,i}(t)} b_i(t, \omega, \mathcal{E}_{\Phi,i}(t, \omega) W_i(t, \omega))}_{=: b_i^*(t, \omega, W_i(t, \omega))} \gamma_i(dt, \omega),$$

$$\tilde{\Delta}C_{ij}^*(dt, \omega) = \underbrace{\frac{1}{\mathcal{E}_{\Phi,i}(t)} b_{ij}(t, \omega, \mathcal{E}_{\Phi,i}(t, \omega) W_i(t, \omega), \mathcal{E}_{\Phi,i}(t, \omega) W_{ij}(t, \omega))}_{=: b_{ij}^*(t, \omega, W_i(t, \omega), W_{ij}(t, \omega))}.$$

Since  $\frac{1}{\mathcal{E}_{\Phi}(t)}$  is bounded on finite intervals, it follows that  $(b_i^*, (b_{ij}^*)_{j:j \neq i})_{i \in \mathcal{J}}$  satisfy Assumption 4.3.9 and Theorem 4.3.10 yields existence and uniqueness of  $W$ .

Set  $X(t) = \mathcal{E}_{\Phi}(t)W(t)$ . Since  $X_i(t) = \mathcal{E}_{\Phi,i}(t)W_i(t)$  and  $X_{ij}(t) = \mathcal{E}_{\Phi,i}(t)W_{ij}(t)$  we get

$$\begin{aligned} R_{ij}(t) &= \tilde{\Delta}C_{ij}(t) + X_{ij}(t) - X_i(t) \\ &= \mathcal{E}_{\Phi,i}(t)(\tilde{\Delta}C_{ij}^*(t) + W_{ij}(t) - W_i(t)) = \mathcal{E}_{\Phi,i}(t)R_{ij}^W(t) \end{aligned}$$

and since  $\mathcal{E}_{\Phi}(t)$  is strictly positive, we have that  $\text{sign}(R_{ij}^W(t)) = \text{sign}(R_{ij}(t))$ , thus allowing us to conclude that  $\Delta_{ij}^W = \Delta_{ij}$ . Using this, and an integration by parts argument as in the proof of Theorem 4.3.6 it follows that  $X$  is a solution of (4.3.6) if and only if  $X(t) = \mathcal{E}_{\Phi}(t)W(t)$ , where  $W$  is as just described. As  $W$  and  $\mathcal{E}_{\Phi}$  both exist and are unique, so is  $X$ , and since  $W$  and  $\mathcal{E}_{\Phi}$  both are càdlàg, of finite variation and bounded on finite intervals, so is  $X$ .  $\square$

### 4.3.3 Proof of Theorem 4.3.10

The key to the proof of Theorem 4.3.10 is to use the, by Proposition 4.3.7 given, equivalence between the existence and uniqueness of a solution  $X$  of (4.3.5) and existence and uniqueness of the following system of Lebesgue-Stieltjes integral equations satisfied by the jump-count representation  $(f^n)_{n \in \mathbb{N}_0}$  of  $X$ :

$$\begin{aligned} f^n(dt, \omega^n) &= -b_{z_n}^n(t, \omega^n, f^n(t, \omega^n))\gamma_{z_n}^n(dt, \omega^n) + \sum_{j:j \neq z_n} R_{z_n j}^{n+1}(t, \omega^n) \Delta_{z_n j}^n(dt, \omega^n), \\ f^n(T, \omega^n) &= Y^n(\omega^n), \quad t \in (0, T], \omega^n \in \Omega^n, n \in \mathbb{N}_0, \end{aligned} \tag{4.3.7}$$

where  $R_{ij}^{n+1}(t, \omega^n) = b_{ij}^n(t, \omega^n, f^n(t, \omega_i^n), f^{n+1}(t, \omega_{t,ij}^{n+1})) + f^{n+1}(t, \omega_{t,ij}^{n+1}) - f^n(t, \omega_i^n)$  and where  $(\Delta_{ij}^n)_{i \neq j}$  are given by one of the following two variants:

$$\begin{aligned} \Delta_{ij}^n(dt, \omega^n) &= \mathbf{1}_{(R_{ij}^{n+1}(t, \omega^n) \geq 0)} U_{ij}^n(dt, \omega^n) + \mathbf{1}_{(R_{ij}^{n+1}(t, \omega^n) < 0)} L_{ij}^n(dt, \omega^n), \quad \forall n \in \mathbb{N}_0, \\ \Delta_{ij}^n(dt, \omega^n) &= \mathbf{1}_{(R_{ij}^{n+1}(t, \omega^n) < 0)} U_{ij}^n(dt, \omega^n) + \mathbf{1}_{(R_{ij}^{n+1}(t, \omega^n) \geq 0)} L_{ij}^n(dt, \omega^n), \quad \forall n \in \mathbb{N}_0. \end{aligned}$$

Here  $(L_{ij}^n)_{n \in \mathbb{N}_0}$ ,  $(U_{ij}^n)_{n \in \mathbb{N}_0}$ ,  $(\gamma_i^n)_{n \in \mathbb{N}_0}$ ,  $(b_i^n)_{n \in \mathbb{N}_0}$  and  $(b_{ij}^n)_{n \in \mathbb{N}_0}$  denote the jump-count representation of the processes  $(L_{ij})_{i \neq j}$ ,  $(U_{ij})_{i \neq j}$ ,  $(\gamma_i)_{i \in \mathcal{J}}$ ,  $(b_i)_{i \in \mathcal{J}}$  and  $(b_{ij})_{i \neq j}$  and

by Proposition 4.2.2 and Remark 4.2.4 conditions (i), (iii) and (iv) of Assumption 4.3.9 remain valid for the functions of the relevant jump-count decomposition. We could thus equivalently state Assumption 4.3.9 in terms of the corresponding jump-count decomposition by adding a superscript  $n$  and replacing  $\omega \in \Omega$  with  $\omega^n \in \bar{\Omega}^n$  in the relevant places.

For a function  $f : [0, T] \rightarrow \mathbb{R}$  let  $|f|(t)$  denote the variation of  $f$  on the interval  $[0, t]$  and set  $\text{BV}_{[0, T]} : \{f : [0, T] \rightarrow \mathbb{R} : |f|(T) < \infty, f \text{ is càdlàg}\}$ . Let  $\|\cdot\|_{BV}$  denote the finite variation norm given by  $\|f\|_{BV} = |f|(T)$  and note that  $(\text{BV}_{[0, T]}, \|\cdot\|_{BV})$  is a Banach space, see also Appendix 4.A. In the proofs, we will also use a family of equivalent norms

$$\|f\|_{BV}^{c, \nu} := |f|(T) + \int_{(0, T]} e^{-c(\nu(T) - \nu(t-))} |f|(dt),$$

where  $\nu : [0, T] \rightarrow [0, \infty)$  is any non-decreasing càdlàg function and  $c > 0$  is an arbitrary constant, see also Appendix 4.A.

The proof that (4.3.7) has a unique solution is now done in a sequence of lemmas. First we prove in Lemma 4.3.12 that for each  $n \in \mathbb{N}_0$  the function  $f^n$  exists, given that  $f^{n+1}$  is known. In order to then obtain existence and uniqueness of the infinite dimensional system (4.3.7), we first construct a sequence  $((f^{n, M})_{n \in \mathbb{N}_0})_{M \in \mathbb{N}}$  of finite dimensional systems and prove in Lemma 4.3.13 that each member of this sequence has a unique solution. This is followed by Lemma 4.3.14, showing that for each  $n \in \mathbb{N}_0$  the sequence  $(f^{n, M})_{M \in \mathbb{N}}$  has a limit  $f^n$ , and Lemma 4.3.15, which shows that  $(f^n)_{n \in \mathbb{N}_0}$  is a solution of the system (4.3.7). Finally, we conclude with Lemma 4.3.16, which shows that (4.3.5) has a unique solution.

**Notation:** Throughout the proofs in this subsection, we will often suppress the dependence on  $\omega^n$  in order to ease the notational burden. In that case we will use the shorthand notation  $f^n(t)$  for  $f^n(t, \omega^n)$  and  $f_{t, ij}^{n+1}(t)$  for  $f^{n+1}(t, \omega_{t, ij}^{n+1}(\omega^n))$ .

**Lemma 4.3.12.** *Fix  $n \in \mathbb{N}_0$  and assume that  $f^{n+1} : [0, T] \times \Omega^{n+1} \rightarrow \mathbb{R}$  is a known càdlàg function of finite variation such that*

$$\forall i \neq j : \sup_{\omega^{n+1} \in \Omega^{n+1}} \|f^{n+1}\|_{BV}(\omega^{n+1}) < \infty$$

*Then (4.3.7) considered for fixed  $n$  has a unique solution.*

*Proof.* Let  $\omega^n \in \Omega^n$  be given and fixed and set  $K_\beta := \sup_{t \in [0, T]} \beta(t)$ . The goal is to apply the Banach Fixed Point Theorem for the Banach space  $\text{BV}_{[0, T]}$  with the norm  $\|\cdot\|_{BV}^{c, \nu}$ , where  $c > 0$  and

$$\nu(t, \omega^n) := (1 + K_\beta) \left( |\gamma_{z_n}^n|(t, \omega^n) + \sum_{j: j \neq z_n} U_{z_n j}(t, \omega^n) \right).$$

Now fix an arbitrary  $\omega^n \in \Omega^n$ . During the rest of the proof, we will suppress the dependence on  $\omega^n$  for ease of notation. Let  $f \in \text{BV}_{[0,T]}$  and define the mapping  $F : \text{BV}_{[0,T]} \rightarrow \text{BV}_{[0,T]}$  by

$$F(f)(t) := Y^n + \int_{(t,T]} b_{z_n}^n(s, f(s)) \gamma_{z_n}^n(ds) - \sum_{j:j \neq z_n} \int_{(t,T]} R_{t,z_nj}^{n+1}(t) \Delta_{z_nj}^n(dt),$$

where  $R_{t,z_nj}^{n+1}(t) := b_{z_nj}^n(t, f(t), f_{t,z_nj}^{n+1}(t)) + f_{t,z_nj}^{n+1}(t) - f(t)$ . First, we have to check that  $T(f)$  actually is of finite variation, given that  $f$  is. Using that  $x^+ := \mathbb{1}_{(x>0)}x$  and  $x^- := \mathbb{1}_{(x<0)}x$  can be written as

$$x^+ = \frac{x + |x|}{2} \quad \text{and} \quad x^- = \frac{x - |x|}{2},$$

we can rewrite  $R_{t,ij}^{n+1} \Delta_{ij}^n(dt)$  for any  $i \neq j$  as

$$\begin{aligned} R_{t,ij}^{n+1}(t) \Delta_{ij}^n(dt) &= \frac{R_{t,ij}^{n+1}(t) + |R_{t,ij}^{n+1}(t)|}{2} U_{ij}^n(dt) + \frac{R_{t,ij}^{n+1}(t) - |R_{t,ij}^{n+1}(t)|}{2} L_{ij}^n(dt) \\ &= R_{t,ij}^{n+1}(t) \left( \frac{U_{ij}^n + L_{ij}^n}{2} \right) (dt) + |R_{t,ij}^{n+1}(t)| \left( \frac{U_{ij}^n - L_{ij}^n}{2} \right) (dt), \end{aligned}$$

for  $t \in (0, T]$ . Applying the triangle inequality and the fact that  $U_{ij}^n + L_{ij}^n$  and  $U_{ij}^n - L_{ij}^n$  are increasing, we arrive at the bound

$$|R_{t,ij}^{n+1}(t)| |\Delta_{ij}^n| (dt) \leq |R_{t,ij}^{n+1}(t)| U_{ij}^n(dt), \quad \forall i, j \in \mathcal{J} : i \neq j$$

Furthermore by Assumption 4.3.9 we obtain the bounds

$$\begin{aligned} |b_{z_n}^n(t, f(t))| |\gamma_{z_n}^n| (dt) &\leq \beta(t)(1 + |f(t)|) |\bar{\gamma}| (dt) \leq K_\beta(1 + \|f\|_{\text{BV}}) |\bar{\gamma}| (dt) \\ |b_{z_nj}^n(t, f_{t,z_nj}^{n+1}(t), f(t))| &\leq \beta(t)(1 + |f(t)| + |f_{t,z_nj}^{n+1}(t)|) \\ &\leq K_\beta \left( 1 + \|f\|_{\text{BV}} + \sup_{\omega^{n+1} \in \Omega^{n+1}} \|f^{n+1}\|_{\text{BV}}(\omega^{n+1}) \right). \end{aligned}$$

By the triangle inequality and Proposition 4.A.2(iv) we thus have

$$\begin{aligned} \|F(f)\|_{\text{BV}} &\leq (1 + K_\beta) \left( 1 + \|f\|_{\text{BV}} + \sup_{\omega^{n+1} \in \Omega^{n+1}} \|f^{n+1}\|_{\text{BV}}(\omega^{n+1}) \right) \sum_{j:j \neq z_n} |U_{z_nj}^n|(T) \\ &\quad + |Y^n| + K_\beta(1 + \|f\|_{\text{BV}}) |\gamma_{z_n}^n|(T) < \infty. \end{aligned}$$

Next, the goal is to show that the mapping  $F$  is a contraction. Let  $f, g \in \text{BV}_{[0,T]}$  and note that

$$\begin{aligned} F(f)(t) - F(g)(t) &= \int_{(t,T]} b_{z_n}^n(s, f(s)) - b_{z_n}^n(s, g(s)) \gamma_{z_n}^n(ds) \\ &\quad + \sum_{j:j \neq z_n} \int_{(t,T]} |R_{t,z_nj}^{n+1,f}(s)| - |R_{t,z_nj}^{n+1,g}(s)| \left( \frac{U_{z_nj}^n - L_{z_nj}^n}{2} \right) (ds) \\ &\quad + \sum_{j:j \neq z_n} \int_{(t,T]} R_{t,z_nj}^{n+1,f}(s) - R_{t,z_nj}^{n+1,g}(s) \left( \frac{U_{z_nj}^n + L_{z_nj}^n}{2} \right) (ds), \end{aligned}$$

for all  $t \in [0, T]$ . Let now  $0 \leq t_1 < t_2 \leq T$ . By applying the triangle inequality, the reverse triangle inequality, the definitions of  $R_{t,ij}^{n+1,f}$  and  $R_{t,ij}^{n+1,g}$ , and Assumption 4.3.9(iii)-(iv), we arrive at

$$\begin{aligned} |F(f)(t_2) - F(g)(t_2)) - (F(f)(t_1) - F(g)(t_1))| &\leq \int_{(t_1, t_2]} K_\beta |f(s) - g(s)| |\gamma^n|(ds) \\ &+ \sum_{j: j \neq z_n} \int_{(t_1, t_2]} (1 + K_\beta) |f(s) - g(s)| U_{z_n j}^n(ds). \end{aligned}$$

By inserting this into the definition of the variation, we find that this above bound with  $t_1 = 0$  and  $t_2 = T$  applies to  $\|f_i^{n,k+1} - f_i^{n,k}\|_{BV,T}$ . Using Proposition 4.A.2(iii) and the definition of  $\nu$  we get

$$\begin{aligned} \|F(f) - F(g)\|_{BV}^{c,\nu} &\leq \int_{(0,T]} e^{-c(\nu(T)-\nu(s-))} \left( |f(T) - g(T)| + \int_{(s,T]} |f - g|(du) \right) \nu(ds) \end{aligned}$$

which, after an application of Tonelli's Theorem, yields

$$\begin{aligned} \|F(f) - F(g)\|_{BV}^{c,\nu} &\leq |f(T) - g(T)| \int_{(0,T]} e^{-c(\nu(T)-\nu(s-))} \nu(ds) \\ &+ \int_{(0,T]} \int_{(0,u)} e^{-c(\nu(T)-\nu(s-))} \nu(ds) |f - g|(du). \end{aligned}$$

An application of Lemma 4.A.6 yields

$$\|F(f) - F(g)\|_{BV}^{c,\nu} \leq \frac{1}{c} \|f - g\|_{BV}^{c,\nu},$$

and by choosing  $c > 1$ , we thus have shown that  $F$  is a contraction. An application of the Banach fixed-point theorem finishes the proof.  $\square$

Since  $f^n$  depends on  $f^{n+1}$ , the system (4.3.7) is coupled upwards in  $n$ , and since  $n \in \mathbb{N}_0$  there is no natural starting point from which to iterate backwards. Therefore, we consider a truncated version of (4.3.7). For  $M \in \mathbb{N}$ , set

$$\begin{aligned} L_{ij}^{n,M}(t) &:= \mathbb{1}_{(n < M)} L_{ij}^n(t), \quad n \in \mathbb{N}_0 \\ U_{ij}^{n,M}(t) &:= \mathbb{1}_{(n < M)} U_{ij}^n(t), \quad n \in \mathbb{N}_0 \end{aligned}$$

and use this to define the system

$$\begin{aligned} f^{n,M}(dt, \omega^n) &= -b_{z_n}^n(t, \omega^n, f^{n,M}(t, \omega^n)) \gamma_{z_n}^n(dt, \omega^n) \\ &\quad - \sum_{j: j \neq z_n} R_{z_n j}^{n+1,M}(t, \omega^n) \underline{\Lambda}_{z_n j}^{n,M}(dt, \omega^n) \\ f^{n,M}(T, \omega^n) &= Y^n(\omega^n), \end{aligned} \tag{4.3.8}$$

for  $t \in (0, T]$ ,  $\omega^n \in \Omega^n$  and  $n \in \mathbb{N}_0$ , where

$$\begin{aligned} R_{ij}^{n+1,M}(t) &= b_{ij}^n(t, f^{n,M}(t), f_{t,ij}^{n+1,M}(t)) + f_{t,ij}^{n+1,M}(t) - f^{n,M}(t) \\ \Delta_{ij}^{n,M}(dt) &= \mathbb{1}_{(R_{t,ij}^{n+1,M}(t) \geq 0)} U_{ij}^{n,M}(dt) + \mathbb{1}_{(R_{t,ij}^{n+1,M}(t) < 0)} L_{ij}^{n,M}(dt). \end{aligned}$$

Since for  $n \geq M$  the function  $f^{n,M}$  does not depend on  $f^{n+1,M}$ , we have successfully constructed a system in which only finitely many  $f^n$  are coupled.

From now on we let  $K_\beta := \sup_{t \in [0, T]} \beta(t)$ ,  $K_Y := \sup_{\omega \in \Omega} |Y(\omega)|$  and define  $\kappa := 1 + K_\beta$  and  $K_1 := 1 + 2K_Y$ . We let  $c > 1$  and set

$$\nu(t) := (1 + K_\beta) \left( \bar{\gamma}(t) + \sum_{i,j:i \neq j} \alpha_{ij}(t) \right).$$

Note that  $\nu$  does not depend on  $\omega^n$  for any  $n \in \mathbb{N}_0$ . We now have the following result:

**Lemma 4.3.13.** *It holds that*

$$\sup_{M \in \mathbb{N}_0} \sup_{n \in \mathbb{N}_0, \omega^n \in \Omega^n} \|f^{n,M}\|_{BV}(\omega^n) \leq K_Y + e^{(4e^{2\nu(T)}+1)\nu(T)} K_1,$$

and there exists a unique solution of the system (4.3.8) for all  $M \in \mathbb{N}_0$ .

*Proof.* Fix  $M \in \mathbb{N}_0$  and let  $n \geq M$ . In this case, we have that

$$f^{n,M}(t) = Y^n + \int_{(t,T]} b_{z_n}^n(s, f^{n,M}(s)) \gamma_{z_n}(ds), \quad \forall t \in (0, T]$$

Since  $f^{n,M}$  does not depend on  $f^{n+1,M}$ , Lemma 4.3.12 immediately yields existence and uniqueness. For any  $t \in [0, T]$  similar arguments as in the proof of Lemma 4.3.12 yield

$$\begin{aligned} \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} |f^{n,M}|(ds) &\leq \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} (1 + K_Y) \nu(ds) \\ &\quad + \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} \int_{(s,T]} |f^{n,M}|(du) \nu(ds) \\ &\leq \frac{c}{c-1} \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} K_1 \nu(ds) \leq \frac{1}{c-1} K_1, \end{aligned}$$

for all  $t \in [0, T]$ . Choosing  $c = 2$ , we consequently obtain

$$\|f^{n,M}\|_{BV}(\omega^n) \leq K_Y + e^{2\nu(T)} K_1 \leq K_Y + e^{(4e^{2\nu(T)}+1)\nu(T)} K_1 \quad \forall \omega^n \in \Omega^n.$$

which implies the boundedness condition.

For  $n < M$  we will prove existence, uniqueness and the boundedness condition by proving the auxiliary result that for each  $n \in \mathbb{N}_0$ , there exists a non-decreasing function  $g^{n,M} \in \text{BV}_{[0,T]}$  such that

$$\int_{(t,T]} |f^{n,M}|(\omega^n)(ds) \leq \int_{(t,T]} g^{n,M}(ds), \quad \forall t \in [0, T], \forall \omega^n \in \Omega^n, \quad (4.3.9)$$

which we will prove by backwards induction starting at  $n = M$ . For  $n \geq M$  we can set  $c = 2$  and use the previous calculations to obtain (4.3.9) with  $g^{n,M}(ds) = 2e^{2\nu(T)}K_1\nu(ds)$ . Let now  $n < M$  and suppose that  $f^{n+1,M}$  satisfies (4.3.9), exists and is unique. Then since the bound (4.3.9) does not depend on  $\omega^{n+1}$ , Lemma 4.3.12 immediately yields existence and uniqueness of  $f^{n,M}$ . By similar calculations as before, we obtain

$$\begin{aligned} \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} |f^{n,M}|(ds) &\leq \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} \int_{(s,T]} |f^{n,M}|(du) \nu(ds) \\ &+ \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} \left( K_1 \nu(ds) + \sum_{j:j \neq z_n} \int_{(s,T]} |f_{s,z_n j}^{n+1,M}|(du) \kappa U_{z_n j}(ds) \right), \end{aligned}$$

for all  $t \in [0, T]$ . Using the induction hypothesis, switching the order of integration by virtue of Tonelli's Theorem, using Lemma 4.A.6 and rearranging yields

$$\begin{aligned} \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} |f^{n,M}|(ds) \\ \leq \frac{c}{c-1} \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} \left( K_1 + \int_{(s,T]} g^{n+1,M}(du) \right) \nu(ds), \end{aligned}$$

for all  $t \in [0, T]$ . Setting  $c = 2$  and rearranging we obtain (4.3.9) with

$$g^{n,M}(ds) = 2e^{2\nu(T)} \left( K_1 + \int_{(s,T]} g^{n+1,M}(du) \right) \nu(ds).$$

We thus established that (4.3.9) is valid for all  $n < M$  and that existence and uniqueness hold for all  $n < M$  as well. Using this in conjunction with switching the order of integration by virtue of Tonelli's Theorem and Lemma 4.A.6 to iterate

upwards from  $n$  to  $M$ , we obtain:

$$\begin{aligned}
& \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} |f^{n,M}|(ds) \\
& \leq \frac{c}{c-1} \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} \left( K_1 + \int_{(s,T]} g^{n+1,M}(du) \right) \nu(ds) \\
& \leq \frac{1}{c-1} K_1 + \frac{1}{c-1} \int_{(0,T]} e^{-c(\nu(T)-\nu(s-))} g^{n+1,M}(ds) \\
& \leq \frac{1}{c-1} K_1 + \frac{2e^{2\nu(T)}}{c-1} \int_{(0,T]} e^{-c(\nu(T)-\nu(s-))} \left( K_1 + \int_{(s,T]} g^{n+2,M}(du) \right) \nu(ds) \\
& \leq \frac{2e^{2\nu(T)}}{c-1} K_1 + \left( \frac{2e^{2\nu(T)}}{c-1} \right)^2 K_1 + \left( \frac{2e^{2\nu(T)}}{c-1} \right)^2 \int_{(0,T]} e^{-c(\nu(T)-\nu(s-))} g^{n+2,M}(ds) \\
& \leq \dots \leq K_1 \sum_{\ell=1}^{M-n} \left( \frac{2e^{2\nu(T)}}{c-1} \right)^\ell \leq K_1 \sum_{\ell=1}^{\infty} \left( \frac{2e^{2\nu(T)}}{c-1} \right)^\ell, \quad \forall t \in [0, T].
\end{aligned}$$

Setting  $c = 4e^{2\nu(T)} + 1$  yields

$$\int_{(0,T]} |f^{n,M}|(ds, \omega^n) \leq e^{(4e^{2\nu(T)}+1)\nu(T)} K_1 \sum_{\ell=1}^{\infty} \frac{1}{2^\ell} = e^{(4e^{2\nu(T)}+1)\nu(T)} K_1, \quad \forall \omega^n \in \Omega^n.$$

As this bound is independent of  $M$ ,  $n$ , and  $\omega^n$ , taking the supremums yields the desired result.  $\square$

**Lemma 4.3.14.** *For each  $M \in \mathbb{N}_0$  let  $(f^{n,M})_{n \in \mathbb{N}_0}$  be a solution of the system (4.3.8). For any  $n \in \mathbb{N}_0$  the sequence  $(f^{n,M})_{M \in \mathbb{N}_0}$  is a Cauchy sequence in  $(BV_{[0,T]}, \|\cdot\|_{BV})$ .*

*Proof.* Fix  $n \in \mathbb{N}_0$  and suppose that  $n \leq N < M$ . We will start by proving the auxiliary result that for each  $n < N$ , there exists a non-decreasing function  $g_n^{N,M} \in BV_{[0,T]}$  such that

$$\int_{(t,T]} |f^{n,M} - f^{n,M}|(ds, \omega^n) \leq \int_{(t,T]} g_{n+1}^{N,M}(ds), \quad \forall t \in [0, T], \forall \omega^n \in \Omega^n. \quad (4.3.10)$$

As in the proof of Lemma 4.3.13, we will use backwards induction. For  $n = N$ , we can, by similar calculations, Tonelli's Theorem, and an application of Lemma 4.3.12

and Lemma 4.A.6, arrive at

$$\begin{aligned}
& \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} |f^{N,M} - f^{N,N}|(ds) \\
& \leq \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} \int_{(s,T]} |f^{N,M} - f^{N,N}|(du) \nu(ds) \\
& \quad + \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} \sum_{j:j \neq z_n} \kappa(1 + \|f_{s,z_n,j}^{N+1,M}\|_{BV} + \|f^{N,M}\|_{BV}) U_{z_n,j}(ds) \\
& \leq \frac{c}{c-1} \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} (1 + 2K_2) \nu(ds), \quad \forall t \in [0, T], \forall \omega^N \in \Omega^N,
\end{aligned}$$

where  $K_2 := K_Y + e^{(4e^{2\nu(T)}+1)\nu(T)} K_1$ . Thus for  $c = 2$ , we have that (4.3.10) is satisfied with  $g_N^{N,M}(ds) = 2e^{2\nu(T)}(1 + 2K_2)\nu(ds)$ . Assume now that (4.3.10) holds for  $n + 1$ . Then calculations similar to those in the proof of Lemma 4.3.12 and the induction hypothesis yield

$$\begin{aligned}
& \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} |f^{n,M} - f^{n,M}|(ds) \\
& \leq \frac{c}{c-1} \int_{(t,T]} e^{-c(\nu(T)-\nu(s-))} \int_{(s,T]} g_{n+1}^{N,M}(du) \nu(ds),
\end{aligned}$$

for all  $t \in [0, T]$ , which for  $c = 2$  yields (4.3.10) with

$$g_n^{N,M}(ds) = 2e^{2\nu(T)} \int_{(s,T]} g_{n+1}^{N,M}(du) \nu(ds).$$

Iterating forwards from  $n$  to  $N$  as in the proof of Lemma 4.3.13 yields

$$\begin{aligned}
& \int_{(0,T]} e^{-c(\nu(T)-\nu(s-))} |f^{N,M} - f^{N,N}|(ds) \leq \frac{1}{c-1} \int_{(0,T]} e^{-c(\nu(T)-\nu(s-))} g_{n+1}^{N,M}(ds) \\
& \leq \frac{2e^{2\nu(T)}}{c-1} \int_{(0,T]} 2e^{2\nu(T)} \int_{(s,T]} g_{n+2}^{N,M}(du) \nu(ds) \\
& \leq \left( \frac{2e^{2\nu(T)}}{c-1} \right)^2 \int_{(0,T]} e^{-c(\nu(T)-\nu(s-))} g_{n+2}^{N,M}(ds) \\
& \leq \dots \leq \left( \frac{2e^{2\nu(T)}}{c-1} \right)^{N-n} (1 + 2K_2)
\end{aligned}$$

Setting  $c = 4e^{2\kappa\nu(T)} + 1$  and rearranging yields the bound

$$\|f^{n,M} - f^{n,N}\|_{BV}(\omega^n) \leq \frac{1}{2^{N-n}} (1 + 2K_2) e^{(4e^{2\nu(T)}+1)\nu(T)}, \quad \forall \omega^n \in \Omega^n,$$

which vanishes for  $N \rightarrow \infty$ , thus finishing the proof.  $\square$

**Lemma 4.3.15.** *Assume that for all  $n \in \mathbb{N}_0$  the sequence  $(f^{n,M})_{M \in \mathbb{N}_0}$  has the limit  $f^n$  in  $BV_{[0,T]}$ . Then  $(f^n)_{n \in \mathbb{N}_0}$  is a solution of the system (4.3.7) and the process  $X$  with jump-count representation  $(f^n)_{n \in \mathbb{N}_0}$  is bounded on finite intervals.*

*Proof.* Let  $n \in \mathbb{N}_0$  be given. Let  $f^n$  be the limit of  $(f^{n,M})_{M \in \mathbb{N}_0}$  and set

$$\tilde{f}^n(t) := Y^n + \int_{(t,T]} b_{z_n}^n(t, f^n(t)) \gamma_{z_n}^n(ds) - \sum_{j: j \neq z_n} \int_{(t,T]} R_{s, z_n j}^{n+1}(s) \Lambda_{z_n j}^n(ds),$$

for all  $t \in [0, T]$ . Let  $M$  be large enough such that  $n < M$ . Then  $L_{ij}^{n,M} = L_{ij}^n$  and  $U_{ij}^{n,M} = U_{ij}^n$  by definition and thus by similar arguments as in the previous proofs, we get

$$\|f^{n,M} - \tilde{f}^n\|_{BV} \leq \nu(T) \|f^{n,M} - f^n\|_{BV} + \sum_{j: j \neq z_n} \int_{(0,T]} \|f_{t, z_n j}^{n+1,M} - f_{t, z_n j}^{n+1}\|_{BV} \nu(dt)$$

By assumption  $\lim_{n \rightarrow \infty} \|f^{n,M} - f^n\|_{BV} = 0$  and  $\lim_{n \rightarrow \infty} \|f_{t, z_n j}^{n+1,M} - f_{t, z_n j}^{n+1}\|_{BV} = 0$  for all  $t \in [0, T]$ . Thus, the first term vanishes, and the bound in Lemma 4.3.12 allows us to apply the dominated convergence theorem to the second term, which allows us to conclude that

$$\lim_{M \rightarrow \infty} \|f^{n,M} - \tilde{f}^n\|_{BV} = 0.$$

Thus the uniqueness of limits yields  $(\tilde{f}^n)_{n \in \mathbb{N}_0} = (f^n)_{n \in \mathbb{N}_0}$ , which means that  $(f^n)_{n \in \mathbb{N}_0}$  indeed is a solution of (4.3.7). Finally, as the limit operator preserves bounds, we have by Lemma 4.3.13 that

$$\sup_{n \in \mathbb{N}_0, \omega^n \in \Omega^n} \|f^n\|_{BV}(\omega^n) \leq K_Y + e^{(4e^{2\nu(T)}+1)\nu(T)} K_1,$$

which implies that  $X(t) = \sum_{n \in \mathbb{N}_0} I^n(t) f^n(t)$  is bounded on finite intervals.  $\square$

**Lemma 4.3.16.** *Let  $X$  be a solution of (4.3.5) that is càdlàg, of finite variation, and bounded on finite intervals. Then  $X$  is unique.*

*Proof.* Assume that  $X_1, X_2$  both are solutions of (4.3.5), bounded on finite intervals, with jump-count representations  $(f^n)_{n \in \mathbb{N}_0}$  and  $(g^n)_{n \in \mathbb{N}_0}$ . Then the process  $X_1 - X_2$  is bounded on finite intervals, and thus there exists a constant  $K > 0$  such that

$$\sup_{t \in [0, T]} \sup_{n \in \mathbb{N}_0, \omega^n \in \Omega^n} |f^n(t, \omega^n) - g^n(t, \omega^n)| \leq K.$$

By a backwards induction argument starting at any  $M \in \mathbb{N}_0$  followed by a forward iteration from any  $n < M$  to  $M$ , exactly as in the proof of Lemma 4.3.14, we obtain that for all  $M \in \mathbb{N}$  and all  $n < M$  it holds that

$$\int_{(0,T]} e^{-c(\nu(T)-\nu(s-))} |f^n - g^n|(ds) \leq \left( \frac{2e^{2\nu(T)}}{c-1} \right)^{M-n} K.$$

Choosing  $c = 4e^{2\nu(T)} + 1$  yields for any  $M \in \mathbb{N}_0$  and  $n < M$  the bound

$$\|f^n - g^n\|_{BV}(\omega^n) \leq \frac{1}{2^{M-n}} h(T) e^{(4e^{2\nu(T)}+1)\nu(T)}, \quad \forall \omega^n \in \Omega^n.$$

By taking the infimum over  $M$  on both sides we obtain  $\|f^n - g^n\|_{BV}(\omega^n) = 0$  for all  $\omega^n \in \Omega^n$  and  $n \in \mathbb{N}_0$ , thus yielding the desired uniqueness.  $\square$

## 4.4 The martingale optimality principle

We now formulate our robust utility maximisation problem and show the associated martingale optimality principle, which is a sufficient condition for the existence of a solution to our posed problem. Both the problem of determining worst- and best-case prospective reserves and the robust consumption-insurance problem are special cases of the problem presented in this section. Consider again the filtered canonical measurable space  $(\Omega, \mathcal{F}, \mathbb{F})$ . Let  $\pi$  be a known and fixed initial distribution and assume that the cumulative transition rates  $\Lambda = (\Lambda_{ij})_{i \neq j}$  determining the true measure  $\mathbb{P}^\Lambda$  are unknown. Instead, statistical methods or expert judgement provide known upper and lower bounds for the true cumulative transition rates, which we assume are given by the cumulative transition rates  $(L_{ij})_{i \neq j}$  and  $(U_{ij})_{i \neq j}$  satisfying Assumption 4.2.12 such that for all  $i, j \in \mathcal{J} : i \neq j$ :

$$L_{ij}(s, \omega) - L_{ij}(t, \omega) \leq U_{ij}(s, \omega) - U_{ij}(t, \omega), \quad \forall 0 \leq t \leq s, \forall \omega \in \Omega. \quad (4.4.1)$$

We are only interested in biometric scenarios within these bounds, and thus, all admissible cumulative transition rates are defined as follows:

**Definition 4.4.1.** *The processes  $\Lambda = (\Lambda_{ij})_{i \neq j}$  are admissible cumulative transition rates if they satisfy Assumption 4.2.12 and if for all  $i, j \in \mathcal{J} : i \neq j$ :*

$$L_{ij}(s, \omega) - L_{ij}(t, \omega) \leq \Lambda_{ij}(s, \omega) - \Lambda_{ij}(t, \omega) \leq U_{ij}(s, \omega) - U_{ij}(t, \omega), \quad (4.4.2)$$

for all  $0 \leq t \leq s$  and  $\omega \in \Omega$ . The set of admissible  $\Lambda$  is denoted by  $\mathcal{M}$ .

