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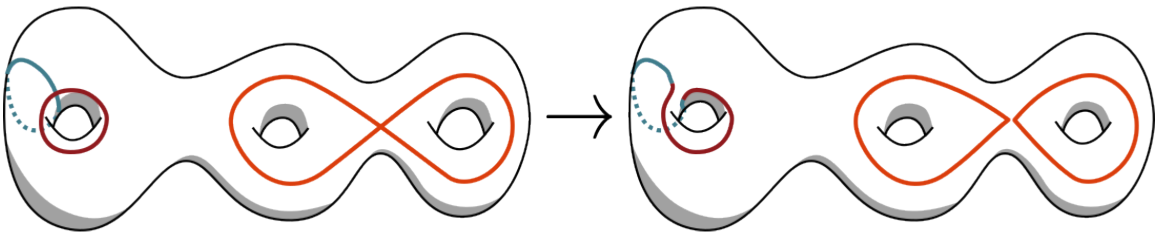

PhD thesis

String topology, operads and Morse theory

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Thesis statement

This thesis is based on the following manuscripts:

Part I is based on an article that has been published in *The Quarterly Journal of Mathematics* 76.4 (2025), pp. 1105–1157.

Part II has appeared in this form on the preprint server arXiv under arXiv:2510.23812.

Part III is based on a manuscript for which an earlier version has appeared on the preprint server arXiv under arXiv:2510.23813.

Summary

This thesis is a synopsis consisting of three articles that study different aspects of string topology – an area of research intersecting geometry and topology. The first article studies graded Frobenius algebras. They are an algebraic structure that appears in, but not only in, string topology. We give an algebraic gadget, a PROP, that encodes and explains this algebraic structure.

The second article studies the string topology coproduct on the subspace of based loop. We prove that the coproduct is better behaved on this subspace. We use this understanding to give explicit computations for certain manifolds.

The third and final article of this synopsis studies the string topology coproduct via Morse homology with differential graded coefficients. Using this Morse theoretic framework, we reconstruct the coproduct on the free loop space from the coproduct on the based loop space.

Sammenfatning

Denne afhandling er en synpose bestående af tre artikler, der undersøger forskellige aspekter af strengtopologi – et forskningsområde, som krydser geometri og topologi. Den første artikel undersøger graderede Frobenius-algebraer. Disse udgør en algebraisk struktur, der forekommer i, men ikke udelukkende i, strengtopologi. Vi præsenterer et algebraisk redskab, et PROP, der kodificerer og forklarer denne algebraiske struktur.

Den anden artikel undersøger strengtopologiens koprodukt på det baserede løkkerum. Vi beviser, at koproduktet opfører sig bedre på underrummet af baserede løkker. Vi bruger denne indsigt til at give eksplicitte beregninger for visse mangfoldigheder.

Den tredje og sidste artikel i denne synopse undersøger strengtopologiens koprodukt via Morse-homologi med differentielt graderede koefficienter. Ved hjælp af denne Morse-teoretiske ramme rekonstruerer vi koproduktet på det frie løkkerum ud fra koproduktet på det baserede løkkerum.

Acknowledgements

I cannot overstate how lucky I am to have done my PhD in this beautiful environment. The biggest impact on this wonderful setting has my supervisor Nathalie Wahl. Therefore, my first thanks belongs to her. With her insights in mathematical and non-mathematical matters, her thoughtful, precise and empathetic communication style and her unwavering support, she was not only an incredible supervisor to me but continues to be an inspiration.

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I would also like to thank Jesper Grodal, Florian Naef and Alexandru Oancea for being part of my thesis committee. I am especially indebted to Alexandru for hosting me in Strasbourg during the spring of last year. I also want to thank Robin Riegel for spirited and helpful discussions during this time. A lot of the insights in this thesis took shape in Strasbourg due to their help.

Zu guter Letzt möchte ich mich bei meiner Familie und meinen Freund*innen in Zürich bedanken. Ihr habt mich vor und während dieser Zeit tatkräftig und durchgehend unterstützt. Ihr habt dieses grosse Abenteuer überhaupt möglich gemacht. Insbesondere gilt mein Dank Mami, Papi, Viviane, Cornelia, Richi, Grossvati und meiner wunderbaren Freundin Janine, die mir auch in schweren Zeiten unermüdlich Kraft gab und mit der ich meine Freuden und Erfolge immer als Erstes teilen konnte.

Jonathan Laurent Clivio
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Contents

Introduction

This thesis consists of three parts that provide different perspectives on string topology: an operadic perspective in Part I, a perspective via the based loop space in Part II and a perspective via Morse theory in Part III.

String topology was introduced by Chas and Sullivan in 1999 [CS99] as the study of certain operations on the space of loops on a manifold – the free loop space. Despite the seemingly simple geometric description in terms of cutting up and gluing together loops (see the cover image), it has proven difficult to give an explicit description of the operations for most manifolds. This is especially true for the Goresky-Hingston coproduct which is a key operation in string topology and constitutes the main object of interest for Parts II and III.

The three articles making up the three parts of this thesis give tools to deepen our understanding of these operations. Furthermore, they provide methods and results that we hope are of interest beyond the context of string topology.

The first article takes an operadic approach to string topology. The operations in string topology do in general not preserve homological degree. This can lead to surprising signs when studying the relations that these operations satisfy. In Part I, we give an explanation for these signs by encoding the operations of a graded Frobenius algebras in an algebraic gadget called a PROP. We obtain the signs and relations that the operation of graded Frobenius algebra satisfy via a graphical calculus as in Figure 1.

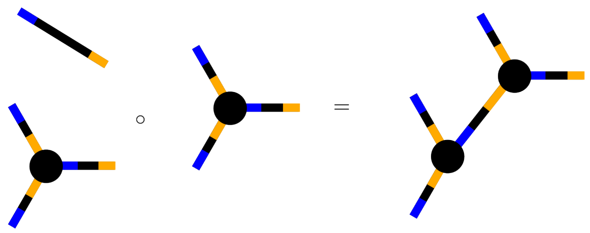

FIGURE 1. A graphical calculus of this form encodes the associativity of graded Frobenius algebras.

Graded Frobenius algebras are a structure that the string topology product and coproduct have in some settings. This algebraic structure also appears in other areas than string topology. Notably, graded Frobenius algebras also describe certain 2-dimensional topological quantum field theories.

In the second article, we focus our attention on the based loop space. We show that the Goresky-Hingston coproduct on the based loop space is much nicer behaved than its bigger brother on the free loop space. The fact that we only allow for one fixed start and end point of our loops means that to understand the behaviour around a self-intersection you only need to understand what happens in an n-dimensional ball (see Figure 2). This is the simplest possible setting when working with n-dimensional spaces. This simpler geometry allows us to prove invariance formulas for the based coproduct that fail for the free loop space. We use these simpler invariance formulas to give concrete computations of the coproduct on the based loop space.

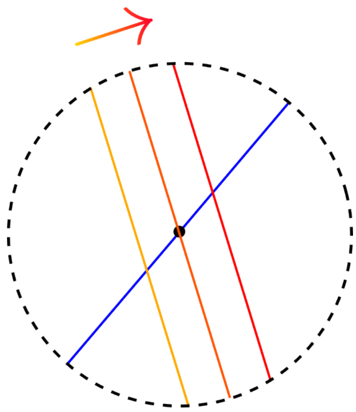

FIGURE 2. A family of loops that contain a loop with one self-intersection.

In the last article, we rebuild the coproduct on the free loop space from the coproduct on the based loop space. We do this in the formalism of Morse homology with differential graded coefficients. This formalism shows that given a Morse function on our manifold, we can understand a class of loops in the free loop space as a sum of classes of based loops. These classes of based loops are based at the critical points of the Morse functions. We show that under certain assumptions on the manifold or coefficients, we can compute the coproduct on the free loop space via the coproduct of the based loops (see Figure 3).

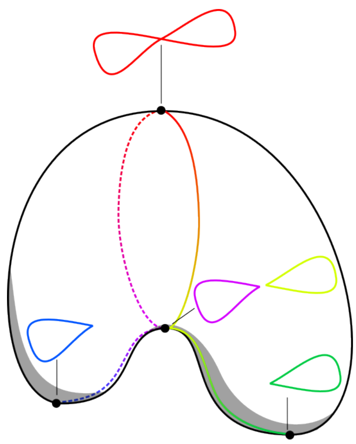

FIGURE 3. The loop at the top represents a family of loops. The coproduct of this family is computed by cutting it up and flowing it down to the loops at the bottom.

The hope for future work is that we can combine Parts II and III: Part II shows that the coproduct on the based loop space is computable and Part III shows that the coproduct on the free loop space can be glued together from the coproduct on the based loop space. It thus seems doable to transport insights and computations from the based loop space to the free loop space. At the moment, this remains work in progress. The bottleneck has moved from understanding the string topology of a manifold to understanding its Morse theory.

Part I

Graded Frobenius algebras

Abstract. We construct a PROP which encodes 2D-TQFTs with a grading. This defines a graded Frobenius algebra as algebras over this PROP. We also give a description of graded Frobenius algebras in terms of maps and relations. This structure naturally arises as the cohomology of manifolds, loop homology and Hochschild homology of Frobenius algebras. In addition, we give a comprehensive description of the signs that arise in suspending algebras over PROPs.

Contents

1. Introduction

The goal of this paper is to clarify what is meant by a graded Frobenius algebra. To this end, we introduce a PROP that describes a Frobenius algebra with multiplication in a fixed degree c and comultiplication of a fixed degree d.

There is a need for such a definition, because the straight-forward attempt of considering Frobenius algebras in graded abelian groups runs into problems. Defining a Frobenius algebra in graded abelian groups is unproblematic if all maps are degree-preserving, i.e., they are maps in degree 0. But this attempt fails to describe some non-trivial examples if maps of different degrees are considered: for example, we consider a graded abelian group A with a unital multiplication $\mu\colon A\otimes A\to A$ and a counital comultiplication $\nu\colon A\to A\otimes A$ on it satisfying the usual Frobenius relation

$$(\mu\otimes\mathrm{id})\circ(\mathrm{id}\otimes\nu)=\nu\circ\mu=(\mathrm{id}\otimes\mu)\circ(\nu\otimes\mathrm{id}). \tag{1}$$

We now assume that μ and ν are of some degree $|\mu|=c$ and $|\nu|=d$. If $c+d$ is odd and we work over a field where $1\neq -1$, the only example that is described is the trivial algebra $A=0$. We prove this in Proposition 3.7.

Another attempt to describe examples of Frobenius algebras with maps in non-zero degree is to suspend a Frobenius algebra. Suppose A is a Frobenius algebra in graded abelian groups with multiplication μ and ν in degree 0. Suspending A gives an algebra ΣA with a multiplication $\Sigma\mu$ in

degree -1 and a comultiplication $\Sigma\nu$ in degree 1. Suspension and desuspension of algebras therefore leaves the sum of the degree of the multiplication and the degree of the comultiplication invariant. Thus an example of A with a multiplication μ in degree 0 and a comultiplication in degree $d \neq 0$ cannot be described as the suspension of an algebra with only operations in degree 0.
Moreover, the canonical map

$$\Sigma\colon \mathrm{Hom}(A^{\otimes k}, A^{\otimes l}) \to \mathrm{Hom}((\Sigma A)^{\otimes k}, (\Sigma A)^{\otimes l}) \tag{2}$$

is only functorial and symmetric monoidal up to sign. We compute these signs in Lemma 5.10. These signs also illustrate why Frobenius algebras in graded abelian groups are not the natural concept to consider. For example if μ and ν satisfy the Frobenius relation as in (1), the signs in the functoriality and monoidality of Σ in (2) induce the following sign in the Frobenius relation:

$$(\Sigma\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Sigma\nu) = (-1)\Sigma\nu \circ \Sigma\mu = (\mathrm{id} \otimes \Sigma\mu) \circ (\Sigma\nu \otimes \mathrm{id}).$$

Therefore, the natural concept of graded Frobenius algebras will have signs appearing in the relations.

1.1. The graded Frobenius PROP. In this paper, we give a definition of graded Frobenius algebras with a multiplication μ in degree c and a comultiplication ν in degree d for any choice $c, d \in \mathbb{Z}$ (see Definition 4.4). This definition solves the two problems described above: we can describe non-trivial examples for any choice of $c, d \in \mathbb{Z}$ and the signs that appear in this definition are stable under suspending algebras. We make this second claim precise when we discuss the relations in subsection 1.3.
We describe a category of graph cobordisms in Definition 2.12. This lets us model every cobordism by an equivalence class of graphs.
For a graph G representing a cobordism, we consider a twisting by the determinant $\mathbf{det}(G, \partial_{in})$ of the homology relative to the incoming boundary. The determinant $\mathbf{det}(G, \partial_{in})$ is a graded rank one free abelian group concentrated in degree $-\chi(G, \partial_{in})$, the negative Euler characteristic. Analogously, we consider $\mathbf{det}(G, \partial_{out})$.
This lets us define the categories of graded graph cobordism $\mathrm{GCob}_{c,d}$, fat graph cobordisms $\mathrm{fGCob}_{c,d}$ and planar graph cobordisms $\mathrm{pGCob}_{c,d}$ in Definition 4.3. These categories have morphisms given by elements in the following graded abelian groups

$$\mathbf{det}_{c,d}(G) := \mathbf{det}(G, \partial_{in})^{\otimes c} \otimes \mathbf{det}(G, \partial_{out})^{\otimes d}.$$

In these categories, there is a morphism $* \sqcup * \to *$. It models a multiplication of the graded Frobenius algebra $*$. It is given by a generator $\omega(\succ\!\!-) \in \mathbf{det}_{c,d}(\succ\!\!-)$. Here $\succ\!\!-$ is the graph with two inputs to the left, one output to the right and one three-valent vertex in the middle. It has relative Euler characteristic $-\chi(\succ\!\!-, \partial_{in}) = 1$ and $-\chi(\succ\!\!-, \partial_{out}) = 0$. Therefore $\mathbf{det}_{c,d}(\succ\!\!-)$ is concentrated in degree c and the multiplication $\omega(\succ\!\!-)$ is in degree c. Analogously, the comultiplication is given by a generator $\omega(-\!\!\prec) \in \mathbf{det}_{c,d}(-\!\!\prec)$ in degree d.
The categories $\mathrm{GCob}_{c,d}$, $\mathrm{fGCob}_{c,d}$ and $\mathrm{pGCob}_{c,d}$ let us thus attain any combination in degrees c, d. Moreover, $\mathrm{GCob}_{c,d}$ is modelled by graphs and describes (c,d)-graded *commutative* Frobenius algebras, $\mathrm{fGCob}_{c,d}$ has morphisms given by fat graphs and describes (c,d)-graded *symmetric* Frobenius algebras and $\mathrm{pGCob}_{c,d}$ has morphisms given by planar graphs and describes (c,d)-graded Frobenius algebras

This twisting was previously described by Godin in [God07] and by Wahl and Westerland in [WW16]. However, in contrast to Godin who models commutative Frobenius algebras by closed surfaces, we work with graphs. In particular, twisting a surface S by $\mathbf{det}(S, \partial_{in})$ corresponds to a twisting by $\mathbf{det}(G, \partial_{in}) \otimes \mathbf{det}(G, \partial_{out})$ where G is a graph modelling the cobordism S. We prove this relation between the twistings in Proposition 6.4. In particular, working with graphs rather than closed surfaces for commutative graded Frobenius algebras allows us to describe examples of algebras where $c + d$ is odd.

1.2. Suspending PROPs. Above we observed that suspending an algebra with multiplication in degree 0 and comultiplication in degree 0 gives an algebra with multiplication in degree -1 and a comultiplication in degree 1. These shifts in degree hold for any choice in degrees c, d.
This can be seen on the level of PROPs: we define the usual suspension PROP Σ in Definition 4.14. This PROP has the following universal property: a graded abelian group A has the structure of an algebra over a PROP P if and only if ΣA has the structure of an algebra over $\Sigma \otimes P$ (see Proposition 4.15). On the other hand, we give natural isomorphisms of PROPs

$$\begin{aligned}\Sigma \otimes \mathrm{GCob}_{c,d} &\cong \mathrm{GCob}_{c-1,d+1},\\ \Sigma \otimes \mathrm{fGCob}_{c,d} &\cong \mathrm{fGCob}_{c-1,d+1},\\ \Sigma \otimes \mathrm{pGCob}_{c,d} &\cong \mathrm{pGCob}_{c-1,d+1}\end{aligned}$$

in Proposition 4.17. This, in particular, shows that suspending a (c, d)-graded Frobenius algebra gives a $(c-1, d+1)$-graded Frobenius algebra.
This is in contrast to the relation between (c, d)-graded Frobenius algebras and $(c+1, d)$-graded Frobenius algebras. In Remark 4.19, we give a construction that produces a $(c+1, d)$-graded Frobenius algebra from a (c, d)-graded Frobenius algebra. This construction could be called a one-sided suspension of graded Frobenius algebras as it only affects the degree of the multiplication and not the coproduct. However in contrast to the usual suspension, this one-sided suspension is not an equivalence of categories.

1.3. Maps, relations and signs. In the ungraded case, there are several equivalent descriptions of a Frobenius algebra. We contrast two different types of definitions:

(1) Descriptions in terms of maps with relations: an object A together with a selection of the following maps: a multiplication μ, a unit η, a comultiplication ν and a counit ε that satisfy certain relations.
(2) Descriptions in terms of PROPs: a 2-dimensional topological quantum field theory (TQFT). This is a (symmetric) monoidal functor out of a 2-dimensional cobordism category.

There is a myriad of different variants of the two flavours of definitions with optional commutativity, unitality, counitality. The equivalence between these two types of definitions is described in [Abr96; LP08; Koc04].
We explained above that we define a (c, d)-graded Frobenius algebra in terms of a PROP. In our main theorem, we relate our definition to a description in terms of maps and a choice of signs appearing in the relations:

Main Theorem (Theorem 5.1). *Let $c, d \in \mathbb{Z}$ and let $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ be a (strict) monoidal category enriched over graded abelian groups and denote $\mathcal{C}_n(X, Y)$ the abelian group of morphisms from X to Y of degree n.*
The data $(A, \mu, \eta, \nu, \varepsilon)$ where

$$\begin{aligned}&A \in \mathcal{C}, &&\\ &\mu \in \mathcal{C}_c(A \otimes A, A), &&\eta \in \mathcal{C}_{-c}(\mathbb{1}, A),\\ &\nu \in \mathcal{C}_d(A, A \otimes A), &&\varepsilon \in \mathcal{C}_{-d}(A, \mathbb{1})\end{aligned}$$

a monoidal functor out of $\mathrm{pGCob}_{c,d}$ into $\mathcal{C}$ uniquely up to isomorphism defines if and only if the following relations are satisfied:

(i) graded associativity $\mu \circ (\mu \otimes \mathrm{id}) = (-1)^c \mu \circ (\mathrm{id} \otimes \mu)$;

(ii) graded unitality $(-1)^c \mu \circ (\eta \otimes \mathrm{id}) = (-1)^{\frac{c(c-1)}{2}} \mathrm{id} = \mu \circ (\mathrm{id} \otimes \eta)$*;*
(iii) graded coassociativity $(\mathrm{id} \otimes \nu) \circ \nu = (-1)^d (\nu \otimes \mathrm{id}) \circ \nu$*;*
(iv) graded counitality $(-1)^d (\mathrm{id} \otimes \varepsilon) \circ \nu = (-1)^{\frac{d(d-1)}{2}} \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \nu$*;*
(v) graded Frobenius relation $(\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \nu) = (-1)^{cd} \nu \circ \mu = (\mathrm{id} \otimes \mu) \circ (\nu \otimes \mathrm{id})$.

If $\mathcal{C}$ *is also symmetric, the data* $(A, \mu, \eta, \nu, \varepsilon)$ *defines a symmetric monoidal functor out of* $\mathrm{GCob}_{c,d}$ *into* $\mathcal{C}$ *uniquely up to isomorphism if and only if the maps satisfy (i)-(v) and*

(vi) graded commutativity $\mu \circ \tau = (-1)^c \mu$

and it defines a symmetric monoidal functor out of $\mathrm{fGCob}_{c,d}$ *into* $\mathcal{C}$ *uniquely up to isomorphism if and only if the maps satisfy (i)-(v) and*

(vi') graded symmetry $\varepsilon \circ \mu \circ \tau = (-1)^c \varepsilon \circ \mu$.

These signs are *not* canonical. They depend on choices of generators $\omega(\succ\!\!-) \in \mathbf{det}_{c,d}(\succ\!\!-)$ and $\omega(-\!\!\prec) \in \mathbf{det}_{c,d}(-\!\!\prec)$. We explain in Remark 5.3 our choices. However, no choice can make the signs disappear.
We remarked above that our signs are stable under suspending algebras. This means that our choices of signs are such that if $(A, \mu, \eta, \nu, \varepsilon)$ satisfies the relations of (c, d)-graded Frobenius algebra, then $(\Sigma A, \Sigma\mu, \Sigma\eta, \Sigma\nu, \Sigma\varepsilon)$ satisfies the relations of a $(c-1, d+1)$-graded Frobenius algebra. This is proved in Proposition 5.11.
Our signs almost coincide with the signs found in the definition of a biunital coFrobenius bialgebra as defined in [CO22b]. We compare this in Remark 5.4.

1.4. Examples. In the literature, graded Frobenius algebras appear with different characterisations. Some have a more PROP-theoretic description and some have a description in terms of maps and relations. Our main theorem lets us relate the two types of description. In section 6, we discuss a selection of the examples found in the literature.
The first example we discuss is the cohomology ring of an oriented manifold M of dimension d (see Example 6.1). The intersection coproduct defines a degree d comultiplication on cohomology. In particular, we discuss different sign conventions for the intersection coproduct found in the literature.
A second example, we discuss, is the Hochschild homology of a graded Frobenius algebra A (see Example 6.3). Wahl and Westerland give a description of a Frobenius algebra on $\overline{HH}_*(A)$ twisted by $\mathbf{det}(S, \partial_{in})^{\otimes d}$. As discussed above, the twisting by the relative homology of the surface gives a (d, d)-graded Frobenius algebra.
In a similar vein, we discuss the homology of the loop space with the Chas-Sullivan product and the trivial coproduct. Godin gives operations which are twisted by $\mathbf{det}(S, \partial_{in})^{\otimes d}$ in [God07]. By our Proposition 6.4, this corresponds to a twisting by $\mathbf{det}(G, \partial_{in})^{\otimes d} \otimes \mathbf{det}(G, \partial_{out})^{\otimes d}$ (see Example 6.5).
Finally, we discuss the relation between the Chas-Sullivan product and the Goresky-Hingston coproduct in Rabinowitz loop homology as in the work by Cieliebak, Hingston and Oancea in [CHO20; CO22b; CO22a; CHO23; CO24].
We discuss the language of Tate vector spaces as presented in [CO24]. Concretely for A a graded Tate vector space, we show that the endomorphisms $\mathrm{End}_A(X, Y) = \mathrm{Hom}(A^{\widehat{\otimes}^* X}, A^{\widehat{\otimes}^! Y})$ have a canonical dioperad structure in Proposition 6.6.
On the other hand, we define the canonical subdioperad of $\mathrm{GCob}_{c,d}$ in Definition 4.21. This dioperad has the same operations as the full PROP in the case of a field of characteristic $\neq 2$. This is a consequence of our description of the subdioperad in Proposition 4.22. We can thus describe the Rabinowitz loop homology as an algebra over the subdioperad in Example 6.8.

Organisation of the paper. In Section 2, we describe a category of graph cobordisms modelling the category of 2-dimensional cobordisms. In Section 3, we recall the classical relations of Frobenius algebras and show that they force algebras to be trivial in the odd dimensional graded case. In Section 4, we construct the categories $\mathrm{GCob}_{c,d}$, $\mathrm{fGCob}_{c,d}$ and $\mathrm{pGCob}_{c,d}$ and study graded Frobenius algebras in terms of PROPs. In section 5, we state the main theorem and use it to study graded Frobenius algebras in terms of maps and relations. In Section 6, we review and relate examples from the literature of graded Frobenius algebras. In Section 7, we set up machinery that helps us to prove the main theorem.

Conventions. Throughout this paper, we use the convention that maps act from the left. This in particular implies that for maps of graded abelian $f\colon A \to A'$ and $g\colon B \to B'$ evaluated on homogeneous elements $a \in A$ and $b \in B$, we have the following signs

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b).$$

Moreover, that the canonical composition map on morphism spaces is order-reversing

$$\begin{aligned}\mathrm{Hom}(Y,Z) \otimes \mathrm{Hom}(X,Y) &\to \mathrm{Hom}(X,Z),\\ f \otimes g &\mapsto f \circ g.\end{aligned}$$

We define the suspension of a graded abelian A by ΣA. It is defined by $(\Sigma A)_{i+1} = A_i$.
We denote by $\mathbb{1}$ the unit of the category of graded abelian groups: $\mathbb{Z}$ concentrated in degree 0. We denote the dual of a graded abelian group by $A^\vee$ defined as $A_k^\vee \coloneqq \mathrm{Hom}_\bullet(A_{-k}, \mathbb{1})$.
The natural isomorphism from the dual of a tensor product to the product of the duals is order-reversing, we have

$$\begin{aligned}(A \otimes B)^\vee &\cong B^\vee \otimes A^\vee,\\ (a \otimes b)^\vee &\mapsto b^\vee \otimes a^\vee.\end{aligned}$$

This can be seen as $b^\vee \otimes a^\vee$ evaluated on $a \otimes b$ gives 1. But $a^\vee \otimes b^\vee$ evaluated on $a \otimes b$ gives $(-1)^{|a||b|}$.

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2. Cobordisms and graphs

In this section, we recall the definition of closed, open and planar 2D-cobordism categories. Moreover, we describe an equivalent category coming from a 2-category of graph cobordism.

2.1. Cobordisms.

Definition 2.1. The category of *closed 2D-cobordisms* $\mathrm{Cob}_2^{\mathrm{closed}}$ has objects

$$\mathrm{Ob}(\mathrm{Cob}_2^{\mathrm{closed}}) \coloneqq \left\{ \coprod_X S^1 \,\middle|\, X \text{ is a finite set} \right\}.$$

A morphism between $\coprod_X S^1$ and $\coprod_Y S^1$ is given by an equivalence class of pairs (S, φ) where

- S is an oriented, compact topological 2-manifold possibly with boundary;
- φ is a homeomorphism

$$\varphi\colon \coprod_X S^1 \sqcup \coprod_Y S^1 \to \partial S$$

and (S,φ) and (S',φ') are equivalent if there exists a homeomorphisms $\psi : S \to S'$ such that $\psi \circ \varphi = \varphi'$. See Figure 4.

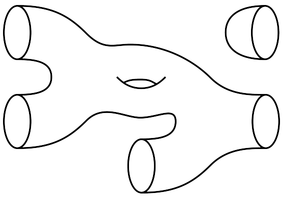


FIGURE 4. An example of a closed cobordism from $S^1 \sqcup S^1 \sqcup S^1$ to $S^1 \sqcup S^1$. We use the convention of drawing the incoming boundary to the left and the outgoing boundary to the right.

DEFINITION 2.2. The category of *open 2D-cobordisms* $\mathrm{Cob}_2^{\mathrm{open}}$ has objects

$$\mathrm{Ob}(\mathrm{Cob}_2^{\mathrm{open}}) = \left\{ \coprod_X [0,1] \,\middle|\, X \text{ is a finite set} \right\}.$$

A morphism between $\coprod_X [0,1]$ and $\coprod_Y [0,1]$ is given by an equivalence class of pairs (S,φ) where

- S is an oriented, compact topological 2-manifold such that every connected component has non-empty boundary;
- φ is an embedding

$$\varphi\colon \coprod_X [0,1] \sqcup \coprod_Y [0,1] \to \partial S.$$

The equivalence relation is given by $(S,\varphi) \sim (S',\varphi)$ if there exists a homeomorphisms $\psi : S \to S'$ such that $\psi \circ \varphi = \varphi'$. We call $\partial_{free} S = \partial S \setminus \mathrm{im}(\varphi)$ the free part of the boundary. See Figure 5.

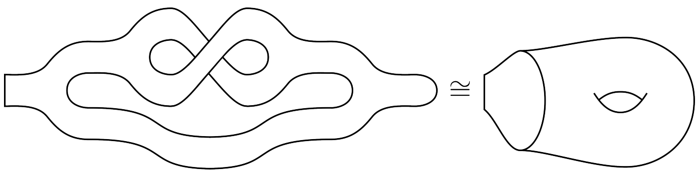


FIGURE 5. An example of an open cobordism from $[0,1]$ to $\varnothing$. The two surfaces are homeomorphic as they both have one boundary and Euler characteristic -1.

We fix some orientation on $[0,1] \times \mathbb{R}$.

DEFINITION 2.3. The category of *planar 2D-cobordisms* $\mathrm{Cob}_2^{\mathrm{planar}}$ has objects

$$\mathrm{Ob}(\mathrm{Cob}_2^{\mathrm{planar}}) = \left\{ \coprod_X [0,1] \,\middle|\, X \text{ is a finite set with linear ordering} \right\}.$$

A morphism between $\coprod_X [0,1]$ and $\coprod_Y [0,1]$ is given by an open cobordisms

$$[(S,\varphi)] \in \mathrm{Cob}_2^{\mathrm{open}}\left(\coprod_X [0,1], \coprod_Y [0,1]\right)$$

such that there exists an oriented embedding $\psi\colon S \hookrightarrow [0,1]\times\mathbb{R}$ satisfying that

- $\psi \circ \varphi|_{\coprod_X [0,1]}$ lands in $\{0\}\times\mathbb{R}$ and the image of the connected components is ordered (with respect to the canonical ordering on $\{0\}\times\mathbb{R}$) according to the linear ordering of X;
- $\psi \circ \varphi|_{\coprod_Y [0,1]}$ lands in $\{1\}\times\mathbb{R}$ and the image of the connected components is ordered in $\{1\}\times\mathbb{R}$ according to the linear ordering of Y.

See Figure 6.

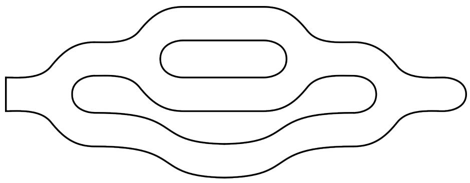


FIGURE 6. A planar cobordism between $[0,1]$ and $\varnothing$.

2.2. Graphs. We later give a description of those three categories using graphs. For this, we introduce what we mean by a graph.

DEFINITION 2.4. A *graph* $G = (V, H, \sigma, s, L_{in}, L_{out})$ consists of the following data:

- a finite set $V(G) \coloneqq V$ called the *vertices*;
- a finite set $H(G) \coloneqq H$ called the *half-edges*;
- an involution $\sigma\colon H \to H$ with no fixed points, the orbits $\{h, \sigma h\}$ are called the *edges* and we denote the set of orbits $H_\sigma \eqqcolon E \eqqcolon E(G)$;
- a surjective map $s\colon H \to V$, where for a vertex $v \in V$ the cardinality $|s^{-1}(v)|$ is called its *arity*;
- subsets $L_{in}(G) \coloneqq L_{in}, L_{out}(G) \coloneqq L_{out} \subseteq V(G)$ of the set of vertices of arity ≤ 1 such that $L_{in} \cap L_{out}$ only contains vertices of arity 0.

A *fat graph* is a graph together with a cyclic ordering of the half-edges in $s^{-1}(v)$ for all v.

We call the vertices in L_{in} and L_{out} the *external vertices*. The other vertices are called the *internal vertices*. In our conventions, we allow for internal vertices of arity ≤ 1.

A graph defines a 1-dimensional CW-complex, called the geometric realization, denoted $|G|$: the zero-cells are the vertices and the one-cells are the edges. Choosing an orientation on each one-cell corresponds to an ordering of the two half-edges making up the edge. Then the map s describes how to glue the edges to the vertices.
We give some additional terminology which is useful, when talking about graphs.

DEFINITION 2.5. Let G be a graph.

- The graph G is *connected* if $|G|$ is connected.
- The graph G is a *forest* if $\pi_1(|G|) = 1$ at every basepoint.
- An edge $e = \{h, \sigma h\}$ is called a *tadpole* if $s(h) = s(\sigma h)$.

Definition 2.6. A *morphism of graphs* $f\colon G \to G'$ is a cellular map of the underlying CW-complexes $|G| \to |G'|$ which is on edges either the collapse or the identity and is a bijection on L_{in} and L_{out}.

Remark 2.7. Our definition of morphism of graphs is such that every morphism of graph is given by a sequence of edge collapses and a graph isomorphism. Importantly, the edges that are collapsed are not tadpoles and are not adjacent to an external vertex.

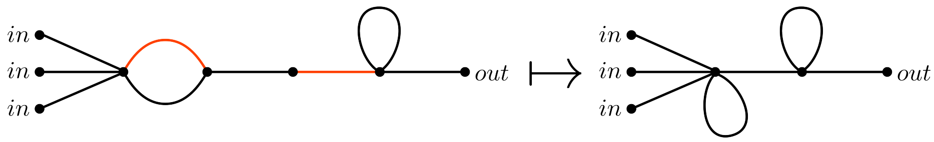


Figure 7. A morphism of graphs sends L_{in} to L_{in} and L_{out} to L_{out} and possibly collapses edges. Here the graph morphism collapses two edges.

Definition 2.8. A *morphism of fat graphs* is a morphism of graphs $f\colon G \to G'$ that can be written as a sequence of edge collapses of edges

$$G =: G_0 \to G/e_1 =: G_1 \to G_1/e_2 =: G_2 \to \cdots \to G_{k-1}/e_k =: G_k$$

and a graph isomorphism

$$\varphi\colon G_k \to G'$$

such that

- if we have for $e_i = \{h_i, \sigma h_i\}$ the cyclic orderings $(h_i\ a_1 \dots a_m) = (a_1 \dots a_m\ h_i)$ on $s^{-1}(s(h_i))$ and $(\sigma h_i\ b_1 \dots b_n)$ on $s^{-1}(s(\sigma h_i))$, then the vertex in G_{i-1}/e_i corresponding to e_i has cyclic ordering
$$(a_1 \dots a_m\ b_1 \dots b_n);$$
- φ commutes with the cyclic ordering.

2.3. Graph cobordisms. We can now construct the 2-categories $\underline{\mathrm{GCob}}$, $\underline{\mathrm{fGCob}}$ and $\underline{\mathrm{pGCob}}$. This gives models for the cobordism categories.

