UNIVERSITY OF COPENHAGEN
FACULTY OF SCIENCE

PhD thesis

Spectral estimates for Wiener-Hopf operators with applications to area laws

Martin Dam Larsen

Advisor: Jan Philip Solovej

Submitted: March 31, 2026

This thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen

Martin Dam Larsen
Centre for the Mathematics of Quantum Theory
Department of Mathematical Sciences
University of Copenhagen
Universitetsparken 5, DK-2100, Copenhagen Ø, Denmark
mdl@math.ku.dk

Thesis title:	Spectral estimates for Wiener-Hopf Operators with applications to area laws
Date of submission:	31 March, 2026
Advisor	**Jan Philip Solovej** University of Copenhagen, Copenhagen, Denmark
Co-advisor	**Søren Fournais** University of Copenhagen, Copenhagen, Denmark
Assessment committee	**Albert Werner** (chair) University of Copenhagen, Copenhagen, Denmark **Alexander V. Sobolev** University College London, London, United Kingdom **Peter Müller** Ludwig-Maximilians-Universität München München, Germany
ISBN:	978-87-7629-240-9

This thesis has been submitted to the PhD School of The Faculty of Science, University of Copenhagen.
It was supported by the Villum Fondation (Grant No.10059).

Abstract

This thesis studies spectral asymptotics for Wiener–Hopf operators with discontinuous symbols. We establish a two-term asymptotic formula for traces of the form $\mathrm{Tr}[f(1_{\alpha\Omega}(x)P(-i\nabla)1_{\alpha\Omega}(x))]$ as $\alpha \to \infty$ under essentially optimal regularity conditions, namely for P a function of bounded variation and Ω a set of finite perimeter, and show that these assumptions are sharp if P is an indicator function.

We also study time-frequency limiting operators, which arise as the special case of Wiener–Hopf operators where the symbol is an indicator function, and obtain sharp uniform bounds on the plunge region when one of the underlying sets is a finite disjoint union of parallelepipeds. As an application, we extend the two-term asymptotic formula to very rough spectral functions f.

Résumé

Denne afhandling studerer spektral asymptotik for Wiener-Hopf operatorer med diskontinuerte symboler. Vi etablerer en toleddet asymptotisk formel for spor af formen $\mathrm{Tr}[f(1_{\alpha\Omega}(x)P(-i\nabla)1_{\alpha\Omega}(x))]$ når $\alpha \to \infty$ under essentielt optimale regularitetsantagelser, nemlig når P er en funktion med begrænset variation og Ω en mængde med endelig perimeter, og vi viser at disse regularitetsantagelse er skarpe når P er en indikatorfunktion.

Vi studerer også tids-frekvensbegrænsende operatorer, som optræder som specialtilfældet af Wiener-Hopf operatorer, hvor symbolet et givet ved en indikatorfunktion, og opnår uniforme skarpe estimater på plunge regionen når én af de underlæggende mængder en en endelig forening af parallelepipeder. Som anvendelse udvider vi den toleddet asymptotiske formel til meget grove spektrale funktioner f.

Acknowledgements

Many people have influenced and shaped my three year journey who deserve to be acknowledged. I would like to start by expressing my gratitude to my advisor, Jan Philip Solovej, for firmly and patiently guiding me through my PhD. Your calm and open approach to research and leadership has been genuinely inspirational.

A well deserved thanks must also be expressed to my co-advisor Søren Fournais, for your kindness and for being consistently supportive throughout my time at QMATH.

I would also like to thank my collaborator Aleksei Kulikov. Working alongside you has been the most rewarding and exciting part of the last months of my PhD. Thank you for suffering with me and for making writing papers an almost enjoyable task.

I want to express my gratitude to Robert Seiringer and his group at ISTA for their hospitality during my stay in the spring of 2025. Thank you for the insightful discussions and collaborations.

I would like to thank all current and former members of QMATH. You have made my journey a remarkable and unforgettable one. In particular, a special thanks goes to Tommaso, Gabriele, Harold, Lukas, and Elias for keeping me sane and making me want to come to the office every day.

I also want to thank my family, for always having reassuring words and for believing in me through this process.

Finally, I want to express my gratitude to my girlfriend, Garbiñe. You are my favorite colleague and the happiness and joy of my life. Thank you for making every day special and exciting.

Copenhagen, 31 March 2026
Martin Dam Larsen

List of publications

This thesis is based on the following completed manuscripts:

[18] S. Fournais, M. D. Larsen, R. Seiringer, and J. P. Solovej. *The Widom Conjecture with optimal regularity assumptions*. 2026.

[33] A. Kulikov and M. D. Larsen. *Sharp estimates for eigenvalues of localization operators with applications to area laws*. 2026. arXiv: 2603.23832.

Contents

Chapter 1

Introduction

This thesis studies asymptotic spectral problems for operators obtained by truncating translation-invariant momentum operators to spatial regions. We refer broadly to such operators as Wiener–Hopf operators, and the principal questions we address are how spectral quantities, such as traces and eigenvalue counting functions, behave as the spatial region becomes large.

Wiener–Hopf operators arise naturally in many contexts, including harmonic analysis, operator theory, pseudodifferential calculus, signal processing, and mathematical physics. As a guiding example, such operators appear when one restricts from an infinite translation-invariant quantum system to a finite spatial subsystem. In this setting, one fundamental spectral problem is the computation of the bipartite entanglement entropy of translation-invariant quasi-free fermionic ground states. This quantity can be phrased directly as the trace of an entropy function of a Wiener–Hopf operator whose underlying momentum symbol is the projection onto the Fermi sea of the one-particle Hamiltonian. The asymptotic behavior of the entanglement entropy is referred to as the (enhanced) area law and has been studied extensively in both the physics and mathematics literature [12, 21, 38, 45, 73]. In the simplest possible setting of a particle-number-preserving Hamiltonian in the ground state, the corresponding momentum symbol of the Wiener-Hopf operator is a sharp indicator function. The main theme of this thesis is that symbols with such jump discontinuities give rise to intricate boundary spectral effects, which govern the sub-volume spectral asymptotics.

We now describe the mathematical problems that we consider more precisely. Let $\Omega \subseteq \mathbb{R}^d$ be a spatial region and let $P : \mathbb{R}^d \to \mathcal{M}_n$ be a bounded momentum symbol taking values in $\mathcal{M}_n$, the space of n-by-n complex matrices. The corresponding Wiener-Hopf operator on $L^2(\mathbb{R}^d;\mathbb{C}^n)$ is given by

$$S(P;\Omega) = 1_\Omega P 1_\Omega, \tag{1.1}$$

where $1_\Omega = 1_\Omega(x)$ is the projection onto the subspace $L^2(\Omega;\mathbb{C}^n)$ and $P = P(p) = P(-i\nabla)$ denotes the corresponding Fourier multiplier. We consistently abuse notation by using 1_Ω, $1_\Omega(x)$, P, and $P(p)$ for the functions, their function values, and the corresponding multipliers. Unpacking the definition, the operator $S(P;\Omega)$ acts on a function $\phi \in L^2(\mathbb{R}^d;\mathbb{C}^n)$ as

$$S(P;\Omega)\phi(x) = 1_\Omega(x)\Big(P\,(1_\Omega\phi)^\wedge\Big)^\vee(x), \tag{1.2}$$

where $\wedge$ and $\vee$ denote the Fourier transform and the inverse Fourier transform. For the first three chapters we use the conventions

$$\mathcal{F}f(p) = \widehat{f}(p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot p} f(x)\,dx,$$
$$\mathcal{F}^{-1}P(x) = \check{P}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot p} P(p)\,dp.$$

A different normalization is introduced in Chapter 4 on time-frequency limiting operators to be consistent with the relevant literature. It is clear that $S(P;\Omega)$ is bounded with $\|S(P;\Omega)\|_\infty \leq \|P\|_\infty$, and that $S(P;\Omega)$ is self-adjoint (non-negative) if P takes values in self-adjoint (non-negative) matrices.

Operators of the form (1.1) are higher dimensional analogues of truncations of one-dimensional Wiener-Hopf operators, which in turn are continuous analogues of Toeplitz matrices. The principal spectral problem, going back to Szegő's limit theorems [62, 63], is to study traces of the form $\operatorname{Tr}\left[f(S(P;\alpha\Omega))\right]$ in the limit $\alpha \to \infty$ for different classes of functions f. Historically most interest was devoted to $f(x) = \log(x)$, which is equivalent to computing the determinant of $S(P;\alpha\Omega)$. For other classical developments we refer to [22, 28, 29, 64]. In its simplest form, the (weak) Szegő limit theorem is a volume law. For a polynomial f satisfying $f(0) = 0$ and P and Ω satisfying weak integrability assumptions, the leading order term scales with the volume:

$$\operatorname{Tr}_{L^2(\mathbb{R}^d;\mathbb{C}^n)}\left[f(S(P;\alpha\Omega))\right] = \left(\frac{\alpha}{2\pi}\right)^d |\Omega| \int_{\mathbb{R}^d} \operatorname{Tr}_{\mathbb{C}^n}\left[f(P(p))\right]dp + o(\alpha^d), \tag{1.3}$$

as $\alpha \to \infty$, where $|\Omega|$ denotes the Lebesgue measure of Ω. An elementary verification using (2.6), for instance, shows that the volume term in (1.3) is exactly given by $\operatorname{Tr}[S(f(P);\alpha\Omega)]$. Since the entirety of this thesis is concerned with the sub-volume behavior, it is convenient to subtract the volume term and introduce

$$S_f(P;\alpha\Omega) = 1_{\alpha\Omega} f(S(P;\alpha\Omega))1_{\alpha\Omega} - S(f(P);\alpha\Omega), \tag{1.4}$$

which, by (1.3), satisfies

$$\operatorname{Tr}_{L^2(\mathbb{R}^d;\mathbb{C}^n)}[S_f(P;\alpha\Omega)] = o(\alpha^d). \tag{1.5}$$

We put the projections $1_{\alpha\Omega}$ around $f(S(P;\alpha\Omega))$ only to avoid assuming $f(0) = 0$. For the special case $f(x) = x^m$, $m \geq 2$, we write $S_m(P;\alpha\Omega)$ instead. Intuitively, the operator $S_f(P;\alpha\Omega)$ should only depend on boundary/higher order effects in P and Ω. It can be verified that $S_f(P;\alpha\Omega)$ can be a trace class operator even when $f(P) \notin L^1$ or $|\Omega| = \infty$, which offers some flexibility.

Further terms in the asymptotic expansion (1.5) critically depend on the regularity of the symbol P, the set Ω, and the function f, and we refer to [71] for a full expansion into powers α^{d-k}, $k \geq 1$, in the fully smooth setting. Focusing only on the leading order term, it is instructive to first consider the smooth, or rather smoother, case without jump discontinuities. This is classically the setting of the strong Szegő limit theorem on Toeplitz operators, and we refer to [30, 67] for sharp results. The corresponding problem for Wiener-Hopf operators was first investigated by Widom [65], later refined and generalized in [68, 71], followed by the works of Roccaforte [51], and more recently Sobolev [60]. In the most abstract setting from [68], the leading order boundary coefficient is given by a reduction to one-dimensional co-tangent sections

$$\operatorname{Tr}_{L^2(\mathbb{R}^d;\mathbb{C}^n)}\left[S_f(P;\alpha\Omega)\right] = \left(\frac{\alpha}{2\pi}\right)^{d-1} \int_{T^*(\partial\Omega)} \operatorname{Tr}_{L^2(\mathbb{R};\mathbb{C}^n)}\left[S_f(P_X;\mathbb{R}_+)\right]dX + o(\alpha^{d-1}). \tag{1.6}$$

We refer to Section 2.1 for precise definitions. In the scalar valued case, the remaining co-tangent traces can be computed explicitly by the remarkable trace formula of Widom [69]:

$$\operatorname{Tr}_{L^2(\mathbb{R};\mathbb{C})}\Big[S_f(Q;\mathbb{R}_+)\Big] = \frac{1}{8\pi^2}\iint_{\mathbb{R}\times\mathbb{R}} \frac{U_f(Q(\xi_1),Q(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2. \tag{1.7}$$

For polynomial f the right hand side of (1.7) diverges for all indicator functions, which carries over to (1.6). More precisely, in the one-dimensional case it was observed by Widom in [69], and earlier in [30, 67] in the Toeplitz setting, that $P \in \dot{H}^{1/2}$ is the minimal smoothness condition to ensure area scaling, which in particular rules out jump discontinuities along co-dimension 1 hypersurfaces. In Chapter 3 we establish (1.6) under the minimal assumptions that Ω is a set of finite perimeter and $P \in \dot{H}^{1/2} \cap L^\infty$. The geometric mechanism behind the boundary term is discussed in Section 2.1, and we also refer to this section for precise statements and for a comparison to relevant literature.

The situation changes dramatically for symbols P with jump discontinuities. In the Toeplitz case, this is classically the setting of symbols with Fisher-Hartwig root-like and jump singularities, whose early development was closely tied to the study of spontaneous magnetization in the two-dimensional Ising model [6]. In contrast to the smooth case, such singularities modify the asymptotic behavior and produce an additional logarithmic factor. It is not our intention to thoroughly review the vast literature on Fisher-Hartwig singularities here; we refer to [2, 17, 66] for early developments and to [9, 11] for the current state of the art. In the Wiener-Hopf setting, jump discontinuities similarly alter the leading order behavior of $\operatorname{Tr}[S_f(P;\alpha\Omega)]$. Here the leading order term is logarithmically enhanced:

$$\operatorname{Tr}\Big[S_f(P;\alpha\Omega)\Big] = \Big(\frac{\alpha}{2\pi}\Big)^{d-1}\log(\alpha)\mathcal{A}_f(\partial\Omega,P^+,P^-) + o(\alpha^{d-1}\log(\alpha)) \tag{1.8}$$

as $\alpha \to \infty$, for a certain boundary coefficient $\mathcal{A}_f(\partial\Omega,P^+,P^-)$ only dependent on the symbol P through its one-sided jump values P^+ and P^- along the jump set. Widom's formula (1.8) for symbols with jump discontinuities was conjectured by Widom in 1982 [70] with partial results by Widom [70, 72] and Landau-Widom [36] in particular for $d = 1$. We also refer to [20, 23] for results on the special case $f(x) = x^2$. Later in 2013, (1.8) was established by Sobolev [54] for regular f and a broad class of regions Ω and symbols P with jumps. See also [3, 55, 58] for later refinements. We establish (1.8) in Chapter 3 under the essentially minimal regularity assumptions that Ω is a set of finite perimeter and P a function of bounded variation. The logarithmic correction arises naturally as a singular limit of the smooth case (1.6), and this mechanism is discussed in Section 2.2.

The jump discontinuous setting is also the one relevant for the entanglement entropy problem mentioned in the beginning of this Introduction. Namely, for a translation invariant, particle-number-preserving quasi-free fermionic ground state, the bipartite entanglement entropy relative to the spatial region Ω is

$$\operatorname{Tr}\Big[h(S(1_\Lambda;\Omega))\Big], \quad h(x) = -x\log(x) - (1-x)\log(1-x), \tag{1.9}$$

where Λ is the Fermi sea of the underlying one-body Hamiltonian. Since $h(0) = h(1) = 0$, the volume term in (1.3) vanishes, and Widom's formula (1.8) therefore predicts an enhanced area law. This connection, first made explicit by Gioev and Klich in 2006 [21], brought renewed interest to the study of Wiener-Hopf operators with discontinuous symbols [3, 24, 38, 39, 41, 54].

We now turn to the simplest, and in many ways most important, class of discontinuous symbols, namely indicator functions. For a measurable set $\Lambda \subseteq \mathbb{R}^d$, the operator $S(1_\Lambda;\Omega) = S_{\Lambda,\Omega}$ is commonly referred to as a time-frequency limiting operator. The study of the spectral properties of such operators was initiated by Landau, Pollak, and Slepian [34, 35, 53], and has since attracted considerable interest both from the mathematical and signal processing communities. In the present context, time-frequency limiting operators provide the basic model for discontinuous Wiener-Hopf symbols. Despite the elementary form of the symbol, these operators exhibit a remarkably rich yet comparably more transparent spectral structure. The eigenvalues of $S(1_\Lambda;\alpha\Omega)$, which agree with those of $S(1_\Omega;\alpha\Lambda)$, are expected to exhibit a sharp transition: the first $\approx |\Omega||\Lambda|\alpha^d$ eigenvalues are close to 1, followed by $\asymp \alpha^{d-1}\log(\alpha)$ intermediate eigenvalues away from 0 and 1, and then a tail of eigenvalues decaying fast. We refer to the intermediate part of the spectrum as the plunge region, the area regime, or the boundary regime. This picture is particularly clear in dimension 1, where the exact limiting behavior and precise uniform estimates of the eigenvalues are known, see [5, 19, 31, 32, 36].

From the point of view of trace asymptotics, it is precisely the eigenvalues in the boundary regime that contribute to the $\alpha^{d-1}\log(\alpha)$ term in Widom's formula (1.8). The size of the plunge region is quantified by the eigenvalue counting function

$$\Lambda_\varepsilon(\alpha\Omega,\Lambda) = |\{n \in \mathbb{N} \mid \varepsilon < \lambda_n(\alpha\Omega;\Lambda) < 1-\varepsilon\}| = \operatorname{Tr}[S_{f_\varepsilon}(1_\Lambda,\alpha\Omega)], \tag{1.10}$$

where $f_\varepsilon(x) = 1_{(\varepsilon,1-\varepsilon)}(x)$ for $0 < \varepsilon < 1/2$. For fixed ε, Widom's formula (1.8) predicts that, as $\alpha \to \infty$,

$$\Lambda_\varepsilon(\alpha\Omega,\Lambda) = C_{\Omega,\Lambda}\alpha^{d-1}\log(\alpha)\log\left(\frac{1-\varepsilon}{\varepsilon}\right) + o(\alpha^{d-1}\log(\alpha)) \tag{1.11}$$

for a constant $C_{\Omega,\Lambda}$ only dependent on Ω and Λ. A quantitative formulation of the spectral picture described above is that the leading order behavior should hold uniformly for small ε, namely

$$\Lambda_\varepsilon(\alpha\Omega,\Lambda) \lesssim \alpha^{d-1}\log(\alpha)\log\left(\frac{1}{\varepsilon}\right). \tag{1.12}$$

In dimension one this is indeed the case as recently shown by Karnik, Romberg, and Davenport [31], while state of the art in the higher-dimensional case incurs additional $\log(\alpha)$ losses [26, 27, 44]. Sharp bounds on the plunge region of the type (1.12) can be used to extend Widom's formula (1.8) from polynomial f, where it is most accessible, to rough spectral functions with singular behavior at the end points 0 and 1. The need for such extension arguments is apparent from the entanglement entropy computation (1.9), since entropy functions are singular at the endpoints. A substantial part of the recent literature on Wiener-Hopf operators have concerned precisely this extension problem, and the corresponding tools based on uniform Schatten quasi-norm estimates, developed in [56–58], cover spectral functions with Hölder-type singularities. In Chapter 4 we establish sharp uniform estimates on the plunge region of the type (1.12) under suitable geometric assumptions. Based on this, we deduce essentially necessary and sufficient conditions on f for Widom's formula to hold. The underlying extension mechanism is discussed in Section 2.3.

Structure

The rest of the thesis is organized as follows.

Chapter 2 gives precise statements of the main results of the thesis and a brief overview of the main ideas behind their proofs.

Chapter 3 contains the manuscript [18] titled *"The Widom Conjecture with Optimal Regularity Assumptions"*. Here we establish Widom's formula (1.8) for regular f under the assumptions that P is a function of bounded variation and Ω a set of finite perimeter. We argue that these conditions are optimal in the time-frequency limiting setting $P = 1_\Lambda$. As a key step of the proof, we study the transition between smooth and non-smooth matrix valued symbols, which is of interest in the study of the entanglement entropy.

Chapter 4 contains the arXiv preprint [33] titled *"Sharp Estimates for Eigenvalues of Localization Operators with Applications to Area Laws"*. Here we study time-frequency limiting operators and we establish sharp uniform bounds for the number of eigenvalues in the plunge region. As an application and building on the work from Chapter 3, we establish essentially necessary and sufficient conditions on f for the trace formula (1.8) to hold when one of the involved sets is a finite union of parallelepipeds.

Chapter 2

Preliminaries and overview

In this chapter we precisely state the main results of this thesis and we discuss some of the mechanisms behind their proofs. We first consider the smooth case, where the leading order boundary term can be understood geometrically through a reduction to co-tangent sections. We then turn to discontinuous symbols, where the logarithmic enhancement arises as a singular limit of the smooth setting. Finally, we discuss the plunge region for time-frequency limiting operators and its role in extensions of Widom's formula to rough spectral functions.

2.1 Trace asymptotics: smooth case

We begin with the smooth case of symbols without jump discontinuities. As mentioned in the Introduction, in this setting the leading order sub-volume term has area scaling, and the boundary coefficient is a reduction to traces of co-tangent sections. Despite smooth symbols not being the main emphasis of this thesis, the smooth case serves as a key component in the proof of Widom's formula (1.8) for discontinuous symbols presented in Chapter 3. In fact, in our work, the boundary coefficient arises as a singular limit of the smooth case, as will be discussed in Section 2.2. The key takeaway is that the smooth case is significantly more accessible, and the geometric mechanism behind the boundary term is more transparent. It is our intention here to discuss this. We also point out that exactly the same geometric mechanism crops up in related trace computations in different models. This is, for instance, the case for anti-Wick quantized domains [46] and for operators arising from spatial restrictions of (not necessarily translation-invariant) free fermionic states [40, 48, 49].

Throughout this section we restrict our attention to polynomial spectral functions, or more precisely natural powers of the form $f(x) = x^m$, $m \geq 2$. The extension to rougher spectral functions requires, in addition to the polynomial trace asymptotics, separate uniform estimates on the singular values of $1_{\alpha\Omega} P 1_{\alpha\Omega^c}$, which will be discussed in Section 2.3. Accordingly, for a bounded symbol $P : \mathbb{R}^d \to \mathcal{M}_n$ and a measurable set $\Omega \subseteq \mathbb{R}^d$, we consider

$$S_m(P;\alpha\Omega) = \left(1_{\alpha\Omega} P 1_{\alpha\Omega}\right)^m - 1_{\alpha\Omega} P^m 1_{\alpha\Omega} \tag{2.1}$$

for $m \geq 2$ and $\alpha \geq 1$.

Precise statements and definitions

To state our main results in full generality we need to introduce some notation and a few definitions. We write $\dot{H}^{1/2}(\mathbb{R}^d;\mathcal{M}_n)$ for the homogeneous fractional Sobolev space consisting of all $P \in L^1_{loc}(\mathbb{R}^d;\mathcal{M}_n)$ such that

$$\|P\|^2_{\dot{H}^{1/2}} = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|P(p)-P(q)|^2}{|p-q|^{d+1}}\,dp\,dq < \infty.$$

If in addition $P \in L^2$, then $\|P\|^2_{\dot{H}^{1/2}} = C_d \int_{\mathbb{R}^d} |\check{P}(x)|^2|x|\,dx$, see [42, Theorem 7.12] or [10].

We also recall that a set $\Omega \subseteq \mathbb{R}^d$ is said to have finite perimeter if the function 1_Ω has bounded variation, or equivalently that

$$\int_{\mathbb{R}^d} |1_\Omega(x+h) - 1_\Omega(x)|\,dx \le C|h|, \quad h \in \mathbb{R}^d. \tag{2.2}$$

By Lemma A.7 of Chapter 3 one may take $C = \mathrm{Per}(\Omega)$, and conversely, the least such C is comparable to $\mathrm{Per}(\Omega)$. We write $\mathcal{F}\Omega$ for the reduced boundary and ν_Ω for the measure theoretic interior normal. For precise statements we refer to Chapter 3 or the standard textbook references [1, 13, 14, 43]. We note for orientation that if $\Omega \subseteq \mathbb{R}^d$ has compact Lipschitz boundary, then Ω is a set of finite perimeter. In this case $\mathcal{F}\Omega$ agrees with $\partial\Omega$ up to $\mathcal{H}^{d-1}$-null sets, ν_Ω coincides almost everywhere with the classical interior unit normal, and $\mathrm{Per}(\Omega) = \mathcal{H}^{d-1}(\partial\Omega)$.

Let Ω be a set of finite perimeter. For $x \in \mathcal{F}\Omega$, let $\Pi_{\nu_\Omega(x)} = \{\nu_\Omega(x)\}^\perp$ denote the corresponding tangent plane. We write

$$T^*(\mathcal{F}\Omega) = \{X = (x,p) \mid x \in \mathcal{F}\Omega,\, p \in \Pi_{\nu_\Omega(x)}\},$$

equipped with the product measure

$$\int_{T^*(\mathcal{F}\Omega)} F(X)\,dX = \int_{\mathcal{F}\Omega}\int_{\Pi_{\nu_\Omega(x)}} F(x,p)\,d\mathcal{H}^{d-1}(p)\,d\mathcal{H}^{d-1}(x).$$

Finally, for $X = (x,p) \in T^*(\mathcal{F}\Omega)$ and a function P on $\mathbb{R}^d$, we define the corresponding co-tangent section

$$P_X(\xi) = P(p + \xi\nu_\Omega(x)), \quad \xi \in \mathbb{R}.$$

Our main results from Chapter 3 on the smooth case are the following.

Theorem 2.1. *Let $P \in \dot{H}^{1/2}(\mathbb{R}^d;\mathcal{M}_n) \cap L^\infty$ and let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter. Then, for all $m \ge 2$,*

$$\mathrm{Tr}\Big[S_m(P;\alpha\Omega)\Big] = \Big(\frac{\alpha}{2\pi}\Big)^{d-1}\int_{T^*(\mathcal{F}\Omega)} \mathrm{Tr}\Big[S_m(P_X;\mathbb{R}_+)\Big]\,dX + o(\alpha^{d-1}) \tag{2.3}$$

as $\alpha \to \infty$.

Theorem 2.2. *Let $P \in \dot{H}^{1/2}(\mathbb{R};\mathbb{C})$. Then, for all $m \ge 2$,*

$$\mathrm{Tr}\Big[S_m(P;\mathbb{R}_+)\Big] = \frac{1}{8\pi^2}\iint_{\mathbb{R}\times\mathbb{R}} \frac{U_m(P(\xi_1),P(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2. \tag{2.4}$$

Here $U_m : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is the symmetric functional

$$U_m(A,B) = -\frac{1}{2}\sum_{k=1}^{m-1}\Big(\frac{1}{k}+\frac{1}{m-k}\Big)\Big(A^k - B^k\Big)\Big(A^{m-k}-B^{m-k}\Big). \tag{2.5}$$

Widom [69] proved the one-dimensional formula (2.4) under the sharp $\dot{H}^{1/2}$ assumption. Our main contribution in this setting is to provide a different, more elementary proof, better aligned with the higher-dimensional mechanism developed below. Our method, which relies on ideas from the classical work [28] of Kac, is presented in Section 5 from Chapter 3.

The more substantial result is Theorem 2.1, which refines earlier work of Widom [65, 68, 71] and Roccaforte [51] by extending the area scaling formula to the finite perimeter setting and by removing an artificial assumption that $\check{P} \in L^1$. We also mention the recent work of Sobolev [60], who, using the one-dimensional trace formula (2.4), established area scaling for scalar-valued smooth symbols under decay assumptions and for domains with piecewise C^1 boundary, for a broader class of spectral functions f. The rest of this section is devoted to explaining the geometric mechanism behind the leading order term from (2.3).

Kernel expansions and the boundary term

We will now explain the mechanism behind the boundary term in Theorem 2.1. For the sake of exposition we restrict our attention to the maximally regular case $P \in C_c^\infty(\mathbb{R}^d;\mathcal{M}_n)$. This is not a real restriction, since the sharp $\dot{H}^{1/2}$ assumption follows from abstract approximation arguments from this case; we refer to Chapter 3 for the details.

The most natural strategy to compute the trace of a given operator is to find its kernel and integrate along the diagonal. This is precisely possible for the operator (2.1). That the kernel is directly expressible is a key feature unique to natural powers $f(x) = x^m$, and this insight is directly or indirectly central to all work on the smooth case. For a given set $\Omega \subseteq \mathbb{R}^d$, the operator $1_{\alpha\Omega}P1_{\alpha\Omega}$ has kernel

$$K(x,y) = (2\pi)^{-d/2}1_{\alpha\Omega}(x)\check{P}(x-y)1_{\alpha\Omega}(y). \tag{2.6}$$

Formally following the composition rule for kernels, the operator $\left(1_{\alpha\Omega}P1_{\alpha\Omega}\right)^m$ has kernel

$$\begin{aligned} K_m(x,y) &= \int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d} K(x,x_1)K(x_1,x_2)\dots K(x_{m-2},x_{m-1})K(x_{m-1},y)\,dx_1\dots\,dx_{m-1} \\ &= (2\pi)^{-\frac{dm}{2}}\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d} \check{P}(x-x_1)\dots\check{P}(x_{m-1}-y) \\ &\qquad \times 1_{\alpha\Omega}(x)1_{\alpha\Omega}(y)1_{\alpha\Omega}(x_1)\dots 1_{\alpha\Omega}(x_{m-1})\,dx_1\dots dx_{m-1}, \end{aligned} \tag{2.7}$$

and similarly, by the elementary identity $(P^m)^\vee(x-y) = (2\pi)^{-\frac{d(m-1)}{2}}\left(\check{P}\right)^{*m}(x-y)$, the operator $1_{\alpha\Omega}P^m1_{\alpha\Omega}$ has kernel

$$K_m'(x,y) = (2\pi)^{-\frac{dm}{2}}1_{\alpha\Omega}(x)1_{\alpha\Omega}(y)\int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d}\check{P}(x-x_1)\dots\check{P}(x_{m-1}-y)\,dx_1\dots\,dx_{m-1}. \tag{2.8}$$

We here point out that to effectively manipulate the expansions above, the integrals (2.7), (2.8) have to converge absolutely for all $m \geq 2$, and the only natural general condition to assure this is $\check{P} \in L^1$, which renders this approach not applicable for discontinuous symbols.

Expressing the trace of $S_m(P;\alpha\Omega)$ as the integral of the kernel along the diagonal, which can be justified e.g. using the result from [7], we obtain after a change of variables and an application of Fubini's theorem that

$$\begin{aligned}&\mathrm{Tr}\Big[S_m(P;\alpha\Omega)\Big]\\&\quad=-(2\pi)^{-\frac{dm}{2}}\int\dots\int\mathrm{Tr}\Big[\check{P}(x_1)\dots\check{P}(x_{m-1})\check{P}(-x_1-\dots-x_{m-1})\Big]\\&\qquad\times\int 1_{\alpha\Omega}(x)\Big(1-1_{\alpha\Omega}(x-x_1)\dots 1_{\alpha\Omega}(x-x_1\dots-x_{m-1})\Big)\,dx\,dx_1\dots dx_{m-1}.\end{aligned}\tag{2.9}$$

In particular, the α dependence of the trace is determined entirely by the volume term

$$\begin{aligned}F_{\alpha\Omega}(\underline{x})&=\int 1_{\alpha\Omega}(x)\Big(1-1_{\Omega}(x-x_1)\dots 1_{\Omega}(x-x_1\dots-x_{m-1})\Big)\,dx\\&=\Big|\alpha\Omega\setminus\Big(\alpha\Omega\cap(\alpha\Omega+x_1)\dots\cap(\alpha\Omega+x_1+\dots+x_{m-1})\Big)\Big|,\end{aligned}\tag{2.10}$$

where we wrote $\underline{x}=(x_1,\dots,x_1+\dots+x_{m-1})$. We point out that to justify the application of Fubini's theorem in (2.9) it is necessary to minimally control $F_{\alpha\Omega}$. For instance, it is sufficient to assume that either Ω or Ω^c has finite measure or that $|\Omega\cap(\Omega^c+y)|\leq C|y|$. The latter is guaranteed for sets of finite perimeter by the characterization (2.2). Since $F_{\alpha\Omega}(\underline{x})=\alpha^d F_{\Omega}(\underline{x}/\alpha)$, the same characterization also immediately implies the uniform bound

$$\Big|\mathrm{Tr}\Big[S_m(P;\alpha\Omega)\Big]\Big|\leq C\alpha^{d-1}\tag{2.11}$$

if Ω is a set of finite perimeter. Isolating the exact leading order coefficient therefore boils down to studying $F_{\Omega}(\underline{x})$ in the limit $\underline{x}\to 0$. As it turns out,

$$F_{\Omega}(\underline{x})=G_{\Omega}(\underline{x})+o(|\underline{x}|),\tag{2.12}$$

where G_{Ω} is the homogeneous surface integral given by

$$G_{\Omega}(\underline{x})=\int_{\mathcal{F}\Omega}\max_{1\leq k\leq m-1}\{\nu_{\Omega}(x)\cdot(x_1+\dots+x_k)\}\,d\mathcal{H}^{d-1}(x).\tag{2.13}$$

Before discussing (2.12) in greater detail, we first point out that the precise formulation of the boundary coefficient (2.3) now follows easily. Indeed, from (2.9), (2.12), and elementary manipulations of the Fourier transform,

$$\begin{aligned}&\mathrm{Tr}\Big[S_m(P;\alpha\Omega)\Big]\\&\quad=-(2\pi)^{-\frac{dm}{2}}\int\dots\int\mathrm{Tr}\Big[\check{P}(x_1)\dots\check{P}(x_{m-1})\check{P}(-x_1-\dots-x_{m-1})\Big]G_{\alpha\Omega}(\underline{x})\,dx_1\dots dx_{m-1}\\&\qquad+o(\alpha^{d-1})\\&\quad=-\Big(\frac{\alpha}{2\pi}\Big)^{d-1}(2\pi)^{-\frac{m}{2}}\int_{T^*(\mathcal{F}\Omega)}\int\dots\int\mathrm{Tr}\Big[\widecheck{P_X}(t_1)\dots\widecheck{P_X}(t_{m-1})\widecheck{P_X}(-t_1-\dots-t_{m-1})\Big]\\&\qquad\times\max_{1\leq k\leq m-1}\{0,t_1+\dots+t_k\}\,dt_1\dots dt_{m-1}+o(\alpha^{d-1}).\end{aligned}$$

The exact identity

$$F_{\mathbb{R}_+}(\underline{t})=\max_{1\leq k\leq m-1}\{0,t_1+\dots+t_k\}\tag{2.14}$$

gives the claim.

The relation (2.12) was first established by Widom [65] in 1960 for the case where Ω is either a polytope or a convex set. The polytope case has also been studied by

Pfirsch [50] with a different technique. Later in 1980, Widom [68] claimed that (2.12) holds for Ω with C^1 boundary. Shortly thereafter, Roccaforte [51] established (2.12) for Ω with C^3 boundary, where he in fact also found the next term in the expansion. We also refer to the more recent work by Roccaforte [52] for a full expansion in the smooth case, which in turn formally leads to a full expansion of the trace (2.3). All the aforementioned references apply (2.12) in exactly the same way as presented above.

While the proofs of (2.12) from [51, 65] are technical, the basic mechanism behind the expansion is a simple local tangent plane approximation: if $x \in \mathcal{F}\Omega$ and y is small, then the membership $x + y \in \Omega$ is to leading order in $|y|$ determined by $y \cdot \nu_\Omega(x) \geq 0$. This is exact if Ω is a half-plane, which we used in (2.14). If $\mathcal{F}\Omega = \partial\Omega$ is parametrized by C^1 (or Lipschitz) graphs and $|\underline{x}|$ is small, this tangent-plane approximation can be carried out directly in small suitably defined coordinate patches. Concretely, if Ω is parametrized by $\{x_d > \psi(x')\}$ in a box $2Q$ around $\partial\Omega$, $x \in Q$, and $y_1, \dots, y_{m-1}$ are small, then $x \in \Omega \cap (\Omega + y_1) \cap \dots \cap (\Omega + y_{m-1})$ if and only if

$$x_d > \psi(x'), \quad \frac{x_d - \psi(x')}{\sqrt{1 + |\nabla\psi(x')|^2}} - \nu_\Omega(\tilde{x}) \cdot y_i \geq \frac{R_{x'}(x' - y_i')}{\sqrt{1 + |\nabla\psi(x')|^2}}, \quad i = 1, \dots, m-1,$$

where $\tilde{x} = (x', \psi(x'))$ is the 'vertical projection' to $\partial\Omega$, and $R_{x'}(z') = \psi(z') - \psi(x') - \nabla\psi(x')(z' - x')$ is the first order Taylor error. The tangent plane approximation lies in setting $R_{x'} = 0$, which induces local errors of size $|R_{x'}(x' - y_i')| = o(|y_i'|)$.

Summing up local patches of this type is crude and requires some care. Conceptually the same technique, wrapped in pseudodifferential language, is used by Sobolev in [60], although without explicitly deducing the volume term F_Ω. A more refined technique is used by Roccaforte [51], where the coordinate patches are obtained by lifting a partition of unity on $\partial\Omega$ to a tubular neighborhood by the map $\partial\Omega \times (-\varepsilon, \varepsilon) \ni (x, t) \to x + t\nu_\Omega(x)$, which is a C^1 diffeomorphism for small ε if Ω has compact C^2 boundary, which allows sharper bounds.

We turn to the finite perimeter setting. Here the techniques described above, which all rely on local parametrization of the boundary by graphs, are not applicable. Instead, a fundamentally different, coordinate free, purely measure theoretic approach is necessary. Our starting point is the observation that (2.12) is equivalent to the statement that the (classical) derivative of $F_\Omega(\underline{x}) - G_\Omega(\underline{x})$ is equal to zero at zero. By Morrey's inequality [13, Theorem 4.10], this would follow by showing that $F_\Omega - G_\Omega$ is Lipschitz and that zero is a Lebesgue point of $\nabla(F_\Omega - G_\Omega)$. This formulation is already much more approachable. Indeed, the weak derivative of F_Ω can be computed using the Gauss-Green type formula $D1_\Omega = \nu_\Omega d\mathcal{H}^{d-1} \llcorner_{\mathcal{F}\Omega}$ for finite perimeter sets, and the local averaging in the Lebesgue point formulation is exactly needed for blow-up arguments on $\mathcal{F}\Omega$. To see these points, let us consider the simplest case $m = 2$. Here $F_\Omega(y) = 1_\Omega * 1_{-\Omega^c}(y)$, so, formally as distributions,

$$\nabla F_\Omega(y) = 1_\Omega * D1_{-\Omega^c}(y) = \int_{\mathcal{F}\Omega} \nu_\Omega(x) 1_\Omega(x + y)\, d\mathcal{H}^{d-1}(x).$$

Similarly, a direct computation shows that

$$\nabla G_\Omega(y) = \int_{\mathcal{F}\Omega} \nu_\Omega(x) 1_{\{\nu_\Omega(x) \cdot y > 0\}}\, d\mathcal{H}^{d-1}(x).$$

For $x \in \mathcal{F}\Omega$ introduce the tangent-plane defect set

$$D_x = \{z \in \mathbb{R}^d \mid z \in \Omega,\ \nu_\Omega(x) \cdot (z - x) \leq 0\} \cup \{z \in \mathbb{R}^d \mid x \in \Omega^c,\ \nu_\Omega(x) \cdot (z - x) \geq 0\}.$$

Note that D_x exactly consists of all points z such that the intuitive leading order tangent plane approximation explained above fails. By blow-up of the reduced boundary [13, Theorem 5.14] and the dominated convergence theorem,

$$\begin{aligned}\fint_{B_r(0)} |\nabla F_\Omega(y) - \nabla G_\Omega(y)|\,dy &= \fint_{B_r(0)} \left| \int_{\mathcal{F}\Omega} \nu_\Omega(x)\Big(1_\Omega(x+y) - 1_{\{\nu_\Omega(x)\cdot y>0\}}(x)\Big)\,d\mathcal{H}^{d-1}(x)\right| dy \\ &\le \int_{\mathcal{F}\Omega}\fint_{B_r(0)} \Big|1_{\Omega-x}(y) - 1_{\{\nu_\Omega(x)\cdot y>0\}}(y)\Big|\,dy\,d\mathcal{H}^{d-1}(x) \\ &\le \int_{\mathcal{F}\Omega} \frac{|D_x\cap B_r(x)|}{|B_r(x)|}\,d\mathcal{H}^{d-1}(x) \to 0\end{aligned}$$

as $r \to 0^+$, which establishes (2.12) for $m = 2$. We refer to Lemma C.6 of Chapter 3 for a proof in the general case, which is mainly more difficult only in notation.

2.2 Trace asymptotics: discontinuous symbols

We now turn to the case of discontinuous symbols, which is the main setting of this thesis. As discussed in the Introduction, jump discontinuities break the area scaling from Section 2.1. Instead, according to Widom's formula (1.8), the leading order term is logarithmically enhanced. It is our intention in this section to explain how this logarithm arises as a singular limit of the smooth case, which is central to the proof of Widom's formula presented in Chapter 3. More precisely, we consider a family (P_δ) of smooth symbols depending on a smoothness parameter $\delta > 0$ which, in the limit $\delta \to 0^+$, develops a jump discontinuity along a prescribed hypersurface Γ. We refer to such (P_δ) as multiscale symbols, and their limits provide a natural initial class for which Widom's formula (1.8) can be established. For each fixed $\delta > 0$, the symbol P_δ admits area scaling by Theorem 2.1, and the key problem is to uniformly and precisely control the δ dependence in the $o(\alpha^{d-1})$ error term from (2.3). This result, which we discuss below, also has independent interest in the study of the entanglement entropy of translation invariant free fermions. For instance, the parameter δ can represent the temperature of a thermal state, or the pairing gap in a superfluid quasi-free state. The latter is inherently a matrix valued problem, and we carry out the analysis in this setting.

As in Section 2.1, we here restrict our attention to spectral functions f given by natural powers. We refer to Section 2.3 for extensions.

Multiscale symbols and precise statements

We first fix the notation for multiscale symbols and state the main results from Chapter 3. Let $\Gamma \subseteq \mathbb{R}^d$ be a bounded set. For $0 < \delta \le 1/2$, introduce the scale function

$$\ell_\delta(p) = \max\Big\{\delta, \tfrac{1}{2}\min\{d(p,\Gamma), 1\}\Big\},$$

where $d(p,\Gamma)$ is the distance from p to Γ. Since ℓ_δ is Lipschitz with parameter 1/2, it is slowly varying in the sense that

$$|p-u| \le \ell_\delta(u) \Rightarrow \tfrac{1}{2}\ell_\delta(u) \le \ell_\delta(p) \le \tfrac{3}{2}\ell_\delta(u). \tag{2.15}$$

We say that a family (P_δ), $0 < \delta \le 1$, satisfies a $C^\kappa(\mathbb{R}^d;\mathcal{M}_n)$ multiscale estimate on Γ if $(P_\delta) \subseteq C^\kappa(\mathbb{R}^d;\mathcal{M}_n)$ and there is a bounded function $F : \mathbb{R}^d \to [0,\infty)$, not dependent on

δ, such that

$$G(p) = \sup_{|p-q| \le \ell_\delta(p)} F(q) \in L^1, \tag{2.16}$$

and

$$|D^\beta P_\delta(p)| \le F(p)\ell_\delta(p)^{-|\beta|} \quad p \in \mathbb{R}^d, \quad |\beta| \le \kappa. \tag{2.17}$$

The simplest example, which is nonetheless sufficient for the analysis in Chapter 3, is that of smoothed out simple functions of the form

$$P_\delta = \sum_{k=1}^{N} A_k 1_{\Lambda_k} * \phi_\delta,$$

where $(A_k) \subseteq \mathcal{M}_n$ and (ϕ_δ) is a family of mollifiers. If the sets (Λ_k) are bounded, then P_δ satisfies a $C^\kappa(R^d;\mathcal{M}_n)$ multiscale estimate on $\Gamma = \cup_{k=1}^N \partial\Lambda_k$ for all $\kappa \in \mathbb{N}$.

We write $\dot{BV}(\mathbb{R}^d;\mathcal{M}_n)$ for the space of all functions in $L^1_{loc}(\mathbb{R}^d;\mathcal{M}_n)$ with bounded variation. For $P \in \dot{BV}(\mathbb{R}^d;\mathcal{M}_n)$, we denote by J_P the jump set, ν_P a choice of orientation on J_P, and by P^+ and P^- the corresponding one-sided jump values. For precise definitions we refer to Chapter 3 or [1, 13, 14, 43]. We note for orientation that if $P = \sigma_1 1_\Lambda + \sigma_2 1_{\Lambda^c}$ for a set Λ with Lipschitz boundary and Lipschitz functions σ_1, σ_2 satisfying

$$\nabla\sigma_1 \in L^1(\Lambda), \quad \nabla\sigma_2 \in L^1(\Lambda^c), \quad \sigma_1 - \sigma_2 \in L^1(\partial\Lambda),$$

then $P \in \dot{BV}$ and $J_P \subseteq \partial\Lambda$. If we choose ν_P to agree with ν_Λ, then also $P^+ = \sigma_1$ and $P^- = \sigma_2$.

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter and suppose that $\Gamma \subseteq \mathbb{R}^d$ is bounded and contained in a finite union of Lipschitz graphs. Assume that (P_δ) satisfies a $C^1(\mathbb{R}^d;\mathcal{M}_n)$ multiscale estimate on Γ. Then, for all $m \ge 2$,*

$$\operatorname{Tr}\left[S_m(P_\delta;\alpha\Omega)\right] = \left(\frac{\alpha}{2\pi}\right)^{d-1} \int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S_m((P_\delta)_X;\mathbb{R}_+)\right] dX + o(\alpha^{d-1}\log(\delta^{-1})) \tag{2.18}$$

as $\alpha \to \infty$ uniformly for $\alpha\delta \ge 1$.

Theorem 2.4. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter and $P \in \dot{BV}(\mathbb{R}^d;\mathbb{C}) \cap L^\infty$. If $d = 1$ assume in addition that $P \in L^2(\mathbb{R}^d;\mathbb{C}) + \mathbb{C}$ and that either Ω or Ω^c has finite measure. Then, for all $m \ge 2$,*

$$\begin{aligned}\operatorname{Tr}&\left[S_m(P;\alpha\Omega)\right] \\ &= \frac{\alpha^{d-1}\log(\alpha)}{(2\pi)^{d+1}} \int_{\mathcal{F}\Omega} \int_{J_P} U_m(P^+(p),P^-(p))|\nu_\Omega(x)\cdot\nu_P(p)|\, d\mathcal{H}^{d-1}(p)\, d\mathcal{H}^{d-1}(x) \\ &\quad + o(\alpha^{d-1}\log(\alpha))\end{aligned} \tag{2.19}$$

as $\alpha \to \infty$.

Theorem 2.3 provides a sharp uniform formulation of Theorem 2.1 for multiscale symbols, which is the heart of the proof of Theorem 2.4. More precisely, if the symbols (P_δ) are scalar valued, then the remaining co-tangent integral from (2.18) can be rewritten using Widom's half-line trace formula (2.4). Assuming that $\|P_\delta - P\|_1 = O(\delta)$ for a limiting function $P \in \dot{BV}$ with jump discontinuities, an abstract and rather involved computation in $\dot{BV}$ shows that

$$\begin{aligned}&\int_{T^*(\mathcal{F}\Omega)} \mathrm{Tr}\Big[S_m((P_\delta)_X;\mathbb{R}_+)\Big]\,dX \\ &\quad= \frac{\log(\delta^{-1})}{4\pi^2}\int_{\mathcal{F}\Omega}\int_{J_P} U_m(P^+(p),P^-(p))|\nu_\Omega(x)\cdot\nu_P(p)|\,d\mathcal{H}^{d-1}(p)\,d\mathcal{H}^{d-1}(x) \\ &\quad\quad+ o(\log(\delta^{-1}))\end{aligned} \tag{2.20}$$

as $\delta \to 0^+$, which precisely explains the logarithm and yields Widom's formula (2.19) for the symbol P. The full strength of Theorem 2.4 is then obtained by a series of technical approximation arguments. We refer to Chapter 3 for the details.

Let us spend a few words comparing our results to the ones present in the literature. Sobolev also studied the transition between smooth and singular symbols in [59] in the indicator case $P = 1_\Lambda$. His assumptions on (P_δ) and the underlying sets are stronger than ours, but his results cover rougher spectral functions f. In particular, for polynomial f, the asymptotics obtained there are recovered by the present approach. We also note that Sobolev uses his results on Widom's formula from [55] as an input, whereas in our setting, Theorem 2.3 is a key component of the proof of Theorem 2.4.

Our formulation of Widom's formula from Theorem 2.4 is a substantial generalization both of Widom's original conjecture from [70] and Sobolev's results from [54, 55]. In particular, our assumptions on Ω and P are symmetric, and we allow symbols with integrable jumps along arbitrary countably $\mathcal{H}^{d-1}$ rectifiable sets. These sets are possibly unbounded and with infinite measure, and outside the jump set we only require very weak $\dot{W}^{1,1}$-type continuity. In fact, we show in Chapter 3 that our assumptions are sharp in the class of indicator function symbols.

In comparison, Sobolev's results cover bounded sets Ω with piece-wise C^1 boundary and discontinuous symbols of the form $P(p) = \sigma(p)1_\Lambda(p)$, where $\sigma \in W^{d+2,\infty}$ and $\Lambda \subseteq \mathbb{R}^d$ is bounded with piece-wise C^3 boundary. Let us also mention the refinements of Bollmann and Müller [3], who establish a version of Widom's formula for matrix valued symbols with jumps along the boundary of a piece-wise C^3 region. Their results do not address more complicated jump structures, and it is not clear how to extend them to such situations. Returning to our argument, the only obstruction to treating the matrix valued case is the lack of a version of Widom's half-line trace formula (2.4) valid in this setting, which is needed to bridge the gap between Theorem 2.3 and Theorem 2.4 as described above.

Widom's original conjecture and Sobolev's results are formulated in a more general pseudodifferential setting, including discontinuous symbols of the form $\sigma(x,p)1_\Lambda(p)$ for smooth σ. In this thesis we restrict attention to pure momentum symbols, which is the setting relevant for the motivations discussed in Chapter 1.

Multiscale localization

The rest of this section will be devoted to explaining key elements of the proof of Theorem 2.3, which relies on a technical multiscale localization technique. Related multiscale assumptions and localization procedures appear elsewhere in the literature, see for instance [38, 41, 54, 57, 59]. Our aim here is not to compare these techniques in technical detail, but rather to explain and motivate the multiscale localization from

Chapter 3 that underlies the proof of Theorem 2.3. For the sake of exposition we restrict our attention to a family (P_δ) satisfying a $C^\kappa(\mathbb{R}^d;\mathcal{M}_n)$ multiscale estimate on Γ for κ much larger than d. This is not a real restriction, since the C^1 condition from Theorem 2.3 follows from abstract approximation arguments from this case; we refer to Chapter 3 for the details.

Our starting point is the observation that the method laid out in Section 2.1 on smooth symbols can not be applied directly. Indeed, pursuing this approach directly, we inevitably need to estimate the error term

$$\begin{aligned}\Big|\mathrm{Tr}\Big[S_m(P_\delta;\alpha\Omega)\Big]-\Big(\frac{\alpha}{2\pi}\Big)^{d-1}\int_{T^*(\mathcal{F}\Omega)}\mathrm{Tr}\Big[S_m((P_\delta)_X;\mathbb{R}_+)\Big]\,dX\Big|\\ =\Big|\int\dots\int\mathrm{Tr}\Big[\check{P}_\delta(x_1)\dots\check{P}_\delta(x_{m-1})\check{P}_\delta(-x_1-\dots-x_{m-1})\Big](F_{\alpha\Omega}(\underline{x})-G_{\alpha\Omega}(\underline{x}))\,dx_1\dots dx_{m-1}\Big|.\end{aligned} \tag{2.21}$$

The only feasible way to estimate this term directly is to apply the triangle inequality and bound, with $\varepsilon>0$ obtained from (2.12),

$$\begin{aligned}\int\dots\int\Big|\mathrm{Tr}\Big[\check{P}_\delta(x_1)\dots\check{P}_\delta(x_{m-1})\check{P}_\delta(-x_1-\dots-x_{m-1})\Big]\Big|\Big(\varepsilon\alpha^{d-1}|\underline{x}|+C_\varepsilon\alpha^{d-2}|\underline{x}|^2\Big)\,dx_1\dots dx_{m-1}\\ \le C_m\|\check{P}_\delta\|_2\|\check{P}_\delta\|_1^{m-2}\Big(\varepsilon\alpha^{d-1}\|\check{P}_\delta(x)|x|\|_2+C_\varepsilon\alpha^{d-2}\|\check{P}_\delta(x)|x|^2\|_2\Big).\end{aligned} \tag{2.22}$$

The problematic factor here is $\|\check{P}_\delta\|_1$, which, generically, scales like $\delta^{-\frac{d-1}{2}}$ whenever $d\ge 2$. Except in the trivial case $m=2$, it seems that no estimate can hope to overcome this obstruction.

The key to control the δ-dependence correctly is to use the multiscale assumption on (P_δ) to localize to scales defined by ℓ_δ, and then to apply the method laid out in Section 2.1 on the localized symbols. Informally, one freezes the symbol on its local scale and then integrates the resulting contributions over all scales. To motivate this idea, and to explain how the logarithmic δ factor appears, we first consider an idealized scenario where the nonlinear functional $\mathrm{Tr}[S_m(P;\alpha\Omega)]$ localizes in a linear way. To localize, we use a continuously parametrized partition of unity generated by the 'slowly varying metric' ℓ_δ. We remark that the more standard discrete partition of unity from [25, Theorem 1.4.10] could also be used. The following is taken from [61].

Lemma 2.5. *There is a collection of non-negative functions* $(\phi_u)\subseteq C_c^\infty(\mathbb{R}^d)$, $u\in\mathbb{R}^d$, *such that* $(u,p)\to\phi_u(p)$ *is measurable, satisfying*

$$\operatorname{supp}\phi_u\subseteq B_{\ell_\delta(u)}(u),\quad \|D^\beta\phi_u\|_\infty\le C_\beta\ell_\delta(u)^{-|\beta|}$$

for all $\beta\in\mathbb{N}_0^d$, *and*

$$\int\phi_u(p)\ell_\delta(u)^{-d}\,du=1$$

for all $p\in\mathbb{R}^d$.

Completely disregarding the non-linearity of the functional $\mathrm{Tr}[S_m(P_\delta,\alpha\Omega)]$, we first imagine that all m copies of the symbol P_δ appearing in the error term (2.21) can be localized simultaneously around the same point $u\in\mathbb{R}^d$ at the same scale $\ell_\delta(u)$. More precisely, we consider the simplified error term

$$\begin{aligned}\int_{\mathbb{R}^d} \ell_\delta(u)^{-d}\Big|\int\dots\int \operatorname{Tr}\Big[(P_\delta\phi_u)^\vee(x_1)\dots(P_\delta\phi_u)^\vee(x_{m-1})(P_\delta\phi_u)^\vee(-x_1-\dots-x_{m-1})\Big]\\ (F_{\alpha\Omega}(\underline{x})-G_{\alpha\Omega}(\underline{x}))\,dx_1\dots dx_{m-1}\Big|du.\end{aligned} \tag{2.23}$$

Since (P_δ) satisfies a multiscale estimate, it follows that, for all $u\in\mathbb{R}^d$,

$$|D^\beta(P_\delta\phi_u)(p)|\le C_\beta G(u)\ell_\delta(u)^{-|\beta|},\quad p\in\mathbb{R}^d,\quad |\beta|\le\kappa,$$

where G is the amplitude function from (2.16). In particular, all relevant Fourier based norms of the localized symbols can be computed effectively by simple scaling

$$\begin{aligned}\|(P_\delta\phi_u)^\vee\|_2&\le CG(u)\ell_\delta(u)^{d/2},\\ \|(P_\delta\phi_u)^\vee(x)|x|\|_2&\le CG(u)\ell_\delta(u)^{d/2-1},\\ \|(P_\delta\phi_u)^\vee(x)|x|^2\|_2&\le CG(u)\ell_\delta(u)^{d/2-2},\\ \|(P_\delta\phi_u)^\vee\|_1&\le CG(u).\end{aligned} \tag{2.24}$$

Note that all norms only depend on simple powers of $\ell_\delta(u)$. Applying these estimates with the bound from (2.22), we readily deduce the upper bound

$$\begin{aligned}&\int_{\mathbb{R}^d} \ell_\delta(u)^{-d}\Big|\int\dots\int \operatorname{Tr}\Big[(P_\delta\phi_u)^\vee(x_1)\dots(P_\delta\phi_u)^\vee(x_{m-1})(P_\delta\phi_u)^\vee(-x_1-\dots-x_{m-1})\Big]\\ &\qquad(F_{\alpha\Omega}(\underline{x})-G_{\alpha\Omega}(\underline{x}))\,dx_1\dots dx_{m-1}\Big|du\\ &\qquad\le C\varepsilon\alpha^{d-1}\int_{\mathbb{R}^d}\ell_\delta(u)^{-1}G(u)\,du+C_\varepsilon\alpha^{d-2}\int_{\mathbb{R}^d}\ell_\delta(u)^{-2}G(u)\,du.\end{aligned}$$

We stress that the ε comes from the purely geometric expansion (2.12). Since $\varepsilon>0$ is arbitrary, only the scaling in ℓ_δ is relevant here. Finally, since G is bounded independently of δ in $L^1\cap L^\infty$ and since Γ is bounded and contained in a finite union of Lipschitz graphs, it is elementary to verify that

$$\begin{aligned}&\int_{\mathbb{R}^d}\ell_\delta(u)^{-1}G(u)\,du\le C\log(\delta^{-1}),\\ &\int_{\mathbb{R}^d}\ell_\delta(u)^{-2}G(u)\,du\le C\delta^{-1}\le\alpha,\quad \alpha\delta\ge1.\end{aligned}$$

Thus, the logarithm in δ exactly arises as the integral of $\ell_\delta(u)^{-1}$ over a neighborhood of Γ. We have now established Theorem 2.3 under the stylized assumption that the error term (2.21) localizes linearly.

The preceding computations suggest that localizing all m copies of P_δ simultaneously around the same point u is the natural way to establish Theorem 2.3. The problem is that $S_m(\cdot;\alpha\Omega)$ is not linear in the symbol, so such a localization cannot be obtained directly. The way to resolve this is to consider a multilinear extension of S_m and carefully localize one entry at a time.

We make the m-linearity of the functional $\operatorname{Tr}[S_m(\cdot;\alpha\Omega)]$ explicit by introducing the operator

$$S(P^1,\dots,P^m;\alpha\Omega)=1_{\alpha\Omega}P^1 1_{\alpha\Omega}\dots 1_{\alpha\Omega}P^m 1_{\alpha\Omega}-1_{\alpha\Omega}P^1\dots P^m 1_{\alpha\Omega}, \tag{2.25}$$

which is separately linear in each symbol $P^1,\dots,P^m$. Here superscripts should not be confused with powers. Of course $S_m(P;\alpha\Omega)$ is recovered by taking $P^1=\cdots=P^m=P$. Firstly, linearity in P^1 and Lemma 2.5 gives that

$$S_m(P_\delta;\alpha\Omega)=\int_{\mathbb{R}^d}\ell_\delta(u)^{-d}S(P_\delta\phi_u,P_\delta,\dots,P_\delta;\alpha\Omega)\,du \tag{2.26}$$

weakly as operators. To localize the remaining entries we introduce localizing factors $\chi_u^2,\dots,\chi_u^m\in C_c^\infty$ with the following three properties: for all $u\in\mathbb{R}^d$ and $2\le n\le m$,

1. χ_u^n is supported in a ball centered at u with radius $\lesssim D_n\ell_\delta(u)$,
2. $\ell_\delta(p)\gtrsim\frac{\ell_\delta(u)}{D_n}$ for $p\in\operatorname{supp}\chi_u^n$,
3. $|D^\beta\chi_u^n(p)|\lesssim D_n^{|\beta|}\ell_\delta(u)^{-|\beta|}$,

for a sequence (D_n) to be determined later. These properties are realized by a concrete construction in Chapter 3. Geometrically, one should think of $\operatorname{supp}\chi_u^n$ as an enlarged neighborhood of $\operatorname{supp}\phi_u$, which is allowed to spread out to distances $D_n\ell_\delta(u)$, except in directions leading to Γ, where it is cut off once the scale $\ell_\delta(u)/D_n$ is reached. Property 2 and the multiscale assumption on (P_δ), together with the bound from property 3, imply that

$$|D^\beta(P_\delta\chi_u^n)(p)|\le C\|F\|_\infty D_n^{|\beta|}\ell_\delta(u)^{-|\beta|},\quad p\in\mathbb{R}^d,\quad |\beta|\le\kappa. \tag{2.27}$$

With the factors $\chi_u^2,\dots,\chi_u^m$ in place, the idealized computation considered above applies to the term

$$\begin{aligned}\int_{\mathbb{R}^d}\ell_\delta(u)^{-d}\Big|&\operatorname{Tr}\big[S(P_\delta\phi_u,P_\delta\chi_u^2,\dots,P_\delta\chi_u^m;\alpha\Omega)\big]\\&-\Big(\frac{\alpha}{2\pi}\Big)^{d-1}\int_{T^*(\mathcal{F}\Omega)}\operatorname{Tr}\big[S((P_\delta\phi_u)_X,(P_\delta\chi_u^2)_X,\dots,(P_\delta\chi_u^m)_X;\mathbb{R}_+)\big]\,dX\Big|\,du.\end{aligned} \tag{2.28}$$

It is important that all entries are localized on regions where the relevant scale is comparable to $\ell_\delta(u)$ up to $D_2,\dots,D_m$ factors. For this part of the argument, only the scaling in $\ell_\delta(u)$ is relevant, since the small $\varepsilon>0$ parameter comes solely from the geometric expansion (2.12). Strictly speaking, one has to repeat the kernel expansion argument from Section 2.1 in a multilinear setting, but this requires no new ideas; we refer to Chapter 3 for the details.

For the main localized term (2.28), the precise size of D_n plays no significant role. The sequence (D_n) becomes relevant only for the part of the full error (2.21) that is not captured by the well localized part (2.28). Here the simple properties 1-3 of χ_u^n are not enough, and more refined control on the support of $1-\chi_u^n$ is necessary. The sequence (D_n) is then used to control the relative position of the supports of χ_u^{n-1} and $1-\chi_u^n$ and of Γ, which enters in the localization error bounds in a critical way. More precisely, choosing (D_n) such that $D_{n-1}\ll D_n$ makes these successive supports sufficiently well separated, except possibly in a thin region very close to Γ. In Chapter 3 we develop a specialized trace class estimate specifically to handle this regime.

2.3 The plunge region and extensions

We finally discuss one of the more subtle points glossed over in Sections 2.1 and 2.2, namely extensions of the trace formulas from Theorems 2.1, 2.2, 2.3, and 2.4 from polynomials $f(x) = x^m$ to more general spectral functions. Such extension arguments appear in essentially all relevant work on trace formulas, and it is not our intention here to give a review. In this thesis, we emphasize the extension to 'rough' spectral functions, that is, functions that lack classical smoothness at one or more points. The principal motivation is to cover entropy functions. In the general Wiener-Hopf setting, the relevant tools to treat such spectral functions have been developed by Sobolev [58] in an abstract framework, which have since been applied frequently in the literature [3, 4, 15, 16, 39, 59, 60]. This method hinges on trace class bounds roughly of the form

$$\|S_f(P;\alpha\Omega)\|_{\mathcal{S}_1} \leq \|f\|_{\mathcal{D}} \|1_{\alpha\Omega} P 1_{\alpha\Omega^c}\|_{\mathcal{S}_q}^q, \tag{2.29}$$

for a suitable norm $\|f\|_{\mathcal{D}}$ defined on a class of rough functions $\mathcal{D}$ with Hölder-type singularities at finitely many points, and a Schatten exponent $q = q_{\mathcal{D}} < 1$ dependent on this class. Generically, polynomials are dense in $\mathcal{D}$ in a suitable sense. A wide class $\mathcal{D}$ forces the parameter q to be close to zero. With this bound and an appropriate polynomial approximation argument, essentially only two additional ingredients are needed to extend a trace formula of $S_f(P;\alpha\Omega)$ from polynomials to $\mathcal{D}$: i) a uniform estimate of $\|1_{\alpha\Omega} P 1_{\alpha\Omega^c}\|_{\mathcal{S}_q}^q$ which catches the leading order behavior of the corresponding trace formula, and ii) a uniform estimate of the corresponding boundary coefficient in the class $\mathcal{D}$. For instance, in the setting of Widom's formula from Theorem 2.4, the relevant bounds are

$$\|1_{\alpha\Omega} P 1_{\alpha\Omega^c}\|_{\mathcal{S}_q}^q \leq C_{q,P,\Omega} \alpha^{d-1} \log(\alpha) \tag{2.30}$$

and

$$\int_{\mathcal{F}\Omega} \int_{J_P} |U_f(P^+(p), P^-(p))| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x) \leq C_{P,\Omega} \|f\|_{\mathcal{D}}. \tag{2.31}$$

Here U_f is the linearization of U_m (which was defined in (2.5)):

$$U_f(A,B) = \int_0^1 \frac{f(\theta A + (1-\theta)B) - \theta f(A) - (1-\theta) f(B)}{\theta(1-\theta)} \, d\theta, \quad A, B \in \mathbb{R}. \tag{2.32}$$

Relevant quasi-norm bounds of the form (2.30) can be found in [39, 56, 59], for instance. We stress, however, that in the general Wiener–Hopf setting considered here, the picture is still far from sharp. Existing bounds of the type (2.30) use specialized techniques and strengthened assumptions on the symbol P and the set Ω, which do not appear to be optimal. Likewise, Sobolev's abstract approach covers classes of functions with finitely many Hölder-type singularities, but there is little reason to believe that this is the natural endpoint for the extension problem. This raises two questions. First, given an exponent $0 < q \leq 1$, what assumptions on P and Ω are needed in order to obtain the sharp scaling bound (2.30)? Second, under assumptions on P and Ω, what is the largest natural class of spectral functions for which the trace formula remains valid?

The rest of this section concerns time-frequency limiting operators of the form $S(1_\Lambda;\alpha\Omega)$, that is, Wiener–Hopf operators with indicator symbols. In this simplified yet highly relevant setting, the underlying spectral picture becomes much more transparent, and, due to the identity

$$S_f(1_\Lambda;\alpha\Omega) = f(S(1_\Lambda;\alpha\Omega)), \quad f(0) = f(1) = 0,$$

the extension mechanism reduces to spectral estimates on the operator $S(1_\Lambda;\alpha\Omega)$. The key quantity to control is the size of the plunge region. Sharp uniform estimates of the form (1.12) lead to essentially necessary and sufficient conditions on the spectral function f for Widom's formula to hold. These bounds are also of independent interest. In this section we discuss some aspects of the sharp uniform bounds on the plunge region and the extension argument.

Precise statements and admissible functions

We start by introducing relevant notation and definitions. Note that the paper [33] included in Chapter 4 uses a different Fourier transform convention and slightly different notation.

For measurable sets $A, B \subseteq \mathbb{R}^d$, we denote by $S_{A,B} = S(1_B;A)$ the corresponding time-frequency limiting operator. This operator satisfies $0 \leq S_{A,B} \leq 1$, and, if both A and B have finite measure, then it is also trace class. We denote by

$$1 \geq \lambda_1(A,B) \geq \lambda_2(A,B) \geq \cdots \geq 0$$

the eigenvalues, arranged in non-increasing order counted with multiplicity. For $0 < \varepsilon < 1/2$, we introduce the eigenvalue counting functions

$$\begin{aligned} \Lambda_\varepsilon^-(A,B) &= |\{n \mid 1/2 \geq \lambda_n(A,B) > \varepsilon\}|, \\ \Lambda_\varepsilon(A,B) &= |\{n \mid 1-\varepsilon \geq \lambda_n(A,B) > \varepsilon\}|. \end{aligned} \tag{2.33}$$

We also need a geometric condition. We say that a set $\Gamma \subseteq \mathbb{R}^d$ has finite upper Minkowski content if

$$\limsup_{r\to 0^+} \frac{\left|\{p \in \mathbb{R}^d \mid d(p,\Gamma) < r\}\right|}{2r} < \infty. \tag{2.34}$$

A set with finite upper Minkowski content is bounded and has finite $\mathcal{H}^{d-1}$-measure. Moreover, if Γ is bounded and contained in a finite union of Lipschitz graphs, then Γ has finite upper Minkowski content, see [18, Lemma C.1]. In particular, if the boundary of a set A has finite upper Minkowski content, then it has finite perimeter. This geometric condition also appears in Chapter 3.

We finally introduce our class of admissible spectral functions for Widom's formula. For a function $f : [0,1] \to \mathbb{C}$, introduce the maximal functions $M_0 f(x) = \sup_{0\leq t\leq x} |f(t)|$ and $M_1 f(x) = \sup_{1-x\leq t\leq 1} |f(1) - f(t)|$.

Definition 2.6. *Consider a function $f : [0,1] \to \mathbb{C}$. We call f trace class admissible for $L^2(\mathbb{R}^d)$ if there is $\delta > 0$ such that*

$$\int_0^\delta \frac{M_0 f(\varepsilon) \log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}\, d\varepsilon < \infty. \tag{2.35}$$

We call f area law admissible if

$$\int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon}\, d\varepsilon < \infty. \tag{2.36}$$

Note that any trace class admissible function f is continuous at 0 and satisfies $f(0) = 0$. Similarly, if f is area law admissible, then f is continuous at 0 and 1 and $f(0) = 0$.

The following are the main results from Chapter 4.

Theorem 2.7. *Assume that $A \subseteq \mathbb{R}^d$ is a bounded set whose boundary has finite upper Minkowski content, and that B is a finite union of boxes. There exists $K = K_{d,A,B} \geq 4$ such that for all $\alpha \geq 2$ and $K^{-\alpha} < \varepsilon < \frac{1}{2}$,*

$$\Lambda_\varepsilon(\alpha A, B) \lesssim \alpha^{d-1} \log\left(\frac{1}{\varepsilon}\right) \log\left(\frac{\alpha K}{\log\left(\frac{1}{\varepsilon}\right)}\right). \tag{2.37}$$

For $\varepsilon \leq K^{-\alpha}$, there are no eigenvalues larger than $1-\varepsilon$, and, if both A and B have positive measure,

$$\Lambda_\varepsilon^-(\alpha A, B) \asymp \left(\frac{\log\left(\frac{1}{\varepsilon}\right)}{\log\left(\frac{\log\left(\frac{1}{\varepsilon}\right)}{\alpha}\right)}\right)^d. \tag{2.38}$$

Theorem 2.8. *Let $A \subseteq \mathbb{R}^d$ be a bounded set whose boundary has finite upper Minkowski content, and let B be a finite union of boxes. Assume that f is both area law admissible and trace class admissible for $L^2(\mathbb{R}^d)$. There exist $\alpha_0(f, A, B) > 0$ and $C(A, B) > 0$ such that, for $\alpha > \alpha_0(f, A, B)$,*

$$\left|\mathrm{Tr}[f(S_{\alpha A,B})] - \left(\frac{\alpha}{2\pi}\right)^d |A||B| f(1)\right| \leq C(A,B)\alpha^{d-1}\log(\alpha)\int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon}\, d\varepsilon. \tag{2.39}$$

If in addition f is Riemann integrable on $[\varepsilon, 1-\varepsilon]$ for all $0 < \varepsilon < \frac{1}{2}$, then also

$$\begin{aligned} &\mathrm{Tr}[f(S_{\alpha A,B})] \\ &\quad = \left(\frac{\alpha}{2\pi}\right)^d |A||B| f(1) + \alpha^{d-1}\log(\alpha) I(A,B) \int_0^1 \frac{f(\theta) - f(1)\theta}{\theta(1-\theta)}\, d\theta + o(\alpha^{d-1}\log(\alpha)) \end{aligned} \tag{2.40}$$

as $\alpha \to \infty$, where

$$I(A,B) = (2\pi)^{-(d+1)} \int_{\mathcal{F}A}\int_{\mathcal{F}B} |\nu_A(x)\cdot\nu_B(p)|\, d\mathcal{H}^{d-1}(p)\, d\mathcal{H}^{d-1}(x).$$

Let us spend a few words commenting on our results. The two term expansion (2.40) is Widom's formula from Theorem 2.4 after expanding the volume term. Equivalently, (2.40) can be stated as

$$\mathrm{Tr}[S_f(1_B; \alpha A)] = \alpha^{d-1}\log(\alpha) I(A,B) \int_0^1 \frac{f(\theta) - f(1)\theta}{\theta(1-\theta)}\, d\theta + o(\alpha^{d-1}\log(\alpha)).$$

Here the f-dependence of the boundary coefficient is precisely given by (2.32). Clearly, finiteness of this coefficient is very close to the condition that f is area law admissible. Indeed, if $|f|$ is monotone near 0 and 1 and Riemann integrable on $[\varepsilon, 1-\varepsilon]$ for all $0 < \varepsilon < 1/2$, then these are equivalent. Thus, the area law condition on f is very natural and essentially necessary.

The more subtle condition is that of trace class admissibility. As the name suggests, this condition is used to ensure that $f(S_{\alpha A,B})$ is trace class. A perhaps surprising result is that this condition is very precise. In fact, we show the following in chapter 4.

Theorem 2.9. *Let* $A, B \subseteq \mathbb{R}^d$ *be sets with non-empty interiors such that* $S_{A,B}$ *is compact. If* $f : [0,1] \to \mathbb{C}$ *is such that* $|f|$ *is non-decreasing near 0 and* $f(S_{A,B})$ *is trace class, then* f *is trace class admissible on* $L^2(\mathbb{R}^d)$*.*

We conclude that $f(S_{A,B})$ being trace class is purely the consequence of a volume effect in the sets A and B. We also here point out that if $d \geq 2$, trace class admissibility places a stronger restriction at 0 than area law admissibility. In fact, it is now easy to find examples of continuous functions $f : [0,1] \to [0,\infty)$ which are monotone near 0 and 1 such that the boundary coefficient from (2.40) is finite, but $f(S_{\alpha A,B})$ is not trace class. The intuition here, which is transparent from the proof of Theorem 2.8 presented in Chapter 4, is that the leading order $\alpha^{d-1}\log(\alpha)$ contribution arises from the eigenvalues in the plunge region $K^{-\alpha} \leq \lambda_n(\alpha A, B) \leq 1 - K^{-\alpha}$, while the total contribution of the eigenvalues satisfying $\lambda_n(\alpha A, B) \leq K^{-\alpha}$ is lower order in α. The relevant regimes can be read from Theorem 2.7. Area law admissibility is used to control the eigenvalues in the plunge region to the correct order, while trace class admissibility is assumed only to ensure that the tail of eigenvalues is summable. Such summability restrictions are not uncommon in operator theory, see for instance [8, 47].

We have now commented on the sharpness of Theorem 2.8 in terms of the admissible spectral functions f. However, we do not expect the conditions on A and B to be sharp. The geometric restriction here comes from Theorem 2.7, whose proof critically relies on the tensor product structure available for boxes. While we believe that it is sufficient to assume that both sets A and B have boundaries with finite upper Minkowski content, in this case we only establish the corresponding bound (2.38) with an additional $\log(\alpha)$ factor. Since we in this setting do not catch the precise leading order term in α, the plunge region bound is not applicable for trace formulas.