With the admissible biometric scenarios in place, we fix a deterministic terminal time  $T > 0$  and consider the following problem:

$$\max_{\theta \in \mathcal{A}} \min_{\Lambda \in \mathcal{M}} \mathbb{E}^\Lambda \left[ \int_{(0, T]} \frac{1}{\mathcal{E}_\Phi(s)} U^{\theta, \Lambda}(ds) + \frac{1}{\mathcal{E}_\Phi(T)} Y^{\theta, \Lambda} \right]. \quad (4.4.3)$$

The process  $U^{\theta, \Lambda}$  is the cumulative utility process, which measures the total utility gained over time, depending on the control  $\theta$  and the biometric scenario  $\Lambda$ , while the  $\mathcal{F}_T$ -measurable random variable  $Y^{\theta, \Lambda}$  measures the terminal utility. The control  $\theta$  is assumed to be an optional process taking values in a measurable space  $(E, \mathcal{E})$  and the process  $\mathcal{E}_\Phi$  is defined as the generalised exponential

$$\mathcal{E}_\Phi(dt) = \mathcal{E}_\Phi(t-) \Phi(dt), \quad \mathcal{E}_\Phi(0) = 1 \quad \Leftrightarrow \quad \mathcal{E}_\Phi(t) = e^{\Phi^c(t) - \Phi^c(0)} \prod_{0 < s \leq t} (1 + \Delta\Phi(s))$$

for some predictable, càdlàg, finite variation process  $\Phi$  that is bounded on finite intervals and satisfies  $\inf_{(t, \omega) \in [0, T] \times \Omega} \Delta\Phi(t, \omega) > -1$ . The process  $\Phi$  can be interpreted as a subjective cumulative discount rate taking into account the subjective time-value of utility. The class of admissible controls is as follows:

**Definition 4.4.2.** *The optional process  $\theta : [0, T] \times \Omega \rightarrow E$  is admissible if for all  $\Lambda \in \mathcal{M}$  it holds that  $U^{\theta, \Lambda}$  is a càdlàg, finite variation process and*

$$\sup_{t \in [0, T]} \mathbb{E}_{t, i}^{\Lambda} [|U^{\theta, \Lambda}(T) - U^{\theta, \Lambda}(t) + Y^{\theta, \Lambda}|](\omega) < \infty, \quad \forall \omega \in \Omega, \forall i \in \mathcal{I}.$$

*The set of admissible  $\theta$  is denoted by  $\mathcal{A}$ .*

The objective of (4.4.3) is to find an admissible pair  $(\underline{\theta}, \underline{\Lambda})$  such that  $\underline{\theta} \in \mathcal{A}$  is the best possible control in the worst possible biometric scenario  $\underline{\Lambda} \in \mathcal{M}$ . In order to formalise this mathematically, we associate with the problem (4.4.3), the expected utility process  $\mathcal{V}(t, \theta, \Lambda)$  given by

$$\mathcal{V}(t, \theta, \Lambda) := \mathbb{E}_{t, Z(t)}^{\Lambda} \left[ \int_t^T \frac{\mathcal{E}_{\Phi}(t)}{\mathcal{E}_{\Phi}(s)} U^{\theta, \Lambda}(\mathrm{d}s) + \frac{\mathcal{E}_{\Phi}(t)}{\mathcal{E}_{\Phi}(T)} Y^{\theta, \Lambda} \right].$$

Note that  $\mathcal{V}(t, \theta, \Lambda)$  is an optional projection of the discounted form and it can be interpreted as the subjective expected present value of future utility associated with the control  $\theta$  under the biometric scenario  $\Lambda$  at time  $t$ . This process is important because it allows us to define what constitutes optimality.

**Definition 4.4.3.** *The admissible pair  $(\underline{\theta}, \underline{\Lambda})$  constitutes the robust optimal control and the worst-case biometric scenario if and only if for all  $t \in [0, T]$  it holds that*

$$\mathcal{V}(t, \theta, \underline{\Lambda}) \leq \mathcal{V}(t, \underline{\theta}, \underline{\Lambda}) \leq \mathcal{V}(t, \underline{\theta}, \Lambda), \quad \forall \theta \in \mathcal{A}, \Lambda \in \mathcal{M}.$$

We now come to the martingale optimality principle, which serves as a sufficient condition for the existence of robust optimal controls and worst-case biometric scenarios. Let  $\underline{\theta} \in \mathcal{A}$  and  $\underline{\Lambda} \in \mathcal{M}$  be the candidate robust optimal control and worst-case biometric scenario and assume that there exists a càdlàg, finite variation process  $V$ , such that  $V(T) = Y^{\underline{\theta}, \underline{\Lambda}}$ . For arbitrary  $\theta \in \mathcal{A}$  and  $\Lambda \in \mathcal{M}$  let  $M^{\theta, \Lambda}(t)$  be given by

$$M^{\theta, \Lambda}(t) := \int_{(0, t]} \frac{1}{\mathcal{E}_{\Phi}(s)} U^{\theta, \Lambda}(\mathrm{d}s) + \frac{1}{\mathcal{E}_{\Phi}(t)} V(t).$$

The process  $M^{\theta, \Lambda}$  consists of two parts. The first part is the utility accumulated until time  $t$  under the control  $\theta$  and the biometric scenario  $\Lambda$ , discounted back to time zero. The second part is the expected present value of future utility under the control  $\underline{\theta}$  and the biometric scenario  $\underline{\Lambda}$ , discounted back to time zero. We now have the following result.

**Theorem 4.4.4** (Martingale optimality principle). *Let  $(\underline{\theta}, \underline{\Lambda})$  be admissible and let  $V$  be càdlàg and of finite variation satisfying  $V(T) = Y^{\underline{\theta}, \underline{\Lambda}}$ . If*

- (i)  $M^{\underline{\theta}, \underline{\Lambda}}$  is a  $(\mathbb{F}, \mathbb{P}^{\underline{\Lambda}})$ -martingale

- (ii)  $M^{\theta, \Lambda}$  is a  $(\mathbb{F}, \mathbb{P}^\Lambda)$ -supermartingale for all  $\theta \in \mathcal{A}$   
 (iii)  $M^{\theta, \Lambda}$  is a  $(\mathbb{F}, \mathbb{P}^\Lambda)$ -submartingale for all  $\Lambda \in \mathcal{M}$

then  $\underline{\theta}$  and  $\underline{\Lambda}$  are robust optimal controls and

$$V(t) = \sup_{\theta \in \mathcal{A}} \inf_{\Lambda \in \mathcal{M}} \mathcal{V}(t, \theta, \Lambda) = \inf_{\Lambda \in \mathcal{M}} \sup_{\theta \in \mathcal{A}} \mathcal{V}(t, \theta, \Lambda) = \mathcal{V}(t, \underline{\theta}, \underline{\Lambda}).$$

*Proof.* We have to show that  $(\underline{\theta}, \underline{\Lambda})$  is a saddle point of  $\mathcal{V}(t, \theta, \Lambda)$  for all  $t \in [0, T]$ . Since  $M^{\underline{\theta}, \underline{\Lambda}}$  is a  $(\mathbb{F}, \mathbb{P}^{\underline{\Lambda}})$ -martingale, we have that  $M^{\underline{\theta}, \underline{\Lambda}}(t) = \mathbb{E}_{t, Z(t)}^{\underline{\Lambda}}[M^{\underline{\theta}, \underline{\Lambda}}(T)]$ , which yields

$$\begin{aligned} \int_0^t \frac{1}{\mathcal{E}_\Phi(s)} U^{\underline{\theta}, \underline{\Lambda}}(ds) + \frac{1}{\mathcal{E}_\Phi(t)} V(t) \\ = \mathbb{E}_{t, Z(t)}^{\underline{\Lambda}} \left[ \int_0^T \frac{1}{\mathcal{E}_\Phi(s)} U^{\underline{\theta}, \underline{\Lambda}}(ds) + \frac{1}{\mathcal{E}_\Phi(T)} u_{Z(T)}^T(X^{\underline{\theta}, \underline{\Lambda}}(T)) \right]. \end{aligned}$$

Rearranging gives us

$$V(t) = \mathbb{E}_{t, Z(t)}^{\underline{\Lambda}} \left[ \int_t^T \frac{\mathcal{E}_\Phi(t)}{\mathcal{E}_\Phi(s)} U^{\underline{\theta}, \underline{\Lambda}}(ds) + \frac{\mathcal{E}_\Phi(t)}{\mathcal{E}_\Phi(T)} u_{Z(T)}^T(X^{\underline{\theta}, \underline{\Lambda}}(T)) \right] = \mathcal{V}(t, \underline{\theta}, \underline{\Lambda}).$$

Let  $\theta \in \mathcal{A}$  be arbitrary. Since  $M^{\theta, \underline{\Lambda}}$  is a  $(\mathbb{F}, \mathbb{P}^{\underline{\Lambda}})$ -supermartingale, we have that  $M^{\theta, \underline{\Lambda}}(t) \geq \mathbb{E}_{t, Z(t)}^{\underline{\Lambda}}[M^{\theta, \underline{\Lambda}}(T)]$ , which yields

$$\begin{aligned} \int_0^t \frac{1}{\mathcal{E}_\Phi(s)} U^{\theta, \underline{\Lambda}}(ds) + \frac{1}{\mathcal{E}_\Phi(t)} V(t) \\ \geq \mathbb{E}_{t, Z(t)}^{\underline{\Lambda}} \left[ \int_0^T \frac{1}{\mathcal{E}_\Phi(s)} U^{\theta, \underline{\Lambda}}(ds) + \frac{1}{\mathcal{E}_\Phi(T)} u_{Z(T)}^T(X^{\theta, \underline{\Lambda}}(T)) \right]. \end{aligned}$$

Since  $V(t) = \mathcal{V}(t, \underline{\theta}, \underline{\Lambda})$ , rearranging yields

$$\mathcal{V}(t, \underline{\theta}, \underline{\Lambda}) \geq \mathbb{E}_{t, Z(t)}^{\underline{\Lambda}} \left[ \int_t^T \frac{\mathcal{E}_\Phi(t)}{\mathcal{E}_\Phi(s)} U^{\theta, \underline{\Lambda}}(ds) + \frac{\mathcal{E}_\Phi(t)}{\mathcal{E}_\Phi(T)} u_{Z(T)}^T(X^{\theta, \underline{\Lambda}}(T)) \right] = \mathcal{V}(t, \theta, \underline{\Lambda}),$$

for all  $\theta \in \mathcal{A}$ . Let  $\Lambda \in \mathcal{M}$  be arbitrary. Since  $M^{\underline{\theta}, \Lambda}$  is a  $(\mathbb{F}, \mathbb{P}^\Lambda)$ -submartingale, we have that  $M^{\underline{\theta}, \Lambda}(t) \leq \mathbb{E}_{t, Z(t)}^\Lambda[M^{\underline{\theta}, \Lambda}(T)]$ , which yields

$$\begin{aligned} \int_0^t \frac{1}{\mathcal{E}_\Phi(s)} U^{\underline{\theta}, \Lambda}(ds) + \frac{1}{\mathcal{E}_\Phi(t)} V(t) \\ \leq \mathbb{E}_{t, Z(t)}^\Lambda \left[ \int_0^T \frac{1}{\mathcal{E}_\Phi(s)} U^{\underline{\theta}, \Lambda}(ds) + \frac{1}{\mathcal{E}_\Phi(T)} u_{Z(T)}^T(X^{\underline{\theta}, \Lambda}(T)) \right]. \end{aligned}$$

Since  $V(t) = \mathcal{V}(t, \underline{\theta}, \underline{\Lambda})$ , rearranging yields

$$\mathcal{V}(t, \underline{\theta}, \underline{\Lambda}) \leq \mathbb{E}_{t, Z(t)}^\Lambda \left[ \int_t^T \frac{\mathcal{E}_\Phi(t)}{\mathcal{E}_\Phi(s)} U^{\underline{\theta}, \Lambda}(ds) + \frac{\mathcal{E}_\Phi(t)}{\mathcal{E}_\Phi(T)} u_{Z(T)}^T(X^{\underline{\theta}, \Lambda}(T)) \right] = \mathcal{V}(t, \underline{\theta}, \Lambda),$$

for all  $\Lambda \in \mathcal{M}$ . Thus we can conclude that  $(\underline{\theta}, \underline{\Lambda})$  is optimal.  $\square$

If one is interested only in the case of absolutely continuous cumulative transition rates determined by the predictable transition intensities  $(\mu_{ij})_{i \neq j}$ , see (4.2.10), statistical methods or expert judgement can provide upper and lower bounds  $(U_{ij})_{i \neq j}$  and  $(L_{ij})_{i \neq j}$  of the same absolutely continuous form, with transition intensities  $(l_{ij})_{i \neq j}$  and  $(u_{ij})_{i \neq j}$  satisfying Assumption 4.2.17. Thus instead of (4.4.1) we assume that

$$l_{ij}(t, \omega) \leq u_{ij}(t, \omega), \quad \forall (t, \omega) \in [0, \infty) \times \Omega, \quad \forall i, j \in \mathcal{J} : i \neq j, \quad (4.4.4)$$

which of course implies (4.4.1). The class of admissible cumulative transition rates can then be restricted to those of the form (4.2.10) using the following class of admissible predictable transition intensities:

**Definition 4.4.5.** *The processes  $\mu = (\mu_{ij})_{i \neq j}$  are admissible transition intensities if they satisfy Assumption 4.2.17 and if it holds that*

$$l_{ij}(t, \omega) \leq \mu_{ij}(t, \omega) \leq u_{ij}(t, \omega), \quad \forall (t, \omega) \in [0, \infty) \times \Omega, \quad \forall i, j \in \mathcal{J} : i \neq j. \quad (4.4.5)$$

The set of admissible  $\mu$  is denoted by  $\mathcal{M}$ .

In that case the problem (4.4.3) can be stated as

$$\max_{\theta \in \mathcal{A}} \min_{\mu \in \mathcal{M}} \mathbb{E}^\mu \left[ \int_{(0, T]} \frac{1}{\mathcal{E}_\Phi(s)} U^{\theta, \mu}(ds) + \frac{1}{\mathcal{E}_\Phi(T)} u_{Z(T)}^T(X^{\theta, \mu}(T)) \right], \quad (4.4.6)$$

where the biometric scenario is characterised directly by the transition intensities. Even though problem (4.4.6) reduces the class of admissible biometric scenarios compared to the problem (4.4.3), the objective is the same, and the martingale optimality principle of Theorem 4.4.4 remains valid.

## 4.5 Best- and worst-case prospective reserves

In life insurance, the process  $Z$  can be interpreted as the health status of an individual and depending on the individual's state and the transitions between states, they receive contractual payments in accordance with an insurance contract. The accumulated insurance payments are given by a process  $C \in \mathbb{Y}$ . By Proposition 4.2.10, any such process can be decomposed as follows:

$$C(dt) = \sum_{i \in \mathcal{J}} I_i(t-) \left( C_i(dt) + \sum_{j: j \neq i} \tilde{\Delta} C_{ij}(t) N_{ij}(dt) \right).$$

The processes  $(C_i)_{i \in \mathcal{J}}$  can be interpreted as the accumulated sojourn payments, which the individual receives/pays as long as the individual's health status  $Z(t)$  equals  $i$ , while the processes  $(\tilde{\Delta} C_{ij})_{i \neq j}$  can be interpreted as the transition payments, which the individual receives/pays upon the transition of the individual from state

$i$  to state  $j$ . We assume that there is a deterministic terminal time  $T > 0$  at which the insurance contract expires and after which no more payments are made. The present value of future payments is given by

$$L(t, T) := \int_{(t, T]} \frac{\mathcal{E}_\Phi(t)}{\mathcal{E}_\Phi(s)} C(ds),$$

where the cumulative interest rate  $\Phi$  is predictable, càdlàg, of finite variation and bounded on finite intervals, such that  $\inf_{(t, \omega) \in [0, T] \times \Omega} \Delta\Phi(t, \omega) > -1$ . The process  $\mathcal{E}_\Phi$  given by

$$\mathcal{E}_\Phi(t) = e^{\Phi^c(t) - \Phi^c(0)} \prod_{0 < s \leq t} (1 + \Delta\Phi(s)),$$

can be thought of as a savings account. The object of interest for the insurance company is the expected present value of future payments (also called the prospective reserve), defined as the optional projection of the present value of future payments given by

$$\mathcal{V}(t, \Lambda) := \mathbb{E}_{t, Z(t)}^\Lambda[L(t, T)] = \sum_{i \in \mathcal{I}} I_i(t) \mathbb{E}_{t, i}^\Lambda[L(t, T)], \quad t \in [0, T],$$

for the true biometric scenario  $\Lambda$ . The processes  $\tilde{V}_i(t) = \mathbb{E}_{t, i}^\Lambda[L(t, T)]$  are called the state-wise prospective reserves.

In practice, the true biometric scenario  $\Lambda$  is unknown. Statistical methods might provide a best estimate, which can be used for calculations, and confidence bounds, that can quantify the uncertainty on the level of the cumulative transition intensities. In light of this, we are now interested in the question: Given contractual payments  $C \in \mathbb{Y}$ , a cumulative interest rate  $\Phi$ , and the bounds (4.4.1) for the true biometric scenario, what are the worst- and best-case prospective reserves

$$\underline{V}(t) := \sup_{\Lambda \in \mathcal{M}} \mathbb{E}_{t, Z(t)}^\Lambda[L(t, T)] \quad \text{and} \quad \bar{V}(t) := \inf_{\Lambda \in \mathcal{M}} \mathbb{E}_{t, Z(t)}^\Lambda[L(t, T)],$$

and what are the corresponding worst- and best-case biometric scenarios  $\underline{\Lambda}$  and  $\bar{\Lambda}$ , which realise the supremum and infimum? Note that from the insurer's point of view, the prospective reserve is a liability and thus the worst-case value is the largest value this liability can take. From the insured's point of view, it is of course exactly the opposite. Going forward, we will refer to the insurer's point of view.

The two problems posed above can be recognised as a special case of the problem (4.4.3) as follows. The control  $\theta \in \mathbb{Y}$  is an insurance payment process, and the set of admissible controls is  $\mathcal{A} = \{C\}$ . The utility process  $U^{\theta, \Lambda}$  is just equal to the one admissible payment process  $C$  and  $Y^{\theta, \Lambda} = 0$ . Thus, we can apply the martingale optimality principle to obtain the following two results.

**Proposition 4.5.1.** *Assume that  $C \in \mathbb{Y}$  and that  $\Phi$  is predictable, càdlàg, of finite variation and bounded on finite intervals with  $\inf_{(t,\omega) \in [0,T] \times \Omega} \Delta\Phi(t,\omega) > -1$ . Let  $V$  be the solution of the BSDE*

$$V(dt) = V(t-)\Phi(dt) - C(dt) + \sum_{i,j:i \neq j} \bar{R}_{ij}(t)(N_{ij}(dt) - I_i(t-)\bar{\Lambda}_{ij}(dt)), \quad (4.5.1)$$

with terminal condition  $V(T) = 0$ , where  $\bar{R}_{ij}(t) := \tilde{\Delta}C_{ij}(t) + V_{ij}(t) - V_i(t)$  and

$$\bar{\Lambda}_{ij}(dt) = \mathbb{1}_{(\bar{R}_{ij}(t) \geq 0)}U_{ij}(dt) + \mathbb{1}_{(\bar{R}_{ij}(t) < 0)}L_{ij}(dt).$$

Then  $V(t) = \sup_{\Lambda \in \mathcal{M}} \mathbb{E}_{t,Z(t)}^{\bar{\Lambda}}[L(t,T)]$  and  $\bar{\Lambda} = (\bar{\Lambda}_{ij})_{i \neq j}$  is the worst-case scenario.

**Proposition 4.5.2.** *Assume that  $C \in \mathbb{Y}$  and that  $\Phi$  is predictable, càdlàg, of finite variation and bounded on finite intervals with  $\inf_{(t,\omega) \in [0,T] \times \Omega} \Delta\Phi(t,\omega) > -1$ . Let  $\bar{V}$  be a solution of the BSDE*

$$\bar{V}(dt) = \bar{V}(t-)\Phi(dt) - C(dt) + \sum_{i,j:i \neq j} \bar{R}_{ij}(t)(N_{ij}(dt) - I_i(t-)\bar{\Lambda}_{ij}(dt)), \quad (4.5.2)$$

with terminal condition  $\bar{V}(T) = 0$ , where  $\bar{R}_{ij}(t) := \tilde{\Delta}C_{ij}(t) + \bar{V}_{ij}(t) - \bar{V}_i(t)$  and

$$\bar{\Lambda}_{ij}(dt) = \mathbb{1}_{(\bar{R}_{ij}(t) < 0)}U_{ij}(dt) + \mathbb{1}_{(\bar{R}_{ij}(t) \geq 0)}L_{ij}(dt).$$

Then  $\bar{V}(t) = \inf_{\Lambda \in \mathcal{M}} \mathbb{E}_{t,Z(t)}^{\bar{\Lambda}}[L(t,T)]$  and  $\bar{\Lambda} = (\bar{\Lambda}_{ij})_{i \neq j}$  is the best-case scenario.

*Proof.* We start with the proof of Proposition 4.5.2. First, we check whether  $\bar{\Lambda}$  is admissible. By definition of  $\bar{\Lambda}$  condition (4.4.2) is satisfied, and the conditions (ii)+(iii) of Assumption 4.2.12 are inherited from  $(L_{ij})_{i \neq j}$  and  $(U_{ij})_{i \neq j}$ , as they satisfy these exact conditions themselves. The  $i$ -predictability condition is satisfied since  $(L_{ij})_{i \neq j}$  and  $(U_{ij})_{i \neq j}$  themselves are  $i$ -predictable and since  $\bar{R}_{ij}(t)$  is  $i$ -predictable. Thus  $\bar{\Lambda}$  is admissible.

In order to obtain optimality, the martingale optimality principle states that it suffices to show that

$$M^\Lambda(t) := \int_{(0,t]} \frac{1}{\mathcal{E}_\Phi(s)} C(ds) + \frac{1}{\mathcal{E}_\Phi(t)} \bar{V}(t)$$

is martingale for  $\Lambda = \bar{\Lambda}$  and a submartingale for admissible  $\Lambda \neq \bar{\Lambda}$ . This is equivalent to showing that

$$\bar{V}(t) \leq \mathbb{E}_{t,Z(t)}^\Lambda[L(t,T)], \quad \forall t \in [0,T], \forall \Lambda \in \mathcal{M},$$

with equality if and only if  $\Lambda = \bar{\Lambda}$  and since an application of Theorem 4.3.6 yields

$$\bar{V}(t) = \mathbb{E}_{t,Z(t)}^{\bar{\Lambda}}[L(t,T)],$$

we can invoke Theorem 8.4 in Christiansen and Furrer (2024) to conclude that this is the case if

$$\begin{aligned} \mathbb{1}_{(R_{ij}(t) \geq 0)}(\Lambda_{ij} - \bar{\Lambda}_{ij})(dt) &\leq 0, \quad t \geq 0, i, j \in \mathcal{J}, i \neq j \\ \mathbb{1}_{(R_{ij}(t) < 0)}(\Lambda_{ij} - \bar{\Lambda}_{ij})(dt) &\geq 0, \quad t \geq 0, i, j \in \mathcal{J}, i \neq j \end{aligned}$$

for all  $\Lambda \in \mathcal{M}$ , with equality in the inequalities if and only if  $\Lambda = \bar{\Lambda}$ . Plugging in our candidate for  $\bar{\Lambda}$ , we indeed get

$$\begin{aligned} \mathbb{1}_{(\bar{R}_{ij}(t) \geq 0)}(\bar{\Lambda}_{ij} - \Lambda_{ij})(dt) &= \mathbb{1}_{(\bar{R}_{ij}(t) \geq 0)}(L_{ij} - \Lambda_{ij})(dt) \leq 0, \quad t \geq 0, i, j \in \mathcal{J}, i \neq j \\ \mathbb{1}_{(\bar{R}_{ij}(t) < 0)}(\bar{\Lambda}_{ij} - \Lambda_{ij})(dt) &= \mathbb{1}_{(\bar{R}_{ij}(t) < 0)}(U_{ij} - \Lambda_{ij})(dt) \geq 0, \quad t \geq 0, i, j \in \mathcal{J}, i \neq j \end{aligned}$$

for all  $\Lambda \in \mathcal{M}$ , with equality in the inequalities if and only if  $\Lambda = \bar{\Lambda}$ . Thus Theorem 8.4 of Christiansen and Furrer (2024) yields

$$\bar{V}(t) = \sum_{i \in \mathcal{J}} I_i(t) \mathbb{E}_{t,i}^{\bar{\Lambda}}[L(t, T)] \leq \sum_{i \in \mathcal{J}} I_i(t) \mathbb{E}_{t,i}^{\Lambda}[L(t, T)] = \mathbb{E}_{t,Z(t)}^{\Lambda}[L(t, T)],$$

for all  $t \in [0, T]$  and  $\Lambda \in \mathcal{M}$ . Taking the infimum yields the desired result. The worst-case result Proposition 4.5.1 now follows directly from the fact that

$$\sup_{\Lambda \in \mathcal{M}} \mathbb{E}_{t,Z(t)}^{\Lambda} \left[ \int_{(t,T]} \frac{\mathcal{E}_{\Phi}(t)}{\mathcal{E}_{\Phi}(s)} C(ds) \right] = - \inf_{\Lambda \in \mathcal{M}} \mathbb{E}_{t,Z(t)}^{\Lambda} \left[ \int_{(t,T]} \frac{\mathcal{E}_{\Phi}(t)}{\mathcal{E}_{\Phi}(s)} (-C)(ds) \right].$$

□

Note that the two Propositions assume the existence of a solution to the BSDEs (4.5.1) and (4.5.2). They do not prove that such a solution exists. But since they both are of the form (4.3.6), Theorem 4.3.11 gives an existence and uniqueness result which is applicable here, even in the case where the contractual payments are reserve-dependent. Thus, Theorem 4.3.11 in conjunction with Proposition 4.5.1 and Proposition 4.5.2 ensures existence and uniqueness of the worst- and best-case prospective reserves and the accompanying worst- and best-case biometric scenario for a wide range of different life, health and disability insurance products.

Note also that the worst/best-case scenario intuitively makes sense. The quantities  $R_{ij}(t) = \tilde{\Delta}C_{ij}(t) + V_{ij}(t) - V_i(t)$  are the so-called sums-at-risk. Suppose that the insured jumps from state  $i$  to state  $j$  at time  $t$ . In that case, the insurer has to pay out the contractual payments  $\tilde{\Delta}C_{ij}(t)$  to the insured, while changing the prospective reserve associated with the insured from  $V_i(t)$  to  $V_{ij}(t)$ . Thus, if  $R_{ij}(t) > 0$ , the insurer has a positive sum of money at risk and a jump from  $i$  to  $j$  at time  $t$  yields a loss. Therefore, whenever  $R_{ij}(t) > 0$ , the worst-case biometric scenario is the one where jumps from  $i$  to  $j$  are most likely ( $U_{ij}$ ), while the best-case biometric scenario is the one where such jumps are least likely ( $L_{ij}$ ). On the other hand, if  $R_{ij}(t) < 0$ , the insurer has a negative sum of money at risk and a jump from  $i$  to  $j$  at time  $t$  yields a gain. Therefore, the choice of best- and worst-case scenarios is reversed.