Definition 2.9. The *graph cobordism category* GCob is a 2-category, i.e. a category enriched over 1-categories with

$$\mathrm{Ob}(\underline{\mathrm{GCob}}) = \{\text{finite sets}\},$$

and the morphism categories $\underline{\mathrm{GCob}}(X, Y)$ are given by the category of graphs G with labellings $L_{in}(G) \cong X$ and $L_{out}(G) \cong Y$ and 2-morphisms graph morphisms which respect the labellings.
The *fat graph cobordism category* is a 2-category $\underline{\mathrm{fGCob}}$ with

$$\mathrm{Ob}(\underline{\mathrm{fGCob}}) = \{\text{finite sets}\},$$

and the morphism categories $\underline{\mathrm{fGCob}}(X, Y)$ are given by the analogous category of fat graphs with labellings and morphisms are fat graph morphisms which respect the labellings.

Example 2.10. Let $f\colon X \cong Y$ be an isomorphism of finite sets. We define the graph G_f

$$V(G_f) \coloneqq X, \qquad H(G_f) \coloneqq \varnothing, \qquad L_{in}(G_f) \coloneqq V(G_f) =: L_{out}(G_f).$$

This has labellings $L_{in}(G) \cong X$ via the identity and $L_{out}(G) \cong Y$ via f. Moreover, G_f consists only of vertices of valence zero and thus has a canonical fat structure.

These categories are well-defined as the composition along objects is given as follows: let X, Y, Z be finite. We define the composition as

$$\underline{\mathrm{GCob}}(Y, Z) \times \underline{\mathrm{GCob}}(X, Y) \to \underline{\mathrm{GCob}}(X, Z)$$

which sends (G', G) to the graph $G' \circ G := G' \cup_Y G$. This is defined as the graph obtained from the disjoint union of G and G' where we associate the elements in $L_{out}(G)$ with $L_{in}(G')$ via the labelling by Y. In more visual terms, we glue the outgoing legs of G to the incoming legs of G' (as in Figure 8).

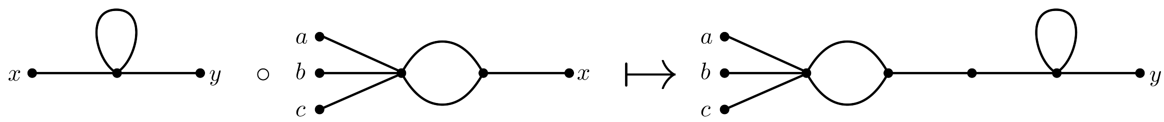


FIGURE 8. Composition of a graph cobordism between $\{a, b, c\}$ and $\{x\}$ and a graph cobordism between $\{x\}$ and $\{y\}$.

We note that this defines a functor as morphisms in $\underline{\mathrm{GCob}}(X, Y)$ and $\underline{\mathrm{GCob}}(Y, Z)$ can be glued together in a similar manner.
The unit $\mathrm{id}_X \in \underline{\mathrm{GCob}}(X, X)$ is given by the graph G_{id} as defined in Example 2.10.
Analogously, we can define composition maps

$$\underline{\mathrm{fGCob}}(Y, Z) \times \underline{\mathrm{fGCob}}(X, Y) \to \underline{\mathrm{fGCob}}(X, Z)$$

for fat graphs.
This works as the newly combined vertices coming from $L_{out}(G) \cong L_{in}(G')$ have arity ≤ 2 and thus there is only one cyclic ordering of the half-edges at these vertices. The fat graph structure of $G' \circ G$ is thus well-defined.
Moreover, the unit G_{id} is also a fat graph.

LEMMA 2.11. *Disjoint unions define a symmetric monoidal structure on* $\underline{\mathrm{GCob}}$ *and* $\underline{\mathrm{fGCob}}$.

PROOF. We note that labellings

$$L_{in}(G) \cong X, \qquad L_{out}(G) \cong Y, \qquad L_{in}(G') \cong X', \qquad L_{out}(G') \cong Y'$$

on (fat) graphs G, G' induce labellings

$$L_{in}(G \sqcup G') \cong X \sqcup X', \qquad L_{out}(G \sqcup G') \cong Y \sqcup Y'.$$

This defines a coherent symmetric monoidal structure because of the following observation: the category $\mathrm{Fin}^{\cong}$ of finite sets with bijections is a canonical subcategory of $\underline{\mathrm{GCob}}$ and $\underline{\mathrm{fGCob}}$ via $f \mapsto G_f$ as in Example 2.10.
It is then a direct calculation that the unit, associator and twist map of the symmetric monoidal structure on $\mathrm{Fin}^{\cong}$ thus also induce unit, associator and twist map on $\underline{\mathrm{GCob}}$ and $\underline{\mathrm{fGCob}}$. □

In [Chi14][Section 1.3], it is described that the inclusion of 1-categories into 2-categories has a left adjoint τ_ι given on a 2-category $\underline{\mathcal{C}}$ by

$$\mathrm{Ob}(\tau_\iota(\underline{\mathcal{C}})) = \mathrm{Ob}(\underline{\mathcal{C}}),$$
$$\tau_\iota(\underline{\mathcal{C}})(X, Y) = \pi_0(\underline{\mathcal{C}}(X, Y)).$$

DEFINITION 2.12. The *associated* 1*-categories* are defined by $\mathrm{GCob} := \tau_\iota(\underline{\mathrm{GCob}})$ and $\mathrm{fGCob} := \tau_\iota(\underline{\mathrm{fGCob}})$.

An analogous argument to Lemma 2.11 shows that disjoint union induces a symmetric monoidal structure on GCob and fGCob.

We can now construct equivalences of categories $\mathrm{GCob} \to \mathrm{Cob}_2^{\mathrm{closed}}$ and $\mathrm{fGCob} \to \mathrm{Cob}_2^{\mathrm{open}}$.
A morphism in GCob (or fGCob) is given by an equivalence class $[G]$ of (fat) graphs up to zig-zags of (fat) graph isomorphisms.
Let $G \in \underline{\mathrm{GCob}}(X,Y)$ be a graph cobordism. We construct the corresponding closed cobordism $\mathrm{Cob}(G)$ as follows.
An internal vertex of arity 0 contributes a copy of S^2 in $\mathrm{Cob}(G)$. A vertex of arity 0 which is only incoming or outgoing contributes a copy of D^2 in $\mathrm{Cob}(G)$ with its boundary labelled by the corresponding element in X or Y. A vertex of arity 0 in $L_{in} \cap L_{out}$ contributes a cylinder $S^1 \times [0,1]$ with its two boundary components labelled by the corresponding elements in X and Y. For a connected component of G consisting of more than just one vertex, each edge gives a copy of the cylinder $S^1 \times [0,1]$, each external vertex of arity 1 is labelled by an element in X or Y. This gives a labelled copy of S^1 which we glue to the corresponding cylinder. Finally, we can glue the edges at internal vertices to copies of $S^2 \setminus (\sqcup_{s^{-1}(v)} D^2)$ (see Figure 9).
The homeomorphism type of this construction is invariant under edge collapses and graph isomorphisms. Therefore $\mathrm{Cob}(G)$ only depends on the class that G represents in $\pi_0(\underline{\mathrm{GCob}}(X,Y))$. We thus get a well defined map

$$\mathrm{GCob}(X,Y) = \pi_0(\underline{\mathrm{GCob}}(X,Y)) \to \mathrm{Cob}_2^{\mathrm{closed}}(\coprod_X S^1, \coprod_Y S^1),$$
$$[G] \mapsto \mathrm{Cob}(G).$$

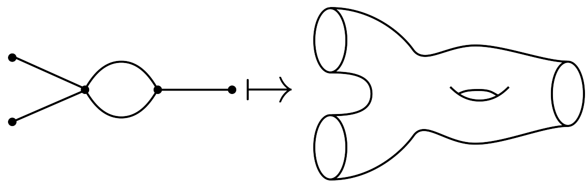


FIGURE 9. The closed cobordism $\mathrm{Cob}(G)$ corresponding to a graph G.

Similarly we can construct for a fat graph cobordism between X and Y an open cobordism $\mathrm{fCob}(G)$ between $\coprod_X [0,1]$ and $\coprod_Y [0,1]$. It is given by the following construction.
Vertices v of arity 0 contribute a copy of D^2 with an embedding of copies of the $[0,1]$ labelled by the elements in X, Y labelling v. For a connected component of G consisting of more than just one vertex, each edge can be thickened to $[0,1] \times [0,1]$ and each external vertex corresponds to a closed interval. This gives us copies of the interval $[0,1]$ labelled by X and Y. Finally, the cyclic ordering at each internal vertex gives a way to glue the thick edges to a closed disc D^2 (see Figure 10).
The homeomorphism type $\mathrm{fCob}(G)$ only depends on the equivalence class $[G] \in \pi_0(\underline{\mathrm{fGCob}}(X,Y))$, because the construction is invariant under fat edge collapses and graph isomorphisms.
We thus have defined a map:

$$\mathrm{fGCob}(X,Y) = \pi_0(\underline{\mathrm{fGCob}}(X,Y)) \to \mathrm{Cob}_2^{\mathrm{open}}\left(\coprod_X [0,1], \coprod_Y [0,1]\right),$$
$$[G] \mapsto \mathrm{fCob}(G).$$

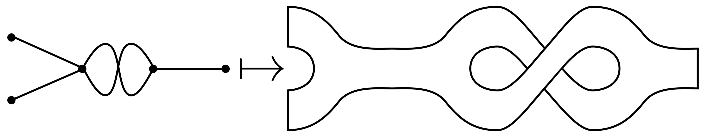


FIGURE 10. The open cobordism fCob(G) corresponding to a fat graph G.

PROPOSITION 2.13. *There are equivalences of categories*

$$\begin{aligned} \mathrm{GCob} &\to \mathrm{Cob}_2^{\mathrm{closed}}, \\ X &\mapsto \coprod_X S^1, \\ [G] &\mapsto \mathrm{Cob}(G) \end{aligned}$$

and

$$\begin{aligned} \mathrm{fGCob} &\to \mathrm{Cob}_2^{\mathrm{open}}, \\ X &\mapsto \coprod_X [0,1], \\ [G] &\mapsto \mathrm{fCob}(G). \end{aligned}$$

PROOF. We first prove that this map commutes with composition: composition of graphs $G \in \mathrm{GCob}(X, Y)$ and $G' \in \mathrm{GCob}(Y, Z)$ is given by gluing the outgoing legs of G to the ingoing legs of G' according to the Y labelling. We do a case distinction for the vertices we glue together.

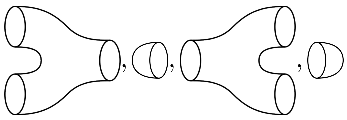
- If one of them is an arity zero vertex which is both incoming and outgoing, gluing the vertex does not change (the isomorphism type of) G. Similarly after applying Cob this corresponds to gluing a cylinder $S^1 \times [0,1]$ to S^1. This does not change the homeomorphism type of Cob(G).
- If one of the vertices is an arity zero vertex v which is either incoming or outgoing, gluing v to G corresponds to forgetting the label at the glued vertex. After applying Cob, this gives $S^2 \setminus D^2$ glued along S^1. This is homeomorphic to gluing $\mathrm{Cob}(v) = D^2 \cong S^2 \setminus D^2$ along S^1.
- If both glued vertices are of arity 1, gluing them gives a valence 2 vertex. Under our construction a valence 2 gives $S^2 \setminus (D^2 \sqcup D^2)$ in $\mathrm{Cob}(G' \circ G)$ to which we glue two cylinders. This is homeomorphic to just gluing the two cylinders together along S^1.

A similar argument to the closed case shows that fCob commutes with composition. For this we note that $[0,1] \times [0,1]$ to a disk D^2 along the boundary is homeomorphic to $[0,1] \times [0,1]$.
The unit in the graph categories are given by G_{id_X} as in Example 2.10. We have

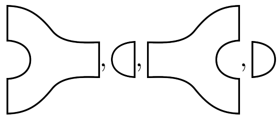
$$\begin{aligned} \mathrm{Cob}\,(G_{\mathrm{id}_X}) &= \coprod_X S^1 \times [0,1] = \mathrm{id}_X, \\ \mathrm{fCob}\,(G_{\mathrm{id}_X}) &= \coprod_X [0,1] \times [0,1] = \mathrm{id}_X \end{aligned}$$

and thus Cob and fCob are functors.
Moreover, the functors are surjective on objects.
It thus remains to show that Cob and fCob are bijective on the set of morphisms. For this, we note that both functors are monoidal because the monoidal structure on the source and the target

is given by disjoint union which commutes with our construction.
Kock shows in [Koc04] that $\mathrm{Cob}_2^{\mathrm{closed}}$ is generated by



as a symmetric monoidal category. These generators are in the image of Cob which shows fullness of Cob.
Similarly, Lauda and Pfeiffer show in [LP08] that $\mathrm{Cob}_2^{\mathrm{open}}$ are generated by



which are in the image. This shows fullness of fCob.
For faithfulness of Cob, we assume that $\mathrm{Cob}(G) = \mathrm{Cob}(G')$. Without loss of generality, we may assume that both graphs are connected and get sent to a surface of genus g. We know that G and G' have the same number of incoming and outgoing legs. We choose trees in $T \subseteq G$ and $T \subseteq G'$ that contain exactly all internal vertices. Collapsing all edges in T and T' gives graphs G/T and G'/T' with one internal vertex. Because $[G] = [G/T]$, we have $\mathrm{Cob}(G/T) = \mathrm{Cob}(G)$ is a surface of genus g. In particular, G/T has g tadpoles. Similarly, we note that G'/T' has g tadpoles. This shows $G/T \cong G'/T'$ and thus $[G] = [G/T] = [G'/T'] = [G']$ shows faithfulness of Cob.
For faithfulness of fCob, we use the category of open-closed fat graphs $\mathcal{F}at^{oc}$ as defined in [Ega15]. Egas shows in [Ega15, Theorem A]

$$\mathcal{F}at^{oc} \simeq \coprod_S B\mathrm{Mod}(S)$$

where the disjoint union runs over all topological types of open-closed cobordisms in which each connected component has at least one boundary component which is not free. Restricting to the open part and taking π_0 shows that if $\mathrm{fCob}(G) = \mathrm{fCob}(G')$ implies $[G] = [G']$ for G and G' where all connected components have at least one external vertex.
For general G and G' with $\mathrm{fCob}(G) = \mathrm{fCob}(G')$, we can add an edge and an external vertex to each connected component and obtain graphs G_+ and G'_+. We have $\mathrm{fCob}(G_+) = \mathrm{fCob}(G'_+)$ by the above argument. We then have a zig-zag of fat graph morphisms connecting G_+ and G'_+. This zig-zag sends the added external vertices to each other. Thus we obtain by restriction a zig-zag connecting G and G' and thus $[G] = [G']$. This shows faithfulness of fCob and concludes the proof. $\square$

We can now define the planar graph cobordism category.

Definition 2.14. The *planar graph cobordism category* is a 2-category $\underline{\mathrm{pGCob}}$ with

$$\mathrm{Ob}(\underline{\mathrm{pGCob}}) = \{\text{finite sets with linear ordering}\},$$

and the morphism categories $\underline{\mathrm{pGCob}}(X, Y)$ are given by the full subcategory of $\underline{\mathrm{fGCob}}(X, Y)$ spanned by all fat graphs G such that $\mathrm{fCob}(G)$ is a planar cobordism in $\mathrm{Cob}_2^{\mathrm{planar}}(X, Y)$.

We denote the associated 1-category $\mathrm{pGCob} := \tau_\iota(\underline{\mathrm{pGCob}})$ with

$$\mathrm{pGCob}(X, Y) = \pi_0(\underline{\mathrm{pGCob}}(X, Y)).$$

The monoidal structure of $\mathrm{Fin}_{\leq}$, the category of finite ordered sets with order preserving maps, induces a monoidal structure on $\underline{\mathrm{pGCob}}$ and pGCob. The argument is analogous to Lemma 2.11 as for any isomorphism of finite ordered sets $f\colon X \cong Y$, the graph G_f as in Example 2.10 is a planar

graph.
However, the monoidal structure is not symmetric. This is because the monoidal structure on $\mathrm{Fin}_{\leq}$ is also not symmetric.

We have a forgetful functor pGCob $\to$ fGCob. The category pGCob is defined such that

$$\mathrm{pGCob} \longrightarrow \mathrm{fGCob} \xrightarrow{\mathrm{fCob}} \mathrm{Cob}_2^{\mathrm{open}}$$

factors through $\mathrm{Cob}_2^{\mathrm{planar}} \to \mathrm{Cob}_2^{\mathrm{open}}$. Moreover, the following Corollary is immediate from Proposition 2.13 and the definitions of pGCob and $\mathrm{Cob}_2^{\mathrm{planar}}$.

Corollary 2.15. *The functor*

$$\begin{aligned} \mathrm{pGCob} &\to \mathrm{Cob}_2^{\mathrm{planar}}, \\ X &\mapsto \coprod_X [0,1], \\ [G] &\mapsto \mathrm{fGCob}(X) \end{aligned}$$

is an equivalence of categories.

3. Frobenius algebras

In this section, we recall the definition of Frobenius algebras. We give definitions both in terms of maps and generators and, on the other hand, as functors out of cobordism categories. Then in 3.1, we apply those definitions to the graded context and show in Proposition 3.7 that in the odd dimensional case, there exist only trivial Frobenius algebras.

Definition 3.1. Let $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ be a (strict) monoidal category.
A *Frobenius algebra* $(A, \mu, \eta, \nu, \varepsilon)$ in $\mathcal{C}$ is an object A in $\mathcal{C}$ together with maps

$$\begin{aligned} &\mu\colon A \otimes A \to A, && \eta\colon \mathbb{1} \to A, \\ &\nu\colon A \to A \otimes A, && \varepsilon\colon A \to \mathbb{1} \end{aligned}$$

satisfying the following relations:
(i) associativity $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$;
(ii) unitality $\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id}_A = \mu \circ (\mathrm{id} \otimes \eta)$;
(iii) coassociativity $(\mathrm{id} \otimes \nu) \circ \nu = (\nu \otimes \mathrm{id}) \circ \nu$;
(iv) counitality $(\mathrm{id} \otimes \varepsilon) \circ \nu = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \nu$;
(v) Frobenius relation $(\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \nu) = \nu \circ \mu = (\mathrm{id} \otimes \mu) \circ (\nu \otimes \mathrm{id})$.

If $\mathcal{C}$ is also symmetric with twist map τ, a *commutative Frobenius algebra* $(A, \mu, \eta, \nu, \varepsilon)$ in $\mathcal{C}$ is an object A in $\mathcal{C}$ together with maps as above satisfying the relations (i)-(v) and

(vi) commutativity $\varepsilon \circ \mu \circ \tau = \mu$

and a *symmetric Frobenius algebra* $(A, \mu, \eta, \nu, \varepsilon)$ in $\mathcal{C}$ is an object A in $\mathcal{C}$ together with maps as above satisfying the relations (i)-(v) and

(vi') symmetry $\varepsilon \circ \mu \circ \tau = \varepsilon \circ \mu$.

A *morphism of Frobenius algebras* $(A, \mu, \eta, \nu, \varepsilon) \to (A', \mu', \eta', \nu', \varepsilon')$ is a morphism $\varphi\colon A \to A'$ in $\mathcal{C}$ such that

- $\mu \circ (\varphi \otimes \varphi) = \varphi \circ \mu'$;
- $\varphi \circ \eta = \eta'$;
- $\nu \circ \varphi = (\varphi \otimes \varphi) \circ \nu'$;
- $\varepsilon = \varepsilon' \circ \varphi$.

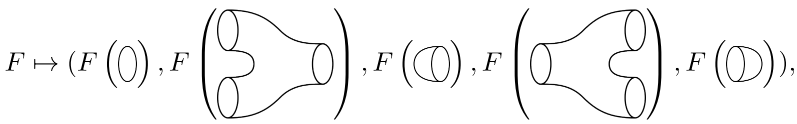
Denote by $\mathrm{Frob}(\mathcal{C})$ the category of Frobenius algebras in $\mathcal{C}$ with morphisms of Frobenius algebras, by $\mathrm{ComFrob}(\mathcal{C})$ the full subcategory spanned by commutative Frobenius algebras and by $\mathrm{SymmFrob}(\mathcal{C})$ the full subcategory spanned by symmetric Frobenius algebras.

It is a well-known fact that an equivalent description of commutative Frobenius algebras is as symmetric monoidal functor out of a closed 2D-cobordism category (see [Abr96; Koc04]). Similarly, a symmetric Frobenius algebra can be characterised as a symmetric monoidal functor out of a open 2D-cobordism category and a Frobenius algebra can be characterised as a monoidal functor out of a planar 2D-cobordism category (see [LP08]).

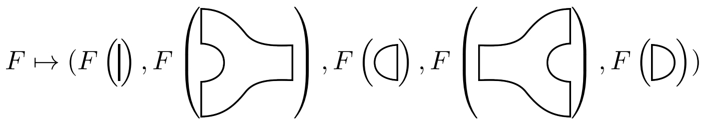
Notation 3.2. For (strict) symmetric monoidal categories $\mathcal{C}, \mathcal{D}$, we denote $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ for the category of (strict) symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ and symmetric monoidal natural transformations between them.
For (strict) monoidal categories $\mathcal{C}, \mathcal{D}$, we denote $\mathrm{Fun}^{\mathrm{mon}}(\mathcal{C}, \mathcal{D})$ for the category of (strict) monoidal functors from $\mathcal{C}$ to $\mathcal{D}$ and monoidal natural transformations between them.

Proposition 3.3 ([LP08, Corollary 4.5, Corollary 4.6, Corollary 4.7.]). *There are natural equivalences of categories*

$$\mathrm{Fun}^{\otimes}(\mathrm{Cob}_2^{\mathrm{closed}}, \mathcal{C}) \to \mathrm{ComFrob}(\mathcal{C}),$$

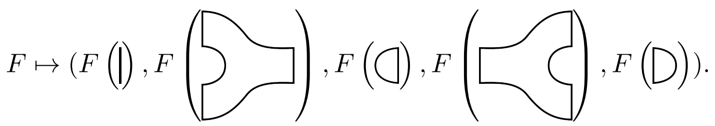


$$\mathrm{Fun}^{\otimes}(\mathrm{Cob}_2^{\mathrm{open}}, \mathcal{C}) \to \mathrm{SymFrob}(\mathcal{C}),$$



and

$$\mathrm{Fun}^{\mathrm{mon}}(\mathrm{Cob}_2^{\mathrm{planar}}, \mathcal{C}) \to \mathrm{Frob}(\mathcal{C}),$$



In particular, a natural transformation gives a morphism of Frobenius algebras.

We can also apply Proposition 2.13 and Corollary 2.15 to get a characterisation in terms of graph cobordisms.

Corollary 3.4. *Let $\mathcal{C}$ be a monoidal category. There exists an equivalence of categories*

$$\mathrm{Fun}^{\mathrm{mon}}(\mathrm{pGCob}, \mathcal{C}) \to \mathrm{Frob}(\mathcal{C}),$$
$$F \mapsto (F(*), F(\succ\!-), F(\bullet\!-), F(-\!\prec), F(-\!\bullet)).$$

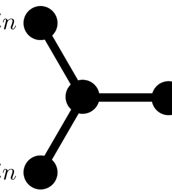
If $\mathcal{C}$ is also symmetric, there are natural equivalences of categories

$$\mathrm{Fun}^{\otimes}(\mathrm{GCob}, \mathcal{C}) \to \mathrm{ComFrob}(\mathcal{C}),$$
$$F \mapsto (F(*), F(\succ\!-), F(\bullet\!-), F(-\!\prec), F(-\!\bullet))$$

and

$$\begin{aligned}\mathrm{Fun}^{\otimes}(\mathrm{fGCob}, \mathcal{C}) &\to \mathrm{SymFrob}(\mathcal{C}),\\ F &\mapsto (F\,(*)\,, F\,(\succ\!\!-)\,, F\,(\bullet\!\!-)\,, F\,(-\!\!\prec)\,, F\,(-\!\!\bullet)).\end{aligned}$$

Here $\succ\!\!-$ denotes the following graph



where we do not denote the external vertices. Similarly, $\bullet\!\!-$ denotes the following graph

without the external vertex. The graphs $-\!\!\prec$ and $-\!\!\bullet$ are defined as their respective duals.
For a functor F out of GCob (or fGCob or pGCob), we define the following notation:

$$\begin{aligned}&A \coloneqq F(*), &&\\ &\mu \coloneqq F(\succ\!\!-), && \eta \coloneqq F(\bullet\!\!-),\\ &\nu \coloneqq F(-\!\!\prec), && \varepsilon \coloneqq F(-\!\!\bullet),\\ &p \coloneqq \varepsilon \circ \mu, && c \coloneqq \nu \circ \eta.\end{aligned}$$

One advantage of the graph models of cobordism categories GCob, fGCob and pGCob compared to the geometric cobordism categories $\mathrm{Cob}_2^{\mathrm{closed}}$, $\mathrm{Cob}_2^{\mathrm{open}}$ and $\mathrm{Cob}_2^{\mathrm{planar}}$ is that it becomes convenient to state that all commutative Frobenius algebras are symmetric and all symmetric Frobenius algebras are Frobenius algebras.

Proposition 3.5. *There is a canonical symmetric monoidal functor* fGCob $\to$ GCob *given by forgetting the fat graph structure. Precomposition with this functor encodes the forgetful functor* $\mathrm{ComFrob}(\mathcal{C}) \to \mathrm{SymFrob}(\mathcal{C})$ *for all symmetric monoidal categories* $\mathcal{C}$.
Moreover, there is a canonical monoidal functor pGCob $\to$ fGCob *given by forgetting the linear ordering on objects. Precomposition with this functor encodes the forgetful functor* $\mathrm{SymFrob}(\mathcal{C}) \to \mathrm{ComFrob}(\mathcal{C})$ *for all monoidal categories* $\mathcal{C}$.

3.1. Graded context. A first attempt to define a Frobenius algebra in a graded context may be to give a functor out of the cobordism categories we described until now. This fails to describe certain interesting examples in the graded context. We first specify what we mean by the graded context.

Let R be a commutative ring and let $A = \{A_i\}_{i\in\mathbb{Z}}$, $B = \{B_i\}_{i\in\mathbb{Z}}$ be two $\mathbb{Z}$-graded R-modules. We define the $\mathbb{Z}$-graded R-module

$$\begin{aligned}R\mathrm{Mod}_\bullet(A, B) &= \{R\mathrm{Mod}_d(A, B)\}_{d\in\mathbb{Z}},\\ R\mathrm{Mod}_d(A, B) &= \{\phi = (\phi_i)_{i\in\mathbb{Z}} \mid \phi_i\colon A_i \to B_{i+d}\ R\text{-linear}\}.\end{aligned}$$

We consider two notions of categories of graded R-module. Firstly, $R\mathrm{Mod}_0$ the category of $\mathbb{Z}$-graded R-modules with morphisms $R\mathrm{Mod}_0(A, B)$. Secondly, $R\mathrm{Mod}_\bullet$ the category with morphisms

$R\mathrm{Mod}_\bullet(A, B)$.
We note that both categories $R\mathrm{Mod}_\bullet$ and $R\mathrm{Mod}_0$ can be enriched over $R\mathrm{Mod}_0$ (or $R\mathrm{Mod}_\bullet$). Furthermore, if R is an S-algebra, both are enriched over $S\mathrm{Mod}_0$ (or $S\mathrm{Mod}_\bullet$).
Both are symmetric monoidal categories with unit

$$\mathbb{1}_d \coloneqq \begin{cases} \mathbb{Z} & \text{if } d = 0, \\ 0 & \text{else,} \end{cases}$$

and the tensor product:

$$(A \otimes B)_k = \bigoplus_{i+j=k} A_i \otimes_R B_j$$

with the twist induced by the Koszul sign

$$\begin{aligned} \tau\colon A_i \otimes B_j &\to B_j \otimes A_i, \\ a \otimes b &\mapsto (-1)^{ij} b \otimes a. \end{aligned}$$

Recall that we denote the shift ΣA as the object with $(\Sigma A)_{i+1} = A_i$.

We fix a (symmetric) monoidal functor $F\colon \mathrm{GCob} \to R\mathrm{Mod}_\bullet$ (or out of fGCob or pGCob) enriched over sets. In particular, we allow all degrees for $\mu = F(\succ\!\!-)$ and $\nu = F(-\!\!\prec)$. We note that unitality implies that $|\mu| + |\eta| = |\mathrm{id}| = 0$ and counitality implies $|\nu| + |\varepsilon| = |\mathrm{id}| = 0$.

Proposition 3.6. *Let R be a principal ideal domain. Let $F\colon \mathrm{pGCob} \to R\mathrm{Mod}_\bullet$ be a monoidal functor. Then A is free of finite rank.*

Proof. A consequence of unitality, counitality and the Frobenius relation is that

$$(p \otimes \mathrm{id}) \circ (\mathrm{id} \otimes c) = \mathrm{id} = (\mathrm{id} \otimes p) \circ (c \otimes \mathrm{id}). \tag{3}$$

This means that A is dualizable which is equivalent to A being projective and finitely generated. As R is a principal ideal domain, this is equivalent to A being free of finite rank. $\square$

Proposition 3.7. *Let R be a principal ideal domain of characteristic not 2.*
Let $F\colon \mathrm{GCob} \to R\mathrm{Mod}_\bullet$, $F\colon \mathrm{fGCob} \to R\mathrm{Mod}_\bullet$ or $F\colon \mathrm{pGCob} \to R\mathrm{Mod}_\bullet$ be a monoidal functor. If $|\mu| - |\nu|$ is odd, then A is the zero module.

Proof. By Proposition 3.5, it suffices to show the statement for pGCob and, by Proposition 3.6, we can fix a basis $\alpha_1, \ldots, \alpha_k$ of A.
We note that $n \coloneqq |p| = |\mu| + |\varepsilon| = |\mu| - |\nu|$ and $|c| = |\nu| + |\eta| = |\nu| - |\mu| = -n$ are both of odd degree. We write

$$c(1) = \sum_j \alpha_j \otimes \beta_j$$

for some $\beta_j \in A$. By duality, $\beta_1, \ldots, \beta_k$ also defines a basis. The fact that $|c| = -n$ shows that $|\beta_i| = -|\alpha_i| - n$ for all i.
If $A \neq 0$, we may assume that $k > 0$ and thus $\alpha_1 \neq 0$. We want to exploit (3) for α_1 and β_1. We first apply the right hand side to α_1:

$$\alpha_1 = (\mathrm{id} \otimes p) \circ (c \otimes \mathrm{id})(\alpha_1) = (\mathrm{id} \otimes p) \left(\sum_j \alpha_j \otimes \beta_j \otimes \alpha_1 \right) = \sum_j (-1)^{n|\alpha_j|} \alpha_j p(\beta_j \otimes \alpha_1).$$

As $\alpha_1, \dots, \alpha_k$ is a basis, we must have

$$p(\beta_1 \otimes \alpha_1) = (-1)^{n|\alpha_1|}. \tag{4}$$

On the other hand, when we apply the left hand side of (3) to β_1, we find

$$\beta_1 = (p \otimes \mathrm{id}) \circ (\mathrm{id} \otimes c)(\beta_1) = (p \otimes \mathrm{id}) \left((-1)^{n|\beta_1|} \beta_1 \otimes \sum_j \alpha_j \otimes \beta_j \right) = (-1)^{n|\beta_1|} \sum_j p(\beta_1 \otimes \alpha_j)\beta_j.$$

As $\beta_1, \dots, \beta_k$ is a basis, we must have

$$p(\beta_1 \otimes \alpha_1) = (-1)^{n(-n-|\alpha_1|)} = (-1)^{n+n|\alpha_1|}. \tag{5}$$

As n is odd and $(-1)^n = -1 \neq 1$ in R, we have the desired contradiction between (4) and (5). Therefore there is no element $\alpha_1 \neq 0$ and thus $A = 0$. $\square$

In section 6, we give examples of structures that look like Frobenius algebras such that $|\mu| - |\nu|$ is odd and are non-zero. In other words, this characterisation of a Frobenius algebra misses key examples.

4. Graded Frobenius algebras: PROPs

The goal of this section is to define graph cobordism categories which already encode the signs and degrees. Concretely, we define categories $\mathrm{GCob}_{c,d}$, $\mathrm{fGCob}_{c,d}$ and pGCob enriched over graded abelian groups where the product has degree c and the coproduct degree d. This gives us a new definition of graded Frobenius algebras as functors out of $\mathrm{GCob}_{c,d}$, $\mathrm{fGCob}_{c,d}$ or $\mathrm{pGCob}_{c,d}$ to a category enriched over graded abelian groups $\mathbb{Z}\mathrm{Mod}_0$.
In Subsection 4.2, we prove some properties immediately from the definition: suspending an algebra over $\mathrm{GCob}_{c,d}$ gives an algebra in $\mathrm{GCob}_{c-1,d+1}$. Tensoring algebras over $\mathrm{GCob}_{c,d}$ and $\mathrm{GCob}_{c',d'}$ gives an algebra over $\mathrm{GCob}_{c+c',d+d'}$. Analogous statements for $\mathrm{fGCob}_{c,d}$ and $\mathrm{pGCob}_{c,d}$ can be proved in the same way.
In Subsection 4.3, we study the canonical subdioperad of $\mathrm{GCob}_{c,d}$ which is generated by forests. If $c+d$ is odd, then this subdioperad has 2-torsion cokernel in the full PROP. This dioperad naturally shows up to describe an infinite-dimensional structure carrying a Frobenius relation as in 6.4.

4.1. Definition. In the following computations and definitions, we heavily use the concept of a top exterior power of an abelian group. Unfortunately, we require multiple slightly different versions of a top exterior power. We hope to ameliorate this situation by giving a clear overview of the versions we use.

- Let M be a free abelian group of rank n, we denote
  - $\mathrm{Or}(M) \coloneqq \Lambda^n M$ the top exterior power as a graded abelian group concentrated in degree 0;
  - $\det(M) \coloneqq \Sigma^{-n}\mathrm{Or}(M)$ the top exterior power as a graded abelian group concentrated in degree $-n$.
- Let A be a free graded abelian group of rank 1. It is invertible with respect to $\otimes$, the tensor product of graded abelian groups, with its inverse $A^{-1} \coloneqq \mathbb{Z}\mathrm{Mod}_\bullet(A, \mathbb{1})$.
- Let $A = \{A_i\}_{i\in\mathbb{Z}}$ be a free graded abelian group of finite rank, we denote

$$\mathbf{det}(A) \coloneqq \bigotimes_{i\in\mathbb{Z}} \det(A_i)^{(-1)^i} \coloneqq \bigotimes_{i=-n}^{n} \det(A_i)^{(-1)^i}$$

  where $n \geq 0$ is such that $A_i = 0$ for $|i| > n$.
- Let G be a graph and $c, d \in \mathbb{Z}$, we denote

– $\mathbf{det}(G, \partial_{in}) \coloneqq \mathbf{det}(H_*(|G|, |L_{in}(G)|))$;
– $\mathbf{det}(G, \partial_{out}) \coloneqq \mathbf{det}(H_*(|G|, |L_{out}(G)|))$;
– $\mathbf{det}_{c,d}(G) \coloneqq \mathbf{det}(G, \partial_{in})^{\otimes c} \otimes \mathbf{det}(G, \partial_{out})^{\otimes d}$.