Plunge region bounds and extensions

We start by indicating how the sharp bounds from Theorem 2.7 can be used to prove Theorem 2.8. On a high level, the extension argument is similar to the abstract mechanism discussed in the beginning of this section, based on Sobolev's result from [58]. In this analogy, the uniform bound (2.39) plays the role of the combination of the bounds (2.29) and (2.30). The relevant class $\mathcal{D}$ consists of all trace class and area law admissible functions on $L^2(\mathbb{R}^d)$, and

$$\|f\|_{\mathcal{D}} = \int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon}\, d\varepsilon.$$

As in Sobolev's approach, polynomials are not dense in $\mathcal{D}$ in the usual sense. The argument proceeds by first splitting f into a localized part G_ε satisfying $\|G_\varepsilon\|_{\mathcal{D}} = o(1)$, and a remainder H_ε supported in $[\varepsilon, 1-\varepsilon]$. Since f is Riemann integrable on $[\varepsilon, 1-\varepsilon]$, the remainder term can be approximated from above and below by simple functions supported away from 0 and 1. These can in turn be approximated in the same way by continuous functions supported away from 0 and 1, and hence by polynomials. Similar arguments are carried out in [36, 37] in the indicator setting, but we stress again that the proof follows the same overall recipe as the abstract argument discussed above. The only essential simplification in the indicator setting is that, away from the singular points 0 and 1, it suffices to assume Riemann integrability rather than smoothness.

It is therefore clear that the essential result is the uniform bound (2.39). We have already commented on how area law and trace class admissibility are used to estimate different regimes. We now provide a few more details. By a linear modification we may assume that $f(0) = f(1) = 0$. We split into the relevant regimes mentioned earlier:

$$\begin{aligned}|\mathrm{Tr}[f(S_{\alpha A,B})]| \leq &\sum_{\lambda_n(\alpha A,B)\leq K^{-\alpha}} |f(\lambda_n(\alpha A,B))| + \sum_{K^{-\alpha}\leq\lambda_n(\alpha A,B)\leq 1-K^{-\alpha}} |f(\lambda_n(\alpha A,B))| \\ &+ \sum_{\lambda_n(\alpha A,B)>1-K^{-\alpha}} |f(\lambda_n(\alpha A,B))|.\end{aligned}$$

The third sum vanishes by Theorem 2.7. We first focus on the intermediate regime. By splitting the sum dyadically

$$2^{-2^{k+1}} \leq \lambda_n \leq 2^{-2^k}, \quad 2^{-2^{k+1}} \leq 1-\lambda_n \leq 2^{-2^k},$$

a little thought shows

$$\sum_{K^{-\alpha}\leq\lambda_n(\alpha A,B)\leq 1-K^{-\alpha}} |f(\lambda_n(\alpha A,B))| \leq \sum_{k=0}^{k_0}\left(M_0 f\left(2^{-2^k}\right) + M_1 f\left(2^{-2^k}\right)\right)\Lambda_{2^{-2^{k+1}}}(\alpha A,B), \tag{2.41}$$

where k_0 satisfies $2^{-2^{k_0+1}} \leq K^{-\alpha} \leq 2^{-2^{k_0}}$. We are in the boundary regime from (2.37), so this bound applies and gives $\Lambda_{2^{-2^{k+1}}}(\alpha A,B) \lesssim \alpha^{d-1}2^k\log(\alpha)$. Since $M_0 f$ and $M_1 f$ are monotone, the simple identity

$$2^k = \frac{2}{\log(2)}\int_{2^{-2^k}}^{2^{-2^{k-1}}} \frac{1}{\varepsilon}\,d\varepsilon$$

allows us to replace the sums in (2.41) with integrals, and we obtain

$$\sum_{K^{-\alpha}\leq\lambda_n(\alpha A,B)\leq 1-K^{-\alpha}} |f(\lambda_n(\alpha A,B))| \lesssim \alpha^{d-1}\log(\alpha)\int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon}d\varepsilon.$$

The corresponding bound of the term

$$\sum_{\lambda_n(\alpha A,B)\leq K^{-\alpha}} |f(\lambda_n(\alpha A,B))|$$

in the bulk or volume regime is more technical. This part only uses that f is trace class admissible on $L^2(\mathbb{R}^d)$ and the bound (2.38). Ultimately, the final estimate reads

$$\sum_{\lambda_n(\alpha A,B)\leq K^{-\alpha}} |f(\lambda_n(\alpha A,B))| \lesssim \log(\alpha)^d \int_0^{K^{-\alpha/2}} \frac{M_0 f(\varepsilon)\log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\log\left(\log\left(\frac{1}{\varepsilon}\right)\right)^d}\,d\varepsilon.$$

As announced above, this term is indeed lower order in α, so taking α large enough depending on A, B, and f leads to the estimate (2.39).

We stress that the assumption that B is a finite union of boxes enters only through Theorem 2.7, and not through the extension argument itself. More precisely, if A and B are bounded sets with finite perimeter, and (2.39) is satisfied for these sets, then Theorem 2.8 follows from the general mechanism discussed above.

Chapter 3

Paper: *The Widom Conjecture with Optimal Regularity Assumptions*

This chapter contains the manuscript entitled “The Widom Conjecture with Optimal Regularity Assumptions”, which is joint work between Søren Fournais, Robert Seiringer, Jan Philip Solovej, and the author.
The paper is included in the form of a final draft – including the title page, abstract and bibliography.

THE WIDOM CONJECTURE WITH OPTIMAL REGULARITY ASSUMPTIONS

SØREN FOURNAIS, MARTIN DAM LARSEN, ROBERT SEIRINGER, AND JAN PHILIP SOLOVEJ

Abstract. In 1982 Widom conjectured a 2-term asymptotic formula for traces of the form $\operatorname{Tr}\left[f\left(1_{\alpha\Omega}(x)g(-i\nabla)1_{\alpha\Omega}(x)\right)\right]$ as α tends to infinity. Sobolev solved this conjecture in 2013 with sufficient regularity assumptions on Ω and g. Our goal is to extend this and establish the conjecture in the most singular cases. For g an indicator function, our regularity assumptions are, indeed, sharp.

1. Introduction

This paper is motivated by the computation of the bipartite entanglement entropy for free Fermionic states. In the case we study, this refers to calculating the entropy of spatially restricted ground states in the scaling limit of large domains. The result of this computation is related to the logarithmically enhanced area law for such systems. It was realized by Gioev and Klich in 2006 [17] that these area laws are a consequence of a 1982 conjecture by Widom [53] on traces of functions of singular Wiener-Hopf operators. The mathematical proof of this conjecture was given in a celebrated paper by Sobolev in 2013 [44]. In the present paper we extend the result of Sobolev on the Widom Conjecture in several directions. Most importantly, we achieve optimal regularity conditions on the spatial domain and the involved operators. Moreover, we extend much of the analysis to matrix valued operators, which is of considerable interest from a physics point of view.

To describe this in greater detail, take a spatial region $\Omega \subseteq \mathbb{R}^d$, $d \geq 1$, and consider a momentum symbol $P : \mathbb{R}^d \to \mathcal{M}_n$ taking values in $\mathcal{M}_n$, the space of n-by-n complex matrices. The focus of the present paper is the study of the eigenvalue distribution of (truncated) Wiener-Hopf operators on $L^2(\mathbb{R}^d;\mathbb{C}^n)$ of the form

$$S(P;\Omega) = 1_\Omega P 1_\Omega \tag{1.1}$$

in the scaling (or semi-classical) limit $\Omega = \alpha\Omega_0$ with $\alpha \to \infty$. Here 1_Ω is the orthogonal projection onto the subspace $L^2(\Omega;\mathbb{C}^n)$ of $L^2(\mathbb{R}^d;\mathbb{C}^n)$, and $P = P(p) = P(-i\nabla)$ denotes the corresponding Fourier multiplier with symbol P. In the following we consistently abuse notation and write 1_Ω, $1_\Omega(x)$, P, $P(p)$, etc. both for the functions and function values and for the corresponding multipliers on L^2.

We call operators of the form (1.1) truncated Wiener-Hopf operators, as they are the higher dimensional analogues of truncations of the one-dimensional Wiener-Hopf operators, which in turn are continuous generalizations of Toeplitz operators. Studying the eigenvalue distribution of Toeplitz operators has a rich history going back to the Szegő limit theorems. In the extensive literature on Szegő limit theorems, the publication most closely related to our work is [2] on jump-type Fisher-Hartwig singularities. See also [12] and [13] and the references therein for more recent developments. For different classes of functions f, the objective is to compute the traces $\operatorname{Tr}\left[S_f(P;\Omega)\right]$ of the operators

$$S_f(P;\Omega) = 1_\Omega f\left(S(P;\Omega)\right)1_\Omega - S\left(f(P);\Omega\right). \tag{1.2}$$

For the special case $f(x) = x^m$ we write $S_m(P;\Omega)$ instead. The operator $S\left(f(P);\Omega\right)$ exactly corresponds to the leading order volume contribution of $1_\Omega f\left(S(P;\Omega)\right)1_\Omega$. Indeed, it can be

---

Date: March 30, 2026.

shown that for polynomial f, say, and weak integrability assumptions on P and Ω,

$$\mathrm{Tr}_{L^2(\mathbb{R}^d;\mathbb{C}^n)}\left[S_f(P;\alpha\Omega)\right] = o(\alpha^d) \tag{1.3}$$

as $\alpha \to \infty$, and

$$\mathrm{Tr}_{L^2(\mathbb{R}^d;\mathbb{C}^n)}\left[S\left(f(P);\alpha\Omega\right)\right] = \left(\frac{\alpha}{2\pi}\right)^d \mathcal{H}^d(\Omega)\int \mathrm{Tr}_{\mathbb{C}^n} f(P(p))\,dp. \tag{1.4}$$

Here and throughout $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure in $\mathbb{R}^d$. We denote other Hausdorff measures similarly.

The entirety of this paper is devoted to the study of the leading order (sub-volume) behavior of $S_f(P;\Omega)$ under minimal conditions on Ω and P for different classes of functions f. It has long been established that the asymptotic behavior critically depends on the regularity of the involved symbols. For smooth P and Ω with smooth boundary, a full expansion of $\mathrm{Tr}\left[S_f(P;\alpha\Omega)\right]$ into powers α^{d-k} is known [55]. Here the leading order term admits area scaling and takes the form

$$\begin{aligned}&\mathrm{Tr}_{L^2(\mathbb{R}^d;\mathbb{C}^n)}\left[S_f(P;\alpha\Omega)\right]\\ &\quad= \left(\frac{\alpha}{2\pi}\right)^{d-1}\int_{T^*(\partial\Omega)} \mathrm{Tr}_{L^2(\mathbb{R};\mathbb{C}^n)}\left[S_f(P_X;\mathbb{R}_+)\right]dX + o(\alpha^{d-1}),\end{aligned} \tag{1.5}$$

where $T^*(\partial\Omega)$ is the co-tangent bundle of $\partial\Omega$ identified with $\{X = (x,p) \mid x \in \partial\Omega,\, p \in \Pi_{\nu_\Omega(x)}\}$, with $\Pi_{\nu_\Omega(x)} = \{\nu_\Omega(x)\}^\perp$ and ν_Ω the interior unit normal to Ω. For $X = (x,p) \in T^*(\partial\Omega)$ and $\xi \in \mathbb{R}$ we denote $P_X(\xi) = P(p + \xi\nu_\Omega(x))$ the corresponding co-tangent section of P. Finally, $T^*(\partial\Omega)$ is equipped with the measure

$$\int_{T^*(\partial\Omega)} F(X)\,dX = \int_{\partial\Omega}\int_{\Pi_{\nu_\Omega(x)}} F(x,p)\,d\mathcal{H}^{d-1}(p)\,d\mathcal{H}^{d-1}(x).$$

For scalar valued symbols, the remaining co-tangent traces in (1.5) can be computed explicitly using the remarkable trace formula [54] of Widom:

$$\mathrm{Tr}_{L^2(\mathbb{R};\mathbb{C})}\left[S_f(Q;\mathbb{R}_+)\right] = \frac{1}{8\pi^2}\iint_{\mathbb{R}\times\mathbb{R}} \frac{U_f(Q(\xi_1),Q(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2. \tag{1.6}$$

In (1.6) we wrote U_f for the symmetric functional

$$U_f(A,B) = \mathrm{Tr}_{\mathbb{C}^n}\int_0^1 \frac{f(\theta A + (1-\theta)B) - \theta f(A) - (1-\theta)f(B)}{\theta(1-\theta)}\,d\theta, \quad A,B \in \mathcal{M}_n, \tag{1.7}$$

which behaves quadratically in $A - B$ for f at least C^2. For the case $f(x) = x^m$ we write U_m instead. The expression (1.5) is due to Widom [52, 55] and Roccaforte [41]. In the most general setting from [54], the expansion is shown for f analytic, Ω bounded with C^1 boundary, and a matrix valued symbol P satisfying $P \in \dot{H}^{1/2} \cap L^1 \cap L^\infty$ such that $\check{P} \in L^1$ (in particular P is assumed to be continuous).

The situation changes drastically for symbols P with jump discontinuities, which can easily be seen from (1.6). In this discontinuous setting Sobolev [44] proved - resolving Widom's conjecture from [53] - that the leading order term picks up an additional logarithmic factor: if P is of the form $P(p) = \sigma(p)1_\Lambda(p)$ where $\sigma \in W^{d+2,\infty}(\mathbb{R}^d;\mathbb{C})$ is a smooth symbol and $\Omega, \Lambda \subseteq \mathbb{R}^d$ bounded sets with C^1 and C^3 boundaries, respectively, then for regular f,

$$\begin{aligned}\mathrm{Tr}\left[S_f(P;\alpha\Omega)\right] = \;&\frac{\alpha^{d-1}\log(\alpha)}{(2\pi)^{d+1}}\int_{\partial\Omega}\int_{\partial\Lambda} U_f\left(\sigma(p),0\right)|\nu_\Omega(x)\cdot\nu_\Lambda(p)|\,d\mathcal{H}^{d-1}(p)\,d\mathcal{H}^{d-1}(x)\\ &+ o(\alpha^{d-1}\log(\alpha)).\end{aligned} \tag{1.8}$$

We say that the operator $S_f(P;\alpha\Omega)$ has (logarithmically) enhanced area scaling. The case $d = 1$ was studied by Widom [53] and Landau-Widom [25]. The expansion (1.8) is now sometimes referred to as the Widom-Sobolev formula. The regularity assumptions on the sets were later slightly relaxed to include piece-wise smooth domains, see [45]. Subsequent work by Sobolev developed the abstract tools to study more singular f with possible Hölder-type singularities, including entropic functions, [46–48], and the transition between smooth and non-smooth symbols was studied in [49]. For applications to the computation of the entanglement entropy for free Fermionic systems we refer to [27–29] by Leschke-Sobolev-Spitzer. For other related work see [4, 5, 16, 35, 36, 50, 56].

We mention that Widom's original conjecture and Sobolev's main result referred to above are stated in the more general setting of pseudo-differential operators. Here the smooth symbol $\sigma = \sigma(x,p)$ is allowed to also depend on the position variable x. While this is an interesting generalization, the pure momentum case is of greatest interest in physics and already contains most of the difficulties.

The main achievement of the present paper is to establish (1.8) for symbols P of bounded variation and sets Ω of finite perimeter, see Theorem 1.1 below, thus providing a generalization of Widom's conjecture and Sobolev's results [44, 45]. These regularity assumptions are sharp in the class of indicator functions, see Theorem 1.5.

Throughout most of the paper we work with polynomial f. This is the foundational case, as it is now clear that refinements to rougher spectral functions require two ingredients: i) the limit theorem for polynomials just described, and ii) uniform estimates on the singular values of $1_{\alpha\Omega}P1_{\alpha\Omega^c}$. Establishing ii) is inherently a separate issue from i) requiring different techniques. Since, however, rough functions are highly physically relevant, e.g. for f corresponding to Rényi entropies, we carry out this extension scheme in Section 7. For the case of P being an indicator function we in particular establish enhanced area scaling for functions f that include Rényi entropies of $\gamma > 1/2$. This, however, requires slightly stronger assumptions on the involved geometries than finite perimeter, see Theorem 1.7.

As a by-product of our method, we also obtain sharp results on the area scaling case (1.5). We show in Theorem 1.13 and Remark 1.15 that the sharp condition on P for area scaling is $H^{1/2}$ smoothness. This refines the results of Widom [52], where it is required that $\check{P} \in L^1$.

Central to our whole analysis is a result on the transition between area and enhanced area scaling in a discontinuous limit. This is analyzed in Section 2, where the key result is Theorem 2.5.

Almost all of our analysis is carried out for matrix valued symbols. However, we are unable to fully bridge the gap between area and enhanced area scaling for matrix valued symbols, since we do not know of a version of the trace formula (1.6) that is valid in this setting.

1.1. **Main results.** The following constitutes our main theorem. Our assumptions are stated in terms of the well-studied function space $\dot{BV}$ of bounded variation functions, and we use standard terminology and notation. See for instance the textbook references [1, 15, 31].

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter and $P \in \dot{BV}(\mathbb{R}^d;\mathbb{C}) \cap L^\infty$. If $d = 1$ assume in addition that $P \in L^2(\mathbb{R}^d;\mathbb{C}) + \mathbb{C}$ and that $\min\{\mathcal{H}^1(\Omega), \mathcal{H}^1(\Omega^c)\} < \infty$. Then, for all $m \geq 2$,*

$$\begin{aligned}
&\operatorname{Tr}\left[S_m(P;\alpha\Omega)\right] \\
&\quad = \frac{\alpha^{d-1}\log(\alpha)}{(2\pi)^{d+1}} \int_{\mathcal{F}\Omega}\int_{J_P} U_m(P^+(p),P^-(p))|\nu_\Omega(x)\cdot\nu_P(p)|\,d\mathcal{H}^{d-1}(p)\,d\mathcal{H}^{d-1}(x) \qquad (1.9)\\
&\quad + o(\alpha^{d-1}\log(\alpha))
\end{aligned}$$

as $\alpha \to \infty$. Here $\mathcal{F}\Omega$ denotes the reduced boundary of Ω and ν_Ω the measure theoretic interior normal, J_P denotes the measure theoretic jump set of P endowed with a choice of orientation ν_P, and the corresponding one-sided jump values are denoted $P^\pm$. These concepts are defined in Appendix A.

Remark 1.2. The right hand side of (1.9) is independent of the choice of orientation ν_P on the jump set J_P since the functional U_m is symmetric. Finiteness of the boundary integral is guaranteed by the assumptions of the Theorem, see (A.1). We here remark that U_m

conveniently can be expressed as

$$U_m(A,B) = -\frac{1}{2}\sum_{k=1}^{m-1}\left(\frac{1}{k}+\frac{1}{m-k}\right)\left(A^k-B^k\right)\left(A^{m-k}-B^{m-k}\right), \quad A,B\in\mathbb{C}. \tag{1.10}$$

Remark 1.3. To illustrate connection to Sobolev's results [44,45], consider the more classical setting $P(p)=\sigma_1(p)1_\Lambda(p)+\sigma_2(p)1_{\Lambda^c}(p)$ with Λ a set with Lipschitz boundary and bounded Lipschitz functions σ_1,σ_2 satisfying

$$\nabla\sigma_1\in L^1(\Lambda), \quad \nabla\sigma_2\in L^1(\Lambda^c), \quad \sigma_1-\sigma_2\in L^1(\partial\Lambda).$$

Here $P\in\dot{BV}(\mathbb{R}^d;\mathbb{C})\cap L^\infty$, $J_P\subseteq\partial\Lambda$, and, if we take ν_P to agree with the (classical) interior normal to Λ, then $P^+=\sigma_1$ and $P^-=\sigma_2$ on J_P. This follows from a direct computation of the distributional derivative DP. Consequently, for any set Ω with Lipschitz boundary satisfying $\mathcal{H}^{d-1}(\partial\Omega)<\infty$, the formula (1.9) reduces to (when $d\geq 2$):

$$\begin{aligned}&\operatorname{Tr}\left[S_m(P;\alpha\Omega)\right]\\ &\quad=\frac{\alpha^{d-1}\log(\alpha)}{(2\pi)^{d+1}}\int_{\partial\Omega}\int_{\partial\Lambda}U_m(\sigma_1(p),\sigma_2(p))|\nu_\Omega(x)\cdot\nu_\Lambda(p)|\,d\mathcal{H}^{d-1}(p)\,d\mathcal{H}^{d-1}(x)\\ &\quad\quad+o(\alpha^{d-1}\log(\alpha)).\end{aligned} \tag{1.11}$$

Using the full strength of Theorem 1.1, it is for this example (with obvious modifications to (1.11)) sufficient to assume that Λ is a set of locally finite perimeter and that $\sigma_1,\sigma_2\in W^{1,1}_{loc}\cap L^\infty$ with $\nabla\sigma_1\in L^1(\Lambda)$, $\nabla\sigma_2\in L^1(\Lambda^c)$, and $\sigma_1^*-\sigma_2^*\in L^1(\mathcal{F}\Lambda)$, where the asterisk denotes the precise representative. It is of course easy to generalize this example to more involved jump structures.

Remark 1.4. The counterexample $P=1_\Omega=1_{\mathbb{R}_+}$ shows that the theorem fails without additional integrability assumptions for $d=1$. The assumption $P\in L^2(\mathbb{R};\mathbb{C})+\mathbb{C}$ is convenient but stronger than necessary. This integrability issue does not appear for $d\geq 2$ by the embedding $\dot{BV}(\mathbb{R}^d;\mathbb{C})\to L^{\frac{d}{d-1}}(\mathbb{R}^d;\mathbb{C})+\mathbb{C}$, see [1, Theorem 3.47].

For the specific case of $P=1_\Lambda$ being an indicator function the limiting coefficient (1.9) takes the form

$$\begin{aligned}\operatorname{Tr}\left[S_m(1_\Lambda;\alpha\Omega)\right]&=\frac{\alpha^{d-1}\log(\alpha)}{(2\pi)^{d+1}}U_m(0,1)\int_{\mathcal{F}\Omega}\int_{\mathcal{F}\Lambda}|\nu_\Omega(x)\cdot\nu_\Lambda(p)|\,d\mathcal{H}^{d-1}(p)\,d\mathcal{H}^{d-1}(x)\\ &\quad+o(\alpha^{d-1}\log(\alpha))\end{aligned} \tag{1.12}$$

whenever Ω and Λ are sets of finite perimeter (and either Ω or Ω^c has finite measure, similarly for Λ, when $d=1$). Note here that the roles of Λ and Ω, as well as the assumptions on them, are interchangeable. This is expected, as an elementary argument shows that the operators $S_m(1_\Lambda;\Omega)$ and $S_m(1_\Omega;\Lambda)$ have the same eigenvalues with equal multiplicities. If Ω, say, is the unit ball $B=B_1$, we exactly recover the perimeter:

$$\lim_{\alpha\to\infty}\frac{1}{\alpha^{d-1}\log(\alpha)}\operatorname{Tr}\left[S_m(1_\Lambda;\alpha B)\right]=K_{1,d}(2\pi)^{-(d+1)}U_m(0,1)\operatorname{Per}(\Lambda), \tag{1.13}$$

where $K_{1,d}=\int_{S^{d-1}}|e\cdot\omega|d\mathcal{H}^{d-1}(\omega)$ for any (all) $e\in S^{d-1}$. In fact the equality (1.13) holds in the following extended sense, which shows that Theorem 1.1 is sharp in the class of indicators.

Theorem 1.5. *Let* $\Lambda\subseteq\mathbb{R}^d$ *be a set such that* $\min\{\mathcal{H}^d(\Lambda),\mathcal{H}^d(\Lambda^c)\}<\infty$. *If, for some* $m\geq 2$,

$$\limsup_{\alpha\to\infty}\frac{1}{\alpha^{d-1}\log(\alpha)}\left|\operatorname{Tr}\left[S_m(1_\Lambda;\alpha B)\right]\right|<\infty, \tag{1.14}$$

then Λ *is a set of finite perimeter and* (1.13) *holds.*

Theorem 1.5 is shown for $m = 2$ in Proposition 3.6 and the general case $m \geq 2$ follows from the elementary inequality $x - x^2 \leq x - x^m \leq (m-1)(x - x^2)$ for $0 \leq x \leq 1$. We can thus see the functional $\Lambda \to \operatorname{Tr}\left[S_m(1_\Lambda; \alpha B)\right]$ as a highly non-linear and non-local determination of the perimeter. In geometric measure theory, such determinations have recently attracted a lot of attention following the publication of the BBM paper [6]. In this context our result is novel. In fact, the main difficulty in our proof is to reduce the analysis to functionals similar to the ones usually studied [10, 37, 38], which we compute in Lemma $A.11$ in the appendix.

As mentioned before, extending the formula (1.9) from polynomials to rough functions f requires an additional ingredient: uniform bounds on the singular values of the operator $1_{\alpha\Omega} P 1_{\alpha\Omega^c}$. This is particularly clear when $P = 1_\Lambda$ is an indicator function due to the exact equality

$$\operatorname{Tr}\left[S_{f_\gamma}(1_\Lambda; \Omega)\right] = \|1_{\alpha\Omega} 1_\Lambda 1_{\alpha\Omega^c}\|_{\mathcal{S}_{2\gamma}}^{2\gamma}, \quad f_\gamma(x) = (x(1-x))^\gamma,$$

where $\mathcal{S}_q$ denotes the usual Schatten-q space, see [43]. In this paper we limit ourselves to establishing a trace class bound on $1_{\alpha\Omega} P 1_{\alpha\Omega^c}$ and carry out the extension procedure with this as an a priori input. To this end, slightly stronger assumptions on the sets Ω and Λ are required.

Definition 1.6. A set $\Gamma \subseteq \mathbb{R}^d$ is said to have finite upper Minkowski content if

$$\mathcal{M}^*(\Gamma) = \limsup_{r \to 0^+} \frac{\mathcal{H}^d(\Gamma + B_r)}{2r} < \infty. \tag{1.15}$$

It follows from [1, Theorem 2.106] that if $\Gamma = f(A)$ for a Lipschitz function $f : \mathbb{R}^{d-1} \to \mathbb{R}$ and a compact set $A \subseteq \mathbb{R}^{d-1}$, then $\mathcal{M}^*(\Gamma) < \infty$. In particular, every compact subset of a finite union of Lipschitz graphs has finite upper Minkowski content. We refer to [1, Section 2.10] and Appendix C for further properties.

Theorem 1.7. *Let $\Omega, \Lambda \subseteq \mathbb{R}^d$ be sets of finite perimeter and assume $\mathcal{F}\Omega$ and $\mathcal{F}\Lambda$ have finite upper Minkowski content. If $d = 1$ assume in addition that $\min\{\mathcal{H}^1(\Omega), \mathcal{H}^1(\Omega^c)\} < \infty$ and $\min\{\mathcal{H}^1(\Lambda), \mathcal{H}^1(\Lambda^c)\} < \infty$. Then,*

$$\|1_{\alpha\Omega} 1_\Lambda 1_{\alpha\Omega^c}\|_{\mathcal{S}_1} = O(\alpha^{d-1} \log(\alpha)). \tag{1.16}$$

Moreover, if $f : [0,1] \to \mathbb{C}$ is Riemann integrable on $[\varepsilon, 1-\varepsilon]$ for all $0 < \varepsilon < 1/2$ and satisfies

$$\limsup_{x \to 0^+} \frac{|f(x) - f(0)|}{x^{1/2}} = 0, \quad \limsup_{x \to 0^+} \frac{|f(1-x) - f(1)|}{x^{1/2}} = 0, \tag{1.17}$$

then

$$\begin{aligned} \operatorname{Tr}\left[S_f(1_\Lambda; \alpha\Omega)\right] = {} & \frac{\alpha^{d-1}\log(\alpha)}{(2\pi)^{d+1}} U_f(0,1) \int_{\mathcal{F}\Omega} \int_{\mathcal{F}\Lambda} |\nu_\Omega(x) \cdot \nu_\Lambda(p)| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x) \\ & + o(\alpha^{d-1} \log \alpha), \end{aligned} \tag{1.18}$$

as $\alpha \to \infty$.

Remark 1.8. It is sufficient to assume that $\partial\Omega$ and $\partial\Lambda$ have finite upper Minkowski content. In this case, $\mathcal{H}^{d-1}(\partial\Lambda) < \infty$ and $\mathcal{H}^{d-1}(\partial\Omega) < \infty$, so both sets will have finite perimeter by [1, Proposition 3.62]. Optimizing Λ and Ω over all measure zero modifications exactly leads to the statement of Theorem 1.7.

Remark 1.9. The assumption that $\mathcal{F}\Lambda$ and $\mathcal{F}\Omega$ have finite upper Minkowski content is used to obtain (1.16), which allows functions f satisfying (1.17). Alternatively, relying on the Hilbert-Schmidt estimate

$$\|1_{\alpha\Omega} 1_\Lambda 1_{\alpha\Omega^c}\|_{\mathcal{S}_2}^2 = O(\alpha^{d-1} \log(\alpha)),$$

which holds without this assumption, a minor modification of the proof yields (1.18) for all f such that the function $\frac{f(x) - xf(1) - (1-x)f(0)}{x(1-x)}$ is Riemann integrable on $[0,1]$. The same condition on f was found by Landau-Widom [25] in the one-dimensional setting.

Remark 1.10. If Ω and Λ are bounded sets with Lipschitz boundaries, the Schatten quasi-norm estimates from [46] allow us to weaken the endpoint regularity assumption (1.17). In this case, it suffices to require that

$$\limsup_{x\to 0^+} \frac{|f(x)-f(0)|}{x^\gamma} = 0, \quad \limsup_{x\to 0^+} \frac{|f(1-x)-f(1)|}{x^\gamma} = 0$$

for some $\gamma > 0$.

Remark 1.11. Schatten quasi-norm estimates alone do not capture the precise spectral behavior of the operator $1_{\alpha\Omega}1_\Lambda 1_{\alpha\Omega^c}$. To obtain sharp results, one must instead analyze the singular value distribution more directly, for instance via the counting function

$$N_\varepsilon = |\{n \in \mathbb{N} \mid \sigma_n\left(1_{\alpha\Omega}1_\Lambda 1_{\alpha\Omega^c}\right) \geq \varepsilon\}|,$$

where σ_n denotes the nth singular value of the corresponding operator. In [23], Kulikov and the second author establish sharp uniform bounds for this counting function in the case where one of the sets Ω and Λ is a finite union of boxes and the other is bounded with boundary of finite upper Minkowski content. As an application, necessary and sufficient conditions on the function f for the asymptotics (1.18) to hold are derived. It is here worth noting that when $d \geq 2$, it is shown in the aforementioned paper that there exist functions f such that the integral defining $U_f(0,1)$ in (1.7) converges absolutely, yet $S_f(1_\Lambda;\Omega)$ is not trace class. In fact, one may take $f(x) = \log\left(\frac{2}{x}\right)^{-\frac{3}{2}}$.

We also obtain sharp results for smooth symbols admitting area scaling (1.5) and (1.6). Compared to the results of Widom [52, 54] and Roccaforte [41], our main achievement is to extend the analysis to the finite perimeter setting and to remove the additional assumption $\check{P} \in L^1$ appearing in these papers. As observed by Widom for the case $\Omega = \mathbb{R}_+$, the fractional Sobolev space $\dot{H}^{1/2}$ exactly describes the smoothness required for area scaling.

Definition 1.12. We write $\dot{H}^{1/2}(\mathbb{R}^d;\mathbb{C}^N)$ for the space of all $P \in L^1_{loc}(\mathbb{R}^d;\mathbb{C}^N)$ such that

$$\|P\|^2_{\dot{H}^{1/2}} = \iint \frac{|P(p)-P(q)|^2}{|p-q|^{d+1}}\, dp\, dq < \infty.$$

It is well-known (see [30, Theorem 7.12]) that if $P \in L^2$, then $P \in \dot{H}^{1/2}$ if and only if $P \in H^{1/2}$, and then

$$\|P\|^2_{\dot{H}^{1/2}} = \frac{1}{\mathcal{H}^d(S^d)} \int |\check{P}(x)|^2 |x|\, dx, \tag{1.19}$$

We only consider the homogeneous case and require no a priori decay.

Theorem 1.13. *Let $P \in \dot{H}^{1/2}(\mathbb{R}^d;\mathbb{C}) \cap L^\infty$ and let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter. Then, for all $m \geq 2$,*

$$\begin{aligned}&\operatorname{Tr}\left[S_m(P;\alpha\Omega)\right] \\ &\quad = \frac{1}{8\pi^2}\left(\frac{\alpha}{2\pi}\right)^{d-1} \int_{T^*(\mathcal{F}\Omega)} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{U_m(P_X(\xi_1),P_X(\xi_2))}{|\xi_1-\xi_2|^2}\, d\xi_1\, d\xi_2\, dX + o(\alpha^{d-1}).\end{aligned} \tag{1.20}$$

as $\alpha \to \infty$. For the homogeneous case $\Omega = \mathbb{R}_\pm$ this reduces to the exact identity:

$$\operatorname{Tr}\left[S_m(P;\mathbb{R}_\pm)\right] = \frac{1}{8\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{U_m(P(\xi_1),P(\xi_2))}{|\xi_1-\xi_2|^2}\, d\xi_1\, d\xi_2. \tag{1.21}$$

Remark 1.14. We give a short and elementary proof of Widom's half-line formula (1.21) in Proposition 5.3. Theorem 1.13 is then a consequence of the following stronger result from Proposition 5.4: if $P \in \dot{H}^{1/2}(\mathbb{R}^d;\mathcal{M}_n) \cap L^\infty$ is a matrix valued symbol and $\Omega \subseteq \mathbb{R}^d$ is a set of finite perimeter, then, for all $m \geq 2$,

$$\operatorname{Tr}\left[S_m(P;\alpha\Omega)\right] = \left(\frac{\alpha}{2\pi}\right)^{d-1}\int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S_m(P_X;\mathbb{R}_+)\right] dX + o(\alpha^{d-1}). \tag{1.22}$$

Remark 1.15. For the special case of $\Omega = B$ the unit ball, the boundary coefficient from equation (1.20) simplifies as

$$\int_{T^*(\mathcal{F}B)} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{U_m(P_X(\xi_1), P_X(\xi_2))}{|\xi_1 - \xi_2|^2} \, d\xi_1 \, d\xi_2 \, dX = - \sum_{k=1}^{m-1} \left(\frac{1}{k} + \frac{1}{m-k} \right) \langle \overline{P}^k, P^{m-k} \rangle_{\dot{H}^{1/2}},$$

where $\langle \cdot, \cdot \rangle_{\dot{H}^{1/2}}$ is the inner product on $\dot{H}^{1/2}(\mathbb{R}^d; \mathbb{C})$. In particular, if P is real-valued and $m = 2$, we exactly recover the $\dot{H}^{1/2}$ norm.

Remark 1.16. Widom showed in [54, Proposition 3] that if $P \in L^\infty$ is real valued, then $S_2(P; \mathbb{R}_\pm)$ is trace class if and only if $P \in \dot{H}^{1/2}$. We argue in Proposition 3.6 that if $P \in L^2(\mathbb{R}^d; \mathbb{R}) + \mathbb{R}$ and

$$\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1}} \left| \mathrm{Tr} \left[S_2(P; \alpha B) \right] \right| < \infty,$$

then $P \in \dot{H}^{1/2}$.

For completeness we also carry out the simplest possible extension argument for general symbols P admitting area or enhanced area scaling. Aside from Theorems 1.1 and 1.13, our only a prori input is a sharp Hilbert-Schmidt estimate on $1_{\alpha\Omega} P 1_{\alpha\Omega^c}$, which is elementary to obtain. In the following, we say that the conclusion of Theorem 1.1 or Theorem 1.13 extend to a function f if, under the same assumptions on P and Ω, either

$$\begin{aligned} &\mathrm{Tr} \left[S_f(P; \alpha\Omega) \right] \\ &\quad = \frac{\alpha^{d-1} \log(\alpha)}{(2\pi)^{d+1}} \int_{\mathcal{F}\Omega} \int_{J_P} U_f(P^+(p), P^-(p)) |\nu_\Omega(x) \cdot \nu_P(p)| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x) \\ &\quad + o(\alpha^{d-1} \log(\alpha)) \end{aligned}$$

or

$$\mathrm{Tr} \left[S_f(P; \alpha\Omega) \right] = \frac{1}{8\pi^2} \left(\frac{\alpha}{2\pi} \right)^{d-1} \int_{T^*(\mathcal{F}\Omega)} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{U_f(P_X(\xi_1), P_X(\xi_2))}{|\xi_1 - \xi_2|^2} \, d\xi_1 \, d\xi_2 \, dX + o(\alpha^{d-1})$$

as $\alpha \to \infty$.

The Besov space $B^2_{\infty,1}(\mathbb{R}, \mathbb{C})$ consists of all $f \in C^{1,1}(\mathbb{R}; \mathbb{C})$ such that

$$\|f\|_{B^2_{\infty,1}} = \|f''\|_\infty + \int_{\mathbb{R}} \frac{\|\Delta_h^3 f\|_\infty}{|h|^3} \, dh, \tag{1.23}$$

where $\Delta_h f(t) = f(t+h) - f(t)$ is the difference operator. See [33]. In particular, $B^2_{\infty,1}$ contains $C_c^{2,\gamma}(\mathbb{R}; \mathbb{C})$ for all $0 < \gamma \leq 1$.

Theorem 1.17. *If P is real valued and either*

a) $f \in B^2_{\infty,1}(\mathbb{R})$

or

b) $P \in L^2(\mathbb{R}^d; \mathbb{R}) + \mathbb{R}$, $\mathcal{H}^d(\Omega) < \infty$, and $f \in C^2(\mathbb{R})$,

then the conclusions of Theorem 1.1 *and Theorem* 1.13 *extend to f. Similarly, for possibly complex valued P, the conclusions of Theorem* 1.1 *and Theorem* 1.13 *extend to f holomorphic on a neighborhood of the numerical range $W(P)$ of P.*

Remark 1.18. The real valued case relies on abstract Schatten class inequalities. The case a) uses the Koplienko–Neidhardt trace formula, while the case b) uses the Berezin-Lieb inequality. The latter requires, essentially, that $1_\Omega(P - P_\infty)$ is Hilbert-Schmidt for some $P_\infty \in \mathbb{R}$, which is ensured by the additional assumptions on P and Ω in b).

Remark 1.19. Peller [32], extending the work of Widom [54], established the one-dimensional area scaling case for $P \in \dot{H}^{1/2}(\mathbb{R}; \mathbb{R})$ and $f \in B^2_{\infty,1}$, which we recover with an additional boundedness assumption. It is worth mentioning that Peller's result was published 16 years before his extension of the Koplienko–Neidhardt trace formula to $B^2_{\infty,1}$ functions, which our argument relies on.

1.2. **Strategy of proof.** The proof of our main theorem, Theorem 1.1, consists of multiple essentially separate parts:

a) One-dimensional reduction estimate for multiscale symbols
b) Computation of boundary coefficient for multiscale symbols
c) Approximation arguments for the domain Ω and $\dot{H}^{1/2}$ and $\dot{BV}$ symbols.

The main step is a), which will be described in greater detail in Section 2. Here we consider appropriately defined smooth multiscale symbols (P_δ) which develop a jump-discontinuity in a controlled way along a prescribed hypersurface Γ in the limit $\delta \to 0^+$. The key problem is to establish (1.20) for P_δ with precise control of the δ dependence in the error term $o(\alpha^{d-1})$ in the critical regime $\alpha\delta \geq 1$. Simply following the proof of Widom [52] and Roccaforte [41] requires us to bound $\check{P}_\delta$ in L^1, which behaves poorly in the singular limit $\delta \to 0$. To overcome this obstruction, the idea is to use the multiscale assumption to localize the symbols P_δ appearing in the operator

$$S_m(P_\delta;\alpha\Omega) = 1_{\alpha\Omega}P_\delta 1_{\alpha\Omega}\dots 1_{\alpha\Omega}P_\delta 1_{\alpha\Omega} - 1_{\alpha\Omega}P_\delta^m 1_{\alpha\Omega},$$

which decomposes $S_m(P_\delta;\alpha\Omega)$ into a nicely localized part and an error term. The localization employed is technical. For the localized part, which contains the main asymptotic term, we can essentially follow the approach of Widom and Roccaforte directly with one additional non-trivial ingredient: the Roccaforte Lemma, [41, Lemma 2.5], for finite perimeter sets, which might be of independent interest. We establish this extension in Lemma $C.6$ in the Appendix.

Next for step b), we reprove Widom's exact trace formula (1.21) by an elementary argument. This is the only part of our argument that breaks down for matrix valued symbols. Assuming that $P - P_\delta = O(\delta)$ in L^1, the boundary coefficient for P_δ reduces to

$$C_d \int_{T^*(\mathcal{F}\Omega)} \iint_{\mathbb{R}\times\mathbb{R}} \frac{U_m\left(P_X(\xi_1),P_X(\xi_2)\right)}{|\xi_1-\xi_2|}\eta_\delta(\xi_1-\xi_2)\,d\xi_1\,d\xi_2\,dX, \tag{1.24}$$

for kernels $\eta_\delta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$\int_0^\infty \eta_\delta(h)\,dh = 1, \qquad \int_\varepsilon^\infty \eta_\delta(h)\,dh \to 0 \text{ as } \delta \to 0^+ \text{ for all } \varepsilon > 0.$$

We compute the leading order term as $\delta \to 0^+$ using abstract tools from the theory of BV functions in Lemma $A.11$ in the Appendix. This supplies us with an initial family of symbols P for which Theorem 1.1 holds, which in particular includes simple functions of the form

$$P(p) = \sum_{k=1}^N a_k 1_{\Lambda_k}(p), \tag{1.25}$$

for $(a_k) \subseteq \mathbb{C}$ and bounded sets Λ_k each having boundary contained in a finite union of Lipschitz graphs. The class defined by (1.25) is dense in $\dot{BV} \cap L^\infty$ in a suitable sense, and the last challenge c) is to establish appropriate forms of uniform continuity of the functional $P \to \operatorname{Tr}\left[S_m(P;\alpha\Omega)\right]$ in $\dot{BV}$, L^∞, and $\dot{H}^{1/2}$. The key here is to reduce the analysis to establishing Hilbert-Schmidt estimates of operators of the form $1_{\alpha\Omega}P1_{\alpha\Omega^c}$. Since the symbol P appears in a power-like fashion in $S_m(P;\alpha\Omega)$, it is essential that we can control pointwise products of $\dot{BV}$ and $\dot{H}^{1/2}$ functions. The relevant tools are developed in Appendix A and Appendix B.

It is at this point appropriate to point out that the outline above has very small overlap with Sobolev's proof from [44]. The main difference is that we do not use any pseudodifferential techniques. Instead we rely to a much larger extent on structural arguments, elementary volume and area bounds, and the work of Widom and Roccaforte mentioned above. For instance, with exactly one exception being Lemma 3.16, we only use Hilbert-Schmidt estimates for the operator $1_{\alpha\Omega}P1_{\alpha\Omega^c}$, which are much more elementary than the trace class bounds employed by Sobolev.

1.3. **Notation and conventions.** We finally fix basic notation used throughout the paper. For $x \in \mathbb{R}^d$ and $r > 0$ we denote by $B_r(x)$ the open ball around x with radius r. If $x = 0$ we sometimes write B_r instead. For a set $\Gamma \subseteq \mathbb{R}^d$, we write $d(p, \Gamma)$ for the distance from p to Γ. We write $|\cdot|$ for the Euclidean (L^2 based) norm on finite dimensional spaces.

We call a family (ϕ_ε) a standard family of mollifiers if $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon)$ for a radial non-negative function $\phi \in C_c^\infty$ supported in B_1 such that $\int_{\mathbb{R}^d} \phi(x)\,dx = 1$.

We use the following conventions for the Fourier transform:

$$\mathcal{F}f(p) = \widehat{f}(p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot p} f(x)\,dx$$

$$\mathcal{F}^{-1}P(x) = \check{P}(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot p} P(p)\,dp.$$

Throughout the paper C, c (with or without subscripts/superscripts) denote positive constants whose precise values are immaterial and may vary from line to line.

2. Multiscaling

As described in the proof strategy, our main technique is to first consider a family of smooth symbols (P_δ) depending on a discontinuity parameter $\delta > 0$, which develops a jump-discontinuity along a nice hypersurface Γ in the limit $\delta \to 0^+$. Since each P_δ is smooth, the operator $S_m(P_\delta; \alpha\Omega)$ satisfies an area law for each fixed $\delta > 0$. The problem is to explicitly control the δ dependence in the area law for P_δ, and in particular to recover the logarithm in a suitable joint limit $\alpha \to \infty$, $\delta \to 0^+$. Our principal motivation of this study is to prove Theorem 1.1, but the problem is also of independent interest from a mathematical and physical perspective. While it is not our intention to elaborate on this fully, in the free Fermionic setting, the parameter δ could naturally represent the temperature of the state or alternatively the spectral gap in a multi-band ground state. We point out that the latter is inherently a matrix valued problem.

We here mention that this problem was studied by Sobolev [49]. His analysis treats δ as the temperature and uses substantially different techniques: his argument relies on the discontinuous case as an input and he treats only the case of $P = 1_\Lambda$ being an indicator function under rather stringent assumptions on the convergence $P_\delta \to P$ and on the approximants P_δ. By contrast, our goal is to derive the discontinuous case itself. See also Remark 2.11 below.

Our argument centrally hinges on precise localization on varying scales ℓ_δ around Γ which we now describe. Consider a fixed compact set $\Gamma \subseteq \mathbb{R}^d$. We set

$$\ell_\delta(p) = \max\{\delta, \tfrac{1}{2}\min\{d(p,\Gamma), 1\}\}.$$

The dependence on Γ will be implicit. Clearly ℓ_δ is Lipschitz with parameter $\frac{1}{2}$, so the prescription $\|p\|_u = |p|/\ell_\delta(u)$ defines a slowly varying metric cf. [20, Definition 1.4.7], that is, for all $p, u \in \mathbb{R}^d$,

$$|p - u| \le \ell_\delta(u) \Rightarrow \frac{1}{2}\ell_\delta(u) \le \ell_\delta(p) \le \frac{3}{2}\ell_\delta(u). \tag{2.1}$$

We refer to ℓ_δ as the scale or scale function. It is perhaps well-known that slowly varying metrics give rise to effective localizations (partitions of unity), see for instance [20, Theorem 1.4.10]. We use instead a continuous parametrization.

Lemma 2.1. *There is a collection of non-negative functions* $(\phi_u) \subseteq C_c^\infty(\mathbb{R}^d)$, $u \in \mathbb{R}^d$, *such that* $(u, p) \to \phi_u(p)$ *is measurable,*

$$\operatorname{supp}\phi_u \subseteq B_{\ell_\delta(u)}(u), \quad \|D^\beta\phi_u\|_\infty \le C_\beta \ell_\delta(u)^{-|\beta|}$$

for all $\beta \in \mathbb{N}_0^d$*, and*

$$\int \phi_u(p)\ell_\delta(u)^{-d}\, du = 1$$

for all $p \in \mathbb{R}^d$*.*

Proof. See [51, Theorem 22]. They assume the scale function is C^1, but the proof easily carries to the Lipschitz case. ■

We now state our assumptions on the family (P_δ). Similar multiscale techniques are employed in various related work. We mention [28, 49].

Definition 2.2. Consider a family of symbols (P_δ), $0 < \delta \leq 1/2$, a compact set $\Gamma \subseteq \mathbb{R}^d$, and $\kappa \in \mathbb{N}$. We say that (P_δ) satisfies a $C^\kappa(\mathbb{R}^d; \mathcal{M}_n)$ multiscale estimate on Γ if the following holds: we have $P_\delta \in C^\kappa(\mathbb{R}^d; \mathcal{M}_n)$ for all $0 < \delta \leq 1/2$, and there exists a function $F : \mathbb{R}^d \to [0, \infty)$, not dependent on δ, satisfying

$$F \in L^\infty, \quad G(p) = \sup_{|p-q| \leq \ell_\delta(p)} F(q) \in L^1, \tag{2.2}$$

such that

$$|D^\beta P_\delta(p)| \leq F(p)\ell_\delta(p)^{-|\beta|}, \quad p \in \mathbb{R}^d, \quad |\beta| \leq \kappa. \tag{2.3}$$

The amplitude function F will often be implicit. If we want to explicitly mention F, we say (P_δ, F) satisfies a $C^\kappa(\mathbb{R}^d; \mathcal{M}_n)$ multiscale estimate on Γ instead.

Remark 2.3. The function G depends on δ, but all L^p norms can be estimated independently of δ. Indeed, clearly $\|G\|_\infty = \|F\|_\infty$, and

$$\int_{\mathbb{R}^d} G(p)\, dp \leq \int_{\mathbb{R}^d} \sup_{|p-q| \leq 1/2} F(q)\, dp \leq \int_{(d,\Gamma) \geq 1} G(p)\, dp + \|F\|_\infty \mathcal{H}^d(\Gamma + B_1).$$

It follows that $G \in L^1$ if and only if $\tilde{G}(p) = \sup_{|p-q| \leq 1/2} F(q) \in L^1$, and $\|G\|_1 \leq \|\tilde{G}\|_1$, which is independent of δ.

It is clear from the definition that if (P_δ, F) satisfies a C^κ multiscale estimate on Γ, then

$$|D^\beta P_\delta(p)| \leq 2^\kappa G(u)\ell_\delta(u)^{-|\beta|}, \quad |p - u| \leq \ell_\delta(u), \quad |\beta| \leq \kappa,$$

which fully captures the locally uniform nature of P_δ on the scale ℓ_δ. In general we think of (P_δ) supported on a neighborhood of Γ and F an indicator function onto a slightly larger neighborhood. With the partition of unity (ϕ_u) supplied by Lemma 2.1,

$$P(p) = \int_{\mathbb{R}^d} \ell_\delta(u)^{-d} P_u(p)\, du, \quad P_u(p) = P(p)\phi_u(p). \tag{2.4}$$

Note that the function P_u is supported on the ball $B_{\ell_\delta(u)}(u)$ with derivatives $D^\beta P_u(p) \sim G(u)\ell_\delta(u)^{-|\beta|}$. In applications, this reduces much of the analysis involving P to simple scaling and volume estimates.

Since the scale function ℓ_δ is a truncated distance function to Γ, we need precise control of local measures in a neighborhood of Γ. Here finite upper Minkowski content $\mathcal{M}^*(\Gamma) < \infty$, which is a global assumption, will not suffice.

Definition 2.4. Let $\Gamma \subseteq \mathbb{R}^d$. We say that Γ has locally uniform upper Minkowski content if

$$\mathcal{M}_u^*(\Gamma) = \sup_{z \in \mathbb{R}^d} \sup_{r > 0} \sup_{0 < \delta < 1} \frac{\mathcal{H}^d((\Gamma + B_\delta) \cap B_r(z))}{\delta r^{d-1}} < \infty.$$

If Γ is bounded and contained in a finite union of Lipschitz graphs, then $\mathcal{M}_u^*(\Gamma) < \infty$, see Lemma C.1 from the Appendix. It is also clear that the condition is stable under taking bi-Lipschitz images.

We study a slightly more general operator than $S_m(P;\alpha\Omega)$. Let $P^1,\dots,P^m \in L^\infty(\mathbb{R}^d;\mathcal{M}_n)$, say, $\Omega \subseteq \mathbb{R}^d$ a measurable set, and $m \geq 2$. We denote

$$\begin{aligned} S(P^1,\dots,P^m;\alpha\Omega) &= S(P^1;\alpha\Omega)\dots S(P^m;\alpha\Omega) - S(P^1\dots P^m;\alpha\Omega) \\ &= 1_{\alpha\Omega}P^1 1_{\alpha\Omega}\dots 1_{\alpha\Omega}P^m 1_{\alpha\Omega} - 1_{\alpha\Omega}P^1\dots P^m 1_{\alpha\Omega}. \end{aligned}$$

Of course, if $P^1 = \dots = P^m$, then $S_m(P;\alpha\Omega) = S(P^1,\dots,P^m;\alpha\Omega)$.

Our main results on multiscale symbols are contained in the next two theorems.

Theorem 2.5. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter, let $\Gamma \subseteq \mathbb{R}^d$ be a set of locally uniform upper Minkowski content, and $m \geq 2$ an integer. Suppose (P_δ^k) satisfies a $C^1(\mathbb{R}^d;\mathcal{M}_n)$ multiscale assumption on Γ for $k = 1,\dots,m$. Then*

$$\begin{aligned} \operatorname{Tr}\left[S(P_\delta^1,\dots,P_\delta^m;\alpha\Omega)\right] &= \left(\frac{\alpha}{2\pi}\right)^{d-1}\int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S((P_\delta^1)_X,\dots,(P_\delta^m)_X;\mathbb{R}_+)\right] dX \\ &\quad + o(\alpha^{d-1}\log(\delta^{-1})), \end{aligned} \tag{2.5}$$

as $\alpha \to \infty$ uniformly for $\alpha\delta \geq 1$.

Remark 2.6. We first prove the Theorem under stronger regularity assumptions on Ω and (P_δ^k), and in this case we obtain a purely quantitative bound, see Proposition 4.1. Theorem 2.5 follows from approximation arguments, so in this case, the error term is not very explicit.

Remark 2.7. It follows from Corollary 3.9 and Lemma D.1 that the boundary coefficient scales correctly in δ:

$$\int_{\mathcal{F}\Omega} \|S((P_\delta^1)_X,\dots,(P_\delta^m)_X;\mathbb{R}_+)\|_{\mathcal{S}_1}\, dX \lesssim \log(\delta^{-1}).$$

Surprisingly, we only use this fact to carry out the approximation argument to deduce Theorem 2.5 from Proposition 4.1.

Theorem 2.8. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter, let $\Gamma \subseteq \mathbb{R}^d$ be a set of locally uniform upper Minkowski content, and $m \geq 2$ an integer. Assume that (P_δ) satisfies a $C^1(\mathbb{R}^d;\mathbb{C})$ multiscale estimate on Γ and that there exists a function $P_0 \in \dot{BV}$ such that $\|P_\delta - P_0\|_1 = o\big(\delta\log(\delta^{-1})\big)$ as $\delta \to 0^+$. Then*

$$\begin{aligned} &\operatorname{Tr}\left[S_m(P_\delta;\alpha\Omega)\right] \\ &\quad = \frac{\alpha^{d-1}\log(\delta^{-1})}{(2\pi)^{d+1}}\int_{\mathcal{F}\Omega}\int_{J_{P_0}} U_m(P_0^+(p),P_0^-(p))|\nu_\Omega(x)\cdot\nu_{P_0}(p)|\, d\mathcal{H}^{d-1}(p)\, d\mathcal{H}^{d-1}(x) \\ &\quad\quad + o(\alpha^{d-1}\log(\delta^{-1})) \end{aligned}$$

as $\delta \to 0^+$ uniformly for $\alpha\delta \geq 1$.

For the case $\alpha\delta \leq 1$, assume in addition that $\min\{\mathcal{H}^1(\Omega),\mathcal{H}^1(\Omega^c)\} < \infty$ *if* $d = 1$. *Then*

$$\begin{aligned} &\operatorname{Tr}\left[S_m(P_\delta;\alpha\Omega)\right] \\ &\quad = \frac{\alpha^{d-1}\log(\alpha)}{(2\pi)^{d+1}}\int_{\mathcal{F}\Omega}\int_{J_{P_0}} U_m(P_0^+(p),P_0^-(p))|\nu_\Omega(x)\cdot\nu_{P_0}(p)|\, d\mathcal{H}^{d-1}(p)\, d\mathcal{H}^{d-1}(x) \\ &\quad\quad + o(\alpha^{d-1}\log(\alpha)) \end{aligned}$$

as $\alpha \to \infty$ uniformly for $\alpha\delta \leq 1$, including the case $\delta = 0$.

Remark 2.9. The quantitative assumption $\|P_\delta - P_0\|_1 = o\big(\delta\log(\delta^{-1})\big)$ is easy to verify in practice but perhaps not intuitive. It arises, essentially, from the condition

$$\limsup_{\delta\to 0^+}\frac{1}{\log(\delta^{-1})}\iint_{\delta\leq|p-q|\leq 1}\frac{|U_m(P_\delta(p),P_\delta(q)) - U_m(P_0(p),P_0(q))|}{|p-q|^{d+1}}\, dp\, dq = 0,$$

which allows us to replace P_δ by P_0 in the co-tangent integrals from Theorem 2.5 (using Theorem 1.13). The resulting expression, which only depends on P_0, is exactly of the form (1.24).

Remark 2.10. The case $\alpha\delta \le 1$ with $\delta = 0$ supplies us with an initial family (P_0) of $\dot{BV}$ symbols for which the conclusion of Theorem 1.1 holds. It is simple to verify that if P_0 is of the form (1.25) and (ϕ_δ) is a standard family of mollifiers, then $P_\delta = P_0 * \phi_\delta$ satisfies all assumptions of Theorem 2.8. In fact, this will be our only application.

Remark 2.11. For the case where $P_0 = 1_\Lambda$ is an indicator function, Theorem 2.8 yields the same transition asymptotics as Sobolev's main results from [49]. Sobolev treats symbols $P_\delta \in C^\infty(\mathbb{R}^d)$ approximating P_0, and allows spectral functions with Hölder-type singularities. By contrast, here we restrict attention to polynomial spectral functions, but allow much more general limiting symbols P_0, require only a C^1 multiscale assumption on (P_δ), and impose substantially weaker assumptions on the convergence $P_\delta \to P_0$.

3. HILBERT-SCHMIDT AND TRACE CLASS ESTIMATES

In this section we establish trace and trace class bounds for the operators $S_m(P;\Omega)$, and, more generally, operators of the form

$$S(P_1,\dots,P_m;\Omega) = 1_\Omega P_1 1_\Omega \dots 1_\Omega P_m 1_\Omega - 1_\Omega P_1 \dots P_m 1_\Omega \tag{3.1}$$

for symbols $P_1,\dots,P_m \in L^\infty(\mathbb{R}^d;\mathcal{M}_n)$, $m \ge 2$, and a measurable set $\Omega \subseteq \mathbb{R}^d$. Most results are abstract in nature and are formulated in terms of well-known function spaces. At the end of the section we establish more specialized estimates to be used in the proof of Theorem 2.5.

3.1. **Basic Hilbert-Schmidt bounds.**

Here we establish basic Hilbert-Schmidt bounds of operators of the form $1_\Omega P 1_{\Omega^c}$ and $1_{\Omega^c} P 1_\Omega$. By the simple identity

$$S_2(P;\Omega) = -1_\Omega P 1_{\Omega^c} P 1_\Omega,$$

this immediately implies (or is equivalent to) trace class bounds of $S_2(P;\Omega)$. We study three cases: area scaling, volume scaling, and enhanced area scaling corresponding to $\dot H^{1/2}$, L^2, and $\dot{BV}$, respectively. These bounds will serve as the foundation for all later results. For the $\dot H^{1/2}$ and $\dot{BV}$ results, we compute the exact asymptotics in Theorem 1.13 and Theorem 1.1, respectively, and we prove a partial converse in Proposition 3.6.

We refer the readers to Appendices A, B, and C for relevant background on $\dot{BV}$, $\dot H^{1/2}$, and sets of finite perimeter.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter and let $\Omega' \subseteq \Omega^c$. Then, for $P \in \dot H^{1/2}(\mathbb{R}^d;\mathcal{M}_n) \cap L^\infty$,*

$$\|1_\Omega P 1_{\Omega'}\|_{\mathcal{S}_2}^2 + \|1_{\Omega'} P 1_\Omega\|_{\mathcal{S}_2}^2 \le C_d \operatorname{Per}(\Omega)\|P\|_{\dot H^{1/2}}^2,$$

and,

$$\int_{\Pi_\nu} \|1_{\mathbb{R}_+} P_{\nu,p} 1_{\mathbb{R}_-}\|_{\mathcal{S}_2}^2 + \|1_{\mathbb{R}_-} P_{\nu,p} 1_{\mathbb{R}_+}\|_{\mathcal{S}_2}^2 \, dp \le C_d \|P\|_{\dot H^{1/2}}^2.$$

uniformly for $\nu \in S^{d-1}$. Here $P_{\nu,p}(\xi) = P(p + \xi\nu)$, $\xi \in \mathbb{R}$.

Proof. Consider first $P \in C_c^\infty$. It is easy to see that the operator $1_\Omega P 1_{\Omega'}$ has kernel $(2\pi)^{-d/2} 1_\Omega(x) \check P(x-y) 1_{\Omega'}(y)$, hence

$$\|1_\Omega P 1_{\Omega'}\|_{\mathcal{S}_2}^2 = (2\pi)^{-d} \iint |\check P(x-y)|^2 1_\Omega(x) 1_{\Omega'}(y)\,dx\,dy = (2\pi)^{-d} \int |\check P(x)|^2 \mathcal{H}^d(\Omega \cap (\Omega' + x))\,dx.$$

Note that $\mathcal{H}^d(\Omega \cap (\Omega' + x)) = 1_\Omega * 1_{-\Omega'}(x)$. We show in Lemma C.3 by a simple argument that $\mathcal{H}^d(\Omega \cap (\Omega' + x)) \le \operatorname{Per}(\Omega)|x|$. Hence, by the Fourier description of $\dot H^{1/2}$ in (1.19),

$$\|1_\Omega P 1_{\Omega'}\|_{\mathcal{S}_2}^2 \le (2\pi)^{-d} \operatorname{Per}(\Omega) \int |\check P(x)|^2 |x|\,dx = \frac{\operatorname{Per}(\Omega)}{\mathcal{H}^d(S^d)(2\pi)^d} \|P\|_{\dot H^{1/2}}^2.$$

The same bound obviously extends to $P \in C_c^\infty + \mathcal{M}_n$ by homogeneity. For general $P \in \dot{H}^{1/2} \cap L^\infty$ it follows from Lemma $B.2$ that there is a sequence $(P_n) \subseteq C_c^\infty + \mathcal{M}_n$ such that $P_n \to P$ in $\dot{H}^{1/2}$ and pointwise almost everywhere, and $\|P_n\|_\infty \leq 3\|P\|_\infty$. In particular $1_\Omega P_n 1_{\Omega'} \to 1_\Omega P 1_{\Omega'}$ strongly, so by Fatou's lemma

$$\|1_\Omega P 1_{\Omega'}\|_{\mathcal{S}_2}^2 \leq \liminf_{n\to\infty} \|1_\Omega P_n 1_{\Omega'}\|_{\mathcal{S}_2}^2 = \liminf_{n\to\infty} \frac{\mathrm{Per}(\Omega)}{\mathcal{H}^d(S^d)(2\pi)^d}\|P_n\|_{\dot{H}^{1/2}}^2 = \frac{\mathrm{Per}(\Omega)}{\mathcal{H}^d(S^d)(2\pi)^d}\|P\|_{\dot{H}^{1/2}}^2.$$

The argument for $1_{\Omega'} P 1_\Omega$ is identical.

The statement on one-dimensional sections follows immediately from the case $\Omega = [0,\infty)$ and Lemma $B.5$. ■

Lemma 3.2. *Let $\Omega, \Omega' \subseteq \mathbb{R}^d$ satisfy $\min\{\mathcal{H}^d(\Omega), \mathcal{H}^d(\Omega')\} < \infty$. Then, for $P \in L^2(\mathbb{R}^d; \mathcal{M}_n)$,*

$$\|1_\Omega P 1_{\Omega'}\|_{\mathcal{S}_2}^2 + \|1_{\Omega'} P 1_\Omega\|_{\mathcal{S}_2}^2 \leq C_d \min\{\mathcal{H}^d(\Omega), \mathcal{H}^d(\Omega')\}\|P\|_2^2.$$

Proof. Arguing like in the proof of Lemma 3.1 we find

$$\begin{aligned}\|1_\Omega P 1_{\Omega'}\|_{\mathcal{S}_2}^2 &= (2\pi)^{-d}\int |\check{P}(x)|^2 \mathcal{H}^d(\Omega \cap (\Omega' + x))\,dx \\ &\leq (2\pi)^{-d}\min\{\mathcal{H}^d(\Omega), \mathcal{H}^d(\Omega')\}\|P\|_2^2.\end{aligned}$$

The argument for $\|1_{\Omega'} P 1_\Omega\|_{\mathcal{S}_2}^2$ is identical. ■

Lemma 3.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite measure. Then, for all $P \in L^1(\mathbb{R}^d; \mathcal{M}_n)$,*

$$\|1_\Omega P 1_\Omega\|_{\mathcal{S}_1} \leq C_d \mathcal{H}^d(\Omega)\|P\|_1.$$

Proof. Let $P = U|P|$ be the polar decomposition. We see from Lemma 3.2 and the Hölder inequality that

$$\|1_\Omega P 1_\Omega\|_{\mathcal{S}_1} \leq \|1_\Omega U |P|^{1/2}\|_{\mathcal{S}_2}\||P|^{1/2} 1_\Omega\|_{\mathcal{S}_2} \leq C_d \mathcal{H}^d(\Omega)\|P\|_1.$$

■

We now show how to interpolate between the area bound from Lemma 3.1 and the volume bound from Lemma 3.2 to achieve bounds for $\dot{BV}$ symbols with enhanced area scaling.

Lemma 3.4. *Let (ϕ_δ) be a family of standard mollifiers and let $P \in \dot{BV}(\mathbb{R}^d; \mathcal{M}_n) \cap L^\infty$. Assume in addition that $P \in L^2 + \mathcal{M}_n$ for $d = 1$. Then, uniformly for $0 < \delta \leq 1/2$,*

$$\|P * \phi_\delta\|_{\dot{H}^{1/2}}^2 \leq C_{\phi,d}\big(\log(\delta^{-1})\|P\|_{\dot{BV}}\|P\|_\infty + \inf_{P_\infty \in \mathcal{M}_n}\|P - P_\infty\|_2^2\big).$$

Proof. Note we have $P \in L^2 + \mathcal{M}_n$ also when $d \geq 2$ by the embedding theorem [1, Theorem 3.47], so we might as well assume $P \in L^2$. Assume that the family (ϕ_δ) is generated by $\phi \in C_c^\infty$, that is $\phi_\delta(h) = \delta^{-d}\phi(h/\delta)$. Then, by the Fourier description of $\dot{H}^{1/2}$,

$$\begin{aligned}\|P * \phi_\delta\|_{\dot{H}^{1/2}}^2 &\leq C_d \int |\check{P}(x)|^2 |\check{\phi_\delta}(x)|^2 |x|\,dx = C_d \int |\check{P}(x)|^2 |\check{\phi}(\delta x)|^2 |x|\,dx \\ &\leq C_d \int_{|x|\leq 1} |\check{P}(x)|^2 + C_d \int_{1\leq|x|\leq 1/\delta} |\check{P}(x)|^2 |x|\,dx \\ &\quad + C_{\phi,d}\delta^{-1}\int_{|x|\geq 1/\delta} |\check{P}(x)|^2\,dx.\end{aligned}$$

We simply bounded $|\check{\phi}(\delta x)| \leq (2\pi)^{-d/2}$ when $|x| \leq 1/\delta$ and $|\check{\phi}(\delta x)| \leq \frac{(2\pi)^{-d/2}}{\delta|x|}\|\nabla\phi\|_1$ when $|x| \geq 1/\delta$. We show in Lemma $A.8$ by an elementary argument that

$$\int_{1\leq|x|\leq 1/\delta} |\check{P}(x)|^2 |x| \leq C_d \log(\delta^{-1})\|P\|_{\dot{BV}}\|P\|_\infty,$$

and

$$\delta^{-1}\int_{|x|\geq 1/\delta} |\check{P}(x)|^2\,dx \leq C_d \|P\|_{\dot{BV}}\|P\|_\infty.$$

The obvious bound $\int_{|x|\leq 1}|\check{P}(x)|^2 \leq \|P\|_2^2$ finishes the proof. ■

Lemma 3.5. *Let* $\Omega \subseteq \mathbb{R}^d$ *be a set of finite perimeter and take* $\Omega' \subseteq \Omega^c$. *Consider* $P \in \dot{BV}(\mathbb{R}^d; \mathcal{M}_n) \cap L^\infty$. *For* $d = 1$ *we assume in addition that* $P \in L^2 + \mathcal{M}_n$ *and that* $\min\{\mathcal{H}^d(\Omega), \mathcal{H}^d(\Omega')\} < \infty$. *Then, uniformly for* $\alpha \geq 2$,

$$\begin{aligned}
&\|1_{\alpha\Omega} P 1_{\alpha\Omega'}\|_{\mathcal{S}_2}^2 + \|1_{\alpha\Omega'} P 1_{\alpha\Omega}\|_{\mathcal{S}_2}^2 \\
&\qquad \leq C_d \operatorname{Per}(\Omega)\alpha^{d-1}\log(\alpha)\|P\|_{\dot{BV}}\|P\|_\infty \\
&\qquad + C_d\alpha^{d-1}\big(\operatorname{Per}(\Omega)\inf_{P_\infty\in\mathcal{M}_n}\|P-P_\infty\|_2^2 + \min\{\mathcal{H}^d(\Omega),\mathcal{H}^d(\Omega')\}\|P\|_{\dot{BV}}\|P\|_\infty\big).
\end{aligned}$$

Proof. Let (ϕ_δ) be a family of standard mollifiers and set $P_\delta = P * \phi_\delta$. It follows from Lemmas 3.1 and 3.4 that

$$\begin{aligned}
\|1_{\alpha\Omega} P_{1/\alpha} 1_{\alpha\Omega'}\|_{\mathcal{S}_2}^2 &\leq C_d \operatorname{Per}(\Omega)\alpha^{d-1}\|P_{1/\alpha}\|_{\dot{H}^{1/2}}^2 \\
&\leq C_d \operatorname{Per}(\Omega)\alpha^{d-1}\big(\log(\alpha)\|P\|_{\dot{BV}}\|P\|_\infty + \inf_{P_\infty\in\mathcal{M}_n}\|P-P_\infty\|_2^2\big).
\end{aligned}$$

On the other hand, by Lemma 3.2,

$$\|1_{\alpha\Omega}(P-P_{1/\alpha})1_{\alpha\Omega'}\|_{\mathcal{S}_2}^2 \leq C_d\alpha^d \min\{\mathcal{H}^d(\Omega),\mathcal{H}^d(\Omega')\}\|P-P_{1/\alpha}\|_\infty\|P-P_{1/\alpha}\|_1.$$

We trivially bound $\|P-P_{1/\alpha}\|_\infty \leq 2\|P\|_\infty$. By the finite difference characterization of $\dot{BV}$ from Lemma $A.7$,

$$\|P-P_{1/\alpha}\|_1 \leq \int \phi_{1/\alpha}(h)\int|P(p+h)-P(p)|\,dp\,dh \leq \|P\|_{\dot{BV}}\int\phi_{1/\alpha}(h)h\,dh \leq C\,\alpha^{-1}\|P\|_{\dot{BV}}.$$

Conclude

$$\begin{aligned}
\|1_{\alpha\Omega}P1_{\alpha\Omega'}\|_{\mathcal{S}_2}^2 &\leq 2\|1_{\alpha\Omega}(P-P_{1/\alpha})1_{\alpha\Omega'}\|_{\mathcal{S}_2}^2 + 2\|1_{\alpha\Omega}P_{1/\alpha}1_{\alpha\Omega'}\|_{\mathcal{S}_2}^2 \\
&\leq C_d \operatorname{Per}(\Omega)\alpha^{d-1}\log(\alpha)\|P\|_{\dot{BV}}\|P\|_\infty \\
&\quad + C_d\alpha^{d-1}\big(\operatorname{Per}(\Omega)\inf_{P_\infty\in\mathcal{M}_n}\|P-P_\infty\|_2^2 + \min\{\mathcal{H}^d(\Omega),\mathcal{H}^d(\Omega')\}\|P\|_{\dot{BV}}\|P\|_\infty\big).
\end{aligned}$$

The argument for $\|1_{\alpha\Omega'}P1_{\alpha\Omega}\|_{\mathcal{S}_2}^2$ is identical.

■

We now present partial converses to Lemma 3.1 and Lemma 3.5 for the specific case of Ω a ball. The proof for the BV case is based on the non-local characterization of bounded variation functions from [6].

Proposition 3.6. *Write* $B = B_1 \subseteq \mathbb{R}^d$ *for the unit ball centered at zero. If* $P \in L^2(\mathbb{R}^d;\mathcal{M}_n) + \mathcal{M}_n$ *and*

$$\|1_{\alpha B}P1_{\alpha B^c}\|_{\mathcal{S}_2}^2 = O(\alpha^{d-1})$$

as $\alpha\to\infty$, *then* $P \in \dot{H}^{1/2}$. *Similarly, if* $\Lambda \subseteq \mathbb{R}^d$ *satisfies* $\min\{\mathcal{H}^d(\Lambda),\mathcal{H}^d(\Lambda^c)\} < \infty$ *and*

$$\|1_{\alpha B}1_\Lambda 1_{\alpha B^c}\|_{\mathcal{S}_2}^2 = O(\alpha^{d-1}\log(\alpha))$$

as $\alpha\to\infty$, *then* Λ *is a set of finite perimeter.*

Proof. We start with the $\dot{H}^{1/2}$ claim. We may assume that $P \in L^2$. Arguing like in Lemma 3.1 we see that

$$\|1_{\alpha B}P1_{\alpha B^c}\|_{\mathcal{S}_2}^2 = (2\pi)^{-d}\int|\check{P}(x)|^2\mathcal{H}^d(\alpha B\cap(\alpha B^c + x))\,dx.$$

It is elementary to verify that

$$\mathcal{H}^d(\alpha B\cap(\alpha B^c+x)) = \alpha^d\mathcal{H}^d(B\cap(B^c+x/\alpha)) \geq c_d\alpha^{d-1}|x| \tag{3.2}$$

for $|x|/\alpha \leq \varepsilon$, with $\varepsilon > 0$ small enough. In fact this follows from Lemma $C.6$ although this reference is overkill. We conclude

$$\begin{aligned}\infty > \limsup_{\alpha\to\infty} \frac{1}{\alpha^{d-1}} \int |\check{P}(x)|^2 \mathcal{H}^d(\alpha B \cap (\alpha B^c + x))\, dx \\ \geq c_1 \limsup_{\alpha\to\infty} \int_{|x|\leq \alpha\varepsilon} |\check{P}(x)|^2 |x|\, dx = c_2 \|P\|^2_{\dot{H}^{1/2}},\end{aligned}$$

giving the claim.

The case of indicator functions is more complicated. We may assume that $\mathcal{H}^d(\Lambda) < \infty$. Note first that by elementary manipulations and the Plancherel theorem

$$\begin{aligned}\|1_{\alpha B} 1_\Lambda 1_{\alpha B^c}\|^2_{\mathcal{S}_2} &= (2\pi)^{-d} \int |\check{1}_\Lambda(x)|^2 \mathcal{H}^d(\alpha B \cap (\alpha B^c + x))\, dx \\ &= \frac{1}{2}(2\pi)^{-d} \int |\widehat{1}_{\alpha B}(h)|^2 \int |1_\Lambda(p+h) - 1_\Lambda(p)|^2\, dp\, dh.\end{aligned}$$

The analogous identity holds for general symbols $P \in L^2$. Introduce the kernels

$$\eta_\alpha(h) = \frac{1}{\alpha^{d-1}\log(\alpha)} \frac{1}{C_\alpha} 1_{\{|h|\leq 1\}}(h) |\widehat{1}_{\alpha B}(h)|^2 |h| \quad C_\alpha = \frac{1}{\log(\alpha)} \int_{|h|\leq\alpha} |\widehat{1}_B(h)|^2 |h|\, dh.$$

We see that η_α is non-negative, radial, and

$$\int \eta_\alpha(h)\, dh = 1, \qquad \int_{|h|\geq\delta} \eta_\alpha(h)\, dh \to 0 \text{ as } \alpha\to\infty \text{ for all } \delta > 0,$$

and by construction

$$\frac{1}{\alpha^{d-1}\log(\alpha)} \|1_{\alpha B} 1_\Lambda 1_{\alpha B^c}\|^2_{\mathcal{S}_2} \geq \frac{1}{2}(2\pi)^{-d} C_\alpha \int \eta_\alpha(h) \frac{1}{|h|} \int |1_\Lambda(p+h) - 1_\Lambda(p)|\, dp\, dh.$$

Note that the right hand side is of the form (1.24) mentioned in the Introduction.