**Example 4.5.3** (Semi-Markov model). We now return to the smooth semi-Markov process of Example 4.2.18, and we assume that the following bounds for the transition intensities are given

$$l_{ij}(t, t - \tau_i(t)) \leq u_{ij}(t, t - \tau_i(t)), \quad \forall i, j \in \mathcal{J} : i \neq j,$$

for bounded and measurable functions  $l_{ij}, u_{ij} : [0, T] \times [0, T] \rightarrow [0, \infty)$ . The contractual sojourn and transition payments are assumed to be of the form

$$\begin{aligned} C_i(dt) &= b_i(t, t - \tau_i(t), V_i(t))dt \\ \tilde{\Delta}C_{ij}(dt) &= b_{ij}(t, t - \tau_i(t), V_i(t), V_{ij}(t)), \end{aligned}$$

for measurable functions

$$\begin{aligned} b_i &: [0, T] \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}, \quad i \in \mathcal{J} \\ b_{ij} &: [0, T] \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \quad i, j \in \mathcal{J} : i \neq j, \end{aligned}$$

where each  $b_i$  is bounded in its first two arguments and Lipschitz in its third argument, and each  $b_{ij}$  is bounded in its first two arguments Lipschitz in its third and fourth argument. Finally, the cumulative interest rate  $\Phi$  is given by,

$$\Phi(dt) = \sum_{i \in \mathcal{J}} I_i(t-) r_i(t, t - \tau_i(t)) dt$$

for bounded and measurable functions

$$r_i : [0, T] \times [0, T] \rightarrow \mathbb{R}, \quad i \in \mathcal{J}.$$

Under these conditions, Theorem 4.3.11 applies, and we get existence and uniqueness of the worst- and best-case prospective reserves. Taking a closer look at the worst-case prospective reserve, Proposition 4.5.1 and Proposition 4.3.8 gives us that its jump-count representation satisfies the following system of differential equations:

$$\begin{aligned} \frac{d}{dt} f^n(t, \omega^n) &= \mathbf{1}_{(t \geq t_n)} \left( r_{z_n}(t, t - t_n) f^n(t, \omega^n) - b_{z_n}(t, t - t_n, f^n(t, \omega^n)) \right. \\ &\quad \left. + \sum_{j: j \neq z_n} R_{t, z_n j}^{n+1}(t, \omega^n) \underline{\mu}_{z_n j}(t, \omega^n) \right), \end{aligned}$$

with  $f^n(T, \omega^n) = 0$  for  $t \in [0, T]$ ,  $\omega^n \in \Omega^n$  and  $n \in \mathbb{N}_0$ , and where

$$\begin{aligned} R_{t, z_n j}^{n+1}(t, \omega^n) &= b_{z_n j}(t, t - t_n, f^{n+1}(t, \omega_{t, z_n j}^{n+1}(\omega^n)), f^n(t, \omega^n)) \\ &\quad + f^{n+1}(t, \omega_{t, z_n j}^{n+1}(\omega^n)) - f^n(t, \omega^n) \\ \underline{\mu}_{z_n j}(t, \omega^n) &= \mathbf{1}_{(R_{z_n j}(t, \omega^n) \geq 0)} u_{z_n j}(t, t - t_n) + \mathbf{1}_{(R_{z_n j}(t, \omega^n) < 0)} l_{z_n j}(t, t - t_n). \end{aligned}$$

By our assumptions, Theorem 4.3.11 implies that there exists a unique solution of this system. Since all the driving functions depend on  $\omega^n$  only via the most

recent jump time and state  $(t_n, z_n)$  and are independent of  $n$ , it can be verified by insertion that this system of equations has a solution given by

$$f^n(t, \omega^n) = \sum_{i \in \mathcal{J}} \mathbb{1}_{(z_n=i)} V_i(t, t - t_n), \quad \forall n \in \mathbb{N}_0,$$

where the functions  $V_i : [0, T] \times [0, T] \rightarrow \mathbb{R}$  are a solution of the system

$$\begin{aligned} \frac{d}{dt} V_i(t, t - \tau) &= \mathbb{1}_{(t \geq \tau)} \left( r_i(t, t - \tau) V_i(t, t - \tau) - b_i(t, t - \tau, f_i(t, t - \tau)) \right. \\ &\quad \left. + \sum_{j: j \neq i} R_{ij}(t, t - \tau) \mu_{ij}(t, t - \tau) \right), \\ R_{ij}(t, t - \tau) &:= b_{ij}(t, t - \tau, V_j(t, 0), V_i(t, t - \tau)) + V_j(t, 0) - V_i(t, t - \tau), \end{aligned}$$

with  $V_i(T, T - \tau) = 0$  for  $t \in [0, T]$ ,  $i \in \mathcal{J}$  and  $\tau \in [0, T]$ . The key to seeing this is to realise that

$$f^{n+1}(t, \omega_{t, ij}^{n+1}(\omega^n)) = V_j(t, t - t) = V_j(t, 0), \quad \forall t \in [0, T], \forall n \in \mathbb{N}_0.$$

We can thus conclude that the worst-case prospective reserve is given by

$$\begin{aligned} \underline{V}(t) &= \sum_{n \in \mathbb{N}_0} I^n(t) \sum_{i \in \mathcal{J}} \mathbb{1}_{(\zeta_n=i)} V_i(t, t - \tau_n) \\ &= V_{Z(t)} \left( t, t - \sum_{n \in \mathbb{N}_0} I^n(t) \tau_n \right) = V_{Z(t)}(t, U(t)), \end{aligned}$$

where  $U(t) := t - \sum_{n \in \mathbb{N}_0} I^n(t) \tau_n$  is the duration since the last jump. Thus in conjunction with (4.2.6) and (4.2.5) we obtain the counterfactuals  $\underline{V}_i(t) = V_i(t, t - \tau_i(t))$  and  $\underline{V}_{ij}(t) = V_j(t, 0)$ , which by Proposition 4.2.6(i) and Theorem 4.3.6 implies that the the state-wise prospective reserves are given by

$$\tilde{V}_i(t) = I_i(t-) V_i(t - \tau_i(t)) + \sum_{j: j \neq i} I_j(t-) V_j(t, 0) = \mathbb{E}_{t, i}^\mu[L(t, T)], \quad i \in \mathcal{J}.$$

We note that this form resembles the expressions obtained in Section 9 of Christiansen and Furrer (2021). Finally, note that the system of differential equations satisfied by the functions  $V_i : [0, T] \times [0, T] \rightarrow \mathbb{R}$  closely resembles the Thiele differential equations in the smooth semi-Markov model without reserve-dependency of intensities and payments, and that the same numerical methods apply, see Adékambi and Christiansen (2017).

## 4.6 The robust consumption-insurance problem

In the previous section, we were interested in determining the worst- and best-case prospective reserves for a given insurance payment process and the associated worst-

and best-case biometric scenario, given the bounds (4.4.1). In this section, we are interested in finding the consumption process and insurance coverage that maximise an individual's utility in a worst-case biometric scenario. For this, we will restrict ourselves to the absolutely continuous case (4.2.10) with the bounds (4.4.4), and we require admissible transition intensities to satisfy (4.4.5).

The consumption processes that the policyholder can choose from are of the form

$$C(dt) = \sum_{i \in \mathcal{J}} I_i(t-) \left( c_i(t) dt + \sum_{j: j \neq i} c_{ij}(t) N_{ij}(dt) \right),$$

for  $i$ -predictable consumption rates  $(c_i)_{i \in \mathcal{J}}$  consumed continuously and  $i$ -predictable lump sum consumption  $(c_{ij})_{i \neq j}$  consumed upon transition. Under each admissible biometric scenario  $\mu$ , the individual has access to an insurance market, consisting of the coverages

$$M_{ij}^*(dt) = N_{ij}(dt) - I_i(t-) \phi_{ij}(t) \mu_{ij}(t) dt, \quad M_{ij}^*(0) = 0, \quad i \neq j,$$

where  $(\phi_{ij})_{i \neq j}$  are  $i$ -predictable, strictly positive and bounded on finite intervals. If one holds one unit of the coverage  $M_{ij}^*$ , one continuously pays the premium rate  $\mu_{ij}^*(t) := \phi_{ij}(t) \mu_{ij}(t)$  while in state  $i$  and receives a payment of one unit upon transition from  $i$  to  $j$ . Note that the processes  $(\mu_{ij}^*)_{i \neq j}$  constitute a set of transition intensities, which specify a pricing measure  $\mathbb{Q}^\mu$ , and that the processes  $(M_{ij}^*)_{i \neq j}$  are  $(\mathbb{F}, \mathbb{Q}^\mu)$ -martingales. The processes  $(\phi_{ij})_{i \neq j}$  can be viewed as price loadings, where  $\phi_{ij} = 1$  corresponds to the actuarially fair case. Given the insurance market, the individual has to decide on an insurance coverage strategy of the form

$$B(dt) = \sum_{i \in \mathcal{J}} I_i(t-) \sum_{j: j \neq i} b_{ij}(t) M_{ij}^*(dt),$$

where  $(b_{ij})_{i \neq j}$  are  $i$ -predictable. Set  $\theta := ((c_i)_{i \in \mathcal{J}}, (c_{ij})_{i \neq j}, (b_{ij})_{i \neq j})$ . With all this in place, the wealth of the individual is given by

$$X^{\theta, \mu}(dt) = X(t-) \Phi(dt) + B(dt) - C(dt), \quad X^{\theta, \mu}(0) = x_0 > 0,$$

where  $\Phi$  is a predictable cumulative interest rate process of the form

$$\Phi(dt) = r(t) dt, \quad \Phi(0) = 0,$$

for an optional interest rate processes  $r$  that is bounded on finite intervals. Utilising Proposition 4.2.6 and Proposition 4.2.10,  $X$  can also be written as

$$\begin{aligned} X^{\theta, \Lambda}(dt) &= \sum_{i \in \mathcal{J}} I_i(t-) \left( r_i(t) X_i(t-) - c_i(t) - \sum_{j: j \neq i} b_{ij}(t) \mu_{ij}^*(t) \right) dt \\ &\quad + \sum_{i, j: i \neq j} (b_{ij}(t) - c_{ij}(t)) N_{ij}(dt). \end{aligned}$$

The utility process  $U^{\theta,\mu}$  specifying the preferences of the individual is given by

$$U^\theta(dt) = \sum_{i \in \mathcal{I}} I_i(t-) \left( u_i(t, c_i(t)) dt + \sum_{j: j \neq i} u_{ij}(t, c_{ij}(t)) N_{ij}(dt) \right).$$

for suitable  $i$ -predictable  $u_i, u_{ij} : [0, T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ . The processes  $u_i$  determine how much utility the individual obtains from continuous consumption while sojourning in state  $i$ , and the processes  $u_{ij}$  determine how much utility the individual obtains from lump sum consumption when transitioning from state  $i$  to state  $j$ . The problem which the individual faces is given by

$$\max_{\theta \in \mathcal{A}} \min_{\mu \in \mathcal{M}} \mathbb{E}^\mu \left[ \int_{(0,T]} e^{-\int_0^s \rho(u) du} U^\theta(ds) + e^{-\int_0^T \rho(u) du} u_{Z(T)}^T(X^{\theta,\mu}(T)) \right], \quad (4.6.1)$$

where  $\rho$  is an optional process that is bounded on finite intervals and  $u_{Z(T)}^T : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is  $\mathcal{F}_T$ -measurable. Here  $\rho$  is the subjective discount rate, modelling the time-value of utility and  $u_{Z(T)}^T$  is the utility of terminal wealth. This is a problem of the form (4.4.6) and thus the martingale optimality principle is applicable. We will now utilise this in order to derive a verification theorem, inspired by the approach of Rieder and Woppperer (2012), which treats the case of a non-Markovian diffusion process.

Let  $(V_i)_{i \in \mathcal{I}}$  be a family of functions  $V_i : [0, T] \times [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  which are  $C^1$  in all arguments and let  $Y$  be an optional process given by

$$Y(dt) = \sum_{i \in \mathcal{I}} I_i(t-) \left( Y_i(dt) + \sum_{j: j \neq i} Y_{ij}(t) - Y_i(t) N_{ij}(dt) \right), \quad Y(T) = \kappa,$$

where  $\kappa$  is  $\mathcal{F}_T$ -measurable and  $Y_i(dt) = y_i(t)dt$  for  $i$ -predictable processes  $(y_i)_{i \in \mathcal{I}}$ . Define

$$A^{\theta,\mu}(dt) := \sum_{i,j: i \neq j} V_j(t, X_i(t) + b_{ij}(t) - c_{ij}(t), Y_{ij}(t)) - V_i(t, X_i(t), Y_i(t)) M_{ij}^\mu(dt),$$

with  $A^{\theta,\mu}(0) = 0$ , and where  $M_{ij}^\mu(dt) = N_{ij}(dt) - I_i(t-) \mu_{ij}(t)dt$  are the jump process martingales. Furthermore, for each  $(t, x, \omega) \in [0, T] \times \mathbb{R} \times \Omega$  let

$$\begin{aligned} \mathcal{G}_i^{\theta,\mu} V_i(t, x, Y_i(t)) &:= \frac{\partial V_i}{\partial t}(t, x, Y_i(t)) - \rho_i(t) V_i(t, x, Y_i(t)) + y_i(t) \frac{\partial V_i}{\partial y}(t, x, Y_i(t)) \\ &\quad + (x r_i(t) - c_i(t)) \frac{\partial V_i}{\partial x}(t, x, Y_i(t)) + \sum_{j: j \neq i} R_{ij}^V(t, x) \mu_{ij}(t), \\ R_{ij}^V(t, x) &:= V_j(t, x + b_{ij}(t) - c_{ij}(t), Y_{ij}(t)) - V_i(t, x, Y_i(t)) \\ &\quad - \phi_{ij}(t) b_{ij}(t) \frac{\partial V_i}{\partial x}(t, x, Y_i(t)). \end{aligned}$$

Finally we define the predictable process  $\lambda^{\theta,\mu}$  given by

$$\lambda^{\theta,\mu}(t) = \sum_{i \in \mathcal{J}} I_i(t-) \underbrace{\left( u_i(t, c_i(t)) + \sum_{j: j \neq i} u_{ij}(t, c_{ij}(t)) \mu_{ij}(t) \right)}_{=: \lambda_i^{\theta,\mu}(t)}.$$

**Definition 4.6.1.**  $\theta = ((c_i)_{i \in \mathcal{J}}, (c_{ij})_{i \neq j}, (b_{ij})_{i \neq j})$  are admissible if

- (i) For all  $i \in \mathcal{J}$ , the processes  $c_i$ ,  $(c_{ij})_{j: j \neq i}$ , and  $(b_{ij})_{j: j \neq i}$  are  $i$ -predictable.
- (ii)  $X^{\theta,\mu}(t) \geq 0$  for all  $t \in [0, T]$  and  $\mu \in \mathcal{M}$
- (iii)  $A^{\theta,\mu}$  is a  $(\mathbb{F}, \mathbb{P}^\mu)$ -martingale for all  $\mu \in \mathcal{M}$
- (iv) For all  $\mu \in \mathcal{M}$  and  $i \in \mathcal{J}$  it holds that

$$\sup_{t \in [0, T]} \mathbb{E}_{t,i}^\mu \left[ \int_{(t, T]} |\lambda^{\theta,\mu}(s)| ds + |u_{Z(T)}^T(X^{\theta,\mu}(T))| \right] (\omega) < \infty, \quad \forall \omega \in \Omega.$$

**Theorem 4.6.2** (Verification theorem). Suppose that the functions  $(V_i)_{i \in \mathcal{J}}$  and the process  $Y$  satisfy

$$\sup_{\theta \in \mathcal{A}} \inf_{\mu \in \mathcal{M}} \left\{ \lambda_i^{\theta,\mu}(t) + \mathcal{G}_i^{\theta,\mu} V_i(t, x, Y_i(t)) \right\} = 0$$

$$V_i(T, x, Y_i(T)) = u_i^T(x)$$

for each  $i \in \mathcal{J}$  and all  $(t, x, \omega) \in [0, T] \times \mathbb{R} \times \Omega$ , while  $(\theta, \mu)$  is the corresponding unique saddle point. If the pair  $(\theta, \mu)$  is admissible, then  $\theta$  is the unique robust optimal control,  $\mu$  is the worst-case biometric scenario and

$$V_{Z(t)}(t, X^{\theta,\mu}(t), Y(t)) = \sup_{\theta \in \mathcal{A}} \inf_{\mu \in \mathcal{M}} \mathcal{V}(t, \theta, \mu) = \inf_{\mu \in \mathcal{M}} \sup_{\theta \in \mathcal{A}} \mathcal{V}(t, \theta, \mu).$$

*Proof.* Let  $\mathcal{E}_\rho(t) := e^{\int_0^t \rho(u) du}$ . For any admissible  $(\theta, \mu)$  define

$$M^{\theta,\mu}(t) := \int_0^t \frac{1}{\mathcal{E}_\rho(s)} \lambda^{\theta,\mu}(s) ds + \frac{1}{\mathcal{E}_\rho(t)} V_{Z(t)}(t, X^{\theta,\mu}(t), Y(t)).$$

By the change of variables formula for finite variation functions, the fact that  $Z(t) \neq Z(t-)$  on a Lebesgue null set and the boundary condition, we get

$$\begin{aligned} \frac{1}{\mathcal{E}_\rho(t)} V_{Z(t)}(t, X^{\theta,\mu}(t), Y(t)) &= \frac{1}{\mathcal{E}_\rho(T)} u_{Z(T)}^T(X^{\theta,\mu}(T)) - \int_t^T \frac{1}{\mathcal{E}_\rho(s)} A^{\theta,\mu}(ds) \\ &\quad - \int_t^T \frac{1}{\mathcal{E}_\rho(s)} \mathcal{G}_{Z(s)}^{\theta,\mu} W_{Z(s)}(s, X^{\theta,\mu}(s), Y(s)) ds. \end{aligned}$$

Since  $A^{\theta,\mu}$  is a martingale, taking conditional expectation and applying Proposition 4.2.15(iii) yields

$$\begin{aligned} &\frac{1}{\mathcal{E}_\rho(t)} V_{Z(t)}(t, X^{\theta,\mu}(t), Y(t)) \\ &= \mathbb{E}_{t, Z(t)}^\mu \left[ \frac{1}{\mathcal{E}_\rho(T)} u_{Z(T)}^T(X^{\theta,\mu}(T)) - \int_t^T \frac{1}{\mathcal{E}_\rho(s)} \mathcal{G}_{Z(s)}^{\theta,\mu} V_{Z(s)}(s, X^{\theta,\mu}(s), Y(s)) ds \right], \end{aligned}$$

and we thus obtain

$$M^{\theta,\mu}(t) = \int_0^t \frac{1}{\mathcal{E}_\rho(s)} \lambda^{\theta,\mu}(s) ds + \mathbb{E}_{t,Z(t)}^\mu \left[ \frac{1}{\mathcal{E}_\rho(T)} U_{Z(T)}^T(X^{\theta,\mu}(T)) - \int_t^T \frac{1}{\mathcal{E}_\rho(s)} \mathcal{G}_{Z(s)}^{\theta,\mu} V_{Z(s)}(s, X^{\theta,\mu}(s), Y(s)) ds \right].$$

Since  $(\theta, \mu)$  is a saddle point it holds for all  $i \in \mathcal{J}$  that

$$\begin{aligned} \lambda^{\theta,\mu}(t) + \mathcal{G}_i^{\theta,\mu} V_i(t, x, Y_i(t)) &= 0, \quad \forall(t, x, \omega) \in [0, T] \times \mathbb{R} \times \Omega, \\ \lambda^{\theta,\mu}(t) + \mathcal{G}_i^{\theta,\mu} V_i(t, x, Y_i(t)) &\leq 0, \quad \forall(t, x, \omega) \in [0, T] \times \mathbb{R} \times \Omega, \quad \forall \theta \in \mathcal{A} \\ \lambda^{\theta,\mu}(t) + \mathcal{G}_i^{\theta,\mu} V_i(t, x, Y_i(t)) &\geq 0, \quad \forall(t, x, \omega) \in [0, T] \times \mathbb{R} \times \Omega, \quad \forall \mu \in \mathcal{M}, \end{aligned}$$

which implies

$$\begin{aligned} \lambda^{\theta,\mu}(t) &= -\mathcal{G}_{Z(t)}^{\theta,\mu} V_{Z(t)}(t, X^{\theta,\mu}(t), Y(t)), \quad \forall(t, \omega) \in [0, T] \times \Omega \\ \lambda^{\theta,\mu}(t) &\leq -\mathcal{G}_{Z(t)}^{\theta,\mu} V_{Z(t)}(t, X^{\theta,\mu}(t), Y(t)), \quad \forall(t, \omega) \in [0, T] \times \Omega, \quad \forall \theta \in \mathcal{A} \\ \lambda^{\theta,\mu}(t) &\geq -\mathcal{G}_{Z(t)}^{\theta,\mu} V_{Z(t)}(t, X^{\theta,\mu}(t), Y(t)), \quad \forall(t, \omega) \in [0, T] \times \Omega, \quad \forall \mu \in \mathcal{M}. \end{aligned}$$

Let  $0 \leq t \leq \tau \leq T$  be arbitrary. By the Tower Property (see Proposition 4.4(ii) of Christiansen and Furrer (2024)) and Proposition 4.2.15(ii) we thus have

$$\begin{aligned} \mathbb{E}_{t,Z(t)}^\mu [M^{\theta,\mu}(\tau)] &= \int_0^t \frac{1}{\mathcal{E}_\rho(s)} \lambda^{\theta,\mu}(s) ds + \mathbb{E}_{t,Z(t)}^\mu \left[ \frac{1}{\mathcal{E}_\rho(T)} u_{Z(T)}^T(X^{\theta,\mu}) \right] \\ &\quad + \mathbb{E}_{t,Z(t)}^\mu \left[ \int_t^\tau \frac{1}{\mathcal{E}_\rho(s)} \lambda^{\theta,\mu}(s) ds - \int_t^\tau \frac{1}{\mathcal{E}_\rho(s)} \mathcal{G}_{Z(s)}^{\theta,\mu} V_{Z(s)}(s, X^{\theta,\mu}(s), Y(s)) ds \right] \\ &= \int_0^t \frac{1}{\mathcal{E}_\rho(s)} \lambda^{\theta,\mu}(s) ds + \frac{1}{\mathcal{E}_\rho(t)} V_{Z(t)}(t, X^{\theta,\mu}(t), Y(t)) = M^{\theta,\mu}(t). \end{aligned}$$

Similarly we obtain  $\mathbb{E}_t^\mu [M^{\theta,\mu}(\tau)] \leq M^{\theta,\mu}(t)$  for all  $\theta \in \mathcal{A}$  and  $\mathbb{E}_t^\mu [M^{\theta,\mu}(\tau)] \geq M^{\theta,\mu}(t)$  for all  $\mu \in \mathcal{M}$ . The martingale optimality principle yields the desired result.  $\square$

#### 4.6.1 Power utility

We now consider the problem (4.6.1) with power utility preferences. For  $i, j \in \mathcal{J}$  with  $i \neq j$  we choose the utility functions

$$u_i(t, c) = \frac{w_i^\gamma(t)}{1-\gamma} c^{1-\gamma} \quad \text{and} \quad u_{ij}(t, c) = \frac{w_{ij}^\gamma(t)}{1-\gamma} c^{1-\gamma} \quad \text{and} \quad u_i^T(x) = \frac{w_{i,T}^\gamma}{1-\gamma} x^{1-\gamma}$$

for some  $\gamma \in (0, 1) \cup (1, \infty)$ , which is the relative risk aversion parameter. For each  $i \in \mathcal{J}$ , the non-negative processes  $(w_i, (w_{ij})_{j:j \neq i})$  are  $i$ -predictable and can be interpreted as utility weights. The larger  $w_i$ , the more utility the individual

gains from consumption while in state  $i$  and the larger  $w_{ij}$ , the more utility the individual gains from consumption upon transition from state  $i$  to state  $j$ . The random variables  $(w_{i,T})_{i \in \mathcal{J}}$  are non-negative and  $\mathcal{F}_T$ -measurable and determine the utility gained from terminal wealth in state  $i$ . Under the following boundedness assumptions, the problem with these power utility preferences can be solved explicitly.

**Assumption 4.6.3.** *Assume that*

- (i) *The processes  $(w_i, (w_{ij})_{j:j \neq i})_{i \in \mathcal{J}}$  are bounded on finite intervals*
- (ii) *The random variables  $(w_{i,T})_{i \in \mathcal{J}}$  are bounded and  $\inf_{\omega \in \Omega} w_{i,T}(\omega) > 0$*
- (iii) *The processes  $r$  and  $\rho$  are bounded on finite intervals*
- (iv) *The processes  $(\phi_{ij})_{i \neq j}$  satisfy  $\inf_{\omega \in \Omega} \phi_{ij}(\omega) > 0$ .*

For each  $i \in \mathcal{J}$  let  $V_i : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$V_i(x, y) = \frac{1}{1-\gamma} y^\gamma x^{1-\gamma}, \quad i \in \mathcal{J}$$

and let the process  $Y$  be given by the BSDE

$$\begin{aligned} Y(dt) &= \hat{r}(t)Y(t-)dt - W(dt) + \sum_{i,j:i \neq j} R_{ij}(t)(N_{ij}(dt) - I_i(t-)\mu_{ij}(t)dt), \\ Y(T) &= w_{Z(T)}(T), \end{aligned} \tag{4.6.2}$$

where  $R_{ij}(t) = (\phi_{ij}^{1-\frac{1}{\gamma}}(t)w_{ij}(t) + (1-\hat{\phi}_{ij}(t))Y_i(t) + (\phi_{ij}^{1-\frac{1}{\gamma}}(t)-1)Y_{ij}(t) + Y_{ij}(t) - Y_i(t))$ . The processes  $\hat{r}_i$  and  $\hat{\phi}_{ij}$  are given by

$$\hat{r}_i(t) = \frac{1}{\gamma}\rho_i(t) + \left(1 - \frac{1}{\gamma}\right)r_i(t) \quad \text{and} \quad \hat{\phi}_{ij}(t) = \frac{1}{\gamma} + \left(1 - \frac{1}{\gamma}\right)\phi_{ij}(t),$$

while the process  $W$  is given by

$$\begin{aligned} W(dt) &= \sum_{i \in \mathcal{J}} I_i(t-) \left( w_i(t)dt + \sum_{j:j \neq i} \tilde{\Delta}W_{ij}(t)N_{ij}(dt) \right), \quad W(0) = 0, \\ \tilde{\Delta}W_{ij}(t) &:= \phi_{ij}^{1-\frac{1}{\gamma}}(t)w_{ij}(t) + (1-\hat{\phi}_{ij}(t))Y_i(t) + (\phi_{ij}^{1-\frac{1}{\gamma}}(t)-1)Y_{ij}(t). \end{aligned}$$

Then define the intensities  $(\mu_{ij})_{i \neq j}$  by

$$\mu_{ij}(t) = \begin{cases} l_{ij}(t) & \text{if } \frac{\gamma}{1-\gamma} R_{ij}(t) \geq 0 \\ u_{ij}(t) & \text{if } \frac{\gamma}{1-\gamma} R_{ij}(t) < 0 \end{cases},$$

the consumption processes  $(c_i, (c_{ij})_{j:j \neq i})_{i \in \mathcal{J}}$  by

$$c_i(t) = w_i(t) \frac{X_i(t)}{Y_i(t)} \quad \text{and} \quad c_{ij}(t) = \phi_{ij}^{-\frac{1}{\gamma}} w_{ij}(t) \frac{X_i(t)}{Y_i(t)}, \quad i, j \in \mathcal{J} : i \neq j,$$

and the insurance allocation  $((\underline{b}_{ij})_{j \neq i})_{i \in \mathcal{I}}$  by

$$\underline{b}_{ij}(t) = \left( \phi_{ij}^{-\frac{1}{\gamma}}(t) \frac{w_{ij}(t) + Y_{ij}(t)}{Y_i(t)} - 1 \right) X_i(t), \quad i, j \in \mathcal{I} : i \neq j.$$

**Proposition 4.6.4.** *Assume that Assumption 4.6.3 holds. Then the processes  $(\underline{c}_i, (\underline{c}_{ij})_{j \neq i}, (\underline{b}_{ij})_{j \neq i})_{i \in \mathcal{I}}$  are the robust optimal controls, the processes  $(\underline{\mu}_{ij})_{i \neq j}$  constitute the biometric worst-case scenario and*

$$\begin{aligned} & V_{Z(t)}(X^{\theta, \mu}(t), Y(t)) \\ &= \sup_{\theta \in \mathcal{A}} \inf_{\mu \in \mathcal{J}} \mathbb{E}_{t, Z(t)}^{\mu} \left[ \int_{(t, T]} e^{-\int_t^s \rho(u) du} U^{\theta}(ds) + e^{-\int_t^T \rho(u) du} u_{Z_T}^T(X^{\theta, \mu}(T)) \right]. \end{aligned}$$

*Proof.* See Subsection 4.6.2. □

From (4.6.2) we see that the process  $Y$  satisfies a BSDE of the form (4.3.6), and our assumptions and Theorem 4.3.11 yield that  $Y$  exists and is unique. Theorem 4.3.6 yields that

$$Y(t) = \mathbb{E}_{t, Z(t)}^{\mu} \left[ \int_{(t, T]} e^{-\int_t^s \hat{r}(u) du} W(ds) + e^{-\int_t^T \hat{r}(u) du} w_{Z(T), T} \right],$$

which means that  $Y$  can be interpreted as the expected present value of the future flow of utility, where the expectation is taken under the worst-case measure. The quantity  $R_{ij}$  is the utility at risk, and if  $R_{ij}$  is positive, then the individual gains utility when transitioning from  $i$  to  $j$ , and if  $R_{ij}$  is negative, the individual loses utility when transitioning from  $i$  to  $j$ . Interestingly, the worst-case biometric scenario not only depends on the sign of  $R_{ij}$ , but also on whether  $\gamma > 1$  or  $\gamma < 1$ . If  $\gamma < 1$  and  $R_{ij}(t) > 0$ , then the worst-case biometric scenario is  $l_{ij}$ , where transitions from  $i$  to  $j$  are less likely, while if  $R_{ij}(t) < 0$  then the worst-case biometric scenario is  $u_{ij}$ , where the transitions from  $i$  to  $j$  are most likely. When  $\gamma > 1$ , then the situation is reversed. If  $R_{ij}(t) > 0$ , then the worst-case biometric scenario is  $u_{ij}$  and if  $R_{ij}(t) < 0$ , the worst-case biometric scenario is  $l_{ij}$ . This indicates that, depending on the risk aversion parameter, the perception of the worst-case changes. When  $\gamma > 1$  we can see that (4.6.2) is of the form (4.5.1) and if  $\gamma < 1$  we see that (4.6.2) is of the form (4.5.2). By Proposition 4.5.1 and Proposition 4.5.2 we can thus conclude that

$$\begin{aligned} Y(t) &= \sup_{\mu \in \mathcal{M}} \mathbb{E}_{t, Z(t)}^{\mu} \left[ \int_{(t, T]} e^{-\int_t^s \hat{r}(u) du} W(ds) + e^{-\int_t^T \hat{r}(u) du} w_{Z(T), T} \right], \quad \gamma > 1 \\ Y(t) &= \inf_{\mu \in \mathcal{M}} \mathbb{E}_{t, Z(t)}^{\mu} \left[ \int_{(t, T]} e^{-\int_t^s \hat{r}(u) du} W(ds) + e^{-\int_t^T \hat{r}(u) du} w_{Z(T), T} \right], \quad \gamma < 1. \end{aligned}$$

This shows that an individual with  $\gamma > 1$  views the future flow of utility as a liability and therefore maximises the expected present value thereof, which, as a

consequence, reduces all robust optimal consumption processes  $(c_i, (c_{ij})_{j:j \neq i})_{i \in \mathcal{J}}$  compared to other admissible biometric scenarios. Thus, the worst-case scenario of an individual with  $\gamma > 1$  is to run out of money to spend, and therefore, they consume conservatively. The individual with  $\gamma < 1$  is exactly the opposite. They view the future flow of utility as an asset and therefore minimise the expected present value thereof, which, as a consequence, increases all robust optimal consumption processes  $(c_i, (c_{ij})_{j:j \neq i})_{i \in \mathcal{J}}$  compared to other admissible biometric scenarios. The worst-case scenario of an individual with  $\gamma < 1$  is thus not to get to spend all of their money, and therefore they consume aggressively. Thus if we set  $\mathcal{J} = \{0 = \text{alive}, 1 = \text{dead}\}$  and assume that  $R_{01}(t) < 0$  (most individuals lose utility when dying), then the worst-case scenario of an individual with  $\gamma < 1$  is to die before spending all their money, while the worst-case scenario of an individual with  $\gamma > 1$  is to spend all their money before they die.