LEMMA 4.1. *Let $A, B, C \in \mathbb{Z}\mathrm{Mod}_\bullet$ free of finite total rank. Then the following holds:*

(a) the determinant $\mathbf{det}(A)$ is of rank 1 and concentrated in degree $-\chi(A)$, where $\chi(A)$ is the Euler characteristic of A;

(b) there exists a natural isomorphism

$$\mathbf{det}(A \oplus B) \cong \mathbf{det}(A) \otimes \mathbf{det}(B);$$

(c) there exists a natural isomorphism

$$\mathbf{det}(A \oplus \Sigma A) \cong \mathbb{1}.$$

This implies a natural isomorphism $\mathbf{det}(A)^{-1} \cong \mathbf{det}(\Sigma A)$;

(d) if $0 \to A \to B \to C \to 0$ is a short exact sequence, there exists a canonical isomorphism $\mathbf{det}(A) \otimes \mathbf{det}(C) \cong \mathbf{det}(B)$;

(e) if ∂ a differential on A such that $H_*(A) \coloneqq H_*(A, \partial)$ and A are finitely generated free $\mathbb{Z}$-modules, then there exists a natural isomorphism:

$$\mathbf{det}(A) \cong \mathbf{det}(H_*(A)).$$

PROOF. Statements (a), (b) and (c) follow directly from the definition and the following facts for finitely generated free $\mathbb{Z}$-modules $M_1, \ldots, M_k$:

$$\det\left(\bigoplus_{i=1}^{k} M_i\right) \cong \bigotimes_{i=1}^{k} \det(M_i) \text{ and } \mathbf{det}(\Sigma M_i) \cong \det(M_i)^{-1}.$$

For statement (d), we note that we can choose a splitting $f\colon C \to B$ because C is free. This gives an isomorphism

$$\mathbf{det}(A) \otimes \mathbf{det}(C) \cong \mathbf{det}(A \oplus C) \cong \mathbf{det}(B).$$

This is independent of the choice of splitting because two splittings of $B \to C$ differ by elements in A and thus the difference between the two induced isomorphisms is zero after tensoring with $\mathbf{det}(A)$. For statement (e), the short exact sequence

$$0 \to \mathrm{im}(\partial\colon A_{i+1} \to A_i) \to \ker(\partial\colon A_i \to A_{i-1}) \to H_i(A) \to 0$$

gives

$$\begin{aligned}
\mathbf{det}(\ker(\partial\colon A_i \to A_{i-1})) &\overset{\text{(d)}}{\cong} \mathbf{det}(\mathrm{im}(\partial\colon A_{i+1} \to A_i)) \otimes \mathbf{det}(H_i(A)) \\
\implies \mathbf{det}(H_i(A)) &\cong \mathbf{det}(\ker(\partial\colon A_i \to A_{i-1})) \otimes \mathbf{det}(\mathrm{im}(\partial\colon A_{i+1} \to A_i))^{-1} \\
&\overset{\text{(c)}}{\cong} \mathbf{det}(\ker(\partial\colon A_i \to A_{i-1})) \otimes \mathbf{det}(\Sigma\, \mathrm{im}(\partial\colon A_{i+1} \to A_i)) \\
&\cong \mathbf{det}(\ker(\partial\colon A_i \to A_{i-1})) \otimes \mathbf{det}(\mathrm{im}(\partial\colon A_i \to A_{i-1}))
\end{aligned}$$

for all $i \in \mathbb{Z}$. On the other hand, the short exact sequence

$$0 \to \ker(\partial\colon A_i \to A_{i-1}) \to A_i \xrightarrow{\partial} \mathrm{im}(\partial\colon A_i \to A_{i-1}) \to 0$$

gives

$$\mathbf{det}(A) \overset{\text{(c)}}{\cong} \mathbf{det}(\ker(\partial\colon A_i \to A_{i-1})) \otimes \mathbf{det}(\mathrm{im}(\partial\colon A_i \to A_{i-1}))$$

which concludes the proof. $\square$

We note that a morphism of graph cobordisms $G \to G'$ specifies homotopy equivalences

$$(|G|, |L_{in}(G)|) \to (|G'|, |L_{in}(G')|),$$
$$(|G|, |L_{out}(G)|) \to (|G'|, |L_{out}(G')|)$$

and thus isomorphisms

$$\mathbf{det}(G, \partial_{in}) \to \mathbf{det}(G', \partial_{in}),$$
$$\mathbf{det}(G, \partial_{out}) \to \mathbf{det}(G', \partial_{out}),$$
$$\mathbf{det}_{c,d}(G) \to \mathbf{det}_{c,d}(G').$$

Therefore $\det_{c,d} \colon \underline{\mathrm{GCob}}(X, Y) \to \mathbb{Z}\mathrm{Mod}_0$ defines a functor for all $c, d \in \mathbb{Z}$ and finite sets X, Y. Moreover, we can glue orientations together:

Lemma 4.2. *Let G, G' be composable graphs in $\underline{\mathrm{GCob}}$, there is a natural isomorphism*

$$\mathrm{comp} \colon \mathbf{det}_{c,d}(G') \otimes \mathbf{det}_{c,d}(G) \cong \mathbf{det}_{c,d}(G' \circ G).$$

Proof. We recall:

$$\begin{aligned}\mathbf{det}_{c,d}(G') \otimes \mathbf{det}_{c,d}(G) &\cong \mathbf{det}(G', \partial_{in})^{\otimes c} \otimes \mathbf{det}(G', \partial_{out})^{\otimes d} \otimes \mathbf{det}(G, \partial_{in})^{\otimes c} \otimes \mathbf{det}(G, \partial_{out})^{\otimes d} \\ &\cong \mathbf{det}(G', \partial_{in})^{\otimes c} \otimes \mathbf{det}(G, \partial_{in})^{\otimes c} \otimes \mathbf{det}(G', \partial_{out})^{\otimes d} \otimes \mathbf{det}(G, \partial_{out})^{\otimes d}\end{aligned}$$

and

$$\mathbf{det}_{c,d}(G' \circ G) \cong \mathbf{det}(G' \circ G, \partial_{in})^{\otimes c} \otimes \mathbf{det}(G' \circ G, \partial_{out})^{\otimes d}.$$

It thus suffices to give isomorphisms

$$\mathbf{det}(G', \partial_{in}) \otimes \mathbf{det}(G, \partial_{in}) \cong \mathbf{det}(G' \circ G, \partial_{in})$$
$$\mathbf{det}(G', \partial_{out}) \otimes \mathbf{det}(G, \partial_{out}) \cong \mathbf{det}(G' \circ G, \partial_{out})$$

We give the first isomorphism, the second one is constructed analogously.
We can canonically embed $|G| \subseteq |G' \circ G|$. Let $C_*(|G|, |\partial_{in}|)$, $C_*(|G'|, |\partial_{in}|)$, $C_*(|G' \circ G|, |\partial_{in}|)$ and $C_*(|G' \circ G|, |G|)$ be the cellular chain complexes of the pairs. We note that the inclusion

$$C_*(|G' \circ G|, |G|) \to C_*(|G'|, |\partial_{in}|)$$

is an isomorphism.
The triple of spaces $(|G' \circ G|, |G|, |\partial_{in}|)$ gives a short exact sequence of chain complexes:

$$0 \longrightarrow C_*(|G|, |\partial_{in}|) \longrightarrow C_*(|G' \circ G|, |\partial_{in}|) \longrightarrow \overbrace{C_*(|G' \circ G|, |G|)}^{C_*(|G'|, |\partial_{in}|)} \longrightarrow 0.$$

Lemma 4.1 (d) and (e) now give the desired isomorphism. □

Definition 4.3. Let $c, d \in \mathbb{Z}$.
The category $\mathrm{GCob}_{c,d}$ is defined with objects finite sets and morphism spaces

$$\mathrm{GCob}_{c,d}(X, Y) = \operatorname*{colim}_{\underline{\mathrm{GCob}}(X,Y)} \mathbf{det}_{c,d}(G).$$

The category $\mathrm{fGCob}_{c,d}$ is defined with objects finite sets and morphism spaces

$$\mathrm{fGCob}_{c,d}(X, Y) = \operatorname*{colim}_{\underline{\mathrm{fGCob}}(X,Y)} \mathbf{det}_{c,d}(G).$$

The category $\mathrm{fGCob}_{c,d}$ is defined with objects finite sets and morphism spaces

$$\mathrm{pGCob}_{c,d}(X, Y) = \operatorname*{colim}_{\underline{\mathrm{pGCob}}(X,Y)} \mathbf{det}_{c,d}(G).$$

Lemma 4.2 lets us define the composition in these categories.

We can now define a graded Frobenius algebra:

Definition 4.4. Let $\mathcal{C}$ be a monoidal category enriched over $\mathbb{Z}\mathrm{Mod}_0$ and let $c, d \in \mathbb{Z}$. A (c,d)-*graded Frobenius algebra* is a monoidal functor $F\colon \mathrm{GCob}_{c,d} \to \mathcal{C}$ enriched over $\mathbb{Z}\mathrm{Mod}_0$, i.e. degree-preserving on morphism spaces. The category is denoted by

$$\mathrm{grFrob}_{c,d}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{mon}}_{\mathbb{Z}\mathrm{Mod}_0}(\mathrm{pGCob}_{c,d}, \mathcal{C}).$$

If $\mathcal{C}$ is also symmetric, a (c,d)-*graded commutative Frobenius algebra* is a symmetric monoidal functor $F\colon \mathrm{GCob}_{c,d} \to \mathcal{C}$ and the category is denoted by

$$\mathrm{grComFrob}_{c,d}(\mathcal{C}) = \mathrm{Fun}^{\otimes}_{\mathbb{Z}\mathrm{Mod}_0}(\mathrm{GCob}_{c,d}, \mathcal{C}).$$

and a (c,d)-*graded symmetric Frobenius algebra* is defined as such a functor $F\colon \mathrm{GCob}_{c,d} \to \mathcal{C}$ and the category is denoted by

$$\mathrm{grSymFrob}_{c,d}(\mathcal{C}) = \mathrm{Fun}^{\otimes}_{\mathbb{Z}\mathrm{Mod}_0}(\mathrm{fGCob}_{c,d}, \mathcal{C}).$$

4.2. Properties. In this Subsection, we prove some first properties of these category. The definitions of the categories $\mathrm{GCob}_{c,d}$, $\mathrm{fGCob}_{c,d}$ and $\mathrm{pGCob}_{c,d}$ we gave are admittedly hard to work with. However, the morphism spaces can be described in a more convenient way.

Proposition 4.5. *There exist isomorphisms*

$$\mathrm{GCob}_{c,d}(X,Y) \cong \bigoplus_{[G]\in \mathrm{GCob}(X,Y)} \mathbf{det}_{c,d}(G)_{\pi_1(|\underline{\mathrm{GCob}}(X,Y)|,G)},$$

$$\mathrm{fGCob}_{c,d}(X,Y) \cong \bigoplus_{[G]\in \mathrm{fGCob}(X,Y)} \mathbf{det}_{c,d}(G)_{\pi_1(|\underline{\mathrm{fGCob}}(X,Y)|,G)}.$$

and

$$\mathrm{pGCob}_{c,d}(X,Y) \cong \bigoplus_{[G]\in \mathrm{pGCob}(X,Y)} \mathbf{det}_{c,d}(G)_{\pi_1(|\underline{\mathrm{pGCob}}(X,Y)|,G)}.$$

Proof. We note that the functor $\mathbf{det}_{c,d}$ sends all maps in $\underline{\mathrm{GCob}}(X,Y)$ to isomorphisms. Therefore, $\mathbf{det}_{c,d}$ factors through the localisations $\underline{\mathrm{GCob}}(X,Y) \to |\underline{\mathrm{GCob}}(X,Y)|$ and $\underline{\mathrm{fGCob}}(X,Y) \to |\underline{\mathrm{fGCob}}(X,Y)|$. Here $|\mathcal{C}|$ is interpreted as the ∞-groupoid given by the category $\mathcal{C}$ with all morphisms formally inverted.
We can thus compute the colimit giving the morphism space between finite sets X and Y as follows:

$$\operatorname*{colim}_{G\in \underline{\mathrm{GCob}}(X,Y)} \mathbf{det}_{c,d}(G) = \operatorname*{colim}_{G\in |\underline{\mathrm{GCob}}(X,Y)|} \mathbf{det}_{c,d}(G)$$

We denote by $\pi_1(|\underline{\mathrm{GCob}}(X,Y)|, G)$ the fundamental group of $|\underline{\mathrm{GCob}}(X,Y)|$ based at the point corresponding to G. An element in $\pi_1(|\underline{\mathrm{GCob}}(X,Y)|, G)$ is represented by a zig-zag of graph morphisms starting and ending at G:

$$G = G_0 \to G_1 \leftarrow G_2 \to \cdots \leftarrow G_k = G.$$

Such a zig-zag gives a zig-zag of maps

$$\mathbf{det}_{c,d}(G_0) \to \mathbf{det}_{c,d}(G_1) \leftarrow \mathbf{det}_{c,d}(G_2) \to \cdots \leftarrow \mathbf{det}_{c,d}(G_k).$$

But all these maps are isomorphism and we thus obtain an action of $\pi_1(|\underline{\mathrm{GCob}}(X,Y)|, G)$ on $\mathbf{det}_{c,d}(G)$.
The functor $\mathbf{det}_{c,d}$ lands in a 1-category. Therefore, we can compute the colimit on $\tau_{\leq 1}\underline{\mathrm{GCob}}(X,Y)$, the 1-truncation of the ∞-groupoid $|\underline{\mathrm{GCob}}(X,Y)|$.

Moreover, two graphs $G, G' \in \underline{\mathrm{GCob}}(X,Y)$ are equivalent in $|\underline{\mathrm{GCob}}(X,Y)|$ if and only if they represent the same element in $\mathrm{GCob}(X,Y) = \pi_0(\underline{\mathrm{GCob}}(X,Y))$. We thus can compute the colimit defining $\mathrm{GCob}_{c,d}(X,Y)$ by:

$$\mathrm{GCob}_{c,d}(X,Y) \cong \operatorname*{colim}_{G\in\tau_{\leq 1}\underline{\mathrm{GCob}}(X,Y)} \mathbf{det}_{c,d}(G) \cong \bigoplus_{[G]\in\mathrm{GCob}(X,Y)} \mathbf{det}_{c,d}(G)_{\pi_1(|\underline{\mathrm{GCob}}(X,Y)|,G)}$$

The proof for fGCob and pGCob is analogous. $\square$

REMARK 4.6. The group $\pi_1(|\underline{\mathrm{GCob}}(X,Y)|,G)$ is isomorphic to $\pi_0(\mathrm{hAut}(|G|))$: a zig-zag of graphs morphisms is the same as a simple homotopy equivalence from the CW-complex $|G|$ to itself. For the 1-dimensional CW complex $|G|$, any homotopy equivalence is homotopic to a simple homotopy equivalence.
On the other hand, the group $\pi_1(|\underline{\mathrm{fGCob}}(X,Y)|,G)$ is isomorphic to the mapping class group $\pi_0(\mathrm{Homeo}^+(\mathrm{fCob}(G), \partial_{in}(\mathrm{fCob}(G)) \cup \partial_{out}(\mathrm{fCob}(G)))$ that fixes the boundary point-wise. This can be seen for example by the open part of Theorem A in [Ega15].
We note that $\underline{\mathrm{pGCob}}(X,Y)$ is a full subcategory of $\underline{\mathrm{fGCob}}(X,Y)$ where if $G \to G' \in \underline{\mathrm{fGCob}}(X,Y)$ and either G or G' is a planar graph, then so is the other. Therefore, we find also

$$\pi_1(|\underline{\mathrm{pGCob}}(X,Y)|,G) \cong \pi_0(\mathrm{Homeo}^+(\mathrm{fCob}(G), \partial_{in}(\mathrm{fCob}(G)) \cup \partial_{out}(\mathrm{fCob}(G))).$$

EXAMPLE 4.7. In the case $c = d = 0$, we have $\mathbf{det}_{0,0}(G) = \mathbb{1}$ and the automorphisms of G act trivially. We therefore compute:

$$\mathrm{GCob}_{0,0}(X,Y) \cong \bigoplus_{[G]\in\mathrm{GCob}(X,Y)} \mathbb{1},$$
$$\mathrm{fGCob}_{0,0}(X,Y) \cong \bigoplus_{[G]\in\mathrm{fGCob}(X,Y|} \mathbb{1}$$

and

$$\mathrm{pGCob}_{0,0}(X,Y) \cong \bigoplus_{[G]\in\mathrm{pGCob}(X,Y|} \mathbb{1}$$

We find

$$\begin{aligned}\mathrm{grComFrob}_{0,0}(R\mathrm{Mod}_\bullet) &= \mathrm{Fun}^{\otimes}_{\mathbb{Z}\mathrm{Mod}_0}(\mathrm{GCob}_{0,0}, R\mathrm{Mod}_\bullet)\\ &\cong \mathrm{Fun}^{\otimes}(\mathrm{GCob}, R\mathrm{Mod}_0)\\ &\cong \mathrm{ComFrob}(R\mathrm{Mod}_0).\end{aligned}$$

Therefore a $(0,0)$-graded commutative Frobenius algebra coincides with the classical definition of a commutative Frobenius algebras with maps in degree 0. An analogous statement holds for (symmetric) Frobenius algebras.

In the following, we study the relation between $\mathrm{GCob}_{c,d}$ for different choices of $c, d \in \mathbb{Z}$. For this, we use the symmetric monoidal structure on PROPs.

REMARK 4.8. Analogous statements hold for fGCob and pGCob with the caveat that pGCob is not symmetric monoidal and thus not a PROP but a PRO. However, the construction for tensor product of PROs and the suspension PRO is analogous to the PROP case.

NOTATION 4.9. We denote the *tensor product* of two PROPs in $\mathbb{Z}\mathrm{Mod}_0$ for the coordinate-wise tensor product: for PROPs P, Q

$$(P \otimes Q)(X,Y) \coloneqq P(X,Y) \otimes Q(X,Y)$$

for finite sets X, Y with the component-wise composition maps.

Example 4.10. Let $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ be a symmetric monoidal category enriched over $\mathbb{Z}\mathrm{Mod}_0$ and let $A, B \in \mathcal{C}$ be objects. There is an isomorphism of PROPs

$$\mathrm{End}_A \otimes \mathrm{End}_B \cong \mathrm{End}_{A\otimes B}$$

given on maps $f\colon A^{\otimes k} \to A^{\otimes l}$ and $g\colon B^{\otimes k} \to B^{\otimes l}$ by

$$(A \otimes B)^{\otimes k} \to A^{\otimes k} \otimes B^{\otimes k} \xrightarrow{f\otimes g} A^{\otimes l} \otimes B^{\otimes l} \to (A \otimes B)^{\otimes l}.$$

Indeed, this map is equivariant and commutes with composition and the product $\otimes$.

Proposition 4.11. *There exist canonical maps of PROPs or PROs*

$$\mathrm{GCob}_{c+c',d+d'} \to \mathrm{GCob}_{c,d} \otimes \mathrm{GCob}_{c',d'},$$
$$\mathrm{fGCob}_{c+c',d+d'} \to \mathrm{fGCob}_{c,d} \otimes \mathrm{fGCob}_{c',d'}$$

and

$$\mathrm{pGCob}_{c+c',d+d'} \to \mathrm{pGCob}_{c,d} \otimes \mathrm{pGCob}_{c',d'}$$

Proof. There are canonical maps

$$\begin{aligned}(6)\qquad &\mathbf{det}(G, \partial_{in})^{\otimes c+c'} \otimes \mathbf{det}(G, \partial_{out})^{\otimes d+d'} \\ &\qquad\cong \mathbf{det}(G, \partial_{in})^{\otimes c} \otimes \mathbf{det}(G, \partial_{out})^{\otimes d} \otimes \mathbf{det}(G, \partial_{in})^{\otimes c'} \otimes \mathbf{det}(G, \partial_{out})^{\otimes d'}.\end{aligned}$$

This then gives maps

$$\mathbf{det}_{c+c',d+d'}(G) \to \operatorname*{colim}_{G' \in \underline{\mathrm{GCob}}(X,Y)} \mathbf{det}_{c,d}(G') \otimes \operatorname*{colim}_{G' \in \underline{\mathrm{GCob}}(X,Y)} \mathbf{det}_{c',d'}(G')$$

for all $G \in \underline{\mathrm{GCob}}(X, Y)$. Taking the colimit over $\underline{\mathrm{GCob}}(X, Y)$ thus gives the desired map for the GCob. The proof for fGCob and pGCob is analogous. $\square$

Remark 4.12. Even though the map (6) used in the construction is an isomorphism, the resulting map of PROPs is not a isomorphism in general. For example if $c + d$ and $c' + d'$ are odd, Proposition 4.22 shows that $\mathrm{GCob}_{c,d}$ and $\mathrm{GCob}_{c',d'}$ is 2-torsion on summands coming from graphs which are not forests. This is not true for $\mathrm{GCob}_{c+c',d+d'}$.

Corollary 4.13. *Let A be an algebra over* $\mathrm{GCob}_{c,d}$ *and B be an algebra over* $\mathrm{GCob}_{c',d'}$*. Then $A\otimes B$ inherits the structure of an algebra over* $\mathrm{GCob}_{c+c',d+d'}$*. Analogous statements hold for* fGCob *and* pGCob*.*

Proof. We show the statement for $\mathrm{GCob}_{c,d}$. The proof for fGCob and pGCob is analogous. Giving a $\mathrm{GCob}_{c,d}$-algebra structure on A is the same thing as giving a map of PROPs $\mathrm{GCob}_{c,d} \to \mathrm{End}_A$ (see [Mar08]). Similarly, we get a map of PROPs $\mathrm{GCob}_{c',d'} \to \mathrm{End}_B$. Precomposing with the map from Proposition 4.11 and postcomposing with the map from Example 4.10, we thus get a map of PROPs

$$\mathrm{GCob}_{c+c',d+d'} \to \mathrm{GCob}_{c,d} \otimes \mathrm{GCob}_{c',d'} \to \mathrm{End}_A \otimes \mathrm{End}_B \to \mathrm{End}_{A\otimes B}$$

and thus a $\mathrm{GCob}_{c+c',d+d'}$-algebra structure on $A \otimes B$. $\square$

Another relation between $\mathrm{GCob}_{c,d}$-algebras for different c, d comes from suspending algebras. This operation has a corresponding operation on the level of PROPs:

Definition 4.14. The *suspension PROP* in $\mathbb{Z}\mathrm{Mod}_0$ is defined as the endomorphism PROP $\mathrm{End}_{\Sigma\mathbb{1}}$ of $\Sigma\mathbb{1}$ in $\mathbb{Z}\mathrm{Mod}_\bullet$. Explicitly, it is given by

$$\Sigma(X,Y) \coloneqq \mathbb{Z}\mathrm{Mod}_\bullet((\Sigma\mathbb{1})^{\otimes X}, (\Sigma\mathbb{1})^{\otimes Y})$$

for finite sets X, Y.
For a PROP P in $\mathbb{Z}\mathrm{Mod}_0$, the *suspended PROP* is defined as $\Sigma P \coloneqq \Sigma \otimes P$.

Proposition 4.15. *Let P be a PROP in $\mathbb{Z}\mathrm{Mod}_0$ and $A \in \mathbb{Z}\mathrm{Mod}_\bullet$. Then giving a P-algebra structure on A is equivalent to giving a ΣP-algebra structure on ΣA.*

Proof. We again use the characterisation of A being a P-algebra as a PROP maps into End_A. By definition, this is equivalent to giving for all finite sets X, Y equivariant maps

$$P(X,Y) \to \mathcal{C}(A^{\otimes X}, A^{\otimes Y})$$

which commute with composition $\circ$ and the symmetric monoidal product $\otimes$. This is equivalent to giving equivariant maps

$$\mathbb{Z}\mathrm{Mod}_\bullet((\Sigma\mathbb{1})^{\otimes X}, (\Sigma\mathbb{1})^{\otimes Y}) \otimes P(X,Y) \to \mathbb{Z}\mathrm{Mod}_\bullet((\Sigma\mathbb{1})^{\otimes X}, (\Sigma\mathbb{1})^{\otimes Y}) \otimes \mathbb{Z}\mathrm{Mod}_\bullet(A^{\otimes X}, A^{\otimes Y})$$

which commute with $\circ$ and $\otimes$. After postcomposing with the isomorphism from Example 4.10, we get equivariant maps

$$\Sigma P(X,Y) \to \mathbb{Z}\mathrm{Mod}_\bullet((\Sigma A)^{\otimes X}, (\Sigma A)^{\otimes Y})$$

which commute with $\circ$ and $\otimes$. This is by definition equivalent to giving a PROP morphism $\Sigma P \to \mathrm{End}_{\Sigma A}$, which is equivalent to giving a ΣP structure on ΣA. $\square$

Remark 4.16. This construction is completely analogous as for PROs, operads, cyclic operads, dioperads and properads. However in the category of modular operads, there exists no suspension modular operad in $\mathbb{Z}\mathrm{Mod}_0$. Indeed there is a canonical pairing $\Sigma\mathbb{1} \otimes \Sigma\mathbb{1} \to \Sigma\mathbb{1}$ in degree -1. This lets us identify

$$\mathrm{End}_{\Sigma\mathbb{1}}(X) = \mathbb{Z}\mathrm{Mod}_\bullet((\Sigma\mathbb{1})^{\otimes X}, \Sigma\mathbb{1}) \cong \Sigma^{1-|X|}\mathbb{1}.$$

For any elements $x \in X$ and $y \in Y$, this gives a map

$$\mathrm{End}_{\Sigma\mathbb{1}}(X) \otimes \mathrm{End}_{\Sigma\mathbb{1}}(Y) \to \mathrm{End}_{\Sigma\mathbb{1}}((X \setminus x) \sqcup (Y \setminus y))$$

in $\mathbb{Z}\mathrm{Mod}_0$. However, the map for elements $x \neq x' \in X$

$$\mathrm{End}_{\Sigma\mathbb{1}}(X) \to \mathrm{End}_{\Sigma\mathbb{1}}(X \setminus \{x, x'\})$$

is of degree -1 and thus not in $\mathbb{Z}\mathrm{Mod}_0$. One way to amend this is to work with the language of twisted modular operads as introduced in [GK98].

Proposition 4.17. *For integers $c, d \in \mathbb{Z}$, there exist canonical isomorphisms*

$$\Sigma\mathrm{GCob}_{c,d} \cong \mathrm{GCob}_{c-1,d+1},$$
$$\Sigma\mathrm{fGCob}_{c,d} \cong \mathrm{fGCob}_{c-1,d+1}$$

and

$$\Sigma\mathrm{pGCob}_{c,d} \cong \mathrm{pGCob}_{c-1,d+1}$$

Proof. We fix integers $c, d \in \mathbb{Z}$ and finite sets X, Y. The key observation is that for every graph cobordism G between X and Y there is an isomorphism

$$\mathbf{det}_{c-1,d+1}(G) \cong \mathbb{Z}\mathrm{Mod}_\bullet((\Sigma\mathbb{1})^{\otimes X}, (\Sigma\mathbb{1})^{\otimes Y}) \otimes \mathbf{det}_{c,d}(G) \tag{7}$$

which is natural in X and Y.
To see the isomorphism, we observe

$$\begin{aligned}\mathbf{det}_{c-1,d+1}(G) &= \mathbf{det}(G,\partial_{in})^{\otimes c-1} \otimes \mathbf{det}(G,\partial_{out})^{\otimes d+1} \\ &\cong \mathbf{det}(G,\partial_{in})^{-1} \otimes \mathbf{det}(G,\partial_{out}) \otimes \mathbf{det}_{c,d}(G).\end{aligned}$$

We consider the short exact sequences

$$\begin{aligned}0 \longrightarrow C_*(|\partial_{in}|) \longrightarrow C_*(|G|) \longrightarrow C_*(|G|,|\partial_{in}) \longrightarrow 0,\\ 0 \longrightarrow C_*(|\partial_{out}|) \longrightarrow C_*(|G|) \longrightarrow C_*(|G|,|\partial_{out}) \longrightarrow 0.\end{aligned}$$

Then Lemma 4.1 (d) gives the natural isomorphisms

$$\begin{aligned}\mathbf{det}(G,\partial_{in})^{-1} \otimes \mathbf{det}(G,\partial_{out}) &\cong \mathbf{det}(|\partial_{in}|) \otimes \mathbf{det}(|G|)^{-1} \otimes \mathbf{det}(|G|) \otimes \mathbf{det}(|\partial_{out}|)^{-1} \\ &\cong \mathbf{det}(|\partial_{in}|) \otimes \mathbf{det}(|\partial_{out}|)^{-1} \\ &\cong \det(\mathbb{Z}^X) \otimes \det(\mathbb{Z}^Y)^{-1} \\ &\cong ((\Sigma\mathbb{1})^{\otimes X})^{-1} \otimes (\Sigma\mathbb{1})^{\otimes Y} \\ &\cong \mathbb{Z}\mathrm{Mod}_\bullet((\Sigma\mathbb{1})^{\otimes X}, (\Sigma\mathbb{1})^{\otimes Y}) = \Sigma(X,Y).\end{aligned}$$

We recall that, in our convention, the determinant lives in the degree equal to the negative rank. Thus, we have for a finite set X a natural isomorphism $\det(\mathbb{Z}^X) \cong ((\Sigma\mathbb{1})^{\otimes X})^{-1}$.
This gives (7) and thus the proof. □

Combining Proposition 4.15 and Proposition 4.17, we obtain the following:

Corollary 4.18. *Giving a (c,d)-graded (commutative or symmetric) Frobenius algebra structure on $A := F(*) \in \mathbb{Z}\mathrm{Mod}_\bullet$ is equivalent to giving a $(c-1,d+1)$-graded (commutative or symmetric) Frobenius algebra structure on ΣA.*

Remark 4.19. We note that $|\mu| + |\nu| = |\Sigma(\mu)| + |\Sigma(\nu)|$. Therefore this defines an invariant under suspension. In particular if $|\mu| + |\nu| \neq 0$, the algebra is not a suspension of an algebra with operations in degree 0. Adding a factor $\mathbf{det}(G,\partial_{in})$ (or $\mathbf{det}(G,\partial_{out})$) in the definition of the PROP $\mathrm{GCob}_{c,d}$ helps us describe other algebras because this only changes the degree of the multiplication and not the comultiplication (or vice-versa). However this is an operation on the PROP and contrary to suspension does not have an obvious analogue on the level of algebras. See Remark 5.9.

4.3. Canonical dioperad. The PROP $\mathrm{GCob}_{c,d}$ has a canonical subdioperad generated by forests. In Subsection 6.4, we study an example where this dioperad shows up naturally rather than the whole PROP. We study the structure of this canonical subdioperad.

For this, we recall the definition of a dioperad:

Definition 4.20. A *dioperad* consists of the following data a collection $\mathcal{D}(X,Y)$ for all finite sets X, Y with

- a (S_X, S_Y)-bimodule structure on $\mathcal{D}(X,Y)$ where S_X and S_Y denote the symmetric group of X and Y, respectively;
- a unit

$$1\colon \mathbb{1} \to \mathcal{D}(*,*);$$

- composition maps

$$ {}_y\circ_x \colon \mathcal{D}(Y,Z)\otimes\mathcal{D}(W,X)\to\mathcal{D}((Y\setminus\{y\})\sqcup W, Z\sqcup(X\setminus\{x\})) $$

for all finite sets W,X,Y,Z and elements $x\in X$, $y\in Y$.

This data has to satisfy some unitality, associativity and equivariance relations. See [Gan02] for the precise formulation of those relations.

Definition 4.21. The *forest dioperad* $\mathrm{GCob}^{\mathrm{forest}}_{c,d}$ of $\mathrm{GCob}_{c,d}$ is given by the collections

$$ \mathrm{GCob}^{forest}_{c,d}(X,Y)\subseteq\mathrm{GCob}_{c,d}(X,Y) \tag{8} $$

generated by the graph cobordisms between X and Y which are forests with the restricted composition.

In general this dioperad carries less information than the PROP. But in some cases, we have the following proposition:

Proposition 4.22. *If $c,d\in\mathbb{Z}$ are of different parity, then the PROP $\mathrm{GCob}_{c,d}$ is an extension by $\mathbb{Z}/2$ factors of dioperad $\mathrm{GCob}^{\mathrm{forest}}_{c,d}$. In explicit terms, the inclusion* (8) *finite sets X,Y*

$$ \mathrm{GCob}^{\mathrm{forest}}_{c,d}(X,Y)\to\mathrm{GCob}_{c,d}(X,Y) $$

has a cokernel which is 2-torsion, for all finite sets X,Y,Z.

Proof. We fix $c,d\in\mathbb{Z}$ of different parity. We use the description

$$ \mathrm{GCob}_{c,d}(X,Y)\cong\bigoplus_{[G]\in\mathrm{GCob}(X,Y)}\mathbf{det}_{c,d}(G)_{\pi_1(|\underline{\mathrm{GCob}}(X,Y)|,G)} $$

as in Proposition 4.5. We have an analogous description:

$$ \mathrm{GCob}^{\mathrm{forest}}_{c,d}(X,Y)\cong\bigoplus_{\substack{[G]\in\mathrm{GCob}(X,Y)\\ G\text{ a forest}}}\mathbf{det}_{c,d}(G)_{\pi_1(|\underline{\mathrm{GCob}}(X,Y)|,G)}. $$

Let $[G]\in\mathrm{GCob}(X,Y)$ be an equivalence class of graphs cobordisms between X and Y with G not a forest. We want to show that the summand

$$ \mathbf{det}_{c,d}(G)_{\pi_1(|\underline{\mathrm{GCob}}(X,Y)|,G)} $$

is $\mathbb{Z}/2$.
After collapsing some edges, we can assume that G has a tadpole. Then there is a graph automorphism of G flipping the direction of the tadpole. This represents an element in $\pi_1(|\underline{\mathrm{GCob}}(X,Y)|,G)$. This automorphism acts by $(-1)^c$ on $\mathbf{det}(G,\partial_{in})^{\otimes c}$ and by $(-1)^d$ on $\mathbf{det}(G,\partial_{out})^{\otimes d}$. See Subsection 7.1 for a detailed explanation how to derive these signs.
It thus acts by $(-1)^{c+d}=-1$ on $\mathbf{det}_{c,d}(G)$ which is a free group of rank one. This shows $\mathbf{det}_{c,d}(G)_{\pi_1(|\underline{\mathrm{GCob}}(X,Y)|,G)}=\mathbb{Z}/2$ and thus concludes the proof. □

5. Graded Frobenius algebra: maps and relations

In this section, we state the main theorem which gives a description of graded Frobenius algebras in terms of maps and relations. We will use this theorem to study some of the simplest examples of graded Frobenius algebras. Moreover, we discuss the suspension of algebras from the perspective of maps and relations.