To proceed we need to use the fact that $\liminf_{\alpha\to\infty} C_\alpha > 0$. This can be seen from the leading order behavior of the Bessel function $J_{d/2}(r)$ as $r\to\infty$. Alternatively, by Lemma $A.8$ and Lemma $C.4$,

$$\begin{aligned}\frac{1}{\alpha^{d-1}\log(\alpha)} |\mathrm{Tr}\left[S_2(1_B; \alpha B)\right]| &= \frac{(2\pi)^{-d}}{\alpha^{d-1}\log(\alpha)} \int |\widehat{1}_B(h)|^2 \mathcal{H}^d(\alpha B \cap (\alpha B^c + h))\, dh \\ &\leq \frac{\mathrm{Per}(B)}{(2\pi)^d} C_\alpha + O\left(\frac{1}{\log(\alpha)}\right),\end{aligned}$$

so $\liminf_{\alpha\to\infty} C_\alpha > 0$ follows from Theorem 1.1, which is proved independently of this result, but it is of course overkill. We finally mention that it is possible to compute $\lim_{\alpha\to\infty} C_\alpha$ directly using Lemma $C.6$ and Lemma $A.11$. Taking the lower bound for granted we see that

$$\infty > \limsup_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log(\alpha)} \|1_{\alpha B} 1_\Lambda 1_{\alpha B^c}\|^2_{\mathcal{S}_2} \geq c \liminf_{\alpha\to\infty} \int \eta_\alpha(h) \frac{1}{|h|} \int |1_\Lambda(p+h) - 1_\Lambda(p)|\, dp\, dh.$$

The right hand side is exactly a BBM-type expression for the variation of 1_Λ. It is shown in [6, Theorem 3'] that, for η_α satisfying the properties stated above,

$$\liminf_{\alpha\to\infty} \int_\Omega \int_\Omega \frac{|1_\Lambda(p) - 1_\Lambda(q)|}{|p-q|} \eta_\alpha(p-q)\, dp\, dq < \infty$$

if and only if 1_Λ has finite variation in Ω, when Ω is a bounded smooth domain. Applying this with $\Omega = B_R$ for all $R > 0$ we see that Λ is a set of locally finite perimeter. Moreover, the variation bounds from [6, Theorem 3'] shows that

$$\begin{aligned}|D1_\Lambda|(B_R) &\leq C \liminf_{\alpha\to\infty} \int_{B_R} \int_{B_R} \frac{|1_\Lambda(p) - 1_\Lambda(q)|}{|p-q|} \eta_\alpha(p-q)\, dp\, dq \\ &\leq C \liminf_{\alpha\to\infty} \int \eta_\alpha(h) \frac{1}{|h|} \int |1_\Lambda(p+h) - 1_\Lambda(p)|\, dp\, dh < \infty\end{aligned}$$

Taking $R \to \infty$ we conclude that $|D1_\Lambda|(\mathbb{R}^d) < \infty$ or equivalently $\mathrm{Per}(\Lambda) < \infty$, finishing the proof. ■

Remark 3.7. The only special property of indicator functions used in the second part of the proof is the lower bound $|P(p) - P(q)|^2 \geq c|P(p) - P(q)|$. This property also holds for simple functions, so the proof carries over to this setting. Precisely, if $P : \mathbb{R}^d \to \mathbb{R}$ only takes finitely many values, $P \in L^2(\mathbb{R}^d;\mathbb{R}) + \mathbb{R}$, and

$$\limsup_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log(\alpha)} \|1_{\alpha B} P 1_{\alpha B^c}\|_{\mathcal{S}_2}^2 < \infty,$$

then $P \in \dot{BV}$.

3.2. **Trace class bounds for S_m.** We now show how to estimate $S_m(P;\Omega)$, or more generally $S(P_1,\dots,P_m;\Omega)$, in trace class for general $m \geq 2$. We use the product structure to reduce to the already established case of $m = 2$. The following essentially contains the entire idea.

Lemma 3.8. *Let $P_1,\dots,P_m \in L^\infty(\mathbb{R}^d;\mathcal{M}_n)$ and let $\Omega \subseteq \mathbb{R}^d$ be a measurable set. Then, for $1 \leq n \leq m$,*

$$\begin{aligned}
&\|S(P_1,\dots,P_m;\Omega)\|_{\mathcal{S}_1} \\
&\qquad \leq \sum_{k=1}^{n-1} \frac{\|P_1\|_\infty \cdots \|P_m\|_\infty}{\|P_k\|_\infty \cdots \|P_m\|_\infty} \|S(P_k, P_{k+1}\dots P_m;\Omega)\|_{\mathcal{S}_1} \\
&\qquad\quad + \sum_{k=n}^{m-1} \frac{\|P_1\|_\infty \cdots \|P_m\|_\infty}{\|P_n\|_\infty \cdots \|P_{k+1}\|_\infty} \|S(P_n\dots P_k, P_{k+1};\Omega)\|_{\mathcal{S}_1},
\end{aligned} \tag{3.3}$$

where the empty sum is zero. Moreover,

$$\|S(P_1,\dots,P_m;\Omega)\|_{\mathcal{S}_1} \leq \sum_{1\leq i<j\leq m} \frac{\|P_1\|_\infty \cdots \|P_m\|_\infty}{\|P_i\|_\infty \|P_j\|_\infty} \|1_\Omega P_i 1_{\Omega^c}\|_{\mathcal{S}_2} \|1_{\Omega^c} P_j 1_\Omega\|_{\mathcal{S}_2}. \tag{3.4}$$

Proof. Repeated use of the identities

$$\begin{aligned}
S(P_1,\dots,P_m;\Omega) &= S(P_1,\dots,P_{m-1};\Omega)1_\Omega P_m 1_\Omega + S(P_1\dots P_{m-1}, P_m;\Omega) \\
S(P_1,\dots,P_m;\Omega) &= 1_\Omega P_1 1_\Omega S(P_2,\dots,P_m;\Omega) + S(P_1, P_2\dots P_m;\Omega),
\end{aligned} \tag{3.5}$$

and the elementary bound $\|1_\Omega P_i 1_\Omega\| \leq \|P_i\|_\infty$ readily leads to (3.3) for $n = 1$ and $n = m$. For $1 < n < m$ similarly expand

$$S(P_1,\dots,P_m;\Omega) = 1_\Omega P_1 1_\Omega \dots 1_\Omega P_{n-1} 1_\Omega S(P_n,\dots,P_m;\Omega) + S(P_1,\dots,P_{n-1}, P_n\dots P_m;\Omega),$$

and apply the established extreme cases (with different m) suitably to see (3.3). The point is that P_n always appear inside $S(\cdot,\cdot;\Omega)$.

For the second claim we apply (3.3) for the case $n = 1$. We find

$$\|S(P_1,\dots,P_m;\Omega)\|_{\mathcal{S}_1} \leq \sum_{k=1}^{m-1} \frac{\|P_1\|_\infty \cdots \|P_m\|_\infty}{\|P_1\|_\infty \cdots \|P_{k+1}\|_\infty} \|S(P_1\dots P_k, P_{k+1};\Omega)\|_{\mathcal{S}_1}.$$

By the $m = 2$ estimate

$$\|S(P_1\dots P_k, P_{k+1};\Omega)\|_{\mathcal{S}_1} \leq \|1_\Omega P_1 \dots P_k 1_{\Omega^c}\|_{\mathcal{S}_2} \|1_{\Omega^c} P_{k+1} 1_\Omega\|_{\mathcal{S}_2},$$

we see that it suffices to bound, for all $1 \leq k \leq m-1$,

$$\|1_\Omega P_1 \dots P_k 1_{\Omega^c}\|_{\mathcal{S}_2} \leq \sum_{1\leq i\leq k} \frac{\|P_1\|_\infty \cdots \|P_k\|_\infty}{\|P_i\|_\infty} \|1_\Omega P_i 1_{\Omega^c}\|_{\mathcal{S}_2}.$$

We argue by induction in k with the case $k = 1$ being trivial. Suppose the claim holds for $1 \leq k \leq m-2$. Then, simply writing $1 = 1_\Omega + 1_{\Omega^c}$ between P_k and P_{k+1},

$$\|1_\Omega P_1 \dots P_{k+1} 1_{\Omega^c}\|_{\mathcal{S}_2} \leq \|P_{k+1}\|_\infty \|1_\Omega P_1 \dots P_k 1_{\Omega^c}\|_{\mathcal{S}_2} + \|P_1 \dots P_k\|_\infty \|1_\Omega P_{k+1} 1_{\Omega^c}\|_{\mathcal{S}_2}.$$

This has the correct form by hypothesis. ■

The following is immediate from Lemma 3.8 using Lemmas 3.1, 3.2, and 3.5.

Corollary 3.9. *Let* $P_1, \dots, P_m \in L^\infty(\mathbb{R}^d; \mathcal{M}_n)$ *and let* $\Omega \subseteq \mathbb{R}^d$ *be a measurable set.*

a) If $P_1, \dots, P_m \in \dot{H}^{1/2}$ *and* Ω *is a set of finite perimeter, then*

$$\|S(P_1, \dots, P_m; \Omega)\|_{\mathcal{S}_1} \leq C_d \operatorname{Per}(\Omega) \sum_{1 \leq i < j \leq m} \frac{\|P_1\|_\infty \dots \|P_m\|_\infty}{\|P_i\|_\infty \|P_j\|_\infty} \|P_i\|_{\dot{H}^{1/2}} \|P_j\|_{\dot{H}^{1/2}}.$$

b) If $\min\{\mathcal{H}^d(\Omega), \mathcal{H}^d(\Omega^c)\} < \infty$ *and there exist* $C_1, \dots, C_m \in \mathcal{M}_n$ *such that* $P_i - C_i \in L^2$ *for* $i = 1, \dots, m$*, then*

$$\begin{aligned}&\|S(P_1, \dots, P_m; \Omega)\|_{\mathcal{S}_1} \\ &\quad \leq C_d \min\{\mathcal{H}^d(\Omega), \mathcal{H}^d(\Omega^c)\} \sum_{1 \leq i < j \leq m} \frac{\|P_1\|_\infty \dots \|P_m\|_\infty}{\|P_i\|_\infty \|P_j\|_\infty} \|P_i - C_i\|_2 \|P_j - C_j\|_2.\end{aligned}$$

c) If $P_1, \dots, P_m \in \dot{BV}$ *and* Ω *is a set of finite perimeter, and in addition* $P_1, \dots, P_m \in L^2 + \mathcal{M}_n$ *and* $\min\{\mathcal{H}^1(\Omega), \mathcal{H}^1(\Omega^c)\} < \infty$ *if* $d = 1$*, then*

$$\begin{aligned}&\|S(P_1, \dots, P_m; \alpha\Omega)\|_{\mathcal{S}_1} \\ &\quad \leq C_d \operatorname{Per}(\Omega) \alpha^{d-1} \log(\alpha) \sum_{1 \leq i < j \leq m} \frac{\|P_1\|_\infty \dots \|P_m\|_\infty}{\|P_i\|_\infty \|P_j\|_\infty} \left(\|P_i\|_\infty \|P_i\|_{\dot{BV}} \|P_j\|_\infty \|P_j\|_{\dot{BV}}\right)^{1/2} \\ &\qquad + o(\alpha^{d-1}),\end{aligned}$$

as $\alpha \to \infty$*.*

We also need a L^1-type bound similar to Lemma 3.3. For our proof it is essential that we take trace and not trace norm when $m > 2$.

Lemma 3.10. *Let* $\Omega \subseteq \mathbb{R}^d$ *be a set satisfying* $\min\{\mathcal{H}^d(\Omega), \mathcal{H}^d(\Omega^c)\} < \infty$ *and let* $P_1, \dots, P_m \in L^2(\mathbb{R}^d; \mathcal{M}_n) \cap L^\infty$*,* $m \geq 2$*. Then, for every* $1 \leq k \leq m$*,*

$$\left|\operatorname{Tr}\left[S(P_1, \dots, P_m; \Omega)\right]\right| \leq C_{d,m} \min\{\mathcal{H}^d(\Omega), \mathcal{H}^d(\Omega^c)\} \frac{\|P_1\|_\infty \dots \|P_m\|_\infty}{\|P_k\|_\infty} \|P_k\|_1.$$

Proof. If $\mathcal{H}^d(\Omega) < \infty$, then the claim follows immediately from Lemma 3.3. Indeed, simply bound using the Hölder inequality

$$\begin{aligned}\left|\operatorname{Tr}\left[S(P_1, \dots, P_m; \Omega)\right]\right| &\leq \|S(P_1, \dots, P_m; \Omega)\|_{\mathcal{S}_1} \\ &\leq \frac{\|P_1\|_\infty \dots \|P_m\|_\infty}{\|P_k\|_\infty} \|1_\Omega P_k 1_\Omega\|_1 + C_d \mathcal{H}^d(\Omega) \|P_1 \dots P_m\|_1 \\ &\leq C_d \mathcal{H}^d(\Omega) \frac{\|P_1\|_\infty \dots \|P_m\|_\infty}{\|P_k\|_\infty} \|P_k\|_1.\end{aligned}$$

The case $\mathcal{H}^d(\Omega^c) < \infty$ is less straight forward. In the expression

$$S(P_1, \dots, P_m; \Omega) = 1_\Omega P_1 1_\Omega \dots 1_\Omega P_m 1_\Omega - 1_\Omega P_1 \dots P_m 1_\Omega,$$

expand in every instance $1_\Omega = 1 - 1_{\Omega^c}$. Note that the terms

$$P_1 \dots P_m, \quad 1_{\Omega^c} P_1 \dots P_m, \quad P_1 \dots P_m 1_{\Omega^c} \tag{3.6}$$

will cancel. The resulting expression will be a sum of operators of the type

$$\pm R_1 1_{\Omega^c} \dots 1_{\Omega^c} R_n \tag{3.7}$$

for $2 \leq n \leq m$ and symbols $R_1, \dots R_n$. Setting $P_0 = P_{m+1} = 1$, the (R_i)'s take the form

$$R_i = P_{\ell_{i-1}} \dots P_{\ell_i - 1}, \quad 1 \leq i \leq n, \quad 0 = \ell_0 < \ell_1 < \dots < \ell_n = m + 2.$$

Since the terms (3.6) are excluded (corresponding to $n = 2$ and $\ell_1 = m + 1$ or $\ell_1 = 1$), at least two of the (R_i)'s will appear, and

$$\operatorname{Tr}[R_1 1_{\Omega^c} R_2] = \operatorname{Tr}[1_{\Omega^c} R_2 R_1 1_{\Omega^c}],$$

and

$$\operatorname{Tr}[R_1 1_{\Omega^c} \dots 1_{\Omega^c} R_n] = \operatorname{Tr}[1_{\Omega^c} R_2 1_{\Omega^c} \dots 1_{\Omega^c} R_{n-1} 1_{\Omega^c} R_n R_1 1_{\Omega^c}], \quad n \geq 3.$$

The L^2 condition ensures that all operators above are trace class by Lemma 3.2. The traces can now be handled like in the case $\mathcal{H}^d(\Omega) < \infty$. ■

3.3. **Multilinear properties of S_m.** The product structure of $S(P_1, \dots, P_m; \Omega)$ induces multilinear structure in both the P_i's and Ω. For instance,

$$\begin{aligned} &S(P_1, \dots, P_m; \Omega) - S(Q_1, \dots, Q_m; \Omega) \\ &\quad = S(P_1 - Q_1, P_2, \dots, P_m; \Omega) + S(Q_1, P_2 - Q_2, P_3, \dots, P_m; \Omega) \\ &\qquad + \dots + S(Q_1, \dots, Q_{m-1}, (P_m - Q_m); \Omega). \end{aligned} \tag{3.8}$$

This allows effective bounds of

$$S(P_1, \dots, P_m; \Omega) - S(Q_1, \dots, Q_m; \Omega), \quad S(P_1, \dots, P_m; \Omega) - S(P_1, \dots, P_m; \Omega').$$

assuming either $P_i \sim Q_i$ or $\Omega \sim \Omega'$.

Lemma 3.11. *Let $\Omega \subseteq \mathbb{R}^d$ be a measurable set and consider symbols $P_1, Q_1, \dots, P_m, Q_m \in L^\infty(\mathbb{R}^d; \mathcal{M}_n)$ for $m \geq 2$. For $1 \leq n \leq m$ introduce*

$$\begin{aligned} L_0^n &= 1, \quad L_k^n = Q_{n-k} \dots Q_{n-1}, \quad 1 \leq k \leq n-1, \\ R_0^n &= 1, \quad R_k^n = P_{n+1} \dots P_{n+k}, \quad 1 \leq k \leq m-n, \end{aligned}$$

and set $K_P = \max_i\{\|P_i\|_\infty\}$ and $K_Q = \max_i\{\|Q_i\|_\infty\}$. Then

$$\begin{aligned} &\|S(P_1, \dots, P_m; \Omega) - S(Q_1, \dots, Q_m; \Omega)\|_{\mathcal{S}_1} \\ &\quad \leq \sum_{n=1}^{m} \Big(\sum_{k=1}^{n-1} K_Q^{k-1} \|1_\Omega Q_k 1_{\Omega^c}\|_{\mathcal{S}_2} \|1_{\Omega^c} L_{n-1-k}^n (P_n - Q_n) R_{m-n}^n 1_\Omega\|_{\mathcal{S}_2} \\ &\qquad + \sum_{k=1}^{m-n} K_Q^{n-1} K_P^{k-1} \|1_\Omega (P_n - Q_n) R_{m-n-k}^n 1_{\Omega^c}\|_{\mathcal{S}_2} \|1_{\Omega^c} P_{m+1-k} 1_\Omega\|_{\mathcal{S}_2} \Big), \end{aligned}$$

with the empty sum seen as zero.

Proof. Simply expand the m-linear forms as

$$\begin{aligned} &S(P_1, \dots, P_m; \Omega) - S(Q_1, \dots, Q_m; \Omega) \\ &\quad = S(P_1 - Q_1, P_2, \dots, P_m; \Omega) + S(Q_1, P_2 - Q_2, P_3, \dots, P_m; \Omega) \\ &\qquad + \dots + S(Q_1, \dots, Q_{m-1}, (P_m - Q_m); \Omega). \end{aligned}$$

and apply Lemma 3.8. Concretely, if $P_n - Q_n$ appears in the n'th entry, apply (3.3) with this n. ■

For the application of Lemma 3.11 in approximation arguments, it is not enough to simply assume that $P_i \sim Q_i$ for all i in $\dot{BV}$ or $\dot{H}^{1/2}$. Instead we need to handle arbitrary pointwise products $P_{i_1} \dots P_{i_k} \sim Q_{i_1} \dots Q_{i_k}$. The relevant tools are developed in Lemma A.15 and Lemma B.4 in the Appendix. This will play a significant role towards the end of the present paper.

Lemma 3.12. *Let $\Omega \subseteq \mathbb{R}^d$ be a set satisfying $\min\{\mathcal{H}^d(\Omega), \mathcal{H}^d(\Omega^c)\} < \infty$ and consider $P_1, Q_1, \dots, P_m, Q_m \in L^2(\mathbb{R}^d; \mathcal{M}_n) \cap L^\infty$ for $m \geq 2$. Set $K = \max\{\|P_i\|_\infty, \|Q_i\|_\infty\}$. Then*

$$\begin{aligned}&\big|\operatorname{Tr}\big[S(P_1,\dots,P_m;\Omega) - S(Q_1,\dots,Q_m;\Omega)\big]\big|\\ &\qquad\leq C_{d,m}K^{m-1}\min\{\mathcal{H}^d(\Omega),\mathcal{H}^d(\Omega^c)\}\sum_{i=1}^m \|P_i - Q_i\|_1.\end{aligned}$$

Proof. Immediate from Lemma 3.10 and the multilinear expansion (3.8) employed in the proof of Lemma 3.11. ■

We have a similar result for the domain Ω, here simplified to the case $P_1, \dots, P_m \in \dot{H}^{1/2}$.

Lemma 3.13. *Let $\Omega, \Omega' \subseteq \mathbb{R}^d$ be sets of finite perimeter and $P_1, \dots, P_m \in \dot{H}^{1/2}(\mathbb{R}^d; \mathcal{M}_n) \cap L^\infty$. Then*

$$\begin{aligned}&\|S(P_1,\dots,P_m;\Omega) - S(P_1,\dots,P_m;\Omega')\|_{\mathcal{S}_1}\\ &\qquad\leq C_{m,d}\big(\operatorname{Per}(\Omega\setminus\Omega') + \operatorname{Per}(\Omega'\setminus\Omega)\big)^{1/2}\big(\operatorname{Per}(\Omega) + \operatorname{Per}(\Omega')\big)^{1/2}\\ &\qquad\quad\times\max\{\|P_i\|_\infty\}^{m-2}\max\{\|P_i\|_{\dot{H}^{1/2}}\}^2.\end{aligned}$$

Proof. We argue inductively in m. Firstly for $m = 2$

$$\|S(P_1,P_2;\Omega) - S(P_1,P_2;\Omega')\|_{\mathcal{S}_1} = \|1_\Omega P_1 1_{\Omega^c} P_2 1_\Omega - 1_{\Omega'} P_1 1_{\Omega'^c} P_2 1_{\Omega'}\|_{\mathcal{S}_1}.$$

In every occurrence we expand into the disjoint union $\Omega = (\Omega \cap \Omega') \cup (\Omega \setminus \Omega')$, similarly for Ω', Ω^c, and Ω'^c. We get

$$\begin{aligned}&\|S(P_1,P_2;\Omega) - S(P_1,P_2;\Omega')\|_{\mathcal{S}_1}\\ &\quad\leq \|1_{\Omega\setminus\Omega'}P_1 1_{\Omega^c}P_2 1_\Omega\|_{\mathcal{S}_1} + \|1_{\Omega\cap\Omega'}P_1 1_{\Omega'\setminus\Omega}P_2 1_\Omega\|_{\mathcal{S}_1} + \|1_{\Omega\cap\Omega'}P_1 1_{\Omega^c\cap\Omega'^c}P_2 1_{\Omega\setminus\Omega'}\|_{\mathcal{S}_1}\\ &\qquad + \|1_{\Omega'\setminus\Omega}P_1 1_{\Omega'^c}P_2 1_{\Omega'}\|_{\mathcal{S}_1} + \|1_{\Omega\cap\Omega'}P_1 1_{\Omega\setminus\Omega'}P_2 1_{\Omega'}\|_{\mathcal{S}_1} + \|1_{\Omega\cap\Omega'}P_1 1_{\Omega^c\cap\Omega'^c}P_2 1_{\Omega'\setminus\Omega}\|_{\mathcal{S}_1}.\end{aligned}$$

Note that the $1_{\Omega\cap\Omega'}P_1 1_{\Omega^c\cap\Omega'^c}P_2 1_{\Omega\cap\Omega'}$ contributions cancel. All terms can be bounded with the stated bounds using Lemma 3.1 and the Hölder inequality.

Assume that the claim holds for some $m - 1 \geq 2$. By the identity (3.5) we can estimate

$$\begin{aligned}&\|S(P_1,\dots,P_m;\Omega) - S(P_1,\dots,P_m;\Omega')\|_{\mathcal{S}_1}\\ &\qquad\leq \|P_m\|_\infty\|S(P_1,\dots,P_{m-1};\Omega) - S(P_1,\dots,P_{m-1};\Omega')\|_{\mathcal{S}_1}\\ &\qquad\quad + \|S(P_1,\dots,P_{m-1};\Omega')\big(1_\Omega P_m 1_\Omega - 1_{\Omega'}P_m 1_{\Omega'}\big)\|.\end{aligned}$$

The first term $\|P_m\|_\infty\|S(P_1,\dots,P_{m-1};\Omega) - S(P_1,\dots,P_{m-1};\Omega')\|_{\mathcal{S}_1}$ is bounded correctly by hypothesis. For the second term we further expand

$$\begin{aligned}&\|S(P_1,\dots,P_{m-1};\Omega')\big(1_\Omega P_m 1_\Omega - 1_{\Omega'}P_m 1_{\Omega'}\big)\|\\ &\qquad\leq \|P_1\|_\infty\dots\|P_{m-2}\|_\infty\|1_{\Omega'}P_{m-1}1_{\Omega'}\big(1_\Omega P_m 1_\Omega - 1_{\Omega'}P_m 1_{\Omega'}\big)\|\\ &\qquad\quad + \|1_{\Omega'}P_1\dots P_{m-1}1_{\Omega'}\big(1_\Omega P_m 1_\Omega - 1_{\Omega'}P_m 1_{\Omega'}\big)\|.\end{aligned}$$

Both terms on the right hand side of course take the same form. We expand $1_\Omega P_m 1_\Omega - 1_{\Omega'}P_m 1_{\Omega'}$ like for the $m = 2$ case:

$$\begin{aligned}1_\Omega P_m 1_\Omega - 1_{\Omega'}P_m 1_{\Omega'} &= 1_{\Omega\cap\Omega'}P_m\big(1_{\Omega\setminus\Omega'} - 1_{\Omega'\setminus\Omega}\big) + \big(1_{\Omega\setminus\Omega'} - 1_{\Omega'\setminus\Omega}\big)P_m 1_{\Omega\cap\Omega'}\\ &\quad + 1_{\Omega\setminus\Omega'}P_m 1_{\Omega\setminus\Omega'} + 1_{\Omega'\setminus\Omega}P_m 1_{\Omega'\setminus\Omega}.\end{aligned}$$

When paired with a term of the form $1_{\Omega'}Q1_{\Omega'}$ on the left, every contribution can be bounded correctly using Lemma 3.1. ■

Remark 3.14. It is possible to obtain a slightly stronger result if the trace norm is replaced with the absolute value of the trace:

$$\begin{aligned}&\big|\operatorname{Tr}\big[S(P_1,\dots,P_m;\Omega) - S(P_1,\dots,P_m;\Omega')\big]\big|\\ &\qquad\leq C_{m,d}\big(\operatorname{Per}(\Omega\setminus\Omega') + \operatorname{Per}(\Omega'\setminus\Omega)\big)\max\{\|P_i\|_\infty\}^{m-2}\max\{\|P_i\|_{\dot{H}^{1/2}}\}^2.\end{aligned}$$

The proof also simplifies somewhat in this setting.

3.4. **Specialized bounds in trace class.** We turn to more specialized statements requiring both smoothness and localization of P. Our argument is based on the Fefferman-de la Llave decomposition, see [18].

Lemma 3.15. *For all $k > 0$ there is $g_k : \mathbb{R}_+ \to \mathbb{R}$ such that, for all $x, y \in \mathbb{R}^d$,*

$$\big(1 + |x-y|^2\big)^{-k/2} = \int_0^\infty g_k(r) \int 1_{B_r(z)}(x) 1_{B_r(z)}(y)\, dz\, dr$$

Moreover there is $C_{k,d} > 0$ such that $|g_k(r)| \le C_{k,d} \max\{r^{1-d}, r^{-d-1-k}\}$.

Proof. The case $d \ge 2$ is covered by [18]. For $d = 1$ one can verify directly that $g(r) = 2V''(2r)$ with $V(r) = (1+r^2)^{-k/2}$ works. ∎

Our application of Lemma 3.15 is the following. For $P \in C_c^k(\mathbb{R}^d; \mathcal{M}_n)$ and domains Ω, Ω', the operator $1_\Omega P 1_{\Omega'}$ has kernel

$$\begin{aligned} K(x,y) &= (2\pi)^{-d/2} 1_\Omega(x) 1_{\Omega'}(y) \check{P}(x-y) \\ &= (2\pi)^{-d/2} 1_\Omega(x) 1_{\Omega'}(y) (1+|x-y|^2)^{-k/2} [(1-\Delta)^{k/2} P]^\vee (x-y) \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}} g_k(r) \int 1_{\Omega \cap B_r(z)}(x) [(1-\Delta)^{k/2} P]^\vee (x-y) 1_{\Omega' \cap B_r(z)}(y)\, dz\, dr. \end{aligned}$$

Hence, weakly as operators,

$$1_\Omega P 1_{\Omega'} = \int_0^\infty g_k(r) \int 1_{\Omega \cap B_r(z)} [(1-\Delta)^{k/2} P] 1_{\Omega' \cap B_r(z)}\, dz\, dr. \tag{3.9}$$

The formal computations are easily justified for $P \in C_c^k$. Note that (3.9) may be viewed as a finite-range decomposition, similar in spirit to a Littlewood–Paley decomposition. The latter could equally well be used below, with only minor modifications. We use (3.9) because localization to balls is more convenient in the present setting than localization to annuli.

Below we think of P as a smooth function supported in a ball of radius $\sim \lambda$ with derivatives scaling like $D^\beta P \sim \lambda^{-|\beta|}$. For our application we localize in such a way that $d(\operatorname{supp} P, \operatorname{supp} Q) \ge c_1 \lambda$ for c_1 not too small, and such that the overlap $\operatorname{supp} P \cap (\operatorname{supp} Q + h)$ has small measure when $|h| \le a\lambda$ for large a.

Lemma 3.16. *Let $\Omega \subseteq \mathbb{R}^d$ be a set whose boundary $\partial\Omega$ is compact and contained in a finite union of Lipschitz graphs. Suppose $P \in C_c^k(\mathbb{R}^d; \mathcal{M}_n)$ for $k \ge d$ an even integer, and assume that $Q \in L^\infty$ satisfies* $\operatorname{supp} P \cap \operatorname{supp} Q = \emptyset$. *Then, for all $\alpha, a, \lambda > 0$,*

$$\begin{aligned} &\|1_{\alpha\Omega} P 1_{\alpha\Omega} Q\|_{\mathcal{S}_1} \\ &\quad \le C_{\Omega,d,k} \|Q\|_\infty \Big(\lambda^{-d/2} \|(1-\lambda\Delta)^{k/2} P\|_2\Big) \Big(1 + (\alpha\lambda)^{d-1}\Big) \\ &\quad\quad \times \left(\left(\frac{\mathcal{H}^d(\operatorname{supp} P)}{a\lambda^d}\right)^{1/2} + \left(a\lambda^{-(d-2)} \sup_{|h| \le a\lambda} \frac{\mathcal{H}^d(\operatorname{supp} P \cap (\operatorname{supp} Q + h))}{|h|^2} \right)^{1/2} \right). \end{aligned}$$

Proof. Note $1_{\alpha\Omega} P 1_{\alpha\Omega} Q = -1_{\alpha\Omega} P 1_{\alpha\Omega^c} Q$ since P and Q have disjoint supports. By a unitary equivalence

$$\|1_{\alpha\Omega} P 1_{\alpha\Omega^c} Q\|_{\mathcal{S}_1} = \|1_{\alpha\lambda\Omega} P_\lambda 1_{\alpha\lambda\Omega^c} Q_\lambda\|_{\mathcal{S}_1},$$

where $P_\lambda(p) = P(\lambda p)$ and $Q_\lambda(p) = Q(\lambda p)$. We bound the latter. Using (3.9) and the triangle inequality we find

$$\|1_{\alpha\lambda\Omega} P_\lambda 1_{\alpha\lambda\Omega^c} Q_\lambda\|_{\mathcal{S}_1} \le \int |g_k(r)| \int \|1_{\alpha\lambda\Omega \cap B_r(z)} [(1-\Delta)^{-k/2} P_\lambda] 1_{\alpha\lambda\Omega^c \cap B_r(z)} Q_\lambda\|_{\mathcal{S}_1}\, dz\, dr. \tag{3.10}$$

Note that the z-integral in (3.10) is non-zero only if $\alpha\lambda\Omega \cap B_r(z) \neq \emptyset$ and $\alpha\lambda\Omega^c \cap B_r(z) \neq \emptyset$, or equivalently $d(z, \alpha\lambda\partial\Omega) \leq r$. Since $\partial\Omega$ is compact and contained in a finite union of Lipschitz graphs, it follows from Lemma C.1 that

$$\mathcal{H}^d(\alpha\lambda\partial\Omega + B_r) \leq C_\Omega\big(r(\alpha\lambda)^{d-1} + r^d\big).$$

Hence

$$\begin{aligned}&\int \|1_{\alpha\lambda\Omega\cap B_r(z)}[(1-\Delta)^{-k/2}P_\lambda]1_{\alpha\lambda\Omega^c\cap B_r(z)}Q_\lambda\|_{\mathcal{S}_1}\,dz\\ &\quad\leq C_\Omega\big(r(\alpha\lambda)^{d-1} + r^d\big)\sup_{z\in\mathbb{R}^d}\|1_{\alpha\lambda\Omega\cap B_r(z)}[(1-\Delta)^{-k/2}P_\lambda]1_{\alpha\lambda\Omega^c\cap B_r(z)}Q_\lambda\|_{\mathcal{S}_1}.\end{aligned} \tag{3.11}$$

We bound the trace norm uniformly in z. By the Hölder inequality and Lemma 3.2:

$$\begin{aligned}&\|1_{\alpha\lambda\Omega\cap B_r(z)}[(1-\Delta)^{-k/2}P_\lambda]1_{\alpha\lambda\Omega^c\cap B_r(z)}Q_\lambda\|_{\mathcal{S}_1}\\ &\quad\leq \|1_{\alpha\lambda\Omega\cap B_r(z)}[(1-\Delta)^{-k/2}P_\lambda]\|_{\mathcal{S}_2}\|1_{\operatorname{supp}P_\lambda}1_{\alpha\lambda\Omega^c\cap B_r(z)}Q_\lambda\|_{\mathcal{S}_2}\\ &\quad\leq C_d r^{d/2}\|Q\|_\infty\|(1-\Delta)^{k/2}P_\lambda\|_2\|1_{\operatorname{supp}P_\lambda}1_{\alpha\lambda\Omega^c\cap B_r(z)}1_{\operatorname{supp}Q_\lambda}\|_{\mathcal{S}_2}.\end{aligned} \tag{3.12}$$

We used that $\operatorname{supp}(1-\Delta)^{k/2}P_\lambda \subseteq \operatorname{supp}P_\lambda$ since k is an even integer. The remaining Hilbert-Schmidt norm is of the form treated in Lemma 3.5, but since $\operatorname{supp}P_\lambda$ and $\operatorname{supp}Q_\lambda$ are separated, we can do better. Proceeding like in the proof of Lemma 3.1,

$$\|1_{\operatorname{supp}P_\lambda}1_{\alpha\lambda\Omega^c\cap B_r(z)}1_{\operatorname{supp}Q_\lambda}\|_{\mathcal{S}_2}^2 \leq C_d\int|\widehat{1}_{\alpha\lambda\Omega^c\cap B_r(z)}(h)|^2\mathcal{H}^d(\operatorname{supp}P_\lambda\cap(\operatorname{supp}Q_\lambda + h))\,dh.$$

We split the integral region into $|h| \leq a$ and $|h| \geq a$. For the $|h| \geq a$ region we apply Lemma A.8:

$$\begin{aligned}&\int_{|h|\geq a}|\widehat{1}_{\alpha\lambda\Omega^c\cap B_r(z)}(h)|^2\mathcal{H}^d(\operatorname{supp}P_\lambda\cap(\operatorname{supp}Q_\lambda + h))\,dh\\ &\quad\leq \frac{C_d}{a}\operatorname{Per}(\alpha\lambda\Omega^c\cap B_r(z))\mathcal{H}^d(\operatorname{supp}P_\lambda).\end{aligned}$$

We bound the $|h| \leq a$ region similarly:

$$\begin{aligned}&\int_{|h|\leq a}|\widehat{1}_{\alpha\lambda\Omega^c\cap B_r(z)}(h)|^2\mathcal{H}^d(\operatorname{supp}P_\lambda\cap(\operatorname{supp}Q_\lambda + h))\,dh\\ &\quad\leq \Big(\sup_{|h|\leq a}\frac{\mathcal{H}^d(\operatorname{supp}P_\lambda\cap(\operatorname{supp}Q_\lambda + h))}{|h|^2}\Big)\int_{|h|\leq a}|\widehat{1}_{\alpha\lambda\Omega^c\cap B_r(z)}(h)|^2|h|^2\,dh\\ &\quad\leq C_d\,a\operatorname{Per}(\alpha\lambda\Omega^c\cap B_r(z))\left(\sup_{|h|\leq a}\frac{\mathcal{H}^d(\operatorname{supp}P_\lambda\cap(\operatorname{supp}Q_\lambda + h))}{|h|^2}\right).\end{aligned}$$

Since $\partial\Omega = \partial\Omega^c$ is contained in a finite union of Lipschitz graphs it follows from Lemma C.1 that

$$\operatorname{Per}(\alpha\lambda\Omega^c\cap B_r(z)) \leq C_\Omega r^{d-1}.$$

We conclude

$$\begin{aligned}&\|1_{\operatorname{supp}P_\lambda}1_{\alpha\lambda\Omega^c\cap B_r(z)}1_{\operatorname{supp}Q_\lambda}\|_{\mathcal{S}_2} \leq C_{d,\Omega}r^{(d-1)/2}\\ &\quad\times\left(\left(\frac{\mathcal{H}^d(\operatorname{supp}P)}{a\lambda^d}\right)^{1/2} + \left(a\lambda^{-(d-2)}\sup_{|h|\leq a\lambda}\frac{\mathcal{H}^d(\operatorname{supp}P\cap(\operatorname{supp}Q + h))}{|h|^2}\right)^{1/2}\right).\end{aligned} \tag{3.13}$$

Combining (3.10), (3.11), (3.12), and (3.13) we finally arrive at

$$\begin{aligned}&\|1_{\alpha\lambda\Omega}P_\lambda 1_{\alpha\lambda\Omega^c}Q_\lambda\|_{\mathcal{S}_1}\\ &\quad\leq C_{\Omega,d}\,\text{const.}\int_0^\infty|g_k(r)|(r(\alpha\lambda)^{d-1} + r^d)r^{d/2}r^{(d-1)/2}\,dr \leq C_{\Omega,d,k}\,\text{const.}(1 + (\alpha\lambda)^{d-1}),\end{aligned}$$

where

$$\begin{aligned}\text{const.} &= \|Q\|_\infty \left(\lambda^{-d/2}\|(1-\lambda\Delta)^{k/2}P\|_2\right)\\ &\quad\times\left(\left(\frac{\mathcal{H}^d(\operatorname{supp}P)}{a\lambda^d}\right)^{1/2}+\left(a\lambda^{-(d-2)}\sup_{|h|\le a\lambda}\frac{\mathcal{H}^d(\operatorname{supp}P\cap(\operatorname{supp}Q+h))}{|h|^2}\right)^{1/2}\right).\end{aligned}$$

The r-integral was computed using the bound $|g_k(r)|\le C_{k,d}\max\{r^{-d+1},r^{-d-1-k}\}$ from Lemma 3.15. The integral is finite since $k>d-1/2$. ■

We have the corresponding result for one-dimensional sections. We remind the reader that for $P:\mathbb{R}^d\to\mathcal{M}_n$, $\nu\in S^{d-1}$ and $p\in\{\nu\}^\perp$ we denote $P_{\nu,p}(\xi)=P(p+\xi\nu)$, $\xi\in\mathbb{R}$. Note that the stated bound below is exactly Lemma 3.16 when $d=1$.

Lemma 3.17. *Let $P\in C_c^2(\mathbb{R}^d;\mathcal{M}_n)$ and $Q\in L^\infty$ be such that $\operatorname{supp}P\cap\operatorname{supp}Q=\emptyset$. Then, uniformly for all $\nu\in S^{d-1}$ and all $a,\lambda>0$,*

$$\begin{aligned}&\int_{\Pi_\nu}\|1_{\mathbb{R}_+}P_{\nu,p}1_{\mathbb{R}_+}Q_{\nu,p}\|_{\mathcal{S}_1}\,dp\\ &\quad\le C\|Q\|_\infty\left(\lambda^{-1/2}\|(1-\lambda\Delta)P\|_2\right)\\ &\quad\quad\times\left(\left(\frac{\mathcal{H}^d(\operatorname{supp}P)}{a\lambda}\right)^{1/2}+\left(a\lambda\sup_{|h|\le a\lambda}\frac{\mathcal{H}^d(\operatorname{supp}P\cap(\operatorname{supp}Q+h))}{|h|^2}\right)^{1/2}\right).\end{aligned}$$

Proof. The proof is essentially the same as that of Lemma 3.16 for $d=1$. Following the same arguments we see that

$$\begin{aligned}&\int_{\Pi_\nu}\|1_{\mathbb{R}_+}P_{\nu,p}1_{\mathbb{R}_+}Q_{\nu,p}\|_{\mathcal{S}_1}\,dp\\ &\quad\le\int_0^\infty|g_2(r)|\int_{|z|\le r}\Big(\int_{\Pi_\nu}\|1_{\mathbb{R}_+\cap B_r(z)}[(1-\Delta)(P_{\nu,p})_\lambda]\|_{\mathcal{S}_2}^2\,dp\Big)^{1/2}\\ &\quad\quad\times\Big(\int\|1_{\operatorname{supp}(P_{\nu,p})_\lambda}1_{\mathbb{R}_-\cap B_r(z)}(Q_{\nu,p})_\lambda\|_{\mathcal{S}_2}^2\,dp\Big)^{1/2}\,dz\,dr.\end{aligned}$$

Arguing like in Lemma $B.5$ we find

$$\int_{\Pi_\nu}\|(1-\Delta)(P_{\nu,p})_\lambda\|_2^2\,dp=\lambda^{-1}\int_{\Pi_\nu}\|(1-\lambda\Delta)P_{\nu,p}\|_2^2\,dp\le\lambda^{-1}\|(1-\lambda\Delta)P\|_2^2,$$

hence

$$\int_{\Pi_\nu}\|1_{\mathbb{R}_+\cap B_r(z)}[(1-\Delta(P_{\nu,p})_\lambda]\|_{\mathcal{S}_2}^2\,dp\le Cr\lambda^{-1}\|(1-\lambda\Delta)P\|_2^2.$$

Similarly, like in the proof of Lemma 3.16,

$$\begin{aligned}&\int\|1_{\operatorname{supp}(P_{\nu,p})_\lambda}1_{\mathbb{R}_-\cap B_r(z)}(Q_{\nu,p})_\lambda\|_{\mathcal{S}_2}^2\,dp\\ &\quad\le C\lambda^{-1}\|Q\|_\infty\int_{\Pi_\nu}\int|\widehat{1}_{\mathbb{R}_-\cap B_r(z)}(\xi)|^2\mathcal{H}^1(\operatorname{supp}P_{\nu,p}\cap(\operatorname{supp}Q_{\nu,p}+\lambda\xi))\,d\xi\,dp.\end{aligned}$$

Since $\operatorname{supp}P_{\nu,p}\subseteq\{\xi\mid p+\xi\nu\in\operatorname{supp}P\}$, similarly for Q, we see that

$$\int_{\Pi_\nu}\mathcal{H}^1(\operatorname{supp}P_{\nu,p}\cap(\operatorname{supp}Q_{\nu,p}+\lambda\xi))\,d\xi\,dp\le\mathcal{H}^d(\operatorname{supp}P\cap(\operatorname{supp}Q+\lambda\xi\nu)).$$

The argument is now exactly the same as that of Lemma 3.16 for $d=1$. ■

We need one more variation which will only be used to prove the trace class bound (1.16).

Lemma 3.18. *Let $\Omega \subseteq \mathbb{R}^d$ be set whose boundary $\partial\Omega$ has finite upper Minkowski content. Let $P \in C_c^k(\mathbb{R}^d; \mathcal{M}_n)$ for an integer $k \geq d$. Then*

$$\|1_{\alpha\Omega} P 1_{\alpha\Omega^c}\|_{\mathcal{S}_1} \leq C_{d,k,\Omega}(1 + \alpha^{d-1})\|(1-\Delta)^{k/2} P\|_1$$

Proof. The argument is significantly simpler than that of Lemma 3.16. We have

$$\begin{aligned}\|1_{\alpha\Omega} P 1_{\alpha\Omega^c}\|_{\mathcal{S}_1} \leq \int_0^\infty |g_k(r)| \int_{d(z,\alpha\partial\Omega)\leq r} &\|1_{\alpha\Omega\cap B_r(z)}[(1-\Delta)^{k/2}P]^{1/2}\|_{\mathcal{S}_2} \\ &\times \|[(1-\Delta)^{k/2}P]^{1/2} 1_{\alpha\Omega^c\cap B_r(z)}\|_{\mathcal{S}_2}\, dz\, dr.\end{aligned}$$

From here it is a simple computation using Lemma 3.2. Note here that we can bound $\mathcal{H}^d(\alpha\partial\Omega + B_r) \leq C_\Omega(r\alpha^{d-1} + r^d)$ since $\partial\Omega$ has finite upper Minkowski content, see (C.2). ■

4. One-dimensional reduction for multiscale symbols

In this section we prove the one-dimensional reduction estimate, Theorem 2.5, for multiscale symbols. We first establish the result under stronger regularity assumptions on the involved symbols and the domain, which will be removed using approximation arguments at the end.

We remind the reader that for $\Omega \subseteq \mathbb{R}^d$ a set of finite perimeter, we identify $T^*(\mathcal{F}\Omega)$ with points $X = (x,p)$ where $x \in \mathcal{F}\Omega$ and $p \in \Pi_{\nu_\Omega(x)}$. The space $T^*(\mathcal{F}\Omega)$ carries a natural measure:

$$\int_{T^*(\mathcal{F}\Omega)} f(X)\, dX = \int_{\mathcal{F}\Omega} \int_{\Pi_{\nu_\Omega(x))}} f(x,p)\, dp\, d\mathcal{H}^{d-1}(x).$$

In general, for a function f on $\mathbb{R}^d$ and $X = (x,p) \in T^*(\mathcal{F}\Omega)$, we denote $f_X(\xi) = f(p + \xi\nu_\Omega(x))$, $\xi \in \mathbb{R}$, for the corresponding co-tangent section.

Proposition 4.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a set whose boundary $\partial\Omega$ is compact and contained in a finite union of Lipschitz graphs, let $\Gamma \subseteq \mathbb{R}^d$ be a set of locally uniform upper Minkowski content, and $m \geq 2$ an integer. There is a constant $C_{d,m,\Omega,\Gamma} > 0$ such that the following holds: Suppose (P_δ^k, F) satisfies a $C^{d+1}(\mathbb{R}^d; \mathcal{M}_n)$ multiscale estimate on Γ for $k = 1, \dots, m$. Then, for all $\varepsilon > 0$, there is $C_{\varepsilon,d,m,\Omega} > 0$ such that uniformly for all $\alpha \geq 2$ and $0 < \delta \leq 1/2$ with $\alpha\delta \geq 1$,*

$$\begin{aligned}\Bigg|\operatorname{Tr}\left[S(P_\delta^1, \dots, P_\delta^m; \alpha\Omega)\right] - \left(\frac{\alpha}{2\pi}\right)^{d-1} \int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S((P_\delta^1)_X, \dots, (P_\delta^m)_X; \mathbb{R}_+)\right] dX\Bigg| \\ \leq C_{d,m,\Omega,\Gamma}\|F\|_\infty^{m-1}\big(\|F\|_\infty + \|G\|_1\big)\left(\varepsilon\log(\delta^{-1})\alpha^{d-1} + C_{\varepsilon,d,m,\Omega}\alpha^{d-2}\delta^{-1}\right).\end{aligned} \tag{4.1}$$

Here $G(p) = \sup_{|p-q|\leq \ell_\delta(p)} F(q)$.

We slowly proceed with the proof, so fix $m \geq 2$, Ω, and Γ according to the statement of Proposition 4.1. Following Definition 2.2 we take families of symbols $(P_\delta^k) \subseteq C^{d+1}(\mathbb{R}^d; \mathcal{M}_n)$, $0 < \delta \leq 1/2$, $k = 1, \dots, m$, satisfying the multiscale estimate

$$|D^\beta P_\delta^k(p)| \leq F(p)\ell_\delta(p)^{-|\beta|}, \quad |\beta| \leq d+1, \quad p \in \mathbb{R}^d, \tag{4.2}$$

for a fixed bounded amplitude function $F : \mathbb{R}^d \to \mathbb{R}_+$ such that $G(p) = \sup_{|p-q|\leq\ell_\delta(p)} F(q) \in L^1$.

In the following we consider $0 < \delta \leq 1/2$ and $\alpha \geq 2$ fixed with $\alpha\delta \geq 1$. Constants are initially only allowed to depend on d, m, Ω, and Γ, which adheres to the logic of Proposition 4.1. We forget the subscript δ in P_δ^k consistently and simply write P^k.

We start with the key technical step of the proof, namely to localize each symbol $P^1, \dots, P^m$ appearing in

$$S(P^1, \dots, P^m; \alpha\Omega) = 1_{\alpha\Omega}P^1 1_{\alpha\Omega} \dots 1_{\alpha\Omega}P^m 1_{\alpha\Omega} - 1_{\alpha\Omega}P^1 \dots P^m 1_{\alpha\Omega}$$

to scales defined by ℓ_δ. Here the main difficulty is that we need to localize all symbols $P^1, \ldots, P^m$ to roughly the same scale around the same point simultaneously. We rely on the continuously parametrized partition of unity (ϕ_u) introduced in Lemma 2.1. We remind the reader that for all $u \in \mathbb{R}^d$, ϕ_u is non-negative, smooth, supported in the ball $B_{\ell_\delta(u)}(u)$ and satisfies

$$1 = \int_{\mathbb{R}^d} \ell_\delta(u)^{-d} \phi_u(p)\, du, \quad p \in \mathbb{R}^d, \tag{4.3}$$

and the multiscale estimate

$$|D^\beta \phi_u(p)| \leq C_\beta \ell_\delta(u)^{-|\beta|}. \tag{4.4}$$

We first localize the symbol P^1. By (4.3):

$$P^1(p) = \int \ell_\delta(u)^{-d} \phi_u(p) P^1(p)\, du. \tag{4.5}$$

Consequently, weakly as operators,

$$S(P^1, \ldots, P^m; \alpha\Omega) = \int \ell_\delta(u)^{-d} S(\phi_u P^1, P^2, \ldots, P^m; \alpha\Omega)\, du. \tag{4.6}$$

To control the localization error, we need to localize the remaining terms $P^2, \ldots, P^m$ to larger and larger scales around u. To this end we introduce the sets

$$Y_n(u) = \{v \in \mathbb{R}^d \mid |u - v| \leq D_n \ell_\delta(u),\, \ell_\delta(v) \geq D_n^{-1} \ell_\delta(u)\} \tag{4.7}$$

for a sequence (D_n) satisfying

$$D_1 = 1, \quad D_n > 3D_{n-1} + 4, \quad m \geq n \geq 2. \tag{4.8}$$

We also introduce a sequence (a_n) satisfying

$$0 < a_n \leq \frac{D_{n-1}}{2}(D_n - 3D_{n-1} - 4), \quad m \geq n \geq 2. \tag{4.9}$$

We think of $D_{n-1} \ll a_n \ll D_n$. Then set, for $2 \leq n \leq m$,

$$\begin{aligned} P_u^n(p) &= \int_{Y_n(u)} \ell_\delta(v)^{-d} \phi_v(p) P^n(p)\, dv, \\ Q_u^n(p) &= \int_{Y_n(u)^c} \ell_\delta(v)^{-d} \phi_v(p) P^n(p)\, dv \end{aligned}$$

and

$$P_u^1(p) = \phi_u(p) P^1(p).$$

Clearly $P^n = P_u^n + Q_u^n$ for $2 \leq n \leq m$. The first condition $|u - v| \leq D_n \ell_\delta(u)$ in $Y_n(u)$ controls the size of the support of P_u^n, while the condition $\ell_\delta(v) \geq D_n^{-1} \ell_\delta(u)$ ensures that we do not get too close to Γ.

The structure of the proof is now simple. Introduce the errors $\mathcal{E}_1, \mathcal{E}_2$, and $\mathcal{E}_3$ given by

$$\begin{aligned} \mathcal{E}_1 &= \operatorname{Tr}\left[S(P^1, \ldots, P^m; \alpha\Omega)\right] - \int \ell_\delta(u)^{-d} \operatorname{Tr}\left[S(P_u^1, \ldots, P_u^m; \alpha\Omega)\right] du, \\ \mathcal{E}_2 &= \int \ell_\delta(u)^{-d} \Big(\operatorname{Tr}\left[S(P_u^1, \ldots, P_u^m; \alpha\Omega)\right) \\ &\quad - \Big(\frac{\alpha}{2\pi}\Big)^{d-1} \int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S((P_u^1)_X, \ldots, (P_u^m)_X; \mathbb{R}_+)\right] dX \Big) du \\ \mathcal{E}_3 &= \Big(\frac{\alpha}{2\pi}\Big)^{d-1} \int \ell_\delta(u)^{-d} \int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S((P_u^1)_X, \ldots, (P_u^m)_X; \mathbb{R}_+)\right] dX du \\ &\quad - \Big(\frac{\alpha}{2\pi}\Big)^{d-1} \int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S(P_X^1, \ldots, P_X^m; \mathbb{R}_+)\right] dX. \end{aligned} \tag{4.10}$$

Then

$$\operatorname{Tr}\big[S(P^1,\dots,P^m;\alpha\Omega)\big]-\Big(\frac{\alpha}{2\pi}\Big)^{d-1}\int_{T^*(\mathcal{F}\Omega)}\operatorname{Tr}\big[S(P^1_X,\dots,P^m_X;\mathbb{R}_+)\big]\,dX=\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3. \tag{4.11}$$

The localization errors $\mathcal{E}_1$ and $\mathcal{E}_3$ will be treated using Lemmas 3.16 and 3.17. The localization is made specifically to control the error $\mathcal{E}_2$, which arises from a tangent plane approximation of Ω (the Roccaforte Lemma). In fact, showing that

$$\operatorname{Tr}\big[S(P^1,\dots,P^m;\alpha\Omega)\big]=\Big(\frac{\alpha}{2\pi}\Big)^{d-1}\int_{T^*(\mathcal{F}\Omega)}\operatorname{Tr}\big[S(P^1_X,\dots,P^m_X;\mathbb{R}_+)\big]\,dX+o(\alpha^{d-1})$$

for smooth symbols $P^1,\dots,P^m$ is straightforward following the procedure laid out by Roccaforte and Widom [41, 52]. The difficulty lies in explicitly controlling the discontinuity parameter δ, and simply following the obvious procedure directly would lead to wrong scaling in δ.

Before proceeding with the estimates for $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ we establish a key technical lemma on the supports and scaling properties of P^n_u and Q^n_u.

Lemma 4.2. *For* (D_n) *and* (a_n) *as in* (4.8) *and* (4.9)*, the following holds for all* $u\in\mathbb{R}^d$*:*

a) $\operatorname{supp}P^n_u\subseteq B_{(\frac{3}{2}D_n+1)\ell_\delta(u)}(u)$ *for* $n=1,\dots,m$.

b) Let $n\geq 2$. *Then* $\operatorname{supp}P^{n-1}_u\cap\operatorname{supp}Q^n_u=\emptyset$. *Moreover, if* $p\in\operatorname{supp}P^{n-1}_u$ *and* $q\in\operatorname{supp}Q^n_u$ *with* $|p-q|\leq a_nD^{-1}_{n-1}\ell_\delta(u)$*, then*

$$|p-q|\geq\ell_\delta(u)(D^{-1}_{n-1}-3D^{-1}_n)>0$$

and $\ell_\delta(q)\leq\frac{3}{2}D^{-1}_n\ell_\delta(u)$.

c) There is a constant $C>0$ *such that*

$$\|D^\beta P^1_u\|_\infty\leq CG(u)\ell_\delta(u)^{-|\beta|}$$

and

$$\|D^\beta P^n_u\|_\infty\leq C\|F\|_\infty D^{|\beta|}_n\ell_\delta(u)^{-|\beta|}$$

for all $2\leq n\leq m$ *and all* $|\beta|\leq d+1$.

Proof. We start with a). The case $n=1$ is trivial, so let $n\geq 2$ and $p\in\operatorname{supp}P^n_u$. Then there is $v\in Y_n(u)$ such that $\phi_v(p)P^n(p)\neq 0$, and therefore $|v-p|\leq\ell_\delta(v)$. Hence

$$|p-u|\leq|p-v|+|v-u|\leq\ell_\delta(v)+D_n\ell_\delta(u)\leq\ell_\delta(u)+\tfrac{1}{2}|u-v|+D_n\ell_\delta(u)\leq(\tfrac{3}{2}D_n+1)\ell_\delta(u).$$

Note that the same argument could be used formally for $n=1$ by defining $Y_1(u)$ by (4.7) with $D_1=1$. We use this in the proof of b) now.

For b), let $n\geq 2$ and take $p\in\operatorname{supp}P^{n-1}_u$ and $q\in\operatorname{supp}Q^n_u$. As in a) we can take $v\in Y_{n-1}(u)$ and $w\in Y_n(u)^c$ such that $|p-v|\leq\ell_\delta(v)$ and $|q-w|\leq\ell_\delta(w)$. Note first that

$$\begin{aligned}|w-u|&\leq|w-q|+|q-p|+|p-v|+|v-u|\leq\ell_\delta(w)+|p-q|+\ell_\delta(v)+D_{n-1}\ell_\delta(u)\\&\leq\tfrac{1}{2}|w-u|+\tfrac{1}{2}|v-u|+|p-q|+(D_{n-1}+2)\ell_\delta(u)\\&\leq\tfrac{1}{2}|w-u|+|p-q|+(\tfrac{3}{2}D_{n-1}+2)\ell_\delta(u),\end{aligned}$$

and therefore

$$|w-u|\leq 2|p-q|+(3D_{n-1}+4)\ell_\delta(u). \tag{4.12}$$

Similarly

$$\begin{aligned}\ell_\delta(w)&\geq\ell_\delta(v)-\tfrac{1}{2}\big(|v-p|+|p-q|+|q-w|\big)\\&\geq\tfrac{1}{2}\ell_\delta(v)-\tfrac{1}{2}\ell_\delta(w)-\tfrac{1}{2}|p-q|,\end{aligned}$$

hence

$$\ell_\delta(w)\geq\tfrac{1}{3}\ell_\delta(v)-\tfrac{1}{3}|p-q|. \tag{4.13}$$

Now we split into cases. Either $|w-u| > D_n \ell_\delta(u)$, and then by (4.12)

$$|p-q| > \frac{1}{2}\big(D_n - 3D_{n-1} - 4\big)\ell_\delta(u) > 0. \tag{4.14}$$

Else $\ell_\delta(w) < D_n^{-1}\ell_\delta(u)$, and then by (4.13)

$$|p-q| > \ell_\delta(v) - \frac{3}{D_n}\ell_\delta(u) \geq \big(D_{n-1}^{-1} - 3D_n^{-1}\big)\ell_\delta(u) > 0.$$

In either case we see that $\operatorname{supp} P_u^{n-1}$ and $\operatorname{supp} Q_u^n$ are separated. For the final point, assume that $|p-q| \leq a_n D_{n-1}^{-1}\ell_\delta(u)$. By (4.9) it follows that

$$|p-q| \leq \frac{1}{2}\left(D_n - 3D_{n-1} - 4\right)\ell_\delta(u),$$

and therefore that $|w-u| \leq D_n\ell_\delta(u)$ by (4.14). Since $w \in Y_n(u)^c$ we conclude $\ell_\delta(w) < D_n^{-1}\ell_\delta(u)$, and thus

$$\ell_\delta(q) \leq \frac{3}{2}\ell_\delta(w) < \frac{3}{2}D_n^{-1}\ell_\delta(u).$$

Finally we consider c). Fix $p \in \mathbb{R}^d$ and $|\beta| \leq d+1$. For $n=1$ we compute directly using the multiscale estimates (4.4) and (4.2) on ϕ_u and P^1:

$$\begin{aligned}|D^\beta P_u^1(p)| &\leq \sum_{\gamma\leq\beta}\binom{\beta}{\gamma}|D^\gamma P^1(p)||D^{\beta-\gamma}\phi_u(p)| \\ &\leq CF(p)\sum_{\gamma\leq\beta}\ell_\delta(u)^{-|\beta|+|\gamma|}\ell_\delta(p)^{-|\gamma|} \\ &\leq CG(u)\ell_\delta(u)^{-|\beta|}.\end{aligned}$$

We used that for $p \in \operatorname{supp}\phi_u$ we have $\ell_\delta(u) \leq 2\ell_\delta(p)$, see (2.1), and that $F(p) \leq G(u)$. The argument for $n \geq 2$ is similar:

$$\begin{aligned}|D^\beta P_u^n(p)| \leq \int_{Y_n(u)}&\ell_\delta(v)^{-d}\sum_{\gamma\leq\beta}\binom{\beta}{\gamma}|D^\gamma P^n(p)||D^{\beta-\gamma}\phi_v(p)|\,dv \\ &\leq CF(p)\int_{Y_n(u)}\ell_\delta(v)^{-d}\sum_{\gamma\leq\beta}\ell_\delta(p)^{-|\gamma|}\ell_\delta(v)^{-|\beta|+|\gamma|}1_{\{|p-v|\leq\ell_\delta(v)\}}\,dv \\ &\leq CF(p)\int_{Y_n(u)}\ell_\delta(v)^{-d}D_n^{|\beta|}\ell_\delta(u)^{-|\beta|}1_{\{|p-v|\leq\ell_\delta(v)\}}\,dv \\ &\leq CF(p)D_n^{|\beta|}\ell_\delta(u)^{-|\beta|}.\end{aligned}$$

Again we used that $\ell_\delta(v)$ and $\ell_\delta(p)$ are comparable when $p \in \operatorname{supp}\phi_v$, and that $\ell_\delta(v) \geq D_n^{-1}\ell_\delta(u)$ for $v \in Y_n(u)$. ■

4.1. **Bounds on $\mathcal{E}_1$ and $\mathcal{E}_3$.** We now handle the localization errors $\mathcal{E}_1$ and $\mathcal{E}_3$. For simplicity let us set, for $2 \leq n \leq m$,

$$\tilde{P}_u^1 = \frac{1}{G(u)}P_u^1, \quad \tilde{P}_u^n = \frac{1}{\|F\|_\infty}P_u^n, \quad \tilde{Q}_u^n = \frac{1}{\|F\|_\infty}Q_u^n.$$

Now expand $P^n = P_u^n + Q_u^n$, $n \geq 2$, in (4.6). Since $\operatorname{supp} P_u^{n-1}$ and $\operatorname{supp} Q_u^n$ are separated by Lemma 4.2, we immediately see that

$$|\mathcal{E}_1| \leq \|F\|_\infty^{m-1}\int \ell_\delta(u)^{-d}G(u)\sum_{n=2}^m \|1_{\alpha\Omega}\tilde{P}_u^{n-1}1_{\alpha\Omega}\tilde{Q}_u^n\|_{\mathcal{S}_1}\,du. \tag{4.15}$$

The bound for $\mathcal{E}_3$ is similar. Taking co-tangent sections in (4.5) we see that for each fixed $X \in T^*(\mathcal{F}\Omega)$,

$$S(P_X^1,\ldots,P_X^m;\mathbb{R}_+) = \int \ell_\delta(u)^{-d}S((P_u^1)_X, P_X^2,\ldots,P_X^m;\mathbb{R}_+)\,du.$$

In general, if $\xi \in \operatorname{supp} P_X$ for $X = (x, p)$, then $p + \nu_\Omega(x)\xi \in \operatorname{supp} P$. Hence $(P_u^{n-1})_X$ and $(Q_u^n)_X$ have separated supports, and the same argument as in (4.15) applied for each fixed X shows that

$$|\mathcal{E}_3| \leq \left(\frac{\alpha}{2\pi}\right)^{d-1} \|F\|_\infty^{m-1} \int \ell_\delta(u)^{-d} G(u) \int_{T^*(\mathcal{F}\Omega)} \sum_{n=2}^{m} \|1_{\mathbb{R}_+}(\tilde{P}_u^{n-1})_X 1_{\mathbb{R}_+}(\tilde{Q}_u^n)_X\|_{\mathcal{S}_1}. \tag{4.16}$$

We can apply Lemmas 3.16 and 3.17 directly to bound both errors (4.15) and (4.16). First for the non-sectional operators in (4.15) we take $\lambda = \lambda_n(u) = D_{n-1}^{-1}\ell_\delta(u)$, $a = a_n$ according to (4.9), and smoothness of degree $k_d = 2\lceil d/2 \rceil \leq d + 1$. The basic bound from Lemma 3.16 reads: There is a constant $C_{\Omega,d}$ only dependent on Ω and d such that for all $u \in \mathbb{R}^d$ and all $2 \leq n \leq m$,

$$\begin{aligned} \|1_{\alpha\Omega}\tilde{P}_u^{n-1} 1_{\alpha\Omega}\tilde{Q}_u^n\|_{\mathcal{S}_1} \leq C_{d,\Omega} \|\tilde{Q}_u^n\|_\infty \big(\lambda_n(u)^{-d/2} \|(1 - \lambda_n(u)^2\Delta)^{k_d/2}\tilde{P}_u^{n-1}\|_2\big)\big(1 + (\alpha\lambda_n(u))^{d-1}\big) \\ \times \left(\left(\frac{\mathcal{H}^d(\operatorname{supp} \tilde{P}_u^{n-1})}{a_n \lambda_n(u)^d} \right)^{1/2} + \left(\frac{a_n}{\lambda_n(u)^{d-2}} I_n(u) \right)^{1/2} \right), \end{aligned} \tag{4.17}$$

where

$$I_n(u) = \sup_{|h| \leq a_n\lambda_n(u)} \frac{\mathcal{H}^d(\operatorname{supp} \tilde{P}_u^{n-1} \cap (\operatorname{supp} \tilde{Q}_u^n + h))}{|h|^2}. \tag{4.18}$$

The bound for the co-tangent operators in (4.16) is similar. We take λ and a as before. By Lemma 3.17 there is a universal constant $C > 0$ such that for all $u \in \mathbb{R}^d$ and all $2 \leq n \leq m$ we have

$$\begin{aligned} &\int_{T^*(\mathcal{F}\Omega)} \|1_{\mathbb{R}_+}(\tilde{P}_u^{n-1})_X 1_{\mathbb{R}_+}(\tilde{Q}_u^n)_X\|_{\mathcal{S}_1}\, dX \\ &\quad \leq C \operatorname{Per}(\Omega) \|\tilde{Q}_u^n\|_\infty \big(\lambda_n(u)^{-1/2} \|(1 - \lambda_n(u)^2\Delta)\tilde{P}_u^{n-1}\|_2\big) \\ &\quad \times \left(\left(\frac{\mathcal{H}^d(\operatorname{supp} \tilde{P}_u^{n-1})}{a_n \lambda_n(u)} \right)^{1/2} + (a_n\lambda_n(u) I_n(u))^{1/2} \right). \end{aligned} \tag{4.19}$$

All terms appearing in (4.17) and (4.19) can be estimated directly by Lemma 4.2 and simple geometric considerations. Indeed, we can bound

$$\mathcal{H}^d(\operatorname{supp} \tilde{P}_u^{n-1}) \leq C_d \ell_\delta(u)^d D_{n-1}^d,$$

and, for integers $k = 1$ or $k = k_d/2$,

$$\|(1 - \lambda_n(u)^2\Delta)^k \tilde{P}_u^{n-1}\|_2 \leq C_d \sum_{|\beta| \leq 2k} \lambda_n(u)^{|\beta|} \|D^\beta \tilde{P}_u^{n-1}\|_2 \leq C_d \ell_\delta(u)^{d/2} D_{n-1}^{d/2}.$$

The term $I_n(u)$ is slightly more involved. Again by Lemma 4.2 we see that if $|h| \leq a_n\lambda_n$ and $p \in \operatorname{supp} \tilde{P}_u^{n-1} \cap (\operatorname{supp} \tilde{Q}_u^n + h)$, then $|h| \geq \ell_\delta(u)\big(D_{n-1}^{-1} - 3D_n^{-1}\big)$ and $\ell_\delta(p - h) \leq \frac{3}{2} D_n^{-1} \ell_\delta(u)$. In particular, since $D_n > 7$, we have

$$p - h \in \big(\Gamma + B_{3D_n^{-1}\ell_\delta(u)}\big) \cap B_{(\frac{3}{2}D_{n-1}+1)\ell_\delta(u)}(u - h).$$

Since Γ has locally uniform upper Minkowski content, the set of all such p has measure at most

$$C_{d,\Gamma} \frac{D_{n-1}^{d-1}}{D_n} \ell_\delta(u)^d.$$

We conclude

$$I_n(u) \leq C_{d,\Gamma} \left(1 - 3\frac{D_{n-1}}{D_n}\right)^{-2} \frac{D_{n-1}^{d+1}}{D_n} \ell_\delta(u)^{d-2}.$$

Plugging into the bounds (4.17) and (4.19) we see, after a bit of bookkeeping,

$$\begin{aligned}\|1_{\alpha\Omega}\tilde{P}_u^{n-1}1_{\alpha\Omega}\tilde{Q}_u^n\|_{\mathcal{S}_1} \leq C_{d,\Omega,\Gamma}D_{n-1}^{2d}&\left(1+\left(\frac{\alpha\ell_\delta(u)}{D_{n-1}}\right)^{d-1}\right)\\ &\times\left(a_n^{-1/2}+\left(\frac{a_n}{D_n}\right)^{1/2}\left(1-3\frac{D_{n-1}}{D_n}\right)^{-1}\right),\end{aligned}\tag{4.20}$$

and

$$\begin{aligned}&\int_{T^*(\mathcal{F}\Omega)}\|1_{\mathbb{R}_+}(\tilde{P}_u^{n-1})_X1_{\mathbb{R}_+}(\tilde{Q}_u^n)_X\|_{\mathcal{S}_1}\,dX\\ &\quad\leq C_{d,\Omega,\Gamma}D_{n-1}^{d+1}\ell_\delta(u)^{d-1}\left(a_n^{-1/2}+\left(\frac{a_n}{D_n}\right)^{1/2}\left(1-3\frac{D_{n-1}}{D_n}\right)^{-1}\right).\end{aligned}\tag{4.21}$$

We apply this directly in (4.15) and (4.16). For convenience we introduce, for $2\leq n\leq m$,

$$K_n = D_{n-1}^{2d}\left(a_n^{-1/2}+\left(\frac{a_n}{D_n}\right)^{1/2}\left(1-3\frac{D_{n-1}}{D_n}\right)^{-1}\right).\tag{4.22}$$

Since Γ has finite upper Minkowski content, it follows from (C.2) and Lemma $C.2$ that

$$\int\ell_\delta(u)^{-1}G(u)\,du\leq C_\Gamma\|F\|_\infty\log(\delta^{-1})+2\|G\|_1,$$

and that, for $d\geq 2$,

$$\int\ell_\delta(u)^{-d}G(u)\,du\leq C_\Gamma\|F\|_\infty\delta^{-(d-1)}+2^d\|G\|_1.$$

Here we simply bound $\delta^{-(d-1)}\leq\alpha^{d-1}\leq C\alpha^{d-1}\log(\delta^{-1})$, which is valid since $\alpha\delta\geq 1$ with $\delta\leq 1/2$. Using these integral estimates, along with (4.20) and (4.21), in (4.15) and (4.16) we conclude

$$|\mathcal{E}_1|,|\mathcal{E}_3|\leq C_{d,\Omega,\Gamma}\|F\|_\infty^{m-1}\big(\|F\|_\infty+\|G\|_1\big)\alpha^{d-1}\log(\delta^{-1})\sum_{n=2}^m K_n.\tag{4.23}$$

Simply choosing $D_{n-1}\ll a_n\ll D_n$, which obviously adheres to the requirements (4.8) and (4.9), we can make the leading order coefficient $C_{d,\Omega,\Gamma}\|F\|_\infty^{m-1}(\|F\|_\infty+\|G\|_1)\sum_2^m K_n$ arbitrary small. We here mention that $\|G\|_1$ can be estimated independently of δ, see Remark 2.3.

4.2. **Bound on $\mathcal{E}_2$.**

We turn to $\mathcal{E}_2$ defined in (4.10) by

$$\begin{aligned}\mathcal{E}_2=\|F\|_\infty^{m-1}\int\ell_\delta(u)^{-d}G(u)\Big(&\operatorname{Tr}\big[S(\tilde{P}_u^1,\dots,\tilde{P}_u^m;\alpha\Omega)\\ &-\big(\frac{\alpha}{2\pi}\big)^{d-1}\int_{T^*(\mathcal{F}\Omega)}\operatorname{Tr}\big[S((\tilde{P}_u^1)_X,\dots,(\tilde{P}_u^m)_X;\mathbb{R}_+)\big]\,dX\Big)\,du.\end{aligned}\tag{4.24}$$

Like for $\mathcal{E}_1$ and $\mathcal{E}_3$, we factored out the F and G dependence. We want to argue that $\mathcal{E}_2=o(\alpha^{d-1}\log(\delta^{-1}))$ when $\alpha\delta\geq 1$. Essentially, the co-tangent integral arises from a local tangent plane approximation of Ω, so the difference will be lower order. Since this effect is purely in Ω, we can be rather crude in estimating norms of the symbols P_u^n, as long as we capture the correct ℓ_δ scaling. In fact, by our localization, we can rely on simple volume and scaling estimates.

We follow the proof strategy of Widom and Roccaforte [41, 52], namely just to write out the kernel directly.

Lemma 4.3. *Let* $P^1, \dots, P^m \in H^s(\mathbb{R}^d; \mathcal{M}_n) \cap L^\infty$ *for* $s > d/2$, *and let* $\Omega \subseteq \mathbb{R}^d$ *be a set of finite perimeter. Then the following identities hold:*

$$\begin{aligned}\operatorname{Tr}\big[S(P^1, \dots, P^m; \Omega)\big] = -(2\pi)^{-\frac{dm}{2}} \int \cdots \int \operatorname{Tr}\Big[\big(P^1\big)^\vee(x_1) \dots \big(P^{m-1}\big)^\vee(x_{m-1}) \\ \times \big(P^m\big)^\vee(-x_1 - \dots - x_{m-1})\Big] F_\Omega(\underline{x})\, dx_1 \dots dx_{m-1},\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}(2\pi)^{-(d-1)} \int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\big[S(P^1_X, \dots, P^m_X; \mathbb{R}_+)\big]\, dX \\ = -(2\pi)^{-dm/2} \int \cdots \int \operatorname{Tr}\Big[\big(P^1\big)^\vee(x_1) \dots \big(P^{m-1}\big)^\vee(x_{m-1}) \\ \times \big(P^m\big)^\vee(-x_1 - \dots - x_{m-1})\Big] G_\Omega(\underline{x})\, dx_1 \dots dx_{m-1}.\end{aligned} \tag{4.26}$$

Here $\underline{x} = (x_1, x_1 + x_2, \dots, x_1 + \dots + x_{m-1})$ *for* $x_1, \dots, x_{m-1} \in \mathbb{R}^d$, *and*

$$\begin{aligned}F_\Omega(\underline{x}) &= \mathcal{H}^d\Big(\Omega \setminus \big(\Omega \cap (\Omega + x_1) \cap (\Omega + x_1 + x_2) \cap \dots \cap (\Omega + x_1 + \dots + x_{m-1})\big)\Big), \\ G_\Omega(\underline{x}) &= \int_{\mathcal{F}\Omega} \max_{1 \le n \le m-1} \{0, \nu_\Omega(x) \cdot (x_1 + \dots + x_n)\}\, d\mathcal{H}^{d-1}(x).\end{aligned} \tag{4.27}$$

For the proof and for later computations, we need the following technical lemma.