### 4.6.2 Verification

We start by noting that the process  $Y$  is given by a BSDE of the form (4.3.6), which by Theorem 4.3.11 has a unique solution, bounded on finite intervals. Thus, our candidate solution is well-defined. Next, we have to check that our candidate optimal controls are admissible. For this note that for all  $i \in \mathcal{J}$ , the processes  $\underline{c}_i$ ,  $(\underline{c}_{ij})_{j:j \neq i}$ ,  $(\underline{b}_{ij})_{j:j \neq i}$  are indeed  $i$ -predictable, while  $\underline{\mu}_{ij}$  satisfies Assumption 4.2.17 and (4.4.5). Thus point  $\underline{\mu}$  is admissible and (i) of Definition 4.6.1 is satisfied. Furthermore, we note that by rewriting (4.6.2) a little, we obtain

$$Y(dt) = \psi(t)Y(t)dt - \hat{W}(dt) + \sum_{i,j:i \neq j} \hat{R}_{ij}(t)(N_{ij}(dt) - I_i(t-)\hat{\mu}_{ij}(t)dt),$$

with termina condition  $Y(T) = w_{Z(T),T}$ , where  $\hat{R}_{ij}(t) = w_{ij}(t) + Y_{ij}(t) - Y_i(t)$ , and  $\psi$  and  $\hat{W}$  are given by

$$\begin{aligned} \hat{W}(dt) &= \sum_{i \in \mathcal{J}} I_i(t-) \left( w_i(t)dt + \sum_{j:j \neq i} w_{ij}(t)N_{ij}(dt) \right), \quad \hat{W}(0) = 0, \\ \psi(t) &= \sum_{i \in \mathcal{J}} I_i(t-) \left( \hat{r}_i(t) - \left( 1 - \hat{\phi}_{ij}(t)\phi_{ij}^{\frac{1}{\gamma}-1}(t) \right) \right). \end{aligned}$$

Since the processes  $(\phi_{ij})_{i \neq j}$  are bounded, bounded away from zero, and  $i$ -predictable, the processes  $\hat{\mu}_{ij} = \phi_{ij}^{1-\frac{1}{\gamma}}\mu_{ij}(t)$  satisfy Assumption 4.2.17 and are therefore valid transition intensities. By Theorem 4.3.6, we thus get that

$$Y(t) = \mathbb{E}_{t,Z(t)}^{\hat{\mu}} \left[ \int_{(t,T]} e^{-\int_t^s \psi(u)du} \hat{W}(ds) + e^{-\int_t^T \psi(u)du} w_{Z(T),T} \right].$$

As all the utility weights are non-negative, the integral in the expectation is always non-negative, and since  $\psi$  is bounded and  $w_{i,T}$  is bounded away from zero, there

exists a constant  $K > 0$  such that

$$\inf_{(t,\omega) \in [0,T] \times \Omega} Y(t, \omega) \geq \inf_{(t,\omega) \in [0,T] \times \Omega} \mathbb{E}_{t,Z(t)}^\mu \left[ e^{-\int_t^T \psi(u) du} w_{Z(T),T} \right] (\omega) \geq K > 0.$$

Thus, we have that  $Y$  is strictly positive and bounded away from zero. For any  $\mu \in \mathcal{M}$  the optimal wealth process is given by

$$\begin{aligned} X^{\theta,\mu}(t) &= x_0 \exp \left( \int_0^t \sum_{i \in \mathcal{J}} I_i(s) \left( r_i(s) - w_i(s) - \sum_{j:j \neq i} \frac{w_{ij}(s) + Y_{ij}(s)}{Y_i(s)} \phi_{ij}^{-\frac{1}{\gamma}}(s) \mu_{ij}^*(s) \right) ds \right) \\ &\quad \cdot \exp \left( \int_{(0,t]} \sum_{i,j:i \neq j} \log \left( \frac{Y_{ij}(s)}{Y_i(s)} \phi_{ij}^{-\frac{1}{\gamma}}(s) \right) N_{ij}(ds) \right). \end{aligned}$$

Since  $\phi_{ij}$  and  $Y$  are strictly positive, we can conclude that  $X^{\theta,\mu}(t) \geq 0$  and condition (ii) of Definition 4.6.1 is satisfied. Next we verify that the process  $A^{\theta,\mu}$  given by

$$A^{\theta,\mu}(dt) = \sum_{i,j:i \neq j} \frac{1}{1-\gamma} Y_i^{\gamma-1}(t) X_i^{1-\gamma}(t) (Y_{ij}(t) \phi_{ij}^{1-\frac{1}{\gamma}} - Y_i(t)) M_{ij}^\mu(dt),$$

with  $A^{\theta,\mu}(0) = 0$  is a  $(\mathbb{F}, \mathbb{P}^\mu)$ -martingale for any  $\mu \in \mathcal{M}$ . By Proposition 4.2 in Christiansen and Furrer (2024), Assumption 4.2.17(ii) and Tonelli's theorem, this is the case if we can show for all  $i, j, k \in \mathcal{J}$  that

$$\int_{(s,t]} \mathbb{E}_{s,i}^\mu \left[ \left| \frac{1}{1-\gamma} Y_j^{\gamma-1}(u) X_j^{1-\gamma}(u) (Y_{jk}(u) \phi_{jk}^{1-\frac{1}{\gamma}}(u) - Y_j(u)) \right| \right] \alpha_{jk}(u) du < \infty,$$

for all  $0 \leq s \leq t \leq T$ . Due to our assumptions,  $\phi_{ij}^{1-\frac{1}{\gamma}}$  is bounded regardless of whether  $\gamma > 1$  or  $\gamma < 1$ . Since  $Y$  is bounded and bounded away from zero, the processes  $Y_i$ ,  $Y_{ij}$  and  $Y_i^{\gamma-1}$  are bounded, regardless of whether  $\gamma > 1$  or  $\gamma < 1$ . The form of  $X^{\theta,\mu}$ , all our boundedness assumptions, and the inequality  $N(s) \leq N(s-)+1$  for all  $s > 0$  allow us to obtain the bound

$$(X^{\theta,\mu})^{1-\gamma}(s) \leq x_0 e^{(K_1(1+N(s)))} \leq x_0 e^{2K_1+K_1N(t-)} e^{K_1(N(s)-N(t))}, \quad 0 \leq t \leq s \leq T,$$

for some constant  $K_1 > 0$ . Using this bound and Proposition 4.2.16, we can for all  $\gamma \in (0, 1) \cup (0, \infty)$  obtain a bound of the form

$$\mathbb{E}_{t,i}[(X_j^{\theta,\mu})^{1-\gamma}(s)](\omega) \leq x_0 e^{2K_1+K_1N(T,\omega)} K_2 < \infty, \quad \forall 0 \leq t \leq s \leq T, \forall \omega \in \Omega,$$

for some constant  $K_2 > 0$ . Thus we can conclude that  $A^{\theta,\mu}$  is a  $(\mathbb{F}, \mathbb{P}^\mu)$ -martingale for all  $\mu \in \mathcal{M}$  and condition (iii) of Definition 4.6.1 is satisfied. Finally, by using the very same bound in conjunction with the other boundedness assumptions and Assumption 4.2.17(ii), we obtain for all  $\mu \in \mathcal{M}$

$$\sup_{t \in [0,T]} \mathbb{E}_{t,i}^\mu \left[ \int_{(t,T]} |\lambda^{\theta,\mu}(s)| ds + |u_{Z(T)}^T(X^{\theta,\mu}(T))| \right] (\omega) < \infty, \quad \forall \omega \in \Omega, \forall i \in \mathcal{J}.$$

Thus, condition (iv) of Definition 4.6.1 is satisfied, and we can conclude that our candidate optimal robust control is admissible as well. Last but not least, we note that

$$\lambda_i^U(x, Y_i(t)) + \mathcal{G}_i^{\theta, \mu} V_i(x, Y_i(t)), \quad i \in \mathcal{J}$$

is concave as a function of  $c_i$ ,  $c_{ij}$  and  $b_{ij}$  and linear (convex-like) on a compact domain  $[l_{ij}(t), u_{ij}(t)]$  in  $\mu_{ij}$ . Thus by Theorem 4.2 of Sion (1958) the problem

$$\sup_{\theta \in \mathcal{A}} \inf_{\mu \in \mathcal{M}} \left\{ \lambda_i^U(x, Y_i(t)) + \mathcal{G}_i^{\theta, \mu} V_i(x, Y_i(t)) \right\}, \quad i \in \mathcal{J}$$

has a unique saddle point, which by simple calculations can be shown to equal the above optimal controls. Furthermore, by inserting everything, it can be shown that

$$\sup_{\theta \in \mathcal{A}} \inf_{\mu \in \mathcal{M}} \left\{ \lambda_i^U(x, Y_i(t)) + \mathcal{G}_i^{\theta, \mu} V_i(x, Y_i(t)) \right\} = 0, \quad i \in \mathcal{J}.$$

Thus, by Theorem 4.6.2, we can conclude that our candidate solution indeed is optimal.

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### 4.A The space of finite variation functions

**Definition 4.A.1.** Let  $f : [0, T] \rightarrow \mathbb{R}$  be a function and let

$$|f|_a^b := \sup \left\{ \sum_{i=1}^n |f(t_i) - f(t_{i-1})| : a = t_0 < t_1 < \dots < t_n = b \right\}$$

be the variation of  $f$  on  $[a, b] \subset [0, T]$ . Then  $f$  is of finite variation on  $[a, b]$ , if  $|f|_a^b < \infty$ .

**Proposition 4.A.2.** For any  $0 \leq a \leq b \leq T$ , the variation  $|f|_a^b$  has the following properties:

- (i)  $t \mapsto |f|_a^t$  is increasing
- (ii) If  $t \mapsto f(t)$  is càdlàg so is  $t \mapsto |f|_a^t$  and  $\Delta |f|_a^t = |\Delta f(t)|$
- (iii)  $|f(t) - f(a)| \leq |f|_a^t$  and  $|f(b) - f(t)| \leq |f|_a^b - |f|_a^t = |f|_t^b$

$$(iv) \quad |f(t)| \leq |f|_t^b + |f(b)| \leq |f|_a^b + |f(b)|$$

Set  $BV_{[0,T]}^{\text{all}} := \{f : [0, T] \rightarrow \mathbb{R} : |f|_0^T < \infty\}$  and define the norms

$$\|f\|_{BV,0} := |f(0)| + |f|_0^T \quad \text{and} \quad \|f\|_{BV,T} := |f(T)| + |f|_0^T.$$

The space  $BV_{[0,T]}^{\text{all}}$  equipped with either  $\|f\|_{BV,0}$  is a Banach space (see Theorem 2.2.2 in Monteiro, Slavík, and Tvrdý (2018)).

**Lemma 4.A.3.** *The norms  $\|f\|_{BV,0}$  and  $\|f\|_{BV,T}$  are equivalent.*

*Proof.* Note that by Proposition 4.A.2(iii)

$$|f(0)| \leq |f(T) - f(0)| + |f(T)| \leq |f|_0^T + |f(T)|$$

and thus  $\|f\|_{BV,0} \leq 2(\|f\|_{BV,T})$ . Similarly

$$|f(T)| \leq |f(T) - f(0)| + |f(0)| \leq |f|_0^T + |f(0)|$$

and thus  $\|f\|_{BV,T} \leq 2(\|f\|_{BV,0})$ . Combining this yields  $\frac{1}{2}\|f\|_{BV,0} \leq \|f\|_{BV,T} \leq 2\|f\|_{BV,0}$ .  $\square$

Let  $BV_{[0,T]}$  be the subspace of finite variation functions that are càdlàg:

$$BV_{[0,T]} := \{f \in BV_{[0,T]}^{\text{all}} : f \text{ is càdlàg}\}.$$

**Lemma 4.A.4.** *The space  $BV_{[0,T]}$  with the norm  $\|f\|_{BV,T}$  is a Banach space.*

*Proof.* Clearly  $BV_{[0,T]}$  is a subspace of  $BV_{[0,T]}^{\text{all}}$ . It remains to show that  $BV_{[0,T]}$  is closed, since closed subspaces of Banach spaces are Banach spaces themselves. Let  $(f_n)_{n \in \mathbb{N}} \subset BV_{[0,T]}$ , such that  $\lim_{n \rightarrow \infty} \|f_n - f\|_{BV,T} = 0$  for some  $f \in BV_{[0,T]}^{\text{all}}$ . Then by Proposition 4.A.2(iv) we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} \sup_{t \in [0,T]} |f_n(t) - f(t)| &\leq \lim_{n \rightarrow \infty} |f_n - f|_0^T + |f_n(T) - f(T)| \\ &= \lim_{n \rightarrow \infty} \|f_n - f\|_{BV,T} = 0, \end{aligned}$$

implying that  $f_n \rightarrow f$  uniformly. Since the limit of a uniformly convergent sequence of càdlàg functions is càdlàg, we can conclude that  $f \in BV_{[0,T]}$  and therefore that  $BV_{[0,T]}$  is closed.  $\square$

We will also use a discounted version of the norm  $\|f\|_{BV,T}$ . Let  $\nu : [0, T] \rightarrow [0, \infty)$  be an increasing càdlàg function, let  $c > 0$  be an arbitrary constant and define

$$\|f\|_{BV,T}^{c,\nu} := |f(T)| + \int_{(0,T]} e^{-c(\nu(T) - \nu(s-))} |f|(ds).$$

**Lemma 4.A.5.** *The norms  $\|f\|_{BV,T}$  and  $\|f\|_{BV,T}^{c,\nu}$  are equivalent.*

*Proof.* As  $\nu$  is increasing, we have that

$$\|f\|_{BV,T}^{c,\nu} \leq |f(T)| + \int_{(0,T]} |f|(ds) = \|f\|_{BV,T}$$

and

$$\|f\|_{BV,T}^{c,\nu} \geq e^{-c(\nu(T)-\nu(0-))} \left( |f(T)| + \int_{(0,T]} |f|(ds) \right) = e^{-c(\nu(T)-\nu(0-))} \|f\|_{BV,T},$$

so  $e^{-c(\nu(T)-\nu(0-))} \|f\|_{BV,T} \leq \|f\|_{BV,T}^{c,\nu} \leq \|f\|_{BV,T}$ .  $\square$

The reason for the usefulness of  $\|\cdot\|_{BV}^{c,\nu}$  is the following bound:

**Lemma 4.A.6.** *Let  $f : [0, T] \rightarrow \mathbb{R}$  be a non-decreasing càdlàg function and let  $c > 0$  be a constant. Then it holds that*

$$\int_{(t_1, t_2)} e^{-c(f(T)-f(t-))} f(dt) \leq \frac{1}{c} e^{-c(f(T)-f(t_2-))}, \quad 0 \leq t_1 \leq t_2 \leq T$$

and

$$\int_{(t, T]} e^{-c(f(T)-f(t-))} f(dt) \leq \frac{1}{c}, \quad 0 \leq t \leq T.$$

*Proof.* Applying the change of variables formula for finite variation functions, we obtain that

$$e^{cf(t_2-)} - e^{cf(t_1)} = \int_{(t_1, t_2)} ce^{cf(t-)} f^c(dt) + \sum_{t_1 < t < t_2} (e^{cf(t)} - e^{cf(t-)}).$$

If  $f$  has a jump at time  $t$ , then  $\Delta f(t) > 0$ , since  $f$  is non-decreasing. By the mean-value theorem there exists  $\xi \in [f(t-), f(t)]$  such that

$$e^{cf(t)} - e^{cf(t-)} = ce^{c\xi} \Delta f(t) \geq ce^{cf(t-)} \Delta f(t).$$

Inserting this in the previous equation and rearranging, we obtain

$$\int_{(t_1, t_2)} e^{cf(t-)} f(dt) \leq \frac{1}{c} (e^{cf(t_2-)} - e^{cf(t_1)}) \leq \frac{1}{c} e^{cf(t_2-)}.$$

Multiplying both sides by  $e^{-cf(T)}$  yields the first result. For the second result, we similarly obtain that

$$\int_{(t, T]} e^{cf(s-)} f(ds) \leq \frac{1}{c} e^{cf(T)},$$

which upon multiplying with  $e^{-cf(T)}$  yields the desired result.  $\square$

Let  $f : [0, T] \rightarrow \mathbb{R}$  be a càdlàg function of finite variation. The Doleà-Dade exponential starting at time  $t \in [0, T]$  is given by

$$\mathcal{E}_f(t, ds) = \mathcal{E}_f(t, s-)f(ds), \quad \mathcal{E}_f(t, t) = 1, \quad s \in [t, T]$$

and has the explicit form

$$\mathcal{E}_f(t, s) = e^{f^c(s) - f^c(t)} \prod_{t < u \leq s} (1 + \Delta f(u)).$$

**Lemma 4.A.7** (Gronwall's inequality). *Let  $f : [0, T] \rightarrow [0, \infty)$  be a non-decreasing càdlàg function, let  $\alpha \geq 0$  be a constant. If  $g : [t, T] \rightarrow [0, \infty)$  satisfies*

$$g(s) \leq \alpha + \int_{(t, s]} g(u-)f(du), \quad \forall s \in [t, T],$$

*then it holds that*

$$g(s) \leq \alpha \mathcal{E}_f(t, s), \quad \forall s \in [t, T].$$

*Proof.* Let  $V(s) := \alpha + \int_{(t, s]} g(u-)f(du)$  and note that since  $g$  is non-negative and  $f$  is non-decreasing, the function  $V$  is non-decreasing. Then, using the integration by parts formula for càdlàg finite variation functions, we get

$$\frac{V(s)}{\mathcal{E}_f(t, s)} - \alpha = \int_{(t, s]} \left( \frac{g(u-)}{\mathcal{E}_f(t, u-)} - \frac{V(u)}{\mathcal{E}_f(t, u)} \right) f(du), \quad \forall s \in [t, T].$$

Using that  $V(u) = V(u-) + g(u-)\Delta f(u)$  and  $\mathcal{E}_f(t, u) = (1 + \Delta f(u))\mathcal{E}_f(t, u-)$  in conjunction with the fact that  $g(u-) \leq V(u-)$ , we obtain

$$\frac{g(u-)}{\mathcal{E}_f(t, u-)} - \frac{V(u)}{\mathcal{E}_f(t, u)} = \frac{g(u-) - V(u-)}{\mathcal{E}_f(t, u)} \leq 0,$$

which implies the desired result. □



## Chapter 5

# Optimal smooth consumption and its trade-offs

This chapter contains the paper *Hornung and Steffensen (2025)*.

### ABSTRACT

We investigate different ways to design smooth pension products based on solutions to optimal consumption and investment problems. The smoothness of a consumption process can be studied from both a pathwise (measured in terms of quadratic variation) and a pointwise (measured in terms of variance) point of view, and we conclude that introducing one type of smoothing does not necessarily improve the other kind of smoothing. Thus, care must be taken when designing smooth pension products. Focusing on pathwise smoothness without disregarding pointwise smoothness, we provide both a qualitative and a quantitative discussion of the trade-offs involved. In the qualitative discussion, we find that to increase smoothness, it is necessary to reduce the starting value, the drift of consumption, or the level of bequest. For the quantitative discussion, we set up an optimal consumption and investment problem, where the first control is the proportion of wealth invested in the risky asset. Still, the second control is not the consumption process itself. Instead, we use the drift and volatility of consumption as controls. The objective is to minimize the quadratic distance to a target drift and volatility while introducing a penalty for the volatility. We derive explicit solutions to this problem using classic dynamic programming methods and apply them to theoretically and numerically study the three trade-offs. All three approaches result in both pointwise and pathwise smoothing compared to the target, but reducing the drift yields better pointwise smoothing for similar levels of pathwise smoothing.

**Keywords:** Stochastic control; quadratic optimization; consumption-investment problems; quadratic variation; pension benefits

## 5.1 Introduction

We formulate and solve a consumption-investment problem in which preferences are specified directly as targets for the drift and the quadratic variation of the consumption rate. Quadratic variation measures the pathwise variation of the consumption rate in contrast to pointwise variation, formalized by the variance of the consumption rate at a specific time point. We discuss and carefully distinguish between the two notions of variation of a dynamic stochastic process. Although we do not base our specification of preferences on classic utility theory, it provides a convenient and immediate framework for studying aspects of consumption smoothness. We formally and numerically analyze a trade-off between four aspects of consumption: consumption quadratic variation, consumption drift, initial consumption, and bequest. The optimal controls are related to controls resulting from ordinary utility maximization. We also link them to special cases with explicit preferences for smoothing, presented in the literature on the smoothness of pension benefits.

The purpose of a pension product for an individual is to finance the individual's consumption after retirement, when their salary can no longer cover this. Therefore, it would be sensible for the benefits of a pension product to emulate the preferred consumption patterns of the individual buying the contract. Drawing inspiration from the vast literature on optimal consumption and investment problems is natural for a product designer. Consequently, the most commonly defined contribution pension product is the pure unit-linked annuity, where the benefit equals the wealth of the individual divided by an annuity. This product directly corresponds to the optimal consumption of the classic constant relative risk aversion (CRRA) power utility individual derived in Merton (1969) and Merton (1971), and it is also in line with the widely accepted Permanent Income Hypothesis of Friedman.

The question of how to design the benefits of a defined contribution pension product could end here if it were not for "The Consumption Smoothing Puzzle," documented by Hansen and Singleton (1983) and Campbell and Deaton (1989), demonstrating that observed consumption patterns are smoother than Merton's optimal consumption.

Various alternative explanations for the Consumption Smoothing Puzzle have been proposed. They are each parameterized by different preference parameters, which determine the structure of the optimal consumption and investment and implicitly determine the variation of consumption over time and the uncertainty of consumption at any time. It isn't easy to compare the properties of the consumption process across classes of preferences using these preference parameters. Therefore, we propose to use the language of stochastic processes instead. Instead of comparing preference parameters such as risk aversion, elasticity of intertemporal substitution,

and the strength and durability of habit formation, we examine the starting value, the drift, and the volatility of consumption. We then distinguish between two types of smoothness: Pathwise smoothness, related to the variation of consumption over time along one consumption path measured in terms of expected quadratic variation, and pointwise smoothness, associated with the uncertainty of consumption across paths at a fixed time point measured in terms of pointwise variance. We can then explicitly measure and compare smoothness properties across different classes of preferences independently of preference parameters. By considering the elements of consumption instead of the preference parameters, we can qualitatively discuss the necessary trade-offs arising from a desire for smoothing on the investment strategy, the aspects of consumption, and the bequest on a much more general level.

We can also discuss these trade-offs quantitatively by formulating an optimal consumption and investment problem, where the first control is the proportion of wealth invested in the risky assets. Still, the second control is not the consumption process itself. Instead, we use the drift and volatility of consumption as controls. The objective is to minimize the quadratic distance to a target drift and volatility while introducing a penalty for the volatility. The quadratic criterion has been extensively used in a pension funding context, for example, see Cairns (2000). The idea of using the components of the consumption process as a control date dates back to Bruhn and Steffensen (2013). They formulate a quadratic criterion that aims to minimize the quadratic distance of the consumption drift to a specified target while setting the consumption volatility to zero. Thus, by including the volatility of consumption as a control instead of setting it to zero, our criterion is a direct generalization of the criterion discussed by Bruhn and Steffensen (2013). Introducing a penalty for the volatility coefficient penalizes the quadratic variation of the consumption process. The inspiration comes from Staden, Dang, and Forsyth (2019), who introduce a penalty for the quadratic variation of wealth in discussing the connection between mean-quadratic variation portfolio optimization and mean-variance optimization. We solve this time-consistent optimal consumption and investment problem using standard dynamic programming techniques and demonstrate the different trade-offs in numerical examples.

The various proposed solutions to the Consumption Smoothing Puzzle rely on changing the market model or using preferences that are not time-separable. We highlight here utility-valued adjustment costs of consumption (Choi, Jeon, and Koo, 2022), multiplicative habit formation (Bilsen, Bovenberg, and Laeven, 2020), additive habit formation with exponential utility (Sundaresan, 1989), with power utility (Constantinides, 1990; Munk, 2008), or with cumulative prospect theory preferences (Bilsen and Laeven, 2020). They share the ambition of indirectly creating a demand for smooth consumption. In contrast, but like Bruhn and Steffensen (2013), we create a demand for smoothing by directly formalizing a smoothing preference.

While we draw on terminology and motivation from optimal consumption-investment problems throughout, our framework and results can also be interpreted in the context of optimal dividend allocation in corporate finance. There, shareholders demand stability in management and earnings, which is reflected in their demand for stable dividends. One can see the stability of dividends as a lack of variation, but again, one must carefully distinguish between pointwise and pathwise variation. Avanzi, Falden, and Steffensen (2022) study an optimal dividend problem with stability objectives related to the pointwise variation of corporate dividend payments. In contrast, all our insights relate to the alternative, pathwise variation of corporate dividends.

For two comprehensive literature reviews regarding "The Consumption Smoothing Puzzle," see Kankaanranta (2006), focusing on the Life Cycle, and Thimme (2017) focusing on the elasticity of intertemporal substitution. Other papers penalizing quadratic variation are Forsyth et al. (2012) and Giacinto, Tebaldi, and Wang (2022), which penalize the terminal quadratic variation of the P&L of a trading strategy. For a smooth pension product inspired by additive habit formation, see Steffensen and Vikkelsøe (2024), and for a numerical comparison of the smooth pension product based on the quadratic criteria of Bruhn and Steffensen (2013) with other standard pension products, see Jørgensen and Linnemann (2011) and Linnemann, Bruhn, and Steffensen (2014). Finally, for a discussion of the effect of smoothing on the assumed interest rate and variable annuities, see Balter and Werker (2020).

In Section 2, we argue that one can view the smoothness of a consumption process from both a pathwise (measured in terms of quadratic variation) and a pointwise (measured in terms of variance) point of view, and we discuss the connection between the two notions. After this, we continue to qualitatively examine the trade-offs one must make to achieve a larger pathwise smoothness and set up our optimal consumption and investment problem. In Section 3, we investigate the smoothing effect of either decreasing the starting value of consumption or the bequest, while keeping the drift fixed, concluding the section with a numerical example. Finally, in Section 4, we maintain the starting value and the bequest and introduce a volatility penalty, while reducing the drift only, and conclude with a numerical example.

## 5.2 General discussion of smoothing

In this section, we start by introducing the paper's theoretical setup and then proceed to discuss in what sense a consumption process can be smooth and, consequently, in what sense one consumption process can be smoother than another. After that, we move on to a general discussion of what trade-offs one must make if one desires smoother consumption. Finally, we introduce an optimization criterion that enables us to control the trade-offs.

Throughout the probabilistic setting is the filtered probability space given by  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ , where the filtration  $\mathbb{F}$  is the smallest augmented filtration generated by a one-dimensional Brownian motion  $W$ . Thus, the usual conditions are satisfied. The financial setting is that of the Black-Scholes market, so there is a risk-free asset  $B$  and a risky asset  $S$  satisfying

$$\begin{aligned} dB_t &= rB_t dt, \quad B_0 = 1, \\ dS_t &= \lambda S_t dt + \sigma S_t dW_t, \quad S_0 = s_0 > 0, \end{aligned}$$

where  $r \in \mathbb{R}$  is the risk-free rate of the risk-free asset,  $\lambda \in \mathbb{R}$  is the expected return, and  $\sigma > 0$  is the volatility of the risky asset. We assume that  $\lambda > r$  and define  $\theta := \frac{\lambda - r}{\sigma}$  to be the market price of risk. Recall that this financial market is arbitrage-free and complete, so there exists a unique equivalent martingale measure  $\mathbb{Q}$ , where the likelihood process  $L_t = \frac{d\mathbb{Q}}{d\mathbb{P}}$  is given by

$$L_t = e^{-\frac{1}{2}\theta^2 t - \theta W_t}.$$

We consider an individual who, over a finite time horizon  $[0, T]$ , invests a proportion  $\pi_t$  of their wealth  $X_t$  into the risky asset and the remaining proportion  $1 - \pi_t$  into the risk-free asset while consuming at rate  $C_t$  financed by their wealth given by

$$dX_t = ((r + \pi_t(\lambda - r))X_t - C_t)dt + \pi_t \sigma X_t dW_t, \quad X_0 = x_0 > 0. \quad (5.2.1)$$

Depending on the context, it may also be convenient to work with the amount of wealth invested in the risky asset, given by  $p_t = \pi_t X_t$ , instead of the proportion. In that case, the wealth reads

$$dX_t = (rX_t + (\lambda - r)p_t - C_t)dt + p_t \sigma dW_t, \quad X_0 = x_0 > 0. \quad (5.2.2)$$

The consumption process itself is assumed to take the form

$$C_t = c_0 + \int_0^t a(s, C_s, X_s)ds + \int_0^t b(s, C_s, X_s)dW_s. \quad (5.2.3)$$

### 5.2.1 Definitions of smoothing

Before discussing how to smooth a consumption process, we must first discuss what smoothness is and how to measure it, as there are two ways to view smoothness, and they do not necessarily align. The first is the pointwise point of view, where one can compare the distributional properties (such as quantiles or the variance) of  $C_t$  for different fixed time points to measure and compare smoothness. For example, one could say that:

**Definition 5.2.1.** *A consumption process  $C^{(1)}$  is pointwise smoother than a consumption process  $C^{(2)}$  at time  $t \in [0, T]$  if*

$$\text{Var}(C_t^{(1)}) \leq \text{Var}(C_t^{(2)}).$$

The second way to consider smoothness is from a pathwise perspective. In this case, the objective is to measure the variability from time point to time point within a consumption path  $t \mapsto C_t(\omega)$ . To measure this, it makes sense to look at the quadratic variation process of  $C$  given by

$$[C]_t := \lim_{\|P_n\| \rightarrow 0} \sum_{k=0}^n (C_{t_k} - C_{t_{k-1}})^2,$$

where for each  $n \in \mathbb{N}$ ,  $P_n = \{0 = t_0 < t_1 < \dots < t_n = t\}$  is a partition of  $[0, t]$  with mesh size  $\|P_n\| = \max_{i=1, \dots, n} |t_i - t_{i-1}|$ , such that  $\lim_{n \rightarrow \infty} \|P_n\| = 0$ . Thus,  $[C]_t(\omega)$  can, in a sense, be understood as measuring the variability accumulated up to time  $t$  in the realized path  $t \mapsto C_t(\omega)$ . Therefore, it would make sense to say that:

**Definition 5.2.2.** *A consumption process  $C^{(1)}$  is pathwise smoother than a consumption process  $C^{(2)}$  at time  $t \in [0, T]$  if*

$$\mathbb{E}[[C^{(1)}]_t] < \mathbb{E}[[C^{(2)}]_t],$$

*and it is perfectly pathwise smooth if  $\mathbb{E}[[C^{(1)}]_t] = 0$ .*

To see how Definition 5.2.1 and 5.2.2 are related, consider the following result:

**Proposition 5.2.3.** *Assume that  $C_t$  has the form (5.2.3). Then, it holds that*

$$\begin{aligned} \text{Var}(C_t) &= \mathbb{E}[[C]_t] + \text{Var} \left( \int_0^t a(s, C_s, X_s) ds \right) \\ &\quad + 2 \text{Cov} \left( \int_0^t a(s, C_s, X_s) ds, \int_0^t b(s, C_s, X_s) dW_s \right) \\ &= \int_0^t 2 \text{Cov}(C_s, a(s, C_s, X_s)) ds + \mathbb{E}[[C]_t], \end{aligned}$$

where  $[C]_t = \int_0^t b(s, C_s, X_s)^2 ds$  is the quadratic variation process of  $C$ .

*Proof.* See Appendix 5.B.2. □

**Remark 5.2.4.** Note that Proposition 5.2.3 also holds for more general consumption processes given by

$$C_t = c_0 + \int_0^t a_s ds + \int_0^t b_s dW_s,$$

where  $c_0 \geq 0$  and  $a$  and  $b$  are suitably adapted stochastic processes.

This shows that the variance of the consumption process (5.2.3) can be decomposed into three components. The first term is  $\mathbb{E}[[C]_t]$ , which measures pathwise smoothness and is directly determined by  $b$ . Therefore, we can also interpret this

term as a measure of volatility risk. The second term is the variance of the drift  $a$ , which we can interpret as the trend risk. The third term is the covariance between the drift of  $C$  and the volatility of  $C$ . Thus, pointwise smoothness measured in terms of variance takes into account all variability (or risk), of which volatility risk is only a part. This shows that the relationship between Definition 5.2.1 and 5.2.2 can be complicated. Just because  $C^{(1)}$  is pathwise smoother than  $C^{(2)}$  does not mean that it is pointwise smoother as well, since either the covariance between drift and volatility or the variance of the drift can be larger. Similarly, just because  $C^{(1)}$  is pointwise smoother than  $C^{(2)}$  does not mean that it is pathwise smoother, as the covariance between drift and volatility or the variance of the drift can be smaller. Therefore, it is possible to decrease one type of smoothness when increasing the other. For example, the appearance of a negative covariance term might reduce the variance (and thus improve pointwise smoothness) while increasing the expected quadratic variation (therefore reducing pathwise smoothness). When designing a smooth pension product, one should always keep in mind the type of smoothness desired (usually primarily the pathwise kind) and be cautious not to compromise one type of smoothness by attempting to improve the other.

### 5.2.2 Trade-offs

Now that we have established precise definitions of smoothness, we can turn to a qualitative discussion of what trade-offs one would make if one wants to achieve pathwise smoothing. For this, we need the following result, which paraphrases the calculations and results of Section 2 in Karatzas, Lehoczky, and Shreve (1987) as follows:

**Proposition 5.2.5.** *Let  $C$  be a consumption process,  $p$  an investment strategy and let  $X$  be given by (5.2.2) which under the risk neutral measure  $\mathbb{Q}$  reads as:*

$$dX_t = (rX_t - C_t)dt + p_t \sigma dW_t^{\mathbb{Q}}, \quad X_0 = x_0 > 0.$$

*Then the following holds:*

1. *The following budget constraint is satisfied:*

$$\mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-r(s-t)} C_s ds + e^{-r(T-t)} X_T \middle| \mathcal{F}_t \right] = X_t, \quad \forall t \in [0, T],$$

2. *Define*

$$X_t^{(1)} := \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-r(s-t)du} C_s ds \middle| \mathcal{F}_t \right] \quad \text{and} \quad X_t^{(2)} := E^{\mathbb{Q}} \left[ e^{-r(T-t)} X_T \middle| \mathcal{F}_t \right].$$

It holds that  $X_t = X_t^{(1)} + X_t^{(2)}$  for all  $t \in [0, T]$  and there exist unique processes  $p_t^{(1)}, p_t^{(2)}$  such that  $p_t = p_t^{(1)} + p_t^{(2)}$  and  $X^{(1)}$  and  $X^{(2)}$  can be written as

$$\begin{aligned} dX_t^{(1)} &= (rX_t^{(1)} - C_t)dt + p_t^{(1)}\sigma dW_t^{\mathbb{Q}}, \quad X_0^{(1)} = x_0^{(1)} > 0 \\ dX_t^{(2)} &= rX_t^{(2)}dt + p_t^{(2)}\sigma dW_t^{\mathbb{Q}}, \quad X_0^{(2)} = x_0^{(2)} > 0. \end{aligned}$$

*Proof.* See Appendix 5.B.2. □

*Remark 5.2.6.* Proposition 5.2.5 also holds for general consumption processes  $C$  in a non-Markovian, complete, and arbitrage-free diffusion market. In such a general setting, it is only possible to prove the existence and uniqueness of the investment strategies  $p^{(1)}$  and  $p^{(2)}$ . It is generally impossible to find them explicitly, except in a Markovian setting. In that case, we can use the Feynman-Kac formula to compute them.