Theorem 5.1. *Let $c, d \in \mathbb{Z}$ and let $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ be a (strict) monoidal category enriched over graded abelian groups.*
The data $(A, \mu, \eta, \nu, \varepsilon)$ where

$$\begin{aligned} &A \in \mathcal{C}, && \\ &\mu \in \mathcal{C}_c(A \otimes A, A), && \eta \in \mathcal{C}_{-c}(\mathbb{1}, A), \\ &\nu \in \mathcal{C}_d(A, A \otimes A), && \varepsilon \in \mathcal{C}_{-d}(A, \mathbb{1}) \end{aligned}$$

defines a monoidal functor out of $\mathrm{pGCob}_{c,d}$ *into $\mathcal{C}$ uniquely up to isomorphism if and only if the following relations are satisfied:*

(i) graded associativity $\mu \circ (\mu \otimes \mathrm{id}) = (-1)^c \mu \circ (\mathrm{id} \otimes \mu)$*;*
(ii) graded unitality $(-1)^c \mu \circ (\eta \otimes \mathrm{id}) = (-1)^{\frac{c(c-1)}{2}} \mathrm{id} = \mu \circ (\mathrm{id} \otimes \eta)$*;*
(iii) graded coassociativity $(\mathrm{id} \otimes \nu) \circ \nu = (-1)^d (\nu \otimes \mathrm{id}) \circ \nu$*;*
(iv) graded counitality $(-1)^d (\mathrm{id} \otimes \varepsilon) \circ \nu = (-1)^{\frac{d(d-1)}{2}} \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \nu$*;*
(v) graded Frobenius relation $(\mu \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \nu) = (-1)^{cd} \nu \circ \mu = (\mathrm{id} \otimes \mu) \circ (\nu \otimes \mathrm{id})$.

If $\mathcal{C}$ is also symmetric, the data $(A, \mu, \eta, \nu, \varepsilon)$ defines a symmetric monoidal functor out of $\mathrm{GCob}_{c,d}$ *into $\mathcal{C}$ uniquely up to isomorphism if and only if the maps satisfy (i)-(v) and*

(vi) graded commutativity $\mu \circ \tau = (-1)^c \mu$

and it defines a symmetric monoidal functor out of $\mathrm{fGCob}_{c,d}$ *into $\mathcal{C}$ uniquely up to isomorphism if and only if the maps satisfy (i)-(v) and*

(vi') graded symmetry $\varepsilon \circ \mu \circ \tau = (-1)^c \varepsilon \circ \mu$.

We postpone the proof of this theorem to section 7 as it involves a lot of technical details. Instead we focus on what are the choices involved in those signs and what are the consequences. The strategy for the theorem is to choose generators of the categories $\mathrm{GCob}_{c,d}$, $\mathrm{fGCob}_{c,d}$ and $\mathrm{pGCob}_{c,d}$ and study the generating relations between them. For this, we choose generators in $\mathbf{det}_{c,d}(G)$ for a list of graphs G such that all other graphs can be decomposed into them.

Definition 5.2. A generator $\omega(G) \in \mathbf{det}_{c,d}(G)$ is called an *orientation* of the graph G.

Remark 5.3 (Choices of Orientations). The relations are *not* canonical. Namely, they depend on a choice of orientations

$$\omega(\succ\!\!-) \in \mathbf{det}_{c,d}(\succ\!\!-),\ \omega(\bullet\!\!-) \in \mathbf{det}_{c,d}(\bullet\!\!-),\ \omega(-\!\!\prec) \in \mathbf{det}_{c,d}(-\!\!\prec),\ \omega(-\!\!\bullet) \in \mathbf{det}_{c,d}(-\!\!\bullet).$$

By definition, this depends on generators of

$$\begin{aligned} &\mathbf{det}(\succ\!\!-, \partial_{in})^{\otimes c} \otimes \mathbf{det}(\succ\!\!-, \partial_{out})^{\otimes d}, && \mathbf{det}(\bullet\!\!-, \partial_{in})^{\otimes c} \otimes \mathbf{det}(\bullet\!\!-, \partial_{out})^{\otimes d}, \\ &\mathbf{det}(-\!\!\prec, \partial_{in})^{\otimes c} \otimes \mathbf{det}(-\!\!\prec, \partial_{out})^{\otimes d}, && \mathbf{det}(-\!\!\bullet, \partial_{in})^{\otimes c} \otimes \mathbf{det}(-\!\!\bullet, \partial_{out})^{\otimes d}. \end{aligned}$$

We thus choose generators $\omega_{in}(G) \in \mathbf{det}(G, \partial_{in})$ and $\omega_{out}(G) \in \mathbf{det}(G, \partial_{out})$ for those four graphs and then define $\omega(G) \coloneqq \omega_{in}(G)^{\otimes c} \otimes \omega_{out}(G)^{\otimes d}$.
We note that $\mathbf{det}(\succ\!\!-, \partial_{out}) \cong \mathbf{det}(\bullet\!\!-, \partial_{out}) \cong \mathbf{det}(-\!\!\prec, \partial_{in}) \cong \mathbf{det}(-\!\!\bullet, \partial_{in}) \cong \mathbb{1}$. We thus choose the canonical generator 1. Moreover for $\mathbf{det}(\bullet\!\!-, \partial_{in})$ and $\mathbf{det}(-\!\!\bullet, \partial_{out})$, we choose the canonical generator coming from the canonical generator of $H_0(\bullet\!\!-, \partial_{in})$ and $H_0(-\!\!\bullet, \partial_{out})$. However, there is no canonical choice for generators

$$\omega_{in}(\succ\!\!-) \in \mathbf{det}(\succ\!\!-, \partial_{in}), \qquad \omega_{out}(-\!\!\prec) \in \mathbf{det}(-\!\!\prec, \partial_{out}).$$

Therefore, there exists no canonical choice of orientation

$$\omega(\succ\!\!-) \coloneqq \omega_{in}(\succ\!\!-)^{\otimes c} \otimes 1^{\otimes d}$$

for c odd and no canonical choice of orientation

$$\omega(\prec) := 1^{\otimes c} \otimes \omega_{out}(\prec)^{\otimes d}$$

for d odd. We make a choice in the proof in 7.2. A different choice would lead to different signs in the unitality and counitality relation.

Remark 5.4. In [CO22b], Cieliebak and Oancea introduce a type of algebra called biunital coFrobenius. In the finite dimensional case, this corresponds to our notion of graded Frobenius algebra (see [CO22b, Proposition 5.14]). The infinite dimensional case is discussed in 6.4.
They define the graded unitality relation

$$\mu \circ (\eta \otimes \mathrm{id}) = \mathrm{id} = (-1)^c \mu \circ (\mathrm{id} \otimes \eta)$$

compared to our graded unitality relation

$$(-1)^c \mu \circ (\eta \otimes \mathrm{id}) = (-1)^{\frac{c(c-1)}{2}} \mathrm{id} = \mu \circ (\mathrm{id} \otimes \eta).$$

The two relations agree for $c \equiv 3, 4 \pmod 4$ and are different for $c \equiv 1, 2 \pmod 4$.
Similarly, the signs in their graded counitality relation agree with our signs for $d \equiv 3, 4 \pmod 4$ and are different for $d \equiv 1, 2 \pmod 4$.
For $(A, \mu, \eta, \nu, \varepsilon)$ satisfying the signs in our relations, the product

$$\mu' := (-1)^{\frac{c(c+1)}{2}} \mu$$

and the coproduct

$$\nu' := (-1)^{\frac{d(d+1)}{2}} \nu$$

together with the same unit and counit satisfy the signs in their relations.

Corollary 5.5. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ be a monoidal category enriched over $\mathbb{Z}\mathrm{Mod}_0$ and let $c, d \in \mathbb{Z}$. Let the data $(A, \mu, \eta, \varepsilon)$ be as in Theorem 5.1. This data uniquely defines a (c, d)-graded Frobenius algebra if and only if it satisfies graded associativity as in* (i), *graded unitality as in* (ii) *and the map*

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{\varepsilon} \mathbb{1}$$

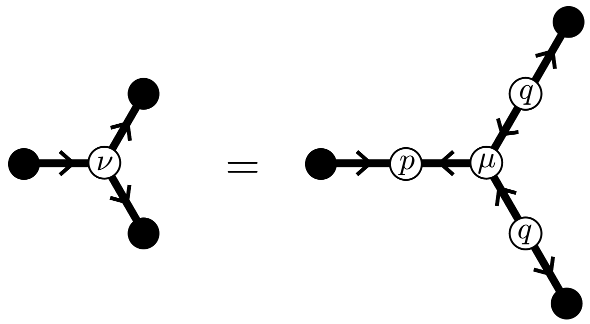
is a non-degenerate pairing.

Proof. The proof strategy is analogous to the ungraded situation as in [Koc04, Section 2.4]: We first use Theorem 5.1. To show that a Frobenius algebra $(A, \mu, \eta, \nu, \varepsilon)$ as in the theorem induces a non-degenerate pairing $p := \varepsilon \circ \mu$, one can show that $q := (-1)^{cd + \frac{c(c+1)}{2} + \frac{d(d+1)}{2}} \nu \circ \eta$ is the corresponding copairing that satisfies

$$(\mathrm{id} \otimes p) \circ (q \otimes \mathrm{id}) = \mathrm{id}.$$

On the other hand, we assume that the data $(A, \mu, \eta, \varepsilon)$ is as in the statement. Let $q \colon \mathbb{1} \to A \otimes A$ be the associated copairing. Then one can show that the comultiplication $\nu := (-1)^{\frac{d(d+1)}{2}} (\mathrm{id} \otimes \mu) \circ (q \otimes \mathrm{id})$ satisfies coassociativity as in (iii) and counitality as in (iv). $\square$

Remark 5.6. Another perspective on graded Frobenius algebras is the following. Assume, we are given a degree 0 multiplication μ and a non-degenerate pairing p with associated copairing q in degree $-d$ and d. Then one can use the language of oriented marked ribbon quivers as in [KTV21, Section 6]. In this language, one can define a coproduct as the following quiver with a 0-orientation:



The signs in coassociativity and the Frobenius relation then come from permutations in the orientations.

We can now study the simplest non-trivial $\mathrm{GCob}_{c,d}$-algebras. The simplest example is given for $c = d = 0$:

EXAMPLE 5.7. In $\mathcal{C}$, the unit $\mathbb{1}$ has a canonical structure of a $\mathrm{GCob}_{0,0}$-algebra given by the canonical maps

$$\mu\colon \mathbb{1}\otimes\mathbb{1}\to\mathbb{1}, \qquad \eta\colon \mathbb{1}\to\mathbb{1},$$
$$\nu\colon \mathbb{1}\to\mathbb{1}\otimes\mathbb{1}, \qquad \varepsilon\colon \mathbb{1}\to\mathbb{1}.$$

EXAMPLE 5.8. Let A be a $\mathrm{GCob}_{c,d}$-algebra in $R\mathrm{Mod}_\bullet$ for some principal ideal domain R. We have unit maps $R\to A_{-c}$ and counit map $A_d\to R$. If A is non-zero, non-degeneracy implies thus that we have a factor R in degree $-c$ generated by $x \coloneqq \eta(1)$ and a factor R in degree d generated by y with $\varepsilon(y) = 1_R$.
We distinguish between two cases: $c + d = 0$ and $c + d \neq 0$.
For $c + d = 0$, we apply Proposition 4.17 to Example 5.7 to find that $\Sigma^d\mathbb{1}$ inherits the structure of a $\mathrm{GCob}_{c,d}$-algebra. This is thus the $\mathrm{GCob}_{c,d}$-algebra in $R\mathrm{Mod}_\bullet$ of smallest rank which is not zero. Moreover, it is the $\mathrm{fGCob}_{c,d}$-algebra of smallest rank bigger which is non-trivial.
For $c+d \neq 0$, the algebra A is not the suspension of a $\mathrm{GCob}_{0,0}$-algebra. Moreover, every non-trivial $\mathrm{GCob}_{c,d}$-algebra in $R\mathrm{Mod}_\bullet$ thus contains at least $\Sigma^{-c}R\oplus\Sigma^d R$. It turns out that we can define a $\mathrm{GCob}_{c,d}$-algebra structure on $R_{c,d} \coloneqq \Sigma^{-c}R\oplus\Sigma^d R$.
We denote the generator of $\Sigma^{-c}R$ by x and the generator of $\Sigma^d R$ by y. By assumption, we have $\eta(1) = x$ and $\varepsilon(y) = 1$. The multiplication μ is then uniquely defined by unitality

$$\mu(x\otimes y) = (-1)^{\frac{c(c+1)}{2}}y, \qquad \mu(y\otimes x) = (-1)^{cd+\frac{c(c-1)}{2}}y, \qquad \mu(x\otimes x) = (-1)^{\frac{c(c+1)}{2}}x$$

and degree reasons

$$\mu(y\otimes y) = 0.$$

On the other hand, the comultiplication ν is defined by counitality as

$$\nu(x) = (-1)^{cd+\frac{d(d+1)}{2}}x\otimes y + (-1)^{\frac{d(d-1)}{2}}y\otimes x, \qquad \nu(y) = (-1)^{\frac{d(d-1)}{2}}y\otimes y.$$

One can check that this satisfies graded associativity, graded coassociativity, the graded Frobenius relation and graded commutativity.
This is also the $\mathrm{fGCob}_{c,d}$- and $\mathrm{pGCob}_{c,d}$-algebra of smallest rank which is non-trivial.

REMARK 5.9. As discussed in Remark 4.19, twisting by $\mathbf{det}(G,\partial_{in})$ and $\mathbf{det}(G,\partial_{out})$ changes only the degree of one operation, either multiplication or comultiplication while suspension changes

degrees of both. On the level of algebras, suspension is simpler than twisting by $\mathbf{det}(G, \partial_{in})$.
We have an equivalence

$$\mathrm{grComFrob}_{c,d}(\mathbb{Z}\mathrm{Mod}_\bullet) \cong \mathrm{grComFrob}_{c-1,d+1}(\mathbb{Z}\mathrm{Mod}_\bullet)$$

given by suspending the algebra. One can think of this in the following way. We tensor an algebra A in $\mathrm{grComFrob}_{c,d}(\mathbb{Z}\mathrm{Mod}_\bullet)$ with $\Sigma\mathbb{1}$ the $\mathrm{GCob}_{-1,1}$-algebra of smallest rank which is not zero. We then apply Corollary 4.13 to get a $\mathrm{GCob}_{c-1,d+1}$-algebra structure on $\Sigma \otimes A \cong \Sigma A$.
We can do an analogous construction to get a functor

$$\mathrm{grComFrob}_{c,d}(\mathbb{Z}\mathrm{Mod}_\bullet) \to \mathrm{grComFrob}_{c,d+1}(\mathbb{Z}\mathrm{Mod}_\bullet).$$

For this we tensor a $\mathrm{GCob}_{c,d}$-algebra A with $\mathbb{Z}_{0,1}$ as defined in Example 5.8. This is the $\mathrm{GCob}_{0,1}$-algebra of smallest rank which is not zero. By Corollary 4.13, this gives a $\mathrm{GCob}_{c,d+1}$-algebra structure on $\mathbb{Z}_{0,1} \otimes A$ which has underlying module $A \oplus \Sigma A$.
In contrast to suspension, this functor is not an equivalence. This can be seen as $\mathbb{Z}_{0,1}$ is not $\otimes$-invertible in contrast to $\mathrm{GCob}_{1,-1}$-algebra $\Sigma\mathbb{1}$.

The suspension of Frobenius algebras in $\mathbb{Z}\mathrm{Mod}_\bullet$ can also be studied on the level of maps and relations. Corollary 4.18 shows that a shifted $\mathrm{GCob}_{c,d}$-algebra is a $\mathrm{GCob}_{c-1,d+1}$-algebra. This can also be stated in terms of maps and relations. However, it is a priori not clear that the suspended maps $\Sigma\mu, \Sigma\eta, \Sigma\nu$ and $\Sigma\eta$ satisfy the signs as in our main theorem because those signs depend on choices of orientations. It turns out that our choices are stable under suspending algebras in this sense.

To make this precise, we describe how a map $\mathbb{Z}\mathrm{Mod}_\bullet(A^{\otimes k}, A^{\otimes l})$ induces a suspended map in $\mathbb{Z}\mathrm{Mod}_\bullet((\Sigma A)^{\otimes k}, (\Sigma A)^{\otimes l})$.
For this, we note that there is a canonical generator $\sigma_{k,l}$ in $\mathbb{Z}\mathrm{Mod}_\bullet((\Sigma\mathbb{1})^{\otimes k}, (\Sigma\mathbb{1})^{\otimes l})$ which sends the canonical generator $s(1) \otimes \cdots \otimes s(1) \in (\Sigma\mathbb{1})^{\otimes k}$ to the canonical generator $s(1) \otimes \cdots \otimes s(1) \in (\Sigma\mathbb{1})^{\otimes l}$.
We thus get a map

$$\begin{aligned}\Sigma\colon \mathbb{Z}\mathrm{Mod}_\bullet(A^{\otimes k}, A^{\otimes l}) &\to \mathbb{Z}\mathrm{Mod}_\bullet((\Sigma\mathbb{1})^{\otimes k}, (\Sigma\mathbb{1})^{\otimes l}) \otimes \mathbb{Z}\mathrm{Mod}_\bullet(A^{\otimes k}, A^{\otimes l}) \cong \mathbb{Z}\mathrm{Mod}_\bullet((\Sigma A)^{\otimes k}, (\Sigma A)^{\otimes l}),\\ f &\mapsto \sigma_{k,l} \otimes f.\end{aligned}$$

We note that this is coherent with our choice that maps (here tensoring with $\sigma_{k,l}$) act from the left. However, this map is functorial and monoidal *only up to sign*:

LEMMA 5.10. *Let $f\colon A^{\otimes k} \to A^{\otimes l}$ and $g\colon A^{\otimes m} \to A^{\otimes n}$ be maps in $\mathbb{Z}\mathrm{Mod}_\bullet$. Then we have*
(a) $\Sigma f \otimes \Sigma g = (-1)^{(n-m)(|f|+k)}\Sigma(f \otimes g)$;
(b) $\Sigma f \circ \Sigma g = (-1)^{(n-m)|f|}\Sigma(f \circ g)$ *if* $n = k$;
(c) $\Sigma \mathrm{id}_{A^{\otimes k}} = \mathrm{id}_{(\Sigma A)^{\otimes k}}$;
(d) $\Sigma\tau_A = -\tau_{\Sigma A}$ *for* $\tau_A\colon A \otimes A \to A \otimes A$ *the twist induced by the Koszul sign.*

PROOF. We first note that the maps $\sigma_{k,l}$ are in degree $l - k$ and satisfy

$$\sigma_{k,l} \otimes \sigma_{m,n}(s(1)^{\otimes k} \otimes s(1)^{\otimes m}) = (-1)^{(n-m)k} s(1)^{\otimes l} \otimes s(1)^{\otimes n} = (-1)^{(n-m)k}\sigma_{k+m,l+n}(s(1)^{\otimes k+m})$$

and

$$\sigma_{k,l} \circ \sigma_{m,k} = \sigma_{m,l}.$$

We can thus compute

$$\begin{aligned}\Sigma f \otimes \Sigma g &= (\sigma_{k,l} \otimes f) \otimes (\sigma_{m,n} \otimes g) = (-1)^{(n-m)|f|}(\sigma_{k,l} \otimes \sigma_{m,n}) \otimes (f \otimes g)\\ &= (-1)^{(n-m)(|f|+k)}(\sigma_{k+m,l+n}) \otimes (f \otimes g) = (-1)^{(n-m)(|f|+k)}\Sigma(f \otimes g)\end{aligned}$$

and if $n = k$

$$\begin{aligned}\Sigma f \circ \Sigma g &= (\sigma_{k,l} \otimes f) \circ (\sigma_{m,n} \otimes g) = (-1)^{|\sigma_{m,n}||f|}(\sigma_{k,l} \circ \sigma_{m,n}) \otimes (f \circ g)\\ &= (-1)^{(n-m)|f|}\sigma_{m,l} \otimes (f \circ g) = (-1)^{(n-m)|f|}\Sigma(f \circ g).\end{aligned}$$

The equation (c) follows directly from (b). For (d), we denote $s(a) \in (\Sigma A)_{i+1}$ for the element coming from $a \in A_i$ and similarly $s(b) \in (\Sigma A)_{j+1}$. The map $\Sigma\tau_A$ is defined as the composition

$$\begin{aligned} s(a) \otimes s(b) &\mapsto (-1)^{|a|} s(1) \otimes s(1) \otimes a \otimes b\\ &\overset{\sigma_{2,2}\otimes\tau_A}{\mapsto} (-1)^{|a||b|+|a|}(s(1) \otimes s(1)) \otimes (b \otimes a)\\ &\mapsto (-1)^{|a||b|+|a|+|b|} s(b) \otimes s(a)\\ &= (-1)^{|s(a)||s(b)|-1} s(b) \otimes s(a) = -\tau_{\Sigma A}(s(a) \otimes s(b)).\end{aligned}$$

□

PROPOSITION 5.11. *Let $(A, \mu, \eta, \nu, \varepsilon)$ in $\mathbb{Z}\mathrm{Mod}_\bullet$ be as in Theorem 5.1 and $c, d \in \mathbb{Z}$. This data satisfies the relations of a (c, d)-graded (commutative or symmetric) Frobenius algebra if and only if $(\Sigma A, \Sigma\mu, \Sigma\eta, \Sigma\nu, \Sigma\varepsilon)$ satisfies the relations of a $(c-1, d+1)$-graded (commutative or symmetric) Frobenius algebra.*

PROOF. This is a direct application of Lemma 5.10. For example, if $(A, \mu, \eta, \nu, \varepsilon)$ satisfies (c, d)-graded left unitality, then we check compute $(c-1, d+1)$-graded left unitality for $(\Sigma A, \Sigma\mu, \Sigma\eta, \Sigma\nu, \Sigma\varepsilon)$:

$$\begin{aligned}(-1)^{c-1}\Sigma\mu \circ (\Sigma\eta \otimes \mathrm{id}_{\Sigma A}) &\overset{(c)}{=} (-1)^{c-1}\Sigma\mu \circ (\Sigma\eta \otimes \Sigma\mathrm{id}_A) \overset{(a)}{=} (-1)^{c-1}\Sigma\mu \circ \Sigma(\eta \otimes \mathrm{id}_A)\\ &\overset{(b)}{=} (-1)^{c-1+c}\Sigma(\mu \circ (\eta \otimes \mathrm{id}_A)) = (-1)^{c-1+\frac{c(c-1)}{2}}\Sigma\mathrm{id}_A\\ &\overset{(c)}{=} (-1)^{\frac{(c-1)(c-1-1)}{2}}\mathrm{id}_{\Sigma A}.\end{aligned}$$

□

6. Examples

In this section, we discuss different examples of graded Frobenius algebras found in the literature. Some of the examples have a description more akin to our Definition 4.4 in terms of PROPs, while some of the examples have a description in terms of maps and relations similar to Theorem 5.1.

Throughout this section, we use Sweedler notation of a coproduct ν on an element a:

$$\nu(a) = \sum_{(a)} a' \otimes a''.$$

6.1. Cohomology of an oriented manifold. Let M be a connected, closed manifold of dimension d and let K be a field. Denote by $H^*(M)$ the cohomology of M with coefficients in K. Assume that M is K-orientable and fix an orientation $[M] \in H_d(M)$.
We denote

$$\begin{aligned}\mu\colon H^*(M) \otimes H^*(M) &\to H^*(M),\\ \alpha \otimes \beta &\mapsto \alpha \cup \beta\end{aligned}$$

and

$$\varepsilon\colon H^*(M) \to K,$$
$$\alpha \mapsto \begin{cases} \alpha([M]) & \text{if } * = d; \\ 0 & \text{else.} \end{cases}$$

The multiplication μ has degree 0 and ε has degree $-d$. Moreover, μ is associative and has unit map

$$\eta\colon K \to H^*(M),$$
$$1 \mapsto 1 \in H^0(M).$$

Moreover, it is graded commutative as it satisfies $\mu\circ\tau = \mu$. Poincaré duality implies that the pairing $\varepsilon \circ \mu$ is non-degenerate. We can thus apply Corollary 5.5 and get the following example

EXAMPLE 6.1. The data $(H^*(M), \mu, \eta, \varepsilon)$ defines a $(0, d)$-graded commutative Frobenius algebra

The associated comultiplication ν is the intersection coproduct. There are two common constructions of the intersection coproduct leading to different signs: we refer to them as the Poincaré coproduct ν_P and the Thom coproduct ν_{Th} (see e.g. [HW17, Appendix B]).
We first define the Poincaré coproduct. As we work with field coefficients, the Künneth maps

$$\times\colon H_*(M) \otimes H_*(M) \to H_*(M^2),$$
$$\times\colon H^*(M) \otimes H^*(M) \to H^*(M^2)$$

are isomorphisms. Denote by θ their inverses. We thus have a coproduct on *homology* given by

$$\nu_{Hom} := \theta \circ \Delta_*\colon H_*(M) \to H_*(M^2) \to H_*(M) \otimes H_*(M).$$

We denote the Poincaré duality by

$$PD\colon H^*(M) \xrightarrow{\cong} H_{d-*}(M),$$
$$\alpha \mapsto \alpha \cap [M].$$

We then define the Poincaré coproduct as Poincaré dual to the homology coproduct ν_{Hom}

$$\nu_P(\alpha) = \sum_{(PD(\alpha))} PD^{-1}(PD(\alpha)') \otimes PD^{-1}(PD(\alpha)'')$$

where the sum comes from ν_{Hom}. This formula in terms of elements seems rather straight forward. But if we describe it as a composition of maps, we need the tensor product without Koszul sign: for graded maps $f\colon V \to V'$ and $g\colon W \to W'$ denote $f\widetilde{\otimes}g$ as the as the map

$$(f\widetilde{\otimes}g)(a \otimes b) = f(a) \otimes g(b).$$

Then we write the Poincaré coproduct as

$$\nu_P := (PD^{-1}\widetilde{\otimes}PD^{-1}) \circ \theta \circ \Delta_* \circ PD.$$

The Thom coproduct is defined via an orientation $[M^2]$ of the manifold M^2. To get the signs as in our main theorem, we define $[M^2] := (-1)^{\frac{n(n-1)}{2}}[M] \times [M]$.
This sign convention might seem surprising. It reflects the fact that we are shuffling the i-th dimension of the first copy of M to be the $2i-1$-th dimension of M^2 and the i-th dimension of the second copy of M to be the $2i$-th dimension of M^2. We call such a shuffle of two ordered sets of the same size a *riffle shuffle*.
In terms of coordinates, take a ordered basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$. We then order the basis

$$\{(v_1, 0), \ldots, (v_n, 0), (0, v_1), \ldots, (0, v_n)\}$$

of $\mathbb{R}^n \oplus \mathbb{R}^n$ as

$$((v_1, 0), (0, v_1), (v_2, 0), \dots, (0, v_n)).$$

This choice has the advantage that for two oriented manifolds M, N of dimension d, d', we have $[(M \times N)^2] = [M^2] \times [N^2]$. Corollary 4.13 then gives that $H^*(M) \otimes H^*(N)$ inherits the structure of a $\mathrm{GCob}_{0,d+d'}$-algebra. Our choice of signs for ν_{Th} then ensures that

$$H^*(M) \otimes H^*(N) \to H^*(M \times N)$$

is a map of coalgebras.
We consider the diagonal $\Delta\colon M \to M \times M$ and fix a tubular neighbourhood $U \subseteq M \times M$ of $\Delta(M)$. We denote $j\colon M \to U$ and $i\colon U \to M \times M$ for the canonical inclusions. We thus have $\Delta = i \circ j$. We define $[U] \in H_{2d}(U, \partial U)$ as the image of $[M^2]$ under the map

$$H_{2d}(M^2) \to H_{2d}(M^2, \Delta(M)^c) \overset{(i_*)^{-1}}{\cong} H_{2d}(U, \partial U).$$

Let $\tau \in H^d(U, \partial U)$ be the Thom class that satisfies

$$\tau \cap [U] = j_*([M]).$$

Now, we can define the Thom coproduct as the following composition

$$\nu_{Th}\colon H^*(M) \cong H^*(U) \overset{\tau\cup}{\to} H^{*+d}(U, \partial U) \cong H^{*+d}(M^2, \Delta(M)^c) \to H^{*+d}(M^2) \cong (H^*(M)^{\otimes 2})^{*+d}.$$

We point the readers attention to the fact that this is coherent with our convention that maps act from the left as we are cupping with τ from the left in the second map.

To compare the signs that arise in the two definitions, we consider this diagram

$$\begin{tikzcd}
H^k(M) \arrow[dd, "\cap[M]"'] & H^k(U) \arrow[l, "j^*"', "\cong"] \arrow[r, bend left, "\tau\cup"] \arrow[r, bend right, "\cup\tau"'] \arrow[r, phantom, "(-1)^{dk}"] \arrow[dd, "\cap(j_*[M])"'] & H^{k+d}(U, \partial U) \arrow[ddl, "\cap[U]"'] & H^{k+d}(M^2, \Delta(M)^c) \arrow[l, "i^*"', "\cong"] \arrow[d] \\
& & H_{d-k}(M^2) \arrow[d, equal] \arrow[r, phantom, "(-1)^{\frac{d(d-1)}{2}}"'] & H^{k+d}(M^2) \arrow[l, "\cap[M^2]"', "\cong"] \arrow[d, equal] \\
H_{d-k}(M) \arrow[r, "j_*"] & H_{d-k}(U) \arrow[r, "i_*"] & H_{d-k}(M^2) \arrow[r, phantom, "(-1)^{d+dk+di}"'] & H^{k+d}(M^2) \arrow[l, "\cap([M]\times[M])"', "\cong"] \\
& & (H_*(M)^{\otimes 2})_{d-k} \arrow[u, "\times", "\cong"'] & (H^*(M)^{\otimes 2})^{k+d}. \arrow[l, "\cap[M]\widetilde{\otimes}\cap[M]"', "\cong"] \arrow[u, "\times", "\cong"']
\end{tikzcd}$$

- The composition along the bottom left gives the Poincaré coproduct ν_P.
- The composition along the top right gives the Thom coproduct ν_{Th}.
- The rectangle on the left hand side and the hexagon on the top right commute due to the naturality of the cap product.
- The triangle in the centre commutes by the definition of the Thom class τ.
- The sign $(-1)^{dk}$ on top comes from graded commutativity of the cup product: $\alpha \cup \beta = (-1)^{|\alpha||\beta|}\beta \cup \alpha$.
- The sign $(-1)^{\frac{d(d-1)}{2}}$ in the rectangle on the right in the middle comes from our definition of $[M^2]$.

- The sign in the bottom right rectangle means that, on elements in $H^i(M) \otimes H^{d+k-i}(M)$, we have the sign $(-1)^{d+dk+di}$ because the cross product and the cap product commute:

$$(\alpha \cap [M]) \times (\beta \cap [M]) = (-1)^{d|\beta|}(\alpha \times \beta) \cap ([M] \times [M]).$$

Combining this, we get on the $(i, d+k-i)$-coordinate in the output the sign

$$\nu_P = (-1)^{\frac{d(d-1)}{2}+d+di}\nu_{Th}. \tag{9}$$

One can check that $(A, \mu, \eta, \nu_{Th}, \varepsilon)$ satisfies the relations of a $(0, d)$-graded commutative Frobenius algebra as in Theorem 5.1. Cieliebak and Oancea prove this with their slightly different sign conventions in [CO22b, Proposition 7.1].
The signs of the Poincaré coproduct can be explained in the following way: as discussed in Proposition 4.17 if A has the structure of a (c, d)-graded commutative Frobenius algebra then ΣA inherits the structure of a $(c-1, d+1)$-graded commutative Frobenius algebra. In this example, this means that

$$\Sigma^{-d}H^*(M) = H^{*+d}(M)$$

inherits the structure of a $(d, 0)$-graded Frobenius algebra. Then the coproduct one obtains is given on an element $s^{-d}(\alpha) \in H^{*+d}(M)$ by:

$$\begin{aligned}
\Sigma^d\nu_{Th}(s^{-d}(\alpha)) &= (\sigma_{1,2}^{\otimes -d} \otimes \nu_{Th})(s(1)^{\otimes -d} \otimes \alpha) \\
&= (-1)^d\sigma_{1,2}^{\otimes -d}(s(1)^{\otimes -d}) \otimes \nu_{Th}(\alpha) \\
&= (-1)^d\sigma_{1,2}(s(1))^{\otimes -d} \otimes \sum_{(\alpha)} \alpha' \otimes \alpha'' \\
&= (-1)^d(s(1)^{\otimes 2})^{\otimes -d} \otimes \sum_{(\alpha)} \alpha' \otimes \alpha'' \\
&= \sum_{(\alpha)} (-1)^{d+|\alpha'|+|s(\alpha')|+\cdots+|s^{d-1}(\alpha')|} s^{-d}(\alpha') \otimes s^{-d}(\alpha'') \\
&= \sum_{(\alpha)} (-1)^{d+d|\alpha'|+\frac{d(d-1)}{2}} s^{-d}(\alpha') \otimes s^{-d}(\alpha'').
\end{aligned}$$

Here $\sigma_{1,2}^{\otimes -d}$ denotes the d-fold tensor product of the canonical map $\Sigma^{-1}\mathbb{1} \to \Sigma^{-1}\mathbb{1} \otimes \Sigma^{-1}\mathbb{1}$.
These are precisely the signs, we observed in (9). We thus conclude $\Sigma^{-d}\nu_{Th} = \nu_P$. Therefore, the Poincaré coproduct is the suspended Thom coproduct on $\Sigma^{-d}H^*(M) \cong H^{*+d}(M)$. In $H^{*+d}(M)$, the coproduct is a degree 0 map, which makes the signs appearing in the graded coassociativity and counitality considerably more comfortable. One pays for this comfort by getting less appealing signs for the associativity and unitality of the corresponding shifted multiplication.