Lemma 4.4.

a) Let $g_1, \dots, g_m$ *be non-negative functions. For any* $1 \le p_n \le \infty$, $n = 1, \dots, m$, *satisfying* $\frac{1}{p_1} + \dots + \frac{1}{p_m} = m - 1$ *we have*

$$\int \cdots \int g_1(x_1) \dots g_{m-1}(x_{m-1}) g_m(-x_1 - \dots - x_{m-1})\, dx_1 \dots dx_{m-1} \le \prod_{n=1}^m \|g_n\|_{p_n}.$$

b) Let $P \in L^2(\mathbb{R}^d; \mathcal{M}_n)$ *with* $\check{P} \in L^1$, *and let* $\Omega \subseteq \mathbb{R}^d$ *be a set of finite perimeter. Then, for any* $s \ge 0$,

$$\int_{T^*(\mathcal{F}\Omega)} \||t|^s \big(P_X\big)^\vee(t)\|_2^2\, dX \le \operatorname{Per}(\Omega) \||x|^s \check{P}(x)\|_2^2,$$

and, for all $X \in T^*(\mathcal{F}\Omega)$,

$$\int_{\mathbb{R}} |\big(P_X\big)^\vee(t)|\, dt \le (2\pi)^{-(d-1)/2} \|\check{P}\|_1.$$

Proof of Lemma 4.4. For a), simply observe that, for all $z \in \mathbb{R}^d$,

$$\begin{aligned}\int \cdots \int g_1(x_1) \dots g_{m-1}(x_{m-1}) g_m(z - x_1 - \dots - x_{m-1})\, dx_1 \dots dx_{m-1} \\ = \int (g_1 * \dots * g_{m-1})(x) g_m(z - x)\, dx \le \|g_m\|_{p_m} \|g_1 * \dots * g_{m-1}\|_{p'_m},\end{aligned} \tag{4.28}$$

where p'_m is the Hölder conjugate. Now repeatedly apply Young's convolution inequality.

For b), the main insight is that orthogonal decomposition $\mathbb{R}^d = \Pi_\nu \oplus (\Pi_\nu)^\perp$ induces a decomposition of the Fourier transform into a tensor. Concretely, for $X = (x, p) \in T^*(\mathcal{F}\Omega)$,

$$P(p + \xi\nu_\Omega(x)) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-it\xi} \left((2\pi)^{-(d-1)/2} \int_{\Pi_{\nu_\Omega(x)}} e^{-iy\cdot p} \check{P}(y + t\nu_\Omega(x))\, dy \right) dt.$$

In particular, by Fourier inversion on $(\Pi_{\nu_\Omega(x)})^\perp$,

$$\big(P_X\big)^\vee(t) = (2\pi)^{-(d-1)/2} \int_{\Pi_{\nu_\Omega(x)}} e^{-iy\cdot p} \check{P}(y + t\nu_\Omega(x))\, dy,$$

so

$$\int_{\mathbb{R}} |\big(P_X\big)^\vee(t)|\,dt \le (2\pi)^{-(d-1)/2}\|\check{P}\|_1.$$

Similarly, applying Plancherel on $\Pi_{\nu_\Omega(x)}$,

$$\begin{aligned}\int_{T^*(\mathcal{F}\Omega)} \||t|^s\big(P_X\big)^\vee(t)\|_2^2\,dX &= \int_{\mathcal{F}\Omega}\int_{\Pi_{\nu_\Omega(x)}}\int_{\mathbb{R}} |t|^{2s}|\check{P}(y+t\nu_\Omega(x))|^2\,dt\,dy\,d\mathcal{H}^{d-1}(x)\\ &\le \operatorname{Per}(\Omega)\||x|^s\check{P}\|_2^2.\end{aligned}$$

■

Proof of Lemma 4.3. Note that if $f \in H^s$ for some $s > d/2$, then $\check{f} \in L^1$. This is well known. It follows that $(P^n)^\vee \in L^1 \cap L^2$ for $n = 1, \dots, m$, which we use below.

We derive (4.25) and (4.26) by expressing the traces as integrals of the corresponding kernels along the diagonal. Recall that if $\Lambda \subseteq \mathbb{R}^d$ is a measurable set and $Q \in L^2(\mathbb{R}^d; \mathcal{M}_n)$, then the operator $1_\Lambda Q 1_\Lambda$ has kernel

$$(2\pi)^{-d/2}1_\Lambda(x)\check{Q}(x-y)1_\Lambda(y).$$

Hence the kernel of the term $1_\Omega P^1 1_\Omega \dots 1_\Omega P^m 1_\Omega$ is obtained by repeated use of the kernel composition formula, while the kernel of $1_\Omega P^1 \dots P^m 1_\Omega$ can be expressed in terms of the convolution $\big(P^1\big)^\vee * \dots * \big(P^m\big)^\vee(x-y)$. We find that the operator $S(P^1, \dots, P^m; \Omega)$ has kernel

$$\begin{aligned}K(x,y) = (2\pi)^{-\frac{dm}{2}} &\int\cdots\int \big(P^1\big)^\vee(x-x_1)\big(P^2\big)^\vee(x_1-x_2)\dots\big(P^m\big)^\vee(x_{m-1}-y)\\ &\times 1_\Omega(x)1_\Omega(x_1)\dots 1_\Omega(x_{m-1})1_\Omega(y)\,dx_1\dots dx_{m-1}\\ &-(2\pi)^{-\frac{dm}{2}}\int\cdots\int \big(P^1\big)^\vee(x-x_1)\big(P^2\big)^\vee(x_1-x_2)\dots\big(P^m\big)^\vee(x_{m-1}-y)\\ &\times 1_\Omega(x)1_\Omega(y)\,dx_1\dots dx_{m-1}\\ = -(2\pi)^{-\frac{dm}{2}}&1_\Omega(x)1_\Omega(y)\int\cdots\int \big(P^1\big)^\vee(x-x_1)\dots\big(P^m\big)^\vee(x_{m-1}-y)\\ &\times\Big(1-1_\Omega(x_1)\dots 1_\Omega(x_{m-1})\Big)\,dx_1\dots dx_{m-1}.\end{aligned}$$

The final integral expression is easily seen to be continuous as a function of (x,y). Indeed, the argument boils down to that of showing the convolution of two L^1 functions is continuous. By the Lebesgue differentiation theorem, we conclude in particular that (x,x) is a Lebesgue point of K for almost all $x \in \mathbb{R}^d$. Since $S(P^1, \dots, P^m; \Omega)$ is trace class by Corollary 3.9, it follows from Brislawn's theorem [8] (applied entry-wise) that the trace is given by integrating along the diagonal:

$$\begin{aligned}&\operatorname{Tr}\big[S(P^1,\dots,P^m;\Omega)\big]\\ &\quad= -(2\pi)^{\frac{dm}{2}}\int\int\cdots\int \operatorname{Tr}\big[\big(P^1\big)^\vee(x-x_1)\dots\big(P^m\big)^\vee(x_{m-1}-x)\big]\\ &\qquad\times 1_\Omega(x)\Big(1-1_\Omega(x_1)\dots 1_\Omega(x_{m-1})\Big)\,dx_1\dots dx_{m-1}dx\\ &\quad= -(2\pi)^{\frac{dm}{2}}\int\cdots\int \operatorname{Tr}\Big[\big(P^1\big)^\vee(x_1)\dots\big(P^{m-1}\big)^\vee(x_{m-1})\big(P^m\big)^\vee(-x_1-\dots-x_{m-1})\Big]\\ &\qquad\times\int 1_\Omega(x)\Big(1-1_\Omega(x-x_1)\dots 1_\Omega(x-x_1\cdots-x_{m-1})\Big)\,dx\,dx_1\dots dx_{m-1}\\ &\quad= -(2\pi)^{\frac{dm}{2}}\int\cdots\int \operatorname{Tr}\Big[\big(P^1\big)^\vee(x_1)\dots\big(P^{m-1}\big)^\vee(x_{m-1})\big(P^m\big)^\vee(-x_1-\dots-x_{m-1})\Big]\\ &\qquad\times F_\Omega(\underline{x})\,dx_1\dots dx_{m-1},\end{aligned}$$

Which gives (4.25). The second equality is a simple change of variables and an application of Fubini's theorem, which we now justify. Since Ω is a set of finite perimeter it follows from Lemma $C.4$ that $F_\Omega(\underline{x}) \leq C_{m,\Omega}\sum_{n=1}^{m-1}|x_n|$. Hence, by Lemma 4.4,

$$\begin{aligned}
&\int \cdots \int \Big| \mathrm{Tr}\left[\big(P^1\big)^\vee(x_1) \dots \big(P^{m-1}\big)^\vee(x_{m-1}) \big(P^m\big)^\vee(-x_1 - \cdots - x_{m-1}) \right] \Big| F_\Omega(\underline{x})\, dx_1 \dots dx_{m-1} \\
&\quad \leq C_{m,\Omega} \sum_{n=1}^{m-1} \| |x| \big(P^n\big)^\vee(x) \|_2 \|P^m\|_2 \prod_{j\neq n,m} \| \big(P^j\big)^\vee \|_1 < \infty.
\end{aligned}$$

The argument for the co-tangent integral (4.26) is similar. By Lemma 4.4 we have $P_X \in H^s$ for almost all $X \in T^*(\mathcal{F}\Omega)$. Hence, the argument above shows that

$$\begin{aligned}
&(2\pi)^{-(d-1)} \int_{T^*(\mathcal{F}\Omega)} \mathrm{Tr}\left[S(P_X^1, \dots, P_X^m; \mathbb{R}_+) \right] dX \\
&\qquad = -(2\pi)^{-m/2-d+1} \int_{T^*(\mathcal{F}\Omega)} \int \cdots \int \mathrm{Tr}\Big[\big(P_X^1\big)^\vee(t_1) \dots \big(P_X^{m-1}\big)^\vee(t_{m-1}) \\
&\qquad\quad \times \big(P_X^m\big)^\vee(-t_1 - \cdots - t_{m-1}) \Big] F_{\mathbb{R}_+}(\underline{t})\, dt_1 \dots dt_{m-1}.
\end{aligned} \tag{4.29}$$

Note that the integral above converges absolutely. Indeed, arguing as above using Lemma 4.4 and the Hölder inequality,

$$\begin{aligned}
&\int_{T^*(\mathcal{F}\Omega)} \int \cdots \int \Big| \mathrm{Tr}\left[\big(P_X^1\big)^\vee(t_1) \dots \big(P_X^{m-1}\big)^\vee(t_{m-1}) \big(P_X^m\big)^\vee(-t_1 - \cdots - t_{m-1}) \right] \Big| \\
&\qquad \times F_{\mathbb{R}_+}(\underline{t})\, dt_1 \dots dt_{m-1} \\
&\qquad \leq C_m \sum_{n=1}^{m-1} \int_{T^*(\mathcal{F}\Omega)} \| |t| \big(P_X^n\big)^\vee(t) \|_2 \|P_X^m\|_2 \prod_{j\neq n,m} \| \big(P_X^j\big)^\vee \|_1 \\
&\qquad \leq C_{m,d} \sum_{n=1}^{m-1} \left(\int_{T^*(\mathcal{F}\Omega)} \| |t| \big(P_X^n\big)^\vee(t) \|_2^2\, dX \right)^{1/2} \left(\int_{T^*(\mathcal{F}\Omega)} \|P_X^m\|_2^2\, dX \right)^{1/2} \prod_{j\neq n,m} \| \big(P^j\big)^\vee \|_1 \\
&\qquad < \infty.
\end{aligned}$$

It is elementary to see that

$$F_{\mathbb{R}_+}(\underline{t}) = \max_{1\leq n\leq m-1} \max\{0, t_1 + \cdots + t_n\}. \tag{4.30}$$

To achieve (4.26) we use the convolution structure in the tangent direction p for $x \in \mathcal{F}\Omega$ fixed:

$$\begin{aligned}
&\int_{\Pi_{\nu_\Omega(x)}} \mathrm{Tr}\left[\big(P_X^1\big)^\vee(t_1) \dots \big(P_X^{m-1}\big)^\vee(t_{m-1}) \big(P_X^m\big)^\vee(-t_1 - \cdots - t_{m-1}) \right] dp \\
&\quad = (2\pi)^{\frac{d-1}{2}} \Big(\mathrm{Tr}\left[\big(P_X^1\big)^\vee(t_1) \dots \big(P_X^{m-1}\big)^\vee(t_{m-1}) \big(P_X^m\big)^\vee(-t_1 - \cdots - t_{m-1}) \right] \Big)^\vee(0) \\
&\quad = (2\pi)^{(d-1)(1-\frac{m}{2})} \int_{\Pi_{\nu_\Omega(x)}} \cdots \int_{\Pi_{\nu_\Omega(x)}} \\
&\qquad \mathrm{Tr}\Big[\big(P^1\big)^\vee(y_1 + t_1\nu_\Omega(x)) \dots \big(P^{m-1}\big)^\vee(y_{m-1} + t_{m-1}\nu_\Omega(x)) \\
&\qquad \times \big(P^m\big)^\vee(-(y_1 + t_1\nu_\Omega(x)) - \cdots - (y_{m-1} + t_{m-1}\nu_\Omega(x))) \Big] dy_1 \dots dy_{m-1}.
\end{aligned}$$

Here, of course, we used the symbol $\vee$ to denote the Fourier transform in three different spaces. Pairing $x_n = y_n + t_n\nu_\Omega(x)$ with $dx_n = dy_n dt_n$ and $t_n = \nu_\Omega(x) \cdot x_n$ in (4.29) finishes the proof. ∎

Using Lemma 4.3 in (4.24) we see that

$$\begin{aligned}\mathcal{E}_2 = -(2\pi)^{-dm/2}\|F\|_\infty^{m-1}&\int \ell_\delta(u)^{-d}G(u)\\ &\times\Bigg(\int\cdots\int \mathrm{Tr}\Big[\big(\tilde P_u^1\big)^\vee(x_1)\dots\big(\tilde P_u^{m-1}\big)^\vee(x_{m-1})\big(\tilde P_u^m\big)^\vee(-x_1-\dots-x_{m-1})\Big]\\ &\times\big(F_{\alpha\Omega}(\underline{x})-G_{\alpha\Omega}(\underline{x})\big)dx_1\dots dx_{m-1}\Bigg)du.\end{aligned} \tag{4.31}$$

Note that $G_\Omega(\underline{x})$ is homogeneous of degree $d-1$ in Ω and degree 1 in $\underline{x}$. In fact, for Ω a set of finite perimeter, the leading order behavior of $F_\Omega(\underline{x})$ as $\underline{x}\to 0$ is exactly captured by $G_\Omega(\underline{x})$:

$$\omega(\rho)=\sup_{|\underline{x}|\le\rho}\frac{1}{|\underline{x}|}|F_\Omega(\underline{x})-G_\Omega(\underline{x})|\to 0 \text{ as } \rho\to 0^+. \tag{4.32}$$

This is shown in Lemma $C.6$ in the Appendix. The same result is used by Roccaforte [41] for Ω with C^3 boundary, where it is shown that $\omega(\rho)=O(\rho)$ (although his proof for this part also works for Ω with C^2 boundary), and in [52] for Ω with C^1 boundary without proof. Essentially, G_Ω arises from F_Ω from local tangent plane approximations: for $x\in\mathcal{F}\Omega$, the membership $x+y\in\Omega$ for $|y|$ small is to leading order determined by $y\cdot\nu_\Omega(x)>0$. We used that this is exact when $\Omega=\mathbb{R}_+$ is a half-plane, see (4.30) above.

By the Lipschitz bounds from Lemmas $C.4$ and $C.5$ we see that $\omega(\rho)\le C_{\Omega,m}$ uniformly. Hence, for any $\rho>0$,

$$|F_{\alpha\Omega}(\underline{x})-G_{\alpha\Omega}(\underline{x})|=\alpha^{d-1}|\underline{x}|\left(\frac{\alpha}{|\underline{x}|}|F_{\alpha\Omega}(\underline{x}/\alpha)-G_{\alpha\Omega}(\underline{x}/\alpha)|\right)\le\alpha^{d-1}\omega(\rho)|\underline{x}|+\frac{C_{\Omega,m}}{\rho}\alpha^{d-2}|\underline{x}|^2. \tag{4.33}$$

Therefore, it suffices to bound

$$\begin{aligned}\int\cdots\int\Big|\mathrm{Tr}\Big[\big(\tilde P_u^1\big)^\vee(x_1)\dots\big(\tilde P_u^{m-1}\big)^\vee(x_{m-1})\big(\tilde P_u^m\big)^\vee(-x_1-\dots-x_{m-1})\Big]\Big|&\\ \times\Big(\alpha^{d-1}\omega(\rho)|\underline{x}|+\frac{C_{\Omega,m}}{\rho}\alpha^{d-2}|\underline{x}|^2\Big)\,dx_1\dots dx_{m-1},&\end{aligned}$$

and we can be rather sloppy with constants since $\alpha^{d-1}\omega(\rho)$ is already lower order.

Let us set $R_u^n(p)=\tilde P_u^n(p\frac{\ell_\delta(u)}{D_n})$. By Lemma 4.2 R_u^n is supported on a ball of radius less than $3D_n^2$, and the first $d+1$ derivatives are bounded by a universal constant. Simple scale invariance shows, for all $s\ge 0$,

$$\||x|^s\big(\tilde P_u^n\big)^\vee\|_p=\left(\frac{\ell_\delta(u)}{D_n}\right)^{d(1-1/p)-s}\||x|^s\big(R_u^n\big)^\vee\|_p.$$

Hence, by the well known estimate $\|\check f\|_1\le C_d\|f\|_{H^{d/2+1}}$,

$$\begin{aligned}\|\big(\tilde P_u^n\big)^\vee\|_1&\le C_dD_n^d,\\ \|\big(\tilde P_u^n\big)^\vee\|_2&\le C_d\ell_\delta(u)^{d/2}D_n^{d/2},\\ \||x|\big(\tilde P_u^n\big)^\vee\|_2&\le C_d\ell_\delta(u)^{d/2-1}D_n^{d/2+1},\\ \||x|^2\big(\tilde P_u^n\big)^\vee\|_2&\le C_d\ell_\delta(u)^{d/2-2}D_n^{d/2+2}.\end{aligned}$$

Note we only need $\max\{2, d/2+1\} \le d+1$ derivatives for all bounds. Using Lemma 4.4 we therefore see that, after bounding $|\underline{x}| \le (m-1)\sum_{n=1}^{m-1}|x_n|$,

$$\begin{aligned}\int\cdots\int&\left|\mathrm{Tr}\left[\big(\tilde P_u^1\big)^\vee(x_1)\dots\big(\tilde P_u^{m-1}\big)^\vee(x_{m-1})\big(\tilde P_u^m\big)^\vee(-x_1-\dots-x_{m-1})\right]\right||\underline{x}|dx_1\dots dx_{m-1}\\ &\le (m-1)\sum_{n=1}^{m-1}\|\big(\tilde P_u^m\big)^\vee\|_2\||x|\big(\tilde P_u^n\big)^\vee(x)\|_2\prod_{i\neq n,m}\|\big(\tilde P_u^i\big)^\vee\|_1\\ &\le C_{d,m}\ell_\delta(u)^{d-1}\sum_{n=1}^{m-1}D_m^{d/2}D_n^{d/2+1}\prod_{i\neq n,m}D_i^d.\end{aligned}$$

Similarly, bounding $|\underline{x}|^2 \le m^2\sum_{n=1}^{m-1}|x_n|^2$,

$$\begin{aligned}\int\cdots\int&\left|\mathrm{Tr}\left[\big(\tilde P_u^1\big)^\vee(x_1)\dots\big(\tilde P_u^{m-1}\big)^\vee(x_{m-1})\big(\tilde P_u^m\big)^\vee(-x_1-\dots-x_{m-1})\right]\right||\underline{x}|^2\,dx_1\dots dx_{m-1}\\ &\le m^2\sum_{n=1}^{m-1}\|\big(\tilde P_u^m\big)^\vee\|_2\||x|^2\big(\tilde P_u^n\big)^\vee(x)\|_2\prod_{i\neq n,m}\|\big(\tilde P_u^i\big)^\vee\|_1\\ &\le C_{d,m}\ell_\delta(u)^{d-2}\sum_{n=1}^{m-1}D_m^{d/2}D_n^{d/2+2}\prod_{i\neq n,m}D_i^d.\end{aligned}$$

Again, since we are not that concerned about constants, we simply bound $D_i \le D_m$ for all $1 \le i \le m$ in agreement with (4.8). Applying these bounds in (4.31) with (4.33) we see that, for all $\rho > 0$,

$$\begin{aligned}|\mathcal{E}_2| &\le C_{d,m}\|F\|_\infty^{m-1}D_m^{d(m-1)+2}\int\ell_\delta(u)^{-d}G(u)\left(\alpha^{d-1}\ell_\delta(u)^{d-1}\omega(\rho)+\frac{C_{\Omega,m}}{\rho}\alpha^{d-2}\ell_\delta(u)^{d-2}\right)du\\ &\le C_{d,m,\Omega,\Gamma}\|F\|_\infty^{m-1}D_m^{d(m-1)+2}\big(\|F\|_\infty+\|G\|_1\big)\left(\alpha^{d-1}\log(\delta^{-1})\omega(\rho)+\frac{1}{\rho}\alpha^{d-2}\delta^{-1}\right).\end{aligned}\tag{4.34}$$

The u-integral is computed using (C.2) and Lemma $C.2$ like for $\mathcal{E}_1$ and $\mathcal{E}_3$.

4.3. **Closing the argument.** We now show how to choose the sequences (D_n) and (a_n) according to (4.8) and (4.9), and the free parameter ρ introduced in (4.33). Concretely, by the final bounds on $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_3$ from (4.23) and (4.34) we get (see (4.11))

$$\begin{aligned}&\left|\mathrm{Tr}\left[S(P^1,\dots,P^m;\alpha\Omega)\right]-\big(\frac{\alpha}{2\pi}\big)^{d-1}\int_{T^*(\mathcal{F}\Omega)}\mathrm{Tr}\left[S(P_X^1,\dots,P_X^m;\mathbb{R}_+)\right]dX\right|\\ &\quad\le C_{d,m,\Omega,\Gamma}\|F\|_\infty^{m-1}\big(\|F\|_\infty+\|G\|_1\big)\\ &\qquad\times\left(\alpha^{d-1}\log(\delta^{-1})\left(\omega(\rho)D_m^{d(m-1)+2}+\sum_{n=2}^{m}K_n\right)+\frac{\alpha^{d-2}\delta^{-1}}{\rho}D_m^{d(m-1)+2}\right),\end{aligned}$$

where K_n is defined in (4.22) by

$$K_n = D_{n-1}^{2d}\left(a_n^{-1/2}+\left(\frac{a_n}{D_n}\right)^{1/2}\left(1-3\frac{D_{n-1}}{D_n}\right)^{-1}\right).$$

For $\varepsilon > 0$ small we choose (D_n) and (a_n) such that $K_n \le C\varepsilon$ for all n. Explicitly,

$$D_n = a_n^2,\quad D_{n-1}^{2d} = \varepsilon D_n^{1/4}$$

does the trick for $\varepsilon \le 1/2$. Indeed, we have

$$D_n - 14D_{n-1} \ge \frac{D_{n-1}^{8d}}{\varepsilon^4} - 14D_{n-1} \ge D_{n-1}\left(\frac{1}{\varepsilon^4}-14\right) \ge 0,$$

hence

$$D_n - 3D_{n-1} - 4 \geq D_n - 7D_{n-1} \geq \frac{1}{2}D_n > 0$$

and also

$$\frac{D_{n-1}}{2}\big(D_n - 3D_{n-1} - 4\big) \geq \frac{D_{n-1}D_n}{4} \geq \frac{a_n^2}{4} \geq a_n.$$

At the end we used that $a_n \geq a_2 \geq 4$. Hence (4.8) and (4.9) are satisfied, and since clearly

$$\left(1 - 3\frac{D_{n-1}}{D_n}\right)^{-1} \leq 2,$$

our choice of D_n and a_n ensures that $K_n \leq 3\varepsilon$ for $n = 2, \dots, m$. Finally, we choose ρ small enough dependent on ε (and the convergence rate stated in (4.32)) such that

$$D_m^{d(m-1)+2}\omega(\rho) \leq \varepsilon.$$

This finishes the proof of Proposition 4.1.

4.4. **Proof of Theorem** 2.5. We need to remove the additional assumptions on Ω and (P_δ^i) from Proposition 4.1. So fix $m \geq 2$, let $\Gamma \subseteq \mathbb{R}^d$ be a set of locally uniform upper Minkowski content, let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter, and assume that (P_δ^i) satisfies a $C^1(\mathbb{R}^d; \mathcal{M}_n)$ multiscale estimate on Γ for $i = 1, \dots, m$. We approximate (P_δ^i) and Ω separately.

First for the symbols. It follows from Lemma $D.2$ that for all $0 < \varepsilon < 1$ we can take a family $(P_{\delta,\varepsilon}^i)$ satisfying a $C^{d+1}(\mathbb{R}^d; \mathcal{M}_n)$ multiscale estimate on Γ, $i = 1, \dots, m$, such that

$$\|P_{\delta,\varepsilon}^i - P_\delta^i\|_\infty \leq C\varepsilon, \quad \|P_{\delta,\varepsilon}^i - P_\delta^i\|_{\dot{H}^{1/2}}^2 \leq C\varepsilon^{1/2}\log(\delta^{-1}). \tag{4.35}$$

uniformly for $0 < \delta \leq 1/2$ and $0 < \varepsilon < 1$. By Lemma $D.1$ we also have

$$\|P_\delta^i\|_{\dot{H}^{1/2}}^2, \|P_{\delta,\varepsilon}^i\|_{\dot{H}^{1/2}}^2 \leq C\log(\delta^{-1}), \quad \|P_\delta^i\|_\infty, \|P_{\delta,\varepsilon}^i\|_\infty \leq C. \tag{4.36}$$

Now for Ω. If $d \geq 2$, it follows from Lemma $C.7$ that we can find sets $\Omega_\mu \subseteq \mathbb{R}^d$, $0 < \mu < 1$, such that, for each μ, the boundary $\partial\Omega_\mu$ is compact and contained in a finite union of Lipschitz graphs, and

$$\mathcal{H}^{d-1}(\mathcal{F}\Omega \setminus C_\mu) \leq \mu, \quad \mathcal{H}^{d-1}(\mathcal{F}\Omega_\mu \setminus C_\mu) \leq \mu, \quad C_\mu = \{x \in \mathcal{F}\Omega \cap \mathcal{F}\Omega_\mu \mid \nu_\Omega(x) = \nu_{\Omega_\mu}(x)\}.$$

In particular

$$\mathrm{Per}(\Omega \setminus \Omega_\mu) \leq 3\mu, \quad \mathrm{Per}(\Omega_\mu \setminus \Omega) \leq 3\mu,$$

see [31, Theorem 16.3]. If $d = 1$ we simply take Ω_μ to be the measure zero modification to Ω such that Ω_μ is given by finite union of separated intervals, see [1, Proposition 3.52].

We need to show that

$$\begin{aligned}\limsup_{\alpha\to\infty}\sup_{\delta\geq 1/\alpha}\frac{1}{\alpha^{d-1}\log(\delta^{-1})}\Big|&\operatorname{Tr}\left[S(P_\delta^1, \dots, P_\delta^m; \alpha\Omega)\right]\\ &-\left(\frac{\alpha}{2\pi}\right)^{d-1}\int_{T^*(\mathcal{F}\Omega)}\operatorname{Tr}\left[S((P_\delta^1)_X, \dots, (P_\delta^m)_X; \mathbb{R}_+)\right]dX\Big| = 0\end{aligned} \tag{4.37}$$

To this end decompose

$$
\begin{aligned}
&\left|\operatorname{Tr}\left[S(P_\delta^1,\dots,P_\delta^m;\alpha\Omega)\right] - \Big(\frac{\alpha}{2\pi}\Big)^{d-1}\int_{T^*(\mathcal{F}\Omega)}\operatorname{Tr}\left[S((P_\delta^1)_X,\dots,(P_\delta^m)_X;\mathbb{R}_+)\right]dX\right|\\
&\quad\le \left|\operatorname{Tr}\left[S(P_\delta^1,\dots,P_\delta^m;\alpha\Omega) - S(P_\delta^1,\dots,P_\delta^m;\alpha\Omega_\mu)\right]\right|\\
&\quad + \left|\operatorname{Tr}\left[S(P_\delta^1,\dots,P_\delta^m;\alpha\Omega_\mu) - S(P_{\delta,\varepsilon}^1,\dots,P_{\delta,\varepsilon}^m;\alpha\Omega_\mu)\right]\right|\\
&\quad + \left|\operatorname{Tr}\left[S(P_{\delta,\varepsilon}^1,\dots,P_{\delta,\varepsilon}^m;\alpha\Omega_\mu)\right] - \Big(\frac{\alpha}{2\pi}\Big)^{d-1}\int_{T^*(\mathcal{F}\Omega_\mu)}\operatorname{Tr}\left[S((P_{\delta,\varepsilon}^1)_X,\dots,(P_{\delta,\varepsilon}^m)_X;\mathbb{R}_+)\right]dX\right|\\
&\quad + \Big(\frac{\alpha}{2\pi}\Big)^{d-1}\left|\int_{T^*(\mathcal{F}\Omega_\mu)}\operatorname{Tr}\left[S((P_{\delta,\varepsilon}^1)_X,\dots,(P_{\delta,\varepsilon}^m)_X;\mathbb{R}_+) - S((P_\delta^1)_X,\dots,(P_\delta^m)_X;\mathbb{R}_+)\right]dX\right|\\
&\quad + \Big(\frac{\alpha}{2\pi}\Big)^{d-1}\left|\Big(\int_{T^*(\mathcal{F}\Omega_\mu)} - \int_{T^*(\mathcal{F}\Omega)}\Big)\operatorname{Tr}\left[S((P_\delta^1)_X,\dots,(P_\delta^m)_X;\mathbb{R}_+)\right]dX\right|\\
&\quad = I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
$$

The I_3 contribution is handled using Proposition 4.1 directly. For all $0 < \varepsilon, \mu < 1$ we have

$$
\begin{aligned}
\limsup_{\alpha\to\infty}\sup_{\delta\ge 1/\alpha}\frac{1}{\alpha^{d-1}\log(\delta^{-1})}&\left|\operatorname{Tr}\left[S(P_{\delta,\varepsilon}^1,\dots,P_{\delta,\varepsilon}^m;\alpha\Omega_\mu)\right]\right.\\
&\left.-\Big(\frac{\alpha}{2\pi}\Big)^{d-1}\int_{T^*(\mathcal{F}\Omega_\mu)}\operatorname{Tr}\left[S((P_{\delta,\varepsilon}^1)_X,\dots,(P_{\delta,\varepsilon}^m)_X;\mathbb{R}_+)\right]dX\right| = 0.
\end{aligned}
$$

Next, we estimate I_2 and I_4. By the simple multilinear expansion (3.8), the $\dot{H}^{1/2}$ trace class estimate from Corollary 3.9, and the bounds (4.35), (4.36) we can bound

$$
I_2 \le C\operatorname{Per}(\Omega_\mu)\alpha^{d-1}\log(\delta^{-1})\varepsilon^{1/2} \le C\alpha^{d-1}\log(\delta^{-1})\varepsilon^{1/2}.
$$

The term I_4 is handled in the same way using Lemma $B.5$ to bound the $\dot{H}^{1/2}$ norms of the co-tangent sections.

For I_1 we simply apply Lemma 3.13 and (4.36). We find

$$
I_1 \le C\alpha^{d-1}\log(\delta^{-1})\mu^{1/2}.
$$

Finally for I_5, we further decompose

$$
\begin{aligned}
I_5 \le C\alpha^{d-1}&\int_{\mathcal{F}\Omega_\mu\setminus C_\mu}\int_{\Pi_{\nu_{\Omega_\mu}(x)}}\left|\operatorname{Tr}\left[S((P_\delta^1)_{(x,p)},\dots,(P_\delta^m)_{(x,p)};\mathbb{R}_+)\right]\right|dp\,d\mathcal{H}^{d-1}(x)\\
&+ C\alpha^{d-1}\int_{\mathcal{F}\Omega\setminus C_\mu}\int_{\Pi_{\nu_{\Omega}(x)}}\left|\operatorname{Tr}\left[S((P_\delta^1)_{(x,p)},\dots,(P_\delta^m)_{(x,p)};\mathbb{R}_+)\right]\right|dp\,d\mathcal{H}^{d-1}(x).
\end{aligned}
$$

Bounding the co-tangent operators in trace norm using Corollary 3.9 and Lemma $B.5$ (like for I_4), we find

$$
I_5 \le C\alpha^{d-1}\log(\delta^{-1})\big(\mathcal{H}^{d-1}(\mathcal{F}\Omega\setminus C_\mu) + \mathcal{H}^{d-1}(\mathcal{F}\Omega_\mu\setminus C_\mu)\big) \le C\alpha^{d-1}\log(\delta^{-1})\mu.
$$

Returning to (4.37) we find

$$
\limsup_{\alpha\to\infty}\sup_{\delta\ge 1/\alpha}\frac{1}{\alpha^{d-1}\log(\delta^{-1})}\Big(I_1 + I_2 + I_3 + I_4 + I_5\Big) \le C\big(\mu^{1/2} + \varepsilon^{1/2}\big).
$$

Taking μ and ε to zero finishes the proof.

5. Widom's half-line trace formula and area scaling

Next we study the area scaling case for a fixed symbol. In relation to the rest of the present paper, the main point of this section is Widom's remarkable trace formula from [54]

$$\operatorname{Tr}\left[S_m(P;\mathbb{R}_+)\right] = \frac{1}{8\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{U_m(P(\xi_1),P(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2, \tag{5.1}$$

for scalar symbols $P \in \dot{H}^{1/2}(\mathbb{R};\mathbb{C}) \cap L^\infty$, which we reprove with an elementary argument in Proposition 5.3. The importance of (5.1) for the rest of the paper is that it turns the co-tangent traces appearing in Theorem 2.5 into explicit double integrals on which we can carry out exact computations.

We also prove (1.22) for matrix valued symbols in Proposition 5.4. Here the hard work has already been carried out in Section 4 and we are left only with simple approximation arguments. Theorem 1.13 is then an immediate consequence of Proposition 5.4 and the trace formula (5.1).

We remind the reader that we defined

$$U_m(A,B) = \int_0^1 \frac{(\theta A + (1-\theta)B)^m - A^m\theta - B^m(1-\theta)}{\theta(1-\theta)}\,d\theta, \quad A,B \in \mathbb{C},$$

It is elementary to verify the following convenient form:

$$U_m(A,B) = -\frac{1}{2}\sum_{k=1}^{m-1}\left(\frac{1}{k}+\frac{1}{m-k}\right)\left(A^k-B^k\right)\left(A^{m-k}-B^{m-k}\right). \tag{5.2}$$

We first establish the trace formula (5.1) under stronger assumptions on the symbol.

Lemma 5.1. *Let $P \in H^s(\mathbb{R};\mathbb{C}) \cap L^\infty$ for $s > 1/2$. Then, for all $m \geq 2$,*

$$\operatorname{Tr}\left[S_m(P;\mathbb{R}_+)\right] = \frac{1}{8\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{U_m(P(\xi_1),P(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2.$$

Our argument is inspired by the classical proof of Kac [21] of the strong Szegő limit theorem. Unlike Widom we do not rely on the Wiener-Hopf decomposition or other complex analytical techniques. Instead, we use the following simple combinatorial lemma. Similar statements can be found in [21], for instance.

Lemma 5.2. *Let $m \geq 1$ and denote by C_m the cyclic permutations on $\{1,\ldots,m\}$. For any $x_1,\ldots,x_m \in \mathbb{R}$ the following holds:*

$$\begin{aligned}\sum_{\sigma\in C_m}\Big(\max_{1\le n\le m}\{0,x_{\sigma_1}+\cdots+x_{\sigma_n}\} - \max_{1\le n\le m-1}\{0,x_{\sigma_1}+\cdots+x_{\sigma_n}\}\Big)&\\ = \max\{0,x_1+\cdots+x_m\}.&\end{aligned}$$

This is seen as a trivial identity when $m = 1$.

Proof. Assume $m \geq 2$. If $x_1+\cdots+x_m < 0$, then trivially

$$\max_{1\le n\le m}\{0,x_{\sigma_1}+\cdots+x_{\sigma_n}\} - \max_{1\le n\le m-1}\{0,x_{\sigma_1}+\cdots+x_{\sigma_n}\} = 0$$

for all $\sigma \in C_m$, in agreement with the claim. Hence, we assume $x_1+\cdots+x_m \geq 0$ and fix $\sigma \in C_m$. Then

$$\max_{1\le n\le m}\{0,x_{\sigma_1}+\cdots+x_{\sigma_n}\} = \max_{1\le n\le m}\{x_{\sigma_1}+\cdots+x_{\sigma_n}\} = x_{\sigma_1} + \max_{2\le n\le m}\{0,x_{\sigma_2}+\cdots+x_{\sigma_n}\}.$$

Summing over all σ we find by telescoping

$$\begin{aligned}
&\sum_{\sigma\in C_m}\Big(\max_{1\leq n\leq m}\{0,x_{\sigma_1}+\cdots+x_{\sigma_n}\}-\max_{1\leq n\leq m-1}\{0,x_{\sigma_1}+\cdots+x_{\sigma_n}\}\Big)\\
&\qquad=\sum_{\sigma\in C_m} x_{\sigma_1}+\Big(\max_{2\leq n\leq m}\{0,x_{\sigma_2}+\cdots+x_{\sigma_n}\}-\max_{1\leq n\leq m-1}\{0,x_{\sigma_1}+\cdots+x_{\sigma_n}\}\Big)\\
&\qquad=x_1+\cdots+x_m.
\end{aligned}$$

■

Proof of Lemma 5.1. As part of the proof of Lemma 4.3 we showed that, under the present assumptions on P,

$$\begin{aligned}
\operatorname{Tr}\big[S_m(P;\mathbb{R}_+)\big]&=-(2\pi)^{-m/2}\int\cdots\int\check{P}(x_1)\dots\check{P}(x_{m-1})\check{P}(-x_1-\cdots-x_{m-1})\\
&\qquad\times\max_{1\leq n\leq m-1}\{0,x_1+\cdots+x_n\}\,dx_1\dots dx_{m-1}\\
&=-(2\pi)^{-m/2}\sum_{k=1}^{m-1}\int\cdots\int\check{P}(x_1)\dots\check{P}(x_{m-1})\check{P}(-x_1-\cdots-x_{m-1})\\
&\qquad\times\Big(\max_{1\leq n\leq k}\{0,x_1+\cdots+x_n\}-\max_{1\leq n\leq k-1}\{0,x_1+\cdots+x_n\}\Big)\,dx_1\dots dx_{m-1}.
\end{aligned}\tag{5.3}$$

The second equality follows by telescoping. The expression

$$\check{P}(x_1)\dots\check{P}(x_{m-1})\check{P}(-x_1-\cdots-x_{m-1})\tag{5.4}$$

is clearly invariant under cyclic permutations in $(x_1,\dots,x_k)$ for any $1\leq k\leq m-1$. This uses that P is scalar valued. Performing the corresponding change of variables in (5.3) and averaging we see that

$$\begin{aligned}
\operatorname{Tr}\big[S_m(P;\mathbb{R}_+)\big]&=-(2\pi)^{-m/2}\sum_{k=1}^{m-1}\frac{1}{k}\int\cdots\int\check{P}(x_1)\dots\check{P}(x_{m-1})\check{P}(-x_1-\cdots-x_{m-1})\\
&\qquad\times\sum_{\sigma\in C_k}\Big(\max_{1\leq n\leq k}\{0,x_{\sigma_1}+\cdots+x_{\sigma_n}\}-\max_{1\leq n\leq k-1}\{0,x_{\sigma_1}+\cdots+x_{\sigma_n}\}\Big)\\
&\qquad\times\,dx_1\dots dx_{m-1}\\
&=-(2\pi)^{-m/2}\sum_{k=1}^{m-1}\frac{1}{k}\int\cdots\int\check{P}(x_1)\dots\check{P}(x_{m-1})\check{P}(-x_1-\cdots-x_{m-1})\\
&\qquad\times\max\{0,x_1+\cdots+x_k\}\,dx_1\dots dx_{m-1}.
\end{aligned}\tag{5.5}$$

We employed Lemma 5.2 at the end. The final expression has explicit convolution structure. A direct computation shows, for all $1\leq k\leq m-1$,

$$\begin{aligned}
&\int\cdots\int\check{P}(x_1)\dots\check{P}(x_{m-1})\check{P}(-x_1-\cdots-x_{m-1})\max\{0,x_1+\cdots+x_k\}\,dx_1\dots dx_{m-1}\\
&\qquad=(2\pi)^{\frac{m-2}{2}}\int\big(P^k\big)^\vee(x)\big(P^{m-k}\big)^\vee(-x)\max\{0,x\}\,dx.
\end{aligned}$$

Hence,

$$\operatorname{Tr}\big[S_m(P;\mathbb{R}_+)\big]=-\frac{1}{2\pi}\sum_{k=1}^{m-1}\frac{1}{k}\int\big(P^k\big)^\vee(x)\big(P^{m-k}\big)^\vee(-x)\max\{0,x\}\,dx.\tag{5.6}$$

We need to symmetrize further. First, since $\operatorname{Tr}\big[S_m(P;\mathbb{R}_+)\big] = \overline{\operatorname{Tr}\big[S_m(\overline{P};\mathbb{R}_+)\big]}$, it follows from (5.6) applied to the conjugate $\overline{P}$ that we also have

$$\begin{aligned}\operatorname{Tr}\big[S_m(P;\mathbb{R}_+)\big] &= -\frac{1}{2\pi}\sum_{k=1}^{m-1}\frac{1}{k}\int \overline{\big(\overline{P}^k\big)^\vee(x)\big(\overline{P}^{m-k}\big)^\vee(-x)}\max\{0,x\}\,dx\\ &= -\frac{1}{2\pi}\sum_{k=1}^{m-1}\frac{1}{k}\int \big(P^k\big)^\vee(-x)\big(P^{m-k}\big)^\vee(x)\max\{0,x\}\,dx\\ &= -\frac{1}{2\pi}\sum_{k=1}^{m-1}\frac{1}{k}\int \big(P^k\big)^\vee(x)\big(P^{m-k}\big)^\vee(-x)\max\{0,-x\}\,dx.\end{aligned}$$

Averaging with the original expression (5.6) gives

$$\begin{aligned}\operatorname{Tr}\big[S_m(P;\mathbb{R}_+)\big] &= -\frac{1}{4\pi}\sum_{k=1}^{m-1}\frac{1}{k}\int \big(P^k\big)^\vee(x)\big(P^{m-k}\big)^\vee(-x)|x|\,dx\\ &= -\frac{1}{4\pi}\sum_{k=1}^{m-1}\frac{1}{2}\left(\frac{1}{k}+\frac{1}{m-k}\right)\int \big(P^k\big)^\vee(x)\big(P^{m-k}\big)^\vee(-x)|x|\,dx.\end{aligned}$$

We averaged out the symmetry in k and $m-k$ at the end. Finally, the latter integrals can be expressed in Fourier-free terms using (1.19):

$$\begin{aligned}&\int \big(P^k\big)^\vee(x)\big(P^{m-k}\big)^\vee(-x)|x|\,dx\\ &\qquad= \frac{1}{2\pi}\iint \frac{\big(P(\xi_1)^k - P(\xi_2)^k\big)\big(P(\xi_1)^{m-k}-P(\xi_2)^{m-k}\big)}{|\xi_1-\xi_2|^2}\,d\xi_1 d\xi_2.\end{aligned}$$

Comparing with (5.2) finishes the proof. ■

We now establish the $\dot{H}^{1/2}$ case by an approximation argument.

Proposition 5.3. *Let $P \in \dot{H}^{1/2}(\mathbb{R};\mathbb{C}) \cap L^\infty$. Then, for all $m \geq 2$,*

$$\operatorname{Tr}\big[S_m(P;\mathbb{R}_+)\big] = \frac{1}{8\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{U_m(P(\xi_1),P(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2. \tag{5.7}$$

Proof. We first claim that (5.7) holds for all $P \in C_c^\infty(\mathbb{R};\mathbb{C}) + \mathbb{C}$. Indeed, if $P = P_0 + P_\infty$ with $P_0 \in C_c^\infty(\mathbb{R};\mathbb{C})$ and $P_\infty \in \mathbb{C}$, then binomial expansion shows

$$S_m(P;\mathbb{R}_+) = \sum_{n=2}^{m}\binom{m}{n}P_\infty^{m-n}S_n(P_0;\mathbb{R}_+)$$

and

$$U_m(A+P_\infty, B+P_\infty) = \sum_{n=2}^{m}\binom{m}{n}P_\infty^{m-n}U_n(A,B), \qquad A,B\in\mathbb{C}.$$

The claim therefore follows from Lemma 5.1.

Consider now $P \in \dot{H}^{1/2}(\mathbb{R};\mathbb{C}) \cap L^\infty$. Using Lemma B.2 we can find a sequence $(P_k) \subseteq C_c^\infty(\mathbb{R};\mathbb{C}) + \mathbb{C}$ such that $P_k \to P$ in $\dot{H}^{1/2}$ and pointwise almost everywhere, and $\|P_k\|_\infty \leq 3\|P\|_\infty$. Since (5.7) holds for P_k for all fixed $k \in \mathbb{N}$, it suffices to show that

$$\limsup_{k\to\infty}\Big|\operatorname{Tr}\big[S_m(P;\mathbb{R}_+)\big] - \operatorname{Tr}\big[S_m(P_k;\mathbb{R}_+)\big]\Big| = 0, \tag{5.8}$$

and

$$\limsup_{k\to\infty}\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{U_m(P(\xi_1),P(\xi_2)) - U_m(P_k(\xi_1),P_k(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2\right| = 0. \tag{5.9}$$

For (5.8), it follows from Lemma 3.1 and the multilinear bound from Lemma 3.11 that it suffices to show that

$$\limsup_{k\to\infty} \|P^j(P-P_k)(P_k)^l\|_{\dot{H}^{1/2}} = 0$$

for all powers $j, l \geq 0$. This in turn is an immediate consequence of Lemma B.4.

The claim (5.9) is similar. We rewrite using the identity (5.2), here with $\langle\cdot,\cdot\rangle_{\dot{H}^{1/2}}$ denoting the inner product on $\dot{H}^{1/2}$,

$$\begin{aligned}&\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{U_m(P(\xi_1),P(\xi_2)) - U_m(P_k(\xi_1),P_k(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2\right| \\ &\qquad \leq \frac{1}{2}\sum_{j=1}^{m-1}\left(\frac{1}{j}+\frac{1}{m-j}\right)\left|\langle \overline{P}^j, P^{m-j}\rangle_{\dot{H}^{1/2}} - \langle \overline{P}_k^j, P_k^{m-j}\rangle_{\dot{H}^{1/2}}\right|.\end{aligned}$$

It therefore suffices to show that $P_k^j \to P^j$ in $\dot{H}^{1/2}$ for all $j \geq 1$, and this is again a consequence of Lemma B.4. ■

We finally establish area scaling for general sets Ω of finite perimeter and matrix valued symbols $P \in \dot{H}^{1/2}$.

Proposition 5.4. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter and let $P \in \dot{H}^{1/2}(\mathbb{R}^d;\mathcal{M}_n)\cap L^\infty$. Then, for any $m \geq 2$,*

$$\operatorname{Tr}\left[S_m(P;\alpha\Omega)\right] = \left(\frac{\alpha}{2\pi}\right)^{d-1}\int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S_m(P_X;\mathbb{R}_+)\right]dX + o(\alpha^{d-1}) \tag{5.10}$$

as $\alpha \to \infty$.

Proof. We first note that (5.10) holds if $P \in C_c^\infty(\mathbb{R}^d;\mathcal{M}_n) + \mathcal{M}_n$. Indeed, this follows from Theorem 2.5 applied to the constant family $P_\delta = P$ with $\Gamma = \{0\}$, say, and $0 < \delta \leq 1/2$ fixed. Alternatively, following the argumentation from subsection 4.2, this is a consequence of Lemma 4.3 and Lemma C.6.

From here the argument is very similar to that of Proposition 5.3. Fix $P \in \dot{H}^{1/2}(\mathbb{R}^d;\mathcal{M}_n)\cap L^\infty$ and take $P_k \in C_c^\infty(\mathbb{R}^d;\mathcal{M}_n)+\mathcal{M}_n$ such that $P_k \to P$ in $\dot{H}^{1/2}$ and pointwise almost everywhere, and $\|P_k\|_\infty \leq 3\|P\|_\infty$. Since (5.10) holds for P_k for each fixed $k \in \mathbb{N}$, it suffices to show that

$$\limsup_{k\to\infty}\limsup_{\alpha\to\infty}\frac{1}{\alpha^{d-1}}\left|\operatorname{Tr}\left[S_m(P;\alpha\Omega)\right] - \operatorname{Tr}\left[S_m(P_k;\alpha\Omega)\right]\right| = 0 \tag{5.11}$$

and that

$$\limsup_{k\to\infty}\int_{T^*(\mathcal{F}\Omega)}\left|\operatorname{Tr}\left[S_m(P_X;\mathbb{R}_+)\right] - S_m((P_k)_X;\mathbb{R}_+)\right]dX\right| = 0. \tag{5.12}$$

The claim (5.11) is handled exactly like (5.8). The argument for (5.12) is similar, but this time we have to rely on Lemma B.5 to bound the $\dot{H}^{1/2}$-norms of the co-tangent sections. Concretely, arguing like in (5.8), it suffices to show that

$$\limsup_{k\to\infty}\int_{T^*(\mathcal{F}\Omega)}\|P_X^j(P_X-(P_k)_X)(P_k)_X^l\|_{\dot{H}^{1/2}}\|Q_X\|_{\dot{H}^{1/2}}\,dX = 0,$$

where either $Q = P$ or $Q = P_k$. By Lemma B.5,

$$\int_{T^*(\mathcal{F}\Omega)}\|P_X^j(P_X-(P_k)_X)(P_k)_X^l\|_{\dot{H}^{1/2}}\|Q_X\|_{\dot{H}^{1/2}}\,dX \leq C_d\operatorname{Per}(\Omega)\|Q\|_{\dot{H}^{1/2}}\|P^j(P-P_k)P_k^l\|_{\dot{H}^{1/2}},$$

so the claim follows from Lemma B.4 as before. ■

Proof of Theorem 1.13. By Proposition 5.4,

$$\operatorname{Tr}\left[S_m(P;\alpha\Omega)\right] = \left(\frac{\alpha}{2\pi}\right)^{d-1}\int_{T^*(\mathcal{F}\Omega)}\operatorname{Tr}\left[S_m(P_X;\mathbb{R}_+)\right]dX + o(\alpha^{d-1}).$$

Note $P_X \in \dot{H}^{1/2}(\mathbb{R};\mathbb{C})$ for almost all $X \in T^*(\mathcal{F}\Omega)$ by Lemma $B.5$, so Proposition 5.3 applies and yields

$$\operatorname{Tr}\left[S_m(P_X;\mathbb{R}_+)\right] = \frac{1}{8\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{U_m(P_X(\xi_1), P_X(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2.$$

The claim follows. ■

6. The boundary coefficient and the Widom formula

We now pass from the double-integral representation to the Widom formula. We first compute the boundary coefficient for multiscale symbols and obtain Theorem 2.8, and then use approximation arguments to prove Theorem 1.1.

6.1. **Boundary coefficient for multiscale symbols.** In this subsection we compute the boundary coefficient:

$$\int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S_m((P_\delta)_X;\mathbb{R}_+)\right] dX$$

for multiscale (P_δ). The main input is Theorem 1.13 and the computation from Lemma $A.11$.

We here remark that, as an immediate consequence of (5.2),

$$|U_m(A,B)| \le \frac{m(m-1)}{2}\max\{|A|,|B|\}^{m-2}|A-B|^2, \tag{6.1}$$

and

$$\begin{aligned}&|U_m(A,B) - U_m(A',B')|\\ &\quad\le \frac{m(m-1)}{2}K^{m-2}\big(|A-A'|+|B-B'|\big)\big(|A-B|+|A'-B'|\big),\end{aligned} \tag{6.2}$$

where $K = \max\{|A|,|A'|,|B|,|B'|\}$.

Proposition 6.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter, let $\Gamma \subseteq \mathbb{R}^d$ be a set of finite upper Minkowski content, and $m \ge 2$ an integer. Assume that (P_δ) satisfies a $C^1(\mathbb{R}^d;\mathbb{C})$ multiscale estimate on Γ and that there exists a function $P \in \dot{BV}(\mathbb{R}^d;\mathbb{C})$ such that $\|P_\delta - P\|_1 = o\big(\delta\log(\delta^{-1})\big)$ as $\delta \to 0^+$. Then,*

$$\begin{aligned}&\int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S_m((P_\delta)_X;\mathbb{R}_+)\right] dX\\ &\quad= \frac{\log(\delta^{-1})}{4\pi^2}\int_{\mathcal{F}\Omega}\int_{J_P} U_m(P^+(p),P^-(p))|\nu_\Omega(x)\cdot\nu_P(p)|\,d\mathcal{H}^{d-1}(p)\,d\mathcal{H}^{d-1}(x)\\ &\qquad+ o(\log(\delta^{-1}))\end{aligned}$$

as $\delta \to 0^+$.

Proof. Take $F \in L^1 \cap L^\infty$ such that (P_δ, F) satisfies the $C^1(\mathbb{R}^d;\mathbb{C})$ multiscale estimate on Γ. The assumptions on P_δ easily ensures that $P_\delta \in \dot{H}^{1/2}(\mathbb{R}^d;\mathbb{C}) \cap L^\infty$ and therefore that $(P_\delta)_X \in \dot{H}^{1/2}(\mathbb{R};\mathbb{C}) \cap L^\infty$ for almost all $X \in T^*(\mathcal{F}\Omega)$ by Lemma $B.5$. Hence, we can apply the trace formula from Theorem 1.13:

$$\begin{aligned}&\int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S_m((P_\delta)_X;\mathbb{R}_+)\right] dX\\ &\quad= \frac{1}{8\pi^2}\int_{T^*(\mathcal{F}\Omega)}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{U_m((P_\delta)_X(\xi_1),(P_\delta)_X(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2\,dX.\end{aligned} \tag{6.3}$$

We split the $d\xi_1\,d\xi_2$ integral into the regions $|\xi_1-\xi_2| \le \delta$, $\delta \le |\xi_1-\xi_2| \le 1$, and $|\xi_1-\xi_2| \ge 1$.

The region $|\xi_1-\xi_2|\leq\delta$ can be bounded using the multiscale assumption on P_δ:

$$\begin{aligned}
&\left|\int_{T^*(\mathcal{F}\Omega)}\iint_{|\xi_1-\xi_2|\leq\delta}\frac{U_m((P_\delta)_X(\xi_1),(P_\delta)_X(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2\,dX\right|\\
&\qquad\leq C\int_{T^*(\mathcal{F}\Omega)}\iint_{|\xi_1-\xi_2|\leq\delta}\frac{|(P_\delta)_X(\xi_1)-(P_\delta)_X(\xi_2)|^2}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2\,dX\\
&\qquad\leq C\delta\int_{\mathcal{F}\Omega}\int_{\Pi_{\nu_\Omega(x)}}\int_{\mathbb{R}}|\nu\cdot\nabla P_\delta(p+\xi\nu_\Omega(x))|^2\,d\xi\,dp\,d\mathcal{H}^{d-1}(x)\\
&\qquad\leq C\delta\int_{\mathbb{R}^d}F(p)^2\ell_\delta(p)^{-2}\,dp\leq C.
\end{aligned}$$

The first inequality uses (6.1) and that the (P_δ)'s are uniformly bounded. The second inequality is Lemma $A.7$ with $q=2$, and the final integral is bounded by a simple computation using Lemma $C.2$ and that Γ has finite upper Minkowski content.

The region $|\xi_1-\xi_2|\geq 1$ can be handled only using integrability alone. Bounding U_m as before,

$$\begin{aligned}
&\left|\int_{T^*(\mathcal{F}\Omega)}\iint_{|\xi_1-\xi_2|\geq 1}\frac{U_m((P_\delta)_X(\xi_1),(P_\delta)_X(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2\,dX\right|\\
&\qquad\leq\int_{T^*(\mathcal{F}\Omega)}\iint_{|\xi_1-\xi_2|\geq 1}\frac{|(P_\delta)_X(\xi_1)|^2+|(P_\delta)_X(\xi_2)|^2}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2\,dX\\
&\qquad\leq C\int_{T^*(\mathcal{F}\Omega)}\|(P_\delta)_X\|_2^2\,dX\leq C.
\end{aligned}$$

For the final region $\delta\leq|\xi_1-\xi_2|\leq 1$ we use the convergence $P_\delta\to P$ in L^1. Specifically, by (6.2) and uniform boundedness

$$\begin{aligned}
&\left|\int_{T^*(\mathcal{F}\Omega)}\iint_{\delta\leq|\xi_1-\xi_2|\leq 1}\frac{U_m((P_\delta)_X(\xi_1),(P_\delta)_X(\xi_2))-U_m(P_X(\xi_1),P_X(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2\,dX\right|\\
&\qquad\leq C\int_{T^*(\mathcal{F}\Omega)}\iint_{\delta\leq|\xi_1-\xi_2|\leq 1}\frac{|(P_\delta)_X(\xi_1)-P_X(\xi_1)|+|(P_\delta)_X(\xi_2)-P_X(\xi_2)|}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2\,dX\\
&\qquad\leq C\frac{1}{\delta}\int_{T^*(\mathcal{F}\Omega)}\|(P_\delta)_X-P_X\|_1\,dX\leq C\frac{\|P_\delta-P\|_1}{\delta}=o(\log(\delta^{-1}))
\end{aligned}$$

as $\delta\to 0^+$.

Going back to (6.3) we conclude that

$$\begin{aligned}
&\int_{T^*(\mathcal{F}\Omega)}\mathrm{Tr}\left[S_m((P_\delta)_X;\mathbb{R}_+)\right]dX\\
&\qquad=\frac{1}{8\pi^2}\int_{T^*(\mathcal{F}\Omega)}\iint_{\delta\leq|\xi_1-\xi_2|\leq 1}\frac{U_m(P_X(\xi_1),P_X(\xi_2))}{|\xi_1-\xi_2|^2}\,d\xi_1\,d\xi_2\,dX+o(\log(\delta^{-1})),
\end{aligned}\tag{6.4}$$

as $\delta\to 0^+$. Introduce the kernels $\eta_\delta:\mathbb{R}_+\to\mathbb{R}_+$ given by

$$\eta_\delta(h)=\frac{1}{\log(\delta^{-1})}1_{\{\delta\leq h\leq 1\}}(h)\frac{1}{h}.$$

Clearly

$$\int_0^\infty\eta_\delta(h)\,dh=1,\qquad\int_\varepsilon^\infty\eta_\delta(h)\,dh\to 0\text{ as }\delta\to 0^+\text{ for all }\varepsilon>0,$$

and by (6.4),

$$\begin{aligned}&\int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S_m((P_\delta)_X;\mathbb{R}_+)\right] dX \\ &\quad = \frac{\log(\delta^{-1})}{8\pi^2}\int_{T^*(\mathcal{F}\Omega)}\int_{\mathbb{R}}\int_{\mathbb{R}} \eta_\delta(|\xi_1-\xi_2|)\frac{U_m(P_X(\xi_1),P_X(\xi_2))}{|\xi_1-\xi_2|}\,d\xi_1\,d\xi_2\,dX + o(\log(\delta^{-1})),\end{aligned}$$

as $\delta \to 0^+$. We are therefore exactly in a situation covered by Lemma A.11. Using this lemma we conclude that, for all $x \in \mathcal{F}\Omega$,

$$\begin{aligned}&\int_{\Pi_{\nu_\Omega(x)}}\int_{\mathbb{R}}\int_{\mathbb{R}} \eta_\delta(|\xi_1-\xi_2|)\frac{U_m(P_{(x,p)}(\xi_1),P_{(x,p)}(\xi_2))}{|\xi_1-\xi_2|}\,d\xi_1\,d\xi_2\,dp \\ &\qquad \to 2\int_{J_P} U_m(P^+(p),P^-(p))|\nu_\Omega(x)\cdot\nu_P(p)|\,d\mathcal{H}^{d-1}(p)\end{aligned}$$

as $\delta \to 0^+$. By the uniform boundedness statement of Lemma A.11, it therefore follows from the dominated convergence theorem that

$$\begin{aligned}&\int_{T^*(\mathcal{F}\Omega)} \operatorname{Tr}\left[S_m((P_\delta)_X;\mathbb{R}_+)\right] dX \\ &\quad = \frac{\log(\delta^{-1})}{4\pi^2}\int_{\mathcal{F}\Omega}\int_{J_P} U_m(P^+(p),P^-(p))|\nu_\Omega(x)\cdot\nu_P(p)|\,d\mathcal{H}^{d-1}(p)\,d\mathcal{H}^{d-1}(x) \\ &\quad + o(\log(\delta^{-1})),\end{aligned}$$

as $\delta \to 0^+$, finishing the proof. ■

Remark 6.2. By optimizing the splitting in the $d\xi_1\,d\xi_2$ integral, it is actually sufficient to assume that $\|P - P_\delta\|_1 = o(\delta\log(\delta^{-1})^2)$, and this carries over to the case $\alpha\delta \geq 1$ in Theorem 2.8. We are not able to establish the $\alpha\delta \leq 1$ case in Theorem 2.8 under this weaker condition, however.

Proof of Theorem 2.8. The case $\alpha\delta \geq 1$ follows immediately from Proposition 6.1 and Theorem 2.5.

The case $\alpha\delta \leq 1$ is a reduction to the case $\alpha\delta = 1$. Explicitly, by Lemma 3.12,

$$\begin{aligned}&\sup_{0\leq\delta\leq 1/\alpha}\left|\operatorname{Tr}\left[S_m(P_\delta;\alpha\Omega) - S_m(P_{1/\alpha};\alpha\Omega)\right]\right| \\ &\qquad \leq C\alpha^d\min\{\mathcal{H}^d(\Omega),\mathcal{H}^d(\Omega^c)\}\sup_{0\leq\delta\leq 1/\alpha}\|P_\delta - P_{1/\alpha}\|_1 \\ &\qquad \leq C\alpha^d\sup_{0\leq\delta\leq 1/\alpha}\|P_0 - P_\delta\|_1 = o(\alpha^{d-1}\log\alpha)\end{aligned}$$

as $\alpha \to \infty$. We remark here that $\min\{\mathcal{H}^d(\Omega),\mathcal{H}^d(\Omega^c)\} < \infty$ either by assumption when $d = 1$ or as a consequence of the isoperimetric inequality when $d \geq 2$. Hence, from the established case $\alpha\delta \geq 1$ with $\delta = 1/\alpha$ we find

$$\begin{aligned}&\operatorname{Tr}\left[S_m(P_\delta;\alpha\Omega)\right] \\ &\quad = \operatorname{Tr}\left[S_m(P_{1/\alpha};\alpha\Omega)\right] + o(\alpha^{d-1}\log(\alpha)) \\ &\quad = \frac{\alpha^{d-1}\log(\alpha)}{(2\pi)^{d+1}}\int_{\mathcal{F}\Omega}\int_{J_{P_0}} U_m(P_0^+(p),P_0^-(p))|\nu_\Omega(x)\cdot\nu_{P_0}(p)|\,d\mathcal{H}^{d-1}(p)\,d\mathcal{H}^{d-1}(x) \\ &\quad + o(\alpha^{d-1}\log(\alpha))\end{aligned}$$

as $\alpha \to \infty$ uniformly for $\alpha\delta \leq 1$. ■

6.2. **Proof of Theorem** 1.1. We now deduce Theorem 1.1. For a fixed set $\Omega \subseteq \mathbb{R}^d$ of finite perimeter, with $\min\{\mathcal{H}^1(\Omega), \mathcal{H}^1(\Omega^c)\} < \infty$ if $d = 1$, introduce the class $\mathcal{C}$ of all symbols $P \in \dot{BV}(\mathbb{R}^d;\mathbb{C}) \cap L^\infty$, with $P \in L^2 + \mathbb{C}$ if $d = 1$, such that

$$
\begin{aligned}
&\operatorname{Tr}\left[S_m(P;\alpha\Omega)\right] \\
&\quad = \frac{\alpha^{d-1}\log(\alpha)}{(2\pi)^{d+1}} \int_{\mathcal{F}\Omega}\int_{J_P} U_m(P^+(p), P^-(p))|\nu_\Omega(x)\cdot\nu_P(p)|\, d\mathcal{H}^{d-1}(p)\, d\mathcal{H}^{d-1}(x) \\
&\quad + o(\alpha^{d-1}\log(\alpha))
\end{aligned}
\tag{6.5}
$$

as $\alpha \to \infty$ for all $m \geq 2$. We first show that the class $\mathcal{C}$ is closed under suitable convergence in L^∞ or $\dot{BV}$.

Lemma 6.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter and let $P \in \dot{BV}(\mathbb{R}^d;\mathbb{C}) \cap L^\infty$. For $d = 1$ assume in addition that $P \in L^2(\mathbb{R},\mathbb{C}) + \mathbb{C}$ and that $\min\{\mathcal{H}^1(\Omega), \mathcal{H}^1(\Omega^c)\} < \infty$. Consider a sequence $(P_n) \subseteq \mathcal{C}$. If either*

a) $P_n \to P$ in L^∞ and $\sup_n\|P_n\|_{\dot{BV}} < \infty$,

or

b) $P_n \to P$ in $\dot{BV}$ and in L^1_{loc}, and $\sup_n\|P_n\|_{L^\infty} < \infty$,

then also $P \in \mathcal{C}$

Proof. Fix $m \geq 2$. Since (6.5) is satisfied for each P_n, it suffices to show that, under either condition a) or b),

$$
\liminf_{n\to\infty} \limsup_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log(\alpha)} \left|\operatorname{Tr}\left[S_m(P;\alpha\Omega) - S_m(P_n;\alpha\Omega)\right]\right| = 0, \tag{6.6}
$$

and

$$
\begin{aligned}
\liminf_{n\to\infty} \Bigg| &\int_{\mathcal{F}\Omega}\int_{J_P} U_m(P^+(p), P^-(p))|\nu_\Omega(x)\cdot\nu_P(p)|\, d\mathcal{H}^{d-1}(p)\, d\mathcal{H}^{d-1}(x) \\
&- \int_{\mathcal{F}\Omega}\int_{J_{P_n}} U_m(P_n^+(p), P_n^-(p))|\nu_\Omega(x)\cdot\nu_{P_n}(p)|\, d\mathcal{H}^{d-1}(p) d\mathcal{H}^{d-1}(x)\Bigg| \\
= 0.&
\end{aligned}
\tag{6.7}
$$

Note that under either assumption a) or b) the sequence (P_n) is uniformly bounded in L^∞ and in $\dot{BV}$, and $P_n \to P$ in L^1_{loc}. Also, $\min\{\mathcal{H}^d(\Omega), \mathcal{H}^d(\Omega^c)\} < \infty$ and P, $P_n \in L^2(\mathbb{R}^d;\mathbb{C}) + \mathbb{C}$ for all $n \in \mathbb{N}$. This follows by assumption if $d = 1$ and by the embedding $\dot{BV}(\mathbb{R}^d;\mathbb{C}) \to L^{\frac{d}{d-1}} + \mathbb{C}$ for $d \geq 2$, see [1, Theorem 3.47].

We start with the claim (6.6). Using the multilinear bound from Lemma 3.11 along with the Hilbert-Schmidt estimate from Lemma 3.5 we see that it suffices to show that

$$
\liminf_{n\to\infty} \|P_n^j(P - P_n)P^l\|_\infty \|P_n^j(P - P_n)P^l\|_{\dot{BV}} = 0
$$

for all integers $j, l \geq 0$. This is immediate under the assumption a), so assume b) holds. Since $P_n \to P$ in $\dot{BV}$ and in L^1_{loc}, it follows from Lemma A.14 that we can find a subsequence (n_k) such that $P^*_{n_k}(p) \to P^*(p)$ for $\mathcal{H}^{d-1}$ almost all $p \in \mathbb{R}^d$. Here the asterisk denotes the precise representative. We conclude using Lemma A.15 that $P^j_{n_k}(P - P_{n_k})P^l \to 0$ in $\dot{BV}$ for all integers $j, l \geq 0$, giving the claim (6.6).

The argument for (6.7) essentially boils down to the same. Note we have freedom to choose the orientations ν_P and ν_{P_n} as we see fit, so assume that $\nu_{P_n} = \nu_P$ on $J_P \cap J_{P_n}$ for all n. Let ν_n be the orientation on $J_P \cup J_{P_n}$ such that $\nu_n = \nu_P$ on J_P and $\nu_n = \nu_{P_n}$ on J_{P_n}. We at this point remind the reader that by our conventions on the jump values we have $P^+ - P^- = 0$

and $P_n^+ - P_n^- = 0$ outside J_p and J_{P_n}, respectively. We can thus bound

$$\begin{aligned}
&\Big| \int_{\mathcal{F}\Omega} \int_{J_P} U_m(P^+(p), P^-(p)) |\nu_\Omega(x) \cdot \nu_P(p)| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x) \\
&\qquad - \int_{\mathcal{F}\Omega} \int_{J_{P_n}} U_m(P_n^+(p), P_n^-(p)) |\nu_\Omega(x) \cdot \nu_{P_n}(p)| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x) \Big| \\
&\quad \le \int_{\mathcal{F}\Omega} \int_{J_P \cup J_{P_n}} \big| U_m(P^+(p), P^-(p)) - U_m(P_n^+(p), P_n^-(p)) \big| \\
&\qquad\qquad \times |\nu_n(x) \cdot \nu_\Omega(x)| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x) \\
&\quad \le \operatorname{Per}(\Omega) \int_{J_P \cup J_{P_n}} \big| U_m(P^+(p), P^-(p)) - U_m(P_n^+(p), P_n^-(p)) \big| \, d\mathcal{H}^{d-1}(p) \\
&\quad \le C \sum_{k=1}^{m-1} \Big(\frac{1}{k} + \frac{1}{m-k} \Big) \int_{J_P \cup J_{P_n}} |[P^k - P_n^k](p)| |[P^{m-k} + P_n^{m-k}](p)| \, d\mathcal{H}^{d-1}(p).
\end{aligned}$$

We used the notation $[Q](p) = Q^+(p) - Q^-(p)$ at the end. The last equality follows from the concrete expression (5.2) of U_m and our choice of orientations ν_P and ν_{P_n}.

Now assume a) holds. Then also $(P_n^k)^\pm \to (P^k)^\pm$ in L^∞ with respect to the measure $\mathcal{H}^{d-1}$. Since (P_n) is uniformly bounded in $\dot{BV}$ and in L^∞, the measure

$$|[P^{m-k} + P_n^{m-k}](p)| \, d\mathcal{H}^{d-1}(p) = d|D^j(P^{m-k} + P_n^{m-k})|(p)$$

has uniformly bounded total mass. The claim (6.7) easily follows. Assume instead b) holds. Arguing as for the claim (6.6) we can take a subsequence (n_i) such that $P_{n_i}^l \to P^l$ in $\dot{BV}$ for all integers $l \ge 0$. In particular, for all $1 \le k \le m-1$,

$$\int |[P^k - P_{n_i}^k](p)| |[P^{m-k} + P_{n_i}^{m-k}](p)| \, d\mathcal{H}^{d-1}(p) \le C |D^j(P^k - P_{n_i}^k)|(\mathbb{R}^d) \to 0,$$

giving he claim (6.6). ■

Proof of Theorem 1.1. Fix a set $\Omega \subseteq \mathbb{R}^d$ with finite perimeter, and assume in addition that $\min\{\mathcal{H}^1(\Omega), \mathcal{H}^1(\Omega^c)\} < \infty$ if $d = 1$. We show that the class $\mathcal{C}$ contains successively larger classes of functions. Our initial class is obtained from Theorem 2.8.

Step 1: the base class. Let

$$P = \sum_{j=1}^N a_j 1_{\Lambda_j},$$

where $(a_j) \subseteq \mathbb{C}$ and $\Lambda_j \subseteq \mathbb{R}^d$ are bounded sets such that, for each $j = 1, \ldots, N$, the boundary $\partial\Lambda_j$ is contained in a finite union of Lipschitz graphs, and set $\Gamma = \cup_{j=1}^N \partial\Lambda_j$. Then $P \in \dot{BV}$ and Γ is a set of locally finite upper Minkowski content by Lemma *C*.1. Let (ϕ_δ) be a standard family of mollifiers and set $P_\delta = P * \phi_\delta$. It follows from Lemma *A*.7 that

$$\|P - P_\delta\|_1 \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |P(p+h) - P(h)| \phi_\delta(h) \, dh \, dp \le \|P\|_{\dot{BV}} \int |h| \phi_\delta(h) \, dh = O(\delta).$$

Moreover, P_δ is locally constant on $\{d(p, \Gamma) \ge \delta\}$ and satisfies

$$|\nabla P_\delta(p)| \le C \int_{\mathbb{R}^d} |\nabla \phi_\delta(h)| \, dh \le C\delta^{-1} \le C\ell_\delta(p)^{-1}.$$

Since P has compact support, it is now easy to see that (P_δ) satisfies a $C^1(\mathbb{R}^d; \mathbb{C})$ multiscale estimate on Γ. It now follows from Theorem 2.8 that $P \in \mathcal{C}$.

Step 2: simple compactly supported $\dot{BV}$ *symbols.* Let

$$P = \sum_{j=1}^N a_j 1_{\Lambda_j},$$

where $(a_j) \subseteq \mathbb{C}$ and $\Lambda_j \subseteq \mathbb{R}^d$ are bounded sets with finite perimeters. If $d = 1$ then P agrees almost everywhere with a symbol covered in step 1, see [1, Proposition 3.52], so assume $d \geq 2$. By Lemma $C.7$ we can take sequences $(\Lambda_{j,n})$, $j = 1, \ldots, N$, such that, for each j, n, the set $\Lambda_{j,n}$ is bounded with boundary $\partial\Lambda_{j,n}$ contained in a finite union of Lipschitz graphs, and

$$\lim_{n\to\infty} \|1_{\Lambda_j} - 1_{\Lambda_{j,n}}\|_{\dot{BV}} = 0, \quad \lim_{n\to\infty} \|1_{\Lambda_j} - 1_{\Lambda_{j,n}}\|_1 = 0$$

for all $1 \leq j \leq N$. Set $P_n = \sum_{j=1}^N a_j 1_{\Lambda_{j,n}}$. Then $P_n \in \mathcal{C}$ for all n by step 1, and $P_n \to P$ in $\dot{BV}$ and in L^1_{loc}, hence $P \in \mathcal{C}$ by Lemma 6.3.

Step 3: compactly supported symbols in $\dot{BV} \cap L^\infty$. Let $P \in \dot{BV} \cap L^\infty$ be a compactly supported function. By Lemma $A.12$ we can find a sequence $(P_n) \subseteq \dot{BV} \cap L^\infty$ of simple functions such that

$$\lim_{n\to\infty} \|P_n - P\|_\infty = 0, \quad \sup_{n\in\mathbb{N}} \|P_n\|_{\dot{BV}} < \infty, \quad |P_n(p)| \leq C|P(p)|, \quad p \in \mathbb{R}^d.$$

In particular, P_n has compact support for each $n \in \mathbb{N}$. Hence, it follows from step 2 that $P_n \in \mathcal{C}$ for all n, so we conclude that $P \in \mathcal{C}$ by Lemma 6.3.

Step 4: integrable symbols in $\dot{BV} \cap L^\infty$. Let $P \in \dot{BV} \cap L^\infty$ and assume that $P \in L^{\frac{d}{d-1}}$ if $d \geq 2$ and that $P \in L^2$ if $d = 1$. Note that $P \in L^{\frac{d}{d-1}} \cap L^\infty \subseteq L^2$ also when $d \geq 2$, and that $P^*(p) \to 0$ as $|p| \to \infty$ when $d = 1$. By Lemma $A.13$ we can find a sequence $(P_n) \subseteq \dot{BV} \cap L^\infty$ of compactly supported functions such that $P_n \to P$ pointwise and

$$\lim_{n\to\infty} \|P - P_n\|_{\dot{BV}} = 0, \quad |P_n(p)| \leq |P(p)|, \quad p \in \mathbb{R}^d.$$

It follows from step 3 and Lemma 6.3 that $P \in \mathcal{C}$.