The budget constraint of Proposition 5.2.5(1) states that wealth at every time point must be large enough to finance the future consumption and the future bequest, represented by the terminal wealth. In contrast, Proposition 5.2.5(2) states that this joint budget constraint can always be divided into two marginal budget constraints, each with its marginal wealth processes and investment strategies. Therefore,  $X^{(1)}$  and  $p^{(1)}$  can be seen as the wealth and investment strategy associated with consumption only, while  $X^{(2)}$  and  $p^{(2)}$  are the wealth and investment strategy associated with the bequest only. Now using this result and the expression for  $C$  given by (5.2.3) under  $\mathbb{Q}$ , the total budget constraint at time 0 can be rewritten to be

$$\begin{aligned} x_0 = c_0 \int_0^T e^{-\int_0^t r du} dt + x_0^{(2)} \\ + \mathbb{E}^{\mathbb{Q}} \left[ \int_0^T e^{-\int_0^t r du} \left( \int_0^t a(s, C_s, X_s) - \theta b(s, C_s, X_s) ds \right) dt \right]. \end{aligned}$$

Proposition 5.2.3 shows that it is possible to increase the pathwise smoothness by reducing the volatility coefficient  $b$ . Still, this budget constraint now indicates that if one reduces the volatility  $b$ , the right-hand side of the equation will increase and exceed  $x_0$ . To reestablish equality, we can do three things: Decrease the starting value  $c_0$ , decrease the drift of consumption,  $a$ , or decrease the capital allocated to the bequest,  $x_0^{(2)}$ , if  $x_0^{(2)} > 0$ . Of course, we can do all three things simultaneously. In the rest of this paper, we investigate these trade-offs separately.

The first two trade-offs also have another consequence. Under  $\mathbb{P}$  the expectation of  $C$  is given by

$$\mathbb{E}[C_t] = c_0 + \mathbb{E} \left[ \int_0^t a(s, C_s, X_s) ds \right].$$

This clearly shows that if one must reduce the drift or the starting value due to reduced volatility, the expected value of  $C_t$  will also decrease. This is not the case when reducing the bequest.

**Example 5.2.7.** The most common product design is the classical variable annuity, where the payments to the policyholder are given by their wealth divided by a unit annuity. This product design has its origins in the results of Merton (1969) and Merton (1971), who solve the optimal consumption-investment problem of the CRRA power utility individual given by

$$\max_{C, \pi} \mathbb{E} \left[ \int_0^T e^{-\rho t} \frac{1}{1-\gamma} C_s^{1-\gamma} ds \right],$$

subject to the wealth dynamics (5.2.1). Here,  $\rho \geq 0$  is a subjective discount rate, while  $\gamma > 0$  is the relative risk aversion. In this case, we disregard the bequest, so we have  $X^{(2)} \equiv 0$  and thus  $X \equiv X^{(1)}$ . The optimal consumption and investment proportion of this problem is given by

$$C_t = \frac{X_t}{f_1(t)} \quad \text{and} \quad \pi_t = \frac{\theta}{\sigma\gamma}$$

where  $f_1(t) = \int_t^T e^{-(s-t)\tilde{r}} ds$  and  $\tilde{r} = \frac{1}{\gamma}\rho + \left(1 - \frac{1}{\gamma}\right) \left(r + \frac{\theta^2}{2\gamma}\right) \geq r$  is a preference-adjusted interest rate. Using Ito's formula, we can rewrite  $C$  as

$$dC_t = \frac{1}{\gamma} \left( r - \rho + \frac{\theta^2(1+\gamma)}{2\gamma} \right) C_t dt + \frac{\theta}{\gamma} C_t dW_t, \quad C_0 = \frac{x_0}{f_1(0)}.$$

This shows that the optimal consumption follows a Geometric Brownian Motion and that the volatility coefficient decreases if  $\gamma$  increases. Our previous discussion shows that the drift or the starting value must change. As  $\tilde{r}$  decreases in  $\gamma$ , it holds that the larger  $\gamma$ , the smaller  $c_0$ , so decreasing the starting value is the first trade-off. Secondly, the absolute value of the drift decreases towards zero when  $\gamma$  increases.

*Remark 5.2.8.* Note that in the case of the classical power utility preferences used in the Merton consumption-investment problem of Example 5.2.7, the relative risk aversion parameter is also equal to the reciprocal of the elasticity of intertemporal consumption (EIS), which is often used in the economic literature to measure the variation of consumption over time. Epstein and Zin (1989) were the first to disentangle risk aversion and EIS in discrete time by developing recursive preferences with separate parameters for risk aversion and EIS. In contrast, Duffie and Epstein (1992) formalized this in continuous time. One might think that the parameter for EIS should fully determine the variation in consumption over time, but this is not the case. Jensen and Steffensen (2015) showed that the optimal consumption under separation of risk aversion and EIS again is a Geometric Brownian Motion,

where the volatility coefficient is equal to the volatility coefficient of the classic Merton problem discussed in Example 5.2.7. This means that the EIS does not significantly influence the volatility coefficient of consumption, and therefore, does not significantly impact the pathwise smoothness. Consequently, it is not the consumer's preference for variation of consumption over time as formalised by the EIS that determines the pathwise smoothness of consumption, but it is the preference for risk. This shows that the notion of risk and pathwise smoothness are essentially equivalent in this classical setup.

So far, we have only discussed how the desire for smoothing might affect the different elements of the consumption process. What about the investment strategy? After all, risky investments are the underlying source of uncertainty, causing fluctuations in consumption that one might want to smooth out. Hence, it would be interesting to know whether reducing risky investments is necessary to mitigate consumption volatility and whether it is possible to make risky investments without compromising zero consumption volatility. The following result sheds some light on the matter:

**Proposition 5.2.9.** *Assume that  $X \equiv X^{(1)}$  and let  $C$  and  $X$  be given by*

$$\begin{aligned} dC_t &= a(t, C_t, X_t)dt + b(t, C_t, X_t)dW(t), \quad C_0 = c_0 \geq 0 \\ dX_t &= (rX_t + p_t(\lambda - r) - C_t)dt + p_t\sigma dW_t, \quad X_0 = x_0 > 0, \end{aligned}$$

and let  $u(t, c, x)$  be defined as

$$u(t, c, x) := \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-r(s-t)} C_s ds \middle| C_t = c, X_t = x \right].$$

Then the budget condition  $u(t, C_t, X_t) = X_t$  holds for all  $t \in [0, T]$  if and only if

1. If  $p(t, C_t, X_t) \neq 0$  then
  - a)  $p(t, C_t, X_t) = \frac{u_c(t, C_t, X_t)}{\sigma(1 - u_x(t, C_t, X_t))} b(t, C_t, X_t)$  whenever  $b(t, C_t, X_t) > 0$
  - b)  $u_x(t, C_t, X_t) = 1$  whenever  $b(t, C_t, X_t) = 0$ .
2. If  $p(t, C_t, X_t) = 0$ , then  $b(t, C_t, X_t) = 0$  or  $u_c(t, C_t, X_t) = 0$ .

*Proof.* See Appendix 5.B.2. □

Proposition 5.2.9(1a) shows a relationship between the volatility of consumption and the investment strategy if the investment strategy and the volatility of consumption are not zero. Initially, this could indicate that a reduction in  $b$  indeed leads to a reduction in  $p$  as well, but since the derivatives  $u_c$  and  $u_x$  depend on  $b$ , it is generally difficult to say. An example where a reduction in  $b$  directly leads to a reduction in  $p$  is the classic Merton consumption.

**Example 5.2.7** (continued). It holds that the optimal investment proportion is given by  $\pi_t = \frac{\theta}{\sigma\gamma}$ , which means that the amount invested into the risky asset is given by  $p_t = \frac{\theta}{\sigma\gamma}X_t$ . Using  $X_t = f_1(t)C_t$ , this can be rewritten into

$$p_t = \frac{f_1(t)}{\sigma} \frac{\theta}{\gamma} C_t = \frac{f_1(t)}{\sigma} b(C_t).$$

This already resembles Proposition 5.2.9(1) and it is equal to Proposition 5.2.9(1) if  $u_x(t, C_t, X_t) = 0$  and  $u_c(t, C_t, X_t) = f_1(t)$ . Using the explicit expression of the optimal consumption and the fact that the budget constraint holds, we obtain

$$u(t, C_t, X_t) = X_t^{(1)} = X_t = f_1(t)C_t,$$

which confirms that  $u_x(t, c, x) = 0$  and  $u_c(t, c, x) = f_1(t)$ . Note also that as  $\gamma \rightarrow \infty$  we have  $b(C_t) \rightarrow 0$  and  $f_1(t) \rightarrow \int_t^T e^{-r(s-t)} ds$ , which means that  $p_t \rightarrow 0$ . So, in this case, it is indeed so that  $p_t \rightarrow 0$  when  $b(C_t) \rightarrow 0$ . Therefore, reducing volatility will always result in a lower amount being invested in the risky asset. The extreme case  $\gamma = \infty$  yields a constant consumption and no risky investments. Thus, the only way to achieve perfect pathwise smoothing is to avoid risky investments.

*Remark 5.2.10.* Example 5.2.7 shows that in the classical Merton case, there is a one-to-one correspondence between risk preferences (the investment strategy) and the preferences for pathwise smoothness (the volatility of consumption). It is precisely this equivalence that the Consumption Smoothing Puzzle contradicts, as it reveals a mismatch between fluctuations in wealth, determined by risky investments, and fluctuations in consumption. It is therefore exactly the separation of risk preferences and preferences for pathwise smoothness, which could be behind the Puzzle.

Proposition 5.2.9(1b) shows that the investment strategy can, in principle, be arbitrary as long as  $u_x(t, C_t, X_t) = 1$  whenever  $b(t, C_t, X_t) = 0$ . Furthermore, we conclude that if  $b(t, c, x) = 0$  for all  $(t, c, x)$ , then  $a$  must depend on  $c$  and  $x$ . If  $a(t, c, x) = \tilde{a}(t, c)$  for all  $(t, c, x)$ , then  $C$  would be independent of  $X$  and it would hold that

$$\begin{aligned} u(t, c, x) &= \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-r(s-t)} C_s ds \middle| C_t = c, X_t = x \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-r(s-t)} C_s ds \middle| C_t = c \right] \quad \forall (t, c, x), \end{aligned}$$

which would imply that  $u_x(t, c, x) = 0$  for all  $(t, c, x)$ . Another way to interpret this is in light of Proposition 5.2.3. If the volatility risk is zero, then the risk associated with risky investments must be reflected as trend risk. An example of this is the smooth pension product of Bruhn and Steffensen (2013).

**Example 5.2.11.** Bruhn and Steffensen (2013) solve the optimization problem:

$$\min_{a, \pi} E \left[ \int_0^T \frac{1}{2} (a_t - \bar{a}C_t)^2 dt + \frac{B}{2} (X_T - \xi C_T)^2 \right]$$

subject to the wealth process (5.2.1) and

$$dC_t = a_t dt, \quad C_0 = c_0 \geq 0.$$

Here  $\bar{a} \in \mathbb{R}$ . The goal of this criterion is to find a consumption process whose drift  $a_t$  is as close as possible to the target drift  $\bar{a}C_t$ , while  $b \equiv 0$  by assumption, meaning that  $C$  is perfectly pathwise smooth. The optimal drift  $a$  and the amount invested in the risky asset in this case are given by

$$a_t = \bar{a}C_t + \frac{g_2(t)}{f_2(t)} (X_t - g_2(t)C_t) \quad \text{and} \quad p_t = \frac{\theta}{\sigma} (g_2(t)C_t - X_t).$$

We can now clearly see that the drift  $a_t$  now depends on  $X_t$ . We can interpret the quantity  $X_t - g_2(t)C_t$  as surplus wealth relative to the target consumption, in the sense that  $g(t)C_t$  is the expected present value of future consumption and bequest if  $C$  is precisely on target, which means that  $C_s = C_t e^{(s-t)\bar{a}}$  for all  $s \geq t$ . Now, depending on the surplus, the investment strategy can take arbitrary positive or negative values. It can also be zero. This is the case where consumption is exactly on target. In that case, the surplus becomes zero.

Proposition 5.2.9(2) shows that if the investment strategy is zero for a specific  $(t, C_t, X_t)$ , then either the volatility of consumption must be zero, or the current location of  $(t, C_t, X_t)$  constitutes a critical point of  $c \mapsto u(t, c, X_t)$ . Whether such a critical point exists depends on the exact choices of  $a$  and  $b$ . However, for most realistic cases, it seems reasonable to assume that  $u_c > 0$  since a higher current consumption level could reasonably lead to a higher expected present value of future consumption. Hence, in realistic settings, we can conclude that zero investment in risky assets leads to zero volatility in consumption.

This discussion concludes that reducing consumption volatility can reduce risk in the investment strategy, but it is not necessary. Even if the volatility of consumption is zero, it is still possible to make risky investments if one accepts trend risk instead of volatility risk.

In the following, motivated by the above discussion, we extend the preferences used by Bruhn and Steffensen (2013) to investigate how the mentioned trade-offs induce smoothing and how smoothing induces certain trade-offs.

### 5.2.3 General model

As discussed in Remark 5.2.8 and 5.2.10, the usual expected utility representation of preferences failed to produce the desired separation of risk preferences and

preferences for pathwise smoothness. Proposition 5.2.9 (1b) showed that it is mathematically possible to achieve this separation, but the question remains what preferences can produce this separation. The discussion in Subsection 5.2.2 indicates that expressing the preferences in terms of the components of consumption is a natural starting point, and the criterion of Bruhn and Steffensen (2013) (see Example 5.2.11) demonstrates that this approach indeed can be successful in achieving separation. The criterion of Bruhn and Steffensen (2013) formalises preferences in terms of the drift and initial value of consumption, while they fix the volatility of consumption at zero. Therefore, it is natural to consider the extension:

$$\min_{a,b,\pi} \mathbb{E} \left[ \int_0^T \frac{1}{2} w_a(t) (a_t - \bar{a}(t, C_t, X_t))^2 + \frac{1}{2} w_b(t) (b_t - \bar{b}(t, C_t, X_t))^2 + \frac{1}{2} w(t) b_t^2 dt + \frac{B}{2} (X_T - \xi C_T)^2 \right], \quad (5.2.4)$$

subject to

$$dX_t = ((r + \pi_t(\lambda - r))X_t - C_t)dt + \pi_t \sigma X_t dW_t, \quad X_0 = x_0 > 0 \quad (5.2.5)$$

and

$$dC_t = a_t dt + b_t dW_t, \quad C_0 = c_0 \geq 0. \quad (5.2.6)$$

It is assumed that  $w_a(t)$ ,  $w_b(t)$ ,  $w(t)$  are continuous for  $t \in [0, T]$  and  $w_a(t) > 0$ ,  $w_b(t) + w(t) > 0$ ,  $w(t), w_b(t) \geq 0$  for all  $t \in [0, T]$  and  $B, \xi \geq 0$ .

Two sets of parameters specify the preferences for the different consumption components. The first set consists of the starting consumption  $c_0$ , the target drift  $\bar{a}$ , and the target volatility  $\bar{b}$ , which implicitly specify a certain target consumption

$$d\bar{C}_t = \bar{a}(t, \bar{C}_t, \bar{X}_t)dt + \bar{b}(t, \bar{C}_t, \bar{X}_t)dW_t, \quad \bar{C}_0 = \bar{c}_0 > 0, \quad (5.2.7)$$

where  $\bar{a}, \bar{b}$  are sufficiently nice deterministic functions and  $\bar{c}_0$  is determined such that the budget condition holds. Finally, the parameter  $\xi$  specifies the target level of bequest given by  $\bar{X}_T = \xi C_T$ . The parameter  $\xi$  can thus be interpreted as the target multiple of terminal consumption, which is left to the dependents. Note, that by specifying  $\bar{a}, \bar{b}$  and by requiring that  $\bar{c}_0$  satisfies the budget constraint, one implicitly specifies a target investment strategy given by Proposition 5.2.9.

The second set of parameters are the weights  $w_a, w_b, w, B$  and the starting value  $c_0$ . The weights  $w_a, w_b$ , and  $B$  specify to what degree the individual is willing to deviate from the target drift, volatility, and bequest. In contrast, the parameter  $w$  specifies an explicit desire for pathwise smoothness by penalizing the volatility. It can be interpreted as the individual's dislike of volatility, and a special case occurs when this dislike of volatility is constant over time. In that case, the term in (5.2.4) involving the penalty term  $w$  can be rewritten:

$$\mathbb{E} \left[ \int_0^T \frac{1}{2} w b_t^2 dt \right] = \frac{1}{2} w \mathbb{E} \left[ \int_0^T b_t^2 dt \right] = \frac{1}{2} w \mathbb{E} [C_T^2],$$

which corresponds to a direct penalty of expected quadratic variation. Thus, we have introduced an explicit preference for pathwise smoothness. As we saw in Subsection 5.2.2, the preference for the initial consumption  $c_0$  might affect the preferences for pathwise smoothness as well. Therefore, we include  $c_0$  as a parameter, which can be varied. An overview of all parameters can be found in Table 5.2.

*Remark 5.2.12.* Note that the Merton consumption from Example 5.2.7 can be used as target consumption, as it is a time-homogeneous geometric Brownian motion and thus a special case of the assumed target consumption. If one chooses  $w_a = w_b = \infty$ ,  $c_0 = \bar{c}_0$  and  $w = 0$ , then one would recover the Merton consumption as the optimal consumption. This makes sense, as an individual in that case is not willing to deviate from the target at all. If now  $w_a, w_b < \infty$  and  $w > 0$  or  $c_0 \neq \bar{c}_0$ , then this individual has the Merton consumption as a target, but is to a certain degree willing to deviate from this, to accommodate the additional desire for pathwise smoothness. The extreme case is achieved when  $w = \infty$ , which represents the desire for perfect pathwise smoothness. If additionally  $\bar{a}(t) = \bar{a} \in \mathbb{R}$  and  $w_a = 1$ , we recover the criterion of Bruhn and Steffensen (2013) described in Example 5.2.11.

The controls in this case are the consumption drift and diffusion coefficients,  $a$  and  $b$ , and the proportion of wealth invested in the risky asset  $\pi$ . They are assumed to be on feedback form and must satisfy the following admissibility criteria.

**Definition 5.2.13.** Let  $\nu : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $\zeta : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by

$$\begin{aligned}\nu(t, C_t, X_t) &= \begin{pmatrix} a(t, C_t, X_t) \\ rX_t + \pi(t, C_t, X_t)(\lambda - r)X_t - C_t \end{pmatrix} \\ \zeta(t, X_t, C_t) &= \begin{pmatrix} b(t, C_t, X_t) \\ \pi(t, C_t, X_t)\sigma X_t \end{pmatrix}\end{aligned}$$

Then, the controls  $a, b, \pi$  are admissible if the following two conditions are satisfied:

1. There exists  $k > 0$  such that for all  $(c, x) \in \mathbb{R}^2$  and  $t \in [0, T]$  it holds that

$$\|\nu(t, c, x)\| + \|\zeta(t, c, x)\| \leq k(1 + \|(c, x)^T\|).$$

2. There exists  $k > 0$  such that for all  $(c_1, x_1), (c_2, x_2) \in \mathbb{R}^2$  and  $t \in [0, T]$  it holds that

$$\|\nu(t, c_1, x_1) - \nu(t, c_2, x_2)\| + \|\zeta(t, c_1, x_1) - \zeta(t, c_2, x_2)\| \leq k\|(c_1, x_1)^T - (c_2, x_2)^T\|.$$

These conditions ensure the existence and uniqueness of  $C$  and  $X$  by Ito's Existence Theorem. The assumed structure of  $X$  combined with the admissibility conditions ensures that the budget constraint is satisfied.

*Remark 5.2.14.* Note that neither the budget constraint nor the admissibility conditions guarantees the non-negativity of  $C$  or  $X$ . To ensure this, we must place additional constraints on the controls or the starting values of  $C$  and  $X$ .

This problem can be tackled using standard dynamic programming methods (see, for example, Fleming and Soner (2006)) and writing down the Hamilton-Jacobi-Bellmann (HJB) equation for Problem (5.2.4) (suppressing dependence on  $(t, c, x)$  for ease of notation) yields

$$\begin{aligned} 0 = V_t + \inf_{a, b, \pi} & \left\{ \frac{1}{2} w_a (a - \bar{a})^2 + \frac{1}{2} w_b (b - \bar{b})^2 \right. \\ & + \frac{1}{2} w b^2 + (rx + \pi(\lambda - r)x - c)V_x + aV_c \\ & \left. + \frac{1}{2} \sigma^2 \pi^2 x^2 V_{xx} + b\sigma \pi x V_{xc} + \frac{1}{2} b^2 V_{cc} \right\}, \end{aligned} \quad (5.2.8)$$

with terminal condition  $V(T, x, c) = \frac{B}{2}(x - \xi c)^2$ , which yields the following expressions for the optimal controls:

$$\begin{aligned} a &= \bar{a} - \frac{V_c}{w_a} \\ b &= \frac{w_b}{w_b + w + V_{cc} + V_{xc}^2 V_{xx}^{-1}} \bar{b} + \frac{\sigma x V_{xc}}{w_b + w + V_{cc} + V_{xc}^2 V_{xx}^{-1}} \frac{\theta}{\sigma} \frac{V_x}{x V_{xx}} \\ \pi &= -\frac{V_{xc}}{V_{xx} \sigma x} b - \frac{\theta}{\sigma} \frac{V_x}{x V_{xx}} \\ &= -\frac{V_{xc} w_b}{\sigma x (V_{xx} (w_b + w + V_{cc}) - V_{xc}^2)} \bar{b} - \frac{\theta}{\sigma} \frac{V_x}{x V_{xx}} \left( 1 + \frac{V_{xc}^2}{V_{xx} (w_b + w + V_{cc}) - V_{xc}^2} \right). \end{aligned}$$

Even though these expressions are given in terms of the partial derivatives of the value functions, they can show how the different components are interrelated. The first thing to note is that the drift is adjusted upward or downward depending on the sign and value of  $V_c$ , and that

$$a = \bar{a} - \frac{V_c}{w_a} \rightarrow \bar{a} \quad \text{for } w_a \rightarrow \infty$$

provided that  $V_c$  does not explode. Secondly, we can see that the investment proportion  $\pi$  and the volatility of consumption  $b$  each consist of two terms. The first term of both  $\pi$  and  $b$  depends directly on the target volatility of consumption  $\bar{b}$ , while the second term depends on  $\frac{\theta}{\sigma} \frac{V_x}{x V_{xx}}$ . Now, assuming that all involved partial derivatives of  $V$  are well-behaved, it holds that

$$\pi \rightarrow -\frac{\theta}{\sigma} \frac{V_x}{x V_{xx}} \quad \text{and} \quad b \rightarrow 0 \quad \text{for } w \rightarrow \infty$$

and

$$\pi \rightarrow -\frac{V_{xc}}{\sigma x V_{xx}} \bar{b} - \frac{\theta}{\sigma} \frac{V_x}{x V_{xx}} \quad \text{and} \quad b \rightarrow \bar{b} \quad \text{for } w_b \rightarrow \infty.$$

The first conclusion of these two results is that the terms of  $b$  and  $\pi$  containing  $\bar{b}$  can be considered targets, which we obtain perfectly when  $w_b = \infty$  and completely disappear when  $w = \infty$ . The second conclusion that we can draw is that the size of the investment proportion depends on the volatility of consumption one is willing to accept, mirroring Proposition 5.2.9(1a), but it also shows, mirroring Proposition 5.2.9(1b), that the investment proportion need not be zero to have a consumption stream with zero volatility, since the proportion of wealth given by  $-\frac{\theta}{\sigma} \frac{V_x}{xV_{xx}}$  can always be invested in the risky asset. Because of this, the third conclusion is that  $-\frac{\theta}{\sigma} \frac{V_x}{xV_{xx}}$  can be interpreted as a surplus in the sense that it is the inherently speculative proportion of wealth that can be invested in the fluctuating risky asset, while still maintaining a perfectly pathwise smooth consumption. Therefore, we can interpret the second term of  $b$  and  $\pi$  as an adjustment of the target. This shows that these preferences achieve the desired separation of risk preferences and preferences for pathwise smoothness.

The discussion suggests two distinct approaches to smoothing. The first approach is to reduce the target volatility by increasing  $w$  or decreasing  $\bar{b}$ , lowering the investment proportion. Keeping the starting value  $c_0$  fixed, the weights  $w_a$  and  $B$  determine to what degree the drift or the bequest is decreased, thus resulting in two of the three different trade-offs. The last trade-off was to reduce the starting consumption, which leads to the second approach of smoothing. Due to the target plus surplus structure of  $b$  and  $\pi$ , one could try to increase the surplus in that direction to the point where it counteracts the target term of  $b$ . The only alterable variable of the surplus is  $C_t$  through the starting value  $c_0$ , since  $x_0$  is fixed.

In the following, we explicitly solve the optimization problem (5.2.4) in the concrete setting of  $\bar{a}(t, C_t, X_t) = \bar{a}(t)C_t$  and  $\bar{b}(t, C_t, X_t) = \bar{b}(t)C_t$ . In this case, we can find an explicit expression for the value function, and it is possible to investigate the two approaches mentioned above in greater detail. We start by illustrating and discussing the approach of smoothing via the surplus adjustment by considering the special case of  $w \equiv 0$ . This is followed by a presentation and discussion of the trade-offs of smoothing by penalizing volatility by considering the general case  $w > 0$ . A numerical example follows each case.

### 5.3 Reducing the starting value and bequest

In this section, we investigate the impact of reducing either the starting value or the bequest by considering the optimization problem (5.2.4) with  $\bar{a}(t, C_t, X_t) = \bar{a}(t)C_t$  and  $\bar{b}(t, C_t, X_t) = \bar{b}(t)C_t$ , and throughout this section we set  $w \equiv 0$ . We start by presenting the explicit solution to the problem and follow up with a discussion of the theoretical properties of optimal consumption. Then, a numerical example illustrates the smoothing effect of reducing either the starting value of consumption or the bequest.

### 5.3.1 Theoretical properties

In Appendix 5.B.1, we verify that the value function given by

$$V(t, x, c) = \frac{1}{2} \frac{(x - g(t)c)^2}{f(t)} \quad (5.3.1)$$

with functions  $f$  and  $g$  satisfying the differential equations

$$g'(t) = (r - \bar{a}(t) + \theta \bar{b}(t))g(t) - 1, \quad g(T) = \xi \quad (5.3.2)$$

$$f'(t) = (2r - \theta^2)f(t) - g^2(t) \left( \frac{1}{w_a(t)} + \frac{\theta^2}{w_b(t)} \right), \quad f(T) = \frac{1}{B}, \quad (5.3.3)$$

and optimal controls given by

$$a_t = \bar{a}(t)C_t + J_a(t)Z_t \quad (5.3.4)$$

$$b_t = \bar{b}(t)C_t - \theta J_b(t)Z_t \quad (5.3.5)$$

$$\pi_t = \frac{g(t)}{\sigma X_t} \bar{b}_t C_t - \frac{\theta}{\sigma} \frac{Z_t}{X_t} (1 + g(t)J_b(t)), \quad (5.3.6)$$

satisfy the Hamilton-Jacobi-Bellmann equation (5.2.8) and conclude that this is the optimal solution. Here we have defined (for an overview of functions see Table 5.3)

$$J_a(t) := \frac{g(t)}{f(t)w_a(t)}, \quad J_b(t) := \frac{g(t)}{f(t)w_b(t)} \quad \text{and} \quad Z_t := X_t - g(t)C_t. \quad (5.3.7)$$

An inspection of (5.3.4)-(5.3.6) now enables us to see the specific form of the target plus surplus structure already identified in the general case. In each case, the first term is the target term depending on either  $\bar{a}$  or  $\bar{b}$ , while the second term is the surplus term depending on the process  $Z$ . The following result shows that  $Z$  indeed can be interpreted as a surplus:

**Proposition 5.3.1.** *Let  $g(t)$  be defined as in (5.3.2), let  $\bar{C}$  be the target consumption of (5.2.7) and let  $\mathbb{Q}$  be the unique equivalent martingale measure of the Black-Scholes market. Then it holds that:*

$$Z_t = X_t - \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-r(s-t)} \bar{C}_s ds + \xi \bar{C}_T e^{-r(T-t)} \middle| \bar{C}_t = C_t \right],$$

and

$$Z_t = z_0 \exp \left( \int_0^t r - g(s)J_a(s) - \theta^2 \left( \frac{3}{2} + g(s)J_b(s) \right) ds - \theta W_t \right).$$

*Proof.* See Appendix 5.B.3. □

Proposition 5.3.1 shows that  $Z$  can be interpreted as the surplus wealth left over after financing the future target consumption, if positive, or the deficit missing

to be able to finance the future target consumption, if negative. It also shows that the sign of the starting value of  $Z$  determines the sign of  $Z$  throughout the entire period. Hence, it follows that  $a \equiv \bar{a}$  and  $b \equiv \bar{b}$  if and only if  $Z \equiv 0$ , which by Proposition 5.3.1 is the case if and only if  $z_0 = 0$ . This implies that  $C \equiv \bar{C}$  if and only if  $c_0 = \frac{x_0}{g(0)}$ . The fraction  $\frac{x_0}{g(0)}$  also has an important interpretation. By Proposition 5.3.1 the budget constraint for  $\bar{C}$  can be rewritten into

$$x_0 = \mathbb{E}^{\mathbb{Q}} \left[ \int_0^T e^{-r(s-t)} \bar{C}_s ds + \xi \bar{C}_T e^{-r(T-t)} \middle| \bar{C}_0 = c_0 \right] = g(0)c_0,$$

which implies that  $\bar{c}_0 = \frac{x_0}{g(0)}$  is the largest starting value of  $\bar{C}$  that one can afford with wealth  $x_0$ . This generalizes to any time point, meaning that  $\frac{X_t}{g(t)}$  is the largest amount of target consumption one can afford at time  $t$  with wealth  $X_t$ .

If now  $c_0 < \bar{c}_0$ , we have  $z_0 > 0$  and by Proposition 5.3.1 the consequence is that  $Z_t > 0$  for all  $t \in [0, T]$ . This surplus implies that the drift is adjusted upwards relative to the target, while the investment proportion  $\pi$  is adjusted downward, as lower expected returns suffice. This, in turn, results in a lower volatility of consumption. On the level of the consumption process itself, this means that  $C_t < \frac{X_t}{g(t)}$  for all  $t \in [0, T]$ , while  $X_T > \xi C_T$ . This means that even though one makes adjustments, the consumption is still smaller than what one can afford at all times, and that the remaining surplus is added to the bequest.

On the other hand, if now  $c_0 > \bar{c}_0$ , then it holds that  $z_0 < 0$  and by Proposition 5.3.1 the consequence is that  $Z_t < 0$  for all  $t \in [0, T]$ . This deficit implies that the drift  $a$  is adjusted downward relative to the target, while the proportion of risky assets  $\pi$  is adjusted upward to increase the expected returns. This, in turn, results in a higher volatility of consumption. On the level of the consumption process itself, this means that  $C_t > \frac{X_t}{g(t)}$  for all  $t \in [0, T]$ , while  $X_T < \xi C_T$ . This means that even though one makes adjustments, the consumption is still larger than what can be afforded at all times, and that the remaining deficit has to be covered by the bequest. This can lead to bankruptcy if the bequest is insufficient to cover the deficit. We can summarize this discussion in the following proposition.