6.2. Hochschild homology of a Frobenius algebra.

Definition 6.2. Let $\mathrm{GCob}_{c,d}^{\partial_{out} \neq \varnothing}$ be the subcategory of $\mathrm{GCob}_{c,d}$ which only has morphisms coming from graphs G satisfying the plumber's condition, i.e., all connected components of G have at least one outgoing external vertex.
Let $\mathrm{GCob}_{c,d}^{\partial_{in} \neq \varnothing}$ be the subcategory of $\mathrm{GCob}_{c,d}$ which only has morphisms coming from graphs G satisfying the reverse plumber's condition, i.e., all connected components of G have at least one incoming external vertex.

In [WW16], Wahl and Westerland show that if A is a $(0, d)$-graded symmetric Frobenius algebra, then the normalized Hochschild homology $\overline{HH}_*(A)$ and A has the structure of an open-closed TQFT.

Isolating the closed part, we get a TQFT twisted by $\mathbf{det}(S, \partial_{out} S)^{\otimes d}$. We can restate this in our language.

EXAMPLE 6.3. Let A be a $(0, d)$-graded symmetric Frobenius algebra. Then $\overline{HH}_*(A)$ has the structure of a $\mathrm{GCob}_{d,d}^{\partial_{in} \neq \varnothing}$-algebra.

The key input for this is the following observation: the twisting $\mathbf{det}(\mathrm{Cob}(G), \partial_{out}(\mathrm{Cob}(G)))^{\otimes d}$ corresponds to our (d, d)-twisting, $\mathbf{det}(G, \partial_{in})^{\otimes d} \otimes \mathbf{det}(G, \partial_{out})^{\otimes d}$. (Recall that $\mathrm{Cob}(G)$ is the corresponding closed surface cobordism represented by the graph G.)
To make this precise, we recall that $\mathbf{det}_{1,1}$ is a functor from $\underline{\mathrm{GCob}}(X, Y)$ which commutes with composition along X and Y. This can be stated by saying that

$$\begin{aligned} \mathbf{det}_{1,1} \colon \underline{\mathrm{GCob}} &\to B\mathrm{Pic}(\mathbb{Z}), \\ X &\mapsto *, \\ G &\mapsto \mathbf{det}_{1,1}(G) \end{aligned}$$

is a functor of 2-categories. Here $B\mathrm{Pic}(\mathbb{Z})$ denotes the 2-category with one object $*$ and morphism category

$$B\mathrm{Pic}(\mathbb{Z})(*, *) = \mathrm{Pic}(\mathbb{Z})$$

the category of $\otimes$-invertible objects in $\mathbb{Z}\mathrm{Mod}_0$ with isomorphisms between them.
The twistings $\mathbf{det}(\mathrm{Cob}(G), \partial_{in}(\mathrm{Cob}(G)))^{\otimes d}$ and $\mathbf{det}(\mathrm{Cob}(G), \partial_{out}(\mathrm{Cob}(G)))^{\otimes d}$ satisfy analogous functoriality and gluing properties as $\mathbf{det}_{1,1}$ and thus also define 2-functors.
That these twistings correspond to our twisting by $\mathbf{det}_{1,1}$ is thus described in the following proposition:

PROPOSITION 6.4. *The three functors of 2-categories*

$$\mathbf{det}_{1,1}, \mathbf{det}(\mathrm{Cob}(-), \partial_{in}), \mathbf{det}(\mathrm{Cob}(-), \partial_{out}) \colon \underline{\mathrm{GCob}} \to B\mathrm{Pic}(\mathbb{Z})$$

are isomorphic.
Explicitly, there exist natural isomorphisms

$$\begin{aligned} \mathbf{det}(G, \partial_{in}) \otimes \mathbf{det}(G, \partial_{out}) &\cong \mathbf{det}(\mathrm{Cob}(G), \partial_{in}(\mathrm{Cob}(G))) \\ \mathbf{det}(G, \partial_{in}) \otimes \mathbf{det}(G, \partial_{out}) &\cong \mathbf{det}(\mathrm{Cob}(G), \partial_{out}(\mathrm{Cob}(G))) \end{aligned} \tag{10}$$

for all $G \in \underline{\mathrm{GCob}}(X, Y)$ which commute with gluing of graphs.

PROOF. We only prove the first isomorphism. The second one is proved analogously.
Both the left and right hand side of (10) represent functors out of $\underline{\mathrm{GCob}}(X, Y)$:

$$\mathbf{det}(-, \partial_{in}) \otimes \mathbf{det}(-, \partial_{out}), \qquad\qquad \mathbf{det}(\mathrm{Cob}(-), \partial_{in}(\mathrm{Cob}(-))).$$

After precomposition with the forgetful functor $\underline{\mathrm{fGCob}}(X, Y) \to \underline{\mathrm{GCob}}(X, Y)$, they define functors out of $\underline{\mathrm{fGCob}}(X, Y)$. What we construct in the following is a natural isomorphism of the two functors out of $\underline{\mathrm{fGCob}}(X, Y)$.
This lifts to a natural transformation of functors out of $\underline{\mathrm{GCob}}(X, Y)$. This can be seen as all edge collapses can be lifted to an edge collapse in a fat graph. It remains to check that graph isomorphisms can be lifted. For this we note that all graph isomorphisms can be decomposed into a graph isomorphism that can be lifted to a fat graph isomorphism and a graph automorphism. Finally all graph automorphisms act trivially on the left and right hand side of (10). Therefore the following isomorphism can be lifted to a natural transformation of functors out of $\underline{\mathrm{GCob}}(X, Y)$.

Let $G \in \underline{\text{fGCob}}(X, Y)$ be a fat graph. We construct isomorphisms which commute with gluing of graphs:

$$f_G \colon \mathbf{det}(\text{fCob}(G), \partial_{out}) \xrightarrow{\cong} \mathbf{det}(\text{fCob}(G), \partial_{in} \cup \partial_{free});$$

$$g_G \colon \mathbf{det}(\text{fCob}(G), \partial_{in}) \otimes \mathbf{det}(\text{fCob}(G), \partial_{in} \cup \partial_{free}) \xrightarrow{\cong} \mathbf{det}(\text{Cob}(G), \partial_{in}).$$

We recall that ∂_{free} is the part of the boundary which is neither in the incoming boundary nor in the outgoing boundary. Then the proposition follows by taking $g_G \circ (\text{id} \otimes f_G)$.
For f_G, we first note that $H_*(|G|, |L_{out}|)$ is a free graded $\mathbb{Z}$-module. Therefore any isomorphism

$$H_*(|G|, |L_{out}|) \cong (H_*(|G|, |L_{out}|))^\vee \cong H^*(|G|, |L_{out}|)$$

gives an isomorphism

$$\mathbf{det}(G, \partial_{out}) = \mathbf{det}(H_*(|G|, |L_{out}|)) \cong \mathbf{det}(H^*(|G|, |L_{out}|)).$$

Next, we note that the homotopy equivalence of pairs

$$(|G|, |L_{out}|) \simeq (\text{fCob}(G), \partial_{out})$$

gives an isomorphism

$$\mathbf{det}(H^*(|G|, |L_{out}|)) \cong \mathbf{det}(H^*(\text{fCob}(G), \partial_{out})).$$

All those isomorphisms so far give an isomorphism

$$\mathbf{det}(G, \partial_{out}) \cong \mathbf{det}(H^*(\text{fCob}(G), \partial_{out}))$$

which commutes with gluing.
For the next step, we use generalized Lefschetz duality (see for example [Hat02, Theorem 3.43]). This gives an isomorphism

$$H^*(\text{fCob}(G), \partial_{out}) \cong H_{2-*}(\text{fCob}(G), \partial_{in} \cup \partial_{free})$$

induced by capping with $[\text{fCob}(G)]$, the orientation in $H_2(\text{fCob}(G), \partial\text{fCob}(G))$. As the fat graphs induce oriented open cobordisms, this isomorphism also commutes with gluing. We thus get an isomorphism

$$\mathbf{det}(H^*(\text{fCob}(G), \partial_{out})) \cong \mathbf{det}(H_{2-*}(\text{fCob}(G), \partial_{in} \cup \partial_{free})) \cong \mathbf{det}(H_*(\text{fCob}(G), \partial_{in} \cup \partial_{free}))$$

which commutes with gluing. This gives f_G.
For g_G, we consider $\text{fCob}(G)$ as a subspace of $\text{Cob}(G)$. Figuratively speaking, we consider it as the lower half. See Figure 11.

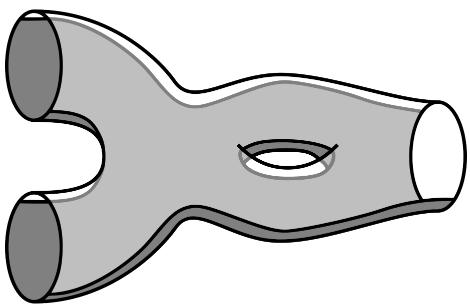


FIGURE 11. Embedding $\text{fCob}(G)$ in grey as the bottom half of $\text{Cob}(G)$.

Consider the triple of spaces $(\mathrm{Cob}(G), \mathrm{fCob}(G) \cup \partial_{in}(\mathrm{Cob}(G)), \partial_{in}(\mathrm{Cob}(G)))$. We note that by excision

$$\begin{aligned} H_*(\mathrm{fCob}(G), \partial_{in}(\mathrm{fCob}(G))) &\cong H_*(\mathrm{fCob}(G) \cup \partial_{in}(\mathrm{Cob}(G)), \partial_{in}(\mathrm{Cob}(G))) \\ H_*(\mathrm{fCob}(G), \partial_{in}(\mathrm{fCob}(G)) \cup \partial_{free}(\mathrm{fCob}(G))) &\cong H_*(\mathrm{Cob}(G), \mathrm{fCob}(G) \cup \partial_{in}(\mathrm{Cob}(G))). \end{aligned}$$

We thus have

$$\begin{aligned} &\mathbf{det}(\mathrm{fCob}(G), \partial_{in}(\mathrm{fCob}(G))) \otimes \mathbf{det}(\mathrm{fCob}(G), \partial_{in}(\mathrm{fCob}(G)) \cup \partial_{free}(\mathrm{fCob}(G))) \\ &\qquad \xrightarrow{\cong} \mathbf{det}(\mathrm{fCob}(G) \cup \partial_{in}(\mathrm{Cob}(G)), \partial_{in}(\mathrm{Cob}(G))) \otimes \mathbf{det}(\mathrm{Cob}(G), \mathrm{fCob}(G) \cup \partial_{in}(\mathrm{Cob}(G))). \end{aligned}$$

We note that collapsing the incoming boundary and the lower half in $\mathrm{Cob}(G)$ is the same space as collapsing the free and incoming boundary of the upper half. See Figure 11.
The short exact sequence of the triple of spaces and Lemma 4.1 (d) then gives

$$\begin{aligned} &\mathbf{det}(\mathrm{fCob}(G) \cup \partial_{in}(\mathrm{Cob}(G)), \partial_{in}(\mathrm{Cob}(G))) \otimes \mathbf{det}(\mathrm{Cob}(G), \mathrm{fCob}(G) \cup \partial_{in}(\mathrm{Cob}(G))) \\ &\qquad \xrightarrow{\cong} \mathbf{det}(\mathrm{Cob}(G), \partial_{in}(\mathrm{Cob}(G))). \end{aligned}$$

This gives g_G and thus concludes the proof. □

We note that the category $\mathrm{GCob}_{d,d}^{\partial_{in} \neq \varnothing}$ is generated by $\omega(\succ\!\!-)$, $\omega(-\!\!\prec)$ and $\omega(-\!\bullet)$. A symmetric monoidal functor $F\colon \mathrm{GCob}_{d,d}^{\partial_{in} \neq \varnothing} \to \mathbb{Z}\mathrm{Mod}_\bullet$ is uniquely determined up to natural isomorphism by an object $F(*)$ and

$$\begin{aligned} \mu_{HH} &\coloneqq F(\omega(\succ\!\!-)) \in \mathbb{Z}\mathrm{Mod}_d(F(*) \otimes F(*), F(*)), \\ \nu_{HH} &\coloneqq F(\omega(-\!\!\prec)) \in \mathbb{Z}\mathrm{Mod}_d(F(*), F(*) \otimes F(*)), \\ \varepsilon_{HH} &\coloneqq F(\omega(-\!\bullet)) \in \mathbb{Z}\mathrm{Mod}_{-d}(F(*), \mathbb{1}). \end{aligned}$$

Those maps satisfy graded associativity, coassociativity, counitality and commutativity and the graded Frobenius relation as in Theorem 5.1.
Following [WW16, Section 6.5] using our sign conventions that maps act from the left, we have the description

$$\begin{aligned} \mu_{HH}((a_0 \otimes \cdots \otimes a_k) \otimes (b_0 \otimes \cdots \otimes b_l)) &= \begin{cases} 0 & \text{if } k > 0, \\ \sum_{(a_0)} (-1)^{(|a_0'|+d)|a_0''|} a_0'' a_0' b_0 \otimes b_1 \otimes \cdots \otimes b_l & \text{if } k = 0, \end{cases} \\ \nu_{HH}(a_0 \otimes \cdots \otimes a_k) &= \sum_{\substack{i=0,\dots,k \\ (a_0)}} (-1)^{(|a_0'|+k-i)(|a_0''|+|a_1|+\cdots+|a_i|)} \\ &\qquad (a_0'' \otimes a_1 \otimes \cdots \otimes a_i) \otimes (a_0' \otimes a_{i+1} \otimes \cdots \otimes a_k), \\ \varepsilon_{HH}(a_0 \otimes \cdots \otimes a_k) &= \begin{cases} 0 & \text{if } k > 0, \\ \varepsilon(a_0) & \text{if } k = 0. \end{cases} \end{aligned}$$

We note that these are different signs compared to [WW16]. One reason is that they have a different sign convention that maps act from the right. On the other hand, the signs in their coproducts are missing the Koszul sign coming from moving a_0' past $a_0'' \otimes a_1 \otimes \cdots \otimes a_i$.

6.3. Homology of the loop space. Let M be a connected, closed, oriented manifold of dimension d and let K be a field. Let $LM = \mathrm{Map}(S^1, M)$ be its loop space.
In [CG04], Cohen and Godin define operations μ_S on $H_*(LM) \coloneqq H_*(LM; K)$ for every closed cobordism S such that each connected component has at least one outgoing boundary. In [God07], Godin moreover explains that this fits into a positive boundary open-closed TQFT which is twisted

by $\mathbf{det}(S, \partial_{in} S)^{\otimes -d}$. This minus sign comes from our convention that the determinant is in negative degree while Godin has the convention that the determinant is in positive degree.
We only focus on the closed underived part of this TQFT. Using the first isomorphism in Proposition 6.4, we can rephrase this in our language as the following example:

EXAMPLE 6.5. The homology of the loop space $H_*(LM)$ has the structure of a $\mathrm{GCob}_{-d,-d}^{\partial_{out}\neq\varnothing}$-algebra with operations given by the Chas-Sullivan product and the trivial coproduct.

We thus get a multiplication on $H_*(LM)$, that we denote by μ_{Go}, and a comultiplication on $H_*(LM)$, that we denote by ν_{Go}.
We can compare this to the results of Tamanoi in [Tam10]. Tamanoi defines a coproduct

$$\Psi\colon H_{*+d}(LM) \to H_*(LM \times LM).$$

As we work over a field, we can assume that the Künneth map $\times\colon H_*(LM) \otimes H_*(LM) \to H_*(LM \times LM)$ is invertible. We therefore can denote $\nu_{Ta} \coloneqq \times^{-1} \circ \Psi$ a comultiplication in our sense. Tamanoi moreover defines multiplications

$$\begin{aligned} \mu_{Ta}\colon H_*(LM) \otimes H_*(LM) &\to H_*(LM), \\ \cdot\colon H_*(LM) \otimes H_*(LM \times LM) &\to H_*(LM \times LM), \\ \cdot\colon H_*(LM \times LM) \otimes H_*(LM) &\to H_*(LM \times LM) \end{aligned}$$

which are defined such that for $a, b, c \in H_*(LM)$, we have

$$\begin{aligned} a \cdot (b \times c) &= \mu_{Ta}(a \otimes b) \times c \\ (a \times b) \cdot c &= a \times \mu_{Ta}(b \otimes c). \end{aligned}$$

In [Tam10, Theorem 2.2], Tamanoi observes the following Frobenius relation

$$\Psi(\mu_{Ta}(a \otimes b)) = (-1)^{d(|a|-d)} a \cdot \Psi(b) = \Psi(a) \cdot b \in H_*(LM \times LM).$$

We interpret everything as living in $H_{*+d}(LM)$ and $H_{*+d}(LM \times LM)$. Then the Künneth map is a degree d map and its inverse a degree $-d$ map. We can thus rewrite the Frobenius relation as

$$\nu_{Ta} \circ \mu_{Ta} = (\mu_{Ta} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \nu_{Ta}) = (\mathrm{id} \otimes \mu_{Ta}) \circ (\nu_{Ta} \otimes \mathrm{id}).$$

This is exactly the relation, we expect for a the $\mathrm{GCob}_{0,-2d}^{\partial_{out}\neq\varnothing}$-algebra structure on $H_{*+d}(LM) = \Sigma^{-d} H_*(LM)$. This structure is predicted by shifting the $\mathrm{GCob}_{-d,-d}^{\partial_{out}\neq\varnothing}$-structure in Example 6.5 and applying Proposition 4.17.

It is worth noting that the coproducts, ν_{Go} and its shifted version ν_{Ta}, are the so called trivial coproduct on $H_*(LM)$. This is in contrast to the Goresky-Hingston coproduct constructed in [GH09; HW17].
Explicitly, Hingston and Wahl give a definition for the Chas-Sullivan product and the Goresky-Hingston product in [HW17].
Their definition of the product is graded associative, unital and commutative. This lifts from the respective properties of the intersection product ν_{Th} discussed in 6.1 (see the proof of [HW17, Theorem 2.5]). This is in line with Example 6.5.
For the coproduct, we slightly amend their definition. Hingston and Wahl define in [HW17, Definition 1.4] the loop coproduct as the degree $1 - n$ map given for a chain $A \in C_*(LM, M)$ by

$$\mathrm{AW}(\mathrm{cut}(R_{\mathrm{GH}}([\tau_{\mathrm{GH}} \cap](A \times I)))).$$

For a detailed explanation of this notation, see [HW17]. We note that the first map in this composition $\times I$ acts from the right while the second map $[\tau_{\mathrm{GH}}\cap]$ acts from the left.
As we chose the convention that maps act from the left, we define the map as the composition

$$\vee(A) \coloneqq \mathrm{AW}(\mathrm{cut}(R_{\mathrm{GH}}([\tau_{\mathrm{GH}}\cap](I \times A)))).$$

An analogous proof strategy as for [HW17, Theorem 2.14] shows that $\vee$ is a degree $1-n$ coproduct that satisfies the graded coassociativity and cocommutativity relation as in Theorem 5.1.

However, the Chas-Sullivan product and the Goresky-Hingston coproduct do not satisfy a Frobenius relation in this setting. In particular, the product is an operation on $H_*(LM)$ while the coproduct is an operation on $H_*(LM, M)$ the homology relative the constant loops.

6.4. Rabinowitz loop homology. In the series of articles [CHO20; CO22b; CO22a; CHO23; CO24], Cieliebak, Hingston and Oancea give an interpretation where the Chas-Sullivan product and the Goresky-Hingston coproduct can be interpreted in a Frobenius type structure, what Cieliebak and Oancea call a biunital coFrobenius bialgebra. As discussed in Remark 5.4, the signs in Theorem 5.1 correspond to the signs appearing in the definition of biunital coFrobenius algebras. This gives a correspondence of their notion of a biunital coFrobenius bialgebras and our graded Frobenius algebras in the finite dimensional case. But for the key example of Rabinowitz loop homology, Cieliebak and Oancea need infinite dimensional algebras. For this they use the language of graded Tate vector spaces to get a Frobenius relation.
They give two tensor products on graded Tate vector spaces: $\widehat{\otimes}^*, \widehat{\otimes}^!$. All maps that appear in their relations have source $A^{\widehat{\otimes}^* k}$ and target $A^{\widehat{\otimes}^! l}$ for some $k, l \geq 0$. The framework of PROPs set up so far fails to describe those algebras. Indeed for a graded Tate vector space A, the collection

$$\mathrm{End}_A(X, Y) \coloneqq \mathrm{Hom}(A^{\widehat{\otimes}^* X}, A^{\widehat{\otimes}^! Y})$$

for X, Y finite sets does not have a canonical structure of a PROP. This is because in general there is no canonical map $A^{\widehat{\otimes}^! X} \to A^{\widehat{\otimes}^* X}$ and thus no composition.
Our remedy for this problem is to work with dioperads and the following proposition:

Proposition 6.6. *Let A be a graded Tate vector space. The collection* $\mathrm{End}_A(X, Y)$ *has a canonical structure of a dioperad in* $\mathbb{Z}\mathrm{Mod}_0$.

Proof. We recall from Subsection 4.3 that a dioperad structure consists of a (S_X, S_Y)-bimodule structure, a unit and composition maps.
The objects $\mathrm{End}_A(X, Y)$ have a canonical (S_X, S_Y)-bimodule structure given by permuting the coordinates.
The unit is given by $\mathrm{id}_A \in \mathrm{End}_A(*, *) = \mathrm{Hom}(A, A)$.
The composition map uses the map

$$\alpha\colon A\widehat{\otimes}^*(B\widehat{\otimes}^! C) \to (A\widehat{\otimes}^* B)\widehat{\otimes}^! C$$

for graded Tate vector spaces A, B, C as described in [CO24, Proposition 5.5].
We fix some sets W, X, Y, Z and elements $x \in X$, $y \in Y$ and an element $f \in \mathrm{End}_A(W, X)$ and $g \in \mathrm{End}_A(Y, Z)$. We define $g_y \circ_x f \in \mathrm{End}_A((Y \setminus \{y\}) \sqcup W, Z \sqcup (X \setminus \{x\}))$ as the composition

$$\begin{aligned} A^{\widehat{\otimes}^*(Y\setminus\{y\})\sqcup W} \cong & (A^{\widehat{\otimes}^* Y\setminus\{y\}})\widehat{\otimes}^*(A^{\widehat{\otimes}^* W}) \xrightarrow{f} (A^{\widehat{\otimes}^* Y\setminus\{y\}})\widehat{\otimes}^*(A^{\widehat{\otimes}^! X}) \\ & \xrightarrow{\alpha} (A^{\widehat{\otimes}^* Y})\widehat{\otimes}^!(A^{\widehat{\otimes}^! X\setminus\{x\}}) \xrightarrow{g} (A^{\widehat{\otimes}^! Z})\widehat{\otimes}^!(A^{\widehat{\otimes}^! X\setminus\{x\}}) \cong A^{\widehat{\otimes}^! Z\sqcup(X\setminus\{x\})}. \end{aligned}$$

Checking that this data satisfies the unitality, associativity and equivariance relations is analogous to checking that the endomorphism dioperad of an object in any symmetric monoidal category is a dioperad. □

This allows us to define a Tate vector space having the structure of an algebra over a dioperad:

DEFINITION 6.7. Let $\mathcal{D}$ a dioperad in $\mathbb{Z}\mathrm{Mod}_0$ and let A be a Tate vector space. A *$\mathcal{D}$-algebra structure* on A is a morphism of dioperads

$$\mathcal{D} \to \mathrm{End}_A$$

enriched over $\mathbb{Z}\mathrm{Mod}_0$, i.e. a morphism of dioperads which is linear and preserves degree.

We recall that Proposition 4.22 shows that if c, d are of different parity working with the forest dioperad does not lose operations compared to the PROP in characteristic $\neq 2$. The main example of Rabinowitz loop homology is such a case. We can thus state the closed part of [CO22b, Corollary 9.3 (c)] in the following example.

EXAMPLE 6.8. Let M be an oriented d-dimensional manifold. The Rabinowitz loop homology $\widehat{H}_*(LM)$ has the structure of a $\mathrm{GCob}^{\mathrm{forest}}_{d,d+1}$-algebra.

REMARK 6.9 (Graded Open-Closed TQFTs). Cieliebak, Hingston and Oancea also use graded open-closed TQFTs. See for example [CHO20, Theorem 1.3.]. As far as we are aware, there does not exist a graph description of graded open-closed TQFTs which describes [CHO20, Theorem 1.3.]. One structure that describes some graded open-closed TQFTs is given in [WW16, Subsection 6.3.]. Taking the bottom homology of the PROP $\mathcal{OC}_d$ describes graded open-closed TQFTs. However, it has the two following restrictions:

- $H_{bot}(\mathcal{OC}_d)$ does not describe a closed unit;
- the degree of the open multiplication is zero, while the degree of the open comultiplication, the closed multiplication and the closed comultiplication are all equal to a fixed $d \in \mathbb{Z}$.

Especially, the second restriction means that this PROP does not describe [CHO20, Theorem 1.3.] even after possibly shifting the algebras.
This leaves the question open whether there is a graph cobordism category describing a PROP or dioperad encoding the graded open-closed TQFTs as in [CHO20, Theorem 1.3.]. One promising outline of a construction may be the following. One constructs a dioperad analogous to $\mathrm{GCob}^{\mathrm{forest}}_{c,d}$ but with two types of vertices. Vertices of *open type*, that have cyclic ordering on $s^{-1}(v)$, and vertices of *closed type*, that do not have a cyclic ordering. Moreover to make this well-defined, one stipulates that edges between vertices of different type may not be collapsed by graph morphisms. Such edges correspond to zippers and cozippers in [LP08; CHO20; CO22b].
Then one can consider a twist by

$$\mathbf{det}(G, \partial^o_{in})^{c_o} \otimes \mathbf{det}(G, \partial^c_{in})^{c_c} \otimes \mathbf{det}(G, \partial^o_{out})^{d_o} \otimes \mathbf{det}(G, \partial^c_{out})^{d_c}$$

where ∂^o_{in} is the in-coming boundary of vertices of open type, ∂^c_{in} is the in-coming boundary of vertices of closed type and so on. This is a dioperad with open and closed part whose multiplication and comultiplication may have any degree.
However, an open-closed TQFT as in [LP08; CHO20; CO22b] satisfies some relations involving the zipper and cozipper. For example, the zipper is an algebra morphism and the cozipper is a coalgebra morphism. These relations are not satisfied in the dioperad as we have described it. While it seems possible to force these relations in the dioperad, making this precise is beyond the scope of this article.

7. Proof of main theorem

In this section, we introduce some computational tools which lets us more easily compute orientations in $\mathbf{det}_{c,d}(G)$. With this machinery we can prove the main theorem in Subsection 7.2.

7.1. Computational tools. We recall that a generator $\omega(G) \in \mathbf{det}_{c,d}(G)$ is called an orientation. For the proof, we need to be able to do three things:

(I) compute how (fat) graph isomorphisms act on orientations;
(II) compute how edge collapses act on orientations;
(III) compute the composition of two orientations.

We recall that the composition in (III) was defined in Lemma 4.2 using a splitting of the short exact sequence:

$$0 \longrightarrow C_*(|G|, |\partial_{in}|) \longrightarrow C_*(|G' \circ G|, |\partial_{in}|) \longrightarrow C_*(|G'|, |\partial_{in}|) \longrightarrow 0.$$

To explicitly compute (I), (II) and (III), we use Proposition 7.1 inspired by [GK98, Proposition 4.14] and [God07, Lemma 14]. This gives us an expression of orientations in terms of top exterior powers of edges, half-edges and vertices:

For X a finite set, we use the following notation

- $\mathrm{Or}(X) \coloneqq \mathrm{Or}(\mathbb{Z}^X)$ the top exterior concentrated in degree 0;
- $\det(X) \coloneqq \det(\mathbb{Z}^X)$ the top exterior concentrated in degree $-|X|$.

Proposition 7.1. *Let G be a graph. There is a natural isomorphism*

$$\mathbf{det}(G, \partial_{in}) \cong \det(E)^{-1} \otimes \mathrm{Or}(H) \otimes \det(V \setminus L_{in}).$$

Moreover, an analogous statement for $\mathbf{det}(G, \partial_{out})$ holds after replacing L_{in} by L_{out}.

Proof. We set $L \coloneqq L_{in}$ and $\partial \coloneqq \partial_{in}$ or $L \coloneqq L_{out}$ and $\partial \coloneqq \partial_{out}$. Consider the cellular chain complex $C_*(|G|, |L|)$ for $|G|$ relative $|L|$:

$$0 \longrightarrow \bigoplus_{e \in E} \mathrm{Or}(\{h_e, \sigma h_e\}) \longrightarrow \mathbb{Z}^V/\mathbb{Z}^L \longrightarrow 0,$$

$$h_e \wedge \sigma h_e \longmapsto s(\sigma h) - s(h).$$

We have that the homology of this complex is $H_*(|G|, |L|)$. By Lemma 4.1, we have a sequence of natural isomorphisms

$$\begin{aligned}
\mathbf{det}(G, \partial) &\overset{\text{(e)}}{\cong} \mathbf{det}(C_*(G, L)) \\
&\overset{\text{(b)}}{\cong} \mathbf{det}\left(\Sigma \bigoplus_{e \in E} \mathrm{Or}(\{h_e, \sigma h_e\})\right) \otimes \det(\mathbb{Z}^V/\mathbb{Z}^L) \\
&\cong \mathbf{det}\left(\Sigma \bigoplus_{e \in E} \mathrm{Or}(\{h_e, \sigma h_e\})\right) \otimes \det(V \setminus L) \\
&\overset{\text{(c)}}{\cong} \det\left(\bigoplus_{e \in E} \mathrm{Or}(\{h_e, \sigma h_e\})\right)^{-1} \otimes \det(V \setminus L) \\
&\overset{\text{(b)}}{\cong} \bigotimes_{e \in E} \det(\mathrm{Or}(\{h_e, \sigma h_e\}))^{-1} \otimes \det(V \setminus L)
\end{aligned}$$

We note that $\det(\mathrm{Or}(\{h_e, \sigma h_e\}))^{-1}$ is a free group of rank 1 concentrated in degree 1. Choosing a generator of this group is equivalent to choosing a generator in the degree 0 group $\mathrm{Or}(\{h_e, \sigma h_e\})$. We thus find

$$\det(\mathrm{Or}(\{h_e, \sigma h_e\})^{-1} \cong \det(\{e\})^{-1} \otimes \mathrm{Or}(\{h_e, \sigma h_e\})$$

where tensoring with $\det(\{e\})^{-1}$ ensures that the right hand side is in degree 1. We thus find

$$\begin{aligned} \mathbf{det}(G, \partial) &\cong \bigotimes_{e\in E} (\det(\{e\})^{-1} \otimes \mathrm{Or}(\{h_e, \sigma h_e\})) \otimes \det(V \setminus L) \\ &\overset{\text{(b)}}{\cong} \det(E)^{-1} \otimes \mathrm{Or}(H) \otimes \det(V \setminus L) \end{aligned}$$

where the last isomorphism comes from the fact that $H = \sqcup_{e\in E}\{h_e, \sigma h_e\}$. $\square$

REMARK 7.2. Explicitly, for $E(G) = \{e_1, \dots, e_\alpha\}$, $e_i = \{h_i, \sigma h_i\}$ and $V(G)\setminus L_{in}(G) = \{v_1, \dots, v_\beta\}$ a generator of $\mathbf{det}(G, \partial_{in})$ can be written as

$$(e_1 \wedge \dots \wedge e_\alpha)^{-1} \otimes h_1 \wedge \sigma h_1 \wedge \dots \wedge \sigma h_\alpha \otimes v_1 \wedge \dots \wedge v_\beta. \tag{11}$$

A graph isomorphism induces a map of edges, half-edges and vertices. This lets us compute isomorphism on orientations which gives (I) on our wish list. For computing edge collapses as in (II), we use Lemma 7.3 which lets us compute an edge collapse on an orientation expressed in terms of edges, half-edges and vertices. For computing compositions as in (III), we use Lemma 7.4. The proofs of both lemmas are quite technical. But once they are proven doing (I), (II) and (III) are straight-forward computations.

LEMMA 7.3. *Let G be a graph and $E = \{e_0, e_1, \dots, e_\alpha\}$, $e_i = \{h_i, \sigma h_i\}$ and $V = \{v_0, v_1, \dots, v_\beta\}$. Assume that e_0 is not a tadpole and $s(\sigma h_0) = v_0 \notin L_{in}(G)$.*
The canonical map $c\colon |G| \to |G|/e_0$ does the following to generators as in (11):

$$\begin{aligned} \mathbf{det}(c)\colon\ & \det(E(G))^{-1} \otimes \mathrm{Or}(H(G)) \otimes \det(V(G) \setminus L_{in}(G)) \\ & \qquad \to \det(E(G/e_0))^{-1} \otimes \mathrm{Or}(H(G/e_0)) \otimes \det(V(G/e_0) \setminus L_{in}(G/e_0)) \\ & (e_0 \wedge e_1 \wedge \dots \wedge e_\alpha)^{-1} \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \dots \wedge \sigma h_\alpha \otimes v_0 \wedge v_1 \wedge \dots \wedge v_\beta \\ & \qquad \mapsto (e_1 \wedge \dots \wedge e_\alpha)^{-1} \otimes h_1 \wedge \dots \wedge \sigma h_\alpha \otimes v_1 \wedge \dots \wedge v_\beta. \end{aligned}$$

Here we identify the newly created vertex in G/e_0 with $s(h_0)$ which gives an identification

$$V(G) \setminus \{v_0\} \cong V(G/e_0)$$

together with the canonical identification

$$E(G) \setminus \{e_0\} \cong E(G/e_0), \qquad\qquad H(G) \setminus \{h_0, \sigma h_0\} \cong H(G/e_0).$$

Moreover, an analogous statement for $\mathbf{det}(G, \partial_{out})$ holds after replacing L_{in} by L_{out}.