Step 5: the general case. Let $P \in \dot{BV} \cap L^\infty$ and assume in addition that $P \in L^2 + \mathbb{C}$ if $d = 1$. Then, either by assumption or by the embedding from [1, Theorem 3.47], there exists a constant $P_\infty \in \mathbb{C}$ such that $P - P_\infty \in L^{\frac{d}{d-1}}$ if $d \geq 2$ and $P - P_\infty \in L^2$ if $d = 1$. Set $P_0 = P - P_\infty$. The claim $P \in \mathcal{C}$ now follows from the binomial expansions

$$\operatorname{Tr}\big[S_m(P; \alpha\Omega)\big] = \sum_{k=2}^m \binom{m}{k} P_\infty^{m-k} \operatorname{Tr}\big[S_k(P_0; \alpha\Omega)\big],$$

and

$$U_m(A + P_\infty, B + P_\infty) = \sum_{k=2}^m \binom{m}{k} P_\infty^{m-k} U_k(A, B), \quad A, B \in \mathbb{C},$$

and the fact that $P_0 \in \mathcal{C}$ by step 4.

■

7. Extensions to general spectral functions

In this section we carry out the extension procedure from polynomial f to more general spectral functions to deduce Theorem 1.7 and Theorem 1.17. As mentioned in the Introduction, carrying out this procedure requires, in addition to the limit theorems for polynomials, uniform estimates on $1_{\alpha\Omega} P 1_{\alpha\Omega^c}$. For general symbols P, the bridge between these two ingredients are given by abstract trace inequalities.

For the case of Theorem 1.17 our argument follows roughly those of Sobolev [44] and Widom [54]. We only rely on Hilbert-Schmidt estimates on $1_\Omega P 1_{\Omega^c}$, which we have already established in Lemma 3.1 and Lemma 3.5. The case of indicators Theorem 1.7 is more precise. Here our argument is a simplified version of the much more precise extension arguments from [23] by Kulikov and the second author.

We start by establishing the abstract operator bounds required for Theorem 1.17.

7.1. **Analytic case.** We first treat holomorphic f. The basic input is a Cauchy integral representation of $S_f(P;\Omega)$, which yields a trace class bound in terms of $\|1_\Omega P 1_{\Omega^c}\|_{\mathcal{S}_2}^2$. We remind the reader that for a bounded operator A on a Hilbert space $\mathcal{H}$ the numerical range $W(A)$ is defined by

$$W(A) = \{\langle \psi, A\psi\rangle \mid \psi \in \mathcal{H},\ \|\psi\| = 1\}.$$

Lemma 7.1. *Let P be a bounded operator on a Hilbert space $\mathcal{H}$. The numerical range has the following properties:*

a) $W(P)$ is compact, convex, and contains $\sigma(P)$ the spectrum of P.

b) If $\chi : \mathcal{H}' \to \mathcal{H}$ is an isometry between Hilbert spaces, then $W(\chi^* P \chi) \subseteq W(P)$.

c) If $z \in W(P)^c$ then

$$\|(P-z)^{-1}\|_\infty \leq \frac{1}{d(z, W(P))}.$$

d) If P is a multiplication operator on $\mathcal{H} = L^2(\mathbb{R}^d;\mathbb{C}^n)$ with symbol $P \in L^\infty$, then $W(P(p)) \subseteq W(P)$ for almost all $p \in \mathbb{R}^d$.

Proof. The statement a) is the Hausdorff–Toeplitz theorem and b) is obvious from the definition. For c) simply note that for any $\psi \in \mathcal{H}$ with $\|\psi\| = 1$:

$$d(z, W(P)) \leq |\langle \psi, P\psi\rangle - z| \leq \|(P-z)\psi\|.$$

Taking inf over all such ψ, conclude $d(z, W(P)) \leq \|(P-z)^{-1}\|_\infty^{-1}$.

Finally for d) assume that P acts by multiplication. Take $p \in \mathbb{R}^d$ a Lebesgue point of P; almost all p will do. We show that $W(P(p)) \subseteq W(P)$, so take $\lambda \in W(P(p))$ and $x \in \mathbb{C}^n$ such that $\langle x, P(p)x\rangle = \lambda$. Let $\psi_\varepsilon(q) = \frac{1}{\mathcal{H}^d(B_\varepsilon)^{1/2}} 1_{B_\varepsilon(p)}(q)x$. Clearly $\|\psi_\varepsilon\|_2 = 1$ for all $\varepsilon > 0$, and by the assumption that p is a Lebesgue point,

$$\langle \psi_\varepsilon, P\psi_\varepsilon\rangle = \fint_{B_\varepsilon(p)} \langle x, P(q)x\rangle\, dq \to \langle x, P(p)x\rangle = \lambda$$

as $\varepsilon \to 0^+$. Since $W(P)$ is closed by a), it follows that $\lambda \in W(P)$. ■

Lemma 7.2. *Let $\Omega \subseteq \mathbb{R}^d$ be a measurable set and let $P \in L^\infty(\mathbb{R}^d;\mathcal{M}_n)$. Let U be an open set containing $W(P)$ and $\Gamma \subseteq U \setminus W(P)$ a rectifiable closed curve with $\operatorname{ind}_\Gamma(z) = 1$ for $z \in W(P)$. Such Γ always exist. Then, for any function f holomorphic on U,*

$$\|S_f(P;\Omega)\|_{\mathcal{S}_1} \leq \frac{1}{2\pi} \sup_{z\in\Gamma} |f(z)| \frac{\mathcal{H}^1(\Gamma)}{d(\Gamma, W(P))^3} \|1_\Omega P 1_{\Omega^c}\|_{\mathcal{S}_2} \|1_{\Omega^c} P 1_\Omega\|_{\mathcal{S}_2}.$$

Proof. For the existence of Γ, one can take Γ to be the boundary of the convex set $W(P) + B_\varepsilon$ for ε small enough. Given Γ, it follows from the Cauchy integral formula that

$$S_f(P;\Omega) = -\frac{1}{2\pi i} \int_\Gamma f(z)\big((1_\Omega P 1_\Omega - z)^{-1} - 1_\Omega (P-z)^{-1} 1_\Omega\big)\, dz.$$

Note all inverses make sense by Lemma 7.1. We have the following useful identity:

$$\begin{aligned} &1_\Omega(P-z)^{-1}1_\Omega - (1_\Omega(P-z)1_\Omega)^{-1} \\ &\quad = (1_\Omega(P-z)1_\Omega)^{-1}\,(1_\Omega P 1_{\Omega^c})\,(1_{\Omega^c}(P-z)1_{\Omega^c})^{-1}\,(1_{\Omega^c} P 1_\Omega)\,\big(1_\Omega(P-z)^{-1}1_\Omega\big). \end{aligned} \tag{7.1}$$

One can see this, for instance, by writing P as a two-by-two block operator in the decomposition $L^2(\mathbb{R}^d;\mathbb{C}^n) = L^2(\Omega;\mathbb{C}^n) \oplus L^2(\Omega^c;\mathbb{C}^n)$ and apply the resolvent identity to the diagonal and off-diagonal part. With this at hand we can estimate using Lemma 7.1:

$$\|1_\Omega(P-z)^{-1}1_\Omega - (1_\Omega(P-z)1_\Omega)^{-1}\|_{\mathcal{S}_1} \leq \frac{1}{d(z, W(P))^3} \|1_\Omega P 1_{\Omega^c}\|_{\mathcal{S}_2} \|1_{\Omega^c} P 1_\Omega\|_{\mathcal{S}_2}.$$

We conclude

$$\begin{aligned}\|S_f(P;\Omega)\|_{\mathcal{S}_1} &\leq \frac{1}{2\pi}\int_\Gamma \frac{|f(z)|}{d(z,W(P))^3}\|1_\Omega P 1_{\Omega^c}\|_{\mathcal{S}_2}\|1_{\Omega^c}P1_\Omega\|_{\mathcal{S}_2}\,dz\\ &\leq \frac{1}{2\pi}\sup_{z\in\Gamma}|f(z)|\frac{\mathcal{H}^1(\Gamma)}{d(\Gamma,W(P))^3}\|1_\Omega P\chi_{\Omega^c}\|_{\mathcal{S}_2}\|1_{\Omega^c}P\chi_\Omega\|_{\mathcal{S}_2},\end{aligned}$$

finishing the proof. ■

Remark 7.3. In this proof we strictly speaking ignored the contribution of the kernel arising from Ω^c by identifying $1_\Omega P 1_\Omega$ with $\chi_\Omega^* P\chi_\Omega$, where $\chi_\Omega : L^2(\Omega;\mathcal{M}_n) \to L^2(\mathbb{R}^d;\mathcal{M}_n)$ is the inclusion map. Since, however, in the decomposition $L^2(\mathbb{R}^d;\mathcal{M}_n) = L^2(\Omega;\mathcal{M}_n)\oplus L^2(\Omega^c;\mathcal{M}_n)$,

$$1_\Omega f(1_\Omega P 1_\Omega)1_\Omega - 1_\Omega f\circ P 1_\Omega = \big(f(\chi_\Omega^* P\chi_\Omega) - \chi_\Omega^* f\circ P\chi_\Omega\big)\oplus 0$$

for all f such that both the right and left hand sides makes sense, this is without loss.

7.2. **Real case.** We turn to the case of self-adjoint P. We prove the required trace class bounds for two classes: $f \in B^2_{\infty,1}$, using the Koplienko–Neidhardt trace formula, and $f\in C^2$, using the Berezin–Lieb inequality. The Besov space $B^2_{\infty,1}$ was introduced in (1.23).

Lemma 7.4. *Let $P\in L^\infty(\mathbb{R}^d;\mathcal{M}_n)$ take values in self-adjoint matrices and let $\Omega\subseteq\mathbb{R}^d$ be a measurable set. Assume that $1_\Omega P 1_{\Omega^c}\in\mathcal{S}_2$. If $f\in B^2_{\infty,1}(\mathbb{R};\mathbb{C})$, then $S_f(P;\Omega)\in\mathcal{S}_1$, and*

$$|S_f(P;\Omega)|_{\mathcal{S}_1} \leq C\|f\|_{B^2_{\infty,1}}\|1_\Omega P 1_{\Omega^c}\|^2_{\mathcal{S}_2},$$

for a universal constant $C>0$.

***Proof*.** We intend to apply the Koplienko–Neidhardt trace formula [33], or specifically the boundedness statement from [33, Theorem 4.6]. To verify its applicability, we first notice that

$$S_f(P;\Omega) = -1_\Omega\left(f(P) - f(1_\Omega P 1_\Omega + 1_{\Omega^c}P1_{\Omega^c})\right)1_\Omega.$$

The expression is essentially perturbative since

$$K = P - (1_\Omega P 1_\Omega + 1_{\Omega^c}P 1_{\Omega^c}) = 1_\Omega P 1_{\Omega^c} + 1_{\Omega^c}P1_\Omega \in \mathcal{S}_2$$

by assumption. Let us write $D = 1_\Omega P 1_\Omega + 1_{\Omega^c}P1_{\Omega^c}$ for the diagonal part. We claim that, for all Lipschitz functions f on $\mathbb{R}$,

$$1_\Omega \frac{d}{ds}f(D+sK)|_{s=0}1_\Omega = 0. \tag{7.2}$$

Here the derivative is taken in the Hilbert space $\mathcal{S}_2$, see [33, (2.6)]. As is well known from the theory of double operator integrals, we have the explicit formula

$$\frac{d}{ds}f(D+sK)|_{s=0} = \iint_{\mathbb{R}\times\mathbb{R}}\frac{f(x)-f(y)}{x-y}dE_D(x)KdE_D(y).$$

Here dE_D is the spectral resolution for the self-adjoint operator D. Since D is diagonal and K off-diagonal in the decomposition $L^2(\mathbb{R}^d;\mathbb{C}^n) = L^2(\Omega;\mathbb{C}^n)\oplus L^2(\Omega^c;\mathbb{C}^n)$, the claim (7.2) follows. We refer to [34] and the references therein for background on double operator integrals and related topics.

It now follows directly from [33] that

$$\|S_f(P;\Omega)\| \leq \|f(D+K) - f(D) - \frac{d}{ds}f(D+sK)|_{s=0}\|_{\mathcal{S}_1} \leq C\|f\|_{B^2_{\infty,1}}\|K\|^2_{\mathcal{S}_2},$$

which finishes the proof. ■

Under the additional assumptions that $P\in L^2(\mathbb{R}^d;\mathbb{R})+\mathbb{R}$ and $\mathcal{H}^d(\Omega)<\infty$, it is easy to verify that $S_f(P;\Omega)\in\mathcal{S}_1$ for $f\in C^2$. This removes the main difficulty from applying the Koplienko–Neidhardt trace formula. In this case, we can use the extension to the Berezin-Lieb inequality from [26] instead, which was also used by Sobolev in [44].

Lemma 7.5. *Let $P \in L^\infty(\mathbb{R}^d;\mathbb{R})$ be real valued and let $\Omega \subseteq \mathbb{R}^d$ be measurable. Assume in addition that $P \in L^2(\mathbb{R}^d;\mathbb{R}) + \mathbb{R}$ and that $\mathcal{H}^d(\Omega) < \infty$. If $f \in W^{2,\infty}_{loc}$ with $f'' \in L^\infty$, then $S_f(P;\Omega) \in \mathcal{S}_1$, and*

$$|\mathrm{Tr}\, S_f(P;\Omega)| \leq \frac{\|f''\|_\infty}{2} \|1_\Omega P 1_{\Omega^c}\|^2_{\mathcal{S}_2}.$$

Proof. Take $P_\infty \in \mathbb{R}$ such that $P_0 = P - P_\infty \in L^2$ and set $\tilde{f}(x) = f(x + P_\infty)$. Then clearly

$$S_f(P;\Omega) = S_{\tilde{f}}(P_0;\Omega).$$

Since $P_0 1_\Omega \in \mathcal{S}_2$ by Lemma 3.2, it follows from the generalization to the Berezin-Lieb inequality [26, Corollary 17] that $S_f(P;\Omega) \in \mathcal{S}_1$ and that

$$|\mathrm{Tr}\, S_f(P;\Omega)| = |\mathrm{Tr}\, S_{\tilde{f}}(P_0;\Omega)| \leq \frac{\|\tilde{f}''\|_\infty}{2} \|1_\Omega P_0 1_{\Omega^c}\|^2_{\mathcal{S}_2} = \frac{\|f''\|_\infty}{2} \|1_\Omega P 1_{\Omega^c}\|^2_{\mathcal{S}_2}.$$

■

7.3. **Proof of Theorem** 1.17**.** We first establish the holomorphic extension for (possibly) complex valued symbols, say for the case of $\dot{BV}$ symbols from Theorem 1.1. So let $P \in \dot{BV}(\mathbb{R}^d;\mathbb{C}) \cap L^\infty$ and let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter. If $d = 1$ we assume in addition that $P \in L^2(\mathbb{R}^d;\mathbb{C}) + \mathbb{C}$ and that $\min\{\mathcal{H}^1(\Omega), \mathcal{H}^1(\Omega^c)\} < \infty$. Let f be holomorphic on a neighborhood $U \supseteq W(P)$. We need to show that

$$\begin{aligned}
&\mathrm{Tr}\left[S_f(P;\alpha\Omega)\right] \\
&\quad = \frac{\alpha^{d-1}\log(\alpha)}{(2\pi)^{d+1}} \int_{\mathcal{F}\Omega}\int_{J_P} U_f(P^+(p), P^-(p)) |\nu_\Omega(x) \cdot \nu_P(p)| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x) \\
&\qquad + o(\alpha^{d-1}\log(\alpha))
\end{aligned}$$

as $\alpha \to \infty$. Take a rectifiable curve $\Gamma \subseteq U \setminus W(P)$ as in Lemma 7.2, and let K be the compact region enclosed Γ. By Runge's theorem we can find a sequence of polynomials (f_n) such that $f_n \to f$ uniformly on K. Since the polynomial case is covered by Theorem 1.1, it suffices to show that

$$\limsup_{n\to\infty} \limsup_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log\alpha} \left|\mathrm{Tr}\left[S_f(P;\alpha\Omega) - S_{f_n}(P;\alpha\Omega)\right]\right| = 0,$$

and

$$\begin{aligned}
\limsup_{n\to\infty} \Big| &\int_{\mathcal{F}\Omega}\int_{J_P} U_f(P^+(p), P^-(p)) |\nu_\Omega(p) \cdot \nu_P(p)| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x) \\
&- \int_{\mathcal{F}\Omega}\int_{J_P} U_{f_n}(P^+(p), P^-(p)) |\nu_\Omega(p) \cdot \nu_P(p)| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x)\Big| = 0.
\end{aligned}$$

The first claim is immediate from Lemma 7.2 and Lemma 3.5 since

$$\begin{aligned}
\left|\mathrm{Tr}\left[S_f(P;\alpha\Omega) - S_{f_n}(P;\alpha\Omega)\right]\right| &\leq \|S_{f-f_n}(P;\alpha\Omega)\|_{\mathcal{S}_1} \\
&\leq C \sup_{z\in K} |f(z) - f_n(z)| \|1_{\alpha\Omega} P 1_{\alpha\Omega^c}\|_{\mathcal{S}_2} \|1_{\alpha\Omega^c} P 1_{\alpha\Omega}\|_{\mathcal{S}_2} \\
&\leq C\alpha^{d-1}\log(\alpha) \sup_{z\in K} |f(z) - f_n(z)|,
\end{aligned}$$

for $\alpha \geq \alpha_{P,\Omega}$ large enough. The second claim follows from the basic inequality, for $A, B \in W(P)$,

$$|U_f(A,B) - U_{f_n}(A,B)| \leq \frac{1}{2} \sup_{z\in W(P)} |f''(z) - f_n''(z)| |A - B|^2, \tag{7.3}$$

and dominated convergence. The bound (7.3) is easily seen after integrating by parts twice, see also [54, (3')]. We also note that $P^\pm(p) \in W(P)$ for $\mathcal{H}^{d-1}$ almost all $p \in J_P$, which can be shown by similar argument as Lemma 7.1 d), and that $f_n^{(k)} \to f$ uniformly on $W(P)$ for all $k \in \mathbb{N}$, a consequence of the global Cauchy integral formula.

The extension argument for Theorem 1.13 for holomorphic f is identical using Lemma 3.1 instead of Lemma 3.5.

We turn to the real case, and let us this time focus on $\dot{H}^{1/2}$. So assume that $P \in \dot{H}^{1/2}(\mathbb{R}^d;\mathbb{R}) \cap L^\infty$ and that $\Omega \subseteq \mathbb{R}^d$ is a set of finite perimeter. Assume that either

a) $f \in B^2_{\infty,1}$,

or that

b) $P \in L^2(\mathbb{R}^d;\mathbb{R}) + \mathbb{R}$, $\mathcal{H}^d(\Omega) < \infty$, and that $f \in C^2(\mathbb{R})$.

We want to show in either case that

$$\operatorname{Tr}\left[S_f(P;\alpha\Omega)\right] = \frac{1}{8\pi^2}\left(\frac{\alpha}{2\pi}\right)^{d-1} \int_{T^*(\mathcal{F}\Omega)} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{U_f(P_X(\xi_1), P_X(\xi_2))}{|\xi_1 - \xi_2|^2}\, d\xi_1\, d\xi_2\, dX + o(\alpha^{d-1})$$

as $\alpha \to \infty$. Assume at first that a) holds. We can take a sequence $(f_n) \subseteq B^2_{\infty,1}$ of real analytic functions such that $f_n \to f$ in $B^2_{\infty,1}$. In fact, we can choose f_n such that the Fourier transforms $\hat{f}_n$ have compact support. This follows from the Littlewood-Paley description of $B^2_{\infty,1}$. Note in particular that $\|f'' - f_n''\|_\infty \to 0$. Since we have already established the limit theorem for (real) analytic functions, it suffices to show that

$$\limsup_{n\to\infty} \limsup_{\alpha\to\infty} \frac{1}{\alpha^{d-1}} \left|\operatorname{Tr}\left[S_f(P;\alpha\Omega) - S_{f_n}(P;\alpha\Omega)\right]\right| = 0,$$

and

$$\limsup_{n\to\infty} \Big| \int_{T^*(\mathcal{F}\Omega)} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{U_f(P_X(\xi_1), P_X(\xi_2))}{|\xi_1 - \xi_2|^2}\, d\xi_1\, d\xi_2\, dX - \int_{T^*(\mathcal{F}\Omega)} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{U_{f_n}(P_X(\xi_1), P_X(\xi_2))}{|\xi_1 - \xi_2|^2}\, d\xi_1\, d\xi_2\, dX \Big| = 0.$$

The first claim follows from Lemma 7.4 and Lemma 3.1, and the second from the bound (7.3). The case b) is easier. Since P is bounded we may assume that $f \in C^2_c(\mathbb{R})$. We approximate f by polynomials (f_n) such that $f_n'' \to f''$ uniformly on the support of f. Such a sequence is supplied, for instance, by the Stone-Weierstrass theorem. The argument is now the same, this time referring to Lemma 7.5 instead.

The extension argument for Theorem 1.1 for real valued P is handled in similarly.

7.4. **The case of indicators.** We here prove Theorem 1.7. We first establish the uniform bound (1.16).

Lemma 7.6. *Let $\Omega, \Lambda \subseteq \mathbb{R}^d$ be sets of finite perimeter and assume $\mathcal{F}\Omega$ and $\mathcal{F}\Lambda$ have finite upper Minkowksi content. If $d = 1$ assume in addition that* $\min\{\mathcal{H}^1(\Omega), \mathcal{H}^1(\Omega^c)\} < \infty$ *and* $\min\{\mathcal{H}^1(\Lambda), \mathcal{H}^1(\Lambda^c)\} < \infty$. *Then*

$$\|1_{\alpha\Omega} 1_\Lambda 1_{\alpha\Omega^c}\|_{\mathcal{S}_1} \le C_{\Omega,\Lambda} \alpha^{d-1} \log(\alpha)$$

for α large enough.

Proof. Like in the proof of Lemma 3.5, the argument is an interpolation between a volume bound and an area bound. Here Lemma 3.18 will serve as our area bound. For the volume bound, we recall the following elementary result of Birman-Solomyak:

$$\|f(x)g(p)\|_{\mathcal{S}_1} \le C\|f\|_{\ell^1(L^2)}\|g\|_{\ell^1(L^2)}, \quad \|h\|_{\ell^1(L^2)} = \sum_{z\in\mathbb{Z}^d} \|1_{Q(z)}h\|_2, \tag{7.4}$$

where $Q(z)$ is the cube centered at z with side length 1. See [43, Theorem 4.5] for a simple proof.

We proceed with the proof. First of all, after possibly modifying Ω and Λ by a measure zero set, we may assume that $\partial\Omega = \overline{\mathcal{F}\Omega}$ and $\partial\Lambda = \overline{\mathcal{F}\Lambda}$. This follows from [31, Proposition 12.19 and Remark 15.3]. In particular, $\partial\Omega$ and $\partial\Lambda$ are compact and sets of finite upper Minkowski content. For $d \ge 2$ this implies that either Ω or Ω^c are bounded, similarly for Λ.

The same conclusion holds for $d = 1$ since $\min\{\mathcal{H}^1(\Omega), \mathcal{H}^1(\Omega^c)\} < \infty$, similarly for Λ. Since $\|1_{\alpha\Omega}1_\Lambda 1_{\alpha\Omega^c}\|_{\mathcal{S}_1} = \|1_{\alpha\Omega^c}1_{\Lambda^c}1_{\alpha\Omega}\|_{\mathcal{S}_1}$, we may without loss assume that Λ and Ω are bounded sets.

Let (ϕ_δ) be a family of standard mollifiers and set $P = 1_\Lambda$ and $P_\delta = 1_\Lambda * \phi_\delta$. It follows from the volume bound (7.4) and a change of variables that

$$\|1_{\alpha\Omega}(P - P_{1/\alpha})1_{\alpha\Omega^c}\|_{\mathcal{S}_1} \leq \|1_\Omega D_{1/\alpha}(P - P_{1/\alpha})\|_{\mathcal{S}_1} \leq \|1_\Omega\|_{\ell^1(L^2)}\|D_{1/\alpha}(P - P_\delta)\|_{\ell^1(L^2)}.$$

Here $D_{1/\alpha}(P - P_{1/\alpha})(p) = P(p/\alpha) - P_{1/\alpha}(p/\alpha)$ is supported in $\alpha\partial\Lambda + B_1$ and uniformly bounded by 1. Denote $Z = \{z \in \mathbb{Z}^d \mid Q(z) \cap (\alpha\partial\Lambda + B_1) \neq \emptyset\}$. Then, for α large enough,

$$\|D_{1/\alpha}(P - P_{1/\alpha})\|_{\ell^1(L^2)} \leq \sum_{z\in Z}\int_{Q(z)} 1\,dp \leq \mathcal{H}^d(\alpha\Lambda + B_{1+2\sqrt{d}}) \leq C_\Lambda \alpha^{d-1},$$

where we used that $\partial\Lambda$ has finite upper Minkowski content at the end. Since Ω is bounded, we can simply bound $\|1_\Omega\|_{\ell^1(L^2)} \leq C_\Omega < \infty$. We conclude that

$$\|1_{\alpha\Omega}(P - P_{1/\alpha})1_{\alpha\Omega^c}\|_{\mathcal{S}_1} \leq C\alpha^{d-1}. \tag{7.5}$$

It therefore suffices to estimate $\|1_{\alpha\Omega}P_{1/\alpha}1_{\alpha\Omega^c}\|_{\mathcal{S}_1}$.

Applying Lemma 3.18 directly to $P_{1/\alpha}$ would lead to wrong scaling bound due to the number of derivatives needed. Instead we localize the symbol $P_{1/\alpha}$ using the partition of unity (ϕ_u) supplied by Lemma 2.1 with $\Gamma = \partial\Lambda$:

$$P_{1/\alpha}(p) = \int_{\mathbb{R}^d} \ell_{1/\alpha}(u)^{-d}P_{u,1/\alpha}(p)\,du, \quad P_{u,1/\alpha} = \phi_u(p)P_{1/\alpha}.$$

It is easy to verify that, for a suitable non-negative function $G \in L^1 \cap L^\infty$,

$$|D^\beta P_{u,1/\alpha}(p)| \leq G(u)\ell_{1/\alpha}(u)^{-|\beta|}, \quad p \in \mathbb{R}^d, \quad |\beta| \leq d + 1, \tag{7.6}$$

uniformly for all $u \in \mathbb{R}^d$. By the triangle inequality and another change of variables we find

$$\begin{aligned}\|1_{\alpha\Omega}P_{1/\alpha}1_{\alpha\Omega^c}\|_{\mathcal{S}_1} &\leq \int \ell_{1/\alpha}(u)^{-d}\|1_{\alpha\Omega}P_{u,1/\alpha}1_{\alpha\Omega^c}\|_{\mathcal{S}_1}\,du\\ &\leq \int \ell_{1/\alpha}(u)^{-d}\|1_{\ell_{1/\alpha}(u)\alpha\Omega}\left(D_{\ell_{1/\alpha}(u)}P_{u,1/\alpha}\right)1_{\ell_{1/\alpha}(u)\alpha\Omega^c}\|_{\mathcal{S}_1}\,du.\end{aligned}$$

Here, as before, $D_{\ell_{1/\alpha}(u)}P_{u,1/\alpha}(p) = P_{u,1/\alpha}(\ell_{1/\alpha}(u)p)$, which is supported in a ball of radius 1 and satisfies

$$|D^\beta D_{\ell_{1/\alpha}(u)}P_{u,1/\alpha}(p)| \leq G(u), \quad p \in \mathbb{R}^d, \quad |\beta| \leq d + 1,$$

by the multiscale estimate (7.6). In particular, it follows from Lemma 3.18 that

$$\|1_{\ell_{1/\alpha}(u)\alpha\Omega}\left(D_{\ell_{1/\alpha}(u)}P_{u,1/\alpha}\right)1_{\ell_{1/\alpha}(u)\alpha\Omega^c}\|_{\mathcal{S}_1} \leq CG(u)\big(1 + (\alpha\ell_{1/\alpha}(u))^{d-1}\big).$$

Integrating over $u \in \mathbb{R}^d$, we conclude using Lemma C.2 that

$$\|1_{\alpha\Omega}P_{1/\alpha}1_{\alpha\Omega^c}\|_{\mathcal{S}_1} \leq C\int \ell_{1/\alpha}(u)^{-d}\big(1 + (\alpha\ell_{1/\alpha}(u))^{d-1}\big)G(u)\,du \leq C\alpha^{d-1}\log(\alpha),$$

finishing the proof. ■

Proof of Theorem 1.7. With the uniform bound (1.16) already established in Lemma 7.6, we only need to prove (1.18), that is

$$\lim_{\alpha\to\infty}\frac{1}{\alpha^{d-1}\log(\alpha)}\operatorname{Tr}\left[S_f(1_\Lambda;\alpha\Omega)\right] = \frac{U_f(0,1)}{(2\pi)^{d+1}}\int_{\mathcal{F}\Omega}\int_{\mathcal{F}\Lambda}|\nu_\Omega(x)\cdot\nu_\Lambda(p)|\,d\mathcal{H}^{d-1}(p)\,d\mathcal{H}^{d-1}(x). \tag{7.7}$$

We may without loss assume that f is real valued by splitting into real and imaginary parts. We establish (7.7) for successively larger classes of functions f.

Step 1: continuous functions supported away from 0 *and* 1. Assume $f : [0,1] \to \mathbb{R}$ is continuous with support in $[\varepsilon, 1-\varepsilon]$ for some $0 < \varepsilon < 1/2$. Then $g(x) = \frac{f(x)}{x(1-x)}$ is also continuous, so we can find a sequence (g_n) of polynomials such that, for all $n \in \mathbb{N}$,

$$\sup_{x\in[0,1]} |g_n(x) - g(x)| \leq 1/n$$

If we denote $a_n(x) = x(1-x)(g_n(x) - \frac{1}{n})$ and $b_n(x) = x(1-x)(g_n(x) + \frac{1}{n})$, then

$$a_n(x) \leq f(x) \leq b_n(x),$$

and therefore, as operators,

$$a_n(1_{\alpha\Omega}1_\Lambda 1_{\alpha\Omega}) \leq f(1_{\alpha\Omega}1_\Lambda 1_{\alpha\Omega}) \leq b_n(1_{\alpha\Omega}1_\Lambda 1_{\alpha\Omega}).$$

Since a_n, b_n, and f vanishes at 0 and 1, this is the same as

$$S_{a_n}(1_\Lambda; \alpha\Omega) \leq S_f(1_\Lambda; \alpha\Omega) \leq S_{b_n}(1_\Lambda; \alpha\Omega).$$

Since the polynomial case is covered by Theorem 1.1 by linearity, we conclude

$$\begin{aligned}
&\frac{U_{a_n}(0,1)}{(2\pi)^{d+1}} \int_{\mathcal{F}\Omega}\int_{\mathcal{F}\Lambda} |\nu_\Omega(x)\cdot\nu_\Lambda(p)|\, d\mathcal{H}^{d-1}(p)\, d\mathcal{H}^{d-1}(x) \\
&\leq \liminf_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log(\alpha)} \operatorname{Tr}\left[S_f(1_\Lambda;\alpha\Omega)\right] \\
&\leq \limsup_{\alpha\to\infty} \frac{1}{\alpha^{d-1}\log(\alpha)} \operatorname{Tr}\left[S_f(1_\Lambda;\alpha\Omega)\right] \\
&\leq \frac{U_{b_n}(0,1)}{(2\pi)^{d+1}} \int_{\mathcal{F}\Omega}\int_{\mathcal{F}\Lambda} |\nu_\Omega(x)\cdot\nu_\Lambda(p)|\, d\mathcal{H}^{d-1}(p)\, d\mathcal{H}^{d-1}(x).
\end{aligned}$$

Since clearly $U_{a_n}(0,1), U_{b_n}(0,1) \to U_f(0,1)$ as $n \to \infty$, the claim (7.7) follows for this f.

Step 2: indicators away from 0 *and* 1. Consider $f = 1_{[a,b]}$ with $0 < a < b < 1$. The argument is essentially the same as in step 1. This time set

$$a_\varepsilon(x) = \left(1 - \frac{d(x,[a,b])}{\varepsilon}\right)_+, \quad b_\varepsilon(x) = \left(1 - \frac{d(x,[a+\varepsilon, b-\varepsilon])}{\varepsilon}\right)_+.$$

Clearly a_ε and b_ε are continuous functions supported away from 0 and 1 for $\varepsilon > 0$ small enough, and $a_\varepsilon(x) \leq f(x) \leq b_\varepsilon(x)$. We can thus repeat the argument above, this time using that (7.7) holds for a_ε and b_ε by step 1.

Step 3: general case Let $f : [0,1] \to \mathbb{C}$ be Riemann integrable on $[\varepsilon, 1-\varepsilon]$ for all $0 < \varepsilon < 1/2$ and assume that

$$\limsup_{x\to 0^+} \frac{|f(x) - f(0)|}{x^{1/2}} = 0, \quad \limsup_{x\to 0^+} \frac{|f(1-x) - f(1)|}{x^{1/2}} = 0. \tag{7.8}$$

We may assume that $f(0) = f(1) = 0$. Indeed, simply note that

$$S_f(1_\Lambda;\alpha\Omega) = S_{\tilde{f}}(1_\Lambda;\alpha\Omega), \quad U_f(0,1) = U_{\tilde{f}}(0,1),$$

where $\tilde{f}(x) = f(x) - (1-x)f(0) - xf(1)$, and clearly f is Riemann integrable on $[\varepsilon, 1-\varepsilon]$ for all $0 < \varepsilon < 1/2$ and satisfies (7.8) if and only if the same is true for $\tilde{f}$.

For all $0 < \varepsilon < 1/2$ we can find Riemann upper and lower sums $a_\varepsilon \leq f \leq b_\varepsilon$ on $[\varepsilon, 1-\varepsilon]$ such that

$$\int_\varepsilon^{1-\varepsilon} \frac{b_\varepsilon(\theta) - a_\varepsilon(\theta)}{\theta(1-\theta)}\, d\theta < \varepsilon. \tag{7.9}$$

Here a_ε and b_ε are finite linear combinations of indicators of the form $1_{[a,b]}$ with $\varepsilon \leq a < b \leq 1-\varepsilon$. For the part close to the end points we introduce the maximal operators

$$M_0 f(\varepsilon) = \sup_{0<x\leq\varepsilon} \frac{|f(x)|}{x^{1/2}}, \quad M_1 f(\varepsilon) = \sup_{0<x\leq\varepsilon} \frac{|f(1-x)|}{x^{1/2}},$$

and employ the elementary bound

$$|f(x)| \leq \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{(1-\varepsilon)^{1/2}} (x(1-x))^{1/2}, \quad x \in [0,1] \setminus (\varepsilon, 1-\varepsilon).$$

We set $g_\varepsilon(x) = \frac{M_0 f(\varepsilon)+M_1 f(\varepsilon)}{(1-\varepsilon)^{1/2}}(x(1-x))^{1/2}$ so that $a_\varepsilon - g_\varepsilon \leq f \leq b_\varepsilon + g_\varepsilon$ on $[0,1]$. Going through the monotonicity argument one final time it suffices to show that

$$\begin{aligned}
&\liminf_{\varepsilon \to 0^+} \liminf_{\alpha \to \infty} \frac{1}{\alpha^{d-1}\log(\alpha)} \operatorname{Tr}\left[S_{a_\varepsilon}(1_\Lambda; \alpha\Omega) - S_{g_\varepsilon}(1_\Lambda; \alpha\Omega)\right] \\
&\qquad = \frac{U_f(0,1)}{(2\pi)^{d+1}} \int_{\mathcal{F}\Omega} \int_{\mathcal{F}\Lambda} |\nu_\Omega(x) \cdot \nu_\Lambda(p)| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x),
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{\varepsilon \to 0^+} \limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1}\log(\alpha)} \operatorname{Tr}\left[S_{b_\varepsilon}(1_\Lambda; \alpha\Omega) + S_{g_\varepsilon}(1_\Lambda; \alpha\Omega)\right] \\
&\qquad = \frac{U_f(0,1)}{(2\pi)^{d+1}} \int_{\mathcal{F}\Omega} \int_{\mathcal{F}\Lambda} |\nu_\Omega(x) \cdot \nu_\Lambda(p)| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x).
\end{aligned}$$

Since (7.7) holds for a_ε and b_ε for all $0 < \varepsilon < 1/2$ by step 2, and clearly

$$U_{a_\varepsilon}(0,1), U_{b_\varepsilon}(0,1) \to U_f(0,1)$$

by (7.9), it suffices to show that

$$\limsup_{\varepsilon \to 0^+} \limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1}\log(\alpha)} \left|\operatorname{Tr}\left[S_{g_\varepsilon}(1_\Lambda; \alpha\Omega)\right]\right| = 0.$$

By the exact identity

$$1_{\alpha\Omega} 1_\Lambda 1_{\alpha\Omega} \left(1 - 1_{\alpha\Omega} 1_\Lambda 1_{\alpha\Omega}\right) = \left(1_{\alpha\Omega} 1_\Lambda 1_{\alpha\Omega^c}\right) \left(1_{\alpha\Omega} 1_\Lambda 1_{\alpha\Omega^c}\right)^*,$$

we conclude using the a priori bound from Lemma 7.6 and the assumption (7.8) that

$$\begin{aligned}
&\limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1}\log(\alpha)} \left|\operatorname{Tr}\left[S_{g_\varepsilon}(1_\Lambda; \alpha\Omega)\right]\right| \\
&\qquad = \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{(1-\varepsilon)^{1/2}} \limsup_{\alpha \to \infty} \frac{1}{\alpha^{d-1}\log(\alpha)} \|1_{\alpha\Omega} 1_\Lambda 1_{\alpha\Omega^c}\|_{\mathcal{S}_1} \\
&\qquad \leq C_{\Omega,\Lambda} \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{(1-\varepsilon)^{1/2}} \to 0
\end{aligned}$$

as $\varepsilon \to 0^+$, finishing the proof. ■

Appendix A. Results on bounded variation functions

In this Appendix we gather definitions and various results for bounded variation functions of a more abstract character.

A.1. **Definitions and fine structure of BV functions.** In this subsection we recall the definition of bounded variation functions and the fine structure of such functions. We follow the standard textbook reference [1], and we refer to this book for background and discussions. We also refer to [14, 15, 31].

Definition A.1. Let $P \in L^1_{loc}(\mathbb{R}^d; \mathbb{C}^n)$. The variation of P on an open set $A \subseteq \mathbb{R}^d$ is

$$\operatorname{Var}(P; A) = \sup\left\{ \left| \sum_{k=1}^n \int P_k(p) \operatorname{div} \phi_k(p) \, dp \right| : \Phi = (\phi_1, \ldots, \phi_n) \in C_c^\infty(A; \mathbb{C}^d)^n, \ |\Phi(p)| \leq 1 \right\}.$$

We say that P has bounded variation if $\operatorname{Var}(P; \mathbb{R}^d) = \|P\|_{\dot{BV}} < \infty$ and denote by $\dot{BV}(\mathbb{R}^d; \mathbb{C}^n)$ the corresponding space of bounded variation functions. We say that P has locally finite variation if $\operatorname{Var}(P; A) < \infty$ for all open bounded sets A and denote by $BV_{loc}(\mathbb{R}^d; \mathbb{C}^n)$ the corresponding space.

Remark A.2. It is non-standard to allow functions with values in a complex vector space. The concerned reader can safely identify $\mathbb{C} \sim \mathbb{R}^2$ in every instance below.

As a consequence of Riesz representation theorem, $P \in \dot{BV}(\mathbb{R}^d; \mathbb{C}^n)$ if and only if the distributional derivative DP is a finite $n \times d$-matrix valued measure, and $\mathrm{Var}(P; A) = |DP|(A)$ for $A \subseteq \mathbb{R}^d$ open, where $|DP|$ is the total variation measure.

Definition A.3. The perimeter of a measurable set $\Omega \subseteq \mathbb{R}^d$ is the variation of the function 1_Ω, that is $\mathrm{Per}(\Omega) = \mathrm{Var}(1_\Omega; \mathbb{R}^d)$. We say that Ω is a set of finite perimeter if $1_\Omega \in \dot{BV}$ and a set of locally finite perimeter if $1_\Omega \in BV_{loc}$.

Definition A.4. The reduced boundary $\mathcal{F}\Omega$ of a set Ω of locally finite perimeter is the set of all points $x \in \operatorname{supp} |D1_\Omega|$ such that the limit

$$\nu_\Omega(x) = \lim_{r \to 0+} \frac{D1_\Omega(B_r(x))}{|D1_\Omega|(B_r(x))}$$

exists and satisfies $|\nu_\Omega(x)| = 1$. The function $\nu_\Omega : \mathcal{F}\Omega \to S^{d-1}$ is called the (measure theoretic) interior normal to Ω.

If Ω is a set of locally finite perimeter, then the reduced boundary $\mathcal{F}\Omega$ is countably $\mathcal{H}^{d-1}$ rectifiable and the distributional derivative of 1_Ω satisfies $D1_\Omega = \nu_\Omega d\mathcal{H}^{d-1} \llcorner_{\mathcal{F}\Omega}$, see [14, Theorem 5.15]. The latter can be seen as a Gauss-Green formula on Ω.

If $\Omega \subseteq \mathbb{R}^d$ is a set with Lipschitz boundary, then Ω has locally finite perimeter. This follows from the Gauss-Green theorem for such sets. If moreover $\mathcal{H}^{d-1}(\partial\Omega) < \infty$ then Ω is a set of finite perimeter. In this case $\mathcal{F}\Omega \subseteq \partial\Omega$ (this is always true), $\mathcal{H}^{d-1}(\partial\Omega \setminus \mathcal{F}\Omega) = 0$, and ν_Ω agrees with the measure theoretic interior almost everywhere.

Definition A.5. Let $P \in L^1_{loc}(\mathbb{R}^d; \mathbb{C}^n)$.

a) We call $p \in \mathbb{R}^d$ a continuity point of P if there is $\tilde{P}(p) \in \mathbb{C}^n$ such that

$$\fint_{B_r(p)} |P(q) - \tilde{P}(q)| \, dq \to 0 \text{ as } r \to 0^+.$$

The singular set S_P of P is the complement of all continuity points.

b) We call $p \in \mathbb{R}^d$ a jump point if there is a triple $(P^+(p), P^-(p), \nu_P(p)) \in \mathbb{C}^n \times \mathbb{C}^n \times S^{d-1}$ such that $P^+(p) \neq P^-(p)$ and

$$\fint_{B_r^\pm(p;\nu_P(p))} |P(q) - P^\pm(p)| \, dq \to 0 \text{ as } r \to 0^+,$$

where $B_r^\pm(p; \nu_P(p)) = \{q \in \mathbb{R}^d \mid |p - q| < r, \pm(q - p) \cdot \nu_P(p) > 0\}$. The triple is uniquely determined down to a change of sign:

$$(P^+(p), P^-(p), \nu_P(p)) \sim (P^-(p), P^+(p), -\nu_P(p)).$$

The jump set J_P of P is the set of all jump points equipped with an implicit choice of (measurable) orientation $\nu_P : J_P \to S^{d-1}$. We extend $P^\pm$ to be defined on all of $\mathbb{R}^d \setminus (S_P \setminus J_P)$ by setting $P^\pm(p) = \tilde{P}(p)$ on $\mathbb{R}^d \setminus S_P$. The precise representative of P is given by $P^* = \frac{1}{2}(P^+ + P^-)$ on $\mathbb{R}^d \setminus (S_P \setminus J_P)$.

It follows from the Federer-Vol'pert theorem [1, Theorem 3.78] that for $P \in \dot{BV}(\mathbb{R}^d; \mathbb{C}^n)$, the jump set J_P is countably $\mathcal{H}^{d-1}$ rectifiable with ν_P a measure theoretic normal, $\mathcal{H}^{d-1}(S_P \setminus J_P) = 0$, and

$$\int_{J_P} |P^+(p) - P^-(p)| \, d\mathcal{H}^{d-1}(p) = |DP|(J_P) \leq |DP|(\mathbb{R}^d) < \infty. \tag{A.1}$$

It follows that the normal ν_P is determined (down to sign $\mathcal{H}^{d-1}$ almost everywhere) by J_P. Given an orientation ν_P, the functions $P^\pm$ are thus uniquely defined $\mathcal{H}^{d-1}$ almost everywhere.

It is possible to make (A.1) more precise. The measure DP splits

$$DP = \nabla P dp + D^c P + (P^+ - P^-) \otimes \nu_P \, d\mathcal{H}^{d-1} \llcorner_{J_P}$$

into an absolutely continuous part $D^a P = \nabla P dp$, a Cantor part $D^c P$, and the jump part $D^j P = (P^+ - P^-) \otimes \nu_P \, d\mathcal{H}^{d-1} \llcorner_{J_P}$. In what follows it suffices to note that the Cantor part $D^c P$ is singular with respect to $\mathcal{H}^d$ and $\mathcal{H}^{d-1}$. See [1, Proposition 3.92] for a more precise statement.

In many proofs it will be convenient to work with one-dimensional sections of $\dot{BV}(\mathbb{R}^d; \mathbb{C}^n)$ functions. For $\nu \in S^{d-1}$ and $p \in \Pi_\nu$ we write $P_{\nu,p}(\xi) = P(p+\xi\nu)$, $\xi \in \mathbb{R}$, for the corresponding section. We will only need the following.

Proposition A.6. *Let $P \in \dot{BV}(\mathbb{R}^d; \mathbb{C}^n)$. For almost all $p \in \Pi_\nu$ the following holds:*

a) $P_{\nu,p} \in \dot{BV}(\mathbb{R}; \mathbb{C}^n)$ and

$$\int_{\Pi_\nu} |DP_{\nu,p}|(\mathbb{R}) \, dp \leq |DP|(\mathbb{R}^d).$$

b) $J_{P_{\nu,p}} = \{\xi \in \mathbb{R} \mid p + \xi\nu \in J_P\}$ and $\nu \cdot \nu_P(p+\xi\nu) \neq 0$ for $\xi \in J_{P_{\nu,p}}$. Moreover, with the orientation $\nu_{P_{\nu,p}}(\xi) = \operatorname{sgn}(\nu \cdot \nu_P(p+\xi\nu))$ on $J_{P_{\nu,p}}$, we have

$$(P_{\nu,p})^\pm(\xi) = (P^\pm)_{\nu,p}(\xi) = P^\pm(p+\xi\nu)$$

for all $\xi \in \mathbb{R}$.

c) $|(P_{\nu,p})^\pm(\xi)| \leq \|P\|_\infty$ for all $\xi \in \mathbb{R}$.

Proof. The statement a) follows from [1, Theorem 3.103] while b) is [1, Theorem 3.108]. For c), simply note that $\|P_{\nu,p}\|_\infty \leq \|P\|_\infty$ for $\mathcal{H}^{d-1}$ almost all $p \in \Pi_\nu$, and that $|(P_{\nu,p})^\pm(\xi)| \leq \|P_{\nu,p}\|_\infty$ everywhere. ■

The fine structure of $P \in \dot{BV}(\mathbb{R}; \mathbb{C}^n)$ in one real variable is well known. We recall the following:

$$\begin{aligned} P^*(\xi_\pm) &= P^\pm(\xi) = P^\mp(\xi_\pm) \text{ if } \nu_P(\xi) = 1, \\ P^*(\xi_\pm) &= P^\mp(\xi) = P^\pm(\xi_\mp) \text{ if } \nu_P(\xi) = -1, \end{aligned} \tag{A.2}$$

where we denoted $g(\xi_\pm) = \lim_{\varepsilon \to 0^+} g(\xi \pm \varepsilon)$. On the other hand,

$$P^+(t) = P^-(t) = P^*(t)$$

are continuous for $t \in \mathbb{R} \setminus J_P$. The jump set J_P is at most countable. We refer to [1, Theorem 3.28] for the details.

A.2. **Uniform bounds in truncated $\dot{H}^s$ norms.** In this subsection we obtain simple estimates for $\dot{BV}$ functions in truncated $\dot{H}^s$ norms. These are applied with $s = 1/2$ in Section 3 (and elsewhere) to bound the operator $1_{\alpha\Omega} P 1_{\alpha\Omega^c}$ in Hilbert-Schmidt norm for $P \in \dot{BV}$.

We first recall the finite difference (Besov type) characterization of $\dot{BV}$ and $\dot{W}^{1,q}$, $q > 1$.

Lemma A.7. *Let $P \in L^1_{loc}(\mathbb{R}^d; \mathbb{C}^n)$. If $q > 1$ and $\nabla P \in L^q$ then*

$$\Big(\int |P(p+h) - P(p)|^q \, dp \Big)^{1/q} \leq \|\nabla P\|_q |h|, \quad h \in \mathbb{R}^d.$$

Similarly, if $P \in \dot{BV}(\mathbb{R}^d; \mathbb{C}^n)$, then

$$\int |P(p+h) - P(p)| \, dq \leq |DP|(\mathbb{R}^d) |h|, \quad h \in \mathbb{R}^d.$$

Conversely, for $q > 1$, if there is $C_{P,q}$ such that

$$\Big(\int |P(p+h) - P(p)|^q \, dq \Big)^{1/q} \leq C_{P,q} |h|$$

for all $h \in \mathbb{R}^d$, then $\nabla P \in L^q$. The analogous result holds for $\dot{BV}$.

Proof. The case $q=\infty$ is the Lipschitz characterization of $\dot{W}^{1,\infty}$. Consider $1<q<\infty$ and let (ϕ_ε) be a family of standard mollifiers. Say $\nabla P \in L^q$ and set $P_\varepsilon = P * \phi_\varepsilon$. By Fatou's lemma and Jensen's inequality

$$\begin{aligned}\int |P(p+h)-P(p)|^q\,dq &\le \liminf_{\varepsilon\to 0^+}\int |P_\varepsilon(p+h)-P_\varepsilon(p)|^q\,dp\\ &= \liminf_{\varepsilon\to 0^+}\int \left|\int_0^1 \nabla P_\varepsilon(p+th)h\,dt\right|^q dp\\ &\le |h|^q \liminf_{\varepsilon\to 0^+}\|\nabla P_\varepsilon\|_q^q = |h|^q\|\nabla P\|_q^q.\end{aligned}$$

The case $q=1$ is entirely similar. If $P\in \dot{BV}$ there is P_ε smooth such that $\|\nabla P_\varepsilon\|_1 \to |Df|(\mathbb{R}^d)$ and $P_\varepsilon \to P$ almost everywhere, see [1, Theorem 3.9]. The argument is now the same.

For the converse, let $D_\nu^t P(p) = \frac{1}{t}(P(p+t\nu)-P(p))$ for $t\in\mathbb{R}$ and $\nu\in S^{d-1}$. Assume that

$$\Big(\int |P(p+h)-P(p)|^q\,dp\Big)^{1/q} \le C_{P,q}|h|.$$

Then clearly $\|D_\nu^t P\|_q \le C_{P,q}$ is uniformly bounded as $t\to 0$, so there is a subsequence $t_n\to 0$ such that $D_\nu^{t_n}P \to g_\nu$ weakly in L^q and $\|g_\nu\|_q \le C_{P,q}$. Clearly $g_\nu = \nabla P\nu$ in the distributional sense, so summing over an orthonormal basis we conclude $\nabla P\in L^q$.

The $\dot{BV}$ case is more direct: For $\Phi=(\phi_1,\dots,\phi_n)\in C_c^\infty(\mathbb{R}^d;\mathbb{C}^d)^n$, we can write $\operatorname{div}\phi_i = \lim_{t\to 0}\sum_k \nu_k\cdot D_{\nu_k}^t\phi_i$ for an orthonormal basis (ν_k), and plug directly into the definition of the variation. We leave the details. ■

Our proof below is inspired by [19] and the integration by parts argument from [3]. The latter paper also contains the exact leading order asymptotics for the case $s=1$.

Lemma A.8. *Let $P\in\dot{BV}(\mathbb{R}^d;\mathbb{C}^n)\cap L^\infty\cap L^2$. There is a constant C_d such that*

$$\int_{|x|\le R}|x|^{2s}|\check{P}(x)|^2\,dx \le \frac{C_d}{2s-1}R^{2s-1}\,\|P\|_{\dot{BV}}\|P\|_\infty, \tag{A.3}$$

and

$$\int_{|x|\ge R}|\check{P}(x)|^2\,dx \le C_d\,R^{-1}\|P\|_{\dot{BV}}\|P\|_\infty. \tag{A.4}$$

uniformly for $s>1/2$ and $R>0$. Also

$$\int_{R^{-1}\le|x|\le R}|x||\check{P}(x)|^2\,dx \le C_d\log R\,\|P\|_{\dot{BV}}\|P\|_\infty \tag{A.5}$$

for $R\ge 1$.

Proof. We first prove (A.3) with $s=1$ from which the rest will follow. Fix $R>0$, $\theta\in S^{d-1}$, and $h\in\mathbb{R}$ with $|h|=\frac{\pi}{2}R^{-1}$. By the Plancherel theorem and Lemma $A.7$ we find

$$\begin{aligned}\int|\check{P}(x)|^2\sin(hx\cdot\theta)^2\,dx \le \int|\check{P}(x)|^2|1-\exp(ihx\cdot\theta)|^2\,dx &= \int|P(p)-P(p-h\theta)|^2\,dp\\ &\le 2|h|\|P\|_{\dot{BV}}\|P\|_\infty.\end{aligned}$$

Now if $|x|\le R$ then $|hx\cdot\theta|\le\pi/2$ and so $\sin(hx\cdot\theta)^2 \ge \frac{4}{\pi^2}|h|^2|x\cdot\theta|^2 = R^{-2}|x\cdot\theta|^2$ by concavity. Restricting the integral on the left hand side to $|x|\le R$ and integrating over $\theta\in S^{d-1}$, conclude

$$\frac{1}{R}\int_{|x|\le R}|\check{P}(x)|^2|x|^2\,dx \le C_d\|P\|_{\dot{BV}}\|P\|_\infty,$$

where, explicitly,

$$C_d = \pi\mathcal{H}^{d-1}(S^{d-1})\Big(\int_{S^{d-1}}|\theta_d|^2\,d\mathcal{H}^{d-1}(\theta)\Big)^{-1} = \pi d.$$

We now show the general case $s > 1/2$, $s \neq 1$. Let $L > 0$ and note that (A.3) for $s = 1$ gives

$$\int_0^L R^{2s-3} \int_{|x|\leq R} |x|^2 |\check{P}(x)|^2 \, dx \, dR \leq C_d \frac{L^{2s-1}}{2s-1} \|P\|_\infty \|P\|_{\dot{BV}}.$$

Integrating the left hand side by parts in R we find

$$0 \leq \frac{1}{2s-2} \Big(L^{2s-2} \int_{|x|\leq L} |x|^2 |\check{P}(x)|^2 \, dx - \int_{|x|\leq L} |x|^{2s} |\check{P}(x)|^2 \, dx \Big) \leq C_d \frac{L^{2s-1}}{2s-1} \|P\|_\infty \|P\|_{\dot{BV}}.$$

Note the boundary contribution from $L = 0$ vanishes. We conclude

$$\int_{|x|\leq L} |x|^{2s} |\check{P}(x)|^2 \, dx \leq C_d \big(\tfrac{|2s-2|}{2s-1} + 1 \big) L^{2s-1} \|P\|_\infty \|P\|_{\dot{BV}} \leq \frac{C_d}{2s-1} L^{2s-1} \|P\|_\infty \|P\|_{\dot{BV}}.$$

The case $s = 1/2$ is identical. By (A.3) for $s = 1$ we find for $L \geq 1$

$$\int_{L^{-1}}^L R^{-2} \int_{|x|\leq R} |\check{P}(x)|^2 |x|^2 \, dx \, dR \leq C_d \log L \|P\|_{\dot{BV}} \|P\|_\infty.$$

Integrating by parts

$$\begin{aligned} &\int_{L^{-1}}^L R^{-2} \int_{|x|\leq R} |\check{P}(x)|^2 |x|^2 \, dx \, dR \\ &\quad = -L^{-1} \int_{|x|\leq L} |\check{P}(x)|^2 |x|^2 \, dx + L \int_{|x|\leq L^{-1}} |\check{P}(x)|^2 |x|^2 \, dx + \int_{L^{-1}\leq|x|\leq L} |\check{P}(x)|^2 |x| \, dx, \end{aligned}$$

and the claim follows from (A.3).

Finally (A.4) follows from

$$\int_L^\infty R^{-3} \int_{|x|\leq R} |x|^2 |\check{P}(x)|^2 \, dx \, dR \leq C_d R^{-1} \|P\|_{\dot{BV}} \|P\|_\infty$$

after integrating by parts. ■

Remark A.9. The condition $P \in \dot{BV} \cap L^\infty$ can be relaxed to $P \in \dot{B}^{1/2}_{2,\infty}$, which is characterized by the seminorm

$$\|P\|^2_{\dot{B}^{1/2}_{2,\infty}} = \sup_{h\in\mathbb{R}^d} \frac{1}{|h|} \int |P(p+h) - P(p)|^2 dp < \infty.$$

A.3. **Deriving the boundary coefficient.** In this subsection we compute the leading order asymptotics as $\varepsilon \to 0^+$ of integrals of the form (1.24):

$$\int_{\Pi_\nu} \iint_{\mathbb{R}\times\mathbb{R}} \eta_\varepsilon(|\xi_1 - \xi_2|) \frac{U(P(p+\xi_1\nu), P(p+\xi_2\nu))}{|\xi_1 - \xi_2|} \, d\xi_1 \, d\xi_2 \, dp,$$

which directly appear in the proof of Theorem 2.8. The result of this computation will exactly be the boundary coefficient from Theorem 1.1. We refer to [6,11,39] for related computations. Since the kernels η_ε concentrate around 0, the leading order term is determined by the behavior of $U(A, B)$ near the diagonal $A = B$. The key point is that $U(A, B) = o(|A - B|)$, which decouples the continuous (or diffuse) part of the derivative from the jump part.

We first establish an auxiliary result.

Lemma A.10. *Consider a function* $U : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^M$. *Assume that* U *is continuous, symmetric, and that there is a function* ω *such that* $U(A, B) = \omega(A, B)|A - B|$ *with*

$$\rho_K(r) = \sup\{|\omega(A, B)| \mid |A - B| \leq r,\ |A|, |B| \leq K\} = o(1)$$

as $r \to 0^+$ *for all* $K > 0$. *Then, for all* $P \in \dot{BV}(\mathbb{R}^d; \mathbb{C}^n) \cap L^\infty$ *and all* $\nu \in S^{d-1}$,

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} \int_{\mathbb{R}^d} U(P(p+\varepsilon\nu), P(p)) \, dp = \int_{J_P} U(P^+(p), P^-(p)) |\nu_P(p) \cdot \nu| \, d\mathcal{H}^{d-1}(p).$$

***Proof*.** We first consider $d = 1$ with $\nu = 1$. Note $P \in L^\infty$, so we simply write $\rho = \rho_{\|P\|_\infty}$. Fix $\delta > 0$ and take $t_1 < t_2 < \cdots < t_n$ such that

$$\sum_{p \in J_P \setminus \{t_1,\ldots,t_n\}} |P^+(p) - P^-(p)| \le \delta, \quad |DP|(t_{k-1}, t_k) \le \delta, \quad k = 1, \ldots, n, \tag{A.6}$$

with $t_0 = -\infty$ and $t_{n+1} = \infty$. Note that we do not require $t_i \in J_P$. For $\varepsilon > 0$ small enough we can decompose

$$\begin{aligned}
&\frac{1}{\varepsilon} \int U(P(p+\varepsilon), P(p))\, dp \\
&\quad = \sum_{k=1}^{n+1} \frac{1}{\varepsilon} \int_{t_{k-1}}^{t_k - \varepsilon} U(P^*(p+\varepsilon), P^*(p))\, dp + \sum_{k=1}^{n} \frac{1}{\varepsilon} \int_{t_k - \varepsilon}^{t_k} U(P^*(p+\varepsilon), P^*(p))\, dp,
\end{aligned} \tag{A.7}$$

of course with the understanding $t_{n+1} - \varepsilon = \infty$. We used that $P = P^*$ almost everywhere. We first handle the second term in (A.7). To this end, since U is continuous and symmetric, it follows from the left/right continuity statement (A.2) that, for all $1 \le k \le n$,

$$\begin{aligned}
&\sup\{|U(P^*(p+\varepsilon), P^*(p)) - U(P^+(t_k), P^-(t_k))| \mid t_k - \varepsilon < p < t_k\} \\
&\qquad \le \sup\{|U(P^*(p), P^*(q)) - U(P^*(t_{k,+}), P^*(t_{k,-}))| \mid t_k - \varepsilon < p < t_k < q < t_k + \varepsilon\} \to 0
\end{aligned}$$

as $\varepsilon \to 0^+$. Of course, if $t_k \in \mathbb{R} \setminus J_P$, this is simply a continuity statement. We conclude

$$\sum_{k=1}^{n} \frac{1}{\varepsilon} \int_{t_k - \varepsilon}^{t_k} U(P^*(p+\varepsilon), P^*(p))\, dp \to \sum_{k=1}^{n} U(P^+(t_k), P^-(t_k)).$$

We now handle the other term in (A.7). Fix $1 \le k \le n+1$. Let Q_k be the extension of $P|_{(t_{k-1}, t_k - \varepsilon)}$ to $\mathbb{R}$ constant on $(-\infty, t_{k-1}]$ and $[t_k, \infty)$ without creating any new jumps. Clearly $|DQ_k|(\mathbb{R}) = |DP|(t_{k-1}, t_k - \varepsilon)$. In particular $|Q_k^*(p+\varepsilon) - Q_k^*(p)| \le |DP|(t_{k-1}, t_k - \varepsilon) \le \delta$ by construction (A.6). It now follows from Lemma $A.7$ that

$$\begin{aligned}
&\sum_{k=1}^{n+1} \frac{1}{\varepsilon} \int_{t_{k-1}}^{t_k - \varepsilon} |U(P^*(p+\varepsilon), P^*(p))|\, dp \\
&\qquad \le \sum_{k=1}^{n+1} \frac{1}{\varepsilon} \int_{\mathbb{R}} |Q_k^*(p+\varepsilon) - Q_k^*(p)| |\omega(Q_k^*(p+\varepsilon), Q_k^*(p))|\, dp \\
&\qquad \le \sum_{k=1}^{n+1} \rho(\delta) |DP|(t_{k-1}, t_k - \varepsilon) \le |DP|(\mathbb{R}) \rho(\delta).
\end{aligned}$$

Summing up, using the construction (A.6) again,

$$\begin{aligned}
&\limsup_{\varepsilon \to 0^+} \left| \frac{1}{\varepsilon} \int U(P(p+\varepsilon), P(p))\, dp - \sum_{p \in J_P} U(P^+(p), P^-(p)) \right| \\
&\qquad \le \limsup_{\varepsilon \to 0^+} \left| \frac{1}{\varepsilon} \int U(P(p+\varepsilon), P(p))\, dp - \sum_{p \in \{t_1,\ldots,t_n\}} U(P^+(p), P^-(p)) \right| \\
&\qquad\quad + \sum_{p \in J_P \setminus \{t_1,\ldots,t_n\}} |U(P^+(p), P^-(p))| \\
&\qquad \le |DP|(\mathbb{R}) \rho(\delta) + \rho(|DP|(\mathbb{R})) \delta.
\end{aligned}$$

Letting $\delta \to 0^+$ gives the result for $d = 1$ and $\nu = 1$.

The general case $d \ge 1$ is a simple consequence of the $d = 1$ case. For fixed $\nu \in S^{d-1}$ write

$$\int_{\mathbb{R}^d} U(P(p + \varepsilon\nu), P(p))\, dp = \int_{\Pi_\nu} \int_{\mathbb{R}} U(P_{\nu,p}(\xi + \varepsilon), P_{\nu,p}(\xi))\, d\xi\, dp.$$

By Proposition $A.6$, the established case $d = 1$, and the dominated convergence theorem, we find

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\Pi_\nu}\int_{\mathbb{R}} U(P_{\nu,p}(\xi+\varepsilon), P_{\nu,p}(\xi))\,d\xi\,dp &\to \int_{\Pi_\nu}\sum_{p+\xi\nu\in J_p} U(P^+(p+\xi\nu), P^-(p+\xi\nu))\,dp\\ &= \int_{J_P} U(P^+(p), P^-(p))|\nu\cdot\nu_P(p)|\,d\mathcal{H}^{d-1}(p).\end{aligned}$$

The last equality follows from the area formula, see the remark following [1, Theorem 2.71] (the Jacobian is computed in exercise 2.18). Dominated convergence is easy to justify by Lemma $A.7$ and Proposition $A.6$. ■

We now apply Lemma $A.10$ to the functionals of the form (1.24).

Lemma A.11. *Let U be as in Lemma $A.10$. Consider kernels $\eta_\varepsilon : \mathbb{R}_+ \to \mathbb{R}$ satisfying*

$$\|\eta_\varepsilon\|_1 \le C, \qquad \int_0^\infty \eta_\varepsilon(r)\,dr = 1, \qquad \int_\delta^\infty |\eta_\varepsilon(r)|\,dr \to 0 \text{ for all } \delta > 0.$$

Then, for all $\nu \in S^{d-1}$ and $P \in \dot{BV}(\mathbb{R}^d;\mathbb{C}^N)\cap L^\infty$ we have

$$\begin{aligned}\int_{\Pi_\nu}\iint \eta_\varepsilon(|\xi_1-\xi_2|)\frac{U(P(p+\xi_1\nu), P(p+\xi_2\nu))}{|\xi_1-\xi_2|}\,d\xi_1\,d\xi_2\,dp&\\ \to 2\int_{J_P} U(P^+(p), P^-(p))|\nu\cdot\nu_P(p)|\,d\mathcal{H}^{d-1}(p)&\end{aligned}\tag{A.8}$$

as $\varepsilon \to 0^+$. Moreover, there is $C = C_{P,U,\eta}$ such that

$$\int_{\Pi_\nu}\iint |\eta_\varepsilon(|\xi_1-\xi_2|)|\frac{|U(P(p+\xi_1\nu), P(p+\xi_2\nu))|}{|\xi_1-\xi_2|}\,d\xi_1\,d\xi_2\,dp \le C_{P,U,\eta}.$$

uniformly for $\varepsilon > 0$ and $\nu \in S^{d-1}$.

Proof. We can write

$$\begin{aligned}&\int_{\Pi_\nu}\iint \eta_\varepsilon(|\xi_1-\xi_2|)\frac{U(P(p+\xi_1\nu), P(P+\xi_2\nu))}{|\xi_1-\xi_2|}\,d\xi_1\,d\xi_2\,dp\\ &\quad = 2\int_{\Pi_\nu}\int_0^\infty \eta_\varepsilon(h)\frac{1}{h}\int_{\mathbb{R}} U(P_{\nu,p}(\xi+h), P_{\nu,p}(\xi))\,d\xi\,dh\,dp.\end{aligned}$$

We first handle the uniform bound. With $\rho = \rho_{\|P\|_\infty}$ as in the proof of Lemma $A.10$ it follows from Lemma $A.7$ that

$$\begin{aligned}&\int_0^\infty |\eta_\varepsilon(h)|\frac{1}{h}\int_{\mathbb{R}}|U(P_{\nu,p}(\xi+h), P_{\nu,p}(\xi))|\,d\xi\,dh\\ &\quad\le \rho(2\|P\|_\infty)\int_0^\infty |\eta_\varepsilon(h)|\frac{1}{h}\int_{\mathbb{R}}|P_{\nu,p}(\xi+h)-P_{\nu,p}(\xi)|\,d\xi\,dh\\ &\quad\le C_\eta\rho(2\|P\|_\infty)|DP_{\nu,p}|(\mathbb{R})\end{aligned}\tag{A.9}$$

for almost all $p \in \Pi_\nu$. By Proposition $A.6$ we conclude

$$\int_{\Pi_\nu}\iint \eta_\varepsilon(|\xi_1-\xi_2|)\frac{U(P(p+\xi_1\nu), P(p+\xi_2\nu))}{|\xi_1-\xi_2|}\,d\xi_1\,d\xi_2\,dp \le 2C_\eta\rho(2\|P\|_\infty)|DP|(\mathbb{R}^d).$$

Hence we can take $C_{P,U,\eta} = 2C_\eta\rho(2\|P\|_\infty)|DP|(\mathbb{R}^d)$.

We now compute the limit. Fix $p \in \Pi_\nu$ satisfying a), b), and c) from Proposition A.6. Almost all $p \in \Pi_\nu$ will do. By Lemma A.10 we then have

$$\begin{aligned}\frac{1}{h}\int_{\mathbb{R}} U(P_{\nu,p}(\xi+h), P_{\nu,p}(\xi))\,d\xi &\to \sum_{\xi \in J_{P_{\nu,p}}} U((P_{\nu,p})^+(\xi), (P_{\nu,p})^-(\xi)) \\ &= \sum_{p+\xi\nu \in J_P} U(P^+(p+\xi\nu), P^-(p+\xi\nu))\end{aligned}$$

as $\varepsilon \to 0^+$. Let $\varepsilon' > 0$ and take $\delta' > 0$ such that

$$\left|\frac{1}{h}\int_{\mathbb{R}} U(P_{\nu,p}(\xi+h), P_{\nu,p}(\xi))\,d\xi - \sum_{p+\xi\nu \in J_P} U(P^+(p+\xi\nu), P^-(p+\xi\nu))\right| \le \varepsilon'$$

for $0 < h < \delta'$. Then, arguing like in (A.9),

$$\begin{aligned}&\limsup_{\varepsilon\to 0^+}\left|\int_0^\infty \eta_\varepsilon(h)\frac{1}{h}\int_{\mathbb{R}} U(P_{\nu,p}(\xi+h), P_{\nu,p}(\xi))\,d\xi\,dh - \sum_{p+\xi\nu \in J_P} U(P^+(p+\xi\nu), P^-(p+\xi\nu))\right| \\ &\quad\le \varepsilon' \limsup_{\varepsilon\to 0^+}\int_0^{\delta'} |\eta_\varepsilon(h)|\,dh + 2\rho(2\|P\|_\infty)|DP_{\nu,p}|(\mathbb{R}) \limsup_{\varepsilon\to 0^+}\int_{\delta'}^\infty |\eta_\varepsilon(h)|\,dh \\ &\quad\le C_\eta \varepsilon'.\end{aligned}$$

Take $\varepsilon' \to 0^+$ to conclude

$$\int_0^\infty \eta_\varepsilon(h)\frac{1}{h}\int_{\mathbb{R}} U(P_{\nu,p}(\xi+h), P_{\nu,p}(\xi))\,d\xi \to \sum_{p+\xi\nu \in J_P} U(P^+(p+\xi\nu), P^-(p+\xi\nu))$$

for almost all $p \in \Pi_\nu$. We finally arrive at

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\int_{\Pi_\nu}\iint \eta_\varepsilon(|\xi_1-\xi_2|)\frac{U(P(p+\xi_1\nu), P(p+\xi_2\nu))}{|\xi_1-\xi_2|}\,d\xi_1\,d\xi_2\,dp \\ &\quad= 2\int_{\Pi_\nu}\sum_{p+\xi\nu \in J_P} U(P^+(p+\xi\nu), P^-(p+\xi\nu))dp \\ &\quad= 2\int_{J_P} U(P^+(p), P^-(p))|\nu\cdot\nu_P(p)|\,d\mathcal{H}^{d-1}(p).\end{aligned}$$

The application of dominated convergence in the first equality is justified by (A.9). The second equality follows from the area formula like the final step in the proof of Lemma A.10 ■

A.4. **Approximation.** For the final step of the proof of Theorem 1.1 we need a few approximation results for $\dot{BV}$ functions. Our first two results are known. We include proofs for completeness.

Lemma A.12. *Let $P \in \dot{BV}(\mathbb{R}^d;\mathbb{C}^N) \cap L^\infty$. There is a sequence $(P_n) \subseteq \dot{BV}$ of simple functions such that*

$$\|P - P_n\|_\infty \to 0, \quad |P_n| \le C_N|P|, \quad |DP_n|(\mathbb{R}^d) \le C_{d,N}|DP|(\mathbb{R}^d).$$

Proof. Splitting into components, real and imaginary parts, and then positive and negative parts we can assume that $0 \le P \le 1$ almost everywhere. For $n \in \mathbb{N}$ and $0 \le j \le n-1$ we can choose $j/n < t_{j,n} < (j+1)/n$ such that

$$|D1_{\{P>t_{j,n}\}}|(\mathbb{R}^d) \le n\int_{j/n}^{(j+1)/n} |D1_{\{P>t\}}|(\mathbb{R}^d)\,dt.$$

Set $E_{j,n} = \{t_{j,n} < P \le t_{j+1,n}\}$ for $0 \le j \le n-1$ with $t_{n,n} = 1$, and define $P_n = \sum_{j=0}^{n-1} \frac{j}{n} 1_{E_{j,n}}$. Clearly $P_n \le P \le P_n + 2/n$. By the coarea formula for BV functions, see [1, Theorem 3.40], we also have

$$\begin{aligned} |DP_n|(\mathbb{R}^d) &= \sum_{j=1}^{n-1} \int_{(j-1)/n}^{j/n} |D1_{\{P_n > t\}}|(\mathbb{R}^d)\, dt = \sum_{j=1}^{n-1} \frac{1}{n} |D1_{\{P > t_{j,n}\}}|(\mathbb{R}^d) \\ &\le \sum_{j=1}^{n-1} \int_{j/n}^{(j+1)/n} |D1_{\{P>t\}}|(\mathbb{R}^d)\, dt \le |DP|(\mathbb{R}^d). \end{aligned}$$

■

Lemma A.13. *Suppose* $P \in \dot{BV}(\mathbb{R}^d; \mathbb{C}^N) \cap L^{d/(d-1)}$ *if* $d \ge 2$ *and* $P \in \dot{BV}(\mathbb{R}, \mathbb{C}^N)$ *with* $P^*(p) \to 0$ *as* $|p| \to \infty$ *if* $d = 1$. *If* $(\chi_R) \subseteq C_c^\infty$ *is a sequence of cutoffs such that*

$$0 \le \chi_R \le 1, \quad \chi_R = 1 \text{ on } B_R, \quad \operatorname{supp} \chi_R \subseteq B_{2R}, \quad |\nabla \chi_R| \le C/R,$$

then

$$|D(1-\chi_R)P|(\mathbb{R}^d) \to 0$$

as $R \to \infty$.

Proof. We can compute the distributional derivative

$$D(1-\chi_R)P = (1-\chi_R)DP - (\nabla \chi_R)DP,$$

hence

$$|D(1-\chi_R)P|(\mathbb{R}^d) \le |DP|(B_R^c) + \int |P(p)||\nabla \chi_R(p)|\, dp.$$

Clearly $|DP|(B_R^c) \to 0$ as $R \to \infty$. If $d \ge 2$, then

$$\int |P(p)||\nabla \chi_R(p)|\, dp \le \|\nabla \chi_R\|_d \|1_{\{R \le |p| \le 2R} P\|_{d/(d-1)} \le C \|1_{\{R \le |p| \le 2R} P\|_{d/(d-1)} \to 0,$$

while for $d = 1$,

$$\int |P(p)||\nabla \chi_R(p)|\, dp \le C \sup_{|p| \ge R} |P^*(p)| \to 0.$$

■

The space $\dot{BV}(\mathbb{R}^d; \mathcal{M}_N) \cap L^\infty$ forms an algebra under pointwise multiplication with

$$\|PQ\|_{\dot{BV}} \le \|P\|_{\dot{BV}} \|Q\|_\infty + \|Q\|_{\dot{BV}} \|P\|_\infty, \quad P, Q \in \dot{BV}(\mathbb{R}^d; \mathcal{M}_N) \cap L^\infty,$$

see [1, (3.10)]. Our final goal in this section is to give a condition for the convergence of pointwise products of the form $P_n^1 \dots P_n^m \to P^1 \dots P^m$ in $\dot{BV}$. It is necessary to control each sequence (P_n^k) pointwise $\mathcal{H}^{d-1}$ almost everywhere, which is achieved by passing to a subsequence. The following fundamental result seems non-standard in the literature on bounded variation functions. We refer to [9, Corollary 4.3] for a proof. The result is also mentioned in [24].