**Proposition 5.3.2.** *It holds for all  $t \in [0, T]$  that*

| Optimal | $z_0 < 0$ | $z_0 = 0$ | $z_0 > 0$ | Target                           |
|---------|-----------|-----------|-----------|----------------------------------|
| $C_t$   | $>$       | $=$       | $<$       | $X_t/g(t)$                       |
| $a_t$   | $<$       | $=$       | $>$       | $\bar{a}(t)C_t$                  |
| $b_t$   | $>$       | $=$       | $<$       | $\bar{b}(t)C_t$                  |
| $\pi_t$ | $>$       | $=$       | $<$       | $(g(t)/\sigma X_t)\bar{b}(t)C_t$ |
| $X_T$   | $<$       | $=$       | $>$       | $\xi C_T$                        |

If we take this discussion into a pension context, then one can think of  $X_t$  as being the assets of a pensioner, which, due to the budget constraint, have to equal

the market price of the pensioner's total future benefits (the reserve). The quantity  $g(t)C_t$  can be thought of as the market price (or reserve) for financing the target benefit throughout the rest of the contractual period, given the current level of benefit. Thus, the surplus  $Z$  can be interpreted as the difference between the pensioner's assets and the market price (or reserve) for financing the future target benefits. We can thus conclude that if  $z_0 > 0$ , then the assets being larger than the liability of financing the target benefits allows the pensioner to reduce  $b_t$  and  $\pi_t$ , while increasing  $a_t$  and  $X_T$  according to their preferences. Thus, potential preferences for smoothing can only be accommodated if the assets are larger than the reserve for the target benefit.

We can interpret the quantities  $J_a$  and  $J_b$  as adjustment weights that determine how the surplus is distributed. The larger they are, the more surplus is used to adjust that particular quantity. They also determine the drift of the surplus (compare with the explicit form of  $Z_t$  in Proposition 5.3.1). The size of  $J_a$  and  $J_b$  is significantly determined by  $w_a$ ,  $w_b$ , and  $B$ . Hence, if  $w_a$  is large and  $w_b$  is small, then  $J_a$  is small and  $J_b$  is large, and the drift of the surplus is negative, because most of the surplus is used to adjust the volatility of consumption. On the other hand, large  $w_b$  and small  $w_a$  lead to small  $J_b$  and large  $J_a$ , which still implies a negative drift of the surplus, but now most of the surplus is used to adjust the drift of consumption. If both  $w_a$  and  $w_b$  are large, then both  $J_a$  and  $J_b$  are small, which means that the surplus is largely preserved throughout the entire period and is then used to adjust the bequest, since  $X_T = \xi C_T + Z_T$ . Finally, if  $B$  is large compared to  $w_a$  and  $w_b$ , then  $J_a$  and  $J_b$  become large towards time  $T$  as  $\lim_{t \uparrow T} J_a(t) = B\xi/w_a$  and  $\lim_{t \uparrow T} J_b(t) = B\xi/w_b$ , with the convention  $\infty \cdot 0 := \infty$ . This forces  $Z_T$  towards zero and thus  $X_T$  towards  $\xi C_T$ . So, if we want to maximize the smoothing effect of the surplus,  $w_b$  should be small compared to  $w_a$  so that most of the surplus is used to reduce volatility.

Choosing  $w_a$  large also has the side effect that the expected quadratic variation completely determines the variance. To see this insert (5.3.4) into the second form of Proposition 5.2.3, which yields

$$\text{Var}(C_t) = 2 \int_0^t \bar{a}(s) \text{Var}(C_s) + J_a(s) \text{Cov}(C_s, Z_s) ds + \mathbb{E}[[C]_t], \quad (5.3.8)$$

and for large  $w_a$  means

$$\text{Var}(C_t) \approx 2 \int_0^t \bar{a}(s) \text{Var}(C_s) ds + \mathbb{E}[[C]_t] = \int_0^t e^{\int_s^t 2\bar{a}(u) du} \frac{d}{dt} \mathbb{E}[[C]_t] \Big|_{t=s} ds.$$

Therefore, reducing  $b$  will always lead to a reduction in the investment proportion, and perfect pathwise and pointwise smoothing can only be obtained if  $b \equiv 0$ , which implies  $\pi \equiv 0$ .

Now, we have established that pathwise smoothing is possible by having a positive surplus, the level of which is determined by the starting surplus  $z_0 = x_0 - g(0)c_0$ . As  $x_0$  is fixed, a positive value of  $z_0$  can be achieved by reducing the starting consumption  $c_0$  or reducing the level of bequest  $\xi$ , upon which  $g(0)$  depends. The question remaining is how to choose this starting surplus. Can this be chosen optimally, and what values of  $c_0$  and  $\xi$  would this correspond to? To answer that, define the processes  $\Phi$ ,  $V$  and  $\Psi$  given by

$$\begin{aligned} d\Phi_t &= \bar{a}(t)\Phi_t dt + \bar{b}(t)\Phi_t dW_t, \quad \Phi_0 = 1, \\ d\Psi_t &= (r - g(t)J_a(t) - \theta^2(1 + g(t)J_b(t))\Psi_t)dt - \theta\Psi_t dW_t, \quad \Psi_0 = 1, \\ dV_t &= (\bar{a}(t)V_t + J_a(t)\Psi_t)dt + (\bar{b}(t)V_t - \theta J_b(t)\Psi_t)dW_t, \quad V_0 = 0. \end{aligned} \quad (5.3.9)$$

Clearly  $\Phi_t = \frac{\bar{C}_t}{c_0}$  and  $\Psi_t = \frac{Z_t}{z_0}$ . The process  $V$  corresponds to the optimal consumption, with initial value 0. By using the following result, these processes can be used to determine a suitable starting surplus.

**Proposition 5.3.3.** *The optimal consumption can be written as  $C_t = c_0\Phi_t + z_0V_t$ . Additionally, it holds that*

$$\begin{aligned} \text{Var}(C_t) &= z_0^2 \text{Var}\left(\frac{\Phi_t}{g(0)} - V_t\right) + x_0^2 \text{Var}\left(\frac{\Phi_t}{g(0)}\right) \\ &\quad + 2z_0x_0\left(\text{Cov}\left(\frac{\Phi_t}{g(0)}, V_t\right) - \text{Var}\left(\frac{\Phi_t}{g(0)}\right)\right) \end{aligned}$$

for all  $t \in [0, T]$ , implying that for each  $t \in [0, T]$  there exists a unique

$$z_0^*(t) = \frac{\text{Var}(\Phi_t g(0)^{-1}) - \text{Cov}(\Phi_t g(0)^{-1}, V_t)}{\text{Var}(\Phi_t g(0)^{-1} - V_t)} x_0,$$

which minimises  $\text{Var}(C_t)$ .

*Proof.* See Appendix 5.B.3. □

Proposition 5.3.3 contains two interesting statements. The first is that an optimal value of the surplus wealth exists for each time point  $t$ , minimizing the variance of consumption at time  $t$ . Whether this value  $z_0^*$  is positive, negative, or zero depends on  $\text{Cov}(\Phi_t, V_t)$ , which depends on the model parameters. Large values of  $w_a$  or  $\theta$  lead to  $\text{Cov}(\Phi_t, V_t) < 0$ , which implies  $z_0^*(t) > 0$ , while  $\theta \leq 0$  implies  $\text{Cov}(\Phi_t, V_t) > 0$ , which might result in  $z_0^*(t) < 0$ . If  $\bar{b} \equiv 0$ , then  $\text{Cov}(\Phi_t, V_t) = 0$  for all  $t \in [0, T]$ .

The second statement of Proposition 5.3.3 is that the variance of the optimal consumption is a convex second-order polynomial in  $z_0$ . This shows that both pointwise and pathwise smoothing is achieved if one chooses  $z_0 \in (0, 2z_0^*(t))$ , assuming that  $z_0^*(t) > 0$  (which it is in most realistic situations) because  $\text{Var}(C_t) <$

$\text{Var}(\bar{C}_t)$  and because the positive surplus reduces the volatility coefficient. It also shows that if one chooses the starting value of the surplus wrongly, then the opposite can happen, and one introduces more variance. The variance increases for  $z_0$ , moving far away from zero in both directions due to the investment strategy. If the surplus becomes more and more positive, then the investment proportion eventually turns into a growing short position, whereas an increasing negative surplus increases the existing long position. Thus, at some point, the target consumption is both pointwise and pathwise smoother than the optimal consumption.

These results show that it is possible to find a range for the starting surplus in which one achieves smoothing, and that one should not choose a value outside this range. Now, Proposition 5.3.3 can be used to find an explicit expression for the variance minimizing starting consumption:

$$c_0^*(t) = \left( 1 - \frac{\text{Var}(\Phi_t g(0)^{-1}) - \text{Cov}(\Phi_t g(0)^{-1}, V_t)}{\text{Var}(\Phi_t g(0)^{-1} - V_t)} \right) \frac{x_0}{g(0)}. \quad (5.3.10)$$

Note that the optimal starting value depends on the time point  $t \in [0, T]$  for which the variance is minimized, which means that one has to choose. The variances and the covariance can be calculated based on the first, second, and product moments of  $V$  and  $\Phi$ , which satisfy a system of linear first-order differential equations (see Appendix 5.B.3).

The variance-minimizing level of the bequest  $\xi^*(t)$  can be determined similarly. However, since  $g(0)$  depends on the relative level of the bequest, an explicit expression cannot be obtained. In addition, one must impose the restriction  $\xi \geq 0$  to avoid a negative bequest.

### 5.3.2 Numerical example

Now, we consider a numerical example in which we simulate 50,000 paths of two different consumption processes, in addition to the target consumption. The first is the optimal consumption with the starting value  $c_0 = c_0^*(T)$  calculated using (5.3.10), which minimizes the variance of consumption at the terminal time  $T$ . The second consumption process is optimal when  $c_0 = \bar{c}_0$ , but the bequest parameter  $\xi$  is set to zero. We make the simulations using the Euler scheme. All other parameters are given in Table 5.1.

*Remark 5.3.4.* The market parameters are chosen similarly to other publications, see Bruhn and Steffensen (2013) and Steffensen and Vikkelsøe (2024). In contrast, the target volatility  $\bar{b}$  is chosen to match the volatility of the optimal Merton consumption with a relative risk aversion  $\gamma = 2$ . The target drift  $\bar{a}$  is chosen to be zero, which matches the Merton consumption with relative risk aversion  $\gamma = 2$  and impatience rate  $\rho = r + \frac{1}{2\gamma}(1 + \gamma)\theta^2 \approx 0.037$ . This aligns with Steffensen and Vikkelsøe (2024), who work with  $\gamma = 2$  and  $\rho = 4$ . For the values of the weights

| Parameter  | Description                                    | Value  |
|------------|------------------------------------------------|--------|
| $x_0$      | Starting wealth                                | 100    |
| $r$        | Risk free rate                                 | 0.02   |
| $\lambda$  | Expected return of stock                       | 0.05   |
| $\sigma$   | Volatility of stock                            | 0.2    |
| $T$        | End of time period                             | 15     |
| $\bar{a}$  | Target drift of consumption                    | 0      |
| $\bar{b}$  | Target volatility of consumption               | 0.075  |
| $B$        | Weight of bequest $X_T$                        | 1      |
| $\xi$      | Proportion of $C$ determining target for $X_T$ | 0 or 1 |
| $w_a$      | Weight of drift $a$                            | 1,000  |
| $w_b$      | Weight of volatility $b$                       | 1      |
| $N$        | Number of simulations                          | 50,000 |
| $\Delta t$ | Stepsize in the Euler scheme                   | 0.01   |

**Table 5.1:** *Parameters used in the numerical examples.*

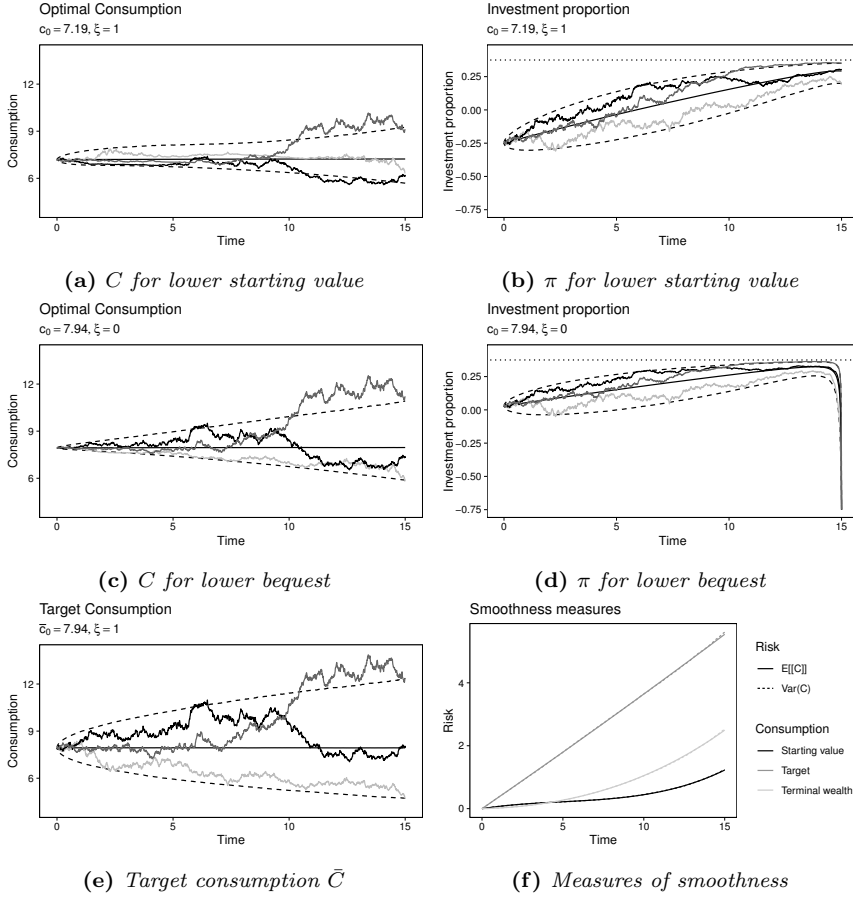
$w_a$ ,  $w_b$ ,  $w$ , and  $B$ , there are no empirical values in the literature, but Bruhn and Steffensen (2013) implicitly assumes  $w_a = 1$ . Thus, we use the value one as the benchmark value for the weights and explore the parameter space from there.

*Remark 5.3.5.* Note that as  $\bar{a} = 0$  and  $w_a = 1000$  we have  $a_t \approx 0$ , which by Proposition 5.2.3 implies  $\text{Var}(C_t) \approx \mathbb{E}[[C]_t]$ . Therefore, in this case, pointwise and pathwise smoothing are the same in the sense of Definitions 5.2.1 and 5.2.2. This would not be the case if  $\bar{a} \neq 0$ .

These parameters imply that  $\bar{c}_0 = 7.94$  and (5.3.10) implies that  $c_0^*(T) = 7.19$  which is a reduction of 9.4% compared to  $\bar{c}_0$ , but it causes the surplus to be positive. As  $\bar{a} = 0$  and  $a \approx 0$ , the drift remains unchanged, and consequently it is not only the starting consumption that is reduced by 9.4%, but it is also the expected consumption throughout the whole period and the expected target bequest that is reduced by 9.4%. Thus, these are the trade-offs made when reducing the initial consumption, but as Figures 5.1a and 5.1f show, the positive surplus is indeed used to achieve a significant degree of pointwise and pathwise smoothing.

When reducing the bequest, the initial consumption and therefore also the expected consumption at any time point is the same as the target, but the target bequest becomes zero. The surplus becomes positive and as Figures 5.1c and 5.1f show, this achieves a significant degree of pointwise and pathwise smoothing.

Figures 5.1b and 5.1d show the respective investment proportion, which in both cases is smaller than the target investment proportion due to the positive surplus, but increasing towards the target most of the time. The reason for this is that the surplus is gradually decreased over time, thus increasing the investment proportion. In the case of the reduced bequest, the investment proportion rapidly decreases to -0.75 (see 5.1d). The mathematical reason is, that as  $\xi = 0$  we have  $\lim_{t \uparrow T} g(t) = 0$



**Figure 5.1:** Figures 5.1a, 5.1c and 5.1e show the simulation results in the form of the mean (drawn black line), 5% and 95% quantiles (dashed lines) of the distribution of  $C_t$ , while Figures 5.1b and 5.1d show the same for the investment proportions, where the dotted line marks the target investment proportion. In addition, these figures also show three sample paths, which are realizations of the same underlying stock market trajectory. Hence, they can be directly compared across figures. Finally, Figure 5.1f shows the pointwise variance and the expected quadratic variation for each of the three consumption processes.

implying

$$\lim_{t \uparrow T} X_t = \lim_{t \uparrow T} (Z_t + g(t)C_t) = Z_T, \quad \text{a.s.}$$

This, in turn, implies

$$\lim_{t \uparrow T} \pi_t = \lim_{t \uparrow T} \left( \frac{g(t)}{\sigma X_t} \bar{b}_t C_t - \frac{\theta}{\sigma} (1 + g(t)J_b(t)) \frac{X_t - g(t)C_t}{X_t} \right) = -\frac{\theta}{\sigma} = -0.75, \quad \text{a.s.}$$

The intuitive reason is that the surplus is positive, but the target bequest is zero and reaching that target would necessitate  $Z_T = 0$ . This is furthered by adopting an investment strategy with a low expected return. This behaviour is thus a consequence of the quadratic criteria (see Remark 5.3.6).

*Remark 5.3.6.* The investment strategy might sometimes seem counterintuitive, since it can turn negative. This is a well-known mathematical consequence of the quadratic criteria, see Cairns (2000) and Bruhn and Steffensen (2013). The quadratic criteria minimize the quadratic distance to a target, and in monetary terms, this distance is quantified by the surplus. If the surplus is larger than zero, there is a natural desire to reduce the surplus to zero, which can be achieved faster by adopting an investment strategy with low expected returns. Therefore, if the surplus is sufficiently large, the investment strategy can turn negative, as this yields a worse expected return. This behaviour is also reflected in the implied relative risk aversion given by

$$\gamma(t, C_t, X_t) := \frac{\theta}{\sigma\pi(t, C_t, X_t)} = \frac{\theta X_t}{g(t)C_t\bar{b}(t) - \theta(1 + g(t)J_b(t))Z_t}$$

Thus, a sufficiently large surplus causing  $\pi(t, C_t, X_t)$  to turn negative will lead to a negative value of  $\gamma(t, C_t, X_t)$ , meaning that the individual becomes risk-loving. This makes sense, since the individual favours a short position in the risky asset over a long position in the risk-free asset, even though the short position has a lower expected return and more risk. Over time, the surplus decreases, indicating that the investment strategy may become positive. When this happens, the implied relative risk aversion will take large negative values, approaching  $-\infty$  when  $\pi$  is approaching zero from below, and will switch to large positive values, approaching  $\infty$  when  $\pi$  approaches zero from above. After this transition, the levels of implied relative risk aversion will decrease towards the implied relative risk aversion of the target investment strategy, as the surplus decreases towards zero.

When directly comparing the three different consumption processes, we can observe the following:

- The lines for the pointwise 5% and 95% quantiles in Figures 5.1a+5.1c narrow compared to Figure 5.1e, indicating a clear smoothing effect, but they narrow in different ways. In Figure 5.1c, they form an almost perfect cone, while they form a rounded cone in Figure 5.1a. This difference also manifests itself in Figure 5.1f, where the variances of the two approaches intersect at approximately  $t = 4$ . This is because  $z_0^*(t)$  depends on the choice of the time point  $t$ . In the case of this numerical example,  $t \mapsto z_0^*(t)$  increases in  $t$ , which means that a greater starting surplus is required to minimize the variance at the terminal time  $T$  instead of earlier times. As the starting surplus in the case of Figure 5.1c is smaller than the starting surplus required to minimize the variance at time  $T$ , consumption is smoothed more at earlier times and less at later ones. This would change if a higher level of relative bequest could be reduced and turned into a surplus. This is reversed in Figure 5.1a. As one chooses the starting surplus to minimize the variance at time  $T$ , it is less

efficient at earlier time points, and it would change if one chooses a larger starting consumption instead.

- The light gray and black sample paths in Figure 5.1a switch places compared to Figure 5.1e. Rearranging (5.3.6) yields

$$b_t = \frac{\sigma X_t \pi_t}{g(t)} + \theta \frac{Z_t}{g(t)}.$$

As the investment proportion is negative at the start of the period (see Figure 5.1b), it appears that it is sufficiently negative to turn  $b_t$  negative, causing  $C_t$  to fluctuate opposite to the market. The light gray path in Figure 5.1e shows that it decreases more or less during the whole period. In Figure 5.1a, the light gray path increases in the first years, after which it slowly turns into a decline as the investment proportion increases and becomes positive (see Figure 5.1b). This effect is not present in Figure 5.1c, as the investment proportion here rarely falls below zero (see Figure 5.1d).

In conclusion, both approaches yield similar smoothing mechanisms and could generate similar levels of smoothing if they generate similar levels of surplus. The difference is that one approach preserves the starting value and the mean, but not the relative level of bequest. In contrast, the other approach preserves the relative level of bequest, but not the starting value or the mean.

## 5.4 Reducing the drift

In this section, we investigate the impact of reducing the drift by considering the optimization problem (5.2.4) with fixed starting value  $c_0 = \bar{c}_0$  and targets  $\bar{a}(t, C_t, X_t) = \bar{a}(t)C_t$  and  $\bar{b}(t, C_t, X_t) = \bar{b}(t)C_t$ . Contrary to Section 5.3 we now require  $w > 0$ . This is necessary, since choosing  $c_0 = \bar{c}_0$  yields a surplus of zero, and the desire for smoothing can therefore only come from the penalty of volatility  $w$ . We start by presenting the explicit solution to the problem and follow up with a discussion of the theoretical properties of the optimal controls and the optimal consumption. Here, we profit from a direct connection to the problem considered in Section 5.3. After this, we demonstrate some of these theoretical properties in a numerical example, using the same parameters as in Section 5.3.

### 5.4.1 Theoretical properties

In Appendix 5.B.1, we verify that the value function given by

$$V(t, x, c) = \frac{1}{2} \frac{(x - g(t)c)^2}{f(t)} + \frac{1}{2} l(t)c^2, \quad (5.4.1)$$

where the functions  $f$  and  $g$  satisfy the differential equations

$$g'(t) = (r - \tilde{a}(t) + \theta \tilde{b}(t))g(t) - 1, \quad g(T) = \xi \quad (5.4.2)$$

$$f'(t) = (2r - \theta^2)f(t) - g^2(t) \left( \frac{1}{w_a(t)} + \frac{\theta^2}{k(t)} \right), \quad f(T) = \frac{1}{B}, \quad (5.4.3)$$

$$l'(t) = \frac{l^2(t)}{w_a(t)} - 2l(t)\tilde{a}(t) - \tilde{b}(t)^2 \frac{w_b(t)(w(t) + l(t))^2 + w_b(t)^2(w(t) + l(t))}{k^2(t)}, \quad (5.4.4)$$

$$l(T) = 0$$

and optimal controls

$$a_t = \tilde{a}(t)C_t + J_a(t)Z_t \quad (5.4.5)$$

$$b_t = \tilde{b}(t)C_t - \theta J_k(t)Z_t \quad (5.4.6)$$

$$\pi_t = \frac{g(t)}{\sigma X_t} \tilde{b}_t C_t - \frac{\theta}{\sigma} \frac{Z_t}{X_t} (1 + g(t)J_k(t)), \quad (5.4.7)$$

satisfy the Hamilton-Jacobi-Bellmann equation (5.2.8). Here we define  $k(t) := w_b(t) + w(t) + l(t)$ , the adjusted target drift  $\tilde{a}(t) := \bar{a}(t) - \frac{l(t)}{w_a(t)}$ , and the adjusted target drift  $\tilde{b}(t) := \frac{w_b(t)}{k(t)} \bar{b}(t)$ . In addition, we define the adjustment speeds  $J_a(t) := \frac{g(t)}{f(t)w_a(t)}$  and  $J_k(t) := \frac{g(t)}{f(t)k(t)}$  and the surplus  $Z_t := X_t - g(t)C_t$ .

The optimal controls (5.4.5) and (5.4.6) again consist of two terms, the first of which involves the quantities

$$\tilde{a}(t) := \bar{a}(t) - \frac{l(t)}{w_a(t)} \quad \text{and} \quad \tilde{b}(t) := \frac{w_b(t)}{w_b(t) + w(t) + l(t)} \bar{b}(t) \quad (5.4.8)$$

and the second involves the process  $Z$ . The quantities  $\tilde{a}$  and  $\tilde{b}$  can be interpreted as the adjusted target coefficients, which becomes apparent when comparing the optimal controls (5.3.4)-(5.3.6) and (5.4.5)-(5.4.7). We can see that they have the same structure, with the difference that the target coefficients  $\bar{a}$  and  $\bar{b}$  in (5.3.4)-(5.3.6) are now replaced by the adjusted target coefficients  $\tilde{a}$  and  $\tilde{b}$  compared to (5.4.5)-(5.4.7). Similarly, the target coefficients are replaced by the adjusted target coefficients in the differential equation for  $g$  in (5.4.2), compared to (5.3.2) and in the differential equation for  $f$  in (5.4.3)  $w_b$  is replaced by  $k$  compared to (5.3.3). Thus Proposition 5.3.1 shows that the process  $Z_t$  now has the form

$$Z_t := X_t - g(t)C_t = X_t - \mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-(s-t)r} \tilde{C}_s ds + \xi \tilde{C}_T e^{-(T-t)r} \middle| \tilde{C}_t = C_t \right],$$

where  $\tilde{C}$  is the adjusted target consumption given by

$$d\tilde{C}_t = \tilde{a}(t)\tilde{C}_t dt + \tilde{b}(t)\tilde{C}_t dW_t, \quad \tilde{C}_0 = \frac{x_0}{g(0)} =: \tilde{c}_0. \quad (5.4.9)$$

Thus, one can again interpret  $Z$  as a surplus, but now calculated relative to the expected present value of the adjusted target consumption  $\tilde{C}$  and not the

target consumption  $\bar{C}$ . The consequence of this is that the optimal controls of Problem (5.2.4) with  $w > 0$  and target coefficients  $\bar{a}$  and  $\bar{b}$  are the same as the optimal controls of Problem (5.2.4) with  $w \equiv 0$ , target coefficients  $\tilde{a}$  and  $\tilde{b}$  and weight  $w_b(t) = k(t)$ . Thus, Proposition 5.3.3, Proposition 5.3.2 and the connected discussions are directly applicable, with the difference that  $\bar{a}$ ,  $\bar{b}$ , and  $\bar{C}$  are replaced by  $\tilde{a}$ ,  $\tilde{b}$ , and  $\tilde{C}$ . Therefore, the effect of the parameters  $w_a$ ,  $w_b$ , and  $w$  on the surplus and the impact of the surplus on the optimal controls and consumption are the same, and it remains to understand how exactly the target consumption is adjusted.

To do this, note that even though the same optimal controls can be obtained from the two different optimization problems, their solutions are not the same because the value functions (5.3.1) and (5.4.1) differ by the term  $\frac{1}{2}l(t)c^2$ . The function  $l$  can be interpreted as quantifying the utility cost of the conflicting objectives of minimizing the quadratic distance to the target volatility while penalizing the volatility, since all cases, in which  $l \equiv 0$ , correspond to cases where there is no conflict:

- $w \equiv 0$ : This corresponds to the problem solved in Section 5.3, so no trade-off has to be made. From (5.4.4) it follows that  $l \equiv 0$ . A quick glance at (5.4.8) shows that  $\tilde{C} = \bar{C}$  regardless of the values of  $w_a$  and  $w_b$ . If  $C_0 = \frac{X_0}{g(0)}$ , then the surplus  $Z \equiv 0$ , which means that  $C \equiv \bar{C}$ .
- $\bar{b} \equiv 0$ : In that case, it holds that

$$\frac{1}{2}w_b(t)(b_t - \bar{b}(t)C_t)^2 + \frac{1}{2}w(t)b_t^2 = \frac{1}{2}(w_b(t) + w(t))b_t^2.$$

Thus, this reduces to the previous case and therefore  $\tilde{C} \equiv \bar{C}$ , regardless of the values of  $w_a$ ,  $w_b$ , and  $w$ .

- $w_b \equiv 0$ : In this case (5.4.4) implies  $l \equiv 0$ . The consequence is that  $\tilde{a} = \bar{a}$ . Formally, it cannot be concluded that  $\tilde{C} = \bar{C}$  as in principle, it could be possible that  $0 = \tilde{b} \neq \bar{b}$ , but this comparison seems artificial given that  $w_b \equiv 0$ .

In the cases where  $l \neq 0$ , the following result shows that whenever any of the weights are increased while the others remain fixed, the function  $l$  will increase. In other words: Whenever the conflict between competing objectives becomes more pronounced, the function  $l$  captures this, with the most pronounced differences (and the largest values of  $l$ ) occurring when the weights become infinite.

**Proposition 5.4.1.** *It holds for all  $t \in [0, T]$  that*

1.  $l^{w_a}(t) < l^{w'_a}(t)$  whenever  $w_a(s) < w'_a(s)$  for all  $s \in [0, T]$  and there exists a function  $l_a^\infty$  such that  $l^{w_a}(t) < l_a^\infty(t)$  for all  $t \in [0, T]$  and any value  $w_a(t) \in (0, \infty)$  assuming that all other parameters are fixed,

2.  $l^{w_b}(t) < l^{w'_b}(t)$  whenever  $w_b(s) < w'_b(s)$  for all  $s \in [0, T]$  and there exists a function  $l_b^\infty$  such that  $l^{w_b}(t) < l_b^\infty(t)$  for all  $t \in [0, T]$  and any value  $w_b(t) \in [0, \infty)$  assuming that all other parameters are fixed
3.  $l^w(t) < l^{w'}(t)$  whenever  $w(s) < w'(s)$  for all  $s \in [0, T]$  and there exists a function  $l^\infty$  such that  $l^w(t) < l^\infty(t)$  for all  $t \in [0, T]$  and any value  $w(t) \in (0, \infty)$  assuming that all other parameters are fixed.

*Proof.* See Appendix 5.B.4 □

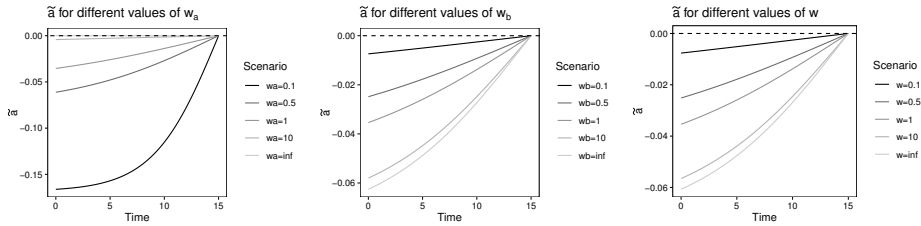
Proposition 5.4.1 can also be used to understand better the influence of the different weights on the adjusted target coefficients  $\tilde{a}$  and  $\tilde{b}$ . In the following fix  $t \in [0, T]$  and assume  $\bar{b}(t), w_a(t), w_b(t), w(t) > 0$ . Then the adjusted drift coefficient  $\tilde{a}$  behaves as follows:

- (a1) From the definition of  $\tilde{a}$  it can be seen that

$$\bar{a}(t) - \tilde{a}(t) = \frac{l(t)}{w_a(t)}.$$

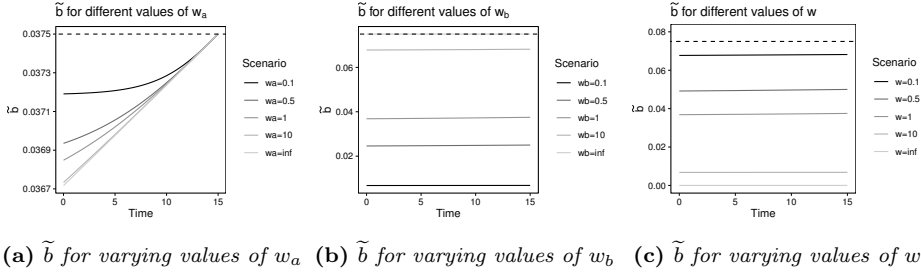
As  $l(t) \geq 0$  for all  $t \in [0, T]$ , it holds that  $\tilde{a}(t) \leq \bar{a}(t)$  for all  $t \in [0, T]$ . Furthermore, since  $\lim_{t \rightarrow T} l(t) = 0$ , it follows that  $\lim_{t \rightarrow T} \tilde{a}(t) = \bar{a}(T)$ .