PROOF. We set $L \coloneqq L_{in}$ or $L \coloneqq L_{out}$ and $\partial \coloneqq \partial_{in}$ or $\partial \coloneqq \partial_{out}$. The map c induces a map of CW-complexes. This in turn induces a map of cellular chain complexes

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{e\in E(G)} \mathrm{Or}(e) \arrow[r] \arrow[d, "c|_E"] & \mathbb{Z}^{V(G)}/\mathbb{Z}^L \arrow[r] \arrow[d, "c|_V"] & 0 \\ 0 \arrow[r] & \bigoplus_{e\in E(G/e_0)} \mathrm{Or}(e) \arrow[r] & \mathbb{Z}^{V(G/e_0)}/\mathbb{Z}^L \arrow[r] & 0 \end{tikzcd}$$

where $c|_E$ sends $\mathrm{Or}(e_0)$ to zero and all other edges to themselves and $c|_V$ sends v_0 to $v \coloneqq s(h_0)$ and all other vertices to themselves. This is a surjective map of chain complexes with kernel:

$$0 \longrightarrow \mathrm{Or}(\{h_0, \sigma h_0\}) \longrightarrow \langle v_0 - v\rangle \longrightarrow 0$$

$$h_0 \wedge \sigma h_0 \longmapsto v_0 - v$$

we denote this complex by C. Because e_0 is not a tadpole $v_0 - v \neq 0$ and thus C has trivial homology. Lemma 4.1 (e) gives the isomorphism

$$\mathbb{1} \cong \mathbf{det}(C) \cong \det(\{e_0\})^{-1} \otimes \mathrm{Or}(e_0) \otimes \det(\langle v_0 - v\rangle)$$

given by

$$1 \mapsto e_0^{-1} \otimes h_0 \wedge \sigma h_0 \otimes (v_0 - v). \tag{12}$$

The geometric map $\mathbf{det}(c)\colon \mathbf{det}(G, \partial) \to \mathbf{det}(G/e_0, \partial)$ is given by the composition

$$\mathbf{det}(H_*(G, \partial)) \cong \mathbf{det}(C) \otimes \mathbf{det}(H_*(G/e_0, \partial)) \cong \mathbb{1} \otimes \mathbf{det}(H_*(G/e_0, \partial)) \cong \mathbf{det}(H_*(G/e_0, \partial)).$$

The isomorphism $\mathbf{det}(H_*(G, \partial)) \cong \mathbf{det}(C) \otimes \mathbf{det}(H_*(G/e_0, \partial))$ induces

$$\begin{aligned}
&\det(E(G))^{-1} \otimes \mathrm{Or}(H(G)) \otimes \det(V(G) \setminus L)\\
&\to \det(E(G/e_0))^{-1} \otimes \det(\{e_0\})^{-1} \otimes \mathrm{Or}(e_0) \otimes \mathrm{Or}(H(G/e_0)) \otimes \det(\{v_0 - v\}) \otimes \det(V(G/e_0) \setminus L)\\
&\to \det(\{e_0\})^{-1} \otimes \mathrm{Or}(e_0) \otimes \det(\{v_0 - v\}) \otimes \det(E(G/e_0))^{-1} \otimes \mathrm{Or}(H(G/e_0)) \otimes \det(V(G/e_0) \setminus L).
\end{aligned}$$

This isomorphism is given by the identifications

$$E(G) = \{e_0\} \sqcup E(G/e_0),\ H(G) = e_0 \sqcup H(G/e_0),\ \mathbb{Z}^{V(G)\setminus L} \cong \mathbb{Z}^{v_0 - v} \oplus \mathbb{Z}^{V(G/e_0)\setminus L}$$

and then permuting the factors. We recall that in our conventions for graded abelian groups, the canonical isomorphism between the dual of the tensor product and the tensor product of the dual is order reversing. Thus the map on edges is order reversing:

$$\begin{aligned}
\det(E(G))^{-1} &\to \det(E(G/e_0)) \otimes \det(\{e_0\})^{-1},\\
(e_0 \wedge e_1 \wedge \cdots \wedge e_\alpha)^{-1} &\mapsto (e_1 \wedge \cdots \wedge e_\alpha)^{-1} \otimes e_0^{-1}.
\end{aligned}$$

Concretely on generators, we thus get the map

$$\begin{aligned}
&(e_0 \wedge e_1 \wedge \cdots \wedge e_\alpha)^{-1} \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \cdots \wedge \sigma h_\alpha \otimes v_0 \wedge v_1 \wedge \cdots \wedge v_\beta\\
&= (e_0 \wedge e_1 \wedge \cdots \wedge e_\alpha)^{-1} \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \cdots \wedge \sigma h_\alpha \otimes (v_0 - v) \wedge v_1 \wedge \cdots \wedge v_\beta\\
&\mapsto (e_1 \wedge \cdots \wedge e_\alpha)^{-1} \otimes e_0^{-1} \otimes h_0 \wedge \sigma h_0 \otimes h_1 \wedge \cdots \wedge \sigma h_\alpha \otimes (v_0 - v) \otimes v_1 \wedge \cdots \wedge v_\beta\\
&\mapsto e_0^{-1} \otimes h_0 \wedge \sigma h_0 \otimes (v_0 - v) \otimes (e_1 \wedge \cdots \wedge e_\alpha)^{-1} \otimes h_1 \wedge \cdots \wedge \sigma h_\alpha \otimes v_1 \wedge \cdots \wedge v_\beta.
\end{aligned}$$

The isomorphism $\mathbf{det}(C) \otimes \mathbf{det}(H_*(G/e_0, \partial)) \cong \mathbf{det}(H_*(G/e_0, \partial))$ is now given by applying (12). Concretely on generators, we compute

$$\begin{aligned}
&e_0^{-1} \otimes h_0 \wedge \sigma h_0 \otimes (v_0 - v) \otimes (e_1 \wedge \cdots \wedge e_\alpha)^{-1} \otimes h_1 \wedge \cdots \wedge \sigma h_\alpha \otimes v_1 \wedge \cdots \wedge v_\beta\\
&\quad\mapsto (e_1 \wedge \cdots \wedge e_\alpha)^{-1} \otimes h_1 \wedge \cdots \wedge \sigma h_\alpha \otimes v_1 \wedge \cdots \wedge v_\beta.
\end{aligned}$$

Composing these two maps thus gives the equation we wanted to compute. □

LEMMA 7.4. *Let G, G' be composable graphs in $\underline{\mathrm{GCob}}$. The composition map*

$$\mathrm{comp}\colon \mathbf{det}(G', \partial_{in}) \otimes \mathbf{det}(G, \partial_{in}) \cong \mathbf{det}(G' \circ G, \partial_{in})$$

given in Lemma 4.2 is computed via

$$\begin{aligned}&\det(E(G'))^{-1}\otimes \mathrm{Or}(H(G'))\otimes \det(V(G')\setminus L_{in}(G'))\\&\qquad\otimes \det(E(G))^{-1}\otimes \mathrm{Or}(H(G))\otimes \det(V(G)\setminus L_{in}(G))\\&\qquad\qquad\cong \det(E(G'\circ G))^{-1}\otimes \mathrm{Or}(H(G'\circ G))\otimes \det(V(G'\circ G)\setminus L_{in}(G'\circ G))\end{aligned}$$

where the map is given by the canonical identifications

$$\begin{gathered}E(G')\sqcup E(G)\cong E(G'\circ G),\\ H(G')\sqcup H(G)\cong H(G'\circ G),\\ (V(G')\setminus L_{in}(G'))\sqcup (V(G)\setminus L_{in}(G))\cong V(G'\circ G)\setminus L_{in}(G'\circ G).\end{gathered}\tag{13}$$

Moreover, an analogous statement holds for $\mathbf{det}(G,\partial_{out})$ *after replacing* L_{in} *by* L_{out}*.*

Proof. We set $L:=L_{in}$ or $L:=L_{out}$ and $\partial:=\partial_{in}$ or $\partial:=\partial_{out}$.
We consider the cellular chain complexes $C_*(G,L), C_*(G',L)$ and $C_*(G'\circ G,L)$. The map comp is given by Lemma 4.1 (d) applied to the short exact sequence of complexes

$$0\to C_*(G,L)\to C_*(G'\circ G,L)\to C_*(G',L)\to 0.$$

One splitting of this short exact sequence is given by the maps on chains coming from the identification of the generating sets as in (13). □

Remark 7.5. We write $E(G)=\{e_1,\dots,e_k\}$, $e_i=\{h_i,\sigma h_i\}$, $V(G)\setminus L_{in}(G)=\{v_1,\dots,v_l\}$, $E(G')=\{e'_1,\dots,e'_m\}$, $e'_i=\{h'_i,\sigma h'_i\}$ and $V(G')\setminus L_{in}(G')=\{v'_1,\dots,v'_n\}$. Then we compute the following Koszul signs:

$$\begin{aligned}&\mathrm{comp}((e'_1\wedge\cdots\wedge e'_m)^{-1}\otimes h'_1\wedge\sigma h'_1\wedge\cdots\wedge\sigma h'_m\otimes v'_1\wedge\cdots\wedge v'_n\\&\qquad\otimes(e_1\wedge\cdots\wedge e_k)^{-1}\otimes h_1\wedge\sigma h_1\wedge\cdots\wedge\sigma h_k\otimes v_1\wedge\cdots\wedge v_l)\\&\quad=(-1)^{(m+n)k}(e'_1\wedge\cdots\wedge e'_m\wedge e_1\wedge\cdots\wedge e_k)^{-1}\otimes h'_1\wedge\cdots\wedge\sigma h'_m\wedge h_1\wedge\cdots\wedge\sigma h_k\\&\qquad\otimes v'_1\wedge\cdots\wedge v'_n\wedge v_1\wedge\cdots\wedge v_l\end{aligned}$$

in $\det(E(G'\circ G))^{-1}\otimes \mathrm{Or}(H(G'\circ G))\otimes\det(V(G'\circ G)\setminus L_{in}(G'\circ G))$.

7.2. Proof. With the calculation tools in Proposition 7.1, Lemma 7.3 and Lemma 7.4, we are in a position to prove Theorem 5.1.

Proof of Theorem 5.1. We recall that Theorem 5.1 gives a description of (c,d)-graded (commutative or symmetric) Frobenius algebras in terms of maps and relations that those maps satisfy. We defined a (c,d)-graded (commutative or symmetric) Frobenius algebra as a (symmetric) monoidal functor preserving degrees out of $\mathrm{GCob}_{c,d}$ (or $\mathrm{fGCob}_{c,d}$ or $\mathrm{pGCob}_{c,d}$).
For the parts of the proof that apply to all three categories, we denote $\mathcal{G}_{c,d}$ for the category $\mathrm{GCob}_{c,d}$, $\mathrm{fGCob}_{c,d}$ or pGCob. This is a category enriched in graded abelian groups. We denote by $\underline{\mathcal{G}}$ the corresponding 2-category of graph cobordisms $\underline{\mathrm{GCob}}$, $\underline{\mathrm{fGCob}}$ or $\underline{\mathrm{pGCob}}$. Finally, we denote by $\mathcal{G}$ the associated 1-categories GCob, fGCob or pGCob where morphisms are equivalence classes of graph cobordisms as in Definition 2.12.

We recall that Proposition 4.5 gives isomorphisms

$$\mathcal{G}_{c,d}(X,Y)\cong\bigoplus_{[G]\in\mathcal{G}(X,Y)}\mathbf{det}_{c,d}(G)_{\pi_1(|\underline{\mathcal{G}}(X,Y)|,G)}.\tag{14}$$

An element $[G]\in\mathcal{G}(X,Y)$ is an equivalence class of a graph cobordism up to zig-zag of graph morphisms. Such an equivalence class can be represented by a graph that can be written as a composition of the four elementary graphs $\succ\!\!-, \bullet\!\!-, -\!\!\prec, -\!\!\bullet$ (and possibly the twist map if $\mathcal{G}$ is GCob or fGCob). Indeed, expand the graph such that it has only graphs of valence ≤ 3 and no two valence 3-vertex are next to each other. Direct the edges such that no vertex of valence 2 or 3 has only incoming or only outgoing edges. To make this possible, one has to possibly introduce some units or counits. See Figure 12.

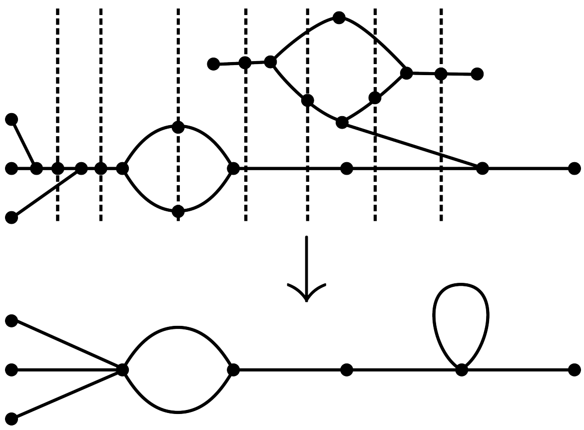


FIGURE 12. The graph on the bottom can be lifted to a graph which can be decomposed into the four elementary graphs. This graph can for example be written as $(-\!\!\bullet \otimes \succ\!\!-) \circ (\succ\!\!- \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes -\!\!\prec \otimes \mathrm{id}) \circ (-\!\!\prec \otimes \mathrm{id}) \circ (\bullet\!\!- \otimes \succ\!\!-) \circ -\!\!\prec \circ \succ\!\!- \circ (\succ\!\!- \otimes \mathrm{id})$. (Note the order-reversing convention for the composition $\circ$.)

This shows that $\mathrm{pGCob}_{c,d}$ is generated under composition and monoidal product by chosen generators

$$\omega(\succ\!\!-) \in \mathbf{det}_{c,d}(\succ\!\!-)_{\pi_1(|\underline{\mathcal{G}}(*\sqcup *,*)|,\succ\!\!-)}, \qquad \omega(\bullet\!\!-) \in \mathbf{det}_{c,d}(\bullet\!\!-)_{\pi_1(|\underline{\mathcal{G}}(\varnothing,*)|,\bullet\!\!-)},$$
$$\omega(-\!\!\prec) \in \mathbf{det}_{c,d}(-\!\!\prec)_{\pi_1(|\underline{\mathcal{G}}(*,*\sqcup *)|,-\!\!\prec)}, \qquad \omega(-\!\!\bullet) \in \mathbf{det}_{c,d}(-\!\!\bullet)_{\pi_1(|\underline{\mathcal{G}}(*,\varnothing)|,-\!\!\bullet)}.$$

The categories $\mathrm{GCob}_{c,d}$ and $\mathrm{fGCob}_{c,d}$ are also generated by those four elements but as a *symmetric* monoidal category rather than just as a monoidal category.
We note that those four elements are non-trivial for all $c, d \in \mathbb{Z}$. This can be seen as there exists examples of (c, d)-graded commutative Frobenius algebras with non-trivial μ, η, ν and ε (see Example 5.8). However, our proof does not use this fact.
Any (symmetric) monoidal functor $F\colon \mathcal{G}_{c,d} \to \mathcal{C}$ is thus uniquely determined up to isomorphism by the images $\mu = F(\omega(\succ\!\!-))$, $\eta = F(\omega(\bullet\!\!-))$, $\nu = F(\omega(-\!\!\prec))$ and $\varepsilon = F(\omega(-\!\!\bullet))$.

We now fix explicit orientations $\omega(\succ\!\!-)$, $\omega(\bullet\!\!-)$, $\omega(-\!\!\prec)$ and $\omega(-\!\!\bullet)$: we label the graphs $\succ\!\!-$, $\bullet\!\!-$, $-\!\!\prec$ and $-\!\!\bullet$ as in Figures 13, 14, 15 and 16. We choose the convention that all half-edges named σh_i are glued to the internal vertex. This is convenient when we later apply Lemma 7.3 to compute edge collapses.

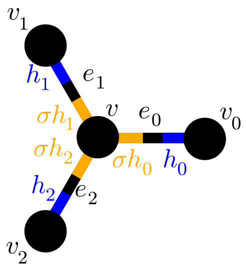


FIGURE 13. The graph $\succ\!\!-$ with v the lone internal vertex, v_0 the lone outgoing vertex and v_1 and v_2 the incoming vertices. The vertex v_1 corresponding to the first coordinate of the input and v_2 to the second coordinate.

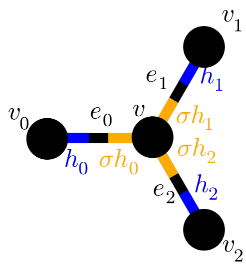


FIGURE 14. The graph $-\!\!\prec$ with v the lone internal vertex, v_0 the lone incoming vertex and v_1 and v_2 the outgoing vertices. The vertex v_1 corresponding to the first coordinate of the output and v_2 to the second coordinate.

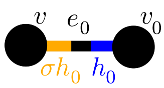


FIGURE 15. The graph $\bullet\!\!-$ with v the lone internal vertex and v_0 the lone outgoing vertex.

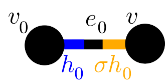


FIGURE 16. The graph $-\!\!\bullet$ with v the lone internal vertex and v_0 the lone incoming vertex.

Using the identification as in Proposition 7.1, we define generators

$$\begin{aligned}
\omega_{in}(\succ\!\!-) &\coloneqq (e_2 \wedge e_1 \wedge e_0)^{-1} \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \sigma h_1 \wedge h_2 \wedge \sigma h_2 \otimes v \wedge v_0 \in \mathbf{det}(\succ\!\!-, \partial_{in}),\\
\omega_{out}(\succ\!\!-) &\coloneqq 1 \in \mathbf{det}(\succ\!\!-, \partial_{out}) \cong \mathbb{1},\\
\omega_{in}(\bullet\!\!-) &\coloneqq e_0^{-1} \otimes h_0 \wedge \sigma h_0 \otimes v \wedge v_0 \in \mathbf{det}(\bullet\!\!-, \partial_{in}),\\
\omega_{out}(\bullet\!\!-) &\coloneqq 1 \in \mathbf{det}(\bullet\!\!-, \partial_{out}) \cong \mathbb{1},\\
\omega_{in}(-\!\!\prec) &\coloneqq 1 \in \mathbf{det}(-\!\!\prec, \partial_{in}) \cong \mathbb{1},\\
\omega_{out}(-\!\!\prec) &\coloneqq (e_2 \wedge e_1 \wedge e_0)^{-1} \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \sigma h_1 \wedge h_2 \wedge \sigma h_2 \otimes v \wedge v_0 \in \mathbf{det}(-\!\!\prec, \partial_{out}),\\
\omega_{in}(-\!\!\bullet) &\coloneqq 1 \in \mathbf{det}(-\!\!\bullet, \partial_{in}) \cong \mathbb{1},\\
\omega_{out}(-\!\!\bullet) &\coloneqq e_0^{-1} \otimes h_0 \wedge \sigma h_0 \otimes v \wedge v_0 \in \mathbf{det}(-\!\!\bullet, \partial_{out}).
\end{aligned}$$

The generator $\omega_{in}(\bullet\!\!-)$ is the canonical generators coming from the isomorphism

$$\begin{array}{ccccc}
\mathbf{det}(H_*(\bullet\!\!-, \partial_{in})) & \overset{4.1(e)}{\to} & \det(C_*(\bullet\!\!-, \partial_{in})) & \overset{7.1}{\to} & \det(E(\bullet\!\!-))^{-1} \otimes \mathrm{Or}(H(\bullet\!\!-)) \otimes \det(V(\bullet\!\!-) \setminus L_{in}(\bullet\!\!-)),\\
v_0 & \mapsto & (h_0 \wedge \sigma h_0)^{-1} \otimes v \wedge v_0 & \mapsto & e_0^{-1} \otimes h_0 \wedge \sigma h_0 \otimes v \wedge v_0.
\end{array}$$

An analogous statement holds for the chosen generator $\omega_{out}(-\!\!\bullet)$.

We note that $\omega_{out}(\succ\!\!-), \omega_{out}(\bullet\!\!-), \omega_{in}(-\!\!\prec)$ and $\omega_{in}(-\!\!\bullet)$ all have degree zero. The generators $\omega_{in}(\succ\!\!-)$ and

$\omega_{out}(\prec)$ are in degree 1 while $\omega_{in}(\bullet\!-)$ and $\omega_{out}(-\!\bullet)$ are in degree -1. Then we set for fixed $c, d \in \mathbb{Z}$ the orientations

$$\omega(\succ) := \omega_{in}(\succ)^{\otimes c} \otimes \omega_{out}(\succ)^{\otimes d} \in \mathbf{det}_{c,d}(\succ), \qquad \omega(\bullet\!-) := \omega_{in}(\bullet\!-)^{\otimes c} \otimes \omega_{out}(\bullet\!-)^{\otimes d} \in \mathbf{det}_{c,d}(\bullet\!-),$$
$$\omega(\prec) := \omega_{in}(\prec)^{\otimes c} \otimes \omega_{out}(\prec)^{\otimes d} \in \mathbf{det}_{c,d}(\prec), \qquad \omega(-\!\bullet) := \omega_{in}(-\!\bullet)^{\otimes c} \otimes \omega_{out}(-\!\bullet)^{\otimes d} \in \mathbf{det}_{c,d}(-\!\bullet).$$

We note that $\omega(\succ)$ is in degree c and $\omega(\bullet\!-)$ is in degree $-c$ while $\omega(\prec)$ is in degree d and $\omega(-\!\bullet)$ is in degree $-d$. As F is enriched over $\mathbb{Z}\mathrm{Mod}_0$, it preserves degrees of maps. We thus have $|\mu| = c$, $|\eta| = -c$, $|\nu| = d$ and $|\varepsilon| = -d$.
We have proven so far that a (symmetric) monoidal functor $F\colon \mathcal{G}_{c,d} \to \mathcal{C}$ is uniquely determined up to isomorphism by the data

$$\begin{aligned} &A = F(*) \in \mathcal{C}, \\ &\mu = F(\omega(\succ)) \in \mathcal{C}_c(A \otimes A, A), && \eta = F(\omega(\bullet\!-)) \in \mathcal{C}_{-c}(\mathbb{1}, A), \\ &\nu = F(\omega(\prec)) \in C_d(A, A \otimes A), && \varepsilon = F(\omega(-\!\bullet)) \in \mathcal{C}_{-d}(A, \mathbb{1}). \end{aligned}$$

The theorem we are proving states that this data defines a functor *if and only if* certain relations are satisfied. We first argue the only if part of the statement assuming the following lemma:

Lemma 7.6. *In the category $\mathcal{G}_{c,d}$, the following identities hold:*

$$\omega(\succ) \circ (\omega(\succ) \otimes 1) = (-1)^c \omega(\succ) \circ (1 \otimes \omega(\succ)) \in \mathbf{det}_{c,d}(\succ\!\succ),$$
$$(-1)^c \omega(\succ) \circ (\omega(\bullet\!-) \otimes 1) = (-1)^{\frac{c(c-1)}{2}} = \omega(\succ) \circ (1 \otimes \omega(\bullet\!-)) \in \mathbf{det}_{c,d}(-\!\bullet\!-) \cong \mathbb{1},$$
$$(\omega(\prec) \otimes 1) \circ \omega(\prec) = (-1)^d (1 \otimes \omega(\prec)) \circ \omega(\prec) \in \mathbf{det}_{c,d}(\prec\!\prec),$$
$$(-1)^d (1 \otimes \omega(-\!\bullet)) \circ \omega(\prec) = (-1)^{\frac{d(d-1)}{2}} = (\omega(-\!\bullet) \otimes 1) \circ \omega(\prec) \in \mathbf{det}_{c,d}(-\!\bullet\!-) \cong \mathbb{1},$$
$$(\omega(\succ) \otimes 1) \circ (1 \otimes \omega(\prec)) = (-1)^{cd} \omega(\prec) \circ \omega(\succ) = (1 \otimes \omega(\succ)) \circ (\omega(\prec) \otimes 1) \in \mathbf{det}_{c,d}(\succ\!\prec).$$

Moreover in the category $\mathrm{GCob}_{c,d}$*, the automorphism of the graph $\succ$ which switches the two incoming edges acts by $(-1)^c$ on $\mathbf{det}_{c,d}(\succ)$ and, in the category* $\mathrm{fGCob}_{c,d}$*, the fat graph automorphism of the fat graph $\succ$ which switches the two incoming edges acts by $(-1)^c$ on $\mathbf{det}_{c,d}(\succ)$.*

These identities are preserved under F because F is linear and monoidal. The lemma thus shows that the relations are satisfied by μ, η, ν and ε if they come from a functor F. This thus concludes the only if part.

Before proving the lemma, we show the if part of the statement that the data $(A, \mu, \eta, \nu, \varepsilon)$ satisfying the relations defines a functor:
Consider a morphism $f \in \mathcal{G}(X, Y)$. As we have

$$\mathcal{G}_{c,d}(X, Y) \cong \bigoplus_{[G] \in \mathcal{G}(X,Y)} \mathbf{det}_{c,d}(G)_{\pi_1(|\underline{\mathcal{G}}(X,Y)|, G)},$$

we may assume that f is a homogeneous generator, i.e., it is given by the $\pi_1(|\underline{\mathcal{G}}(X, Y)|, G)$-orbit of $\omega(G)$ for some generator $\omega(G) \in \mathbf{det}_{c,d}(G)$. We write $[G]$ as a composition of the graphs $\succ$, $\bullet\!-$, $\prec$ and $-\!\bullet$ as in Figure 12. We then define $F(\omega(G))$ as the corresponding composition of μ, η, ν and ε. The map F is by definition linear and degree-preserving, i.e., it is enriched over $\mathbb{Z}\mathrm{Mod}_0$. Moreover up to choosing the right decomposition of G, this is functorial and (symmetric) monoidal. However, our definition of $F(f)$ depends on two choices: the decomposition of G and the choice of representative of the $\pi_1(|\underline{\mathcal{G}}(X, Y)|, G)$-orbit.
We first argue why the definition is independent of the choice of decomposition: in [Koc04; LP08], Kock and Lauda and Pfeiffer describe a normal form for a decomposition of a closed or an open surface. Moreover, they show that any decomposition can be brought into this normal form using the

six generating relations: associativity, unitality, coassociativity, counitality, the Frobenius relation and possibly commutativity or symmetry. Thus any two decompositions are related to each other by a sequence of those six relations.

An analogous argument for (fat) graphs shows that any two decompositions into the four graphs $\succ\!-$, $\bullet\!-$, $-\!\prec$ and $-\!\bullet$ are related by a sequence of those six relations.

Therefore, any two decompositions are related by applying the relations in Lemma 7.6. If the morphisms μ, η, ν and ε satisfy the corresponding relation in $\mathcal{C}$, the definition of $F(\omega(G))$ as above is independent of the choice of decomposition.

It remains to show that our definition of F is independent of choice of representative in the $\pi_1(|\underline{\mathcal{G}}(X,Y)|, G)$-orbit. This is equivalent to showing that our definition of $F(\omega(G))$ is invariant under the $\pi_1(|\underline{\mathcal{G}}(X,Y)|, G)$-action:

Any element in $\pi_1(|\underline{\mathcal{G}}(X,Y)|, G)$ is represented by a zig-zag of edge collapses and (fat) graph isomorphisms:

$$G = G_0 \longrightarrow G_1 \longleftarrow G_2 \longrightarrow G_3 \longleftarrow \ldots \longrightarrow G_{n-1} \longleftarrow G_n = G.$$

Applying one of the relations in Lemma 7.6 to a decomposition G produces a new decomposition G'. The relation then specifies a zig-zag

$$G \longrightarrow G'' \longleftarrow G'$$

where G and G' are given by a different decomposition. We show that all zig-zags can be lifted to a sequence of zig-zags coming from the relations in Lemma 7.6.

We can lift each graph G_{2i} to a decomposition into the four elementary graphs: $\widetilde{G}_{2i} \to G_{2i}$ where $\widetilde{G}_{2i}$ has a decomposition into the four elementary graphs. We fix such a decomposition for all $\widetilde{G}_{2i}$.

We thus have the following diagram:

$$\begin{tikzcd}
\widetilde{G}_0 \arrow[d, equal] & & \widetilde{G}_2 \arrow[d] & & \ldots & & \widetilde{G}_n \arrow[d, equal] \\
G \arrow[r] & G_1 & G_2 \arrow[l] \arrow[r] & G_3 & \ldots \arrow[l] \arrow[r] & G_{n-1} & G. \arrow[l]
\end{tikzcd}$$

Each $\widetilde{G}_{2i} \to G_{2i} \to G_{2i\pm 1}$ gives a decomposition of $G_{2i\pm 1}$. We thus can rewrite the above diagram suggestively as

$$\begin{tikzcd}
\widetilde{G}_0 \arrow[r, equal] \arrow[d, equal] & \widetilde{G}_0 \arrow[d] & \widetilde{G}_2 \arrow[r, equal] \arrow[d] & \widetilde{G}_2 \arrow[r, equal] \arrow[d] & \widetilde{G}_2 \arrow[d] & \ldots & \widetilde{G}_n \arrow[r, equal] \arrow[d] & \widetilde{G}_n \arrow[d, equal] \\
G \arrow[r] & G_1 \arrow[r, equal] & G_1 & G_2 \arrow[l] \arrow[r] & G_3 & \ldots \arrow[l] \arrow[r] & G_{n-1} & G. \arrow[l]
\end{tikzcd}$$

The decompositions $\widetilde{G}_{2i} \to G_{2i+1}$ and $\widetilde{G}_{2i+2} \to G_{2i+1}$ are two decompositions of G_{2i+1}. Therefore $\widetilde{G}_{2i}$ and $\widetilde{G}_{2i+2}$ are related by a sequence of the relations in Lemma 7.6. Thus there exists a zig-zag coming from those relations fitting into the lifting problem

$$\begin{tikzcd}
\widetilde{G}_{2i} \arrow[r, dashed] \arrow[d] & \widetilde{G}_{2i+2} \arrow[d] \\
G_{2i+1} \arrow[r, equal] & G_{2i+1}.
\end{tikzcd}$$

This lifts the whole zig-zag to a zig-zag coming from the relations in Lemma 7.6.
The action of the element of $\pi_1(|\underline{\mathcal{C}}(X,Y)|, G)$ can thus be computed by applying the relations in Lemma 7.6.
Consequently, if μ, η, ν and ε satisfy the corresponding relations in $\mathcal{C}$, our definition of $F(\omega(G))$ does not depend on the representative of the $\pi_1(\underline{\mathcal{G}}(X,Y), G)$-orbit as it is invariant under the $\pi_1(\underline{\mathcal{G}}(X,Y), G)$-action.
This shows that the data μ, η, ν and ε satisfying the relations lets us define a functor F and thus concludes the only if part of the statement. The only thing left to prove is Lemma 7.6. The proof makes up the rest of the paper.

Proof of Lemma 7.6. We prove the seven identities using Lemmas 7.3 and 7.4.

Graded associativity. We show that

$$\omega(\succ) \circ (\omega(\succ) \otimes 1) = (-1)^c \omega(\succ) \circ (1 \otimes \omega(\succ)) \in \mathbf{det}_{c,d}(\not\succ) \tag{15}$$

holds in $\mathcal{G}_{c,d}$.
We start computing

$$\begin{aligned}
\omega(\succ) \circ (\omega(\succ) \otimes 1) &= (\omega_{in}(\succ)^{\otimes c} \otimes \omega_{out}(\succ)^{\otimes d}) \circ (\omega_{in}(\succ)^{\otimes c} \otimes \omega_{out}(\succ)^{\otimes d} \otimes 1) \\
&= \mathrm{comp}_1(\omega_{in}(\succ)^{\otimes c} \otimes \omega_{in}(\succ)^{\otimes c}) \otimes \mathrm{comp}_1(\omega_{out}(\succ)^{\otimes d} \otimes \omega_{out}(\succ)^{\otimes d}) \\
&= (-1)^{\frac{c(c-1)}{2}} \mathrm{comp}_1(\omega_{in}(\succ) \otimes \omega_{in}(\succ))^{\otimes c} \otimes \mathrm{comp}_1(\omega_{out}(\succ) \otimes \omega_{out}(\succ))^{\otimes d}.
\end{aligned}$$

Here comp_1 is the composition we get by gluing one copy of $\succ$ at v_0 to v_1 of the other copy. The sign $(-1)^{\frac{c(c-1)}{2}}$ comes from the fact that we have to reshuffle $\omega_{in}(\succ)^{\otimes c} \otimes \omega_{in}(\succ)^{\otimes c}$ into c pairs of $\omega_{in}(\succ) \otimes \omega_{in}(\succ)$ (via a riffle shuffle). For $\omega_{out}(\succ)$ no such sign appears as they are in degree 0.
Analogously, we compute

$$\omega(\succ) \circ (1 \otimes \omega(\succ)) = (-1)^{\frac{c(c-1)}{2}} \mathrm{comp}_2(\omega_{in}(\succ) \otimes \omega_{in}(\succ))^{\otimes c} \otimes \mathrm{comp}_2(\omega_{out}(\succ) \otimes \omega_{out}(\succ))^{\otimes d}.$$

Here comp_2 is the composition we get by gluing one copy of $\succ$ at v_0 to v_2 of the other copy because v_2 is the vertex corresponding to the second input.
It remains to compare $\mathrm{comp}_1(\omega_{in}(\succ) \otimes \omega_{in}(\succ))$ to $\mathrm{comp}_2(\omega_{in}(\succ) \otimes \omega_{in}(\succ))$ and $\mathrm{comp}_1(\omega_{out}(\succ) \otimes \omega_{out}(\succ))$ to $\mathrm{comp}_2(\omega_{out}(\succ) \otimes \omega_{out}(\succ))$.
We note that because of triviality of the $\omega_{out}(\succ)$ orientations, we have

$$\mathrm{comp}_1(\omega_{out}(\succ) \otimes \omega_{out}(\succ)) = \mathrm{comp}_2(\omega_{out}(\succ) \otimes \omega_{out}(\succ)).$$

Equation (15) then follows from the following identity

$$\mathrm{comp}_1(\omega_{in}(\succ) \otimes \omega_{in}(\succ)) = -\mathrm{comp}_2(\omega_{in}(\succ) \otimes \omega_{in}(\succ)). \tag{16}$$

To show this identity, we start with $\mathrm{comp}_1(\omega_{in}(\succ) \otimes \omega_{in}(\succ))$. For this, we denote the edges, half-edges and vertices of the left copy of $\succ$ by e, e_0, e_1, e_2 and so on, as before. This represents the multiplication that is applied second. We denote the edges, half-edges and vertices of the right copy of $\succ$ by e', e'_0, e'_1, e'_2 and so on. This represents the multiplication that is applied first.
We compute using Lemma 7.4 and Remark 7.5:

$$\begin{aligned}
\mathrm{comp}_1(\omega_{in}(\succ) \otimes \omega_{in}(\succ)) &= \mathrm{comp}_1((e_2 \wedge e_1 \wedge e_0)^{-1} \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \sigma h_1 \wedge h_2 \wedge \sigma h_2 \otimes v \wedge v_0 \\
&\qquad \otimes (e'_2 \wedge e'_1 \wedge e'_0)^{-1} \otimes h'_0 \wedge \sigma h'_0 \wedge h'_1 \wedge \sigma h'_1 \wedge h'_2 \wedge \sigma h'_2 \otimes v' \wedge v'_0) \\
&= -(e_2 \wedge e_1 \wedge e_0 \wedge e'_2 \wedge e'_1 \wedge e'_0)^{-1} \\
&\qquad \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \sigma h_1 \wedge h_2 \wedge \sigma h_2 \wedge h'_0 \wedge \sigma h'_0 \wedge h'_1 \wedge \sigma h'_1 \wedge h'_2 \wedge \sigma h'_2 \\
&\qquad \otimes v \wedge v_0 \wedge v' \wedge v'_0
\end{aligned}$$

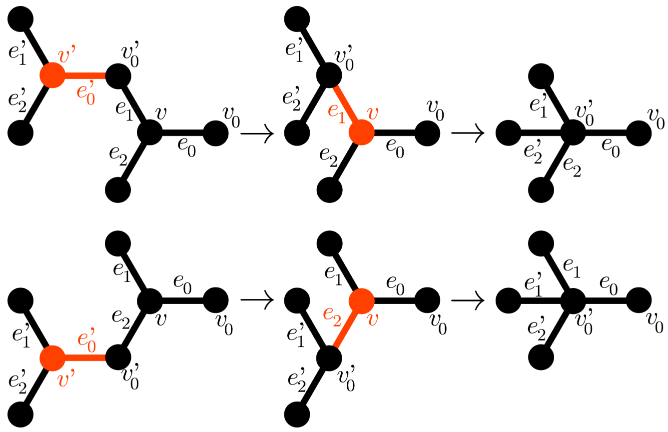


FIGURE 17. The two composition of two multiplications give the same graph after collapsing two edges.