Lemma A.14. *Let* $(P_n) \subseteq \dot{BV}(\mathbb{R}^d; \mathbb{C}^N)$ *and assume that there is* $P \in \dot{BV}$ *such that* $P_n \to P$ *in* $\dot{BV}$ *and in* L^1_{loc}. *Then there is a subsequence* (n_k) *such that* $P_{n_k}^*(p) \to P^*(p)$ *for* $\mathcal{H}^{d-1}$ *almost all* $p \in \mathbb{R}^d$.

Lemma A.15. *Let* $m \ge 1$ *and consider sequences* $(P_n^i) \subseteq \dot{BV}(\mathbb{R}^d; \mathcal{M}_N) \cap L^\infty$, $i = 1, \dots, m$, *with* $\sup_{i,n} \|P_n^i\|_\infty < \infty$. *Assume that* $P_n^i \to P^i$ *in* $\dot{BV}$ *and that* $(P_n^i)^*(p) \to (P^i)^*(p)$ *for* $\mathcal{H}^{d-1}$ *almost every* $p \in \mathbb{R}^d$ *for* $i = 1, \dots, m$. *Then*

$$P_n^1 \dots P_n^m \to P^1 \dots P^m$$

in variation as $n \to \infty$.

Proof. Arguing component wise and splitting into real and imaginary parts we can assume all functions are real valued. Furthermore, we only need to consider the case $m = 2$ by a simple induction argument. So consider sequences $(P_n), (Q_n) \subseteq \dot{BV}(\mathbb{R}^d; \mathbb{R}) \cap L^\infty$ with $\|P_n\|_\infty, \|Q_n\|_\infty \leq K$ for all $n \geq 1$, and assume that $P_n \to P$ and $Q_n \to Q$ in $\dot{BV}$ and that $P_n^*(p) \to P^*(p)$ and $Q_n^*(p) \to Q^*(p)$ for $\mathcal{H}^{d-1}$ almost every p. Note that $|P^*(p)|, |P_n^*(p)| \leq K$ for $\mathcal{H}^{d-1}$ almost all p, similarly for Q.

For general $f \in \dot{BV}$ denote $\tilde{D}f = Df - D^j f$ the diffuse part of the derivative. Note that $\tilde{D}f \ll \mathcal{H}^{d-1}$ always. By the chain rule for $\dot{BV}$ functions, [1, Theorem 3.96], we find

$$\tilde{D}(P_nQ_n - PQ) = \big(P_n^*\tilde{D}Q_n - P^*\tilde{D}Q\big) + \big(Q_n^*\tilde{D}P_n - Q^*\tilde{D}P\big).$$

Taking total variation on both sides we conclude that

$$\begin{aligned}|\tilde{D}(P_nQ_n - PQ)|(\mathbb{R}^d) \leq{}& \tfrac{1}{2}|P^* - P_n^*||\tilde{D}(Q + Q_n)|(\mathbb{R}^d) + \tfrac{1}{2}|P^* + P_n^*||\tilde{D}(Q - Q_n)|(\mathbb{R}^d)\\ &+ \tfrac{1}{2}|Q^* - Q_n^*||\tilde{D}(P + P_n)|(\mathbb{R}^d) + \tfrac{1}{2}|Q^* + Q_n^*||\tilde{D}(P - P_n)|(\mathbb{R}^d).\end{aligned}$$

We bound $|\tilde{D}(Q + Q_n)| \leq 2|\tilde{D}Q| + |\tilde{D}(Q - Q_n)|$ and use that $|Q^*(p)|, |Q_n^*(p)| \leq K$ almost everywhere with respect to all relevant measures, and similarly for P. Conclude

$$\begin{aligned}|\tilde{D}(P_nQ_n - PQ)|(\mathbb{R}^d) &\leq 2K|\tilde{D}(P - P_n)|(\mathbb{R}^d) + 2K|\tilde{D}(Q - Q_n)|(\mathbb{R}^d)\\ &\quad + |P^* - P_n^*||\tilde{D}Q|(\mathbb{R}^d) + |Q^* - Q_n^*||\tilde{D}P|(\mathbb{R}^d)\\ &\to 0\end{aligned}$$

as $n \to \infty$ by dominated convergence.

The argument for the jump part is entirely similar. Firstly we fix orientations ν_P and ν_Q such that $\nu_P = \nu_Q$ on $J_P \cap J_Q$. We then fix $\nu_{P_n} = \nu_{Q_n}$ on $J_{P_n} \cap J_{Q_n} \cap \big(J_P \cup J_Q\big)$ by choosing the normals to agree with either ν_P or ν_Q or both. Finally we choose $\nu_{P_n} = \nu_{Q_n}$ on $(J_{P_n} \cap J_{Q_n}) \setminus (J_P \cup J_Q)$. These technicalities ensure, for instance,

$$(PQ - P_nQ_n)^\pm = P^\pm Q^\pm - P_n^\pm Q_n^\pm.$$

We can now compute directly using the chain rule again:

$$\begin{aligned}|D^j(PQ - P_nQ_n)|(\mathbb{R}^d) &= \int \big|\big(P^+Q^+ - P_n^+Q_n^+\big) - \big(P^-Q^- - P_n^-Q_n^-\big)\big| d\mathcal{H}^{d-1}\\ &\leq \tfrac{1}{2}\int |(P^+ - P_n^+)(Q^+ + Q_n^+) - (P^- - P_n^-)(Q^- + Q_n^-)| d\mathcal{H}^{d-1}\\ &\quad + \tfrac{1}{2}\int |(Q^+ - Q_n^+)(P^+ + P_n^+) - (Q^- - Q_n^-)(P^- + P_n^-)| d\mathcal{H}^{d-1}\\ &\leq \tfrac{1}{2}\int |Q^* + Q_n^*||(P^+ - P_n^+) - (P^- - P_n^-)| d\mathcal{H}^{d-1}\\ &\quad + \tfrac{1}{2}\int |P^* - P_n^*||(Q^+ + Q_n^+) - (Q^- + Q_n^-)|\, d\mathcal{H}^{d-1}\\ &\quad + \tfrac{1}{2}\int |P^* + P_n^*||(Q^+ - Q_n^+) - (Q^- - Q_n^-)| d\mathcal{H}^{d-1}\\ &\quad + \tfrac{1}{2}\int |Q^* - Q_n^*||(P^+ + P_n^+) - (P^- + P_n^-)|\, d\mathcal{H}^{d-1}\end{aligned}$$

Arguing like above this is easily seen to converge to zero as $n \to \infty$. ■

Appendix B. Some properties of $\dot{H}^{1/2}$ functions

In this Appendix we collect several auxiliary results concerning $\dot{H}^{1/2}$. More precisely, we prove an approximation result by smooth compactly supported functions, a convergence result for products, and a uniform bound on one-dimensional sections. We remind the reader that $\dot{H}^{1/2}$ was introduced in Definition 1.12.

For the first result we require the Sobolev embedding $\dot{H}^{1/2}(\mathbb{R}^d;\mathbb{C}^n) \to L^{\frac{2d}{d-1}} + \mathbb{C}^n$, $d \geq 2$. We have not found a reference for this statement in the homogeneous setting, so we include a proof. The argument is based on the usual Sobolev inequality [30, Theorem 8.4], which states that if $d \geq 2$, $P \in \dot{H}^{1/2}(\mathbb{R}^d,\mathbb{C}^N)$, and the set $\{|P| > \varepsilon\}$ has finite measure for all $\varepsilon > 0$,

$$\|P\|_{2d/(d-1)} \leq C_d \|P\|_{\dot{H}^{1/2}}, \tag{B.1}$$

together with the Poincaré inequality, valid for all $d \geq 1$,

$$\int |P(p) - P_R|^2 \, dp \leq C_d R \|P\|^2_{\dot{H}^{1/2}}, \quad P_R = \fint_{B_R} P(p)\, dp. \tag{B.2}$$

The latter is a simple consequence of Jensen's inequality.

Lemma B.1. *Let $d \geq 2$ and suppose $P \in \dot{H}^{1/2}(\mathbb{R}^d;\mathbb{C}^N)$. There is $P_\infty \in \mathbb{C}^N$ such that $P - P_\infty \in L^{2d/(d-1)}$ and $\|P - P_\infty\|_{2d/(d-1)} \leq C_d \|P\|_{\dot{H}^{1/2}}$.*

Proof. Take a family $(\chi_R) \subseteq C_c^\infty$, $R \geq 1$, satisfying

$$0 \leq \chi_R \leq 1, \quad \chi_R = 1 \text{ on } B_R, \quad \operatorname{supp}\chi_R \subseteq B_{2R}, \quad |\nabla \chi_R| \leq C/R.$$

We claim that $\|(P - P_{2R})\chi_R\|_{\dot{H}^{1/2}} \leq C\|P\|_{\dot{H}^{1/2}}$ uniformly in $R \geq 1$. We compute directly:

$$\begin{aligned}
&\|(P - P_{2R})\chi_R\|^2_{\dot{H}^{1/2}} \\
&\quad \leq 2\int_{|p|\leq 2R}\int_{|q|\leq 2R} \frac{|P(p) - P_{2R}|^2|\chi_R(p) - \chi_R(q)|^2 + |P(p) - P(q)|^2}{|p-q|^{d+1}}\, dq\, dp \\
&\quad\quad + 2\int_{|p|\leq 2R}\int_{|q|\geq 2R} \frac{|P(p) - P_{2R}|^2 \chi_R(p)^2}{|p-q|^{d+1}}\, dq\, dp \\
&\quad \leq 2\|P\|^2_{\dot{H}^{1/2}} + 2\int_{|P|\leq 2R} |P(p) - P_{2R}|^2 \int_{\mathbb{R}^d} \frac{|\chi_R(p) - \chi_R(q)|^2}{|p-q|^{d+1}}\, dq\, dp.
\end{aligned}$$

If $|p| \leq 2R$, then

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|\chi_R(p) - \chi_R(q)|^2}{|p-q|^{d+1}}\, dq \\
&\quad \leq \int_{|q|\geq 3R} \frac{1}{|p-q|^{d+1}}\, dq + \int_{|p-q|\leq 5R} \frac{|\chi_R(p) - \chi_R(q)|^2}{|p-q|^{d+1}}\, dq \\
&\quad \leq C_d \int_{|q|\geq 3R} |q|^{-d-1}\, dq + C_\chi R^{-2} \int_{|p-q|\leq 5R} |p-q|^{-(d-1)}\, dq \\
&\quad \leq C_{d,\chi} R^{-1}.
\end{aligned}$$

We used that for $|p| \leq 2R$ and $|q| \geq 3R$ we have $|p - q| \geq \frac{1}{3}|q|$. Conclude using the Poincaré inequality (B.2):

$$\|(P - P_{2R})\chi_R\|^2_{\dot{H}^{1/2}} \leq 2\|P\|^2_{\dot{H}^{1/2}} + C_{d,\chi} R^{-1} \int_{B_{2R}} |P(p) - P_{2R}|^2\, dp \leq C_{d,\chi} \|P\|^2_{\dot{H}^{1/2}}.$$

We next claim that (P_R) has a convergent subsequence. Indeed, for $n \geq 0$,

$$|P_{2^{n+1}} - P_{2^n}|^2 \leq C_d 2^{-2dn} \int_{B_{2^n}} \int_{B_{2^{n+1}}} |P(p) - P(q)|^2\, dp\, dq \leq C_d 2^{-n(d-1)} \|P\|^2_{\dot{H}^{1/2}},$$

so the limit $P_\infty = P_1 + \sum_{n=0}^\infty P_{2^{n+1}} - P_{2^n}$ exists, and $P_\infty - P_{2^N} = \sum_{n=N}^\infty P_{2^{n+1}} - P_{2^n} \to 0$ as $N \to \infty$. Since the Sobolev inequality (B.1) holds for $(P - P_{2R})\chi_R$ we finally conclude

$$\begin{aligned}
\|P - P_\infty\|_{2d/(d-1)} &\leq \liminf_{n\to\infty} \|(P - P_{2^{n+1}})\chi_{2^n}\|_{2d/(d-1)} \leq C_d \liminf_{n\to\infty} \|(P - P_{2^{n+1}})\chi_{2^n}\|_{\dot{H}^{1/2}} \\
&\leq C_{d,\chi} \|P\|_{\dot{H}^{1/2}}.
\end{aligned}$$

■

Lemma B.2. *Let $P \in \dot{H}^{1/2}(\mathbb{R}^d;\mathbb{C}^N)$, and assume in addition that $P \in L^\infty$ if $d = 1$. There is a constant $P_\infty \in \mathbb{C}^N$ and a sequence $(P_n) \subseteq C_c^\infty$ such that $P_n + P_\infty \to P$ in $\dot{H}^{1/2}$ and pointwise almost everywhere. Moreover $\|P_n + P_\infty\|_\infty \leq 3\|P\|_\infty$.*

Proof. Arguing component wise we may assume P is scalar valued. Consider first $d \geq 2$ and take P_∞ from Lemma B.1 such that $P - P_\infty \in L^{2d/(d-1)}$. Let (ϕ_ε) be a family of standard mollifiers and consider a sequence of cutoffs $(\chi_R) \subseteq C_c^\infty$ satisfying

$$0 \leq \chi_R \leq 1, \quad \chi_R = 1 \text{ on } B_R, \quad \operatorname{supp}\chi_R \subseteq B_{2R}, \quad |\nabla\chi_R| \leq C_\chi/R.$$

Set $P_{R,\varepsilon} = \big(\chi_R(P - P_\infty)\big) * \phi_\varepsilon$. By the triangle inequality

$$\begin{aligned}\|P_{R,\varepsilon} - (P - P_\infty)\|_{\dot{H}^{1/2}} \leq{}& \|(P - P_\infty)(1 - \chi_R)\|_{\dot{H}^{1/2}} \\ &+ \|\chi_R(P - P_\infty) - \big(\chi_R(P - P_\infty)\big) * \phi_\varepsilon\|_{\dot{H}^{1/2}}.\end{aligned}$$

Now, for any $n \in \mathbb{N}$, it follows from [7, Lemma B.1] that there is $R_n > 0$ such that

$$\|(P - P_\infty)(1 - \chi_R)\|_{\dot{H}^{1/2}} \leq \frac{1}{2n}$$

for $R \geq R_n$. This part uses $P - P_\infty \in L^{2d/(d-1)}$. Similarly, by [7, Lemma A.1] there is $\varepsilon_n > 0$ such that

$$\|\chi_{R_n}(P - P_\infty) - \big(\chi_{R_n}(P - P_\infty)\big) * \phi_\varepsilon\|_{\dot{H}^{1/2}} \leq \frac{1}{2n}$$

for $\varepsilon \leq \varepsilon_n$. Taking $P_n = P_{R_n,\varepsilon_n}$ gives the desired sequence.

The critical case $d = 1$ is slightly more involved. Since $P \in L^\infty$ we can take a sequence $R_k \to \infty$ such that the limit

$$P_\infty = \lim_{k\to\infty} C_k, \quad C_k = \fint_{B_{R_k^2}} P(p)\,dp$$

exists. Consider this time cutoffs $(\chi_R) \subseteq C_c^\infty$, $R \geq 2$, with

$$0 \leq \chi_R \leq 1, \quad \chi_R = 1 \text{ on } B_R, \quad \operatorname{supp}\chi_R \subseteq B_{R^2}, \quad |\nabla\chi_R| \leq \frac{C_\chi}{R^2}.$$

We set $P_{k,\varepsilon} = \big(\chi_{R_k}(P - C_k)\big) * \phi_\varepsilon$. By the triangle inequality again

$$\begin{aligned}\|P_{k,\varepsilon} - (P - P_\infty)\|_{\dot{H}^{1/2}} \leq{}& \|(P - C_k)(1 - \chi_{R_k})\|_{\dot{H}^{1/2}} \\ &+ \|\chi_{R_k}(P - C_k) - \big(\chi_{R_k}(P - C_k)\big) * \phi_\varepsilon\|_{\dot{H}^{1/2}}.\end{aligned}$$

Using [7, Lemma B.3] instead of [7, Lemma B.1], the argument is now exactly as for $d \geq 2$. ■

Remark B.3. The L^∞ assumption for $d = 1$ is only used to show that $\fint_{B_R} P(p)\,dp$ is uniformly bounded along a sequence $R_k \to \infty$. For this it is sufficient to assume that P has bounded mean oscillation.

It is easy to see that $\dot{H}^{1/2}(\mathbb{R}^d;\mathcal{M}_n) \cap L^\infty$ is an algebra under pointwise multiplication with

$$\|PQ\|_{\dot{H}^{1/2}} \leq \|P\|_{\dot{H}^{1/2}}\|Q\|_\infty + \|Q\|_{\dot{H}^{1/2}}\|P\|_\infty, \quad P, Q \in \dot{H}^{1/2}(\mathbb{R}^d;\mathcal{M}_n) \cap L^\infty. \tag{B.3}$$

We now show that convergence in $\dot{H}^{1/2}$ respects this product structure under very mild additional assumptions, similar to the $\dot{BV}$ case Lemma A.15. A similar claim can be found in [54] for $d = 1$.

Lemma B.4. *Let $m \geq 1$ and consider sequences $(P_n^i) \subseteq \dot{H}^{1/2}(\mathbb{R}^d;\mathcal{M}_N) \cap L^\infty$, $i = 1, \ldots, m$, with $\sup_{i,n}\|P_n^i\|_\infty < \infty$. Assume that $P_n^i \to P^i$ in $\dot{H}^{1/2}$ and pointwise almost everywhere for $i = 1, \ldots, m$. Then*

$$P_n^1 \ldots P_n^m \to P^1 \ldots P^m$$

in $\dot{H}^{1/2}$ as $n \to \infty$.

***Proof*.** Arguing component wise we may assume all symbols are scalar valued. Furthermore, we only need to consider the case $m = 2$ by a simple induction argument. So consider sequences $(P_n), (Q_n) \subseteq \dot{H}^{1/2} \cap L^\infty$ with $\|P_n\|_\infty, \|Q_n\|_\infty \leq K$ for all n, and assume $P_n \to P$ and $Q_n \to Q$ in $\dot{H}^{1/2}$ and pointwise almost everywhere. A tedious rewrite shows, for all $p, q \in \mathbb{R}^d$,

$$\begin{aligned}
&\big(P(p)Q(p) - P_n(p)Q_n(p)\big) - \big(P(q)Q(q) - P_n(q)Q_n(q)\big) \\
&\quad = \frac{1}{2}\Big(\big(P(p) - P_n(p)\big) - \big(P(q) - P_n(q)\big)\Big)\big(Q_n(p) + Q_n(q)\big) \\
&\qquad + \frac{1}{2}\Big(\big(P(p) - P_n(p)\big) + \big(P(q) - P_n(q)\big)\Big)\big(Q(p) - Q(q)\big) \\
&\qquad + \frac{1}{2}\Big(\big(Q(p) - Q_n(p)\big) - \big(Q(q) - Q_n(q)\big)\Big)\big(P_n(p) + P_n(q)\big) \\
&\qquad + \frac{1}{2}\Big(\big(Q(p) - Q_n(p)\big) + \big(Q(q) - Q_n(q)\big)\Big)\big(P(p) - P(q)\big)
\end{aligned}$$

We need to argue that the original expression converges to zero in $L^2(d\mu(p,q))$ where μ is the measure $d\mu(p,q) = |p-q|^{-(d+1)}dp\,dq$. Note μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{2d}$. Hence,

$$\begin{aligned}
&\int \big|\big(P(p) - P_n(p)\big) - \big(P(q) - P_n(q)\big)\big|^2 |Q_n(p) + Q_n(q)|^2 \, d\mu(p,q) \\
&\quad \leq 4K^2 \int \big|\big(P(p) - P_n(p)\big) - \big(P(q) - P_n(q)\big)\big|^2 \, d\mu(p,q) \\
&\quad = 4K^2 \|P - P_n\|^2_{\dot{H}^{1/2}} \to 0,
\end{aligned}$$

and similarly, since $Q(p) - Q(q) \in L^2(d\mu(p,q))$ and $\big(P(p) - P_n(p)\big) + \big(P(q) - P_n(q)\big) \to 0$ for μ-almost all $(p,q) \in \mathbb{R}^d$ it follows by the dominated convergence theorem that

$$\int |\big(P(p) - P_n(p)\big) + \big(P(q) - P_n(q)\big)|^2 |Q(p) - Q(q)|^2 \, d\mu(p,q) \to 0.$$

The remaining two terms are handled by interchanging the roles of P and Q. ■

We finally show a simple yet useful result on one-dimensional sections of $\dot{H}^{1/2}$ functions. We remind the reader that for $\nu \in S^{d-1}$ and $p \in \Pi_\nu$ we write $P_{\nu,p}(\xi) = P(p + \xi\nu)$, $\xi \in \mathbb{R}$, for the corresponding one-dimensional section.

Lemma B.5. *Let $P \in \dot{H}^{1/2}(\mathbb{R}^d; \mathbb{C}^N)$. Then for all $\nu \in S^{d-1}$ and almost all $p \in \Pi_\nu$ we have $P_{\nu,p} \in \dot{H}^{1/2}(\mathbb{R}; \mathbb{C}^N)$, and*

$$\int_{\Pi_\nu} \|P_{\nu,p}\|^2_{\dot{H}^{1/2}} \, dp \leq C_d \|P\|^2_{\dot{H}^{1/2}}.$$

***Proof*.** The statement is trivial for $d = 1$ so assume $d \geq 2$. Assume at first $P \in C_c^\infty$ and fix $\nu \in S^{d-1}$. Then for all $x' \in \Pi_\nu$ and $t \in \mathbb{R}$

$$\check{P}(x' + t\nu) = (2\pi)^{-(d-1)/2} \int_{\Pi_\nu} e^{ix'\cdot p}\Big((2\pi)^{-1/2} \int_{\mathbb{R}} e^{it\xi} P(p + \xi\nu) \, d\xi\Big) dp.$$

Hence, by (1.19) and the Plancherel theorem on Π_ν,

$$\begin{aligned}
\int_{\Pi_\nu} \|P_{\nu,p}\|^2_{\dot{H}^{1/2}} \, dp = 2\pi \int_{\Pi_\nu} \int_{\mathbb{R}} |(P_{\nu,p})^\vee(t)|^2 |t| \, dt = 2\pi \int_{\Pi_\nu} \int_{\mathbb{R}} |\check{P}(x' + t\nu)|^2 |t| \, dt \, dx' \\
2\pi \leq \int_{\mathbb{R}^d} |\check{P}(x)|^2 |x| \, dx = \frac{2\pi}{\mathcal{H}^d(S^d)} \|P\|^2_{\dot{H}^{1/2}}.
\end{aligned} \tag{B.4}$$

The same estimate extends to $P \in C_c^\infty + \mathbb{C}^N$ by homogeneity. Now for general $P \in \dot{H}^{1/2}$ we can by Lemma $B.2$ take $(P_n) \subseteq C_c^\infty + \mathbb{C}^N$ such that $P_n \to P$ in $\dot{H}^{1/2}$ and almost everywhere.

By lower continuity and the established claim (B.4)

$$\begin{aligned}\int_{\Pi_\nu} \|P_{\nu,p}\|^2_{\dot H^{1/2}}\,dp &\leq \int_{\Pi_\nu} \liminf_{n\to\infty} \|(P_n)_{\nu,p}\|^2_{\dot H^{1/2}}\,dp \leq \liminf_{n\to\infty} \int_{\Pi_\nu} \|(P_n)_{\nu,p}\|^2_{\dot H^{1/2}}\,dp \\ &\leq \frac{2\pi}{\mathcal{H}^d(S^d)} \liminf_{n\to\infty} \|P_n\|^2_{\dot H^{1/2}} = \frac{2\pi}{\mathcal{H}^d(S^d)} \|P\|^2_{\dot H^{1/2}}.\end{aligned}$$

■

Appendix C. Geometric estimates and a lemma by Roccaforte

In this Appendix we prove basic volume and area bounds, we establish a variant of [41, Lemma 2.5] by Roccaforte in the finite perimeter setting, and record an approximation result on finite perimeter sets.

C.1. **Basic area and volume bounds.**

We here establish basic volume and area bounds for Lipschitz graphs and sets of finite upper Minkowski. The following is well known. We include a proof for completeness.

Lemma C.1. *Assume $\Gamma \subseteq \mathbb{R}^d$ is contained in a finite union of Lipschitz graphs. Then*

$$\mathcal{H}^{d-1}(\Gamma \cap B_r(z)) \leq C_\Gamma r^{d-1},$$

and

$$\mathcal{H}^d\big((\Gamma + B_\delta) \cap B_r(z)\big) \leq C_\Gamma \delta\, r^{d-1},$$

uniformly for $r > 0$, $z \in \mathbb{R}^d$, and $\delta > 0$. If in addition Γ is bounded, then also

$$\mathcal{H}^d\big(\Gamma + B_\delta\big) \leq C_\Gamma \delta (1+\delta)^{d-1}$$

for all $\delta > 0$.

Proof. For all estimates we may without loss assume that Γ is contained in a single graph, $\Gamma \subseteq \{p = (p', p_d) \in \mathbb{R}^d \mid p_d = \psi(p')\}$ for $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ a Lipschitz function with $\operatorname{Lip}\psi = M$. It follows immediately from the area formula that

$$\begin{aligned}\mathcal{H}^{d-1}(\Gamma \cap B_r(z)) &\leq \int_{\{p' \mid (p', \psi(p')) \in B_r(z)\}} \sqrt{1 + |\nabla\psi(p')|^2}\,dp' \\ &\leq \mathcal{H}^{d-1}(B_1(0'))\sqrt{1+M^2}\, r^{d-1}.\end{aligned} \tag{C.1}$$

A simple cone argument shows that

$$\frac{1}{\sqrt{1+M^2}} |q_d - \psi(q')| \leq d(q, \{p_d = \psi(p')\}) \leq |q_d - \psi(q')|, \quad q \in \mathbb{R}^d.$$

By the coarea formula and the computation from (C.1) we find

$$\begin{aligned}\mathcal{H}^d\big((\Gamma + B_\delta) \cap B_r(z)\big) &\leq \int_{\{|p-z| \leq r,\, |p_d - \psi(p')| \leq \sqrt{1+M^2}\delta\}} dp \\ &= \int_{-\delta\sqrt{1+M^2}}^{\delta\sqrt{1+M^2}} \int_{\{|p-z| \leq r,\, \psi(p') = p_d + s\}} \frac{1}{\sqrt{1+|\nabla\psi(p')|^2}}\, d\mathcal{H}^{d-1}(p)\,ds \\ &\leq \int_{-\delta\sqrt{1+M^2}}^{\delta\sqrt{1+M^2}} \mathcal{H}^{d-1}\big(\{p_d = \psi(p') - s\} \cap B_r(z)\big)\,ds \\ &\leq C_d(1+M^2)\,\delta\, r^{d-1}.\end{aligned}$$

The final claim for bounded Γ follows by taking r large enough. ■

We remind the reader that sets of finite upper Minkowski content are introduced in Definition 1.6 by the condition

$$\mathcal{M}^*(\Gamma) = \limsup_{r\to 0^+} \frac{\mathcal{H}^d(\Gamma + B_r)}{2r} < \infty.$$

It is clear that any set with finite upper Minkowski content is bounded, and $\mathcal{M}^*(\overline{\Gamma}) = \mathcal{M}^*(\Gamma)$. By arguing directly from the definition of the Hausdorff measure $\mathcal{H}^{d-1}$, it is straightforward to show that $\mathcal{H}^{d-1}(\Gamma) \le C_d\mathcal{M}^*(\Gamma)$. We finally point out that if $\mathcal{M}^*(\Gamma) < \infty$, then there is $C_{d,\Gamma} > 0$ such that

$$\mathcal{H}^d(\Gamma + B_r) \le C_{d,\Gamma}\big(r + r^d\big) \tag{C.2}$$

uniformly for all $r > 0$. Indeed, we can take $C_{d,\Gamma} = 4\mathcal{M}^*(\Gamma)$ for r small enough, say $r \le r_0$. For $r \ge r_0$, we can take $R > 0$ such that $\Gamma \subseteq B_R$ and then bound

$$\mathcal{H}^d(\Gamma + B_r) \le C_d(r+R)^d \le c_d r^d\left(1 + \frac{R}{r_0}\right)^d.$$

The claim follows.

We end this subsection with a useful integral estimate involving distance functions.

Lemma C.2. *Let $\Gamma \subseteq \mathbb{R}^d$ be a set of finite upper Minkowski content and let $s \ge 1$. There is $C = C_{s,d,\Gamma} > 0$ such that*

$$\int_{\delta\le d(p,\Gamma)\le 1} d(p,\Gamma)^{-s}\,dp \le C\begin{cases}\log(\delta^{-1}), & s=1\\ \delta^{-(s-1)}, & s>1\end{cases}$$

uniformly for $0 < \delta \le 1/2$.

Proof. We argue using a dyadic decomposition. Fix $0 < \delta \le 1/2$ and an integer $N \ge 1$ such that $2^{-N-1} < \delta \le 2^{-N}$. Then, by the coarea formula and (C.2),

$$\begin{aligned}\int_{\delta\le d(p,\Gamma)\le 1} d(p,\Gamma)^{-s}\,dp &\le \sum_{n=0}^{N}\int_{2^{-n-1}\le d(p,\Gamma)\le 2^{-n}} d(p,\Gamma)^{-s}\,dp\\ &\le 2^s\sum_{n=0}^{N}\mathcal{H}^d(\Gamma + B_{2^{-n}})2^{ns} \le C_{d,\Gamma}2^s\sum_{n=0}^{N}2^{n(s-1)}\\ &\le C_{s,d,\Gamma}\begin{cases}N+1 & s=1\\ 2^{(s-1)N}, & s>1\end{cases}.\end{aligned}$$

The claim follows. ■

C.2. **A lemma of Roccaforte.** Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter. In this subsection we give uniform bounds and compute the leading order asymptotics of the volume term

$$\mathcal{H}^d\big(\Omega \setminus \big(\Omega \cap (\Omega + y_1) \cap \cdots \cap (\Omega + y_m)\big)\big). \tag{C.3}$$

in the limit $y_1, \dots, y_m \to 0$. The term (C.3) appears directly in the integral expansion of the trace of $S_m(P;\Omega)$. We prove that

$$\mathcal{H}^d\big(\Omega \setminus \big(\Omega \cap (\Omega + y_1) \cap \cdots \cap (\Omega + y_m)\big)\big) = \int_{\mathcal{F}\Omega}\max_{1\le k\le m}\{0, \nu_\Omega(x)\cdot y_k\}\,d\mathcal{H}^{d-1}(x) + o(|y|) \tag{C.4}$$

as $\max|y_i| \to 0$. Roccaforte [41] showed (C.4) with $O(|y|^2)$ error for Ω with C^3 boundary using a tubular neighborhood argument, and Widom [52] left it as a tricky exercise for the reader in the C^1 setting. A full expansion of (C.3) for smooth domains similar to the Weyl tube formula can be found in [42]. We also refer to [22] who proved a homogeneous version of (C.4) in the finite perimeter setting with a very different motivation.

For $y = (y_1, \dots, y_m) \in \mathbb{R}^{dm}$ set $\Omega_y = \Omega \cap (\Omega + y_1) \cap \dots \cap (\Omega + y_m)$. We prove (C.4) by showing that $F_\Omega : \mathbb{R}^{dm} \to \mathbb{R}$ given by

$$F_\Omega(y_1, \dots, y_m) = \mathcal{H}^d(\Omega \setminus \Omega_y) - \int_{\mathcal{F}\Omega} \max_{1 \le k \le m} \{0, \nu_\Omega(x) \cdot y_k\} \, d\mathcal{H}^{d-1}(x)$$

is differentiable at $y = 0$ with $\nabla F_\Omega(0) = 0$. Differentiability follows a standard argument: we show $F_\Omega \in W^{1,\infty}$ and then that 0 is a Lebesgue point of the weak gradient ∇F_Ω. The argument is closed using Morrey's inequality. This technique is substantially different than the arguments from [22, 41].

Lemma C.3. *Let $\Omega \subseteq \mathbb{R}^d$ be a set with finite perimeter and let $\Omega' \subseteq \mathbb{R}^d$ be a measurable set such that $\mathcal{H}^d(\Omega \cap \Omega') = 0$. Then the function $f_{\Omega,\Omega'}(y) = \mathcal{H}^d(\Omega \cap (\Omega' + y))$ is Lipschitz continuous with distributional gradient*

$$\nabla f_{\Omega,\Omega'}(y) = \int_{\mathcal{F}\Omega} \nu_\Omega(x) 1_{\Omega'}(x - y) \, d\mathcal{H}^{d-1}(x).$$

In particular $\operatorname{Lip} f_{\Omega,\Omega'} \le \operatorname{Per}(\Omega)$.

Proof. We first claim that $f_{\Omega,\Omega'}$ is locally bounded. For $d \ge 2$ this is a simple consequence of the isoperimetric inequality

$$f_{\Omega,\Omega'}(y) \le \mathcal{H}^d(\Omega \cap (\Omega^c + y)) \le \min\{\mathcal{H}^d(\Omega), \mathcal{H}^d(\Omega^c)\} < \infty.$$

For $d = 1$ we can write $\Omega = \cup_{k=1}^N I_k$ as a finite union of separated intervals (I_k), see [1, Proposition 3.52]. It is then elementary to see that

$$f_{\Omega,\Omega'}(y) \le \sum_{k=1}^N \mathcal{H}^1(I_k \cap (I_k^c + y)) \le N|y|,$$

which gives the claim in this case also.

We can now compute the distributional gradient directly. Let $\phi \in C_c^\infty$ and compute formally, here with $\langle \cdot, \cdot \rangle$ denoting the dual pairing on C_c^∞,

$$\begin{aligned} \langle \nabla f_{\Omega,\Omega'}, \phi \rangle &= -\int\int 1_\Omega(x + y) 1_{\Omega'}(x) \nabla\phi(y) \, dx \, dy = \int 1_{\Omega'}(x) \langle \nabla 1_{\Omega - x}, \phi \rangle \, dx \\ &= \int 1_{\Omega'}(x) \int_{\mathcal{F}\Omega - x} \nu_{\Omega - x}(z) \phi(z) \, d\mathcal{H}^{d-1}(z) \, dx \\ &= \int 1_{\Omega'}(x) \int_{\mathcal{F}\Omega} \nu_\Omega(z) \phi(z - x) \, d\mathcal{H}^{d-1}(z) \, dx \\ &= \int \phi(x) \Big(\int_{\mathcal{F}\Omega} \nu_\Omega(z) 1_{\Omega'}(z - x) \, d\mathcal{H}^{d-1}(z) \Big) \, dx. \end{aligned}$$

Only applications of Fubini's theorem need justification, in particular that

$$\int\int 1_\Omega(x + y) 1_{\Omega'}(x) |\nabla\phi(y)| \, dx \, dy < \infty,$$

$$\int\int_{\mathcal{F}\Omega} 1_{\Omega'}(x) |\phi(z - x)| \, d\mathcal{H}^{d-1}(z) \, dx < \infty.$$

The first follows since $f_{\Omega,\Omega'} \in L^\infty_{loc}$ and the latter since $\mathcal{H}^{d-1}(\mathcal{F}\Omega) < \infty$. ■

Introduce also

$$\Omega_y^i = \Omega \cap \big(\cap_{j \ne i} \Omega + y_j \big), \quad i = 1, \dots, m.$$

Clearly Ω_y and Ω_y^i are sets of finite perimeter.

Lemma C.4. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter and let $f_\Omega : \mathbb{R}^{md} \to \mathbb{R}$ be the function $f_\Omega(y) = \mathcal{H}^d(\Omega \setminus \Omega_y)$. Then f_Ω is Lipschitz continuous and the distributional gradients are given by*

$$\nabla_{y_i} f_\Omega(y) = \int_{\mathcal{F}\Omega} \nu_\Omega(x) 1_{\Omega_y^i}(x + y_i)\, d\mathcal{H}^{d-1}(x), \quad i = 1, \dots, m.$$

Proof. By symmetry it suffices to fix $y' = (y_1, \dots, y_{m-1})$ and show that, in the sense of distributions,

$$\nabla_{y_m} f_\Omega(y) = \int_{\mathcal{F}\Omega} \nu_\Omega(x) 1_{\Omega_y^m}(x + y_m)\, d\mathcal{H}^{d-1}(x).$$

This is a simple consequence of Lemma $C.3$. Indeed we can write

$$f_\Omega(y) = \mathcal{H}^d(\Omega \cap (\Omega^c + y_m)) + \mathcal{H}^d(\tilde{\Omega} \cap (\Omega + y_m)),$$

where $\tilde{\Omega} = \Omega \setminus \Omega_y^m$ is not dependent on y_m. Lemma $C.3$ now gives

$$\begin{aligned}\nabla_{y_m} \mathcal{H}^d(\Omega \cap (\Omega^c + y_m)) &= -\int_{\mathcal{F}\Omega^c} \nu_{\Omega^c}(x) 1_\Omega(x + y_m)\, d\mathcal{H}^{d-1}(x) \\ &= \int_{\mathcal{F}\Omega} \nu_\Omega(x) 1_\Omega(x + y_m)\, d\mathcal{H}^{d-1}(x)\end{aligned}$$

and

$$\nabla_{y_m} \mathcal{H}^d(\tilde{\Omega} \cap (\Omega + y_m)) = -\int_{\mathcal{F}\Omega} \nu_\Omega(x) 1_{\tilde{\Omega}}(x + y_m)\, d\mathcal{H}^{d-1}(x).$$

Hence

$$\begin{aligned}\nabla_{y_m} f_\Omega(y) &= \int_{\mathcal{F}\Omega} \nu_\Omega(x) \big(1_\Omega(x + y_m) - 1_{\tilde{\Omega}}(x + y_m)\big)\, d\mathcal{H}^{d-1}(x) \\ &= \int_{\mathcal{F}\Omega} \nu_\Omega(x) 1_{\Omega_y^m}(x + y_m)\, d\mathcal{H}^{d-1}(x),\end{aligned}$$

giving the claim. ■

Lemma C.5. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter and let $g_\Omega : \mathbb{R}^{dm} \to \mathbb{R}$ be the function*

$$g_\Omega(y) = \int_{\mathcal{F}\Omega} \max_{1 \le i \le m} \{0, y_i \cdot \nu_\Omega(x)\}\, d\mathcal{H}^{d-1}(x).$$

Then g_Ω is Lipschitz continuous with weak gradients

$$\nabla_{y_i} g_\Omega(y) = \int_{\mathcal{F}\Omega} \nu_\Omega(x) 1_{S_y^i}(x)\, d\mathcal{H}^{d-1}(x), \quad i = 1, \dots, m,$$

where

$$S_y^i = \{x \in \mathcal{F}\Omega \mid \nu_\Omega(x) \cdot y_i > \max_{j \ne i} \{0, \nu_\Omega(x) \cdot y_j\}\}.$$

Proof. We compute the distributional gradient directly like in Lemma $C.3$. By symmetry it suffices to compute the gradient in y_m. Formally for $\phi \in C_c^\infty(\mathbb{R}^{dm})$

$$\begin{aligned}\langle \nabla_{y_m} g_\Omega, \phi \rangle &= -\int_{\mathbb{R}^{dm}} \nabla_{y_m} g(y) \int_{\mathcal{F}\Omega} \max_{1 \le i \le m} \{0, \nu_\Omega(x) \cdot y_i\}\, d\mathcal{H}^{d-1}(x)\, dy \\ &= \int_{\mathcal{F}\Omega} \langle \nabla_{y_m} \max_{1 \le i \le m} \{0, \nu_\Omega(x) \cdot y_i\}, g(y) \rangle\, d\mathcal{H}^{d-1}(x) \\ &= \int_{\mathcal{F}\Omega} \int g(y) 1_{S_y^m}(x) \nu_\Omega(x)\, dy\, d\mathcal{H}^{d-1}(x) \\ &= \int g(y) \int_{\mathcal{F}\Omega} 1_{S_y^m}(x) \nu_\Omega(x)\, d\mathcal{H}^{d-1}(x)\, dy.\end{aligned}$$

Applications of Fubini's theorem are clearly justified. We used the well known fact that if $f \in W^{1,1}_{loc}$ then $\max\{0, f\} \in W^{1,1}_{loc}$ with weak gradient $1_{\{f>0\}} \nabla f$. ■

We are now in a position to prove (C.4).

Lemma C.6. *Let $\Omega \subseteq \mathbb{R}^d$ be a set of finite perimeter and $m \geq 1$. Then*

$$\lim_{y\to 0}\frac{1}{|y|}\left|\mathcal{H}^d(\Omega\setminus\Omega_y) - \int_{\mathcal{F}\Omega}\max_{1\leq i\leq m}\{0,\nu_\Omega(x)\cdot y_i\}\,d\mathcal{H}^{d-1}(x)\right| = 0$$

Proof. We use notation from the proofs of Lemmas $C.4$ and $C.5$. Set $F_\Omega(y) = f_\Omega(y) - g_\Omega(y)$. By Lemmas $C.4$ and $C.5$ the function F_Ω is Lipschitz continuous with weak gradients, for $i = 1, \dots, m$,

$$\nabla_{y_i}F_\Omega(y) = \int_{\mathcal{F}\Omega}\nu_\Omega(x)\big(1_{\Omega_y^i}(x+y_i) - 1_{S_y^i}(x)\big)\,d\mathcal{H}^{d-1}. \tag{C.5}$$

By Morrey's inequality [14, Theorem 4.10] we can estimate

$$|F_\Omega(y)| = |F_\Omega(y) - F_\Omega(0)| \leq C_{d,m}|y|\Big(\fint_{B_{|y|}}|\nabla F_\Omega(z)|^{dm+1}\,dz\Big)^{1/(dm+1)},$$

so it suffices to show that $\fint_{B_{|y|}}|\nabla F_\Omega(z)|^{dm+1}\,dz \to 0$ as $y \to 0$. By Jensen's inequality and (C.5) we further bound

$$\fint_{B_{|y|}}|\nabla F_\Omega(z)|^{dm+1}\,dz \leq C_{d,m,\Omega}\sum_{i=1}^{m}\int_{\mathcal{F}\Omega}\fint_{B_{|y|}}|1_{\Omega_z^i}(x+z_i) - 1_{S_z^i}(x)|\,dz\,d\mathcal{H}^{d-1}(x). \tag{C.6}$$

The symmetric differences appearing in (C.6) have simple geometric descriptions. Fix $1 \leq i \leq m$ and $x \in \mathcal{F}\Omega$. Let us write $z_0 = 0$. If $z \in \mathbb{R}^{md}$ and $|1_{\Omega_z^i}(x+z_i) - 1_{S_z^i}(x)| \neq 0$ then there is $j \neq i$, including the case $j = 0$, such that either

a) $x + z_i - z_j \in \Omega^c$ and $\nu_\Omega(x)\cdot(z_i - z_j) \geq 0$

or

b) $x + z_i - z_j \in \Omega$ and $\nu_\Omega(x)\cdot(z_i - z_j) \leq 0$.

Introduce the tangent half-planes $\Pi_x^\pm = \{v \in \mathbb{R}^d \mid \pm\nu_\Omega(x)\cdot(v - x) \geq 0\}$ and the tangent plane defect set $D_x = \big(\Pi_x^+ \cap \Omega^c\big) \cup \big(\Pi_x^- \cap \Omega\big)$. We conclude that

$$|1_{\Omega_z^i}(x+z_i) - 1_{S_z^i}(x)| \leq \sum_{j\neq i}1_{D_x}(x + z_i - z_j).$$

Returning to (C.6), a simple computation now shows

$$\sum_{i=1}^{m}\int_{\mathcal{F}\Omega}\fint_{B_{|y|}}|1_{\Omega_z^i}(x+z_i) - 1_{S_z^i}(x)|\,dz \leq C_{m,d}\int_{\mathcal{F}\Omega}\frac{\mathcal{H}^d(D_x \cap B_{2|y|}(x))}{|y|^d}\,d\mathcal{H}^{d-1}(x).$$

The reduced boundary $\mathcal{F}\Omega$ is rectifiable so

$$\frac{\mathcal{H}^d(D_x \cap B_{2|y|}(x))}{|y|^d} \to 0 \text{ as } |y| \to 0$$

for all $x \in \mathcal{F}\Omega$, see [14, Theorem 5.14]. An application of the dominated convergence theorem finishes the proof. ■

C.3. **Approximation.** We need a simple but strong approximation result for sets of finite perimeter. The following is a minor modification of [40].

Lemma C.7. *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be a set of finite perimeter. For all $\varepsilon > 0$ there exists a measurable set $L \subseteq \mathbb{R}^d$ such that*

a) ∂L is contained in a finite union of Lipschitz graphs,

b) the set $C = \{x \in \mathcal{F}\Omega \cap \mathcal{F}L \mid \nu_\Omega(x) = \nu_L(x)\}$ satisfies

$$\mathcal{H}^{d-1}(\mathcal{F}\Omega\setminus C) \leq \varepsilon, \quad \mathcal{H}^{d-1}(\mathcal{F}L\setminus C) \leq \varepsilon,$$

c) $|D(1_\Omega - 1_L)|(\mathbb{R}^d) + \|1_\Omega - 1_L\|_1 \leq \varepsilon$,

d) either L or L^c is bounded.

Proof. A construction of a set L with the properties a), b), and c) is given in [40]. If $\mathcal{H}^d(\Omega) < \infty$, then $L \cap B_R$ for $R > 0$ appropriately chosen will work. Else $\mathcal{H}^d(\Omega^c) < \infty$ by the isoperimetric inequality, and we can take a bounded set F satisfying a), b), and c) for Ω^c. Then $L = F^c$ works. We leave the details to the reader. ■

Appendix D. Multiscale symbols

We here prove that multiscale symbols scale logarithmically in the $\dot{H}^{1/2}$ norm, and we give an approximation result. The following proof is inspired by that of [47, Theorem 6.1].

Lemma D.1. *Let $\Gamma \subseteq \mathbb{R}^d$ be a set of finite upper Minkowski content. Suppose (P_δ, F) satisfies a $C^1(\mathbb{R}^d;\mathbb{C}^n)$ multiscale estimate on Γ. Then, for $s > 1$,*

$$\iint \frac{|P_\delta(p) - P_\delta(q)|^s}{|p-q|^{d+1}}\, dp\, dq \le C_{d,s,\Gamma} \|F\|_\infty^s \log(\delta^{-1}) + C_{d,s} \int G(p)^s\, dp.$$

In particular,

$$\|P_\delta\|_{\dot{H}^{1/2}}^2 \le \|P_\delta\|_\infty^{1/2} \Big(C_{d,\Gamma} \|F\|_\infty^{3/2} \log(\delta^{-1}) + C_d \int G(p)^{3/2}\, dp \Big).$$

Here $G(p) = \sup_{|p-q| \le \ell_\delta(p)} F(q)$.

Proof. First split

$$\begin{aligned}\iint & \frac{|P_\delta(p) - P_\delta(q)|^s}{|p-q|^{d+1}}\, dp\, dq \\ &= \iint_{|p-q| \le \ell_\delta(p)} \frac{|P_\delta(p) - P_\delta(q)|^s}{|p-q|^{d+1}}\, dp\, dq + \iint_{|p-q| \ge \ell_\delta(p)} \frac{|P_\delta(p) - P_\delta(q)|^s}{|p-q|^{d+1}}\, dp\, dq.\end{aligned}$$

Now if $|p-q| \le \ell_\delta(p)$ then $|P_\delta(p) - P_\delta(q)| \le C_d |p-q| \ell_\delta(p)^{-1} G(p)$. Hence, using Lemma $C.2$,

$$\begin{aligned}\iint_{|p-q| \le \ell_\delta(p)} \frac{|P_\delta(p) - P_\delta(q)|^s}{|p-q|^{d+1}}\, dp\, dq &\le C_{d,s} \int \frac{G(p)^s}{\ell_\delta(p)}\, dp \\ &\le C_{d,s} \|F\|_\infty^s \int_{d(p,\Gamma) \le 1} \ell_\delta(p)^{-1}\, dp + C_{d,s} \int G(p)^s\, dp \\ &\le C_{d,s,\Gamma} \|F\|_\infty^s \log(\delta^{-1}) + C_{d,s} \int G(p)^s\, dp.\end{aligned}$$

Similarly, if $|p-q| \ge \ell_\delta(p)$, then $|p-q| \ge \frac{2}{3}\ell_\delta(q)$, so

$$\begin{aligned}\iint_{|p-q| \ge \ell_\delta(p)} & \frac{|P_\delta(p) - P_\delta(q)|^s}{|p-q|^{d+1}}\, dp\, dq \\ &\le C_s \iint_{|p-q| \ge \ell_\delta(p)} \frac{|P_\delta(p)|^s}{|p-q|^{d+1}}\, dp\, dq + C_s \iint_{|p-q| \ge \frac{2}{3}\ell_\delta(q)} \frac{|P_\delta(q)|^s}{|p-q|^{d+1}}\, dp\, dq \\ &\le C_{d,s} \int \frac{|P_\delta(p)|^s}{\ell_\delta(p)}\, dp \le C_{d,s,\Gamma} \|F\|_\infty^s \log(\delta^{-1}) + C_{d,s} \int G(p)^s\, dp.\end{aligned}$$

The remaining claim follows by bounding $|P_\delta(p) - P_\delta(q)|^2 \le 2\|P_\delta\|_\infty^{1/2} |P_\delta(p) - P_\delta(q)|^{3/2}$ and applying the bound for $s = 3/2$. ■

We also obtain an approximation result for multiscale symbols.

Lemma D.2. *Let $\Gamma \subseteq \mathbb{R}^d$ be a set of finite upper Minkowski content and assume that (P_δ) satisfies a $C^1(\mathbb{R}^d;\mathbb{C}^n)$ multiscale estimate on Γ. Fix $\kappa \in \mathbb{N}$. For all $0 < \varepsilon < 1$ there is a family $(P_{\delta,\varepsilon})$ satisfying a $C^\kappa(\mathbb{R}^d;\mathbb{C}^n)$ multiscale estimate on Γ such that*

$$\|P_\delta - P_{\delta,\varepsilon}\|_\infty \le C\varepsilon,$$

and

$$\|P_\delta - P_{\delta,\varepsilon}\|_{\dot{H}^{1/2}}^2 \le C\varepsilon^{1/2} \log(\delta^{-1}).$$

uniformly for $0 < \delta \leq 1/2$ *and* $0 < \varepsilon < 1$.

Proof. Take an amplitude function F such that (P_δ, F) satisfies the C^1 multiscale estimate and set $G(p) = \sup_{|p-q|\leq \ell_\delta(p)} F(q)$. We use the partition of unity (ϕ_u) supplied by Lemma 2.1. We can write

$$P_\delta(p) = \int \ell_\delta(u)^{-d}\phi_u(p)P_\delta(p)\,du.$$

Let (η_ε) be a standard family of mollifiers. We define, for $0 < \varepsilon < 1$,

$$P_{\delta,\varepsilon}(p) = \int \ell_\delta(u)^{-d}\phi_u(p)\big(P_\delta * \eta_{\varepsilon\ell_\delta(u)}\big)(p)\,du.$$

It follows from the multiscale assumption that for $C > 0$ only dependent on η and d, for all $|p-u| \leq \ell_\delta(u)$,

$$\big|\big(P_\delta * \eta_{\varepsilon\ell_\delta(u)}\big)(p) - P_\delta(p)\big| \leq \int |P_\delta(p+h) - P_\delta(p)|\eta_{\varepsilon\ell_\delta(u)}(h)\,dh \leq C\varepsilon G(u).$$

Hence, $\|P_\delta - P_{\delta,\varepsilon}\|_\infty \leq C\varepsilon$. We now show that $(P_{\delta,\varepsilon})$ satisfies a C^1 multiscale estimate on Γ uniformly in $0 < \varepsilon < 1$. To this end we bound directly using properties of (ϕ_u) from Lemma 2.1 and the multiscale assumption on P_δ,

$$\begin{aligned}|\nabla P_{\delta,\varepsilon}(p)| &\leq \int \ell_\delta(u)^{-d}\Big(|\nabla\phi_u(p)||P_\delta * \eta_{\varepsilon\ell_\delta(u)}(p)| + \phi_u(p)|(\nabla P_\delta) * \eta_{\varepsilon\ell_\delta(u)}(p)|\Big)\,du\\ &\leq C\int_{|p-u|\leq\ell_\delta(u)} \ell_\delta(u)^{-d}G(u)\ell_\delta(u)^{-1}\,du\\ &\leq C\ell_\delta(p)^{-1}\sup_{|p-u|\leq\ell_\delta(u)} G(u)\end{aligned}$$

We used that $\ell_\delta(p) \sim \ell_\delta(u)$ when $|p-u| \leq \ell_\delta(u)$ at the end. It is elementary to see that $\sup_{|p-u|\leq\ell_\delta(u)} G(u) \leq \sup_{|p-q|\leq 2} F(q) \in L^1 \cap L^\infty$, see Remark 2.3. It now follows from Lemma D.1 that

$$\|P_\delta - P_{\delta,\varepsilon}\|_{\dot{H}^{1/2}}^2 \leq C\|P_\delta - P_{\delta,\varepsilon}\|_\infty^{1/2}\log(\delta^{-1}) \leq C\varepsilon^{1/2}\log(\delta^{-1})$$

uniformly for $0 < \varepsilon < 1$ and $0 < \delta \leq 1/2$.

A similar argument to the one employed above shows that $(P_{\delta,\varepsilon})$ satisfies a C^κ multiscale assumption; simply place all derivatives of $P_\delta * \eta_{\varepsilon\ell_\delta(u)}$ on the η-term. ■

Acknowledgments. Martin Dam Larsen was supported by the VILLUM Centre of Excellence for the Mathematics of Quantum Theory (QMATH) with Grant No.10059.

REFERENCES

[1] Luigi Ambrosio, Nicola Fusco, and Diego Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[2] Estelle Basor, *Asymptotic formulas for Toeplitz determinants*, Trans. Amer. Math. Soc. **239** (1978), 33–65.

[3] Thomas Beretti and Luca Gennaioli, *Fourier transform of bv functions and applications*, arXiv preprint (2024), available at `2407.13347`.

[4] Leon Bollmann and Peter Müller, *The Widom-Sobolev formula for discontinuous matrix-valued symbols*, J. Funct. Anal. **287** (2024), no. 12, Paper No. 110651, 54.

[5] ______, *Enhanced area law in the Widom-Sobolev formula for the free Dirac operator in arbitrary dimension*, Pure Appl. Anal. **7** (2025), no. 3, 595–613.

[6] Jean Bourgain, Haim Brezis, and Petru Mironescu, *Another look at Sobolev spaces*, Optimal control and partial differential equations, 2001, pp. 439–455.

[7] Lorenzo Brasco, David Gómez-Castro, and Juan Luis Vázquez, *Characterisation of homogeneous fractional Sobolev spaces*, Calc. Var. Partial Differential Equations **60** (2021), no. 2, Paper No. 60, 40.

[8] Chris Brislawn, *Kernels of trace class operators*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1181–1190.

[9] Alessandro Cucinotta, *On the properties of the set where a generalized function of bounded variation takes an infinite value*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **34** (2023), no. 4, 845–862.

[10] J. Dávila, *On an open question about functions of bounded variation*, Calc. Var. Partial Differential Equations **15** (2002), no. 4, 519–527.

[11] ———, *On an open question about functions of bounded variation*, Calc. Var. Partial Differential Equations **15** (2002), no. 4, 519–527.

[12] Percy Deift, Alexander Its, and Igor Krasovsky, *Asymptotics of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher-Hartwig singularities*, Ann. of Math. (2) **174** (2011), no. 2, 1243–1299.

[13] Torsten Ehrhardt, *A status report on the asymptotic behavior of Toeplitz determinants with Fisher-Hartwig singularities*, Recent advances in operator theory (Groningen, 1998), 2001, pp. 217–241.

[14] Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Revised, Textbooks in Mathematics, CRC Press, Boca Raton, FL, 2015.

[15] Herbert Federer, *Geometric measure theory.*, Springer-Verlag New York, Inc., New York, 1969.

[16] Fabrizio Ferro, Paul Pfeiffer, and Wolfgang Spitzer, *Entanglement entropy in the ground state of non-interacting massless Dirac fermions in dimension one*, Adv. Math. **482** (2025), Paper No. 110562, 37.

[17] Dimitri Gioev and Israel Klich, *Entanglement entropy of fermions in any dimension and the Widom conjecture*, Phys. Rev. Lett. **96** (2006), no. 10, 100503, 4.

[18] Christian Hainzl and Robert Seiringer, *General decomposition of radial functions on $\mathbb{R}^n$ and applications to N-body quantum systems*, Lett. Math. Phys. **61** (2002), no. 1, 75–84.

[19] C. S. Herz, *Fourier transforms related to convex sets*, Ann. of Math. (2) **75** (1962), 81–92.

[20] Lars Hörmander, *The analysis of linear partial differential operators. I*, Second, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 256, Springer-Verlag, Berlin, 1990. Distribution theory and Fourier analysis.

[21] M. Kac, *Toeplitz matrices, translation kernels and a related problem in probability theory*, Duke Math. J. **21** (1954), 501–509.

[22] Markus Kiderlen and Jan Rataj, *Dilation volumes of sets of finite perimeter*, Adv. in Appl. Probab. **50** (2018), no. 4, 1095–1118.

[23] Aleksei Kulikov and Martin Dam Larsen, *Sharp estimates for eigenvalues of localization operators with applications to area laws*, arXiv preprint (2026), available at `2603.23832`.

[24] Panu Lahti, *A notion of fine continuity for BV functions on metric spaces*, Potential Anal. **46** (2017), no. 2, 279–294.

[25] H. J. Landau and H. Widom, *Eigenvalue distribution of time and frequency limiting*, J. Math. Anal. Appl. **77** (1980), no. 2, 469–481.

[26] A. Laptev and Yu. Safarov, *A generalization of the Berezin-Lieb inequality*, Contemporary mathematical physics, 1996, pp. 69–79.

[27] Hajo Leschke, Alexander V. Sobolev, and Wolfgang Spitzer, *Large-scale behaviour of local and entanglement entropy of the free Fermi gas at any temperature*, J. Phys. A **49** (2016), no. 30, 30LT04, 9.

[28] ———, *Trace formulas for Wiener-Hopf operators with applications to entropies of free fermionic equilibrium states*, J. Funct. Anal. **273** (2017), no. 3, 1049–1094.

[29] ———, *Rényi entropies of the free Fermi gas in multi-dimensional space at high temperature*, Toeplitz operators and random matrices—in memory of Harold Widom, [2022] ©2022, pp. 477–508.

[30] Elliott H. Lieb and Michael Loss, *Analysis*, Second, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.

[31] Francesco Maggi, *Sets of finite perimeter and geometric variational problems*, Cambridge Studies in Advanced Mathematics, vol. 135, Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[32] V. V. Peller, *When is a function of a Toeplitz operator close to a Toeplitz operator?*, Toeplitz operators and spectral function theory, 1989, pp. 59–85.

[33] ———, *An extension of the Koplienko-Neidhardt trace formulae*, J. Funct. Anal. **221** (2005), no. 2, 456–481.

[34] ———, *Multiple operator integrals in perturbation theory*, Bull. Math. Sci. **6** (2016), no. 1, 15–88.

[35] Paul Pfeiffer and Wolfgang Spitzer, *Entanglement entropy of ground states of the three-dimensional ideal Fermi gas in a magnetic field*, Ann. Henri Poincaré **25** (2024), no. 8, 3649–3698.

[36] ———, *Logarithmically enhanced area-laws for fermions in vanishing magnetic fields in dimension two*, Integral Equations Operator Theory **96** (2024), no. 4, Paper No. 28, 64.

[37] Nicola Picenni, *New estimates for a class of non-local approximations of the total variation*, J. Funct. Anal. **287** (2024), no. 1, Paper No. 110419, 21.

[38] Augusto C. Ponce and Daniel Spector, *On formulae decoupling the total variation of BV functions*, Nonlinear Anal. **154** (2017), 241–257.

[39] ———, *On formulae decoupling the total variation of BV functions*, Nonlinear Anal. **154** (2017), 241–257.

[40] Thierry Quentin de Gromard, *Strong approximation of sets in* $\mathrm{BV}(\Omega)$, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), no. 6, 1291–1312.

[41] Raymond Roccaforte, *Asymptotic expansions of traces for certain convolution operators*, Trans. Amer. Math. Soc. **285** (1984), no. 2, 581–602.

[42] ______, *The volume of non-uniform tubular neighborhoods and an application to the n-dimensional Szegö theorem*, J. Math. Anal. Appl. **398** (2013), no. 1, 61–67.

[43] Barry Simon, *Trace ideals and their applications*, Second, Mathematical Surveys and Monographs, vol. 120, American Mathematical Society, Providence, RI, 2005.

[44] A. V. Sobolev, *Pseudo-differential operators with discontinuous symbols: Widom's conjecture*, Mem. Amer. Math. Soc. **222** (2013), no. 1043, vi+104.

[45] ______, *Wiener-Hopf operators in higher dimensions: the Widom conjecture for piece-wise smooth domains*, Integral Equations Operator Theory **81** (2015), no. 3, 435–449.

[46] Alexander V. Sobolev, *On the Schatten-von Neumann properties of some pseudo-differential operators*, J. Funct. Anal. **266** (2014), no. 9, 5886–5911.

[47] ______, *On a coefficient in trace formulas for Wiener-Hopf operators*, J. Spectr. Theory **6** (2016), no. 4, 1021–1045.

[48] ______, *Functions of self-adjoint operators in ideals of compact operators*, J. Lond. Math. Soc. (2) **95** (2017), no. 1, 157–176.

[49] ______, *Quasi-classical asymptotics for functions of Wiener-Hopf operators: smooth versus non-smooth symbols*, Geom. Funct. Anal. **27** (2017), no. 3, 676–725.

[50] ______, *On Szegő formulas for truncated Wiener-Hopf operators*, Integral Equations Operator Theory **91** (2019), no. 1, Paper No. 1, 28.

[51] Jan Philip Solovej and Wolfgang L. Spitzer, *A new coherent states approach to semiclassics which gives Scott's correction*, Comm. Math. Phys. **241** (2003), no. 2-3, 383–420.

[52] Harold Widom, *Szegő's limit theorem: The higher-dimensional matrix case*, J. Functional Analysis **39** (1980), no. 2, 182–198.

[53] ______, *On a class of integral operators with discontinuous symbol*, Toeplitz centennial (Tel Aviv, 1981), 1982, pp. 477–500.

[54] ______, *A trace formula for Wiener-Hopf operators*, J. Operator Theory **8** (1982), no. 2, 279–298.

[55] ______, *Asymptotic expansions for pseudodifferential operators on bounded domains*, Lecture Notes in Mathematics, vol. 1152, Springer-Verlag, Berlin, 1985.

[56] ______, *On a class of integral operators on a half-space with discontinuous symbol*, J. Funct. Anal. **88** (1990), no. 1, 166–193.

Søren Fournais
University of Copenhagen, Department of Mathematical Sciences, Universitetsparken 5, 2100 Copenhagen, Denmark,
`fournais@math.ku.dk`

Martin Dam Larsen,
University of Copenhagen, Department of Mathematical Sciences, Universitetsparken 5, 2100 Copenhagen, Denmark,
`mdl@math.ku.dk`

Robert Seiringer,
Institute of Science and Technology Austria, Am Campus 1, 3400 Klosterneuburg, Austria,
`robert.seiringer@ist.ac.at`

Jan Philip Solovej,
University of Copenhagen, Department of Mathematical Sciences, Universitetsparken 5, 2100 Copenhagen, Denmark,
`solovej@math.ku.dk`

Chapter 4

Paper: *Sharp Estimates for Eigenvalues of Localization Operators with Applications to Area Laws*

This chapter contains the manuscript entitled "Sharp Estimates for Eigenvalues of Localization Operators with Applications to Area Laws", which is joint work between Aleksei Kulikov and the author.
The paper is included in its preprint form – including the title page, abstract and bibliography.

SHARP ESTIMATES FOR EIGENVALUES OF LOCALIZATION OPERATORS WITH APPLICATIONS TO AREA LAWS

ALEKSEI KULIKOV AND MARTIN DAM LARSEN

Abstract. We study the eigenvalues of the localization operator $S_{A,B} = P_A\mathcal{F}^{-1}P_B\mathcal{F}P_A$, where $\mathcal{F}$ is the Fourier transform and $A = cA_0, B = B_0$ for some fixed sets $A_0, B_0 \subset \mathbb{R}^d$ and a large parameter $c > 0$. For the counting function of the eigenvalues $|\{n : \varepsilon < \lambda_n(A,B) \le 1-\varepsilon\}|$ we obtain a sharp uniform upper bound if one of the sets is a finite disjoint union of parallelepipeds and a bound which is only a single logarithm off the conjectural optimal bound in the general case. These bounds are applied to the estimation of traces $\operatorname{Tr} f(S_{A,B})$ for functions f with a very low regularity, in particular establishing an enhanced area law in the former case.

1. Introduction

1.1. **Time-frequency localization operator.** For a measurable set $A \subset \mathbb{R}^d$ by P_A we denote the projection onto A and by Q_A we denote the Fourier projection onto A

$$Q_A = \mathcal{F}^{-1}P_A\mathcal{F},$$

where $\mathcal{F}$ is the Fourier transform

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)\,dx,$$

whose inverse is

$$\mathcal{F}^{-1}f(x) = \check{f}(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} f(\xi)\,d\xi.$$

For a pair of measurable sets $A, B \subset \mathbb{R}^d$ by $S_{A,B}$ we denote the time-frequency localization operator

$$S_{A,B} = P_AQ_BP_A.$$

This is a non-negative definite self-adjoint operator on $L^2(\mathbb{R}^d)$ with the integral kernel

$$S_{A,B}(x,y) = 1_A(x)\check{1}_B(x-y)1_A(y), \quad x,y \in \mathbb{R}^d. \tag{1.1}$$

If the measures of A and B are finite, $S_{A,B}$ is a trace class operator with $\operatorname{Tr}(S_{A,B}) = |A||B|$. In particular, it is compact and as such it has a sequence of eigenvalues

$$1 > \|S_{A,B}\| = \lambda_1(A,B) \ge \lambda_2(A,B) \ge \ldots \ge 0.$$

We note for future reference that $S_{A,B}$ obviously does not change if we modify A and B by sets of measure zero.

It can be shown that the eigenvalues remain the same if we swap A and B, that is $\lambda_n(A,B) = \lambda_n(B,A)$. The fact that $\lambda_1(A,B)$ is strictly less than 1 says that there does not exist a non-zero function such that it and its Fourier transform have supports of finite measure. This is known as the uncertainty principle of Benedicks and Amrein–Berthier [2, 3]. In this paper we will be studying the finer properties of the distribution of these eigenvalues.

The usual regime that is considered is when the sets A and B are scalings of some fixed sets A_0, B_0. So, we will assume that $A = cA_0$ and $B = B_0$ for a large parameter c. Note that we do not put a scaling onto B, as it can be transferred to the scaling on A by an affine change of variables without altering the eigenvalues. More general affine changes of variables will play a role in our proofs later.

The goal of the present paper is to study the distribution of eigenvalues $\lambda_n(cA_0, B_0)$. We expect that the eigenvalues exhibit a phase transition: the first $\approx c^d|A_0||B_0|$ eigenvalues are very close to 1, then there are only $\asymp c^{d-1}\log c$ intermediate eigenvalues away from 0 and 1, and after that the eigenvalues decay to 0 extremely fast. The intermediate part of the spectrum is usually referred to as the plunge region. This picture is particularly clear in dimension 1 in the case when A_0 and B_0 are fixed intervals, where the exact limiting behaviour and very precise uniform estimates of the eigenvalues are known [6, 10, 18, 21, 22], see also Section 2 below.

In higher dimensions much less is known as the geometry of sets A_0, B_0 starts to play a significant role. To describe the results, let us introduce the counting functions for the eigenvalues.

$$\begin{aligned} N_\varepsilon(A,B) &= |\{n : \lambda_n(c) > \varepsilon\}|, \\ \Lambda_\varepsilon^+(A,B) &= \left|\left\{n : 1-\varepsilon > \lambda_n(c) > \frac{1}{2}\right\}\right|, \\ \Lambda_\varepsilon^-(A,B) &= \left|\left\{n : \frac{1}{2} \geq \lambda_n(c) > \varepsilon\right\}\right|, \\ \Lambda_\varepsilon(A,B) &= \Lambda_\varepsilon^+(A,B) + \Lambda_\varepsilon^-(A,B) = |\{n : 1-\varepsilon \geq \lambda_n(c) > \varepsilon\}|. \end{aligned} \tag{1.2}$$

We chose this arrangement of strict and non-strict inequalities to ensure that for $0 < \varepsilon < \frac{1}{2}$ we have

$$N_{1-\varepsilon}(A,B) = N_{1/2}(A,B) - \Lambda_\varepsilon^+(A,B), \quad N_\varepsilon(A,B) = N_{1/2}(A,B) + \Lambda_\varepsilon^-(A,B). \tag{1.3}$$

Of course, for all applications whether we have strict or non-strict inequalities is immaterial.

Previous results established bounds on $N_\varepsilon(cA_0, B_0)$ and $\Lambda_\varepsilon(cA_0, B_0)$ under strong enough assumptions on the boundaries ∂A_0 and ∂B_0. In particular, Sobolev [26] established a two-term asymptotic formula for $N_\varepsilon(cA_0, B_0)$ for fixed $\varepsilon > 0$. There are also several uniform estimates on $N_\varepsilon(cA_0, B_0)$ and $\Lambda_\varepsilon(cA_0, B_0)$ with $\varepsilon \to 0$ as $c \to \infty$ [12, 14, 17, 23].

In this work we establish estimates on $\Lambda_\varepsilon(cA_0, B_0)$ which are on the one hand better than all of the previously obtained estimates, in particular being sharp in some cases, and on the other hand requiring much weaker and more natural geometric conditions on the sets A_0, B_0.

The form of our results depends on the assumptions we put on the sets A_0, B_0. If both of them are axis-parallel boxes then we can leverage known estimates in the one-dimensional case to establish precise two-sided estimates on $\Lambda^\pm_\varepsilon(cA_0, B_0)$. If one of the sets is a disjoint finite union of parallelepipeds and the other satisfies a very weak boundary regularity condition then we have sharp uniform bounds on $\Lambda^\pm_\varepsilon(cA_0, B_0)$ from above. This in particular allows us to find two-term asymptotics for $\mathrm{Tr} f(S_{A,B})$ for very rough functions f. For the general case, the bound we obtain is off the conjectural one by at most a single logarithm. We will now present the cases we consider in the increasing order of generality.

1.2. **Case of two boxes.** In this section we consider the case $A = [0, a]^d$, $B = [0, b]^d$. Since the variables separate, the operator $S_{A,B}$ splits into a tensor power $S_{A,B} = \left(S_{[0,a],[0,b]}\right)^{\otimes d}$. In particular, the eigenvalues depend only on the product $c = ab$, and we get that the eigenvalues of $S_{A,B}$ are all the products of d eigenvalues of the operator $S_{[0,a],[0,b]}$ counted with multiplicity. Using this we can generalize all of the estimates on the counting functions from dimension 1 to higher dimensions with a simple combinatorial argument.

Theorem 1.1. *Consider $A = B = [0, 1]^d$, $d \geq 1$, and $c \geq 2$. There exists $\alpha_d \geq 4$ such that*

$$\Lambda^\pm_\varepsilon(cA, B) \lesssim c^{d-1} \log(\tfrac{1}{\varepsilon}) \log\left(\frac{\alpha_d\, c}{\log\left(\frac{1}{\varepsilon}\right)}\right) \tag{1.4}$$

uniformly for all $\alpha_d^{-c} < \varepsilon < 1/2$. Moreover, if $\varepsilon < c^{-\alpha_d}$ then we also have

$$\Lambda^\pm_\varepsilon(cA, B) \gtrsim c^{d-1} \log(\tfrac{1}{\varepsilon}) \log\left(\frac{\alpha_d\, c}{\log\left(\frac{1}{\varepsilon}\right)}\right) \tag{1.5}$$

If $\varepsilon \leq \alpha_d^{-c}$ then there are no eigenvalues larger than $1 - \varepsilon$ and

$$\Lambda^-_\varepsilon \asymp \left(\frac{\log(\frac{1}{\varepsilon})}{\log\left(\frac{\log(\frac{1}{\varepsilon})}{c}\right)}\right)^d. \tag{1.6}$$

The same tensor product structure can also be employed if A and B are axis-parallel boxes with not necessarily equal side lengths. In this way, by the same combinatorial argument, we get the following slightly more general result.

Proposition 1.2. *Let $d \geq 1$ and $I_1, \dots, I_d$ and $J_1, \dots, J_d$ be fixed closed and finite intervals. Set $A = I_1 \times \cdots \times I_d$ and $B = J_1 \times \cdots \times J_d$. Then the conclusion of Theorem 1.1 holds, except α_d now also depends on the intervals (I_k), (J_k).*

1.3. **General case.** For general sets A, B the operator $S_{A,B}$ no longer splits into a tensor product, and the precision of our results will depend on the geometries of A and B. Specifically, we will either assume that they are disjoint finite unions of parallelepipeds, or that they are bounded and their boundaries have finite upper Minkowski content.

Definition 1. *A set $\Gamma \subset \mathbb{R}^d$ is said to be of finite upper Minkowski content if*

$$\limsup_{r \to 0^+} \frac{|\{x \in \mathbb{R}^d : \operatorname{dist}(x, \Gamma) < r\}|}{2r} < \infty.$$

It is well known that for ∂A regular enough, the limit $\lim_{r \to 0^+} \frac{|\{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial A) < r\}|}{2r}$ exactly agrees with the perimeter of A. Here "regular enough" includes any bounded set with Lipschitz (or C^1) boundary, see [1, Theorem 2.106]. Note that since $2r$ goes to 0 when $r \to 0^+$, any set of finite upper Minkowski content has zero Lebesgue measure.

It is also not hard to see that, in general, if Γ has finite upper Minkowski, then Γ must must be bounded. For $d \geq 2$ the boundary of the set $A \subset \mathbb{R}^d$ is bounded if and only if either A or $\mathbb{R}^d \backslash A$ is bounded; if $d = 1$ then either A or $\mathbb{R}^d \backslash A$ is bounded, or each of them contains a ray. In either case, the only option for A to have finite measure is if A is bounded, so we will always assume that all of our sets are bounded.