- (a2) Let  $w_b(t), w(t)$  be fixed and vary  $w_a(t)$ . As  $l(t) < l_a^\infty(t) < \infty$  it holds that  $\lim_{w_a(t) \rightarrow \infty} \tilde{a}(t) = \bar{a}(t)$ . Intuitively, this makes perfect sense, as one places more weight on matching the target drift. Conversely, a smaller value of  $w_a$  will lead to a smaller value of  $\tilde{a}$ , as one places less weight on matching the target drift. This can be seen in Figure 5.2a.
- (a3) Let  $w_a(t)$  be fixed and vary  $w_b(t)$  or  $w(t)$ . In that case,  $\tilde{a}$  only varies as  $l$  varies, which means that  $\tilde{a}$  decreases when  $w_b$  or  $w$  increases (see Figures 5.2b and 5.2c). This makes intuitive sense, as the closeness of the drift to the target becomes relatively less important than matching or reducing the volatility.



(a)  $\tilde{a}$  for varying values of  $w_a$  (b)  $\tilde{a}$  for varying values of  $w_b$  (c)  $\tilde{a}$  for varying values of  $w$

**Figure 5.2:**  $\tilde{a}$  for varying values of  $w_a$ ,  $w_b$  and  $w$ . All parameters not varied are fixed at the values given in Table 5.1. The dashed black line marks  $\bar{a}$ .



**Figure 5.3:**  $\tilde{l}$  for varying values of  $w_a$ ,  $w_b$  and  $w$ . All parameters not varied are fixed at the values given in Table 5.1. In the case of Figure 5.3a, the dashed line shows  $\frac{w_b}{w_b + w} \bar{l}$ , while it shows  $\bar{l}$  in the other two cases.

The adjusted volatility coefficient behaves as follows:

- (b1) From the definition of  $\tilde{l}$  it can be seen that

$$\bar{l}(t) - \tilde{l}(t) = \bar{l}(t) \left( 1 - \frac{w_b(t)}{w_b(t) + w(t) + l(t)} \right).$$

As  $w(t) \geq 0$  and  $l(t) \geq 0$  for all  $t \in [0, T]$  it holds that the fraction is bounded from above by one and as  $l(t) < l_b^\infty(t) < \infty$  for all  $t \in [0, T]$  the fraction is bounded from below by 0, implying  $\tilde{l}(t) \leq \bar{l}(t)$  for all  $t \in [0, T]$ . Furthermore, as  $\lim_{t \rightarrow T} l(t) = 0$ , we have  $\lim_{t \rightarrow T} \tilde{l}(t) = \frac{w_b(T)}{w_b(T) + w(T)} \bar{l}(T)$ . This development can be seen in Figure 5.3a.

- (b2) Let  $w_b(t), w(t)$  be fixed and vary  $w_a(t)$ . Here  $\tilde{l}(t)$  only varies as the function  $l$  varies, and as  $l$  is increasing in  $w_a(t)$ , it holds that  $\tilde{l}(t)$  is decreasing in  $w_a$ . Figure 5.3 shows that this does not necessarily have a large impact numerically.
- (b3) Let  $w_a(t), w(t)$  be fixed and vary  $w_b(t)$ . In that case  $\tilde{l}(t) \rightarrow \bar{l}(t)$  for  $w_b(t) \rightarrow \infty$  as  $l(t) < l_b^\infty(t) < \infty$  for all  $t \in [0, T]$ . At the same time,  $\tilde{l}(t) \rightarrow 0$  as  $w_b(t) \rightarrow 0$ , since the penalty on volatility becomes relatively more important than closeness to the target (see Figure 5.3b).
- (b4) Let  $w_a(t), w_b(t)$  be fixed and vary  $w(t)$ . In that case  $\tilde{l}(t) \rightarrow 0$  for  $w(t) \rightarrow \infty$  as  $l(t) < l^\infty(t) < \infty$  for all  $t \in [0, T]$ . At the same time,  $\tilde{l}(t) \rightarrow \frac{w_b(t)}{w_b(t) + l(t)} \bar{l}(t)$  as  $w(t) \rightarrow 0$ , since the penalty on volatility becomes relatively less important than closeness to the target (see Figure 5.3c). The limit will be equal to  $\bar{l}$  if and only if  $l(t) = 0$ , which is the case if and only if  $w(s) = 0$  for all  $s \geq t$ .

By comparing Figure 5.3a with Figures 5.3b-5.3c, we see that the function  $l$  does not appear to have a large influence on  $\tilde{l}$ , compared to the size of the weights  $w_b$  and  $w$ . In contrast, Figure 5.2 shows that the function  $l$  significantly impacts  $\tilde{a}$ .

Bullet (b1) also has the consequence that the target investment proportion under the penalty of volatility is now smaller than without penalty, since  $\bar{l}$  in

(5.3.6) is replaced by  $\tilde{b}$  in (5.4.7). Therefore, penalizing volatility not only reduces consumption volatility but also reduces the amount invested in risky assets.

Due to the bullets (a1) and (b1) and (5.4.9) it holds for any choice of  $w_a$ ,  $w_b$  and  $w$  that

$$\mathbb{E}[\tilde{C}_t] = c_0 e^{\int_0^t \tilde{a}(u) du} \leq c_0 e^{\int_0^t \bar{a}(u) du} = E[\bar{C}_t], \quad \forall t \in [0, T]$$

and

$$\begin{aligned} \text{Var}(\tilde{C}_t) &= c_0^2 e^{2 \int_0^t \tilde{a}(u) du} \left( e^{\int_0^t \tilde{b}^2(u) du} - 1 \right) \\ &\leq c_0^2 e^{2 \int_0^t \bar{a}(u) du} \left( e^{\int_0^t \bar{b}^2(u) du} - 1 \right) = \text{Var}(\bar{C}_t), \quad \forall t \in [0, T] \end{aligned}$$

This shows that penalizing volatility always leads to an adjusted target consumption with a lower mean and a lower variance than the original target (assuming the same starting value).

As Proposition 5.3.2 is still valid with the target consumption replaced by the adjusted target consumption, this means that the amount  $\frac{X_t}{g(t)}$  now is the maximal amount of adjusted target consumption that can be purchased with wealth  $X_t$  at time  $t$ . Especially it means that if  $z_0 = 0$ , then  $C \equiv \tilde{C}$  and not  $\bar{C}$ . As we have fixed  $c_0 = \bar{c}_0$  we obtain

$$z_0 = x_0 - g(0)\bar{c}_0 = x_0 \left( 1 - \frac{g(0)}{\bar{g}(0)} \right) \neq 0,$$

as  $g(0) \neq \bar{g}(0)$ . Here,  $\bar{g}$  satisfies the differential equation (5.4.2) with  $\tilde{a}$  and  $\tilde{b}$  replaced by  $\bar{a}$  and  $\bar{b}$ . Thus, a surplus will arise, and to avoid obstructing the smoothing achieved by the penalty, we will have to ensure that the surplus is not negative.

We can see that  $z_0 < 0$  if  $g(0) > \bar{g}(0)$  and  $z_0 > 0$  if  $g(0) < \bar{g}(0)$ . To get an idea of how the weights  $w_a$ ,  $w_b$ , and  $w$  affect this relationship, we take a closer look at the function  $g$ , which is determined by the differential equation (5.4.2) and has an explicit solution

$$g(t) = \int_t^T e^{-\int_t^s r - \tilde{a}(u) + \theta \tilde{b}(u) du} ds + \xi e^{-\int_t^T r - \tilde{a}(u) + \theta \tilde{b}(u) du}.$$

As  $\bar{g}$  has the same structure, with  $\tilde{a}$  and  $\tilde{b}$  replaced by  $\bar{a}$  and  $\bar{b}$ , we can conclude that

$$\begin{aligned} z_0 < 0 &\Leftrightarrow \int_0^t \bar{a}(s) - \tilde{a}(s) ds < \int_0^t \theta(\bar{b}(s) - \tilde{b}(s)) ds \quad \text{for enough } t \in [0, T] \\ z_0 > 0 &\Leftrightarrow \int_0^t \bar{a}(s) - \tilde{a}(s) ds > \int_0^t \theta(\bar{b}(s) - \tilde{b}(s)) ds \quad \text{for enough } t \in [0, T]. \end{aligned}$$

This shows that the surplus is negative if  $\tilde{a}$  is close to  $\bar{a}$ , which by bullet (a2) is the case if  $w_a$  is sufficiently large and if  $\tilde{b}$  is sufficiently far from  $\bar{b}$ , which by bullet (b4) is the case for  $w$  sufficiently large. In short, if one insists on the target drift while also wanting low volatility, the surplus becomes negative.

Similarly, the surplus becomes positive if  $\tilde{a}$  is sufficiently far from  $\bar{a}$ , which by bullet (a1) is the case if  $w_a$  is small and if  $\tilde{b}$  is sufficiently close to  $\bar{b}$ , which by bullet (b3) is the case for  $w_b$  sufficiently large. In short, if one adjusts the drift while allowing for higher volatility, the surplus becomes positive.

By bullet (b1) it holds that  $0 \leq \bar{b}(t) - \tilde{b}(t) \leq \bar{b}(t)$ , while by bullet (a1) it holds that  $0 \leq \bar{a}(t) - \tilde{a}(t) < \infty$ . This means that the values of  $w_b$  and  $w$  only have a limited impact on the two inequalities and that the value of  $w_a$  is the main determinant of the sign of the surplus. Hence, we must be sufficiently tolerant of adjustments to the drift.

This has another effect. If  $w_a$  is small, then (5.3.8) can be rewritten to become

$$\text{Var}(C_t) = \int_0^t e^{\int_s^t 2\tilde{a}(u)du} \left( J_a(s) \text{Cov}(C_s, Z_s) + \frac{d}{dt} \mathbb{E}[C]_t \Big|_{t=s} \right) ds.$$

This means that the variance of  $C$  is not only determined by the expected quadratic variation, but also by the covariance between the consumption and the surplus. If this covariance is negative, then additional pointwise smoothing can be achieved. One can make additional risky investments without decreasing the pathwise smoothness.

The two smoothing approaches discussed in this Section and Section 5.3 are both extremes. In Section 5.3, we create a desire for smoothing by creating a positive surplus through a reduction of  $c_0$  or  $\xi$ , while we in this Section keep  $c_0$  and  $\xi$  fixed and create a desire for smoothing by penalising volatility, which is accommodated by adjusting the drift. It is possible to combine these two approaches, but the question then becomes how to choose  $z_0$ . In Section 5.3, we could use Proposition 5.3.3 to guide us towards a choice of  $z_0$ , and since  $w_a$  was large, pathwise and pointwise smoothing were maximized for the same value of  $z_0$ . Now, since  $w_a$  is small, Proposition 5.3.3 can only help us to find a  $z_0$ , which increases pointwise smoothing, but not necessarily pathwise smoothing. Luckily, a similar result can be obtained.

Similarly to (5.3.9), define

$$\begin{aligned} d\Phi_t &= \tilde{a}(t)\Phi_t dt + \tilde{b}(t)\Phi_t dW_t, & \Phi_0 &= 1, \\ d\Psi_t &= (r - g(t)J_a(t) - \theta^2(1 + g(t)J_k(t))\Psi_t)dt - \theta\Psi_t dW_t, & \Psi_0 &= 1, \\ dV_t &= (\tilde{a}(t)V_t + J_a(t)\Psi_t)dt + (\tilde{b}(t)V_t - \theta J_k(t)\Psi_t)dW_t, & V_0 &= 0. \end{aligned} \quad (5.4.10)$$

Then the following counterpart to Proposition 5.3.3 can be derived.

**Proposition 5.4.2.** *The optimal consumption can be written as  $C_t = c_0\Phi_t + z_0V_t$ . Additionally, it holds that*

$$\begin{aligned}\mathbb{E}[[C]_T] = & z_0^2 \mathbb{E} \left[ \left[ \frac{\Phi_T}{g(0)} - V \right]_T \right] + x_0^2 \mathbb{E} \left[ \left[ \frac{\Phi}{g(0)} \right]_T \right] \\ & + 2z_0x_0 \left( \mathbb{E} \left[ \left[ \frac{\Phi}{g(0)}, V \right]_T \right] - \mathbb{E} \left[ \left[ \frac{\Phi}{g(0)} \right]_T \right] \right),\end{aligned}$$

implying that there exists a unique

$$z_0^* = \frac{\mathbb{E}[[\Phi g(0)^{-1}]_T] - \mathbb{E}[[\Phi g(0)^{-1}, V]_T]}{\mathbb{E}[[\Phi g(0)^{-1} - V]_T]} x_0$$

which minimizes  $\mathbb{E}[[C]_T]$ .

*Proof.* See Appendix 5.B.4 □

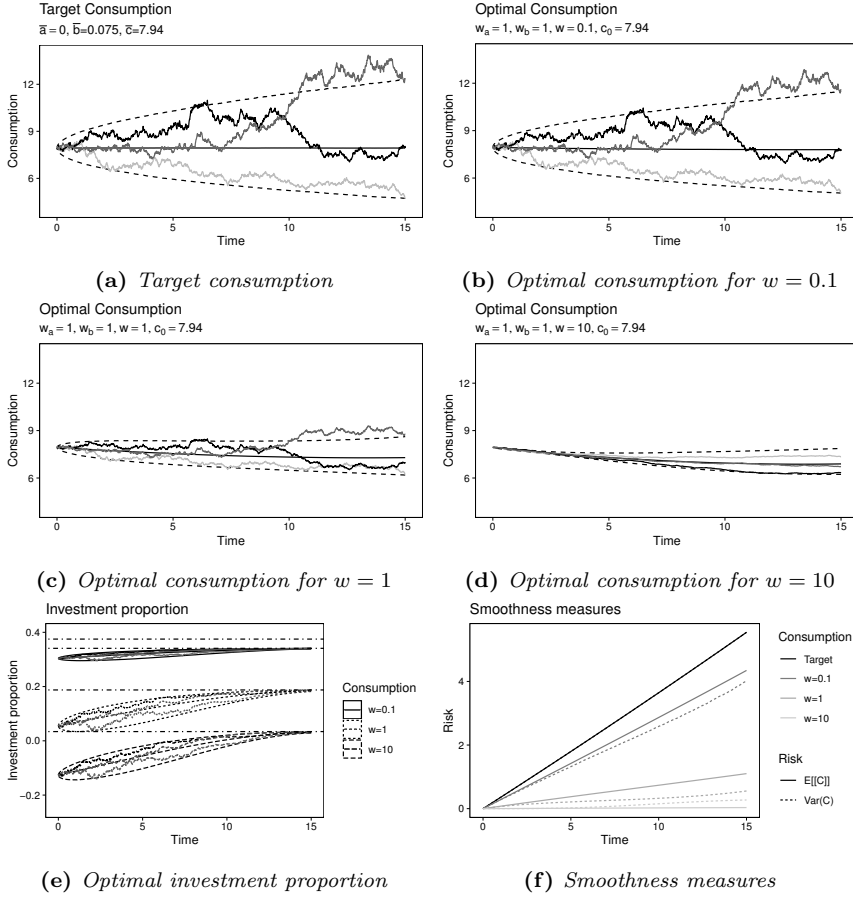
This result provides us with the means to find a unique value  $z_0^*$ , which minimizes the expected quadratic variation and that can serve as a guideline when mixing the two approaches. The value  $z_0^*$  can be calculated by solving a system of linear differential equations (see Appendix 5.B.4).

### 5.4.2 Numerical example

In this example, we simulate the optimal consumption for different values of the penalty parameter  $w$ . The other parameters are the same as in Table 5.1, except for  $w_a$ , now equal to 1, and  $\xi$ , now fixed at 1. The starting value of consumption is chosen as the starting value of the target consumption of the previous numerical example in Section 5.3. Note also that all sample paths depicted in Figure 5.4 are calculated based on precisely the same underlying realization as those depicted in Figure 5.1 and are therefore directly comparable.

Looking now at Figures 5.4a-5.4d we can see that an increasing value of  $w$  indeed leads to an increased amount of both pathwise and pointwise smoothing, but we can also see that  $\mathbb{E}[C_t] < \mathbb{E}[\bar{C}_t]$  for  $w = 0.1, 1, 10$  and the difference increases with the value of  $w$ . This shows that even though the surplus is positive in all three cases, it is not enough to eliminate the drift reduction caused by the desire for smoothing. Furthermore,  $\mathbb{E}[C_t]$  decreases in  $t$  for all values of  $w$  and the larger  $w$  is, the more it decreases (see Figures 5.4b-5.4d). This is a consequence of the fact that  $w \mapsto \tilde{a}$  decreases (see bullet (a3)) and shows that the greater the desire for smoothing, the greater the drift reduction.

We can also see that this desire for smoothing leads to a smaller investment proportion (see 5.4e) than the target. As in Section 5.3, the investment strategy is increasing because there is a positive surplus, which is reduced over time. The



**Figure 5.4:** Figures 5.4a-5.4d show the results of the simulation in the form of the mean (drawn black line), 5% and 95% quantiles (dashed lines) of  $C_t$ , while Figure 5.4e shows the same for the investment proportions, where the dotted line marks the target investment proportion and the dashed-dotted lines mark the adjusted target investment proportion. In addition, these figures show three sample paths corresponding to the same three underlying stock market trajectories as in Figure 5.1. Hence, they can be directly compared across figures. Finally, Figure 5.4f shows the pointwise variance and the expected quadratic variation for each of the four consumption processes.

difference is now that as  $t$  approaches  $T$  the investment proportion in each case approaches  $\frac{w_b}{w+w_b} \frac{\bar{b}}{\sigma} < \frac{\bar{b}}{\sigma}$ . Therefore, the penalty of volatility results in a smaller investment proportion. Again, it is possible for the investment proportion to turn negative as a consequence of a large surplus. The explanation for this is unchanged and discussed in Remark 5.3.6.

Looking at the sample paths in Figures 5.4a-5.4d, we can see that an increase in  $w$  yields a higher pathwise smoothness, and at the same time the sample paths in Figure 5.4d are switched compared to the sample paths in Figures 5.4a-5.4c. The

reason for this is, as in Section 5.3, a negative investment proportion due to a large surplus (see Figure 5.4e). In Figure 5.4d, the sample paths almost stay at the same pointwise quantile during the last half of the period. This is because the investment proportion is close to zero during the second half of the period (see Figure 5.4e). While this means that the level of consumption is relatively stable, it also means that the losses (gains) made during the first half of the period cannot be recovered (lost) during the second half of the period. It is a good illustration of the effects of an investment strategy, which decreases investment risk over time. It is also a demonstration of the trade-off of transforming volatility risk into trend risk due to the desire for almost perfect pathwise smoothing. This becomes clear when looking at Figure 5.4f and recalling the decomposition of the variance stated in Proposition 5.2.3. Since the expected quadratic variation is almost zero, the risk measured by the variance must consist mainly of trend risk.

In Figure 5.4f, we can see that the variance is lower than the expected quadratic variation for  $w = 0.1$  and  $w = 1$ . This is because the covariance term of Proposition 5.2.3 is negative, as both the drift and the volatility term of the optimal consumption depend on the surplus, but with opposite signs. This means that, in addition to achieving pathwise smoothing relative to the target, this approach introduces an extra layer of pointwise smoothing. Looking at the smoothness measures for  $w = 10$ , this changes. Although a much higher degree of pathwise smoothness is achieved, pointwise smoothness increases to a much lesser degree because the underlying risk does not disappear. Instead, it is converted into trend risk.

So, in this progression of increasing  $w$ , three things happen that lead to smoothing. The first is that the investment proportion is decreased, resulting in less risk being taken, which leads to pointwise smoothing. The second thing is that the risk is redistributed from volatility risk to trend risk, resulting in pathwise smoothing. Lastly, the surplus introduces a negative correlation between the drift and volatility of  $C$ , further increasing pointwise smoothing.

Finally, comparing the smoothing mechanism of Section 5.3 shown in Figures 5.1a and 5.1c with Figures 5.4c and 5.4d we can see that reducing the drift instead of the starting value seems to be more efficient at smoothing. The reason for this is that the investment proportions in Figure 5.4e are consistently closer to zero than the investment proportions in Figures 5.1b and 5.1d, meaning that one takes less risk during the entire period. Another thing to note is that the reduction of the drift yields pointwise smoother consumption, as the correlation between the drift and volatility term becomes negative.

## 5.5 Conclusion

In this paper, we have established that when discussing smooth pension products, one should always distinguish between pathwise (variability within one consumption path) and pointwise (variability between paths) smoothing. If one wants to improve pathwise smoothness, which is often the goal of a smooth pension product, then one of the following trade-offs will have to be made:

1. At least one of the starting values of consumption, the drift of consumption, or the level of bequest has to be reduced, which in the case of the first two will always yield a lower expected consumption.
2. A reduction of the volatility of consumption will imply either a reduction of the risky investments or a larger trend risk. This improves pathwise smoothing, but may or may not reduce pointwise smoothing, depending on how the covariance between the drift and volatility components of the consumption behaves.

Subsequently, we have solved a novel consumption-investment problem to investigate the effect of these trade-offs. Using the optimal controls, we have found that reducing the starting value or the bequest creates a surplus, which can then be used to reduce the volatility of consumption and the investment proportion, thereby leading to pathwise and pointwise smoothing. We have developed methods to determine the starting values of consumption or the levels of bequest that lead to sensible values of the starting surplus.

After this, we solved the same consumption investment problem with an added penalty of volatility, and we showed that the optimal controls of this penalized problem are equal to the optimal controls of the unpenalized problem with adjusted targets. This adjusted target consumption has a lower drift and volatility than the original target. This time, we fixed the starting value of consumption and the level of bequest and adjusted the drift.

Both approaches were analyzed using numerical examples, and we found that all investment proportions are lower than the target and increase with time as the surplus slowly reduces. We also found that keeping the drift fixed and adjusting the starting value or bequest has the disadvantage that the variance of consumption is entirely determined by the expected quadratic variation, meaning that one can achieve an increase in smoothing if and only if one lowers the investment proportion. The advantage is that one can achieve smoothing while maintaining the target drift. If, on the other hand, the starting value and bequest are fixed, the drift is reduced. The advantage is that it is possible to reduce volatility without reducing risky investments, and that the surplus induces a negative correlation between the drift and volatility of consumption, leading to additional pointwise smoothing.

Interesting future work could be to look at actual smooth pension products and to find out what choice of parameters  $w_a$ ,  $w_b$ , and  $w$  minimizes the utility loss. One could imagine the following procedure. Find a company with a smooth pension product. This company most likely also offers a pure unit-linked annuity, which is designed based on classic power utility preferences. Thus, this pure unit-linked annuity could be used to define  $\bar{a}$  and  $\bar{b}$ . If the company does not offer such a product, a pure unit-linked annuity from a different company could be used. After this, one can attempt to find the corresponding  $a$  and  $b$  values for the smooth pension product. Finally, one could simulate different paths of the financial market and then find the values of  $w_a$ ,  $w_b$ , and  $w$  that minimize (5.2.4), where  $a$  and  $b$  are considered fixed. This procedure could be repeated for different companies and their smooth pension products. This would enable the comparison of different smooth pension products within the same system of preferences.

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### 5.A Overview of parameters and functions

| Parameter   | Description                         |
|-------------|-------------------------------------|
| $\bar{a}$   | Target drift of consumption         |
| $\bar{b}$   | Target volatility of consumption    |
| $\bar{c}_0$ | Target initial consumption          |
| $\bar{C}$   | Target consumption process          |
| $\xi$       | Factor determining target bequest   |
| $w_a$       | Weight for importance of drift      |
| $w_b$       | Weight for importance of volatility |
| $w$         | Penalty of volatility               |
| $B$         | Weight for importance of bequest    |

**Table 5.2:** *Overview of parameters*

| Notation    | Location            | Definition                         |
|-------------|---------------------|------------------------------------|
| $f(t)$      | Eq. (5.3.3)+(5.4.3) | Part of $V(t, x, c)$               |
| $g(t)$      | Eq. (5.3.2)+(5.4.2) | Part of $V(t, x, c)$               |
| $l(t)$      | Eq. (5.4.4)         | Part of $V(t, x, c)$               |
| $Z_t$       | Eq. (5.3.7)         | $X_t - g(t)C_t$                    |
| $J_a(t)$    | Eq. (5.3.7)         | $\frac{g(t)}{f(t)w_a(t)}$          |
| $J_b(t)$    | Eq. (5.3.7)         | $\frac{g(t)}{f(t)w_b(t)}$          |
| $k(t)$      | Sec. 5.4            | $w_b(t) + w(t) + l(t)$             |
| $J_k(t)$    | Sec. 5.4            | $\frac{g(t)}{f(t)k(t)}$            |
| $\tilde{a}$ | Eq. (5.4.8)         | $\bar{a}(t) - \frac{l(t)}{w_a(t)}$ |
| $\tilde{b}$ | Eq. (5.4.8)         | $\frac{w_b(t)}{k(t)}\bar{b}(t)$    |

**Table 5.3:** Overview of functions

## 5.B Proofs

### 5.B.1 Verification of optimal controls and value function

The candidate optimal value function is

$$V(t, x, c) = \frac{1}{2} \frac{(x - g(t)c)^2}{f(t)} + \frac{1}{2} l(t)c^2,$$

where  $f, g$  and  $l$  are given by the following differential equations

$$\begin{aligned} g'(t) &= \left( r - \bar{a}(t) + \frac{l(t)}{w_a(t)} + \theta \frac{w_b(t)}{k(t)} \bar{b}(t) \right) g(t) - 1, \quad g(T) = \xi, \\ f'(t) &= (2r - \theta^2)f - g(t)^2 \left( \frac{1}{w_a(t)} + \frac{\theta^2}{k(t)} \right), \quad f(T) = \frac{1}{B}, \\ l'(t) &= \frac{l^2(t)}{w_a(t)} - 2l(t)\bar{a}(t) - \bar{b}(t)^2 \frac{w_b(t)(w(t) + l(t))^2 + w_b(t)^2(w(t) + l(t))}{k(t)^2}, \end{aligned}$$

with  $l(T) = 0$ . Here  $k(t) := w(t) + w_b(t) + l(t)$ .  $V$  has partial derivatives

$$\begin{aligned} V_t(t, x, c) &= -\frac{(x - g(t)c)}{f(t)} \left( g'(t)c + \frac{x - g(t)c}{2f(t)} f'(t) \right) + \frac{1}{2} l'(t)c^2 \\ V_x(t, x, c) &= \frac{x - g(t)c}{f(t)} \\ V_c(t, x, c) &= -g(t) \frac{x - g(t)c}{f(t)} + l(t)c \\ V_{xx}(t, x, c) &= \frac{1}{f(t)} \\ V_{xc}(t, x, c) &= -\frac{g(t)}{f(t)} \\ V_{cc}(t, x, c) &= \frac{g(t)^2}{f(t)} + l(t) \end{aligned}$$

In the following, we suppress all dependence on  $t, x, c$ , for ease of notation. Let  $J_k := \frac{g}{fk}$  and  $J_a := \frac{g}{fw_a}$ . The HJB (5.2.8) turns into

$$0 = V_t + \inf_{a,b,\pi} \left\{ \frac{1}{2}w_a(a - \bar{a}c)^2 + \frac{1}{2}w_b(b - \bar{b}c)^2 + \frac{1}{2}wb^2 \right. \\ \left. + (rx + \pi(\lambda - r)x - c)V_x + aV_c \right. \\ \left. + \frac{1}{2}\sigma^2\pi^2x^2V_{xx} + b\sigma\pi xV_{xc} + \frac{1}{2}b^2V_{cc} \right\}, \quad V(T, x, c) = \frac{B}{2}(x - \xi c)^2.$$

The necessary conditions for the infimum are

$$\begin{aligned} 0 &= w_a(a - \bar{a}c) + V_c \\ 0 &= b(w_b + w) - w_b\bar{b}c + \sigma\pi xV_{xc} + bV_{cc} \\ 0 &= (\lambda - r)xV_x + \pi\sigma^2x^2V_{xx} + b\sigma xV_{xc} \end{aligned}$$

Rearranging and then inserting the second equation into the third and then the third back into the second, as well as inserting the partial derivatives of  $V$ , yields the following stationary point

$$\begin{aligned} a^* &= \left( \bar{a} - \frac{l}{w_a} \right) c + J_a(x - gc) \\ b^* &= \frac{w_b}{k}\bar{b}c - \theta J_k(x - gc) \\ \pi^* &= \frac{w_b}{k} \frac{g}{\sigma x} \bar{b}c - \frac{\theta}{\sigma} (1 + gJ_k) \frac{(x - gc)}{x} \end{aligned}$$

### Second order condition

To ensure that this stationary point  $(a^*, b^*, \pi^*)$  indeed is a global minimum, we find the Hessian of the expression in the infimum

$$H(a, b, \pi) = \begin{pmatrix} w_a & 0 & 0 \\ 0 & w + w_b + V_{cc} & \sigma x V_{xc} \\ 0 & \sigma x V_{xc} & \sigma^2 x^2 V_{xx} \end{pmatrix}.$$

Note that  $H(a, b, \pi) = H$  is constant. Hence, it is enough to check whether the matrix  $H$  is positive definite for all  $(t, x, c)$ . The characteristic polynomial is given by

$$(w_a - \lambda)((w + w_b + V_{cc} - \lambda)(\sigma^2 x^2 V_{xx} - \lambda) - \sigma^2 x^2 V_{xc}^2) = 0$$

We can directly see that the first root is  $\lambda = w_a > 0$ . We can find the other two by finding the roots of

$$\lambda^2 - \lambda(w + w_b + V_{cc} + \sigma^2 x^2 V_{xx}) + \sigma^2 x^2 ((w + w_b + V_{cc})V_{xx} - V_{xc}^2) = 0.$$

This yields

$$\begin{aligned} \lambda_{\pm} &= \frac{1}{2}(w + w_b + V_{cc} + \sigma^2 x^2 V_{xx}) \\ &\quad \pm \frac{1}{2}\sqrt{(w + w_b + V_{cc} + \sigma^2 x^2 V_{xx})^2 - 4\sigma^2 x^2 ((w + w_b + V_{cc})V_{xx} - V_{xc}^2)}. \end{aligned}$$

Now, check if the expression in the square root is positive. Rewriting the term inside the square root yields

$$\begin{aligned}
&= (w + w_b + V_{cc} + \sigma^2 x^2 V_{xx})^2 - 4\sigma^2 x^2 ((1 + w + V_{cc})V_{xx} - V_{xc}^2) \\
&= (w + w_b + V_{cc})^2 - 2(w + w_b + V_{cc})\sigma^2 x^2 V_{xx} + (\sigma^2 x^2 V_{xx})^2 + 4\sigma^2 x^2 V_{xc}^2 \\
&= (w + w_b + V_{cc} - \sigma^2 x^2 V_{xx})^2 + 4\sigma^2 x^2 V_{xc}^2 > 0 \quad \forall (t, x, c).
\end{aligned}$$

This means there are two real roots. Plugging in the derivatives of  $V$ , it holds that

$$(w + w_b + V_{cc})V_{xx} - V_{xc}^2 = \left( w + w_b + \frac{g^2}{f} + l \right) \frac{1}{f} - \frac{g^2}{f^2} = \frac{w + w_b}{f} + \frac{l}{f} > 0,$$

since  $f$  and  $l$  are non-negative, which proves that

$$\begin{aligned}
&\frac{1}{2}(w + w_b + V_{cc} + \sigma^2 x^2 V_{xx}) \\
&> \frac{1}{2}\sqrt{(w + w_b + V_{cc} + \sigma^2 x^2 V_{xx})^2 - 4\sigma^2 x^2 ((w + w_b + V_{cc})V_{xx} - V_{xc}^2)}.
\end{aligned}$$

This means that both remaining roots are positive, which implies convexity of  $V$  and, thus, the existence of a global minimum.