As we glued v_0' to v_1, we can collapse e_0' and e_1 to obtain the graph $\rightarrowtail$ (see top of Figure 17). We recall the naming convention that $s(\sigma h_i)$ is an internal vertex. Therefore when collapsing e_0', we can apply Lemma 7.3 using that $s(\sigma h_0') = v'$ is the vertex we want to delete from the orientation. Thereafter deleting e_1, we note that $s(\sigma h_1) = v$ is the vertex we delete in the process. Importantly to apply Lemma 7.3, we must reshuffle the edge and vertex, we want to delete to the first position. This gives some signs. We do the reshuffle explicitly in the first computation and thereafter only denote the corresponding sign after the edge collapse. We find:

$$\begin{aligned}
&-(e_2 \wedge e_1 \wedge e_0 \wedge e_2' \wedge e_1' \wedge e_0')^{-1}\\
&\qquad \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \sigma h_1 \wedge h_2 \wedge \sigma h_2 \wedge h_0' \wedge \sigma h_0' \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \otimes v \wedge v_0 \wedge v' \wedge v_0'\\
&\qquad = (e_0' \wedge e_2 \wedge e_1 \wedge e_0 \wedge e_2' \wedge e_1')^{-1}\\
&\qquad\quad \otimes h_0' \wedge \sigma h_0' \wedge h_0 \wedge \sigma h_0 \wedge h_1 \wedge \sigma h_1 \wedge h_2 \wedge \sigma h_2 \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \otimes v' \wedge v \wedge v_0 \wedge v_0'\\
&\qquad \mapsto (e_2 \wedge e_1 \wedge e_0 \wedge e_2' \wedge e_1')^{-1}\\
&\qquad\quad \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \sigma h_1 \wedge h_2 \wedge \sigma h_2 \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \otimes v \wedge v_0 \wedge v_0'\\
&\qquad \mapsto \; -(e_2 \wedge e_0 \wedge e_2' \wedge e_1')^{-1} \otimes h_0 \wedge \sigma h_0 \wedge h_2 \wedge \sigma h_2 \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \otimes v_0 \wedge v_0'.
\end{aligned}$$

The edges of $\rightarrowtail$ are here labelled (from top to bottom) by e_1', e_2' and e_2 as in the top right of Figure 17.
Analogously, we compute

$$\begin{aligned}
\mathrm{comp}_2(\omega_{in}(\succ) \otimes \omega_{in}(\succ)) &= -(e_2 \wedge \mathrm{comp}_2(\omega_{in}(\succ) \otimes \omega_{in}(\succ))\\
&= -(e_2 \wedge e_1 \wedge e_0 \wedge e_2' \wedge e_1' \wedge e_0')^{-1}\\
&\quad \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \sigma h_1 \wedge h_2 \wedge \sigma h_2 \wedge h_0' \wedge \sigma h_0' \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2'\\
&\quad \otimes v \wedge v_0 \wedge v' \wedge v_0'.
\end{aligned}$$

But as we glued v_0' to v_2, we can now collapse e_0' and e_2 (see bottom of Figure 17). In this process, we first delete the vertex $s(\sigma h_0') = v'$ and then the vertex $s(\sigma h_2) = v$. We thus find

$$
\begin{aligned}
&-(e_2 \wedge e_1 \wedge e_0 \wedge e_2' \wedge e_1' \wedge e_0')^{-1} \\
&\qquad \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \sigma h_1 \wedge h_2 \wedge \sigma h_2 \wedge h_0' \wedge \sigma h_0' \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \otimes v \wedge v_0 \wedge v' \wedge v_0' \\
&\qquad \mapsto (e_2 \wedge e_1 \wedge e_0 \wedge e_2' \wedge e_1')^{-1} \\
&\qquad\quad \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \sigma h_1 \wedge h_2 \wedge \sigma h_2 \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \otimes v \wedge v_0 \wedge v_0' \\
&\qquad \mapsto (e_1 \wedge e_0 \wedge e_2' \wedge e_1')^{-1} \otimes h_0 \wedge \sigma h_0 \wedge h_1 \wedge \sigma h_1 \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \otimes v_0 \wedge v_0'.
\end{aligned}
$$

The edges of $\succ\!\!-$ are here labelled (from top to bottom) by e_1, e_1' and e_2' as in the bottom right of Figure 17. This differs from the naming scheme of the other composition by an even permutation. This shows that the two orientations differ by the sign -1. We have proved (16) and thus (15).

Graded unitality. We want to compute

(17) $$(-1)^c \omega(\succ\!\!-) \circ (\omega(\bullet\!-) \otimes 1) = (-1)^{\frac{c(c-1)}{2}} = \omega(\succ\!\!-) \circ (1 \otimes \omega(\bullet\!-)) \in \mathbf{det}_{c,d}(-\!\bullet\!-) \cong \mathbb{1}.$$

As for graded associativity, we compute the signs from the riffle shuffle

$$\omega(\succ\!\!-) \circ (\omega(\bullet\!-) \otimes 1) = (-1)^{\frac{c(c-1)}{2}} \mathrm{comp}_1(\omega_{in}(\succ\!\!-) \otimes \omega_{in}(\bullet\!-))^{\otimes c} \otimes \mathrm{comp}_1(\omega_{out}(\succ\!\!-) \otimes \omega_{out}(\bullet\!-))^{\otimes d}$$

and

$$\omega(\succ\!\!-) \circ (1 \otimes \omega(\bullet\!-)) = (-1)^{\frac{c(c-1)}{2}} \mathrm{comp}_2(\omega_{in}(\succ\!\!-) \otimes \omega_{in}(\bullet\!-))^{\otimes c} \otimes \mathrm{comp}_2(\omega_{out}(\succ\!\!-) \otimes \omega_{out}(\bullet\!-))^{\otimes d}.$$

We note that, by triviality of the ω_{out}-orientations, we have

$$\mathrm{comp}_1(\omega_{out}(\succ\!\!-) \otimes \omega_{out}(\bullet\!-)) = 1 = \mathrm{comp}_2(\omega_{out}(\succ\!\!-) \otimes \omega_{out}(\bullet\!-)) \in \mathbf{det}(-\!\bullet\!-, \partial_{out}).$$

The equation (17) then follows from the following identity

(18) $$-\mathrm{comp}_1(\omega_{in}(\succ\!\!-) \otimes \omega_{in}(\bullet\!-)) = 1 = \mathrm{comp}_2(\omega_{in}(\succ\!\!-) \otimes \omega_{in}(\bullet\!-)) \in \mathbf{det}(-\!\bullet\!-, \partial_{in}) \cong \mathbb{1}.$$

To show this, we denote the edges, half-edges and vertices of $\bullet\!-$ by e_0 and so on. On the other hand, we denote the edges, half-edges and vertices of $\succ\!\!-$ by e_0', e_1', e_2' and so on. We thus compute by Lemma 7.4 and Remark 7.5:

$$
\begin{aligned}
\mathrm{comp}_1(\omega_{in}(\succ\!\!-) \otimes \omega_{in}(\bullet\!-)) &= \mathrm{comp}_1((e_2' \wedge e_1' \wedge e_0')^{-1} \otimes h_0' \wedge \sigma h_0' \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \otimes v' \wedge v_0' \\
&\qquad \otimes (e_0)^{-1} \otimes h_0 \wedge \sigma h_0 \otimes v \wedge v_0) \\
&= -(e_2' \wedge e_1' \wedge e_0' \wedge e_0)^{-1} \\
&\qquad \otimes h_0' \wedge \sigma h_0' \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \wedge h_0 \wedge \sigma h_0 \otimes v' \wedge v_0' \wedge v \wedge v_0.
\end{aligned}
$$

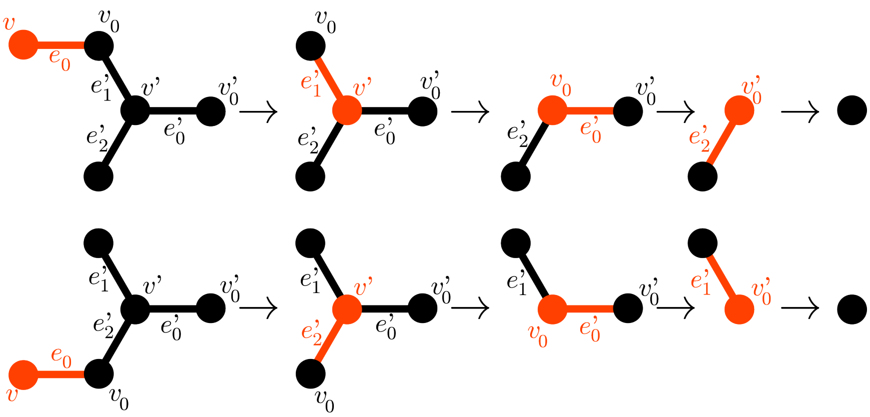


FIGURE 18. The two composition of a unit and a multiplications give the graph with one vertex representing id after collapsing four edges.

Now we can collapse e_0 and e_1' as in the top of Figure 18. We use Lemma 7.3 and note that when we collapse e_0, we have to delete $s(\sigma h_0) = v$. If we then collapse e_1', we note that $s(\sigma h_1') = v'$. We compute:

$$\begin{aligned} &-(e_2' \wedge e_1' \wedge e_0' \wedge e_0)^{-1} \otimes h_0' \wedge \sigma h_0' \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \wedge h_0 \wedge \sigma h_0 \otimes v' \wedge v_0' \wedge v \wedge v_0 \\ &\qquad \mapsto (e_2' \wedge e_1' \wedge e_0')^{-1} \otimes h_0' \wedge \sigma h_0' \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \otimes v' \wedge v_0' \wedge v_0 \\ &\qquad \mapsto -(e_2' \wedge e_0')^{-1} \otimes h_0' \wedge \sigma h_0' \wedge h_2' \wedge \sigma h_2' \otimes v_0' \wedge v_0. \end{aligned}$$

Next we collapse e_0' which deletes $s(\sigma h_0') = v_0$:

$$\begin{aligned} &-(e_2' \wedge e_0')^{-1} \otimes h_0' \wedge \sigma h_0' \wedge h_2' \wedge \sigma h_2' \otimes v_0' \wedge v_0 \\ &\qquad \mapsto -(e_2')^{-1} \otimes h_2' \wedge \sigma h_2' \otimes v_0'. \end{aligned}$$

Finally collapsing e_2', we delete $s(\sigma h_2') = v_0'$ and find:

$$-(e_2')^{-1} \otimes h_2' \wedge \sigma h_2' \otimes v_0' \mapsto -1.$$

This shows the left equality in (18).
For the other equation, we compute:

$$\begin{aligned} \mathrm{comp}_2(\omega_{in}(\succ) \otimes \omega_{in}(\bullet\!-)) = &-(e_2' \wedge e_1' \wedge e_0' \wedge e_0)^{-1} \\ &\otimes h_0' \wedge \sigma h_0' \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \wedge h_0 \wedge \sigma h_0 \otimes v' \wedge v_0' \wedge v \wedge v_0. \end{aligned}$$

But now we collapse first e_0 and at the same time deleting $s(\sigma h_0) = v$. Then we collapse e_2' and delete $s(\sigma h_2') = v'$. Then we delete e_0' and delete $s(\sigma h_0') = v_0$. Finally, we collapse e_1' and delete $s(\sigma h_1') = v_0'$ as in the bottom of Figure 18. We thus find

$$-(e_0 \wedge e_2' \wedge e_1' \wedge e_0')^{-1} \otimes h_0 \wedge \sigma h_0 \wedge h_0' \wedge \sigma h_0' \wedge h_1' \wedge \sigma h_1' \wedge h_2' \wedge \sigma h_2' \otimes v \wedge v_0 \wedge v' \wedge v_0' \mapsto 1.$$

One can see that by either doing the calculation explicitly. Or one can notice that we are deleting the vertices in the same order as before but switched the order of the two edges e_1' and e_2' in which we delete them (see bottom of Figure 18). This shows (18) and thus (17).

Graded coassociativity. This computation is dual to the computation of graded associativity, we thus obtain:

$$(\omega(\prec) \otimes 1) \circ \omega(\prec) = (-1)^d (1 \otimes \omega(\prec)) \circ \omega(\prec) \in \mathbf{det}_{c,d}(-\!\!\prec).$$

Graded counitality. This computation is dual to the computation of graded unitality, we thus obtain:

$$(-1)^d(1\otimes\omega(-\!\bullet))\circ\omega(-\!\!\prec) = (-1)^{\frac{d(d-1)}{2}} = (\omega(-\!\bullet)\otimes 1)\circ\omega(-\!\!\prec) \in \mathbf{det}_{c,d}(-\!\bullet\!-) \cong \mathbb{1}.$$

We note that the dual term to $\omega(\succ\!\!-)\circ(\omega(\bullet\!-)\otimes 1)$ is $(1\otimes\omega(-\!\bullet))\circ\omega(-\!\!\prec)$.

The graded Frobenius relation. We want to show that

$$(\omega(\mu)\otimes 1)\circ(1\otimes\omega(\nu)) = (-1)^{cd}\omega(\nu)\circ\omega(\mu) = (1\otimes\omega(\mu))\circ(\omega(\nu)\otimes 1) \in \mathbf{det}_{c,d}(\succ\!\!\prec). \tag{19}$$

We start computing

$$\begin{aligned}(\omega(\mu)\otimes 1)\circ(1\otimes\omega(\nu)) &= (\omega_{in}(\mu)^{\otimes c}\otimes\omega_{out}(\mu)^{\otimes d}\otimes 1)\circ(1\otimes\omega_{in}(\nu)^{\otimes c}\otimes\omega_{out}(\nu)^{\otimes d})\\ &= (\omega_{in}(\mu)^{\otimes c}\otimes 1)\circ(1\otimes\omega_{in}(\nu)^{\otimes c})\otimes(\omega_{out}(\mu)^{\otimes d}\otimes 1)\circ(1\otimes\omega_{out}(\nu)^{\otimes d})\\ &= \mathrm{comp}_{1,2}(\omega_{in}(\mu)\otimes\omega_{in}(\nu))^{\otimes c}\otimes\mathrm{comp}_{1,2}(\omega_{out}(\mu)\otimes\omega_{out}(\nu))^{\otimes d}.\end{aligned}$$

Here $\mathrm{comp}_{1,2}$ is the composition given by gluing together the first outgoing vertex of $-\!\!\prec$ to the second incoming vertex of $\succ\!\!-$.
Analogously, we compute

$$(1\otimes\omega(\mu))\circ(\omega(\nu)\otimes 1) = \mathrm{comp}_{2,1}(\omega_{in}(\mu)\otimes\omega_{in}(\nu))^{\otimes c}\otimes\mathrm{comp}_{2,1}(\omega_{out}(\mu)\otimes\omega_{out}(\nu))^{\otimes d}.$$

Here $\mathrm{comp}_{2,1}$ is the composition given by gluing together the second outgoing vertex of $-\!\!\prec$ to the first incoming vertex of $\succ\!\!-$.
In contrast, we compute

$$\begin{aligned}\omega(\nu)\circ\omega(\mu) &= (\omega_{in}(\nu)^{\otimes c}\otimes\omega_{out}(\nu)^{\otimes d})\circ(\omega_{in}(\mu)^{\otimes c}\otimes\omega_{out}(\mu)^{\otimes d})\\ &= (-1)^{cd}(\omega_{in}(\nu)^{\otimes c}\circ\omega_{in}(\mu)^{\otimes c})\otimes(\omega_{out}(\nu)^{\otimes d}\circ\omega_{out}(\mu)^{\otimes d}).\end{aligned}$$

The sign $(-1)^{cd}$ comes from the fact that we have to move $\omega_{out}(\nu)^{\otimes d}$ which is in degree d past $\omega_{in}(\mu)^{\otimes c}$ which is in degree c. By definition, we have:

$$\begin{aligned}&(-1)^{cd}(\omega_{in}(\nu)^{\otimes c}\circ\omega_{in}(\mu)^{\otimes c})\otimes(\omega_{out}(\nu)^{\otimes d}\circ\omega_{out}(\mu)^{\otimes d})\\ &\qquad = (-1)^{cd}\mathrm{comp}(\omega_{in}(\nu)\otimes\omega_{in}(\mu))^{\otimes c}\otimes\mathrm{comp}(\omega_{out}(\nu)\otimes\omega_{out}(\mu))^{\otimes d}.\end{aligned}$$

The equation (19) now follows from the following identities It holds that

$$\mathrm{comp}_{1,2}(\omega_{in}(\mu)\otimes\omega_{in}(\nu)) = \mathrm{comp}(\omega_{in}(\nu)\otimes\omega_{in}(\mu)) = \mathrm{comp}_{2,1}(\omega_{in}(\mu)\otimes\omega_{in}(\nu))$$

and

$$\mathrm{comp}_{1,2}(\omega_{out}(\mu)\otimes\omega_{out}(\nu)) = \mathrm{comp}(\omega_{out}(\nu)\otimes\omega_{out}(\mu)) = \mathrm{comp}_{2,1}(\omega_{out}(\mu)\otimes\omega_{out}(\nu)).$$

This can be seen either by doing the hands-on computation using Lemmas 7.3 and 7.4 or by noting that $\omega_{in}(\nu)$ and $\omega_{out}(\mu)$ are trivial. Composition with them thus gives the trivial map.

Graded commutativity. We want to show that the automorphism of the graph $\succ\!\!-$ which switches the two incoming edges acts by $(-1)^c$ on $\omega(\succ\!\!-)$. For this, we note that this automorphism acts trivially on $\mathbf{det}(\succ\!\!-,\partial_{out})\cong\mathbb{1}$. On the other hand, we compute the action $\mathbf{det}(\succ\!\!-,\partial_{in})$ as

$$\begin{aligned}&\mathbf{det}(\succ\!\!-,\partial_{in}) \to \mathbf{det}(\succ\!\!-,\partial_{in}),\\ &(e_2\wedge e_1\wedge e_0)^{-1}\otimes h_0\wedge\sigma h_0\wedge h_1\wedge\sigma h_1\wedge h_2\wedge\sigma h_2\otimes v\wedge v_0\\ &\qquad\mapsto (e_1\wedge e_2\wedge e_0)^{-1}\otimes h_0\wedge\sigma h_0\wedge h_2\wedge\sigma h_2\wedge h_1\wedge\sigma h_1\otimes v\wedge v_0.\end{aligned}$$

It only changes the order of e_1 and e_2 and their half-edges. In other words, it acts by -1 on $\mathbf{det}(\succ\!\!-,\partial_{in})$. This combines to a $(-1)^c$-action on $\mathbf{det}_{c,d}(\succ\!\!-)$.

Graded symmetry. We note that flipping the two incoming vertices of $\succ\!\!-$ is not a fat graph automorphism. As before, we call the edges, half-edges and vertices of $\succ\!\!-$ e_0, e_1, e_2 and so on. On the other hand, we call the edges, half-edges and vertices of $\multimap$ e'_0, $h_0, \sigma h_0$ and v'_0, v'. We glue v_0 to v'_0, then we can collapse the edge e_0 and e'_0. On the resulting graph $\succ$, flipping v_1 with v_2 is a fat graph automorphism. This acts on $\mathbf{det}(\succ, \partial_{in})$ by -1 and on $\mathbf{det}(\succ, \partial_{out})$ trivially. This combines to a $(-1)^c$-action on $\mathbf{det}_{c,d}(\succ)$.
Therefore, we have computed all signs appearing in the identities in Lemma 7.6. This concludes the proof of Lemma 7.6, the proof of Theorem 5.1 and thus the paper. □

□

Part II

The Goresky-Hingston coproduct on based loop spaces

Abstract. We construct a lift of the Goresky-Hingston coproduct on the based loop space. Using this lift, we produce formulas for the interaction of the coproduct with the π_1-action and with maps of manifolds. These results lets us compute the torsion-free part of the coproduct on the free loop space for manifolds of the form S^n/G.

Contents

1. Introduction

In this article, we study the Goresky-Hingston coproduct on the based loop space of a pointed, closed, oriented, n-dimensional manifold (M, x_0):

$$\vee_{\mathrm{GH}} \colon H_*(\Omega M, x_0) \to H_{1-n+*}((\Omega M, x_0) \times (\Omega M, x_0)).$$

This coproduct was defined in [GH09] and [Sul04].

In this article, we construct a slightly different coproduct, the *avoiding-stick coproduct*

$$\vee_{\Omega M} \colon H_*(\Omega M) \to H_{1-n+*}(\Omega M \times \Omega M).$$

The avoiding-stick coproduct is a lift of the Goresky-Hingston coproduct (see Proposition 2.12).

Because this coproduct does not have any relative terms, it is easier to work with. We prove an array of results for the avoiding-stick coproduct that directly descend to results for the Goresky-Hingston coproduct. The first result is an invariance result under a π_1-action:

Theorem A (Corollary 3.11). *Assume that the dimension is $n > 1$. The avoiding-stick coproduct $\vee$ factors through the fixed points $H_*(\Omega M \times \Omega M)^{\pi_1(M,x_0)}$ where $\pi_1(M, x_0)$ acts on $\Omega M \times \Omega M$ via*

$$[\tau] \cdot (\gamma, \gamma') := (\gamma \cdot \tau, \tau^{-1} \cdot \gamma').$$

This immediately implies that the avoiding-stick and thus the Goresky-Hingston coproduct both are zero if $\pi_1(M, x_0)$ is infinite (see Corollary 3.12).

Our second main result is a formula for how the avoiding-stick coproduct interacts with maps between manifolds $f \colon M \to N$. It generalizes a result in a note by Nancy Hingston [Hin10, Lemma 3] to the case where f is of any degree and N has arbitrary fundamental group:

Theorem B (Corollary 4.3). *Let $f \colon (M, x_0) \to (N, y_0)$ be a map of closed, oriented, pointed manifolds of dimension $n > 1$. Assume that y_0 is a regular value with $f^{-1}(y_0) = \{x_0, \dots, x_k\}$. Let τ_i be paths from x_i to x_0. For the avoiding-stick coproducts $\vee_{\Omega M}$ and $\vee_{\Omega N}$, it holds*

$$\vee_{\Omega N} \circ f_* = \sum_{i=0}^{k} \deg(f)_{x_i} (r_{[f_*(\tau_i)]} \times l_{[f_*(\tau_i)]^{-1}}) \circ (f \times f)_* \circ \vee_{\Omega M}$$

where r_γ and l_γ denote right and left multiplication with an element $\gamma \in \pi_1(N, y_0)$.

This has as immediate consequence that the avoiding-stick and Goresky-Hingston coproduct on the based loop space commute with degree 1 maps (see Corollary 4.4). This is in contrast to the Goresky-Hingston coproduct on the free loop space (see [Nae21; NS24; Wah19; KP24]).

Applying Theorem B to the case where f is a universal covering lets us describe the coproduct fully in terms of its universal covering:

Theorem C (Remark 4.7 and Proposition 4.8). *Let (M, x_0) be a pointed, closed, oriented manifold with finite fundamental group and universal covering $p\colon (\widetilde{M}, y_0) \to (M, x_0)$. Any element $\alpha \in H_*(\Omega M)$ can be uniquely written as*

$$\alpha = \sum_{g \in \pi_1(M, x_0)} g \cdot p_*(\beta_g)$$

for some $\beta_g \in H_*(\Omega\widetilde{M})$. If the avoiding-stick coproduct on $H_*(\Omega\widetilde{M})$ is computed by

$$\vee_{\Omega\widetilde{M}}(\beta_g) = \sum_i \gamma_{i,g} \times \delta_{i,g},$$

then it holds

$$\vee_{\Omega M}(\alpha) = \sum_{g,h \in \pi_1(M)} \sum_i g \cdot p_*(\gamma_{i,g}) \cdot h \times h^{-1} \cdot p_*(\delta_{i,g}).$$

Because we can compute the coproduct on spheres (Example 3.14), we get a full description of the coproduct on $M = S^n/G$ where G is a finite group acting freely on S^n:

Theorem D (Proposition 5.1 and Corollary 5.2). *Let G be a finite group that acts on S^n freely and orientation-preserving for some $n \geq 2$ and let $M = S^n/G$. The Pontryagin ring on ΩM is given by*

$$H_*(\Omega M) \cong \mathbb{Z}[G][x]$$

where x is a central element of degree $n-1$.

The avoiding-stick coproduct extends to a map $H_{n-1+*}(\Omega M) \to H_*(\Omega M) \otimes H_*(\Omega M)$ which is computed as

$$\vee(gx^k) = \sum_{i+j=k-1} \sum_{h \in G} gh^{-1}x^i \otimes hx^j.$$

This coproduct helps us understand the Leray-Serre spectral sequence associated to the fibration:

$$\begin{tikzcd} \Omega M \arrow[r] & \Lambda M \arrow[d, "\mathrm{ev}_0"] \\ & M \end{tikzcd} \tag{20}$$

where $\Lambda M := \mathrm{Map}(S^1, M)$ denotes the free loop space. Theorem D lets us show that all differentials on the E^r-page for $r \geq 2$ are zero and we find the following result:

Theorem E (Corollary 5.4). *Let G be a non-trivial, finite group that acts on S^n freely and orientation-preserving for some $n \geq 2$ and let $M = S^n/G$. The homology of the free loop space ΛM has connected components $\Lambda_{[g]}M$ corresponding to conjugacy classes $[g]$ in G. The homology of $\Lambda_{[g]}M$ fits in the following short exact sequence:*

$$0 \longrightarrow H_{i-n-1}(S^n/C_G(g)) \longrightarrow H_{i+k(n-1)}(\Lambda_{[g]}M) \longrightarrow H_i(S^n/C_G(g)) \longrightarrow 0$$

for $k \geq 0$ and $1 \leq i \leq n$. This short exact sequence splits for all $i \neq n-1$.

In [Mei11, Theorem 4.8], Meier gave a description of the Goresky-Hingston coproduct on the Leray-Serre spectral sequence associated to (20) in terms of the coproduct on the based loop space.

Together with Theorem D, we use this to give a formula for the coproduct on $H_*(\Lambda M;\mathbb{Q})$ for $M = S^n/G$:

Theorem F (Theorem 5.8). *Let G be a non-trivial, finite group acting freely and orientation-preserving on S^n for $n > 1$. The relative homology of the free loop space of $M = S^n/G$ with rational coefficients is given by*

$$H_*(\Lambda M;\mathbb{Q}) \cong \bigoplus_{[g]\in \mathrm{ccl}(G)} \bigoplus_{k\geq 0} \mathbb{Q}x_{[g],k} \oplus \mathbb{Q}y_{[g],k}$$

with $x_{[g],k}$ in degree $k(n-1)$ and $y_{[g],k}$ in degree $k(n-1)+n$. The homology $H_*(\Lambda M, M;\mathbb{Q})$ has the same description except missing the $[g]=[1]$ and $k=0$ summand.

The Goresky-Hingston coproduct lifts to a map

$$\widehat{\vee}\colon H_*(\Lambda M;\mathbb{Q}) \to H_*(\Lambda M;\mathbb{Q}) \otimes H_*(\Lambda M;\mathbb{Q})$$

computed by

$$\widehat{\vee}(x_{[g],k}) = \sum_{i+j=k-1} \sum_{h\in G} x_{[gh^{-1}],i} \otimes x_{[h],j}$$

and

$$\widehat{\vee}(y_{[g],k}) = \sum_{i+j=k-1} \sum_{h\in G} (x_{[gh^{-1}],i} \otimes y_{[h],j} + y_{[gh^{-1}],i} \otimes x_{[h],j}).$$

We note that this description only depends on G and not on the action itself. This is in contrast to the coproduct with integer coefficients which "knows" about the simple homotopy of S^n/G: in [Nae21], Naef computes part of the coproduct on the lens spaces which is an example of $S^3/(\mathbb{Z}/7\mathbb{Z})$ to give an example of a coproduct which does not commute with homotopy equivalences.

1.1. The avoiding-stick coproduct. We briefly sketch the idea behind the avoiding-stick coproduct and why it does not produce any relative terms. In Subsection 2.3, this is done in full detail.

We first recall a sketch of the definition of the Goresky-Hingston coproduct (see Subsection 2.2 for more details). Roughly speaking, the coproduct on $\alpha \in C_*(\Omega M)$ is computed as follows: multiplying with the fundamental class of the interval $I=[0,1]$ gives a class $I \times \alpha \in C_{*+1}(I\times \Omega M, \partial I \times \Omega M)$. We then "intersect" this class with the subspace of self-intersections: all pairs $(t,\gamma) \in I \times \Omega M$ with $\gamma(t) = \gamma(0)$. Finally we apply the cutting map on the space of self-intersections: a pair (γ, t) gets sent to the pair of loops $(t,\gamma) \mapsto (\gamma|_{[0,t]}, \gamma|_{[t,1]}) \in \Omega M$.

All loops $\gamma \in \Omega M$ have trivial self-intersections for $t=0$ and $t=1$. Cutting at a trivial self-intersection gives a pair of loops $(\gamma_1,\gamma_2) \in \Omega M \times \Omega M$ where γ_1 or γ_2 is the constant loop x_0. This is why the Goresky-Hingston coproduct gives a map

$$C_*(\Omega M) \to C_{1-n+*}(\Omega M \times \Omega M, \{x_0\}\times \Omega M \cup \Omega M \times \{x_0\}).$$

The first advantage of the avoiding-stick coproduct is that it isolates any trivial self-intersection. It is defined analogously to the Goresky-Hingston coproduct but only on a subspace of ΩM: we fix a path $\sigma\colon [0,\varepsilon] \to M$ a stick such that $\sigma(t) \neq x_0$ for $t > 0$ and $\sigma(0) = x_0$. We consider the subspace $\Omega^\sigma M \subseteq \Omega M$ of all paths that follow σ on $[0,\varepsilon]$ and $[1-\varepsilon, 1]$ (see Figure 19). This subspace is homotopic to the whole space ΩM. We define the avoiding-stick coproduct on $\Omega^\sigma M$ instead of ΩM.

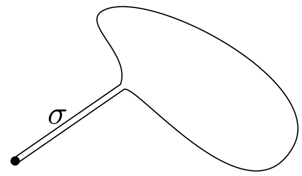


FIGURE 19. The space $\Omega^\sigma M$ consists of all loops that start and end with the path σ.

Working with the space $\Omega^\sigma M$ has the advantage that the trivial self-intersections are isolated because for all $t \in (0,\varepsilon] \cup [1-\varepsilon, 1)$ and $\gamma \in \Omega^\sigma M$, we have $\gamma(t) \neq x_0$. All non-trivial self-intersections are at times $t \in (\varepsilon, 1-\varepsilon)$. A cycle $\alpha \in C_*(I \times \Omega^\sigma M, \partial I \times \Omega^\sigma M)$ which represents some non-trivial self-intersections is thus also a non-relative cycle. Running the rest of the machinery of the Goresky-Hingston coproduct then produces no relative terms.

The avoiding-stick coproduct thus gives a map

$$C_*(\Omega M) \cong C_*(\Omega^\sigma M) \to C_{1-n+*}(\Omega M \times \Omega M)$$

which lifts the Goresky-Hingston coproduct.

Another advantage of the space $\Omega^\sigma M$ is that we have time at both ends of the loop to manipulate the loops. We can take advantage of this in the following way: let $H\colon [0,\varepsilon] \times M \to M$ be a 1-parameter family of homeomorphisms such that $H(0,x) = x$ and $\widetilde{\sigma}(t) = H(t, \sigma(t))$ is also a stick avoiding x_0. We define the following homotopy equivalence:

$$\Phi\colon \Omega^\sigma M \to \Omega^{\widetilde{\sigma}} M,$$
$$(\gamma\colon [0,1] \to M) \mapsto t \mapsto \begin{cases} H(t,\gamma(t)) & \text{if } 0 \leq t \leq \varepsilon; \\ H(\varepsilon, \gamma(t)) & \text{if } \varepsilon \leq t \leq 1-\varepsilon; \\ H(1-t, \gamma(t)) & \text{if } 1-\varepsilon \leq t \leq 1. \end{cases}$$

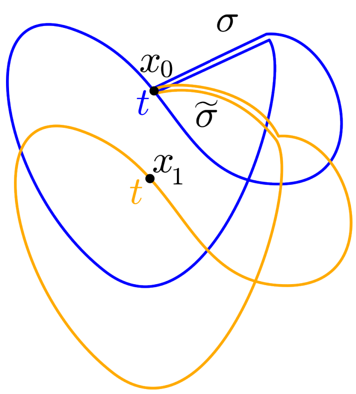


FIGURE 20. In this picture, the 1-parameter family H is given by translation downwards. The blue loop γ is in $\Omega^\sigma M$ and has a self-intersection at t. Its image $\Phi(\gamma)$ in yellow then is in $\Omega^{\widetilde{\sigma}} M$ and at t it crosses $x_1 = \phi^\varepsilon(x_0)$.

The map Φ sends the non-trivial self-intersections $(t,\gamma) \in (\varepsilon, 1-\varepsilon) \times \Omega^\sigma M$ to all pairs $(t, \gamma') \in (\varepsilon, 1-\varepsilon) \times \Omega^\sigma M$ with $\gamma'(t) = x_1 := H(\varepsilon, x_0)$ (see Figure 20). In fact, it gives a bijection. Thus we can detect self-intersections at the point x_1. This is the insight that lets us prove Proposition 3.7 which is the key technical result unlocking the other statements of this article.

Organisation of the paper. In Section 2, we define and compare the trivial coproduct, the Goresky-Hingston coproduct and the avoiding-sticks coproduct on the based loop space of a manifold. In Section 3, we study what happens when we cut at a different point rather than the basepoint and relate it to the coproducts from the previous section. In Section 4, we describe how the different coproducts interact with maps of manifold. We apply this to the example of S^n/G in Section 5.