Definition 2. *We call a set A a parallelepiped if there exists an invertible $d \times d$ matrix M and a vector $b \in \mathbb{R}^d$ such that $A = M[0,1]^d + b$.*

Theorem 1.3. *Assume that A is a bounded set whose boundary has finite upper Minkowski content and B is a finite union of parallelepipeds with disjoint interiors such that both A and B have positive measures. There exists $\alpha = \alpha(d, A, B) \geq 4$ such that for all $c \geq 2$ and $\alpha^{-c} < \varepsilon < \frac{1}{2}$ we have*

$$\Lambda_\varepsilon(cA, B) \lesssim c^{d-1} \log(\tfrac{1}{\varepsilon}) \log\left(\frac{\alpha\, c}{\log\left(\frac{1}{\varepsilon}\right)}\right). \tag{1.7}$$

For $\varepsilon \leq \alpha^{-c}$ there are no eigenvalues larger than $1-\varepsilon$ and

$$\Lambda_\varepsilon^-(cA,B) \asymp \left(\frac{\log(\frac{1}{\varepsilon})}{\log\left(\frac{\log(\frac{1}{\varepsilon})}{c}\right)} \right)^d . \tag{1.8}$$

Remark 1.4. *If we restrict to finite unions of axis-parallel boxes then we do not have to assume that their interiors are disjoint, as we can do a finite subdivision to make them disjoint.*

Remark 1.5. *We need to assume that A and B have positive measures only to ensure that we have a lower bound in* (1.8)*, since if A or B has zero measure then $S_{A,B} = 0$ and all of the eigenvalues are zero.*

Theorem 1.6. *Assume that A and B are bounded sets whose boundaries have finite upper Minkowski content and such that both A and B have positive measures. There exists $\alpha = \alpha(d,A,B) \geq 4$ such that for all $c \geq 2$ and $\alpha^{-c} < \varepsilon < \frac{1}{2}$ we have*

$$\Lambda_\varepsilon(cA,B) \lesssim c^{d-1} \log(\tfrac{1}{\varepsilon}) \log^2 \left(\frac{\alpha\, c}{\log\left(\frac{1}{\varepsilon}\right)} \right) .$$

For $\varepsilon \leq \alpha^{-c}$ there are no eigenvalues larger than $1-\varepsilon$ and

$$\Lambda_\varepsilon^-(cA,B) \asymp \left(\frac{\log(\frac{1}{\varepsilon})}{\log\left(\frac{\log(\frac{1}{\varepsilon})}{c}\right)} \right)^d .$$

When we pass from Theorem 1.3 to Theorem 1.6 we have to put an additional square on the term $\log\left(\frac{\alpha\, c}{\log\left(\frac{1}{\varepsilon}\right)}\right)$. While we believe that this effect does not actually take place, we are currently unable to remove it.

Remark 1.7. *In both of these results we are also able to vary A and B with c as long as they remain uniformly bounded, contain uniformly bounded from below balls inside of them and we have uniform bounds in the definition of finite upper Minkowski content for their boundaries for all $0 < r < 1$, say. We leave these routine generalizations to the interested reader.*

Arguments similar to our proof of Theorem 1.1 have already appeared in the literature, see e.g. [12] and [17]. However, both of them relied on estimates of Karnik, Romberg and

Davenport [18], which are not the strongest available bounds in all regimes in dimension 1, see Section 2 below. In particular, in the regime $\varepsilon < \alpha_d^c$, while we obtain at worst

$$\Lambda_\varepsilon^-(cA, B) \lesssim \left(\log \frac{1}{\varepsilon}\right)^d,$$

their upper bound is $\left(\log c \log \frac{1}{\varepsilon}\right)^d$, that is, they are losing a factor of at least $(\log c)^d$.

For the general case, our assumption of boundaries having finite upper Minkowski content is weaker than all of the assumptions previously appearing in the literature, particularly the maximally Ahlfors regular boundary assumption in [14, 23], see [1, Theorem 2.104]. Moreover, our estimates in all regimes are at least one logarithm better than the bounds obtained in these papers, in particular being sharp in the setting of Theorem 1.3. In the proof strategy below we will highlight the key new estimates which allowed us to get this improvement.

The previous best estimate in the case of Theorem 1.3 is due to Israel and Mayeli [17]. They used wave packet basis to get a bound of the form

$$\Lambda_\varepsilon(cA, B) \lesssim_\delta c^{d-1}(\log c)^{2+\delta}$$

in the regime $\varepsilon \sim \frac{1}{c^s}$ with fixed $s > 0$ for any $\delta > 0$, thus being only $(\log c)^\delta$ away from our result. In the proof they relied on Gevrey classes, that is on the possible decay of the Fourier transform of the compactly supported functions. Thus, due to the Beurling–Malliavin theorem [4], it is not possible to use their approach to match our result even in this regime. The same applies to the recent work [15] compared to our Theorem 1.6.

1.4. **Proof strategy.** The first step in all of our arguments is the $S - S^2$ trick. We have $\varepsilon < \lambda < 1 - \varepsilon$ if and only if $\varepsilon(1-\varepsilon) < \lambda(1-\lambda)$. For the time-frequency localization operator $S_{A,B}$ we have

$$S_{A,B} - S_{A,B}^2 = P_A Q_B Q_B P_A - P_A Q_B P_A Q_B P_A = P_A Q_B (1 - P_A) Q_B P_A = P_A Q_B P_{A^c} Q_B P_A,$$

where A^c is the complement of A. This latter expression is equal to T^*T where $T = P_{A^c} Q_B P_A$. Therefore, we have to bound from above the number of singular values of T larger than $\sqrt{\varepsilon(1-\varepsilon)}$.

Next, we want to partition the sets A and B into smaller sets and write T as a sum of operators

$$T = \sum_{k,l} P_{A^c} Q_{B_l} P_{A_k}.$$

In our initial argument we applied Weyl's inequality to this sum, following the approach in [18]. However, there the authors decomposed T as a sum of only 3 operators, so it was enough for them to divide ε by 3. In our setting the number of terms is of order c^d, so this approach gave sharp results only for $\varepsilon < \frac{1}{c^d}$. Instead of applying Weyl's inequality, we will use the p-Schatten quasi-norm approach from [14].

Let $K : H \to H$ be a compact operator from a Hilbert space H to itself. Let $\sigma_1(K) \geq \sigma_2(K) \geq \ldots \geq 0$ be the sequence of its singular values. For $0 < p < \infty$ we define its p-Schatten quasi-norm by

$$\|K\|_p^p = \sum_{k=1}^{\infty} \sigma_k(K)^p.$$

If K is not compact then we define $\|K\|_p = \infty$.

For $p = \frac{1}{2\log\frac{1}{\delta}}$ with $0 < \delta < \frac{1}{2}$ we have

$$\|K\|_p^p \geq \sum_{k:\sigma_k(K)\geq\delta} \sigma_k(K)^p \geq \frac{1}{\sqrt{e}}|\{k : \sigma_k(K) \geq \delta\}|, \tag{1.9}$$

where in the second step we used that $\delta^p = \frac{1}{\sqrt{e}}$. Thus, the number of singular values larger than δ is at most $\sqrt{e}\|K\|_p^p$. Note that for all $0 < \delta < \frac{1}{2}$ we have $0 < p < 1$. The key non-trivial fact [5, Theorem 11.5.9] about the p-Schatten quasi-norms for $0 < p < 1$ is that they satisfy the following subadditivity property

$$\|K_1 + K_2\|_p^p \leq \|K_1\|_p^p + \|K_2\|_p^p. \tag{1.10}$$

Thus, it suffices to estimate the p-Schatten quasi-norms of the operators $P_{A^c}Q_{B_l}P_{A_k}$ with $p = \frac{1}{\log(\frac{1}{\varepsilon(1-\varepsilon)})}$.

For clarity we will describe our decomposition only when $B = [0,1]^d$ and A is a set with boundary of finite upper Minkowski content. In this case we do not partition B. The partition (A_k) of A will arise from the Whitney decomposition of the interior of A into dyadic cubes. We keep all the cubes in the Whitney decomposition with side lengths larger than some threshold 2^{-D}, and cover the remaining part by sets contained in cubes with side length 2^{-D}. This gives (A_k). From the assumption that ∂A has finite upper Minkowski content we can estimate the number of cubes with side length 2^t by $C2^{-(d-1)t}$ for a constant C.

For a single term of the form $P_U Q_V P_W$ it is easy to see that if we enlarge U or W then the singular values can only increase. For the cube A_k from our partition we will enlarge A^c to the complement of $(2A_k)$ which contains A^c by the properties of the Whitney decomposition. In the case of cubes at the threshold we enlarge A^c to the whole $\mathbb{R}^d$ and also enlarge A_k to the corresponding cube with side length 2^{-D} containing it.

Splitting $(2A_k)^c$ additionally into $2^d - 1$ regions, we can reduce the analysis of $P_{(2A_k)^c}Q_BP_{A_k}$ and $Q_BP_{A_k}$ to the study of tensor products of the one-dimensional operators

$$I_r = Q_{[0,1]}P_{[0,r]} \quad \text{and} \quad J_r = P_{(-\infty,-2r]\cup[2r,+\infty)}Q_{[0,1]}P_{[-r,r]}.$$

Note that I_r is just the usual time-frequency localization operator, while J_r comes from the separation in the Whitney decomposition. For these operators we show the following estimates on the singular values.

Lemma 1.8. *There exist $\tau > 0$, $C > 0$, and $r_0 > 0$ such that for all $r > r_0$ and all $n \in \mathbb{N}$ we have*

$$\sigma_n(I_r) \le \begin{cases} 1, & n < 10r, \\ Ce^{-\tau n}, & n \ge 10r, \end{cases}$$

$$\sigma_n(J_r) \le Ce^{-\tau n}.$$

The estimate for $\sigma_n(I_r)$ follows from Theorem 2.5 stated below (and is also explicitly or implicitly contained in many previous publications, see e.g. [7, 24]), but we will present a self-contained complex-analytic proof inspired by [21].

The proof of the estimate for $\sigma_n(J_r)$ uses the precise structure of the kernel of J_r. Specifically, it is possible to decompose $J_r = \sum_{n=0}^{\infty} J_{r,n}$ into a sum of rank 2 operators $J_{r,n}$ with $\|J_{r,n}\| \lesssim 2^{-n}$, from which the result follows. Let us remark that our initial argument was more involved and proceeded through the low displacement-rank structure and Zolotorev numbers estimates employed by Karnik, Romberg and Davenport in [18].

The fact that the estimate for $\sigma_n(J_r)$ does not depend on r is the key new ingredient in our proof compared to the argument in [14]. It ultimately stems from our use of the separation condition in the Whitney decomposition which was not used in the previous works.

Doing the combinatorial argument similar to the proof of Theorem 1.1, combining all of our summands by means of (1.10) and carefully choosing the threshold D for the Whitney decomposition we get the desired estimate for $\varepsilon > \alpha^{-c}$. For $\varepsilon \le \alpha^{-c}$ we can simply compare with the case of two boxes, as this is exactly the regime when the boundary term becomes proportional to the volume term.

1.5. **Area laws.** We are interested in the traces $\mathrm{Tr} f(S_{A,B})$ for general functions $f : [0,1] \to \mathbb{C}$. For a compact self-adjoint operator $T : H \to H$ with eigenvalues $1 \ge \lambda_1 \ge \lambda_2 \ge \ldots \ge 0$ and corresponding normalized eigenvectors $v_1, v_2, \ldots$ we define $f(T)(v) := \sum_{n=1}^{\infty} f(\lambda_n)\langle v, v_n\rangle v_n$. This operator is trace class whenever $\mathrm{Tr} f(T) = \sum_{n=1}^{\infty} f(\lambda_n)$ converges absolutely. We will always assume that $f(0) = 0$ for convenience since all operators that we consider have infinite-dimensional kernels. First, we establish results guaranteeing that $f(T)$ is trace class.

For a function $f : [0,1] \to \mathbb{C}$ we let $M_0 f(t) = \sup_{0\le x\le t} |f(x)|$ and $M_1 f(t) = \sup_{1-t\le x\le 1} |f(1) - f(x)|$.

Definition 3. *Consider a function $f : [0,1] \to \mathbb{C}$. We call f trace class admissible for $L^2(\mathbb{R}^d)$ if there is $\delta > 0$ such that*

$$\int_0^\delta \frac{M_0 f(\varepsilon) \log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon \left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d} \, d\varepsilon < \infty. \tag{1.11}$$

Note that for any trace class admissible for $L^2(\mathbb{R}^d)$ function f we must necessarily have $\lim_{x\to 0^+} f(x) = f(0) = 0$.

Theorem 1.9. *If $A, B \subseteq \mathbb{R}^d$ are bounded sets and f is trace class admissible for $L^2(\mathbb{R}^d)$, then $f(S_{A,B})$ is trace class.*

Theorem 1.10. *Let $A, B \subseteq \mathbb{R}^d$ be such that $S_{A,B}$ is compact and such that their interiors are non-empty. If $f : [0,1] \to \mathbb{C}$ is such that $|f(x)|$ is non-decreasing near 0 and $f(S_{A,B})$ is trace class then f is trace class admissible for $L^2(\mathbb{R}^d)$.*

We now turn to our main application of the uniform bounds from Theorem 1.1 and Theorem 1.3, namely to compute a two-term asymptotic formula for $\operatorname{Tr} f(S_{cA,B})$ in the limit $c \to \infty$ for extremely general functions f. As an input we use the corresponding formula for polynomials f. For $d = 1$ this is supplied by the work of Landau and Widom [22]. The higher dimensional case is considerably more difficult. The formula was conjectured by Widom [29] and much later established by Sobolev [26] for regular enough sets A and B. For the context of the present paper, we rely on the sharp generalization of Sobolev's result from the recent work [9] by Fournais, Seiringer, Solovej, and the second author. It was established that, for polynomials f and sets A, B with finite measure and finite perimeter,

$$\begin{aligned}&\operatorname{Tr} f(S_{cA,B}) \\ &\quad = c^d |A||B| f(1) + c^{d-1}\log(c) I(A,B) \int_0^1 \frac{f(\theta) - f(1)\theta}{\theta(1-\theta)} \, d\theta + o(c^{d-1}\log(c)),\end{aligned} \tag{1.12}$$

as $c \to \infty$, for a certain boundary coefficient $I(A,B)$. If A and B are sets with C^1 boundaries, then

$$I(A,B) = \frac{1}{4\pi^2} \int_{\partial A} \int_{\partial B} |\nu_A(x) \cdot \nu_B(p)| \, d\mathcal{H}^{d-1}(p) \, d\mathcal{H}^{d-1}(x),$$

where ν_A and ν_B denote the interior normals to A and B, respectively, and $d\mathcal{H}^{d-1}$ is the $d-1$-dimensional Hausdorff measure. Tools from geometric measure theory are necessary to describe the coefficient in the finite perimeter setting. We refer to [9] for the details.

For our work it is important to note that if the boundary of a set has finite upper Minkowksi content then this set has finite perimeter, thus (1.12) applies in our case. This is so because any set of finite upper Minkowski content has finite $d-1$-dimensional Hausdorff measure (see e.g. Lemma 4.5 below) and the fact that if the boundary of a set has finite $d-1$-dimensional Hausdorff measure then it has finite perimeter [1, Proposition 3.62]. Note that even in this case the factor $I(A,B)$ can only be defined with the help of geometric measure theory.

Since Sobolev's original result, significant effort has been put to extend the two-term asymptotic formula (1.12) from polynomials to rougher spectral functions [27, 28]. This is in part motivated from physics. Here it was realized by Gioev and Klich [11] that the trace $\mathrm{Tr} f(S_{cA,B})$, with $f(\theta) = -\theta\log(\theta) - (1-\theta)\log(1-\theta)$ or related Rényi entropy functions, appears directly in the expression for the bipartite entanglement entropy for free Fermionic systems in the ground state. In this context, the domain B represents the Fermi sea of the Fermionic operator and A represents the spatial subsystem. Since the entropy functions satisfy $f(1) = 0$, the formula (1.12) expresses that $\mathrm{Tr}\, f(S_{cA,B}) \sim c^{d-1}\log(c)$, which is referred to as an enhanced area law in the physics literature. See [8] for relevant background.

Previously, Sobolev carried out the extension argument relying on uniform Schatten quasi-norm estimates of the involved operators [28], which allowed f with Hölder-type singularities. With the much more precise spectral bounds from Theorem 1.1 and Theorem 1.3, we are able to push this much further, in particular obtaining essentially necessary and sufficient conditions on the function f.

Definition 4. *Consider a function $f : [0,1] \to \mathbb{C}$. We call f area law admissible if*

$$\int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon}\, d\varepsilon < \infty. \tag{1.13}$$

Theorem 1.11. *Let $A \subseteq \mathbb{R}^d$ be a bounded set whose boundary has finite upper Minkowski content and let B be a finite union of parallelepipeds with disjoint interiors. Assume that f is both area law admissible and trace class admissible for $L^2(\mathbb{R}^d)$. There exist $c_0(f,A,B)$ and $C(A,B)$ such that for $c > c_0(f,A,B)$ we have*

$$|\,\mathrm{Tr}\, f(S_{cA,B}) - c^d|A||B|f(1)| \le C(A,B)c^{d-1}\log(c) \int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon}\, d\varepsilon. \tag{1.14}$$

If in addition f is Riemann integrable on $[\varepsilon, 1-\varepsilon]$ for all $0 < \varepsilon < \frac{1}{2}$ then we also have

$$\begin{aligned} &\operatorname{Tr} f(S_{cA,B}) \\ &\quad = c^d|A||B|f(1) + c^{d-1}\log(c)I(A,B)\int_0^1 \frac{f(\theta)-f(1)\theta}{\theta(1-\theta)}\,d\theta + o(c^{d-1}\log(c)), \end{aligned} \tag{1.15}$$

as $c \to \infty$.

Remark 1.12. *It is crucial for us that $C(A,B)$ does not depend on f. On the other hand, $c_0(f,A,B)$ can and will depend on f, specifically on the integral in the definition of trace class admissibility for $L^2(\mathbb{R}^d)$.*

For $d = 1$ every f which is area law admissible is automatically trace class admissible for $L^2(\mathbb{R}^d)$. On the other hand, for every $d \geq 2$ there exists a function which is area law admissible but which is not trace class admissible for $L^2(\mathbb{R}^d)$ and which is monotone near 0. Thus, by Theorem 1.10 for any sets $A, B \subset \mathbb{R}^d$ with non-empty interiors we have $\operatorname{Tr} f(S_{A,B}) = \infty$ even though f is area law admissible.

Example 1. *Let $f(\theta) = \frac{1}{\log(\frac{2}{\theta})^{3/2}}$. Then f is monotone increasing on $[0,1]$, it is area law admissible but it is not trace class admissible for $L^2(\mathbb{R}^d)$ for any $d \geq 2$.*

For general sets A, B our argument gives an upper bound with one extra logarithm.

Theorem 1.13. *Let $A, B \subseteq \mathbb{R}^d$ be bounded sets whose boundaries have finite upper Minkowski content. Assume that f is both area law admissible and trace class admissible for $L^2(\mathbb{R}^d)$. There exist $c_0(f,A,B)$ and $C(A,B)$ such that for $c > c_0(f,A,B)$ we have*

$$|\operatorname{Tr} f(S_{cA,B}) - c^d|A||B|f(1)| \leq C(A,B)c^{d-1}\log(c)^2 \int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon}\,d\varepsilon.$$

1.6. **Lower order terms for $\operatorname{Tr} S^2$.** If both A and B are finite unions of axis-parallel boxes then we are able to go further in the expansion in the simplest non-trivial case $f(\theta) = \theta^2$. We carry out this computation in part out of mathematical curiosity, but also because there seems to be some interest from a physics point of view for further terms in the expansion of $\operatorname{Tr} f(S_{cA,B})$, specifically for f being the entropy function. For instance, Kitaev and Preskill argued in [19] from a physical understanding of the entanglement entropy that the constant order term for $d = 2$ in the expansion of $\operatorname{Tr} f(S_{cA,B})$ should carry topological information. While it is not our intention to place their claims on a rigorous mathematical footing, nor to verify them, we still see it as an interesting mathematical problem to obtain further terms for general functions f.

Theorem 1.14. *Let* $A, B \subseteq \mathbb{R}^d$ *be finite unions of axis-parallel boxes. The trace of* $S^2_{A,B}$ *can be computed in terms of the side lengths of boxes constituting* A *and* B *by means of four standard arithmetic operations, exponentiation, taking logarithms and the exponential integral function* $\mathrm{E}_1(z) = \int_z^\infty \frac{e^{-t}}{t}$.

Unfortunately, if A and B consist of respectively n and m boxes then our procedure gives us of order $9^d n^2 m^2$ terms, making it fairly infeasible to do in practice. So, we will do an explicit computation only in the case $A = [0, c]$, $B = [0, 1]$.

Theorem 1.15. *The following expansion holds:*

$$\begin{aligned}
&\operatorname{Tr} S^2_{[0,c],[0,1]} \\
&\quad = c - \frac{\log(c)}{\pi^2} - \frac{1 + \gamma + \log(2\pi)}{\pi^2} + \left(\frac{2}{\pi} c \left(\mathrm{Si}(2\pi c) - \frac{\pi}{2} \right) + \frac{\cos(2\pi c)}{\pi^2} + \frac{\mathrm{Ci}(2\pi c)}{\pi^2} \right) \\
&\quad = c - \frac{\log(c)}{\pi^2} - \frac{1 + \gamma + \log(2\pi)}{\pi^2} \\
&\qquad - \frac{\cos(2\pi c)}{\pi^2} \sum_{n=1}^{\infty} (-1)^n \frac{(2n)! - (2n-1)!}{(2\pi c)^{2n}} - \frac{\sin(2\pi c)}{\pi^2} \sum_{n=1}^{\infty} (-1)^n \frac{(2n+1)! - (2n)!}{(2\pi c)^{2n+1}},
\end{aligned}$$

where $\mathrm{Si(t)} = \int_0^t \frac{\sin(x)}{x} dx$ *is the sine integral,* $\mathrm{Ci(t)} = -\int_t^\infty \frac{\cos(x)}{x} dx$ *is the cosine integral and* γ *is the Euler–Mascheroni constant. The series is asymptotic, meaning that if we take the first* N *terms in both sums then the error will be* $O\left(\frac{1}{c^{2N+2}}\right)$.

Note that the oscillations in $\operatorname{Tr} S^2_{[0,c],[0,1]}$ start appearing only at $O(\frac{1}{c^2})$ term. Our argument shows that they can potentially appear already in $O(1)$ term, but it turned out that both $O(1)$ and $O(\frac{1}{c})$ oscillating terms cancelled out. We do not have an a priori explanation for this phenomenon.

1.7. **One-term asymptotics.** If we are only interested in the first term in the expansion (1.15) then the conditions on A, B and f can be greatly relaxed, in particular we do not need to assume anything about the geometries of ∂A and ∂B.

Since for any A, B of finite measure $\mathrm{Tr} S_{A,B} = |A||B|$ is finite, for any function $f : [0,1] \to \mathbb{C}$ such that $|f(\theta)| \le C\theta, \theta \in [0,1]$ we have that $f(S_{A,B})$ is trace class. If in addition we assume that f is continuous at 1 then we can establish a one-term asymptotic formula for $\mathrm{Tr} f(S_{cA,B})$.

Theorem 1.16. *Let* $A, B \subseteq \mathbb{R}^d$ *be sets with finite measure. If* $f : [0,1] \to \mathbb{C}$ *is continuous at* 1 *and satisfies* $|f(\theta)| \le C\theta$ *for some* $C > 0$*, then*

$$\operatorname{Tr} f(S_{cA,B}) = c^d |A||B| f(1) + o(c^d), \tag{1.16}$$

as $c \to \infty$*.*

Although the condition $|f(\theta)| \leq C\theta$ might seem restrictive, in this generality it is actually the optimal one.

Proposition 1.17. *Given a function* f *such that* $\lim_{x\to 0^+} \frac{|f(x)|}{x} = \infty$ *and a set* $B \subseteq \mathbb{R}$ *with finite and positive measure, there exists a set* $A \subseteq \mathbb{R}$ *of finite measure such that* $f(S_{A,B})$ *is not trace class.*

In particular, this applies to the entropy function $f(\theta) = -\theta\log(\theta) - (1-\theta)\log(1-\theta)$.

On the other hand, if the sets A and B are bounded then we can replace the assumption $|f(\theta)| \leq C\theta$ with the assumption of f being bounded and trace class admissible for $L^2(\mathbb{R}^d)$ as in Theorem 1.11.

Theorem 1.18. *Let* $A, B \subseteq \mathbb{R}^d$ *be bounded sets and assume that* f *is bounded, trace class admissible for* $L^2(\mathbb{R}^d)$ *and that* f *is continuous at* 1*. Then*

$$\operatorname{Tr} f(S_{cA,B}) = c^d|A||B|f(1) + o(c^d),$$

as $c \to \infty$*.*

2. Previous results on one-dimensional operators

In this section we will collect previous results on the distribution of eigenvalues of $S_{A,B}$ in the simplest case when A and B are one-dimensional intervals and restate them in the form that will be convenient for our use. By rescaling we can always assume that $A = [0, c], B = [0, 1]$ so that we have a sequence $(\lambda_n(c))$ of eigenvalues of $S_{A,B}$. The first result about their behaviour was found by Slepian [25] and rigorously proved by Landau and Widom [22].

Theorem 2.1. *For all bounded intervals* $A, B \subseteq \mathbb{R}$ *and all fixed* $0 < a < 1$ *we have*

$$N_a(A, B) = |A||B| + \tfrac{1}{\pi^2}\log\left(\tfrac{1-a}{a}\right)\log(|A||B|) + o(\log(|A||B|)),$$

as $|A||B| \to \infty$*.*

Note that this result is slightly different from the statement in [22] as we use a different normalization of the Fourier transform. This also applies to some of the following results.

Their proof proceeded by first establishing two-term asymptotics for $\operatorname{Tr} S_{A,B}^n$ for all powers $n \in \mathbb{N}$ (even with $O(1)$ error term) and then approximating characteristic function of an interval by polynomials. However, since their argument had extremely poor uniformity in n and thus extremely poor uniformity in a, it is virtually impossible to use their estimates to bound $N_a(A,B)$ and $\Lambda_a(A,B)$ for varying a. So, there was a lot of research on uniform estimates for these numbers when a is varying [6, 16, 20]. One of the best results, which is at the same time completely uniform, sharp and explicit, was recently obtained by Karnik, Romberg and Davenport [18].

Theorem 2.2. *For all* $0 < \varepsilon < 1/2$ *and all bounded intervals* A *and* B *we have*

$$\Lambda_\varepsilon(A,B) \le \frac{2}{\pi^2}\log(50|A||B|+25)\log\left(\frac{5}{\varepsilon(1-\varepsilon)}\right)+7.$$

Note that the dependence on c and on $\varepsilon(1-\varepsilon)$ in this result is logarithmic just like in Theorem 2.1, while being completely explicit in all constants. In particular, combining Theorem 2.2 with Theorem 2.1 is already enough to establish our version of the enhanced area law for the two-term asymptotics of $\mathrm{Tr} f(S_{A,B})$

$$\mathrm{Tr} f(S_{A,B}) = f(1)|A||B| + \frac{1}{\pi^2}\log(|A||B|)\int_0^1 \frac{f(\theta)-\theta f(1)}{\theta(1-\theta)}\,d\theta + o(\log(|A||B|))$$

for extremely general functions f, see Theorem 1.11 for the precise statement.

This result is essentially sharp for a very wide range of values of ε. However, for very small values of ε (almost exponentially small in $|A||B|$) it is possible to get a better bound. For the eigenvalues $\lambda_n(c)$ with $n < c$ this was done by the first author [21].

Theorem 2.3. *There exist numbers* $c_0, B, \eta, \mu > 0$ *such that for* $c > c_0$ *and* $n < c - B\log^2(c)$ *we have*

$$\exp\left(-\mu\frac{c-n}{\log(\frac{2c}{c-n})}\right) < 1-\lambda_n(c) < \exp\left(-\eta\frac{c-n}{\log(\frac{2c}{c-n})}\right). \tag{2.1}$$

For $n > c$ not only is it possible to obtain better estimates, but even an asymptotic formula for $\lambda_n(c)$ with a tiny relative error was obtained by Bonami and Karoui [6].

Theorem 2.4. *For all* $n \ge c \ge 10$ *we have*

$$\lambda_n(c) = \exp\left(-\frac{\pi^2\left(n+\frac{1}{2}\right)}{2}\int_{\Phi\left(\frac{c}{n+\frac{1}{2}}\right)}^1 \frac{1}{tE(t)^2}dt + O(\log(n))\right),$$

where $E(t) = \int_0^1 \sqrt{\frac{1-t^2x^2}{1-x^2}}dx$ *is the elliptic integral of the second kind and* Φ *is the inverse of the function* $t \to \frac{t}{E(t)}$.

Using known asymptotics of the elliptic integral near 0 and 1 we can restate it in the following (much cruder) form which has an advantage of not involving any special functions.

Theorem 2.5. *There exist numbers* $c_0, B, \eta, \mu > 0$ *such that for* $c > c_0$*,* $2c > n > c + B\log^2(c)$ *we have*

$$\exp\left(-\mu\frac{n-c}{\log(\frac{2c}{n-c})}\right) < \lambda_n(c) < \exp\left(-\eta\frac{n-c}{\log(\frac{2c}{n-c})}\right)$$

while for $n \geq 2c$ *we have*

$$\exp\left(-\mu n\log\left(\frac{n}{c}\right)\right) < \lambda_n(c) < \exp\left(-\eta n\log\left(\frac{n}{c}\right)\right).$$

To actually apply these estimates, we want to reformulate them in terms of counting functions as well. Note that for the lower bound on $N_{1-\varepsilon}([0,1],[0,c])$ and the upper bound on $N_\varepsilon([0,1],[0,c])$ we can combine Theorem 2.3 and Theorem 2.5, respectively, with Theorem 2.2 and Theorem 2.1 to get estimates valid for all $\varepsilon < \frac{1}{2}$. For the remaining bounds we do not have an analogue of Theorem 2.2 at our disposal, so they will only work for $\varepsilon < c^{-A}$ for a large enough constant A. Specifically, we have the following six assertions. Let $\kappa, c_1 > 0$ be large enough constants and $c_2 > 0$ be a small enough constant, independent of c and ε, and we assume that $c > c_1$.

Whenever $2^{-\kappa c} < \varepsilon < \frac{1}{2}$, we have

$$N_{1-\varepsilon}([0,c],[0,1]) \geq c - c_1\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log(\frac{1}{\varepsilon})}\right) \tag{2.2}$$

and

$$N_{\varepsilon}([0,c],[0,1]) \leq c + c_1\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log(\frac{1}{\varepsilon})}\right). \tag{2.3}$$

For $2^{-\kappa c} < \varepsilon < c^{-A}$ we have complementary bounds as well

$$N_{1-\varepsilon}([0,c],[0,1]) \leq c - c_2\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log(\frac{1}{\varepsilon})}\right) \tag{2.4}$$

and

$$N_{\varepsilon}([0,c],[0,1]) \geq c + c_2\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log(\frac{1}{\varepsilon})}\right). \tag{2.5}$$

For $\varepsilon \leq 2^{-\kappa c}$ we have

$$N_{1-\varepsilon}([0,c],[0,1]) = 0$$

and

$$N_{\varepsilon}([0,c],[0,1]) \asymp \frac{\log\left(\frac{1}{\varepsilon}\right)}{\log\left(\frac{4\log(\frac{1}{\varepsilon})}{c}\right)}.$$

Let us, as an illustration, deduce (2.2) from Theorems 2.1, 2.2 and 2.3, and leave the other five cases to the interested reader.

For $\frac{1}{100} \leq \varepsilon < \frac{1}{2}$ we have $N_{1-\varepsilon}([0,c],[0,1]) \geq N_{99/100}([0,c],[0,1]) = c + O(\log c)$ by Theorem 2.1. Next, we notice that if $n_c = [c]$, where $[x]$ is the largest integer not greater than x, and c is large enough then $\frac{1}{3} \leq \lambda_{n_c}(c) \leq \frac{2}{3}$. Indeed, by Theorem 2.1 we have $N_{1/3}([0,c],[0,1]) = c + \frac{\log(2)}{\pi^2}\log(c) + o(\log(c))$ and $N_{2/3}([0,c],[0,1]) = c - \frac{\log(2)}{\pi^2}\log(c) + o(\log(c))$, thus n_c does not satisfy $\lambda_n(c) > \frac{2}{3}$ but does satisfy $\lambda_n(c) \geq \frac{1}{3}$.

If $c^{-A} \leq \varepsilon < \frac{1}{100}$ then n_c is in the set $\{n : \varepsilon < \lambda_n(c) \leq 1-\varepsilon\}$. Note also that this set is clearly a segment of integers. We therefore have

$$N_{1-\varepsilon}([0,1],[0,c]) \geq n_c - |\{n : \varepsilon < \lambda_n(c) < 1-\varepsilon\}| = n_c - \Lambda_\varepsilon([0,1],[0,c]).$$

Applying Theorem 2.2 and $n_c \geq c$ for large enough c we get

$$N_{1-\varepsilon}([0,1],[0,c]) \geq c - 100\log(c)\log\left(\frac{1}{\varepsilon}\right).$$

If $c^{-A} \leq \varepsilon$ and c is large enough then $\log\left(\frac{\kappa c}{\log(\frac{1}{\varepsilon})}\right) \geq \frac{1}{2}\log(c)$, so we have to at most double the constant. In fact, in [21] in the proof of Theorem 2.3 the same argument was used for a much wider range of ε, up to $\varepsilon = \exp(-c^{7/8})$.

If $0 < \varepsilon < c^{-A}$ then first of all $c_1 \log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log(\frac{1}{\varepsilon})}\right) \geq B\log^2(c)$. Thus, it suffices to show that for $n = \left[c - c_1\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log(\frac{1}{\varepsilon})}\right)\right] + 1$ the right-hand side in (2.1) is at most ε (if $n \leq 0$ then the required estimate holds automatically). For large enough c_1 we clearly have $n < c - \frac{c_1}{2}\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log(\frac{1}{\varepsilon})}\right)$. So, it is enough to show that for large enough c_1 we have

$$\log\left(\frac{1}{\varepsilon}\right) \leq \eta\frac{c_1\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log(\frac{1}{\varepsilon})}\right)}{2\log\left(\frac{2c}{c_1\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log(\frac{1}{\varepsilon})}\right)}\right)}.$$

Dividing by $\log\left(\frac{1}{\varepsilon}\right)$ and putting $t = \frac{\kappa c}{\log\left(\frac{1}{\varepsilon}\right)}$ we get an inequality

$$1 \leq \frac{\eta c_1}{2}\frac{\log(t)}{\log(t) - \log\log(t) - \log(\frac{c_1}{2})},$$

which is always true if $c_1 > \frac{2}{\eta} + 2$. For (2.4) we would instead need to show that similar expression with μ and c_2 is at most 1, and for this we would use that $\log\log(t) \leq \frac{\log(t)}{2}$, say.

3. Case of two boxes

In this section we will prove Theorem 1.1. The proof of Proposition 1.2 is similar so we leave its proof to the interested reader.

We begin with the following simple observation that will nevertheless be sufficient for the proof:

$$N_{a^{1/d}}([0,c],[0,1])^d \le N_a([0,c]^d,[0,1]^d) \le N_a([0,c],[0,1])^d. \tag{3.1}$$

Indeed, if the product of d numbers, each of which is between 0 and 1, is larger than a then each of them is also larger than a, which gives us an upper bound. On the other hand, if each of the numbers is larger than $a^{1/d}$ then their product is larger than a.

So, we can apply the estimates for $N_a([0,c],[0,1])$ from the previous section. Note that for $2^{-\kappa c} < \varepsilon < \frac{1}{2}$ we always have $\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log(\frac{1}{\varepsilon})}\right) = O(c)$. When raising to the d'th power, we will use the following simple inequality: if $y = O(x)$ then $(x+y)^d = x^d + O(x^{d-1}y)$. We will only use it with $x = c$.

We begin with estimating $N_{1/2}([0,c]^d,[0,1])$ from (3.1). We get

$$N_{2^{-1/d}}([0,c],[0,1])^d \le N_{1/2}([0,c]^d,[0,1]^d) \le N_{1/2}([0,c],[0,1])^d.$$

By Theorem 2.1 both $N_{1/2}([0,c],[0,1])$ and $N_{2^{-1/d}}([0,c],[0,1])$ are $c + O(\log c)$. Raising this to the power d we get

$$N_{1/2}([0,c]^d,[0,1]^d) = c^d + O(c^{d-1}\log c). \tag{3.2}$$

First, we are going to prove (1.4). For this we will use a lower bound on $N_{1-\varepsilon}([0,c],[0,1])$ and an upper bound on $N_\varepsilon([0,c],[0,1])$, respectively. For $\Lambda^+_\varepsilon([0,c]^d,[0,1]^d)$ we get

$$\begin{aligned}\Lambda^+_\varepsilon([0,c]^d,[0,1]^d) &= N_{1/2}([0,c]^d,[0,1]^d) - N_{1-\varepsilon}([0,c]^d,[0,1]^d)\\ &\le N_{1/2}([0,c]^d,[0,1]^d) - N_{(1-\varepsilon)^{1/d}}([0,c],[0,1])^d.\end{aligned}$$

By Bernoulli's inequality we know that $(1-\varepsilon)^{1/d} \le 1 - \frac{\varepsilon}{d}$ and therefore we have $N_{(1-\varepsilon)^{1/d}}([0,c],[0,1]) \ge N_{1-\varepsilon/d}([0,c],[0,1])$. Plugging in our bound we get for $2^{-\kappa c} < \varepsilon < \frac{1}{2}$

$$\Lambda^+_\varepsilon([0,c]^d,[0,1]^d) \le c^d + O(c^{d-1}\log c) - c^d + O\left(c^{d-1}\log\left(\frac{d}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log(\frac{d}{\varepsilon})}\right)\right).$$

Canceling c^d, noting that the first big-O is dominated by the second and the fact that for $\varepsilon < \frac{1}{2}$ we have $\log\left(\frac{d}{\varepsilon}\right) \asymp \log\left(\frac{1}{\varepsilon}\right)$ we get the desired upper bound after possibly increasing κ to α_d.

For $\Lambda_\varepsilon^-([0,c]^d,[0,1]^d)$ we get by a similar reasoning for $2^{-\kappa c} < \varepsilon < \frac{1}{2}$

$$\Lambda_\varepsilon^-([0,c]^d,[0,1]^d) \le c^d + O\left(c^{d-1}\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\kappa c}{\log\left(\frac{1}{\varepsilon}\right)}\right)\right) - c^d + O(c^{d-1}\log c).$$

Again, canceling c^d and noting that the first big-O dominates the second one we get the desired result.

To prove (1.5) we will instead employ an upper bound on $N_{1-\varepsilon}([0,c],[0,1])$ and a lower bound on $N_\varepsilon([0,c],[0,1])$. For this, just an inequality $(x+y)^d = x^d + O(x^{d-1}y)$ will not be enough, we will need that if $0 \le y \le \frac{1}{2}x$ then $(x-y)^d \le x^d - c_d x^{d-1}y$ and $(x+y)^d \ge x^d + c_d x^{d-1}y$. Note that we can achieve $y \le \frac{1}{2}x$ by making the constant c_2 in (2.4) and (2.5) smaller if necessary. The rest of the argument is essentially the same as in the proof of (1.4), the only changes are that we have to write inequalities explicitly instead of big-O and that in the case of $\Lambda_\varepsilon^-([0,c]^d,[0,1])$ we have $\log\left(\frac{1}{\varepsilon^{1/d}}\right)$ instead of $\log\left(\frac{1}{\varepsilon}\right)$, but since they are proportional to each other we get the same result in the end.

It remains to cover the case $\varepsilon \le 2^{-\kappa c}$. First of all, there are no eigenvalues larger than $1-\varepsilon$ because the right-hand side of (3.1) is zero. For $\Lambda_\varepsilon^-([0,c]^d,[0,1]^d)$ we get

$$\Lambda_\varepsilon^-([0,c]^d,[0,1]^d) = N_\varepsilon([0,c]^d,[0,1]^d) - N_{1/2}([0,c]^d,[0,1]^d).$$

For the upper bound we simply discard the second term and bound

$$\Lambda_\varepsilon^-([0,c]^d,[0,1]^d) \lesssim \left(\frac{\log\left(\frac{1}{\varepsilon}\right)}{\log\left(\frac{4\log\left(\frac{1}{\varepsilon}\right)}{c}\right)}\right)^d.$$

For the lower bound, first we pick some $\gamma_d > 0$ and if $\varepsilon > \gamma_d^{-c}$ then we can use a lower bound for $\varepsilon = 2^{-\kappa c}$ from the previous case (increasing κ if needed) which is proportional to the lower bound in the current case with the obvious monotonicity of $\Lambda_\varepsilon^-(A,B)$ in ε.

Finally, in the case $\varepsilon \le \gamma_d^{-c}$ we first crudely write

$$\Lambda_\varepsilon^-([0,c]^d,[0,1]^d) = N_\varepsilon([0,c]^d,[0,1]^d) - N_{1/2}([0,c]^d,[0,1]^d) \ge N_\varepsilon([0,c]^d,[0,1]^d) - 2c^d$$

and estimate the first term with (3.1) as

$$N_\varepsilon([0,c]^d,[0,1]^d) \ge \left(\frac{\delta\log\left(\frac{1}{\varepsilon^{1/d}}\right)}{\log\left(\frac{4\log\left(\frac{1}{\varepsilon^{1/d}}\right)}{c}\right)}\right)^d$$

for some $\delta > 0$. If $\varepsilon \le \gamma_d^{-c}$ for small enough γ_d then this is proportional to the desired lower bound and is also at least $4c^d$ so we can cancel $-2c^d$ that we had.

4. General case

In this section we will prove Theorem 1.3 and Theorem 1.6. We begin with the first one as it still contains most of our techniques while being slightly easier. So, let $A \subseteq \mathbb{R}^d$ be a bounded set with positive measure and boundary of finite upper Minkowski content and $B = \cup_n B_n$ be a finite union of parallelepipeds with disjoint interiors. We will also only consider the case $\varepsilon > \alpha^{-c}$ for now and cover the other case at the end of the section.

We start with the $S - S^2$ trick. We have

$$S_{cA,B} - S_{cA,B}^2 = P_{cA} Q_B P_{cA^c} Q_B P_{cA} = T^* T,$$

where $T = P_{cA^c} Q_B P_{cA}$. We are interested in the number of eigenvalues of $S_{cA,B} - S_{cA,B}^2$ larger than $\varepsilon(1-\varepsilon)$, that is the number of singular values of T larger than $\sqrt{\varepsilon(1-\varepsilon)}$.

Since ∂B_n has measure 0, we clearly have $Q_B = \sum_{n=1}^N Q_{B_n}$, and therefore

$$P_{cA^c} Q_B P_{cA} = \sum_{n=1}^N P_{cA^c} Q_{B_n} P_{cA}.$$

Let $p = \frac{1}{\log\left(\frac{1}{\varepsilon(1-\varepsilon)}\right)} < 1$. By (1.9) and (1.10) we have

$$\Lambda_\varepsilon(cA, B) \leq \sqrt{e} \|P_{cA^c} Q_B P_{cA}\|_p^p \leq \sqrt{e} \sum_{n=1}^N \|P_{cA^c} Q_{B_n} P_{cA}\|_p^p. \tag{4.1}$$

Using the $S - S^2$ trick in reverse we can rewrite these quasi-norms in terms of the eigenvalues $\lambda_m(cA, B_n)$ as

$$\|P_{cA^c} Q_{B_n} P_{cA}\|_p^p = \sum_m (\lambda_m(cA, B_n)(1 - \lambda_m(cA, B_n)))^{\frac{p}{2}}.$$

We have $B_n = M_n[0,1]^d + b_n$ for some invertible matrix M_n and $b_n \in \mathbb{R}^d$. To reduce to the case $[0,1]^d$ we will use the following lemma.

Lemma 4.1. *Let $A, B \subseteq \mathbb{R}^d$ be sets of finite measure and $M : \mathbb{R}^d \to \mathbb{R}^d$ be an invertible linear map. Then, for all $u, v \in \mathbb{R}^d$ and all $n \in \mathbb{N}$,*

$$\lambda_n(A, B) = \lambda_n(MA + v, M^{-T} B + u).$$

Proof. We start with the case $u = v = 0$. Consider the unitary dilation operator $D_M f(x) = |\det(M)|^{\frac{1}{2}} f(Mx)$ with inverse $D_M^{-1} = D_{M^{-1}}$. We are going to compute $D_M^{-1} S_{A,B} D_M$. To this end, it is sufficient to note that, for any invertible matrix M and all measurable sets X,

$$\mathcal{F} D_M = D_{M^{-T}} \mathcal{F}, \quad \mathcal{F}^{-1} D_{M^{-T}} = D_M \mathcal{F}^{-1}, \quad P_X D_M = D_M P_{MX}$$

It follows that

$$D_M^{-1} S_{A,B} D_M = P_{MA} Q_{M^{-T}B} P_{MA}$$

and therefore $\lambda_n(A, B) = \lambda_n(MA, M^{-T}B)$ by unitary equivalence. Next, we handle general translations u, v. Consider the unitary translation operator $T_v f(x) = f(x - v)$ and the phase shift multiplication operator $W_u f(p) = e^{-2\pi i u \cdot p} f(p)$ with inverses T_{-v} and W_{-u}. As before, the following commutation relations hold:

$$\mathcal{F} T_v = W_v \mathcal{F}, \quad \mathcal{F}^{-1} W_v = T_v \mathcal{F}^{-1}, \quad P_X T_v = T_v P_{X-v}, \quad P_X W_u = W_u P_X.$$

We conclude that $W_u T_v S_{A,B} T_v^{-1} W_u^{-1} = S_{A+v,B+u}$ and therefore that $\lambda_n(A, B) = \lambda_n(A + v, B + u)$.

□

Applying Lemma 4.1 to A, B_n we get $\lambda_m(cA, B_n) = \lambda_m(cM_n^{-T}A, [0,1]^d)$. Observe that $M_n^{-T}A$ is still a bounded set whose boundary has finite upper Minkowski content. Indeed, simply note that for any $r > 0$

$$|M^{-T}A + B_r| \leq \frac{1}{|\det(M)|} \left| A + B_{r\|M^{-T}\|^{-1}} \right|,$$

where + stands for the Minkowski sum of two sets. So, from now on to simplify the notation we will assume that $B = [0,1]^d$ and A is some bounded set whose boundary has finite upper Minkowski content and we want to bound $\|P_{cA^c} Q_{[0,1]^d} P_{cA}\|_p^p$.

Next, we introduce the Whitney decomposition of A. To do this we recall that ∂A has measure 0, hence $P_A = P_{\text{int}A}$ and so we can assume that A is open. Note also that $\partial(\text{int}A) \subseteq \partial A$, hence $\partial(\text{int}A)$ also has finite upper Minkowski content. The following is taken from [13, Appendix J]

Proposition 4.2. *Let $A \subsetneq \mathbb{R}^d$ be an open subset. There exists a family (Q_k) of closed dyadic cubes such that*

(i) *$\bigcup_k Q_k = A$ and the Q_k's have disjoint interiors,*

(ii) *There are constants c_1, c_2 only dependent on d such that*

$$c_1 \operatorname{diam}(Q_k) \leq \operatorname{dist}(Q_k, A^c) \leq c_2 \operatorname{diam}(Q_k).$$

For the future use, we will need to modify this decomposition slightly to make constant c_1 as large as we like.

Proposition 4.3. *Let $A \subsetneq \mathbb{R}^d$ be an open subset. There exists a family (Q_k) of closed dyadic cubes such that*

(i) *$\bigcup_k Q_k = A$ and the Q_k's have disjoint interiors,*

(ii) *There is a constant* c_3 *only dependent on* d *such that*

$$2\,\mathrm{diam}(Q_k) \le \mathrm{dist}(Q_k, A^c) \le c_3\,\mathrm{diam}(Q_k).$$

Proof. Let Q_k be a sequence of cubes from Proposition 4.2 and cut each of them, for some $m \in \mathbb{N}$, into 2^{md} dyadic subcubes $Q_{k,1}, \dots, Q_{k,2^{md}}$ of diameter $2^{-m}\mathrm{diam}(Q_k)$. We have

$$\mathrm{dist}(Q_{k,n}, A^c) \le \mathrm{dist}(Q_k, A^c) + \mathrm{diam}(Q_k) \le (c_2+1)\mathrm{diam}(Q_k) = (c_2+1)2^m\mathrm{diam}(Q_{k,n}),$$

so $c_3 = (c_2+1)2^m$ works. On the other hand,

$$\mathrm{dist}(Q_{k,n}, A^c) \ge \mathrm{dist}(Q_k, A^c) \ge c_1\mathrm{diam}(Q_k) = c_1 2^m\mathrm{diam}(Q_{k,n}),$$

so if we choose m so that $c_1 2^m > 2$ we will get the desired result for the cubes $Q_{k,n}$. ☐

By A_l we denote the collection of cubes in the Whitney decomposition of A with side length 2^{-l}. Since we assume that A is bounded, there is $l_0 \in \mathbb{Z}$ such that $A_l = \varnothing$ for $l < l_0$. We will need the following two lemmas about open, bounded sets A whose boundary has finite upper Minkowski content.

Lemma 4.4. *Let* A *be an open bounded set with boundary of finite upper Minkowski content. There exists a constant* C *depending only on* A *such that*

$$|A_l| \le C2^{(d-1)l}.$$

Lemma 4.5. *Let* A *be an open bounded set with boundary of finite upper Minkowski content. There exists a constant* C *depending only on* A *such that for all small enough* $r > 0$ *we can find a set* N_r *of axis-parallel boxes with side length* r *and with disjoint interiors such that* $|N_r| \le Cr^{1-d}$ *and*

$$\partial A + B_r \subseteq \cup_{Q \in N_r} Q.$$

Proof of Lemma 4.4. Let $X_l = \cup_{Q \in A_l} Q$. We have

$$|A_l| = \sum_{Q \in A_l} 1 = \sum_{Q \in A_l} |Q|2^{dl} = 2^{dl}|X_l|.$$

By a property of the Whitney decomposition, there is $C_d > 0$ such that

$$\mathrm{dist}(x, \partial A) \le C_d 2^{-l}$$

for all $x \in X_l$. Since ∂A has finite upper Minkowski content, there is l_1 only dependent on A such that

$$|X_l| \le |\{x : \mathrm{dist}(x, \partial A) \le C_d 2^{-l}\}| \le C_A 2^{-l}.$$

for $l \ge l_1$ which gives us the desired estimate. If $l \le l_1$ then we simply bound

$$2^{dl}|X_l| \le 2^{dl}|A| \le 2^{(d-1)l}|A|2^{l_1}.$$

□

Proof of Lemma 4.5. For $z \in r\mathbb{Z}^d$ denote by $Q_r(z)$ the cube with side length r centred at z. Let $Z_r \subseteq r\mathbb{Z}^d$ be the collection of points $z \in r\mathbb{Z}^d$ such that $Q_r(z) \cap (\partial A + B_r) \neq \varnothing$. If $\omega \in Q_r(z)$ for $z \in Z_r$, then

$$\operatorname{dist}(\omega, \partial A) \leq \inf_{p \in Q_r(z)} (|\omega - p| + \operatorname{dist}(p, \partial A)) \leq (\sqrt{d} + 1)r.$$

Hence,

$$\left| \bigcup_{z \in Z_r} Q_r(z) \right| \leq |\partial A + B_{(\sqrt{d}+1)r}| \leq Cr$$

for r small enough. Since $Q_r(z)$ for different z are disjoint up to measure 0, we get

$$|Z_r| \leq Cr^{1-d}.$$

Finally, we observe that we have the covering $A + B_r \subseteq \cup_{z \in Z_r} Q_r(z)$, so this covering works.

□

Remark 4.6. *Lemma 4.4 requires only the internal part of the upper Minkowski content of ∂A, that is only $|(\partial A + B_r) \cap A|$, but for Lemma 4.5 we need the full upper Minkowski content.*

Remark 4.7. *Note that Lemma 4.5 in particular implies that if a set has finite upper Minkowski content then it has finite $d-1$-dimensional Hausdorff measure.*

We will pick a small number $\delta > 0$ and consider the threshold $D = \left[\log_2\left(\frac{c}{\delta \log(\frac{1}{\varepsilon})}\right)\right]$. The cubes with side lengths larger than 2^{-D} we will leave as is, and note that their union covers all of A except possibly for the points in $C_d 2^{-D}$-neighborhood of ∂A. Let $r = C_d 2^{-D}$ and consider the cubes N_r from Lemma 4.5. We get

$$A \subseteq (\cup_{l<D} \cup_{Q \in A_l} Q) \cup (\cup_{Q \in N_r} Q).$$

First part of this union is disjoint. We make the second part disjoint by setting, for $X \in N_r$,

$$X' = X \cap A \setminus (\cup_{l<D} \cup_{Q \in A_l} Q).$$

Hence,

$$A = (\cup_{l<D} \cup_{Q \in A_l} Q) \cup (\cup_{Q \in N_r} Q').$$

From this we can write

$$P_{cA^c} Q_{[0,1]^d} P_{cA} = \sum_{l<D} \sum_{Q \in A_l} P_{cA^c} Q_{[0,1]^d} P_{cQ} + \sum_{Q \in N_r} P_{cA^c} Q_{[0,1]^d} P_{cQ'}. \tag{4.2}$$

Next, we want to enlarge these sets to make the computations easier. To see how the singular values change under enlargements we will use the following lemma.

Lemma 4.8. *Consider* $X, Y, Z \subseteq \mathbb{R}^d$ *and assume that* $P_X Q_Y P_Z$ *is compact. If* $X \subset X'$ *and* $P_{X'} Q_Y P_Z$ *is compact, then*

$$\sigma_k(P_X Q_Y P_Z) \leq \sigma_k(P_{X'} Q_Y P_Z).$$

Similarly, if $Z \subseteq Z'$ *and* $P_X Q_Y P_{Z'}$ *is compact, then*

$$\sigma_k(P_X Q_Y P_Z) \leq \sigma_k(P_X Q_Y P_{Z'}).$$

Proof. By the max-min theorem, if T is a compact operator on H, then

$$\sigma_k(T) = \max_{\substack{V \subseteq H \\ \dim H = k}} \min_{\psi \in V \setminus \{0\}} \frac{\|T\psi\|}{\|\psi\|}.$$

For the first inequality we simply have $\|P_{X'} Q_Y P_Z \psi\| \geq \|P_X Q_Y P_Z \psi\|$ for all $\psi \in L^2(\mathbb{R}^d)$. The second inequality follows from the first by taking the adjoint. □

Corollary 4.9. *Let* X, Y, Z *be measurable sets. If* $X \subseteq X'$ *and* $Z \subseteq Z'$ *then* $\|P_X Q_Y P_Z\|_p^p \leq \|P_{X'} Q_Y P_{Z'}\|_p^p$.

We will apply (1.10) and Corollary 4.9 to (4.2). For $P_{cA^c} Q_{[0,1]^d} P_{cQ}$, $Q \in A_l$, we enlarge cA^c to $c\tilde{Q}^c$, where $\tilde{Q}$ is the cube with the same centre as Q and two times larger side length. Note that by the property of the Whitney decomposition 4.3 $\tilde{Q}$ is still contained in A. For $P_{cA^c} Q_{[0,1]^d} P_{cQ'}$, $Q \in N_r$, we simply enlarge cA^c to all of $\mathbb{R}^d$ and cQ' to cQ. We get

$$\|P_{cA^c} Q_{[0,1]^d} P_{cA}\|_p^p \leq \sum_{l<D} \sum_{Q \in A_l} \|P_{c\tilde{Q}^c} Q_{[0,1]^d} P_{cQ}\|_p^p + \sum_{Q \in N_r} \|Q_{[0,1]^d} P_{cQ}\|_p^p. \tag{4.3}$$

After shifting with the help of Lemma 4.1 we can see that the operators in the second sum are simply tensor powers of one-dimensional operators I_{cr} from Lemma 1.8. We will use the fact that the Schatten norms are multiplicative under taking the tensor products.

Lemma 4.10. *Consider Hilbert spaces* $H_1, \dots, H_m$ *and compact operators* $A_j \in H_j$, $j = 1, \dots, m$. *If* T *is the operator* $T = A_1 \otimes \dots \otimes A_m$ *on* $H = H_1 \otimes \dots \otimes H_m$, *then for any* $0 < p < \infty$ *we have*

$$\|T\|_p = \|A_1\|_p \dots \|A_m\|_p.$$

Proof. As multisets, we have

$$\{\sigma_n(T) \mid n \in \mathbb{N}\} = \{\sigma_{n_1}(A_1) \dots \sigma_{n_m}(A_m) \mid n_1, \dots, n_m \in \mathbb{N}\}.$$

Summing over p'th powers of the elements in the left- and right-hand sides, the claim follows. □

By Lemma 1.8 we have

$$\|I_{cr}\|_p^p \leq \sum_{n=1}^{[10cr]} 1^p + \sum_{n=[10cr]}^{\infty} C^p e^{-\tau p n} \leq 10cr + C^p \frac{1}{1-e^{-\tau p}} \leq C_1 \left(cr + \frac{1}{p} \right)$$

for some absolute constant C_1. By our choice of r and p we have $C_2 cr \geq \frac{1}{p}$, thus $\|I_{cr}\|_p^p \leq C_3 cr$ and therefore by 4.10

$$\|Q_{[0,1]^d} P_{cQ}\|_p^p = \|I_{cr}\|_p^{pd} \leq C_3^d c^d r^d.$$

We turn to the operator $P_{c\tilde{Q}} Q_{[0,1]^d} P_{cQ}$ for $Q \in A_l$, $l < D$. To exactly connect it to the one-dimensional operators that we stated in Lemma 1.8, we are going to do further simplifications. Firstly, we can, after a translation by means of an argument similar to the proof of Lemma 4.1, assume that

$$Q = I^d, \quad \tilde{Q} = \tilde{I}^d,$$

where $I = [-2^{-l-1}, 2^{-l-1}]$ and $\tilde{I} = [-2^{-l}, 2^{-l}]$. Then, with the unions being disjoint,

$$\tilde{Q}^c = \left(\tilde{I}^c \times \mathbb{R}^{d-1} \right) \cup \left(\tilde{I} \times \tilde{I}^c \times \mathbb{R}^{d-2} \right) \cup \cdots \cup \left(\tilde{I}^{d-1} \times \tilde{I}^c \right).$$

We write $P_{c\tilde{Q}^c} Q_{[0,1]^d} P_Q$ as a sum with respect to this decomposition of $\tilde{Q}^c$ and enlarge $\tilde{I}$ to all of $\mathbb{R}$ in every single instance. It follows by Lemma 4.8, Lemma 4.10, and (1.10) that

$$\begin{aligned} \|P_{c\tilde{Q}^c} Q_{[0,1]^d} P_{cQ}\|_p^p &\leq d \| \left(Q_{[0,1]} P_{cI} \right)^{\otimes (d-1)} \otimes P_{c\tilde{I}^c} Q_{[0,1]} P_{cI} \|_p^p \\ &= d \|Q_{[0,1]} P_{cI}\|_p^{p(d-1)} \|P_{c\tilde{I}^c} Q_{[0,1]} P_{cI}\|_p^p. \end{aligned}$$

For $Q_{[0,1]} P_{cI} = I_{c2^{-l}}$ as above we have

$$\|Q_{[0,1]} P_{cI}\|_p^p \leq C_1 \left(c2^{-l} + \frac{1}{p} \right).$$

Again, by our choice of D and p we always have $C_2 c 2^{-l} \geq \frac{1}{p}$, hence

$$\|Q_{[0,1]} P_{cI}\|_p^p \leq C_3 c 2^{-l}.$$

For $P_{c\tilde{I}^c} Q_{[0,1]} P_{cI} = J_{c2^{-l-1}}$ it follows from Lemma 1.8 that

$$\|P_{c\tilde{I}^c} Q_{[0,1]} P_{cI}\|_p^p \leq \sum_{n=1}^{\infty} C^p e^{-\tau p} \leq \frac{C_1}{p}.$$

Hence, in total

$$\|P_{c\tilde{Q}^c} Q_{[0,1]^d} P_{cQ}\|_p^p \leq \frac{C}{p} c^{d-1} 2^{-(d-1)l}.$$

Returning to (4.3), we see that

$$\|P_{cA^c}Q_{[0,1]^d}P_{cA}\|_p^p \leq C\sum_{l<D}|A_l|c^{d-1}2^{-(d-1)l} + C|N_r|c^d r^d.$$

It follows from Lemma 4.4 that $|A_l| \leq C2^{(d-1)l}$ and from Lemma 4.5 that $|N_r| \leq Cr^{-(d-1)}$. Also, $|A_l| = 0$ for $l < l_0$. Hence

$$\|P_{cA^c}Q_{[0,1]^d}P_{cA}\|_p^p \leq \frac{C}{p}\sum_{l_0\leq l<D} c^{d-1} + Cc^d r = C\left((D+l_0)c^{d-1}\frac{1}{p} + c^d r\right).$$

By our choice of D, p, r for $\frac{1}{2} > \varepsilon > \alpha^{-c}$ if δ is small enough we get

$$\|P_{cA^c}Q_{[0,1]^d}P_{cA}\|_p^p \lesssim c^{d-1}\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\alpha c}{\log\left(\frac{1}{\varepsilon}\right)}\right). \tag{4.4}$$

Plugging this estimate into (4.1) and using that N is a fixed constant we finally conclude

$$\Lambda_\varepsilon(cA, B) \lesssim c^{d-1}\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\alpha c}{\log\left(\frac{1}{\varepsilon}\right)}\right).$$

Next, we prove Theorem 1.6 where both A and B are bounded sets with boundaries of finite upper Minkowski content in the same regime $\frac{1}{2} > \varepsilon > \alpha^{-c}$. For this we do the $S - S^2$ trick again, and this time apply the Whitney decomposition to the set B with the same threshold $D = \left[\log_2\left(\frac{c}{\delta\log(\frac{1}{\varepsilon})}\right)\right]$. With an obvious adaptation of notation from the set A, the set B decomposes as

$$B = \left(\cup_{l<D}\cup_{Q\in B_l} Q\right) \cup \left(\cup_{Q\in N_r} Q'\right).$$

We have

$$\Lambda_\varepsilon(cA, B) \leq \sqrt{e}\|P_{A^c}Q_B P_A\|_p^p \leq \sqrt{e}\sum_{l<D}\sum_{Q\in B_l}\|P_{cA^c}Q_Q P_{cA}\|_p^p + \sqrt{e}\sum_{Q\in N_r}\|P_{cA^c}Q_{Q'}P_{cA}\|_p^p.$$

For $Q \in B_l$ it follows from Lemma 4.1 that

$$\|P_{cA^c}Q_Q P_{cA}\|_p^p = \|P_{c2^{-l}A^c}Q_{[0,1]^d}P_{c2^{-l}A}\|_p^p,$$

so we can use (4.4) and get

$$\|P_{cA^c}Q_Q P_{cA}\|_p^p \lesssim c^{d-1}2^{-l(d-1)}\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\alpha c 2^{-l}}{\log\left(\frac{1}{\varepsilon}\right)}\right).$$

For the second logarithm we notice that it is uniformly bounded by $\log\left(\frac{\alpha c 2^{-l_0}}{\log\left(\frac{1}{\varepsilon}\right)}\right)$, so, denoting $\alpha' = \alpha 2^{-l_0}$, we get

$$\begin{aligned}\sum_{l<D}\sum_{Q\in B_l}\|P_{cA^c}Q_Q P_{cA}\|_p^p &\lesssim (D+l_0)c^{d-1}\log\left(\frac{1}{\varepsilon}\right)\log\left(\frac{\alpha' c}{\log\left(\frac{1}{\varepsilon}\right)}\right)\\ &\lesssim c^{d-1}\log\left(\frac{1}{\varepsilon}\right)\log^2\left(\frac{\alpha' c}{\log\left(\frac{1}{\varepsilon}\right)}\right),\end{aligned}$$

which is exactly the required bound.

For the boundary layer $P_{cA^c}Q_{Q'}P_{cA}$, $Q\in N_r$, we first enlarge cA^c to the whole $\mathbb{R}^d$ and enlarge cA to cR where R is a fixed cube containing A to reduce the analysis to the operator $Q_{Q'}P_{cR}$ using Corollary 4.9. Then, by the equality $\sigma_n(Q_{Q'}P_{cR}) = \lambda_n(cR, Q')^{\frac{1}{2}}$ and the symmetry $\lambda_n(U,V) = \lambda_n(V,U)$ we get

$$\|P_{cA^c}Q_{Q'}P_{cA}\|_p^p \le \|Q_{Q'}P_{cR}\|_p^p = \|P_{cR}Q_{Q'}P_{cR}\|_{\frac{p}{2}}^{\frac{p}{2}} = \|P_{Q'}Q_{cR}P_{Q'}\|_{\frac{p}{2}}^{\frac{p}{2}} = \|Q_{cR}P_{Q'}\|_p^p.$$

Now, we are able to enlarge Q' to Q and put all of the scaling onto Q by means of Lemma 4.1 to conclude

$$\|P_{cA^c}Q_{Q'}P_{cA}\|_p^p \le \|Q_{cR}P_{Q'}\|_p^p \le \|P_{[0,cr]^d}Q_{[0,1]^d}\|_p^p.$$

By Lemma 4.10 the right-hand side is equal to $\|P_{[0,cr]}Q_{[0,1]}\|_p^{pd}$. Arguing as in the proof of Theorem 1.3 we see

$$\|P_{[0,cr]^d}Q_{[0,1]^d}\|_p^p \lesssim c^d r^d,$$

and therefore

$$\sum_{Q\in N_r}\|P_{cA^c}Q_{Q'}P_{cA}\|_p^p \lesssim |N_r|c^d r^d \lesssim c^d r \lesssim c^{d-1}\log\left(\frac{1}{\varepsilon}\right),$$

which is even smaller than the required bound. This finishes the proof of Theorem 1.6 in the regime $\frac{1}{2} > \varepsilon > \alpha^{-c}$.

Finally, we will deal with the regime $0 < \varepsilon \le \alpha^{-c}$. The proof will be the same for both Theorem 1.3 and Theorem 1.6. Let $U_A \subset A \subset V_A$ and $U_B \subset B \subset V_B$ be some fixed boxes. By Lemma 4.8 and symmetry $\lambda_k(X,Y) = \lambda_k(Y,X)$ we have

$$\lambda_1(cA,B) \le \lambda_1(cV_A,B) \le \lambda_1(cV_A,V_b).$$

In particular, there are no eigenvalues larger than $1-\alpha^{-c}$ for large enough α by Proposition 1.2.

For the count of eigenvalues between ε and $\frac{1}{2}$ we have

$$\Lambda_\varepsilon^-(cA,B) = N_\varepsilon(cA,B) - N_{1/2}(cA,B).$$

For the second term we have

$$N_{1/2}(cA, B) \leq 2c^d|A||B|$$

because $\operatorname{Tr} S_{cA,B} = c^d|A||B|$. By Lemma 4.8 and symmetry we have

$$\lambda_k(cU_A, U_B) \leq \lambda_k(cA, B) \leq \lambda_k(cV_A, V_B),$$

and therefore

$$N_\varepsilon(cU_A, U_B) \leq N_\varepsilon(cA, B) \leq N_\varepsilon(cV_A, V_B).$$

From (1.6) one can check that if $0 < \varepsilon < \alpha^{-c}$ and α is large enough then

$$N_\varepsilon(cU_A, U_B) \geq 4c^d|A||B|.$$

Plugging in the bounds we get

$$\frac{N_\varepsilon(cU_a, U_B)}{2} \leq N_\varepsilon(cU_A, U_B) - N_{1/2}(cA, B) \leq \Lambda_\varepsilon^-(cA, B) \leq N_\varepsilon(cV_A, V_B)$$

and both sides are proportional to the required value by (1.6) when $0 < \varepsilon \leq \alpha^{-c}$ for large enough α.

Remark 4.11. *Note that the final argument gives us a lower bound on eigenvalues for all sets A, B with non-empty interiors and an upper bound on eigenvalues for all bounded sets A, B.*

5. SINGULAR VALUES OF ONE-DIMENSIONAL OPERATORS

In this section we will prove Lemma 1.8. Since the result for I_r can be deduced from Theorem 2.5, we will start with an estimate for $\sigma_k(J_r)$.

5.1. **Singular values of J_r.** We will show that the operator $J_r = P_{[-2r,2r]^c}Q_{[0,1]}P_{[-r,r]}$ for $r > 0$ decomposes into a sum of finite rank operators. This idea is very similar to the one employed by Karnik, Romberg and Davenport [18] who showed that a similar operator had low displacement-rank structure, and for such operators the singular values can be effectively estimated via Zolotarev numbers. By (1.1) the operator J_r has kernel

$$K(x, y) = 1_{[-2r,2r]^c}(x)\check{1}_{[0,1]}(x - y)1_{[-r,r]}(y) = \frac{1}{2\pi i}1_{[-2r,2r]^c}(x)\frac{e^{2\pi i(x-y)} - 1}{x - y}1_{[-r,r]}(y).$$

The variables x and y separate except for the term $\frac{1}{x-y}$. In [18] the authors handled this term by instead considering $xJ_r - J_rx$, which has rank 2. However, simply writing the term $\frac{1}{x-y}$ as a geometric series shows that

$$K(x,y) = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\left(x^{-n-1}1_{[-2r,2r]^c}(x)e^{2\pi ix}\right)\left(y^n 1_{[-r,r]^c}(y)e^{-2\pi iy}\right)$$
$$- \left(x^{-n-1}1_{[-2r,2r]^c}(x)\right)\left(y^n 1_{[-r,r]^c}(y)\right).$$

Note that the series converges absolutely since $\frac{|y|}{|x|} \leq \frac{1}{2}$ whenever the indicators are non-zero. This is the key point where we use the separation. Hence, if we set

$$f_n(x) = x^{-n-1}1_{[-2r,2r]^c}(x), \quad \tilde{f}_n(x) = f_n(x)e^{2\pi ix},$$
$$g_n(y) = y^n 1_{[-r,r]}(y), \quad \tilde{g}_n(y) = g_n(y)e^{2\pi iy},$$

we see that, for $\varphi \in L^2$,

$$J_r\varphi = -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\Big(\langle\varphi, g_n\rangle f_n - \langle\varphi, \tilde{g}_n\rangle\tilde{f}_n\Big).$$

Clearly each term in the sum has rank at most 2. By a well-known characterization of the singular values of a compact operator A,

$$\sigma_{k+1}(A) = \inf\{\|A - P\| \mid P \text{ is an operator of rank at most } k\},$$

we immediately find, for all $N \in \mathbb{N}$,

$$\sigma_{2N+3}(J_r) \leq \frac{1}{2\pi}\sum_{n=N+1}^{\infty}\Big(\|f_n\|\,\|g_n\| + \|\tilde{f}_n\|\,\|\tilde{g}_n\|\Big) = \frac{1}{\pi}\sum_{n=N+1}^{\infty}\|f_n\|\,\|g_n\|.$$

The norms are easily computed exactly:

$$\|f_n\|^2 = \frac{2}{2n+1}(2r)^{-2n-1}, \quad \|g_n\|^2 = \frac{2}{2n+1}r^{2n+1}.$$

Therefore, we get

$$\sigma_{2N+3}(J_r) \leq \frac{\sqrt{2}}{\pi}\sum_{n=N+1}^{\infty}\frac{1}{2n+1}2^{-n} \leq \frac{\sqrt{2}}{\pi}\frac{1}{2N+3}2^{-N} \leq \frac{\sqrt{2}}{\pi}2^{-N},$$

which finishes the proof.

5.2. **Singular values of I_r.** Since I_r is a product of operators with norm at most 1, it is obvious that $\sigma_n(I_r) \leq 1$ for all $n \in \mathbb{N}$, so we will focus on $n > 10r$. As we already mentioned, since $I_r^* I_r = S_{[0,r],[0,1]}$, we can in principle extract the required bound from Theorem 2.5, as well as from many previous results in the literature (in fact, we can replace 10 by any constant larger than 1). To keep our argument self-contained we will instead present a complex-analytic proof following the argument in [21].

Singular values of I_r are square roots of the eigenvalues of $S_{[0,r],[0,1]}$. By Lemma 4.8 if we increase r the eigenvalues can only increase. Thus, it is enough to consider the case $n = 10r$. Since the eigenvalues depend only on the products of length of the intervals, for convenience we will instead consider the eigenvalues of $S_{[-\frac{1}{2},\frac{1}{2}],[-\frac{r}{2},\frac{r}{2}]}$.

Consider first n normalized eigenfunctions $f_1, f_2, \dots, f_n$ of $S_{[-\frac{1}{2},\frac{1}{2}],[-\frac{r}{2},\frac{r}{2}]}$ with eigenvalues $\lambda_1(r) \geq \dots \geq \lambda_n(r)$. If $\lambda_n(r) = 0$ then there is nothing to prove (in fact, it is not hard to see again by the max-min principle that this is never the case), so we assume without loss of generality that $\lambda_n(r) > 0$. It follows immediately that $f_1, \dots, f_n$ are supported on $[-\frac{1}{2}, \frac{1}{2}]$. In particular, their Fourier transforms

$$g_k(z) = \hat{f}_k(z) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_k(t) e^{-2\pi i z t} dt$$

are defined for all $z \in \mathbb{C}$ and they are holomorphic functions of z.

Let us fix $n-1$ distinct complex numbers $z_1, z_2, \dots, z_{n-1}$. By simple linear algebra we can find scalars $a_1, \dots, a_n \in \mathbb{C}$ not all zero such that the linear combination $g(z) = \sum_{k=1}^n a_k g_k(z)$ satisfies $g(z_1) = \dots = g(z_{n-1}) = 0$. By scaling we can assume that $\|g\|_2 = 1$.

First, we have a uniform pointwise bound on $g(x+iy)$:

$$|g(x+iy)| \leq e^{\pi|y|}. \tag{5.1}$$

Indeed, putting $f = \sum a_k f_k$, we have

$$|g(x+iy)| = \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) e^{-2\pi i (x+iy) t} dt \right| \leq e^{\pi|y|} \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(t)| dt \leq e^{\pi|y|},$$

where in the last step we used the Cauchy–Schwarz inequality. By the max-min characterization of the eigenvalues we also have

$$\lambda_n(r)^2 \leq \|S_{[-\frac{1}{2},\frac{1}{2}],[-\frac{r}{2},\frac{r}{2}]} f\|^2 \leq \int_{-r/2}^{r/2} |g(x)|^2 \, dx. \tag{5.2}$$

We want to arrive at a contradiction by choosing an appropriate sequence z_l.

We will assume for now that $n \geq 1000$. The cases $n < 1000$ we will cover at the end. Our sequence will be an arithmetic progression with $z_1 = -r$, $z_{n-1} = r$. Its step is $s = \frac{2r}{n-1}$. Since $n = 10r$, $n \geq 1000$ we have $s \leq \frac{1}{4}$. We want to bound $|g(x_0)|$ for $x_0 \in [-\frac{r}{2}, \frac{r}{2}]$ assuming that $|g(x+iy)| \leq e^{\pi|y|}$ for all $x, y \in \mathbb{R}$ and $g(z_l) = 0$. We will use Jensen's formula for the disk centred at x_0 of radius $\frac{r}{2}$:

$$\log|g(x_0)| = \int_0^1 \log\left| g\left(x_0 + \frac{r}{2} e^{2\pi i t}\right) \right| dt + \sum_{|z - x_0| < \frac{r}{2}, g(z) = 0} \log \frac{2|z - x_0|}{r}.$$

Since each term in the sum is negative, by leaving in it only z_l's we get an inequality

$$\log|g(x_0)| \leq \int_0^1 \log\left|g\left(x_0+\frac{r}{2}e^{2\pi it}\right)\right|dt + \sum_{|z_l-x_0|<\frac{r}{2}} \log\frac{2|z_l-x_0|}{r}. \tag{5.3}$$

For the integral term we have $\left|g\left(x_0+\frac{r}{2}e^{2\pi it}\right)\right| \leq e^{\pi\frac{r}{2}|\sin(2\pi t)|}$ by (5.1), hence

$$\int_0^1 \log\left|g\left(x_0+\frac{r}{2}e^{2\pi it}\right)\right|dt \leq \int_0^1 \frac{\pi r}{2}|\sin(2\pi t)|dt = r.$$

For the sum over z_l's to the right of x_0 we have

$$\begin{aligned}\sum_{x_0<z_l<x_0+\frac{r}{2}} \log\frac{2|z_l-x_0|}{r} &\leq \frac{1}{s}\sum_{x_0<z_l<x_0+\frac{r}{2}} \int_{z_l}^{z_l+s} \log\frac{2(\omega-x_0)}{r}d\omega \\ &= \frac{1}{s}\int_{\min\limits_{z_l>x_0} z_l}^{\max\limits_{z_l<x_0+\frac{r}{2}} z_l+s} \log\frac{2(\omega-x_0)}{r}d\omega \\ &\leq \frac{1}{s}\int_s^{\frac{r}{2}+s} \log\frac{2\omega}{r}d\omega \leq -\frac{1}{4s}r+1.\end{aligned}$$

where in the first step we used monotonicity of the function $\log\frac{2t}{r}$ for $t>0$, in the second step we combined the integrals into one, in the third step we used that the function $\log\frac{2t}{r}$ is negative for $t<\frac{r}{2}$ and positive for $t>\frac{r}{2}$ and that the minimum of z_l's is at most x_0+s and the maximum of z_l's is at least $x_0+\frac{r}{2}-s$ because $x_0+\frac{r}{2}\leq r$, so we can not get past the interval $[-r.r]$, and in the last step we used inequality $\log u \leq u-1$. For the sum over z_l's to the left of x_0 we get exactly the same upper bound.