### Admissibility

Next, we have to check that the candidate optimal controls are admissible. Recall that the admissibility requirements for  $a, b, \pi$  are

1. There exists  $k > 0$  such that for all  $(c, x) \in \mathbb{R}^2$  and  $t \in [0, T]$  it holds that

$$\|\nu(t, c, x)\| + \|\zeta(t, c, x)\| \leq k(1 + \|(c, x)^T\|).$$

2. There exists  $k > 0$  such that for all  $(c_1, x_1), (c_2, x_2) \in \mathbb{R}^2$  and  $t \in [0, T]$  it holds that

$$\|\nu(t, c_1, x_1) - \nu(t, c_2, x_2)\| + \|\zeta(t, c_1, x_1) - \zeta(t, c_2, x_2)\| \leq k\|(c_1, x_1)^T - (c_2, x_2)^T\|.$$

where  $\nu : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and  $\zeta : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$  are given by:

$$\begin{aligned}
\nu(t, C_t, X_t) &= \begin{pmatrix} a(t, C_t, X_t) \\ rX_t + \pi(t, C_t, X_t)(\lambda - r)X_t - C_t \end{pmatrix} \\
\zeta(t, X_t, C_t) &= \begin{pmatrix} b(t, C_t, X_t) \\ \pi(t, C_t, X_t)\sigma X_t \end{pmatrix}
\end{aligned}$$

Inserting the optimal controls it becomes clear that  $\nu(t, c, x) = M_1(t) \binom{c}{x}$  and  $\zeta(t, c, x) = M_2(t) \binom{c}{x}$ , where

$$M_1(t) = \begin{pmatrix} \tilde{a}(t) - g(t)J_a(t) & J_a(t) \\ \theta g(t)(\tilde{b}(t) + \theta(1 + g(t)J_k(t))) - 1 & r - \theta^2(1 + g(t)J_k(t)) \end{pmatrix}$$

$$M_2(t) = \begin{pmatrix} \tilde{b}(t) + \theta g(t)J_k(t) & -\theta J_k(t) \\ g(t)(\tilde{b}(t) + \theta(1 + g(t)J_k(t))) & -\theta(1 + g(t)J_k(t)) \end{pmatrix}.$$

For fixed  $t \in [0, T]$  this means that  $(c, x) \mapsto \nu(t, c, x)$  and  $(c, x) \mapsto \zeta(t, c, x)$  are continuous linear maps, implying that

$$\|\nu(t, c, x)\| \leq K_1(t)\|(c, x)^T\| \quad \text{and} \quad \|\zeta(t, c, x)\| \leq K_2(t)\|(c, x)^T\|,$$

where  $K_l(t) = \left(\sum_{i=1}^2 \sum_{j=1}^2 (m_{l,ij}(t))^2\right)^{\frac{1}{2}}$ ,  $l = 1, 2$  and  $m_{l,ij}(t)$  is the  $(i, j)$ th entry of  $M_l(t)$ . As all entries  $m_{l,ij}(t)$  are continuous on  $[0, T]$ , it holds that  $K_l(t)$ ,  $l = 1, 2$  are continuous on  $[0, T]$  and hence there exists a constant  $K_l^* = \sup_{t \in [0, T]} K_l(t) > 0$  such that  $K_l(t) \leq K_l^*$  for all  $t \in [0, T]$ . Therefore, it holds that

$$\|\nu(t, c, x)\| + \|\zeta(t, c, x)\| \leq (K_1 + K_2)\|(c, x)^T\| \leq (K_1^* + K_2^*)(1 + \|(c, x)^T\|),$$

which is the first condition of admissibility. Utilising this result and the linear structure of  $\nu$  and  $\zeta$  the second admissibility condition follows directly:

$$\begin{aligned} \|\nu(t, c_1, x_1) - \nu(t, c_2, x_2)\| + \|\zeta(t, c_1, x_1) - \zeta(t, c_2, x_2)\| \\ \leq (K_1^* + K_2^*)\|(c_1, x_1)^T - (c_2, x_2)^T\|. \end{aligned}$$

### Verification

We have found that  $(a^*, b^*, \pi^*)$  realizes the infimum and satisfies the admissibility conditions, so we verify that  $V$  satisfies the HJB equation. Inserting the optimal controls and partial derivatives of  $V$  into the HJB equation yields, after some algebra,

$$\begin{aligned} 0 = & c^2 \left( \frac{1}{2} l_t - \frac{1}{2} \frac{l^2}{w_a} + \bar{a}l + \frac{1}{2} \bar{b}^2 \frac{w_b(w + l^2 + w_b)^2 + w_b^2(w + l)}{k^2} \right) \\ & + \frac{c(x - gc)}{f} \left( rg + \theta \frac{w_b}{k} \bar{b}g + \frac{l}{w_a} g - \bar{a}g - 1 - g_t \right) \\ & + \frac{(x - gc)^2}{2f^2} \left( \left( 2r - \frac{(\lambda - r)^2}{\sigma^2} \right) f - g^2 \left( \frac{1}{w_a} + \frac{(\lambda - r)^2}{\sigma^2 k} \right) - f' \right) \end{aligned}$$

This equality is true by the defining differential equations of  $f$ ,  $g$ , and  $l$ . Hence, we have verified that  $V$  satisfies the HJB equation. By the classical Verification Theorem (see, for example, Th. III.8.1 in Fleming and Soner (2006)), it holds that  $V$  is indeed the optimal value function and  $(a^*, b^*, \pi^*)$  are the optimal controls.

### 5.B.2 Proofs of Section 5.2

*Proof of Proposition 5.2.3.* In this proof we use the short-hand notation  $a_t := a(t, C_t, X_t)$  and  $b_t := b(t, C_t, X_t)$ . We easily find the first formula using that

$$C_t = c_0 + \int_0^t a_s ds + \int_0^t b_s dW_s$$

and utilizing basic facts about the variance. The second formula is obtained by utilising that  $\text{Var}(C_t) = \mathbb{E}[C_t]^2 + (\mathbb{E}[C_t])^2$ . Applying Ito's formula to  $C^2$  and taking expectation yields

$$\mathbb{E}[C_t]^2 = c_0^2 + \int_0^t 2\mathbb{E}[a_s C_s] ds + \mathbb{E}[[C_t]].$$

Now note that

$$(\mathbb{E}[C_t])^2 = \left( c_0 + \int_0^t \mathbb{E}[a_s] ds \right)^2 = c_0^2 + 2c_0 \int_0^t \mathbb{E}[a_s] ds + \left( \int_0^t \mathbb{E}[a_s] ds \right)^2.$$

Differentiating yields

$$\frac{d}{dt}(\mathbb{E}[C_t])^2 = 2\mathbb{E}[a_t] \left( c_0 + \int_0^t \mathbb{E}[a_s] ds \right) = 2\mathbb{E}[a_t]\mathbb{E}[C_t], \quad (EC_0)^2 = c_0^2,$$

which implies

$$(\mathbb{E}[C_t])^2 = c_0^2 + \int_0^t 2\mathbb{E}[a_s]\mathbb{E}[C_s] ds.$$

Combining everything now yields

$$\text{Var}(C_t) = \int_0^t 2(\mathbb{E}[a_s C_s] - \mathbb{E}[a_s]\mathbb{E}[C_s]) ds + \mathbb{E}[[C_t]] = \int_0^t 2 \text{Cov}(C_s, a_s) ds + \mathbb{E}[[C]_t].$$

□

*Proof of Proposition 5.2.5:* The wealth  $X$  has an explicit solution

$$X_t = e^{rt} \left( x_0 - \int_0^t e^{-rs} C_s ds + \int_0^t e^{-rs} p_t \sigma dW_s \right),$$

which we can rearrange into

$$e^{-rt} X_t + \int_0^t e^{-rs} C_s ds = x_0 + \int_0^t e^{-rs} p_t \sigma dW_s. \quad (5.B.1)$$

Now, evaluating this for  $t = T$ , taking conditional expectation and rearranging yields

$$\mathbb{E}_{c_0}^{\mathbb{Q}} \left[ \int_t^T e^{-r(s-t)} C_s ds + e^{-r(T-t)} X_T \middle| \mathcal{F}_t \right] = X_t, \quad \forall t \in [0, T].$$

Thus, the budget condition is satisfied. From this budget condition it can immediately be seen that  $X_t = X_t^{(1)} + X_t^{(2)}$ , with

$$X_t^{(1)} := \mathbb{E}_{c_0}^{\mathbb{Q}} \left[ \int_t^T e^{-r(s-t)} C_s ds \middle| \mathcal{F}_t \right] \quad \text{and} \quad X_t^{(2)} := E_{c_0}^{\mathbb{Q}} \left[ e^{-r(T-t)} X_T \middle| \mathcal{F}_t \right].$$

Now evaluating (5.B.1) in  $t = T$  and taking conditional expectation yields

$$x_0 + \int_0^t e^{-rs} p_t \sigma dW_s^{\mathbb{Q}} = \underbrace{\mathbb{E}_{c_0}^{\mathbb{Q}} \left[ \int_0^T e^{-sr} C_s ds \middle| \mathcal{F}_t \right]}_{:= M_t^{(1)}} + \underbrace{\mathbb{E}_{c_0}^{\mathbb{Q}} \left[ e^{-rT} X_T \middle| \mathcal{F}_t \right]}_{:= M_t^{(2)}}. \quad (5.B.2)$$

Both  $M^{(1)}$  and  $M^{(2)}$  are martingales, and thus, by Proposition 2.6 and Corollary 2.9 in Karatzas, Lehoczky, and Shreve (1987), there exist processes  $\phi^{(1)}$  and  $\phi^{(2)}$  such that

$$M_t^{(1)} = M_0^{(1)} + \int_0^t e^{-rs} p_t^{(1)} \sigma dW_s^{\mathbb{Q}} \quad \text{and} \quad M_t^{(2)} = M_0^{(2)} + \int_0^t e^{-rs} p_t^{(2)} \sigma dW_s^{\mathbb{Q}}, \quad (5.B.3)$$

where  $p_t^{(i)} = e^{rt} \sigma^{-1} \phi_t^{(i)}$ ,  $i = 1, 2$ . Rearranging (5.B.3) yields

$$\begin{aligned} X_t^{(1)} &= e^{rt} \left( M_0^{(1)} - \int_0^t e^{-rs} C_s ds + \int_0^t e^{-rs} p_s^{(1)} \sigma dW_s^{\mathbb{Q}} \right) \\ X_t^{(2)} &= e^{rt} \left( M_0^{(2)} + \int_0^t e^{-rs} p_s^{(2)} \sigma dW_s^{\mathbb{Q}} \right). \end{aligned}$$

Finally inserting (5.B.3) into (5.B.2) yields

$$x_0 + \int_0^t e^{-rs} p_t \sigma dW_s^{\mathbb{Q}} = M_0^{(1)} + \int_0^t e^{-rs} p_t^{(1)} \sigma dW_s^{\mathbb{Q}} + M_0^{(2)} + \int_0^t e^{-rs} p_t^{(2)} \sigma dW_s^{\mathbb{Q}},$$

Implying that  $x_0 = M_0^{(1)} + M_0^{(2)}$  and  $p_t = p_t^{(1)} + p_t^{(2)}$ .  $\square$

*Proof of Proposition 5.2.9.* First, note that by the Feynman-Kac Formula, the function  $u$  satisfies the PDE

$$u_t(t, c, x) + \mathcal{A}u(t, c, x) - ru(t, c, x) + c = 0,$$

where  $\mathcal{A}$  is the infinitesimal generator of the two-dimensional diffusion process  $(C, X)$ . Under the measure  $\mathbb{Q}$  the process  $X$  becomes

$$dX_t = (rX_t - C_t)dt + \sigma p(t, C_t, X_t)dW_t^{\mathbb{Q}}, \quad X_0 = x_0.$$

Now multiplying the budget condition  $u(t, C_t, X_t) = X_t$  by  $e^{-rt}$  and applying integration by parts on both sides yields

$$\begin{aligned} u(0, c_0, x_0) - \int_0^t r e^{-rs} u(s, C_s, X_s) ds + \int_0^t e^{-rs} du(s, C_s, X_s) + \int_0^t e^{-rs} C_s ds \\ = x_0 + \int_0^t e^{-rs} \sigma p(t, C_s, X_s) dW_s^{\mathbb{Q}}. \end{aligned}$$

By applying Ito's formula to calculate the dynamics of  $u(t, C_t, X_t)$  and the Feynman-Kac formula, the LHS becomes

$$\begin{aligned} & u(0, c_0, x_0) + \int_0^t e^{-rs} (u_t(s, C_s, X_s) + \mathcal{A}u(s, C_s, X_s) - ru(s, C_s, X_s) + C_s) ds + M_t \\ &= u(0, c_0, x_0) + M_t, \\ M_t &= \int_0^t e^{-rs} (b(s, C_s, X_s)u_c(s, C_s, X_s) + \sigma p(s, C_s, X_s)u_x(s, C_s, X_s)) dW_s^{\mathbb{Q}}. \end{aligned}$$

From this, we can see that the budget constraint holds if and only if  $x_0 = u(0, c_0, x_0)$ . Furthermore, if  $p(t, C_t, X_t) = 0$ , then  $b(t, C_t, X_t) = 0$  or  $u_c(t, C_t, X_t) = 0$  and if  $p(t, C_t, X_t) > 0$  then

$$p(t, C_t, X_t) = \frac{u_c(t, C_t, X_t)}{\sigma(1 - u_x(t, C_t, X_t))} b(t, C_t, X_t) \quad \forall t \in [0, T],$$

if  $b(t, C_t, X_t) > 0$  or if  $b(t, C_t, X_t) = 0$ , then  $u_x(t, C_t, X_t) = 1$ , while  $\pi$  can be arbitrary. This also means that  $a$  must depend on  $X$ , otherwise  $u_x = 0$ .  $\square$

### 5.B.3 Proofs of Section 5.3

*Proof: Proposition 5.3.1.* Note that by the Girsanov Theorem,  $\bar{C}$  under  $\mathbb{Q}$ -is given by

$$d\bar{C}_t = (\bar{a}(t) - \theta\bar{b}(t)) C_t dt + \bar{b}(t) C_t dW_t^{\mathbb{Q}}, \quad \bar{C}_0 = \bar{c}_0.$$

Hence  $\bar{C}$  has explicit solution

$$\bar{C}_t = \bar{c}_0 \exp \left( \int_0^t \bar{a}(s) - \theta\bar{b}(s) - \frac{1}{2}\bar{b}(s)^2 ds + \int_0^t \bar{b}(s) dW_s^{\mathbb{Q}} \right).$$

and we can calculate

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}} [\bar{C}_s | \mathcal{F}_t] \\ &= \bar{c}_0 \exp \left( \int_0^s \bar{a}(u) - \theta\bar{b}(u) du \right) \mathbb{E}^{\mathbb{Q}} \left[ \exp \left( \int_0^s \bar{b}(u) dW_u^{\mathbb{Q}} - \frac{1}{2} \int_0^s \bar{b}(u)^2 du \right) \middle| \mathcal{F}_t \right]. \end{aligned}$$

We can recognize the term in the conditional expectation as a Doléan exponential martingale, and by using the martingale property, we obtain

$$\mathbb{E}^{\mathbb{Q}} [\bar{C}_s | \mathcal{F}_t] = \bar{C}_t \exp \left( \int_t^s \bar{a}(u) - \theta\bar{b}(u) du \right).$$

Now, using this result, we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}} \left[ \int_t^T e^{-(s-t)r} \bar{C}_s ds + e^{-(T-t)r} \bar{X}_T \middle| \mathcal{F}_t \right] \\
= \int_t^T e^{-(s-t)r} \mathbb{E}^{\mathbb{Q}} [\bar{C}_s | \mathcal{F}_t] ds + \xi \mathbb{E}^{\mathbb{Q}} [\bar{C}_T e^{-(T-t)r} | \mathcal{F}_t] \\
= \bar{C}_t \left( \int_t^T e^{-\int_t^s r - \bar{a}(u) + \theta \bar{b}(u) du} ds + \xi e^{-\int_t^T r - \bar{a}(u) + \theta \bar{b}(u) du} \right) \\
= \bar{C}_t g(t).
\end{aligned}$$

We use Ito's formula to find the semimartingale decomposition of  $Z$ , which will be used to prove the next part. Inserting the optimal controls (5.3.4) and (5.3.5) into (5.2.6), yields

$$dC_t = (\bar{a}(t)C_t + J_a(t)Z_t) dt + (\bar{b}(t)C_t - \theta J_b(t)Z_t) dW(t), \quad C_0 = c_0$$

and using integration by parts, we obtain for  $Y_t := C_t g(t)$ :

$$dY_t = ((r + \theta \bar{b}(t))Y_t + g(t)J_a(t)Z_t - C_t)dt + (\bar{b}(t)Y_t - \theta g(t)J_b(t)Z_t)dW_t,$$

with  $Y_0 = c_0 g(0)$ . Inserting the investment strategy (5.3.6) into the wealth (5.2.5) yields

$$\begin{aligned}
dX_t &= (rX_t + \theta \bar{b}(t)Y_t - \theta^2 Z_t(1 + g(t)J_b(t)) - C_t) dt \\
&\quad + (\bar{b}(t)Y_t - \theta Z_t(1 + g(t)J_b(t))) dW(t), \quad X_0 = x_0.
\end{aligned}$$

Applying Ito's formula to calculate  $dZ_t = dX_t - dY_t$  yields

$$dZ_t = (r - g(t)J_a(t) - \theta^2(1 + g(t)J_b(t))) Z_t dt - \theta Z_t dW(t), \quad Z_0 = x_0 - g(0)c_0.$$

This shows that  $Z$  has an explicit solution

$$Z_t = z_0 \exp \left( \int_0^t r - g(s)J_a(s) - \theta^2 \left( \frac{3}{2} + g(s)J_b(s) \right) ds - \theta W_t \right),$$

and the results concerning the sign follow.  $\square$

*Proof: Proposition 5.3.3.* The optimal consumption satisfies the SDE

$$dC_t = (\bar{a}(t)C_t + J_a(t)Z_t) dt + (\bar{b}(t)C_t - \theta J_b(t)Z_t) dW_t, \quad C_0 = c_0 > 0,$$

which has an explicit solution

$$\begin{aligned}
C_t &= c_0 \Phi_t + z_0 \Phi_t \left( \int_0^t \Phi_s^{-1} (J_a(s) + \theta \bar{b}(s)J_b(s)) \Psi_s ds - \int_0^t \Phi_s^{-1} \theta J_b(s) \Psi_s dW_s \right) \\
&= c_0 \Phi_t + z_0 \Phi_t Y_t,
\end{aligned}$$

where  $\Phi_t$  and  $\Psi_t$  are given by

$$\begin{aligned}\Phi_t &:= \exp \left( \int_0^t \bar{a}(s) - \frac{1}{2} \bar{b}(s)^2 ds + \int_0^t \bar{b}(s) dW_s \right) = \frac{\bar{C}_t}{c_0} \\ \Psi_t &:= \exp \left( \int_0^t r - g(s)J_a(s) - \theta^2 \left( \frac{3}{2} + g(s)J_b(s) \right) ds - \theta W_t \right) = \frac{Z_t}{z_0}.\end{aligned}$$

Now define  $V_t := \Phi_t Y_t$ . Note that

$$dY_t = \Phi_t^{-1} \left( J_a(t) + \theta \bar{b}(t) J_k(t) \right) \Psi_t dt - \Phi_t^{-1} \theta J_k(t) \Psi_t dW_t, \quad Y_0 = 0.$$

Applying integration by parts yields

$$dV_t = (J_a(t)\Psi_t + \bar{a}(t)V_t)dt + (V_t\bar{b}(t) - \theta J_k(t)\Psi_t)dW_t, \quad V_0 = 0.$$

Hence, we have identified the processes  $V$  and  $\Phi$  such that  $C_t = c_0\Phi_t + z_0V_t$ .

Now for the variance, note that as  $C_t = c_0\Phi_t + z_0V_t$ , it holds that

$$\text{Var}(C_t) = \text{Var}(c_0\Phi_t + z_0V_t) = c_0^2 \text{Var}(\Phi_t) + 2z_0c_0 \text{Cov}(\Phi_t, V_t) + z_0^2 \text{Var}(V_t).$$

Using the definition of  $z_0$  we can write  $c_0 = \frac{x_0 - z_0}{g(0)}$ , which yields

$$\frac{(x_0 - z_0)^2}{g(0)^2} \text{Var}(\Phi_t) + 2z_0 \frac{x_0 - z_0}{g(0)} \text{Cov}(\Phi_t, V_t) + z_0^2 \text{Var}(V_t).$$

Rearranging yields the desired expression, which we recognize as a convex 2nd-degree polynomial in  $z_0$ . Hence, the minimal value follows.  $\square$

*Calculation of the variance.* Define  $G_V(t) := \mathbb{E}[V_t]$ ,  $G_\Psi(t) := \mathbb{E}[\Psi_t]$  and  $G_\Phi(t) := \mathbb{E}[\Phi_t]$ . By taking expectations of the respective SDE, we obtain the following differential equations for  $G_V$ ,  $G_\Psi$ , and  $G_\Phi$ :

$$\begin{aligned}G'_V(t) &= J_a(t)G_\Psi(t) + \bar{a}(t)G_V(t), \quad G_V(0) = 0 \\ G'_\Psi(t) &= a_\Psi(t)G_\Psi(t), \quad G_\Psi(0) = 1 \\ G'_\Phi(t) &= \bar{a}(t)G_\Phi(t), \quad G_\Phi(0) = 1,\end{aligned}$$

where  $a_\Psi(t) := r - g(t)J_a(t) - \theta^2(1 + g(t)J_b(t))$ .

Define further  $G_{V\Phi}(t) := \mathbb{E}[V_t\Phi_t]$ ,  $G_{\Psi\Phi}(t) := \mathbb{E}[\Psi_t\Phi_t]$ ,  $G_{V\Psi}(t) := \mathbb{E}[V_t\Psi_t]$ ,  $G_{V^2}(t) := \mathbb{E}[V_t^2]$ ,  $G_{\Phi^2}(t) = \mathbb{E}[\Phi_t^2]$  and  $G_{\Psi^2}(t) := \mathbb{E}[\Psi_t^2]$ . Using integration by parts and subsequently taking expectation, we find that these functions satisfy the

following system of differential equations

$$\begin{aligned}
G'_{V\Phi}(t) &= (J_a(t) - \theta\bar{b}(t)J_b(t))G_{\Psi\Phi}(t) + (2\bar{a}(t) + \bar{b}(t)^2)G_{V\Phi}(t), \\
G_{V\Phi}(0) &= 0, \\
G'_{V\Psi}(t) &= (\bar{a}(t) + a_\Psi(t) - \theta\bar{b}(t))G_{V\Psi}(t) + (J_a(t) + \theta^2 J_b(t))G_{\Psi^2}(t), \\
G_{V\Psi}(0) &= 0, \\
G'_{\Psi\Phi}(t) &= (\bar{a}(t) - \theta\bar{b}(t) + a_\Psi(t))G_{\Psi\Phi}(t), \\
G_{\Psi\Phi}(0) &= 1, \\
G'_{V^2}(t) &= (2\bar{a}(t) + \bar{b}(t)^2)G_{V^2}(t) + 2(J_a(t) - \theta\bar{b}(t)J_b(t))G_{V\Psi}(t) + \theta^2 J_b(t)^2 G_{\Psi^2}(t), \\
G_{V^2}(0) &= 0, \\
G_{\Phi^2}(t) &= (2\bar{a}(t) + \bar{b}(t)^2)G_{\Phi^2}(t), \quad G_{\Phi^2}(0) = 1 \\
G_{\Psi^2}(t) &= (2a_\Psi(t) + \theta^2)G_{\Psi^2}(t), \quad G_{\Psi^2}(0) = 1.
\end{aligned} \tag{5.B.4}$$

We must solve this differential equation system to calculate the variance numerically.  $\square$

#### 5.B.4 Proofs of Section 5.4

*Proof of Proposition 5.4.1.* Recall that  $l_t(t) = f(t, l, w_a, w_b, w)$ , where

$$f(t, l, w_a, w_b, w) = \frac{l(t)^2}{w_a(t)} - 2l(t)\bar{a}(t) - \bar{b}(t)^2 h(t, w_b, w, l), \quad l(T) = 0,$$

where

$$h(t, w_b, w, l) := \frac{w_b(t)(w(t) + l(t))^2 + w_b(t)^2(w(t) + l(t))}{(w_b(t) + w(t) + l(t))^2}.$$

1. Assume that  $w(t)$  and  $w_b(t)$  are fixed and assume  $w_a(t) < w'_a(t)$  for all  $t \in [0, T]$ . Then it follows that

$$l_t^{w_a}(t) = f(t, l^{w_a}, w_a, w_b, w) \geq f(t, l^{w'_a}, w'_a, w_b, w).$$

Let now  $l^{w'_a}$  be the solution to  $l_t^{w'_a} = f(t, l^{w'_a}, w'_a, w_b, w)$ . It holds that  $l^{w_a}(T) = l^{w'_a}(T) = 0$ . By Lemma 5.B.1 it holds that  $l^{w_a}(t) < l^{w'_a}(t)$  for all  $t \in [0, T)$ . The dominating function  $l_a^\infty$  is thus the solution of the differential equation (5.4.4) with  $w_a(t) = \infty$  for all  $t \in [0, T]$  given by:

$$(l_a^\infty)'(t) = -2l_a^\infty(t)\bar{a}(t) - \bar{b}(t)^2 \frac{w_b(t)(w(t) + l_a^\infty(t))^2 + w_b(t)^2(w(t) + l_a^\infty(t))}{(w_b(t) + w(t) + l_a^\infty(t))^2},$$

with terminal condition  $l_a^\infty(T) = 0$ .

2. Assume that  $w(t)$  and  $w_a(t)$  are fixed and assume  $w_b(t) < w'_b(t)$  for all  $t \in [0, T]$ . For any fixed  $t \in [0, T]$  it holds that

$$\frac{\partial}{\partial w_b} h(t, w_b, w, l) = \frac{(l(t) + w(t))^2 w_b(t) + (l(t) + w(t))^2}{(w_b(t) + w(t) + l(t))^3} > 0, \quad \forall w_b(t) \geq 0,$$

meaning that  $h$  is increasing in  $w_b$ . The consequence is that

$$l_t^{w_b}(t) = f(t, l^{w_b}, w_a, w_b, w) \geq f(t, l^{w_b}, w_a, w'_b, w).$$

Let now  $l^{w'_b}$  be the solution to  $l_t^{w'_b} = f(t, l^{w'_b}, w_a, w'_b, w)$ . It holds that  $l^{w_b}(T) = l^{w'_b}(T) = 0$ . By Lemma 5.B.1 it holds that  $l^{w_b}(t) < l^{w'_b}(t)$  for all  $t \in [0, T)$ . The dominating function  $l_b^\infty$  is thus the solution of the differential equation (5.4.4) with  $w_b(t) = \infty$  for all  $t \in [0, T]$  given by:

$$(l_b^\infty)'(t) = \frac{l_b^\infty(t)^2}{w_a(t)} - 2l_b^\infty(t)\bar{a}(t) - \bar{b}(t)^2(w(t) + l_b^\infty(t)), \quad l_b^\infty(T) = 0.$$

3. Assume that  $w_a(t)$  and  $w_b(t)$  are fixed and assume  $w(t) < w'(t)$  for all  $t \in [0, T]$ . For any fixed  $t \in [0, T]$  it holds that

$$\frac{\partial}{\partial w} h(t, w_b, w, l) = \frac{(l(t) + w(t))w_b(t) + w_b(t)^2}{(w_b(t) + w(t) + l(t))^3} > 0, \quad \forall w(t) \geq 0,$$

meaning that  $h$  is increasing in  $w$ . The consequence is that

$$l_t^w(t) = f(t, l^w, w_a, w_b, w) \geq f(t, l^w, w_a, w_b, w').$$

Let now  $l^{w'}$  be the solution to  $l_t^{w'} = f(t, l^{w'}, w_a, w_b, w')$ . It holds that  $l^w(T) = l^{w'}(T) = 0$ . By Lemma 5.B.1 it holds that  $l^w(t) < l^{w'}(t)$  for all  $t \in [0, T)$ . The dominating function  $l^\infty$  is thus the solution of the differential equation (5.4.4) with  $w(t) = \infty$  for all  $t \in [0, T]$  given by:

$$(l^\infty)'(t) = \frac{l^\infty(t)^2}{w_a(t)} - 2l^\infty(t)\bar{a}(t) - \bar{b}(t)^2w(t), \quad l^\infty(T) = 0.$$

□

**Lemma 5.B.1.** *Assume the function  $u$  satisfies  $u_t(t) > f(t, u(t))$  for  $t \in [0, T]$  while the function  $v$  satisfies  $v_t(t) = f(t, v(t))$  for  $t \in [0, T]$  and assume  $v(T) = u(T)$ . Then, it holds that*

$$u(t) < v(t) \quad \forall t \in [0, T).$$

*Proof.* First note that as  $v(T) = u(T)$ , it holds that  $u_t(T) > v_t(T)$ , which implies that there exists a  $k > 0$  such that  $u_t(T) - v_t(T) > k$ . This means that there exists a region  $(T - \varepsilon, T)$  where  $u(t) < v(t)$ . To prove this, define

$$L_u(\varepsilon) := \frac{u(T - \varepsilon) - u(T)}{-\varepsilon} \quad \text{and} \quad L_v(\varepsilon) := \frac{v(T - \varepsilon) - v(T)}{-\varepsilon}.$$

It holds that  $\lim_{\varepsilon \rightarrow 0} L_u(\varepsilon) = u_t(T)$  and  $\lim_{\varepsilon \rightarrow 0} L_v(\varepsilon) = v_t(T)$ . Due to this convergence for any  $\delta > 0$  there exists  $\varepsilon > 0$  such that

$$u_t(T) - \delta < L_u(\varepsilon) < u_t(T) + \delta \quad \text{and} \quad v_t(T) - \delta < L_v(\varepsilon) < v_t(T) + \delta.$$

This guarantees that there exists an  $\varepsilon > 0$ , such that the above holds for  $\delta < \frac{k}{2}$ . Using this, one obtains

$$L_u(\varepsilon) - L_v(\varepsilon) > u_t(T) - v_t(T) - 2\delta > k - 2\delta > 0,$$

which directly implies  $u(t) < v(t)$  for  $(T - \varepsilon, T)$ .

Now we have to prove that there cannot exist a  $t_1 \leq T - \varepsilon$ , such that  $u(t_1) \geq v(t_1)$ . Assume for contradiction that such a  $t_1$  exists. As  $u$  and  $v$  are continuous, there must exist a  $t_0$  such that  $t_1 \leq t_0 \leq T - \varepsilon$ , with  $u(t_0) = v(t_0)$ , and  $u(t) < v(t)$  for  $t > t_0$ . Again it holds that  $u_t(t_0) > f(t_0, u(t_0)) = f(t_0, v(t_0)) = v_t(t_0)$ . Now as  $u(t) < v(t)$  for  $t > t_0$  it holds that

$$u(t) - v(t) < 0 \Leftrightarrow \frac{u(t) - u(t_0)}{t - t_0} - \frac{v(t) - v(t_0)}{t - t_0} < 0.$$

Taking the limit  $t \rightarrow t_0^+$  on both sides yields  $u_t(t_0) < v_t(t_0)$ , which is a contradiction.  $\square$

*Proof of Proposition 5.4.2.* The representation  $C_t = c_0\Phi_t + z_0V_t$  is derived exactly as the proof of Proposition 5.3.3. Utilizing this representation, we get

$$\mathbb{E}[C]_T = \mathbb{E}[c_0\Phi + z_0V]_T = c_0^2\mathbb{E}[\Phi]_T + z_0\mathbb{E}[V]_T + 2c_0z_0\mathbb{E}[\Phi, V]_T.$$

Inserting  $c_0 = \frac{x_0 - z_0}{g(0)}$  and rearranging yields the desired result, which can be recognized as a second-degree convex polynomial in  $z_0$ . Thus, a unique minimizer exists.  $\square$

*Calculation of the expected quadratic variation.* All quantities needed for calculating the minimizer  $z_0^*$  of Proposition 5.4.2 can be calculated from  $\mathbb{E}[(\Phi g(0)^{-1})_T]$ ,  $\mathbb{E}[V]_T$  and  $\mathbb{E}[(\Phi g(0)^{-1}, V)_T]$ , since

$$\mathbb{E}[(\Phi g(0)^{-1} - V)_T] = \mathbb{E}[(\Phi g(0)^{-1})_T] + \mathbb{E}[V]_T - 2\mathbb{E}[(\Phi g(0)^{-1}, V)_T].$$

We have that

$$\begin{aligned} \mathbb{E}[(\Phi g(0)^{-1})_T] &= \frac{1}{g(0)^2} \int_0^T \tilde{b}(s) \mathbb{E}[\Phi_s] ds \\ \mathbb{E}[V]_T &= \int_0^T \tilde{b}^2(s) \mathbb{E}[V_s^2] - 2\theta \tilde{b}(s) J_k(s) \mathbb{E}[V_s \Psi_s] + \theta^2 J_k^2(s) \mathbb{E}[\Psi_s^2] ds \\ \mathbb{E}[(\Phi g(0)^{-1}, V)_T] &= \frac{1}{g(0)} \int_0^T \tilde{b}^2(s) \mathbb{E}[\Phi_s V_s] - \theta \tilde{b}(s) J_k(s) \mathbb{E}[\Phi_s \Psi_s] ds, \end{aligned}$$

which can be calculated after solving the system of ODEs (5.B.4), where of course  $\bar{a}, \bar{b}$  and  $J_b$  are replaced by  $\tilde{a}, \tilde{b}$  and  $J_k$ .  $\square$

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