Plugging all our bounds into (5.3) together with $s\leq\frac{1}{4}$ we get

$$|g(x_0)| \leq e^{2-r}.$$

Therefore, by (5.2)

$$\lambda_n(r)^2 \leq \int_{-\frac{r}{2}}^{\frac{r}{2}} |g(x)|^2dx \leq re^{4-2r} \leq e^{4-r},$$

as required.

For $n\leq 1000$ we can simply estimate $\lambda_n(r)\leq 1$, which is consistent with our bound if we increase C.

6. TRACE CLASS CONDITIONS

In this section we prove Theorem 1.9 and Theorem 1.10.

6.1. **Sufficient condition.** We begin with Theorem 1.9, so consider bounded measurable sets $A, B \subseteq \mathbb{R}^d$ and let f be a trace class admissible function, that is

$$\int_0^\delta \frac{M_0 f(\varepsilon) \log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon \left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d} \, d\varepsilon < \infty. \tag{6.1}$$

Since A and B are bounded they have finite measure and so $S_{A,B}$ is a compact operator. We need to show that $\sum_{n=1}^\infty |f(\lambda_n(A,B))| < \infty$. Since $\lambda_n(A,B) \to 0$ there are only finitely many n such that $\lambda_n(A,B) > \frac{\delta}{2}$. So, it is enough to show that the tail $\sum_{\lambda_n(A,B) \le \frac{\delta}{2}} |f(\lambda_n(A,B))|$ is finite. We will also ignore all of the eigenvalues equal to 0 since $f(0) = 0$ for all trace class admissible for $L^2(\mathbb{R}^d)$ functions f. We decompose the sum dyadically and bound f from above by $M_0 f$:

$$\begin{aligned} \sum_{0<\lambda_n(A,B)\le\frac{\delta}{2}} |f(\lambda_n(A,B))| &= \sum_{k=0}^\infty \sum_n 1_{\{\delta 2^{-2^{k+1}} < \lambda_n(A,B) \le \delta 2^{-2^k}\}} |f(\lambda_n(A,B))| \\ &\le \sum_{k=0}^\infty M_0 f\left(\delta 2^{-2^k}\right) \left(N_{\delta 2^{-2^{k+1}}}(A,B) - N_{\delta 2^{-2^k}}(A,B)\right) \\ &\le \sum_{k=0}^\infty M_0 f\left(\delta 2^{-2^k}\right) N_{\delta 2^{-2^{k+1}}}(A,B). \end{aligned}$$

We chose this exact decomposition because on this scale the estimates in (1.6) roughly double at each step. Since A and B are bounded, there exist fixed cubes V_A, V_B such that $A \subseteq V_A$, $B \subseteq V_B$. By Lemma 4.8 we have $N_\varepsilon(A,B) \le N_\varepsilon(V_A, V_B)$ for all $0 < \varepsilon < 1$. Hence,

$$\begin{aligned} \sum_{k=0}^\infty M_0 f\left(\delta 2^{-2^k}\right) N_{\delta 2^{-2^{k+1}}}(A,B) &\le \sum_{k=0}^\infty M_0 f\left(\delta 2^{-2^k}\right) N_{\delta 2^{-2^{k+1}}}(V_A, V_B) \\ &= \sum_{k=0}^\infty M_0 f\left(\delta 2^{-2^k}\right) (N_{1/2}(V_A, V_B) + \Lambda^-_{\delta 2^{-2^{k+1}}}(V_A, V_B)). \end{aligned}$$

Since the integral (6.1) converges, $M_0 f(\varepsilon)$ is finite for all $\varepsilon < \delta$. In particular, it suffices to show that the tail

$$\sum_{k=k_0}^\infty M_0 f\left(\delta 2^{-2^k}\right) (N_{1/2}(V_A, V_B) + \Lambda^-_{\delta 2^{-2^{k+1}}}(V_A, V_B))$$

is finite for some $k_0 > 0$. Note that $N_{1/2}(V_A, V_B)$ is a fixed number while $\Lambda^-_{\delta 2^{-2^{k+1}}}(V_A, V_B)$ tends to infinity, so for k_0 large enough we have $N_{1/2}(V_A, V_B) \le \Lambda^-_{\delta 2^{-2^{k+1}}}(V_A, V_B)$ for $k \ge k_0$. By possibly choosing k_0 larger still we can apply the upper bound (1.6) (note that c, which is the product of side lengths of V_A and V_B, is a constant for the present discussion), and find

$$\sum_{k=k_0}^{\infty} M_0 f\left(\delta 2^{-2^k}\right)\left(N_{1/2}(V_A, V_B) + \Lambda^-_{\delta 2^{-2^{k+1}}}(V_A, V_B)\right)$$

$$\lesssim \sum_{k=k_0}^{\infty} M_0 f\left(\delta 2^{-2^k}\right)\left(\frac{\log\left(\frac{1}{\delta 2^{-2^{k+1}}}\right)}{\log\left(\frac{\log\left(\frac{1}{\delta 2^{-2^{k+1}}}\right)}{c}\right)}\right)^d .$$

A direct computation shows that the expression in brackets is proportional to $\frac{2^k}{k}$. So, it remains to show that $\sum_{k=k_0}^{\infty} M_0 f\left(\delta 2^{-2^k}\right)\frac{2^{kd}}{k^d}$ is finite. We have the following chain of inequalities:

$$\begin{aligned}
\sum_{k=k_0}^{\infty} M_0 f\left(\delta 2^{-2^k}\right)\frac{2^{kd}}{k^d} &= \sum_{k=k_0}^{\infty} M_0 f\left(\delta 2^{-2^k}\right)\frac{2^{kd}}{k^d}\frac{1}{\log(2)2^{k-1}}\int_{\delta 2^{-2^k}}^{\delta 2^{-2^{k-1}}}\frac{1}{\varepsilon}\,d\varepsilon \\
&\lesssim \sum_{k=k_0}^{\infty} M_0 f\left(\delta 2^{-2^k}\right)\int_{\delta 2^{-2^k}}^{\delta 2^{-2^{k-1}}}\frac{\log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}\,d\varepsilon \\
&\leq \sum_{k=k_0}^{\infty}\int_{\delta 2^{-2^k}}^{\delta 2^{-2^{k-1}}}\frac{M_0 f(\varepsilon)\log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}\,d\varepsilon \\
&\leq \int_0^{\delta}\frac{M_0 f(\varepsilon)\log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}\,d\varepsilon < \infty.
\end{aligned}$$

We used that for $\delta 2^{-2^k} < \varepsilon < \delta 2^{-2^{k-1}}$ the term $\frac{\log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}$ is proportional to $\frac{2^{k(d-1)}}{k^d}$ and that $M_0 f$ is non-decreasing. This finishes the proof of Theorem 1.9.

6.2. **Necessary condition.** We turn to the proof of Theorem 1.10, so assume that $A, B \subseteq \mathbb{R}^d$ are sets with non-empty interiors such that $S_{A,B}$ is compact, and consider a function f such that $|f(x)|$ is non-decreasing close to 0, say for $0 \leq x \leq \beta$. Assume that $f(S_{A,B})$ is trace class. We need to show that

$$\int_0^{\delta}\frac{M_0 f(\varepsilon)\log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}\,d\varepsilon < \infty$$

for some $\delta > 0$. Note that for $0 \le \varepsilon \le \beta$ we have $M_0 f(\varepsilon) = |f(\varepsilon)|$, so if $\delta \le \beta$ then

$$\int_0^\delta \frac{M_0 f(\varepsilon) \log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}\, d\varepsilon = \int_0^\delta \frac{|f(\varepsilon)| \log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}\, d\varepsilon.$$

As in the proof of Theorem 1.9, we split $\sum |f(\lambda_n(A,B))|$ into the sum over eigenvalues larger than β and at most β and throw away the first one as it is finite. Thus, our basic assumption is that

$$\sum_{\lambda_n(A,B)\le\beta} |f(\lambda_n(A,B))| < \infty.$$

By definition $\lambda_n(A,B)$ is non-increasing in n so this is the sum from some n_0 to infinity. Since A and B have non-empty interiors, we can find cubes U_A and U_B such that $U_A \subset A$, $U_B \subset B$. By Lemma 4.8 we have $\lambda_n(A,B) \ge \lambda_n(U_A,U_B)$. Using monotonicity of $|f|$ we get

$$\sum_{\lambda_n(A,B)\le\beta} |f(\lambda_n(A,B))| \ge \sum_{n=n_0}^{\infty} |f(\lambda_n(U_A,U_B))|.$$

Next, we do the dyadic decomposition but with a large constant $K \ge 2$ to be fixed later. Pick k_0 so that $2^{-K^{k_0}} \le \lambda_{n_0}(U_A,U_B)$ and write, using monotonicity of $|f|$,

$$\begin{aligned}\sum_{n=n_0}^{\infty} |f(\lambda_n(U_A,U_B))| &\ge \sum_{k=k_0}^{\infty}\sum_n 1_{\{2^{-K^{k+1}}<\lambda_n(U_A,U_B)\le 2^{-K^k}\}} |f(\lambda_n(U_A,U_B))| \\ &\ge \sum_{k=k_0}^{\infty} \left|f\left(2^{-K^{k+1}}\right)\right| \left(N_{2^{-K^{k+1}}}(U_A,U_B) - N_{2^{-K^k}}(U_A,U_B)\right) \\ &= \sum_{k=k_0}^{\infty} \left|f\left(2^{-K^{k+1}}\right)\right| \left(\Lambda^-_{2^{-K^{k+1}}}(U_A,U_B) - \Lambda^-_{2^{-K^k}}(U_A,U_B)\right)\end{aligned}$$

We intend to apply (1.6). Note here that there are $c_1, c_2 > 0$ independent of K such that for all $k \ge k_0(K)$

$$c_1 \frac{K^{kd}}{(\log(K)k)^d} \le \Lambda^-_{2^{-K^k}}(U_A,U_B) \le c_2 \frac{K^{kd}}{(\log(K)k)^d}.$$

We choose K so that $K^d \ge \frac{2^{d+1}c_2}{c_1}$ which fixes k_0 and ensures that

$$\Lambda^-_{2^{-K^{k+1}}}(U_A,U_B) \ge 2\Lambda^-_{2^{-K^k}}(U_A,U_B)$$

whenever $k \geq k_0$. Hence,

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} \left| f\left(2^{-K^{k+1}}\right)\right| \left(\Lambda^-_{2^{-K^{k+1}}}(U_A, U_B) - \Lambda^-_{2^{-K^k}}(U_A, U_B)\right) \\
&\qquad \geq \sum_{k=k_0}^{\infty} \left| f\left(2^{-K^{k+1}}\right)\right| \Lambda^-_{2^{-K^k}}(U_A, U_B) \\
&\qquad \gtrsim \sum_{k=k_0}^{\infty} \left| f\left(2^{-K^{k+1}}\right)\right| \frac{K^{kd}}{k^d}.
\end{aligned}$$

Arguing as in the proof of Theorem 1.9 and using monotonicity of $|f|$ we obtain

$$\sum_{k=k_0}^{\infty} \left| f\left(2^{-K^{k+1}}\right)\right| \frac{K^{kd}}{k^d} \gtrsim \int_0^{2^{-K^{k+1}}} \frac{|f(\varepsilon)| \log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}\, d\varepsilon.$$

Collecting everything, we conclude

$$\infty > \sum_{\lambda_n(A,B)\leq\beta} |f(\lambda_n(A,B))| \gtrsim \sum_{k=k_0}^{\infty} \left| f\left(2^{-K^{k+1}}\right)\right| \frac{K^{kd}}{k^d} \gtrsim \int_0^{2^{-K^{k+1}}} \frac{|f(\varepsilon)| \log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}\, d\varepsilon,$$

finishing the proof.

7. AREA LAWS

In this section we prove Theorem 1.11 and Theorem 1.13.

7.1. **Uniform bounds.** We now prove (1.14) and Theorem 1.13. For both of them we can without loss of generality assume that $f(1) = 0$. Indeed, denoting $\tilde{f}(\theta) = f(\theta) - \theta f(1)$ we have $\tilde{f}(1) = 0$,

$$\mathrm{Tr} f(S_{cA,B}) = \mathrm{Tr}\tilde{f}(S_{cA,B}) + f(1)\mathrm{Tr}(S_{cA,B}) = \mathrm{Tr}\tilde{f}(S_{cA,B}) + c^d|A||B|f(1)$$

and

$$\int_0^1 \frac{\tilde{f}(\theta) - \theta\tilde{f}(1)}{\theta(1-\theta)}\, d\theta = \int_0^1 \frac{f(\theta) - f(1)\theta}{\theta(1-\theta)}\, d\theta.$$

Note that $\tilde{f}$ is also area law admissible and trace class admissible for $L^2(\mathbb{R}^d)$ if f is. Thus, it suffices to establish the behaviour of $\mathrm{Tr}\tilde{f}(S_{cA,B})$ for (1.15). To get the claimed uniform bound (1.14) we also need to show that

$$\int_0^1 \frac{M_0\tilde{f}(\varepsilon) + M_1\tilde{f}(\varepsilon)}{\varepsilon} d\varepsilon \leq C \int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon} d\varepsilon$$

for some absolute constant $C > 0$. We have $M_0\tilde{f}(\varepsilon) \leq M_0 f(\varepsilon) + \varepsilon|f(1)|$ and $M_1\tilde{f}(\varepsilon) \leq M_1 f(\varepsilon) + \varepsilon|f(1)|$, so we just have to show that

$$\lambda \int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon} d\varepsilon \geq |f(1)|$$

for some absolute constant $\lambda > 0$. For $\varepsilon > \frac{1}{2}$ we have $M_0 f(\varepsilon) \geq |f(\frac{1}{2})|$ and $M_1 f(\varepsilon) \geq |f(1) - f(\frac{1}{2})|$, thus $M_0 f(\varepsilon) + M_1 f(\varepsilon) \geq |f(1)|$. Integrating this we get

$$\int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon} d\varepsilon \geq \log(2)|f(1)|,$$

so $\lambda = \frac{1}{\log(2)}$ works.

We start with proving the uniform bound (1.14), so assume that $A \subseteq \mathbb{R}^d$ is a set whose boundary ∂A has finite upper Minkowski content and $B \subseteq \mathbb{R}^d$ is a finite union of parallelepipeds with disjoint interiors, and let f be both area law admissible and trace class admissible for $L^2(\mathbb{R}^d)$ with $f(1) = 0$. We need to show that, for c large enough dependent on A, B, and f,

$$|\operatorname{Tr} f(S_{cA,B})| \leq C(A,B)c^{d-1}\log(c) \int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon} d\varepsilon$$

The proof of Theorem 1.13 will be almost identical. Take α from the statement of Theorem 1.3. We split the eigenvalues into the relevant regimes:

$$\begin{aligned} |\operatorname{Tr} f(S_{cA,B})| &= \left|\sum_{n=1}^{\infty} f(\lambda_n(cA,B))\right| \leq \sum_{n=1}^{\infty} |f(\lambda_n(cA,B))| \\ &= \sum_{\lambda_n(cA,B) > 1-\alpha^{-c}} |f(\lambda_n(cA,B))| + \sum_{1-\alpha^{-c} \geq \lambda_n(cA,B) \geq \alpha^{-c}} |f(\lambda_n(cA,B))| \\ &\quad + \sum_{\alpha^{-c} > \lambda_n(cA,B) > 0} |f(\lambda_n(cA,B))|. \end{aligned} \tag{7.1}$$

We will bound each of these three sums separately. The first sum is empty by Theorem 1.3, since there are no eigenvalues larger than $1-\alpha^{-c}$. We proceed with the bound for the second sum. We split the sum into eigenvalues close to 1 and close to 0 and decompose dyadically like in the proof of Theorem 1.9. Let $k_0(c)$ be such that $2^{-2^{k_0(c)+1}} < \alpha^{-c} \leq 2^{-2^{k_0(c)}}$. We write

$$\begin{aligned}\sum_{1-\alpha^{-c}\geq\lambda_n(cA,B)\geq\alpha^{-c}} |f(\lambda_n(cA,B))| &\leq \sum_{k=0}^{k_0(c)}\sum_n 1_{\{2^{-2^{k+1}}<\lambda_n(cA,B)\leq 2^{-2^k}\}}\,|f(\lambda_n(cA,B))|\\ &\quad+\sum_{k=0}^{k_0(c)}\sum_n 1_{\{2^{-2^{k+1}}<1-\lambda_n(cA,B)\leq 2^{-2^k}\}}\,|f(\lambda_n(cA,B))|\\ &\leq \sum_{k=0}^{k_0(c)}\sum_n 1_{\{2^{-2^{k+1}}<\lambda_n(cA,B)\leq 2^{-2^k}\}}\,M_0f\left(2^{-2^k}\right)\\ &\quad+\sum_{k=0}^{k_0(c)}\sum_n 1_{\{2^{-2^{k+1}}<1-\lambda_n(cA,B)\leq 2^{-2^k}\}}\,M_1f\left(2^{-2^k}\right)\\ &\leq \sum_{k=0}^{k_0(c)}\left(M_0f\left(2^{-2^k}\right)+M_1f\left(2^{-2^k}\right)\right)\Lambda_{2^{-2^{k+1}}}(cA,B).\end{aligned}$$

In the first step we did the dyadic splitting (with possibly overcounting beyond α^{-c}), in the second step we bounded f by M_0f and M_1f, respectively, and in the third step we crudely bounded the number of eigenvalues in the corresponding intervals by $\Lambda_\varepsilon(cA,B)$ for a suitable ε.

For $k<k_0(c)$ we will use (1.7). For $k=k_0(c)$ we have to use (1.8) but one can check that in this regime the estimate is proportional to the one in (1.7) so we will use (1.7) here as well. We get

$$\Lambda_{2^{-2^{k+1}}}(cA,B)\lesssim c^{d-1}2^k\log\left(\alpha c2^{-k}\right)\lesssim c^{d-1}2^k\log(c). \tag{7.2}$$

Applying this we find

$$\begin{aligned}\sum_{1-\alpha^{-c}\geq\lambda_n(cA,B)\geq\alpha^{-c}} |f(\lambda_n(cA,B))| &\leq \sum_{k=0}^{k_0(c)}\left(M_0f\left(2^{-2^k}\right)+M_1f\left(2^{-2^k}\right)\right)\Lambda_{2^{-2^{k+1}}}(cA,B)\\ \lesssim \sum_{k=0}^{k_0(c)}\left(M_0f\left(2^{-2^k}\right)+M_1f\left(2^{-2^k}\right)\right)c^{d-1}2^k\log(c).&\end{aligned}$$

By the simple identity

$$2^k=\frac{2}{\log(2)}\int_{2^{-2^k}}^{2^{-2^{k-1}}}\frac{1}{\varepsilon}\,d\varepsilon,$$

we conclude

$$\begin{aligned}
&\sum_{1-\alpha^{-c}\geq\lambda_n(cA,B)\geq\alpha^{-c}} |f(\lambda_n(cA,B))| \\
&\lesssim \sum_{k=0}^{k_0(c)} \left(M_0 f\left(2^{-2^k}\right) + M_1 f\left(2^{-2^k}\right)\right) c^{d-1}\log(c)\frac{2}{\log(2)}\int_{2^{-2^k}}^{2^{-2^{k-1}}} \frac{1}{\varepsilon}\, d\varepsilon \\
&\leq c^{d-1}\log(c)\frac{2}{\log(2)}\sum_{k=0}^{k_0(c)}\int_{2^{-2^k}}^{2^{-2^{k-1}}} \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon}\, d\varepsilon \\
&\leq c^{d-1}\log(c)\frac{2}{\log(2)}\int_{0}^{2^{-1/2}} \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon} d\varepsilon,
\end{aligned}$$

which is of the required form for (1.14).

We finally turn to the third sum in (7.1). Using the exact same dyadic decomposition we will ultimately get

$$\sum_{\alpha^{-c}>\lambda_n(cA,B)>0} |f(\lambda_n(cA,B))| \leq \sum_{k=k_0(c)}^{\infty} M_0 f\left(2^{-2^k}\right)\Lambda^-_{2^{-2^{k+1}}}(cA,B). \tag{7.3}$$

Since we are now in the regime $\varepsilon < \alpha^{-c}$, it follows from (1.8) that

$$\Lambda^-_{2^{-2^{k+1}}}(cA,B) \lesssim \left(\frac{2^{k+1}}{\log(\frac{2^{k+1}}{c})}\right)^d.$$

We claim that $\left(\frac{2^{k+1}}{\log\left(\frac{2^{k+1}}{c}\right)}\right)^d \leq \frac{2^{(k+1)d}}{(k+1)^d}C(d,\alpha)\log(c)^d$ for $k \geq k_0(c)$, where $C(d,\alpha)$ is some constant depending only on d and α. Indeed, this is equivalent to $\frac{k+1}{\log\left(\frac{2^{k+1}}{c}\right)} \leq C(d,\alpha)^{1/d}\log(c)$. The left-hand side is equal to $\frac{1}{\log(2)-\frac{\log(c)}{k+1}}$ which is clearly a decreasing function of k, so it is enough to verify the inequality for $k = k_0(c)$. Using $2^{-2^{k_0(c)+1}} \leq \alpha^{-c} \leq 2^{-2^{k_0(c)}}$ and $\alpha \geq 4$ we get

$$\frac{k_0(c)+1}{\log\left(\frac{2^{k_0(c)+1}}{c}\right)} \leq \frac{\log_2(c)+\log_2(\log_2(\alpha))+1}{\log(\log_2(\alpha))} \leq \log(c)\frac{\frac{1}{\log(2)}+\log_2(\log_2(\alpha))+1}{\log(2)}.$$

which gives the claim with $C(d,\alpha)^{1/d} = \frac{\frac{1}{\log(2)}+\log_2(\log_2(\alpha))+1}{\log(2)}$.

Plugging this into (7.3) and arguing as in the proof of Theorem 1.9 we get

$$\begin{aligned}\sum_{\alpha^{-c}>\lambda_n(cA,B)>0} |f(\lambda_n(cA,B))| &\lesssim \log(c)^d \sum_{k=k_0(c)}^{\infty} M_0 f\left(2^{-2^k}\right) \frac{2^{(k+1)d}}{(k+1)^d} \\ &\lesssim \log(c)^d \int_0^{2^{-2^{k_0(c)-1}}} \frac{M_0 f(\varepsilon) \log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}\, d\varepsilon.\end{aligned} \tag{7.4}$$

The final integral is finite for c large enough since f is trace class admissible for $L^2(\mathbb{R}^d)$. Moreover, as $c \to \infty$ we have $k_0(c) \to \infty$, thus the integral can be as small as we like. In particular, for $c > c_0(f, A, B)$ we can assume that

$$\int_0^{2^{-2^{k_0(c)-1}}} \frac{M_0 f(\varepsilon) \log\left(\frac{1}{\varepsilon}\right)^{d-1}}{\varepsilon\left(\log\log\left(\frac{1}{\varepsilon}\right)\right)^d}\, d\varepsilon \leq \int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon} d\varepsilon$$

assuming that $\int_0^1 \frac{M_0 f(\varepsilon)+M_1 f(\varepsilon)}{\varepsilon} d\varepsilon$ is non-zero. But it can be zero only if $f \equiv 0$ in which case the theorem is trivial. Thus, for $c > c_0(f, A, B)$ we have

$$|\mathrm{Tr} f(S_{cA,B})| \leq C(A,B)(c^{d-1}\log(c) + \log(c)^d) \int_0^1 \frac{M_0 f(\varepsilon) + M_1 f(\varepsilon)}{\varepsilon} d\varepsilon.$$

For $c > 1$ we have $\log(c) \leq c$, which gives the desired estimate with at most doubling the constant $C(A, B)$.

To prove Theorem 1.13 the only thing that we have to change is that in (7.2) we will have $\log^2(c)$ instead of $\log(c)$, which leads to the final error bound $O(c^{d-1}\log^2(c))$.

Remark 7.1. *It was absolutely crucial for our argument that the third sum in* (7.1) *turned out to be* $o(c^{d-1}\log(c))$*, otherwise it might happen that the bound does not hold. For* $d \geq 2$ *the third sum is* $O(\log(c)^d) = o(c^{d-1}\log(c))$*, so in this case it is enough for us to only know the value of* δ *and the integral in the definition of trace class admissibility for* $L^2(\mathbb{R}^d)$*.*

For $d = 1$ *our proof as written requires us to also know how fast does the integral* $\int_0^t \frac{M_0 f(\varepsilon)}{\varepsilon \log\log(\frac{1}{\varepsilon})} d\varepsilon$ *converges to* 0 *as* $t \to 0$ *to get* $O(\log(c))$ *with as small of a constant as we like. However, coincidentally* $d = 1$ *is also the only case where trace class admissibility for* $L^2(\mathbb{R}^d)$ *is weaker than area law admissibility. In particular,*

$$\int_0^t \frac{M_0 f(\varepsilon)}{\varepsilon \log\log(\frac{1}{\varepsilon})} d\varepsilon \leq \frac{1}{\log\log(\frac{1}{t})} \int_0^t \frac{M_0 f(\varepsilon)}{\varepsilon} d\varepsilon.$$

In this way we would get that the third sum is $O(1)$ *(and even* $o(1)$*) for* $d = 1$*, and the value* $c_0(f, A, B)$ *would even be independent of* f*.*

7.2. **Two-term asymptotics.** It remains to establish the two-term asymptotic expansion (1.15) under the same assumptions on A and B as in the previous subsection, but with the additional assumption that f is Riemann integrable on $[\varepsilon, 1-\varepsilon]$ for all $0 < \varepsilon < 1/2$. We will also assume without loss of generality that f is real-valued, as we can first prove the result for $\mathrm{Re} f$ and $\mathrm{Im} f$ separately, which satisfy all of our assumptions if f satisfies them, and use that both sides of (1.15) are linear in f. Lastly, as before, we will also assume that $f(1) = 0$ by subtracting $\theta f(1)$ from f.

Recall that (1.15) holds for polynomials [9]. We will first extend (1.15) to all continuous functions supported on $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon > 0$ and then deduce from this that (1.15) holds for $f = 1_{[a,b]}$ for all $0 < a < b < 1$. This is the same argument that was used in [22] to prove Theorem 2.1.

Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(\theta) = 0$ if $0 < \theta < \varepsilon$ or $1-\varepsilon < \theta < 1$. The function $g(\theta) = \frac{f(\theta)}{\theta(1-\theta)}$ is clearly continuous on $[0,1]$. Given $\delta > 0$, by the Stone–Weierstrass theorem, we can find a polynomial P such that $|P(\theta) - g(\theta)| \leq \delta$ for all $\theta \in [0,1]$. We clearly have

$$(P(\theta)+\delta)\theta(1-\theta) \geq f(\theta) \geq (P(\theta)-\delta)\theta(1-\theta).$$

Denoting $Q(\theta) = (P(\theta)+\delta)\theta(1-\theta)$, $R(\theta) = (P(\theta)-\delta)\theta(1-\theta)$ we have

$$\operatorname{Tr} Q(S_{cA,B}) \geq \operatorname{Tr} f(S_{cA,B}) \geq \operatorname{Tr} R(S_{cA,B})$$

Dividing by $c^{d-1}\log(c)$, applying (1.15) to Q and R and taking the limit $c \to \infty$ we get

$$\begin{aligned} I(A,B)\int_0^1 \frac{Q(\theta)}{\theta(1-\theta)}d\theta &\geq \limsup_{c\to\infty} \frac{\operatorname{Tr} f(S_{cA,B})}{c^{d-1}\log(c)} \\ &\geq \liminf_{c\to\infty} \frac{\operatorname{Tr} f(S_{cA,B})}{c^{d-1}\log(c)} \geq I(A,B)\int_0^1 \frac{R(\theta)}{\theta(1-\theta)}d\theta \end{aligned}$$

The difference between the left-hand side and the right-hand side is at most $2\delta I(A,B)$, so taking the limit $\delta \to 0$ and using the squeeze theorem we get

$$\lim_{c\to\infty} \frac{\operatorname{Tr} f(S_{cA,B})}{c^{d-1}\log(c)} = I(A,B)\int_0^1 \frac{f(\theta)}{\theta(1-\theta)}d\theta,$$

which establishes (1.15) if f is continuous and vanishes outside of $[\varepsilon, 1-\varepsilon]$.

Now, we turn to $f(\theta) = 1_{[a,b]}(\theta)$ for $0 < a < b < 1$. For small enough $\delta > 0$ consider the continuous functions $Q_\delta(\theta) = \max\left(1 - \frac{\operatorname{dist}(\theta,[a,b])}{\delta}, 0\right)$ and $R_\delta(\theta) = \max\left(1 - \frac{\operatorname{dist}(\theta,[a+\delta,b-\delta])}{\delta}, 0\right)$ which satisfy $Q_\delta(\theta) \geq 1_{[a,b]}(\theta) \geq R_\delta(\theta)$. Letting $\delta \to 0$ and applying the squeeze theorem again we deduce (1.15) for $1_{[a,b]}$.

Now, we turn to the general $f : [0,1] \to \mathbb{R}$ with $f(1) = 0$ which is area law admissible and trace class admissible for $L^2(\mathbb{R})$ and which is Riemann integrable on $[\varepsilon, 1-\varepsilon]$ for all $0 < \varepsilon < \frac{1}{2}$. By the Riemann integrability on $[\varepsilon, 1-\varepsilon]$ we can find Riemann upper and lower sums $Q_\varepsilon \geq f \geq R_\varepsilon$ such that

$$\int_{\varepsilon}^{1-\varepsilon} \frac{Q_\varepsilon(\theta) - R_\varepsilon(\theta)}{\theta(1-\theta)}\, d\theta < \varepsilon. \tag{7.5}$$

Here Q_ε and R_ε are finite linear combinations of indicators of the form $1_{[a,b]}$ for $\varepsilon \leq a < b \leq 1-\varepsilon$. Define

$$G_\varepsilon(\theta) = \begin{cases} M_0 f(\theta), & 0 \leq \theta < \varepsilon, \\ 0, & \varepsilon \leq \theta \leq 1-\varepsilon, \\ M_1 f(1-\theta), & 1-\varepsilon < \theta \leq 1. \end{cases}$$

We have the following chain of inequalities

$$Q_\varepsilon(\theta) + G_\varepsilon(\theta) \geq f(\theta) \geq R_\varepsilon(\theta) - G_\varepsilon(\theta)$$

and therefore, as before,

$$\begin{aligned} &\limsup_{\varepsilon \to 0^+} \left(\limsup_{c\to\infty} \frac{1}{c^{d-1}\log(c)} \operatorname{Tr} Q_\varepsilon(S_{cA,B}) + \limsup_{c\to\infty} \frac{1}{c^{d-1}\log(c)} \operatorname{Tr} G_\varepsilon(S_{cA,B}) \right) \\ &\quad \geq \limsup_{c\to\infty} \frac{1}{c^{d-1}\log(c)} \operatorname{Tr} f(S_{cA,B}) \geq \liminf_{c\to\infty} \frac{1}{c^{d-1}\log(c)} \operatorname{Tr} f(S_{cA,B}) \\ &\quad \geq \liminf_{\varepsilon\to 0^+} \left(\liminf_{c\to\infty} \frac{1}{c^{d-1}\log(c)} \operatorname{Tr} R_\varepsilon(S_{cA,B}) - \limsup_{c\to\infty} \frac{1}{c^{d-1}\log(c)} \operatorname{Tr} G_\varepsilon(S_{cA,B}) \right). \end{aligned}$$

Since Q_ε and R_ε are finite linear combinations of indicators of the form $1_{[a,b]}$ for $0 < a < b < 1$, we know that

$$\begin{aligned} \limsup_{c\to\infty} \frac{1}{c^{d-1}\log(c)} \operatorname{Tr} Q_\varepsilon(S_{cA,B}) &= I(A,B) \int_0^1 \frac{Q_\varepsilon(\theta)}{\theta(1-\theta)}\, d\theta, \\ \liminf_{c\to\infty} \frac{1}{c^{d-1}\log(c)} \operatorname{Tr} R_\varepsilon(S_{cA,B}) &= I(A,B) \int_0^1 \frac{R_\varepsilon(\theta)}{\theta(1-\theta)}\, d\theta. \end{aligned}$$

Taking the limit $\varepsilon \to 0$, it follows from (7.5) that

$$\begin{aligned} \limsup_{\varepsilon\to 0^+} \int_0^1 \frac{Q_\varepsilon(\theta)}{\theta(1-\theta)}\, d\theta &= \int_0^1 \frac{f(\theta)}{\theta(1-\theta)}\, d\theta, \\ \liminf_{\varepsilon\to 0^+} \int_0^1 \frac{R_\varepsilon(\theta)}{\theta(1-\theta)}\, d\theta &= \int_0^1 \frac{f(\theta)}{\theta(1-\theta)}\, d\theta. \end{aligned}$$

Thus, it suffices to establish that

$$\limsup_{\varepsilon\to 0^+}\limsup_{c\to\infty}\frac{1}{c^{d-1}\log(c)}\operatorname{Tr}G_\varepsilon(S_{cA,B})=0. \tag{7.6}$$

Here we use the established bound (1.14). It gives

$$\limsup_{c\to\infty}\frac{1}{c^{d-1}\log(c)}\operatorname{Tr}G_\varepsilon(S_{cA,B})\le C(A,B)\int_0^1\frac{M_0G_\varepsilon(\theta)+M_1G_\varepsilon(\theta)}{\theta}d\theta.$$

For $\theta<\varepsilon$ we have $M_0G_\varepsilon(\theta)=M_0f(\theta)$ and $M_1G_\varepsilon(\theta)=M_1f(\theta)$, thus

$$M_0G_\varepsilon(\theta)+M_1G_\varepsilon(\theta)\le M_0f(\theta)+M_1f(\theta).$$

For $\varepsilon\le\theta<1$ we have

$$M_0G_\varepsilon(\theta)+M_1G_\varepsilon(\theta)\le 2\max(M_0f(\varepsilon),M_1f(\varepsilon))\le 2(M_0f(\theta)+M_1f(\theta)). \tag{7.7}$$

We get that $\frac{M_0G_\varepsilon(\theta)+M_1G_\varepsilon(\theta)}{\theta(1-\theta)}$ is majorized by the L^1-function $\frac{2(M_0f(\theta)+M_1f(\theta))}{\theta(1-\theta)}$. Thus, if we show that it tends to zero pointwise as $\varepsilon\to 0^+$ then the dominated convergence theorem will imply (7.6). By (7.7) it suffices to show that

$$\max(M_0f(\varepsilon),M_1f(\varepsilon))\to 0 \text{ as } \varepsilon\to 0^+. \tag{7.8}$$

Clearly, $\max(M_0f(\varepsilon),M_1f(\varepsilon))$ is non-negative and non-decreasing in ε, thus the right limit $\lim_{\varepsilon\to 0^+}\max(M_0f(\varepsilon),M_1f(\varepsilon))=\nu$ exists and is non-negative. If $\nu>0$ then the integral

$$\int_0^1\frac{M_0f(\theta)+M_1f(\theta)}{\theta}d\theta$$

diverges, contradicting the assumption that f is area law admissible. Thus, $\nu=0$ giving us (7.6).

8. COMPUTATION OF $\operatorname{Tr}S_{A,B}^2$ WHEN A AND B ARE FINITE UNIONS OF BOXES

In this section we prove Theorem 1.14 and Theorem 1.15. The key idea of the computation is that $\operatorname{Tr}S_{A,B}^2=\|S_{A,B}\|_2^2$ is the Hilbert–Schmidt norm squared of $S_{A,B}$. Since $S_{A,B}$ is an operator with the kernel (1.1), we have

$$\|S_{A,B}\|_2^2=\iint_{\mathbb{R}^d\times\mathbb{R}^d}|1_A(x)\check{1}_B(x-y)1_A(y)|^2\,dx\,dy.$$

Assume that $A = \cup_{k=1}^{n} A_k$ and $B = \cup_{l=1}^{m} B_l$ are finite unions of axis-parallel boxes with disjoint interiors. We have

$$\begin{aligned}
&\iint_{\mathbb{R}^d \times \mathbb{R}^d} |1_A(x)\check{1}_B(x-y)1_A(y)|^2 \, dx \, dy \\
&\qquad = \sum_{k_1,k_2=1}^{n} \sum_{l_1,l_2=1}^{m} \iint_{\mathbb{R}^d \times \mathbb{R}^d} 1_{A_{k_1}}(x)1_{A_{k_2}}(y)\check{1}_{B_{l_1}}(x-y)\check{1}_{B_{l_2}}(y-x) \, dx \, dy.
\end{aligned}$$

For each k_1, k_2, l_1, l_2 the variables separate and we reduce to the integrals of the form

$$\iint_{\mathbb{R}\times\mathbb{R}} 1_{I_1}(x)1_{I_2}(y)\check{1}_{J_1}(x-y)\check{1}_{J_2}(y-x)dx \, dy$$

for some intervals $I_1, I_2, J_1, J_2 \subset \mathbb{R}$. Doing the change of variables $x - y = z$ and applying Fubini's theorem we get

$$\iint_{\mathbb{R}\times\mathbb{R}} 1_{I_1}(x)1_{I_2}(x+z)\check{1}_{J_1}(z)\check{1}_{J_2}(-z)dx \, dz = \int_{\mathbb{R}} \check{1}_{J_1}(z)\check{1}_{J_2}(-z)|I_1 \cap (I_2 - z)|dz. \tag{8.1}$$

We do exactly the same on the Fourier side with the intervals J_1 and J_2. For each $z \in \mathbb{R}$

$$\check{1}_{J_1}(z)\check{1}_{J_2}(-z) = \iint_{\mathbb{R}\times\mathbb{R}} 1_{J_1}(u)1_{J_2}(v)e^{2\pi i z(u-v)} \, du \, dv = \int_{\mathbb{R}} e^{2\pi i z w}|J_1 \cap (J_2 - w)| \, dw.$$

Plugging this into (8.1) we get

$$\iint_{\mathbb{R}\times\mathbb{R}} e^{2\pi i z w}|I_1 \cap (I_2 - z)||J_1 \cap (J_2 - w)| \, dz \, dw \tag{8.2}$$

For fixed intervals I, I' the function $\mathbb{R} \ni t \to I \cap (I' - t)$ is zero up to some point p_1, then linear up to some p_2, then constant up to some p_3, linear again up to some p_4 and zero afterwards (if $|I| = |J|$ then $p_2 = p_3$). Thus, separating into the nine cases with respect to the pairs I_1, I_2 and J_1, J_2 we have to compute the integrals of the form

$$\int_a^b \int_c^d (\alpha z + \beta)(\gamma w + \delta)e^{2\pi i z w} dz dw.$$

A direct computation shows

$$\begin{aligned}
&\int_a^b \int_c^d (\alpha z + \beta)(\gamma w + \delta)e^{2\pi i z w} \, dz \, dw \\
&\qquad = \int_a^b (\gamma w + \delta) \left(\frac{(\alpha d + \beta)e^{2\pi i w d} - (\alpha c + \beta)e^{2\pi i w c}}{2\pi i w} + \frac{\alpha\big(e^{2\pi i w d} - e^{2\pi i w c}\big)}{4\pi^2 w^2} \right) dw.
\end{aligned}$$

This integral can be computed explicitly with the use of the exponential integral function $E_1(w)$, because $\int \frac{e^{ir}}{r^2} dr = -\frac{e^{ir}}{r} + i \int \frac{e^{ir}}{r}$. If $0 \in [a, b]$ then one has to exercise a bit of care, removing a small interval $[-\varepsilon, \varepsilon]$, integrating over the resulting segments and letting $\varepsilon \to 0$

with the use of the known asymptotics of $E_1(w)$ for small w which will give the logarithmic terms.

Unfortunately, executing this strategy in practice is rather infeasible due to the number of terms appearing, so we will only do the simplest one-dimensional case $A = [0, c], B = [0, 1]$. Our starting point will be (8.2) with $I_1 = I_2 = [0, c]$ and $J_1 = J_2 = [0, 1]$. We find that

$$\begin{aligned}\operatorname{Tr} S^2_{[0,c],[0,1]} &= \iint_{\mathbb{R}\times\mathbb{R}} e^{2\pi i z w}\ |[0,c]\cap([0,c]-z)|\ |[0,1]\cap([0,1]-w)|\ dz\,dw\\ &= \int_{-1}^{1}\int_{-c}^{c} e^{2\pi i z w}\,(c-|z|)\,(1-|w|)\,dz\,dw.\end{aligned}$$

Using Fubini's theorem and taking the integral in w we get

$$\operatorname{Tr} S^2_{[0,c],[0,1]} = \int_{-c}^{c}(c-|z|)\frac{\sin(\pi z)^2}{\pi^2 z^2}dz = 2\int_0^c (c-z)\frac{\sin(\pi z)^2}{\pi^2 z^2}dz = \lim_{\varepsilon\to 0^+} 2\int_\varepsilon^c (c-z)\frac{\sin(\pi z)^2}{\pi^2 z^2}dz.$$

Using $\sin^2(t) = \frac{1-\cos(2t)}{2}$ and $\int \frac{\cos(t)}{t^2} = -\frac{\cos(t)}{t} - \int \frac{\sin(t)}{t}$ we can find an explicit primitive and get

$$2\int_\varepsilon^c (c-z)\frac{\sin(\pi z)^2}{\pi^2 z^2}dz = \left.\frac{2\pi c z \mathrm{Si}(2\pi z) + c\cos(2\pi z) - c + z\mathrm{Ci}(2\pi z) - z\log(z)}{\pi^2 z}\right|_\varepsilon^c.$$

The limit of the primitive as $\varepsilon \to 0$ is $\frac{\gamma+\log(2\pi)}{\pi^2}$ using the known asymptotics of $\mathrm{Ci}(z)$ for small z. So,

$$\operatorname{Tr} S^2_{[0,c],[0,1]} = c - \frac{\log(c)}{\pi^2} - \frac{1+\gamma+\log(2\pi)}{\pi^2} + \left(\frac{2}{\pi}c\left(\mathrm{Si}(2\pi c) - \frac{\pi}{2}\right) + \frac{\cos(2\pi c)}{\pi^2} + \frac{\mathrm{Ci}(2\pi c)}{\pi^2}\right),$$

as required. Recall the known asymptotics for $\mathrm{Si}(t)$ and $\mathrm{Ci}(t)$ for large t

$$\mathrm{Si}(t) = \frac{\pi}{2} - \cos(t)\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{t^{2n+1}} + \sin(t)\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!}{t^{2n}}, \tag{8.3}$$

$$\mathrm{Ci}(t) = \sin(t)\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{t^{2n+1}} + \cos(t)\sum_{n=1}^{\infty}\frac{(-1)^n(2n-1)!}{t^{2n}}. \tag{8.4}$$

We note that the first term in the first sum in (8.3) cancels with $\frac{\cos(2\pi c)}{\pi^2}$ and the first term in the second sum in (8.3) cancels with the first term in the first sum in (8.4). Thus, the oscillating terms begin with $O(\frac{1}{c^2})$ term. Plugging these series into our formula gives the desired asymptotic expression.

9. One-term asymptotics

To establish one-term asymptotic (1.16) in general we first need to establish it for any two qualitatively different functions directly. For $f(\theta) = \theta$ we already know it, even without any error terms, so we will consider $f(\theta) = \theta^2$.

Claim 9.1. *Let $A, B \subseteq \mathbb{R}^d$ be sets with finite measure. Then $S^2_{cA,B}$ is trace class and*

$$\operatorname{Tr} S^2_{cA,B} = c^d|A||B| + o(c^d)$$

as $c \to \infty$.

Proof. It suffices to show that $\operatorname{Tr}\left[S_{cA,B} - S^2_{cA,B}\right] = o(c^d)$. By the $S - S^2$ trick we have

$$S_{cA,B} - S^2_{cA,B} = \left|P_{cA}Q_BP_{cA^c}\right|^2,$$

so the trace is given by the Hilbert–Schmidt norm squared of $P_{cA}Q_BP_{cA^c}$. Arguing like in the previous section, we find

$$\|P_{cA}Q_BP_{cA^c}\|_2^2 = c^d \int_{\mathbb{R}^d} \left|\check{1}_B(x)\right|^2 \left|A \cap \left(A^c - \frac{x}{c}\right)\right| \, dx.$$

The function $F_A(x) = |A \cap (A^c - x)| = 1_A * 1_{-A^c}(-x)$ is continuous, bounded, and satisfies $F_A(0) = 0$. The claim now follows immediately from the dominated convergence theorem. ☐

Proof of Theorem 1.16. Subtracting $\theta f(1)$ from f we can without loss of generality assume that $f(1) = 0$. Pick a small $\frac{1}{2} > \varepsilon > 0$. We split $\operatorname{Tr} f(S_{cA,B})$ into the terms with $\lambda_n > 1 - \varepsilon$ and $1 - \varepsilon \geq \lambda_n \geq 0$:

$$\operatorname{Tr} f(S_{cA,B}) = \sum_{\lambda_n(cA,B) > 1-\varepsilon} f(\lambda_n(cA,B)) + \sum_{1-\varepsilon \geq \lambda_n(cA,B) \geq 0} f(\lambda_n(cA,B)).$$

For the second sum we simply apply the linear bound $|f(\theta)| \leq C\theta$ and the elementary inequality $\theta \leq \frac{\theta(1-\theta)}{\varepsilon}$, valid for $0 \leq \theta \leq 1 - \varepsilon$. We find

$$\left| \sum_{1-\varepsilon \geq \lambda_n(cA,B) \geq 0} f(\lambda_n(cA,B)) \right| \leq \frac{C}{\varepsilon} \operatorname{Tr}\left[S_{cA,B} - S^2_{cA,B}\right],$$

which is $o(c^d)$ for all fixed $0 < \varepsilon < \frac{1}{2}$ by Claim 9.1.

For the first sum, by continuity of f at 1, for any $\delta > 0$ there is ε small enough such that $|f(\lambda_n(cA,B))| < \delta$ if $\lambda_n(cA,B) > 1 - \varepsilon$. Since $0 < \varepsilon < \frac{1}{2}$, this implies that $|f(\lambda_n(cA,B)| < 2\delta\lambda_n(cA,B)$. Thus, the first sum is at most $2\delta c^d|A||B|$. Since $\delta > 0$ can be arbitrarily small, this gives us the result. ☐

Now, we turn to the proof of Theorem 1.18. This time, we will do a more complicated splitting of $\mathrm{Tr} f(S_{cA,B})$.

Proof of Theorem 1.18. As in the previous proof we can without loss of generality assume that $f(1) = 0$. We also remark that since f is trace class admissible for $L^2(\mathbb{R}^d)$, f must satisfy $\lim_{\theta \to 0^+} f(\theta) = 0$, by an argument similar to the proof of (7.8), just with the trace class admissible for $L^2(\mathbb{R}^d)$ condition instead of the area law admissible condition.

Since A, B are bounded, there exist boxes V_A, V_B such that $A \subseteq V_A$, $B \subseteq V_B$. Fix once and for all a large number D to be determined later (it will only depend on V_A, V_B). Let $0 < \varepsilon < \frac{1}{2}$ be a small number. We have

$$\begin{aligned}\mathrm{Tr} f(S_{cA,B}) = &\sum_{\lambda_n(cA,B) > 1-\varepsilon} f(\lambda_n(cA,B)) + \sum_{1-\varepsilon \geq \lambda_n(cA,B) \geq \varepsilon} f(\lambda_n(cA,B)) \\ &+ \sum_{\substack{0 \leq \lambda_n(cA,B) < \varepsilon \\ n \leq Dc^d}} f(\lambda_n(cA,B)) + \sum_{\substack{0 \leq \lambda_n(cA,B) < \varepsilon \\ n > Dc^d}} f(\lambda_n(cA,B)).\end{aligned}$$

For the first sum for any $\delta > 0$ we can choose $\frac{1}{2} > \varepsilon > 0$ small enough so that it is at most $2\delta |A||B| c^d$, as in the previous proof.

For the second sum we simply use that f is bounded and the elementary inequality $1 \leq \frac{\theta(1-\theta)}{\varepsilon(1-\varepsilon)}$ for $\theta \in [\varepsilon, 1-\varepsilon]$. Thus, the second sum is at most $\frac{\mathrm{Tr}[S_{cA,B} - S_{cA,B}^2]}{\varepsilon(1-\varepsilon)} \sup_{t \in [0,1]} |f(t)|$, which is $o(c^d)$ by Claim 9.1.

For the third sum, we use that $\lim_{\theta \to 0^+} f(\theta) = 0$ to conclude that for any $\delta > 0$ we can choose $\varepsilon > 0$ small enough so that it is at most $\delta D c^d$.

Finally, we turn to the most challenging fourth sum. Firstly, we estimate $|f(\lambda_n(cA,B))|$ by $M_0 f(\lambda_n(cA,B))$. Since $M_0 f$ is non-decreasing, we can upper bound $\lambda_n(cA,B)$ by $\lambda_n(cV_A, V_B)$ using Lemma 4.8. So, it remains to show that

$$\sum_{n > Dc^d} M_0 f(\lambda_n(cV_A, V_B)) = o(c^d).$$

We claim that if D is large enough and $n > Dc^d$, then $\lambda_n(cV_A, V_B) \leq \alpha_d^{-c}$, where α_d is taken from Theorem 1.1. This is equivalent to saying that $N_{\alpha_d^{-c}}(cV_A, V_B) \leq Dc^d$. We have

$$N_{\alpha_d^{-c}}(cV_A, V_B) = N_{1/2}(cV_A, V_B) + \Lambda_{\alpha_d^{-c}}(cV_A, V_B).$$

The first term is $O(c^d)$ by (3.2) and the second term is $O(c^d)$ by (1.6). Thus, $N_{\alpha_d^{-c}}(cV_A, V_B) \leq Dc^d$ holds for large enough D. Given the claim, the computation (7.4) therefore shows that

$$\sum_{n > Dc^d} M_0 f(\lambda_n(cV_A, V_B)) \leq \sum_{\lambda_n(cV_a, V_B) \leq \alpha_d^{-c}} M_0 f(\lambda_n(cV_A, V_B)) = O(\log(c)^d) = o(c^d),$$

which finishes the proof. ☐

9.1. **Counterexample for unbounded sets.** In this subsection we prove Proposition 1.17. First, for convenience we replace f with a monotone subordinate function g.

Claim 9.2. *For any function* $f : [0,1] \to \mathbb{C}$ *such that* $\lim_{x\to 0^+} \frac{|f(x)|}{x} = \infty$ *there exist* $\varepsilon > 0$ *and* $g : [0,1] \to [0,\infty)$ *such that g is non-decreasing,* $\lim_{x\to 0^+} \frac{g(x)}{x} = \infty$ *and* $g(x) \le |f(x)|$ *for* $0 \le x \le \varepsilon$.

Proof. Since $\lim_{x\to 0^+} \frac{|f(x)|}{x} = \infty$, for any $k \in \mathbb{N}$ there exists $y_k > 0$ such that for $0 < x \le y_k$ we have $|f(x)| \ge 2^k x$. Inductively making y_k smaller if necessary we can additionally assume that $y_k \ge 2y_{k+1}$. We define $g(y_k) = 2^{k-1} y_k$, between y_k and y_{k+1} we extend g linearly, and for $x \in [y_1, 1]$ we set $g(x) = g(y_1)$. We are going to show that g is non-decreasing, $\lim_{x\to 0^+} \frac{g(x)}{x} = \infty$ and $g(x) \le |f(x)|$ for $0 < x \le y_1$.

For the first assertion it is enough to check that $g(y_k) \ge g(y_{k+1})$ since linear functions do not change monotonicity. This is equivalent to $y_k \ge 2y_{k+1}$ which is true by our assumption. For the second assertion, we have

$$\inf_{x\in[y_k,y_{k+1}]} \frac{g(x)}{x} = \frac{g(y_k)}{y_k} = 2^{k-1},$$

which tends to infinity as $k \to \infty$, thus $\lim_{x\to 0^+} \frac{g(x)}{x} = \infty$. Analogously, we also have

$$\sup_{x\in[y_k,y_{k+1}]} \frac{g(x)}{x} = \frac{g(y_{k+1})}{y_{k+1}} = 2^k,$$

thus $g(x) \le 2^k x$ for $x \in [y_k, y_{k+1}]$. On the other hand, for $x \in [y_k, y_{k+1}]$ we have $|f(x)| \ge 2^k x$, which gives us the last assertion. Finally, we put $g(0) = 0$ to complete the proof. ☐

For any sets A, B of finite measure we clearly have that if $f(S_{A,B})$ is trace class then $g(S_{A,B})$ is trace class. Thus, given a set B of positive and finite measure, it suffices to construct a set A such that $g(S_{A,B})$ is not trace class, or equivalently $\operatorname{Tr} g(S_{cA,B}) = \infty$.

We will take $A = \bigcup_{k\in\mathbb{N}} A_k$ where each A_k has measure $\frac{1}{2^k}$. By monotonicity, $\operatorname{Tr} g(S_{A,B}) \ge \operatorname{Tr} g(S_{A_k,B})$, thus it suffices to make $\operatorname{Tr} g(S_{A_k,B})$ tend to infinity. Since $\lim_{x\to 0^+} \frac{g(x)}{x} = \infty$, for each k there exists $x_k > 0$ such that $g(x) \ge 4^k x$ for $0 < x \le x_k$. If $S_{A_k,B}$ does not have eigenvalues larger than x_k then we have

$$\operatorname{Tr} g(S_{A_k,B}) = \sum_{n=1}^{\infty} g(\lambda_n(A_k, B)) \ge \sum_{n=1}^{\infty} 4^k \lambda_n(A_k, B) = \frac{4^k}{2^k}|B| = 2^k|B|,$$

which tends to infinity as $k \to \infty$.

So, it remains to construct sets A_k of measure $\frac{1}{2^k}$ such that $S_{A_k,B}$ does not have eigenvalues larger than x_k. This is equivalent to demanding that $\lambda_1(A_k,B) \le x_k$. We clearly have

$$\lambda_1(A_k,B) \le \left(\sum_{n=1}^{\infty} \lambda_n(A_k,B)^2\right)^{1/2} = \left(\mathrm{Tr} S^2_{A_k,B}\right)^{1/2}.$$

Thus, if we can make $\mathrm{Tr} S^2_{A_k,B}$ as small as we like, we will get the desired inequality. We pick a large number N and let A_k be a union of N vastly separated intervals of length $\frac{1}{N2^k}$:

$$A_k = \bigcup_{m=1}^{N} \left[Nm, Nm + \frac{1}{N2^k}\right].$$

We have

$$\mathrm{Tr} S^2_{A_k,B} = \int_{\mathbb{R}} |\check{1}_B(x)|^2 \, |A_k \cap (A_k - x)| \; dx.$$

Direct inspection shows that for $\frac{1}{N2^k} < |x| < N - \frac{1}{N2^k}$ we have $A_k \cap (A_k - x) = \varnothing$ and for x outside this range the measure of the intersection is clearly at most $\frac{1}{2^k}$. Thus,

$$\mathrm{Tr} S^2_{A_k,B} \le \frac{1}{2^k} \int_{|x| < \frac{1}{N2^k}} |\check{1}_B(x)|^2 dx + \frac{1}{2^k} \int_{|x| > N - \frac{1}{N2^k}} |\check{1}_B(x)|^2 dx.$$

As $N \to \infty$, by the dominated convergence theorem both of these integrals go to 0. Thus, we can make $\mathrm{Tr} S^2_{A_k,B}$ as small as we like by taking N large enough, as required.

Acknowledgments. Aleksei Kulikov and Martin Dam Larsen were supported by the VILLUM Centre of Excellence for the Mathematics of Quantum Theory (QMATH) with Grant No.10059.

REFERENCES

[1] Luigi Ambrosio, Nicola Fusco, and Diego Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[2] W. O. Amrein and A. M. Berthier, *On support properties of L^p-functions and their Fourier transforms*, J. Functional Analysis **24** (1977), no. 3, 258–267.

[3] Michael Benedicks, *On Fourier transforms of functions supported on sets of finite Lebesgue measure*, J. Math. Anal. Appl. **106** (1985), no. 1, 180–183.

[4] A. Beurling and P. Malliavin, *On Fourier transforms of measures with compact support*, Acta Math. **107** (1962), 291–309.

[5] M. Sh. Birman and M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*, Mathematics and its Applications (Soviet Series), D. Reidel Publishing Co., Dordrecht, 1987. Translated from the 1980 Russian original by S. Khrushchëv and V. Peller.

[6] Aline Bonami and Abderrazek Karoui, *Spectral decay of time and frequency limiting operator*, Appl. Comput. Harmon. Anal. **42** (2017), no. 1, 1–20.

[7] ———, *Random discretization of the finite fourier transform and related kernel random matrices*, J. Fourier Anal. Appl. **26** (2020), no. 2, Paper No. 29, 29.

[8] J. Eisert, M. Cramer, and M. B. Plenio, *Colloquium: Area laws for the entanglement entropy*, Rev. Modern Phys. **82** (2010), no. 1, 277–306.

[9] Søren Fournais, Martin Dam Larsen, Robert Seiringer, and Jan Philip Solovej, *The Widom conjecture with optimal regularity assumptions.* In preparation.

[10] W. H. J. Fuchs, *On the eigenvalues of an integral equation arising in the theory of band-limited signals*, J. Math. Anal. Appl. **9** (1964), 317–330.

[11] Dimitri Gioev and Israel Klich, *Entanglement entropy of fermions in any dimension and the Widom conjecture*, Phys. Rev. Lett. **96** (2006), no. 10, 100503, 4.

[12] Luis Gomez, Jonathan Jaimangal, Azita Mayeli, and Tasfia Proma, *Eigenvalue distribution analysis of multidimensional prolate matrices* (2025), arXiv:2507.10412.

[13] Loukas Grafakos, *Classical Fourier analysis*, Second, Graduate Texts in Mathematics, vol. 249, Springer, New York, 2008.

[14] Kevin Hughes, Arie Israel, and Azita Mayeli, *On the eigenvalue distribution of spatio-spectral limiting operators in higher dimensions, II*, J. Fourier Anal. Appl. **31** (2025), no. 4, Paper No. 51, 22.

[15] ———, *Wave packets and eigenvalue estimates for limiting operators on the disk* (2026), arXiv:2601.21224.

[16] Arie Israel, *The eigenvalue distribution of time-frequency localization operators* (2015), arXiv:1502.04404.

[17] Arie Israel and Azita Mayeli, *On the eigenvalue distribution of spatio-spectral limiting operators in higher dimensions*, Appl. Comput. Harmon. Anal. **70** (2024), Paper No. 101620, 28.

[18] Santhosh Karnik, Justin Romberg, and Mark A. Davenport, *Improved bounds for the eigenvalues of prolate spheroidal wave functions and discrete prolate spheroidal sequences*, Appl. Comput. Harmon. Anal. **55** (2021), 97–128.

[19] Alexei Kitaev and John Preskill, *Topological entanglement entropy*, Phys. Rev. Lett. **96** (2006), no. 11, 110404, 4.

[20] Aleksei Kulikov, *Exponential lower bound for the eigenvalues of the time-frequency localization operator before the plunge region*, Appl. Comput. Harmon. Anal. **71** (2024), Paper No. 101639, 8.

[21] ———, *Sharp estimates for eigenvalues of localization operators before the plunge region* (2026), arXiv:2603.07407.

[22] H. J. Landau and H. Widom, *Eigenvalue distribution of time and frequency limiting*, J. Math. Anal. Appl. **77** (1980), no. 2, 469–481.

[23] Felipe Marceca, José Luis Romero, and Michael Speckbacher, *Eigenvalue estimates for Fourier concentration operators on two domains*, Arch. Ration. Mech. Anal. **248** (2024), no. 3, Paper No. 35, 31.

[24] Andrei Osipov, *Certain upper bounds on the eigenvalues associated with prolate spheroidal wave functions*, Appl. Comput. Harmon. Anal. **35** (2013), no. 2, 309–340.

[25] David Slepian, *Some asymptotic expansions for prolate spheroidal wave functions*, J. Math. and Phys. **44** (1965), 99–140.

[26] A. V. Sobolev, *Pseudo-differential operators with discontinuous symbols: Widom's conjecture*, Mem. Amer. Math. Soc. **222** (2013), no. 1043, vi+104.

[27] Alexander V. Sobolev, *On the Schatten-von Neumann properties of some pseudo-differential operators*, J. Funct. Anal. **266** (2014), no. 9, 5886–5911.

[28] ———, *Functions of self-adjoint operators in ideals of compact operators*, J. Lond. Math. Soc. (2) **95** (2017), no. 1, 157–176.

[29] Harold Widom, *On a class of integral operators with discontinuous symbol*, Toeplitz centennial (Tel Aviv, 1981), 1982, pp. 477–500.

ALEKSEI KULIKOV,
UNIVERSITY OF COPENHAGEN, DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN, DENMARK,
lyosha.kulikov@mail.ru

MARTIN DAM LARSEN,
UNIVERSITY OF COPENHAGEN, DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITETSPARKEN 5, 2100 COPENHAGEN, DENMARK,
mdl@math.ku.dk

Bibliography

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000, pp. xviii+434.

[2] E. L. Basor and K. E. Morrison. "The Fisher-Hartwig conjecture and Toeplitz eigenvalues". *Linear Algebra Appl.* **202** (1994), pp. 129–142.

[3] L. Bollmann and P. Müller. "The Widom-Sobolev formula for discontinuous matrix-valued symbols". *J. Funct. Anal.* **287** (2024), Paper No. 110651, 54.

[4] L. Bollmann and P. Müller. "Enhanced area law in the Widom-Sobolev formula for the free Dirac operator in arbitrary dimension". *Pure Appl. Anal.* **7** (2025), pp. 595–613.

[5] A. Bonami and A. Karoui. "Spectral decay of time and frequency limiting operator". *Appl. Comput. Harmon. Anal.* **42** (2017), pp. 1–20.

[6] A. Böttcher. "The Onsager formula, the Fisher-Hartwig conjecture, and their influence on research into Toeplitz operators". *J. Statist. Phys.* **78** (1995). Papers dedicated to the memory of Lars Onsager, pp. 575–584.

[7] C. Brislawn. "Kernels of trace class operators". *Proc. Amer. Math. Soc.* **104** (1988), pp. 1181–1190.

[8] C. Coine, C. Le Merdy, D. Potapov, F. Sukochev, and A. Tomskova. "Resolution of Peller's problem concerning Koplienko-Neidhardt trace formulae". *Proc. Lond. Math. Soc. (3)* **113** (2016), pp. 113–139.

[9] P. Deift, A. Its, and I. Krasovsky. "Asymptotics of Toeplitz, Hankel, and Toeplitz + Hankel determinants with Fisher-Hartwig singularities". *Ann. of Math. (2)* **174** (2011), pp. 1243–1299.

[10] E. Di Nezza, G. Palatucci, and E. Valdinoci. "Hitchhiker's guide to the fractional Sobolev spaces". *Bull. Sci. Math.* **136** (2012), pp. 521–573.

[11] T. Ehrhardt. "A status report on the asymptotic behavior of Toeplitz determinants with Fisher-Hartwig singularities". In: *Recent advances in operator theory (Groningen, 1998)*. Vol. 124. Oper. Theory Adv. Appl. Birkhäuser, Basel, 2001, pp. 217–241.

[12] J. Eisert, M. Cramer, and M. B. Plenio. "Colloquium: Area laws for the entanglement entropy". *Rev. Modern Phys.* **82** (2010), pp. 277–306.

[13] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Revised. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015, pp. xiv+299.

[14] H. Federer. *Geometric measure theory*. Springer-Verlag New York, Inc., New York, 1969, pp. xiv+676.

[15] F. Ferro, P. Pfeiffer, and W. Spitzer. "Entanglement entropy in the ground state of non-interacting massless Dirac fermions in dimension one". *Adv. Math.* **482** (2025), Paper No. 110562, 37.

[16] F. Finster, M. Lottner, and A. V. Sobolev. "The fermionic entanglement entropy and area law for the relativistic Dirac vacuum state". *Adv. Theor. Math. Phys.* **28** (2024), pp. 1933–1985.

[17] M. E. Fisher and R. E. Hartwig. "Toeplitz Determinants: Some Applications, Theorems, and Conjectures". In: *Advances in Chemical Physics*. John Wiley & Sons, Ltd, 1969, pp. 333–353.

[18] S. Fournais, M. D. Larsen, R. Seiringer, and J. P. Solovej. *The Widom Conjecture with optimal regularity assumptions*. 2026.

[19] W. H. J. Fuchs. "On the eigenvalues of an integral equation arising in the theory of band-limited signals". *J. Math. Anal. Appl.* **9** (1964), pp. 317–330.

[20] D. Gioev. "Szegö limit theorem for operators with discontinuous symbols and applications to entanglement entropy". *Int. Math. Res. Not.* (2006), Art. ID 95181, 23.

[21] D. Gioev and I. Klich. "Entanglement entropy of fermions in any dimension and the Widom conjecture". *Phys. Rev. Lett.* **96** (2006), pp. 100503, 4.

[22] U. Grenander and G. Szegö. *Toeplitz forms and their applications*. California Monographs in Mathematical Sciences. University of California Press, Berkeley-Los Angeles, 1958, pp. vii+245.

[23] R. Helling, H. Leschke, and W. Spitzer. "A special case of a conjecture by Widom with implications to fermionic entanglement entropy". *Int. Math. Res. Not. IMRN* (2011), pp. 1451–1482.

[24] R. Helling, H. Leschke, and W. Spitzer. "A special case of a conjecture by Widom with implications to fermionic entanglement entropy". *Int. Math. Res. Not. IMRN* (2011), pp. 1451–1482.

[25] L. Hörmander. *The analysis of linear partial differential operators. I*. Second. Vol. 256. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, 1990, pp. xii+440.

[26] K. Hughes, A. Israel, and A. Mayeli. "On the eigenvalue distribution of spatio-spectral limiting operators in higher dimensions, II". *J. Fourier Anal. Appl.* **31** (2025), Paper No. 51, 22.

[27] A. Israel and A. Mayeli. "On the eigenvalue distribution of spatio-spectral limiting operators in higher dimensions". *Appl. Comput. Harmon. Anal.* **70** (2024), Paper No. 101620, 28.

[28] M. Kac. "Toeplitz matrices, translation kernels and a related problem in probability theory". *Duke Math. J.* **21** (1954), pp. 501–509.

[29] M. Kac, W. L. Murdock, and G. Szegö. "On the eigenvalues of certain Hermitian forms". *J. Rational Mech. Anal.* 2 (1953), pp. 767–800.

[30] A. Y. Karlovich. "Asymptotics of determinants and traces of Toeplitz matrices with symbols in weighted Wiener algebras". *Z. Anal. Anwend.* **26** (2007), pp. 43–56.

[31] S. Karnik, J. Romberg, and M. A. Davenport. “Improved bounds for the eigenvalues of prolate spheroidal wave functions and discrete prolate spheroidal sequences”. *Appl. Comput. Harmon. Anal.* **55** (2021), pp. 97–128.

[32] A. Kulikov. “Sharp estimates for eigenvalues of localization operators before the plunge region” (2026). arXiv: `2603.07407`.

[33] A. Kulikov and M. D. Larsen. *Sharp estimates for eigenvalues of localization operators with applications to area laws*. 2026. arXiv: `2603.23832`.

[34] H. J. Landau and H. O. Pollak. “Prolate spheroidal wave functions, Fourier analysis and uncertainty. II”. *Bell System Tech. J.* **40** (1961), pp. 65–84.

[35] H. J. Landau and H. O. Pollak. “Prolate spheroidal wave functions, Fourier analysis and uncertainty. III. The dimension of the space of essentially time- and band-limited signals”. *Bell System Tech. J.* **41** (1962), pp. 1295–1336.

[36] H. J. Landau and H. Widom. “Eigenvalue distribution of time and frequency limiting”. *J. Math. Anal. Appl.* **77** (1980), pp. 469–481.

[37] H. Leschke, A. V. Sobolev, and W. Spitzer. “Scaling of Rényi Entanglement Entropies of the Free Fermi-Gas Ground State: A Rigorous Proof”. *Phys. Rev. Lett.* **112** (16 Apr. 2014), p. 160403.

[38] H. Leschke, A. V. Sobolev, and W. Spitzer. “Large-scale behaviour of local and entanglement entropy of the free Fermi gas at any temperature”. *J. Phys. A* **49** (2016), 30LT04, 9.

[39] H. Leschke, A. V. Sobolev, and W. Spitzer. “Trace formulas for Wiener-Hopf operators with applications to entropies of free fermionic equilibrium states”. *J. Funct. Anal.* **273** (2017), pp. 1049–1094.

[40] H. Leschke, A. V. Sobolev, and W. Spitzer. “Asymptotic growth of the local ground-state entropy of the ideal Fermi gas in a constant magnetic field”. *Comm. Math. Phys.* **381** (2021), pp. 673–705.

[41] H. Leschke, A. V. Sobolev, and W. Spitzer. “Rényi entropies of the free Fermi gas in multi-dimensional space at high temperature”. In: *Toeplitz operators and random matrices—in memory of Harold Widom*. Vol. 289. Oper. Theory Adv. Appl. Birkhäuser/Springer, Cham, [2022] ©2022, pp. 477–508.

[42] E. H. Lieb and M. Loss. *Analysis*. Second. Vol. 14. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001, pp. xxii+346.

[43] F. Maggi. *Sets of finite perimeter and geometric variational problems*. Vol. 135. Cambridge Studies in Advanced Mathematics. An introduction to geometric measure theory. Cambridge University Press, Cambridge, 2012, pp. xx+454.

[44] F. Marceca, J. L. Romero, and M. Speckbacher. “Eigenvalue estimates for Fourier concentration operators on two domains”. *Arch. Ration. Mech. Anal.* **248** (2024), Paper No. 35, 31.

[45] P. Müller, L. Pastur, and R. Schulte. “How much delocalisation is needed for an enhanced area law of the entanglement entropy?” *Comm. Math. Phys.* **376** (2020), pp. 649–679.

[46] J. P. Oldfield. “Two-term Szegő theorem for generalised anti-Wick operators”. *J. Spectr. Theory* **5** (2015), pp. 751–781.

[47] V. V. Peller. “Besov spaces in operator theory”. *Uspekhi Mat. Nauk* **79** (2024), pp. 3–58.

[48] P. Pfeiffer and W. Spitzer. “Entanglement entropy of ground states of the three-dimensional ideal Fermi gas in a magnetic field”. *Ann. Henri Poincaré* **25** (2024), pp. 3649–3698.

[49] P. Pfeiffer and W. Spitzer. “Logarithmically enhanced area-laws for fermions in vanishing magnetic fields in dimension two”. *Integral Equations Operator Theory* **96** (2024), Paper No. 28, 64.

[50] B. Pfirsch. “A Szegő limit theorem for translation-invariant operators on polygons”. *Math. Nachr.* **292** (2019), pp. 1567–1594.

[51] R. Roccaforte. “Asymptotic expansions of traces for certain convolution operators”. *Trans. Amer. Math. Soc.* **285** (1984), pp. 581–602.

[52] R. Roccaforte. “The volume of non-uniform tubular neighborhoods and an application to the n-dimensional Szegö theorem”. *J. Math. Anal. Appl.* **398** (2013), pp. 61–67.

[53] D. Slepian and H. O. Pollak. “Prolate spheroidal wave functions, Fourier analysis and uncertainty. I”. *Bell System Tech. J.* **40** (1961), pp. 43–63.

[54] A. V. Sobolev. “Pseudo-differential operators with discontinuous symbols: Widom’s conjecture”. *Mem. Amer. Math. Soc.* **222** (2013), pp. vi+104.

[55] A. V. Sobolev. “Wiener-Hopf operators in higher dimensions: the Widom conjecture for piece-wise smooth domains”. *Integral Equations Operator Theory* **81** (2015), pp. 435–449.

[56] A. V. Sobolev. “On the Schatten-von Neumann properties of some pseudo-differential operators”. *J. Funct. Anal.* **266** (2014), pp. 5886–5911.

[57] A. V. Sobolev. “On a coefficient in trace formulas for Wiener-Hopf operators”. *J. Spectr. Theory* **6** (2016), pp. 1021–1045.

[58] A. V. Sobolev. “Functions of self-adjoint operators in ideals of compact operators”. *J. Lond. Math. Soc. (2)* **95** (2017), pp. 157–176.

[59] A. V. Sobolev. “Quasi-classical asymptotics for functions of Wiener-Hopf operators: smooth versus non-smooth symbols”. *Geom. Funct. Anal.* **27** (2017), pp. 676–725.

[60] A. V. Sobolev. “On Szegő formulas for truncated Wiener-Hopf operators”. *Integral Equations Operator Theory* **91** (2019), Paper No. 1, 28.

[61] J. P. Solovej and W. L. Spitzer. “A new coherent states approach to semiclassics which gives Scott’s correction”. *Comm. Math. Phys.* **241** (2003), pp. 383–420.

[62] G. Szegö. “Ein Grenzwertsatz über die Toeplitzschen Determinanten einer reellen positiven Funktion”. *Math. Ann.* **76** (1915), pp. 490–503.

[63] G. Szegö. “On certain Hermitian forms associated with the Fourier series of a positive function”. *Comm. Sém. Math. Univ. Lund [Medd. Lunds Univ. Mat. Sem.]* **1952** (1952), pp. 228–238.

[64] H. Widom. “On the eigenvalues of certain Hermitian operators”. *Trans. Amer. Math. Soc.* **88** (1958), pp. 491–522.

[65] H. Widom. “A theorem on translation kernels in n dimensions”. *Trans. Amer. Math. Soc.* **94** (1960), pp. 170–180.

[66] H. Widom. “Toeplitz determinants with singular generating functions”. *Amer. J. Math.* **95** (1973), pp. 333–383.

[67] H. Widom. “Asymptotic behavior of block Toeplitz matrices and determinants. II”. *Advances in Math.* **21** (1976), pp. 1–29.

[68] H. Widom. “Szegő’s limit theorem: The higher-dimensional matrix case”. *J. Functional Analysis* **39** (1980), pp. 182–198.

[69] H. Widom. “A trace formula for Wiener-Hopf operators”. *J. Operator Theory* **8** (1982), pp. 279–298.

[70] H. Widom. “On a class of integral operators with discontinuous symbol”. In: *Toeplitz centennial (Tel Aviv, 1981)*. Vol. 4. Oper. Theory Adv. Appl. Birkhäuser Verlag, Basel-Boston, Mass., 1982, pp. 477–500.

[71] H. Widom. *Asymptotic expansions for pseudodifferential operators on bounded domains*. Vol. 1152. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1985, pp. iv+150.

[72] H. Widom. “On a class of integral operators on a half-space with discontinuous symbol”. *J. Funct. Anal.* **88** (1990), pp. 166–193.

[73] M. M. Wolf. “Violation of the Entropic Area Law for Fermions”. *Phys. Rev. Lett.* **96** (1 Jan. 2006), p. 010404.