Conventions. We denote by I the closed interval $[0,1]$ and $\partial I = \{0,1\}$. For $\sigma\colon [a,b] \to X$ a path, we denote $\overline{\sigma}(t) := \sigma(b-t+a)$ for the path in reverse direction.

Let M be a closed, oriented topological manifold of dimension n and let $x_0 \in M$ be a fixed basepoint. We denote by

$$\Omega M := \{\gamma\colon I \to M \mid \gamma(0) = \gamma(1) = x_0\}$$

the space of based loops in M. We denote by x_0 the constant path at x_0.

We use the convention that for topological spaces $X, Y \subseteq Z$, we write $C_*(X,Y) := C_*(X, X\cap Y)$ for the relative chains, leaving the intersection implicit.

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2. Various coproducts

In this section, we give several definitions of coproducts on the based loop space.

2.1. The trivial coproduct. We start with the so-called trivial coproduct on $C_*(\Omega M)$.

We fix U a contractible neighbourhood of x_0 and a representative $\tau \in C^n(U, \{x_0\}^c)$ of the local orientation. We define the map

$$\begin{aligned} e\colon \Omega M &\to M, \\ \gamma &\mapsto \gamma(\tfrac{1}{2}). \end{aligned}$$

Definition 2.1. The *Figure 8 space* is defined as

$$\mathcal{F} := \mathcal{F}M := e^{-1}(x_0) = \{\gamma \in \Omega M \mid \gamma(\tfrac{1}{2}) = x_0\}.$$

The *cutting map* is given by

$$\begin{aligned} c\colon \mathcal{F} &\to \Omega M \times \Omega M, \\ \gamma &\mapsto (\gamma|_{[0,\frac{1}{2}]}, \gamma|_{[\frac{1}{2},1]}). \end{aligned}$$

The inclusion $C_*(e^{-1}(U), \mathcal{F}^c) \to C_*(\Omega M, \mathcal{F}^c)$ is a homotopy equivalence by excision. By [Hat02, Proposition 2.21], we can fix a map

$$\rho\colon C_*(\Omega M, \mathcal{F}^c) \to C_*(e^{-1}(U), \mathcal{F}^c)$$

that is an inverse on homology.

We fix a homotopy

$$R_U\colon I \times U \to U$$

between id_U and the constant map x_0 that fixes x_0.

Remark 2.2. In [Hin10], this is induced by the Riemannian metric on M. For a generalization to the topological case, we do not assume that M has a Riemannian metric. This in turn means, we have to fix the above R_U.

The map

$$\begin{aligned} R\colon e^{-1}(U) &\to \mathcal{F}, \\ \gamma &\mapsto \begin{cases} \gamma(2t) & \text{if } 0 \le t \le \frac{1}{4}, \\ R_U(4t-1, \gamma(\frac{1}{2}))) & \text{if } \frac{1}{4} \le t \le \frac{1}{2}, \\ R_U(3-4t, \gamma(\frac{1}{2})) & \text{if } \frac{1}{2} \le t \le \frac{3}{4}, \\ \gamma(2t-1) & \text{if } \frac{3}{4} \le t \le 1 \end{cases} \end{aligned}$$

defines a homotopy inverse to the inclusion $\mathcal{F} \to e^{-1}(U)$.

Definition 2.3. The *trivial coproduct* is defined as the composition

$$\begin{aligned} \vee_{\mathrm{triv}}\colon C_*(\Omega M) \to C_*(\Omega M, \mathcal{F}^c) &\xrightarrow{\rho} C_*(e^{-1}(U), \mathcal{F}^c) \\ &\xrightarrow{e^*\tau\cap} C_{-n+*}(e^{-1}(U)) \xrightarrow{R_*} C_{-n+*}(\mathcal{F}) \xrightarrow{c_*} C_{-n+*}(\Omega M \times \Omega M). \end{aligned}$$

Remark 2.4. This is the restriction to the based loop space of (a chain level model of) the loop coproduct studied by Tamanoi in [Tam10]. Tamanoi shows that the coproduct is zero except on $H_n(M) \subseteq H_*(\Lambda M)$. In particular, the trivial coproduct is zero on homology. However, the coproduct on chains is still of interest as it gives rise to the following more interesting coproducts.

2.2. The Goresky-Hingston coproduct. We next turn our attention to the Goresky-Hingston coproduct as introduced in [GH09]. We follow the description as in [HW17].

We define the reparametrization map

$$\begin{aligned} r\colon I \times \Omega M &\to \Omega M, \\ (s, \gamma) &\mapsto \left(t \mapsto \begin{cases} \gamma(2st) & \text{if } 0 \le t \le \frac{1}{2}; \\ \gamma(2(1-s)t - 1 + 2s) & \text{if } \frac{1}{2} \le t \le 1. \end{cases} \right) \end{aligned}$$

that parametrizes γ such that $r(\gamma)(\frac{1}{2}) = \gamma(s)$.

We denote $\mathcal{H}$ as the space of half-constant loops

$$\mathcal{H} := \{\gamma \in \Omega \mid \gamma|_{[0,\frac{1}{2}]} \equiv x_0 \text{ or } \gamma|_{[\frac{1}{2},1]} \equiv x_0\}.$$

We note that r sends $\partial I \times \Omega M$ and $I \times \{x_0\}$ to $\mathcal{H}$ and $\vee_{\mathrm{triv}}$ sends $C_*(\mathcal{H})$ to $C_{-n+*}(\Omega M \times \{x_0\} \cup \{x_0\} \times \Omega M)$. We therefore get the following definition

Definition 2.5. The *Goresky-Hingston coproduct* is defined as the composition

$$\begin{aligned} \vee_{\mathrm{GH}}\colon C_*(\Omega M, x_0) &\xrightarrow{I\times} C_{1+*}(I \times \Omega M, \partial I \times \Omega M \cup I \times \{x_0\}) \\ &\xrightarrow{r_*} C_{1+*}(\Omega M, \mathcal{H}) \xrightarrow{\vee_{\mathrm{triv}}} C_{1-n+*}((\Omega M, x_0) \times (\Omega M, x_0)). \end{aligned}$$

Remark 2.6. This definition is indeed a chain-level model of the original definition as in [GH09] (see [HW17, Theorem 2.13]).

2.3. The avoiding-stick definition. In this subsection, we define a coproduct using sticks avoiding x_0. This construction avoids self-intersections coming from $\partial I \times \Omega M$ and thus allows us to lift this coproduct to a coproduct on $C_*(\Omega M)$.

We fix $0 < \varepsilon < \frac{1}{6}$.

DEFINITION 2.7. Let $x_1, \ldots, x_k \in M$. A *stick avoiding* $x_0, x_1, \ldots, x_k$ is a path $\sigma\colon [0, \varepsilon] \to M$ with $\sigma(0) = x_0$ and there exists a chart $\varphi\colon \mathbb{R}^n \cong U \subseteq M$ such that $\{x_0, x_1, \ldots, x_k\} \cap U = \{x_0\}$ and $\sigma(t) = \varphi(t, 0, \ldots, 0)$.

For a stick σ avoiding $x_0, x_1, \ldots, x_k$, we consider the subspace

$$\Omega^\sigma M := \{\gamma \in \Omega \mid \gamma|_{[0,\varepsilon]} = \sigma, \gamma|_{[1-\varepsilon,1]} = \overline{\sigma}\} \subseteq \Omega M \tag{21}$$

that look like σ at the start and at the end. Here $\overline{\sigma}$ denotes the inverse path to σ.

LEMMA 2.8. *Let $\sigma\colon [0, \varepsilon] \to M$ be a stick avoiding x_0. Then the inclusion $\iota\colon \Omega^\sigma M \hookrightarrow \Omega M$ has homotopy inverse*

$$\begin{aligned} \pi_\sigma\colon \Omega M &\to \Omega^\sigma M, \\ \gamma &\mapsto \sigma \cdot \overline{\sigma} \cdot \gamma \cdot \sigma \cdot \overline{\sigma} \end{aligned}$$

where the composition is parametrized such that γ is followed on $[\varepsilon, \varepsilon]$ in $\frac{1}{1-4\varepsilon}$-speed.

PROOF. We consider the homotopy

$$H(s, \gamma) = \sigma|_{[0,s\varepsilon]} \cdot \overline{\sigma|_{[0,s\varepsilon]}} \cdot \gamma \cdot \sigma|_{[0,s\varepsilon]} \cdot \overline{\sigma|_{[0,s\varepsilon]}}$$

where the composition is parametrized such that γ is followed on $[2s\varepsilon, 1 - 2s\varepsilon]$ with $\frac{1}{1-4s\varepsilon}$ speed. This defines a homotopy between from $\mathrm{id}_{\Omega M}$ to $\iota \circ \pi_\sigma$.

On the other hand, we can write any element of $\Omega^\sigma M$ uniquely as $\sigma \cdot \beta \cdot \overline{\sigma}$ for some $\beta \in \Omega_{\sigma(\varepsilon)} M$. Then

$$G(s, \sigma \cdot \beta \cdot \overline{\sigma}) = \sigma \cdot \overline{\sigma|_{[0,s\varepsilon]}} \cdot \sigma|_{[0,s\varepsilon]} \cdot \beta \cdot \overline{\sigma|_{[0,s\varepsilon]}} \cdot \sigma|_{[0,s\varepsilon]} \cdot \overline{\sigma}$$

is a homotopy between $\mathrm{id}_{\Omega^\sigma M}$ and $\pi_\sigma \circ \iota$. $\square$

We therefore have a homotopic space $\Omega^\sigma M \simeq \Omega M$. This space has the advantage that we can get rid off the half-constant loops $\mathcal{H}$ because they are isolated from the non-trivial self-intersections in $\mathcal{F} \cap (I \times \Omega^\sigma M)$. Therefore we can apply excision to find that we can only study times $(\varepsilon, 1 - \varepsilon)$ to detect all non-trivial self-intersections. In precise terms, we have the following key lemma:

LEMMA 2.9. *Let σ be a stick avoiding x_0. The inclusion*

$$C_*(r((\varepsilon, 1-\varepsilon) \times \Omega^\sigma M), \mathcal{F}^c) \to C_*(r(I \times \Omega^\sigma M), \mathcal{F}^c \cup \mathcal{H})$$

is a quasi-isomorphism.

PROOF. We want to apply excision. We thus must check that the closure of

$$A := r(I \times \Omega^\sigma M) \setminus r((\varepsilon, 1-\varepsilon) \times \Omega^\sigma M)$$

is contained in the interior of $\mathcal{F}^c \cup \mathcal{H}$.

It thus suffices to show the following:

(1) A is closed;
(2) $\mathcal{F}^c \cup \mathcal{H}$ is open;
(3) $A \subseteq \mathcal{F}^c \cup \mathcal{H}$.

For (1), we check that $r|_{I\times\Omega^\sigma M}$ is a homeomorphism onto its image. We first consider $(s,\gamma)\in I\times\Omega^\sigma M$ with $s\notin\partial I$. We note that $r(s,\gamma)(t)=\sigma(2st)\neq x_0$ for $t\in(0,\frac{\varepsilon}{2s})$ and $r(s,\gamma)(t)=\sigma(2(1-s)t-1+2s)\neq x_0$ for $t\in(\frac{1-\varepsilon}{2(1-s)},1)$. This shows that $r(t,\gamma)\notin\mathcal{H}$.

Moreover if $s'\in I$ is such that

$$r(s,\gamma)(2s't)=\sigma(t)$$

for $t\in[0,\varepsilon]$, we have $s=s'$. Therefore we can recover s from $r(s,\gamma)$ and we similarly recover

$$\gamma(t)=\begin{cases} r(s,\gamma)(\frac{t}{2s}) & \text{if } 0\leq t\leq s;\\ r(s,\gamma)(\frac{t+1-2s}{2(1-s)}) & \text{if } s\leq t\leq 1.\end{cases}$$

On the other hand if $r(s,\gamma)\in\mathcal{H}$, we find either $r(s,\gamma)|_{[0,\frac{1}{2}]}\equiv x_0$ or $r(s,\gamma)|_{[\frac{1}{2},1]}\equiv x_0$. Both halves are not constant because otherwise γ would have been constant but γ is not in $\Omega^\sigma M$. If $r(s,\gamma)|_{[0,\frac{1}{2}]}\equiv x_0$, we recover $s=0$ and $\gamma(t)=r(s,\gamma)|_{[\frac{1}{2},1]}(2t-1)$ and, if $r(s,\gamma)|_{[\frac{1}{2},1]}\equiv x_0$, we recover $\gamma(t)=r(s,\gamma)|_{[1,\frac{1}{2}]}(2t)$.

This shows that $r|_{I\times\Omega^\sigma M}$ is a homeomorphism onto its image and thus

$$A=r(I\times\Omega^\sigma M\setminus(\varepsilon,1-\varepsilon)\times\Omega^\sigma M)=r(([0,\varepsilon]\cup[1-\varepsilon,1])\times\Omega^\sigma M).$$

Therefore A is closed because $([0,\varepsilon]\cup[1-\varepsilon,1])\times\Omega^\sigma M$ is closed in $I\times\Omega^\sigma M$.

We now check (2) that $\mathcal{F}^c\cup\mathcal{H}$ is open. On the one hand, $\mathcal{F}^c$ is an open set. On the other hand, the above discussion shows that $\mathcal{H}=r(\partial I\times\Omega^\sigma M)$. This subspace has an open neighbourhood $r(([0,\varepsilon)\cup(1-\varepsilon,1])\times\Omega^\sigma M)$ which is contained in $\mathcal{H}\cup\mathcal{F}^c$ because $r(((0,\varepsilon)\cup(1-\varepsilon,1))\times\Omega^\sigma M)\subseteq\mathcal{F}^c$. This shows that $\mathcal{F}^c\cup\mathcal{H}$ is open.

We finally have to check (3) that A is contained in $e(x_0)^c\cup\mathcal{H}$. This follows from the fact that

$$A=r(\partial I\times\Omega^\sigma M)\cup r\left(((0,\varepsilon]\times[1-\varepsilon,1))\times\Omega^\sigma M\right)$$

and that $r(\partial I\times\Omega^\sigma M)\subseteq\mathcal{H}$ and $r\left(((0,\varepsilon]\times[1-\varepsilon,1))\times\Omega^\sigma M\right)\subseteq\mathcal{F}^c$.

Therefore excision implies that

$$C_*(r(I\times\Omega^\sigma M)\setminus A,\mathcal{F}^c\cup\mathcal{H}\setminus A)\to C_*(r(I\times\Omega^\sigma M),\mathcal{F}^c\cup\mathcal{H})$$

is a homotopy equivalence. The observation

$$\begin{aligned} r(I\times\Omega^\sigma M)\setminus A&=r((\varepsilon,1-\varepsilon)\times\Omega^\sigma M),\\ \mathcal{F}^c\cup\mathcal{H}\setminus A&\subseteq\mathcal{F}^c\end{aligned}$$

concludes the proof. ☐

We fix a map

$$\rho_\sigma\colon C_*(r(I\times\Omega^\sigma M),\mathcal{F}^c\cup\mathcal{H})\to C_*(r((\varepsilon,1-\varepsilon)\times\Omega^\sigma M),\mathcal{F}^c)$$

which is an inverse on homology to the above inclusion (e.g. see [Hat02, Proposition 2.21] for a construction of such a map).

Definition 2.10. Let σ be a stick avoiding x_0. The *avoiding-stick coproduct* is defined as the composition

$$\begin{aligned}\vee_\sigma\colon C_*(\Omega M)&\xrightarrow{(\pi_\sigma)_*}C_*(\Omega^\sigma M)\xrightarrow{I\times}C_{1+*}(I\times\Omega^\sigma M,\partial I\times\Omega^\sigma M)\\ &\xrightarrow{r_*}C_{1+*}(r(I\times\Omega^\sigma M),\mathcal{F}^c\cup\mathcal{H})\\ &\xrightarrow{\rho_\sigma}C_{1+*}(r((\varepsilon,1-\varepsilon)\times\Omega^\sigma M),\mathcal{F}^c)\\ &\xrightarrow{\vee_{\mathrm{triv}}}C_{1-n+*}(\Omega M\times\Omega M).\end{aligned}$$

Remark 2.11. We note that because the first map in the definition of $\vee_{\mathrm{triv}}$ in Definition 2.3 is
$$C_*(\Omega M) \to C_*(\Omega M, \mathcal{F}^c)$$
the avoiding-stick coproduct lands indeed in non-relative chains.

The avoiding-stick coproduct is a lift of the Goresky-Hingston coproduct:

Proposition 2.12. *The map $i\colon (\Omega M, \varnothing) \to (\Omega M, x_0)$ intertwines the Goresky-Hingston coproduct and the avoiding-stick coproduct on homology, that is, the following diagram commutes:*
$$\begin{tikzcd}
H_*(\Omega M) \arrow[r, "i_*"] \arrow[d, "\vee_\sigma"] & H_*(\Omega M, x_0) \arrow[d, "\vee_{\mathrm{GH}}"] \\
H_*(\Omega M \times \Omega M) \arrow[r, "(i\times i)_*"] & H_*((\Omega M, x_0) \times (\Omega M, x_0)).
\end{tikzcd}$$

Proof. We denote ι_σ for the inclusion $\Omega^\sigma M \to \Omega M$ and its induced maps. We consider the diagram
$$\begin{tikzcd}
C_*(\Omega M) \arrow[d, "(\pi_\sigma)_*"] \arrow[dr, "i_*"] & \\
C_*(\Omega^\sigma M) \arrow[r, "(\iota_\sigma)_*"] \arrow[d, "I\times"] & C_*(\Omega M, x_0) \arrow[d, "I\times"] \\
C_{1+*}(I \times \Omega^\sigma M, \partial I \times \Omega^\sigma M) \arrow[r, "(\iota_\sigma)_*"] \arrow[d, "r_*"] & C_{1+*}(I \times \Omega M, \partial I \times \Omega M \cup I \times \{x_0\}) \arrow[d, "r_*"] \\
C_{1+*}(r(I \times \Omega^\sigma M), \mathcal{F}^c \cup \mathcal{H}) \arrow[r, "(\iota_\sigma)_*"] \arrow[d, "\rho_\sigma"] & C_{1+*}(\Omega M, \mathcal{F}^c \cup \mathcal{H}) \arrow[dd, "\vee_{\mathrm{triv}}"] \\
C_{1+*}(r((\varepsilon, 1-\varepsilon) \times \Omega^\sigma M), \mathcal{F}^c) \arrow[ur, "(\iota_\sigma)_*"] \arrow[d, "\vee_{\mathrm{triv}}"] & \\
C_{1-n+*}(\Omega M \times \Omega M) \arrow[r, "(i\times i)_*"] & C_{1-n+*}((\Omega M, x_0) \times (\Omega M, x_0)).
\end{tikzcd}$$
The top triangle commutes in homology because $(\pi_\sigma)_*$ and $(\iota_\sigma)_*$ are homotopy inverses (see Lemma 2.8). The bottom triangle commutes in homology because ρ_σ is a homotopy inverse to an inclusion of chain complexes and inclusions of chain complexes commute. The rest of the diagram commutes because the maps from left to right are come from inclusions of spaces. Therefore the diagram commutes in homology.

Composition along the bottom left computes $(i \times i)_* \circ \vee_\sigma$ and composition along the top left computes $\vee_{\mathrm{GH}} \circ i_*$. ☐

A natural question to ask is now the following:

Question 2.13. Does the absolute coproduct depend on the choice of stick σ?

The answer is yes, if $n = 1$ (see Example 2.14), but no, if $n > 1$ (see Corollary 3.8).

Example 2.14. For $M = S^1$, the lift is not independent of choice of σ. We consider $S^1 \subseteq \mathbb{C}$. We have
$$H_*(\Omega S^1) = \mathbb{Z}[x^{\pm 1}]$$

as a Pontryagin algebra with x in degree 0. Because every group is free, the Künneth map $H_*(\Omega S^1 \times \Omega S^1) \cong H_*(\Omega S^1) \otimes H_*(\Omega S^1)$ is an isomorphism. We can therefore consider the avoiding-stick coproduct as a map $H_*(\Omega S^1) \to H_*(\Omega S^1) \otimes H_*(\Omega S^1)$.

We make the choice that our basepoint is $x_0 := 1 \in \mathbb{C}$. Moreover, we choose

$$\begin{aligned} \gamma_1 \colon I/\partial I &\to S^1, \\ t &\mapsto \exp(2\pi i t) \end{aligned}$$

as a representative of $x \in H_0(\Omega S^1)$.

We fix $\varepsilon := \frac{1}{8}$. Up to homotopy, there are two choices for σ:

$$\begin{aligned} \sigma^+ \colon [0, \tfrac{1}{8}] &\to S^1, \\ t &\mapsto \exp(2\pi i t); \\ \sigma^- \colon [0, \tfrac{1}{8}] &\to S^1, \\ t &\mapsto \exp(-2\pi i t). \end{aligned}$$

We set $x^+ := \sigma^+(\frac{1}{8})$ and $x^- := \sigma^-(\frac{1}{8})$. We compute the coproduct $\vee^+ := \vee_{\sigma^+}$:

- We first compute $\vee^+(1)$. The element $1 = x^0 \in H_*(\Omega S^1)$ is represented by the constant loop. It gets sent to $\sigma^+ \cdot \overline{\sigma^+} \cdot \sigma^+ \cdot \overline{\sigma^+} \in \Omega^{\sigma^+} S^1$ under the map π_{σ^+}. This loop is homotopic to $\gamma := \sigma^+ \cdot c_{x^+} \cdot \overline{\sigma^+}$ as an element in $\Omega^{\sigma^+} S^1$. Here c_{x^+} denotes the constant loop at x^+. This loop has no $t \in [\frac{1}{8}, \frac{7}{8}]$ such that $\gamma(t) = 0$. Therefore, $r_*(I \times \gamma) \in C_*(\mathcal{F}^c \cup \mathcal{H})$ and we have $\vee^+(1) = 0$.
- Next, we compute $\vee^+(t^k)$ for $k > 0$. This element is represented by

  $$\gamma_k(t) = \exp(2\pi i k t).$$

  It gets sent to $\sigma^+ \cdot \overline{\sigma^+} \cdot \gamma_k \cdot \sigma^+ \cdot \overline{\sigma^+} \in \Omega^{\sigma^+} S^1$ under the map π_{σ^+}. This is homotopic to the map

  $$\gamma'_k(t) = \begin{cases} \sigma^+(t) & \text{if } 0 \leq t \leq \frac{1}{8}; \\ \sigma^+(1-t) & \text{if } \frac{7}{8} \leq t \leq 1; \\ \exp\left(\frac{2\pi i (t - \frac{1}{8})}{1 - \frac{1}{4}} + \frac{2\pi i}{8}\right) & \text{if } \frac{1}{8} \leq t \leq \frac{7}{8}; \end{cases}$$

  as an element in $\Omega^{\sigma^+} S^1$.

  There are k times $t \in (\frac{1}{8}, \frac{7}{8})$ such that $\gamma_k(t) = x_0$: concretely, define t_l such that

  $$\frac{2\pi i (t_l - \frac{1}{8})}{1 - \frac{1}{4}} + \frac{2\pi i}{8} = 2\pi i l$$

  for $l = 1, \ldots, k$. These are transversal intersections of the basepoint. Therefore, they get detected by capping with $e^*\tau \in C^0(e^{-1}(U), \mathcal{F}^c)$.

  Cutting at t_k, we get the loop γ_k in the first variable and $\sigma^+ \cdot \overline{\sigma^+} \simeq c_{x_0}$ in the second variable. After cutting at t_l for $l < k$, we get $\sigma^+ \cdot \overline{\sigma^+} \cdot \gamma_l \simeq \gamma_l$ in the first variable and $\gamma_{k-l} \cdot \sigma^+ \cdot \overline{\sigma^+} \simeq \gamma_{k-l}$.

  We therefore find

  $$\vee^+(x^k) = \sum_{l=1}^{k} x^l \otimes x^{k-l}.$$

- An analogous computation for $k<0$ shows
$$\vee^+(x^k) = -\sum_{l=k+1}^{0} x^l \otimes x^{k-l}.$$
The negative sign comes from the fact that all intersections pass x_0 in negative direction.

We summarize our results for $\vee^+$ as
$$\vee^+(t^k) = \begin{cases} 0 & \text{if } k=0; \\ \sum_{l=1}^{k} x^l \otimes x^{k-l} & \text{if } k>0; \\ -\sum_{l=k+1}^{0} x^l \otimes x^{k-l} & \text{if } k<0. \end{cases}$$
An analogous computation for σ^- shows
$$\vee^-(x^k) = \begin{cases} 0 & \text{if } k=0; \\ \sum_{l=0}^{k-1} x^l \otimes x^{k-l} & \text{if } k>0; \\ -\sum_{l=k}^{-1} x^l \otimes x^{k-l} & \text{if } k<0. \end{cases}$$

Remark 2.15. In [CHO23, Subsection 8.4], Cieliebak, Hingston and Oancea study four different lifts of the Goresky-Hingston coproduct. Two of them correspond to an avoiding-stick coproduct: it holds $\vee^+ = \lambda_{v_-,v_-}$ and $\vee^- = \lambda_{v_+,v_+}$ for λ_{v_-,v_-} and λ_{v_+,v_+} as in [CHO23, Remark 8.8].

These coproducts are neither cocommutative or coassociative. In [CHO23, Proposition 8.6], Cieliebak, Hingston and Oancea also compute two coproducts which on the based loop space give
$$\lambda_+(x^k) = \begin{cases} \sum_{l=0}^{k} x^l \otimes x^{k-l} & \text{if } k\geq 0; \\ -\sum_{l=k+1}^{-1} x^l \otimes x^{k-1} & \text{if } k<0; \end{cases}$$
and
$$\lambda_+(x^k) = \begin{cases} \sum_{l=1}^{k-1} x^l \otimes x^{k-l} & \text{if } k>0; \\ -\sum_{l=k}^{0} x^l \otimes x^{k-1} & \text{if } k\leq 0. \end{cases}$$
Such results can be obtained if one replaces $\Omega^\sigma M$ by
$$\{\gamma \in \Omega M \mid \gamma|_{[0,\frac{1}{8}]} = \sigma^+, \gamma|_{[\frac{7}{8},1]} = \overline{\sigma^-}\}$$
which gives λ_+ or by
$$\{\gamma \in \Omega M \mid \gamma|_{[0,\frac{1}{8}]} = \sigma^-, \gamma|_{[\frac{7}{8},1]} = \overline{\sigma^+}\}$$
which gives λ_-.

In general there is no need to restrict to a subspace where $\gamma(t) = \gamma(1-t) = \sigma(t)$ for small t as we do in our definition of the avoiding-path coproduct. One can choose a path for the exit at $0 \leq t \leq \varepsilon$ and another path for the entrance at $1-\varepsilon \leq t \leq 1$ and run the same machinery. However if $n>1$, we show in Corollary 3.8 that the coproduct does not depend on the choice of the paths. For convenience, we thus fix the same path for exit and entrance.

3. Cutting at other points

In this section, we study maps defined by cutting at points other than the basepoint. We relate them to the coproducts we have defined so far. Cutting at other points naturally shows up when we consider maps between manifolds.

3.1. The trivial pre-coproduct. We fix some point $x \in M$. We denote

$$\begin{aligned}\mathcal{P}_{x_0\to x} &:= \{\gamma\colon I \to M \mid \gamma(0) = x_0, \gamma(1) = x\},\\ \mathcal{P}_{x\to x_0} &:= \{\gamma\colon I \to M \mid \gamma(0) = x, \gamma(1) = x_0\},\\ \mathcal{F}_x &:= e^{-1}(x) := \{\gamma \in \Omega M \mid \gamma(\tfrac{1}{2}) = x\}.\end{aligned}$$

We fix some contractible neighbourhood U_x of x. We can define a homotopy inverse

$$\rho_x\colon C_*(\Omega M, \mathcal{F}_x^c) \to C_*(e^{-1}(U_x), \mathcal{F}^c)$$

to the inclusion and a map

$$R_x\colon e^{-1}(U_x) \to \mathcal{F}_x$$

akin to R. Similarly, we can define a cutting map

$$c_x\colon \mathcal{F}_x \to \mathcal{P}_{x_0\to x} \times \mathcal{P}_{x\to x_0}.$$

The orientation of M gives us a local orientation $\tau_x \in C^n(U_x, \{x\}^c)$. Using this, we can define an analogue of the trivial coproduct

Definition 3.1. Let $x \in M$. The *trivial pre-coproduct at x* is given as the composition

$$\begin{aligned}\vee^x_{\text{triv}}\colon C_*(\Omega M) \to C_*(\Omega M, \mathcal{F}_x^c) &\overset{\rho_x}{\to} C_*(e^{-1}(U_x), \mathcal{F}_x^c)\\ &\overset{e^*\tau_x\cap}{\to} C_{-n+*}(e^{-1}(U_x)) \overset{(R_x)_*}{\to} C_{-n+*}(\mathcal{F}_x) \overset{(c_x)_*}{\to} C_{-n+*}(\mathcal{P}_{x_0\to x} \times \mathcal{P}_{x\to x_0}).\end{aligned}$$

Remark 3.2. The definition of the trivial pre-coproduct at x_0 gives the definition of the trivial coproduct as in Definition 2.3.

We now want to relate different pre-coproducts to each other. First, we note that a path τ from x_1 to x_2 defines a homotopy equivalence

$$\begin{aligned}r_\tau \times l_{\overline{\tau}}\colon \mathcal{P}_{x_0\to x_1} \times \mathcal{P}_{x_1\to x_0} &\to \mathcal{P}_{x_0\to x_2} \times \mathcal{P}_{x_2\to x_0},\\ (\gamma_1, \gamma_2) &\mapsto (\gamma_1 \cdot \tau, \overline{\tau} \cdot \gamma_2).\end{aligned}$$

We denote the simplest piece-wise linear map $\chi\colon I \to I$ such that

$$\chi(0) = 0, \qquad \chi|_{[\varepsilon,1-\varepsilon]} \equiv 1, \qquad \chi(1) = 0.$$

We note that $\chi(t) = \chi(1-t)$. (See Figure 21.)

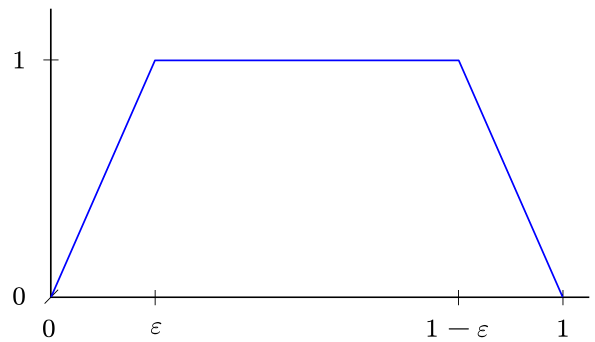

Figure 21. The graph of the function χ.

LEMMA 3.3. *Let $x_1, x_2 \in M$. Let $\phi^s \colon M \to M$ for $s \in [0,1]$ be a 1-parameter family of homeomorphisms such that $\phi^1(x_1) = x_2$.*

Let τ be the path $\tau(t) = \phi^t(x_1)$ and

$$\begin{aligned}\Phi\colon \Omega M &\to \Omega M,\\ \gamma &\mapsto (t \mapsto \phi^{\chi(t)} \circ \gamma(t)).\end{aligned}$$

It holds

$$(r_\tau \times l_{\overline{\tau}})_* \circ \vee^{x_1}_{\mathrm{triv}} = \vee^{x_2}_{\mathrm{triv}} \circ \Phi_*$$

as maps $H_*(\Omega M, \mathcal{F}^c_{x_1}) \to H_{-n+*}(\mathcal{P}_{x_0 \to x_2} \times \mathcal{P}_{x_2 \to x_0})$.

PROOF. The map Φ_* sends $C_*(\Omega M, \mathcal{F}^c_{x_1})$ to $C_*(\Omega M, \mathcal{F}^c_{x_2})$. Indeed, Φ sends $\mathcal{F}^c_{x_1}$ to $\mathcal{F}^c_{x_2}$. Moreover in homology, the maps ρ_{x_i} and R_{x_i} can be defined such that they are intertwined by Φ.

The naturality of the cap product gives that the diagram

$$\begin{array}{ccc} C_*(e^{-1}(U_{x_1}), e(x_1)^c) & \xrightarrow{\Phi_*} & C_*(e^{-1}(U_{x_2}), e(x_2)^c) \\ \Big\downarrow{\scriptstyle \Phi^* e^* \tau_{x_2} \cap} & & \Big\downarrow{\scriptstyle e^* \tau \cap} \\ C_{-n+*}(e^{-1}(U_{x_1})) & \xrightarrow{\Phi_*} & C_{-n+*}(e^{-1}(U_{x_2})). \end{array}$$

We set $\phi := \phi^1$ and find in homology

$$\Phi^* e^* \tau_{x_2} = e^* \phi^* \tau_{x_2} = \deg(\phi)_{x_1} e^* \tau_{x_1}.$$

Because ϕ is homotopic to the identity, we have that $\deg(\phi)_{x_1} = 1$. Therefore Φ intertwines the map $e^* \tau_{x_i} \cap$.

We have shown so far that Φ_* intertwines all but the last map in the definition of the trivial pre-coproduct. We thus have:

$$\vee^{x_2}_{\mathrm{triv}} \circ \Phi_* = (c_{x_2})_* \circ \Phi_* \circ (R_{x_1})_* \circ (e^* \tau_{x_1} \cap) \circ \rho_{x_1}$$

as maps $H_*(\Omega M, \mathcal{F}^c_{x_1}) \to H_{-n+*}(\mathcal{P}_{x_0 \to x_2} \times \mathcal{P}_{x_2 \to x_0})$.

Finally we note that $\Phi \colon \mathcal{F}_{x_1} \to \mathcal{F}_{x_2}$ is homotopic to the map

$$\begin{aligned}\mathcal{F}_{x_1} &\to \mathcal{F}_{x_2},\\ \gamma &\mapsto \gamma|_{[0,\frac{1}{2}]} \cdot \tau \cdot \overline{\tau} \cdot \gamma\end{aligned}$$

through the homotopies

$$\begin{aligned}[\gamma \mapsto \Phi(\gamma)] &\simeq \left[\gamma \mapsto \Phi(\gamma|_{[0,\frac{1}{2}]} \cdot c_{x_1} \cdot c_{x_1} \cdot \gamma|_{[\frac{1}{2},1]}) = \left(t \mapsto \begin{cases} \phi^{\chi(2t)}(\gamma(2t)) & \text{if } 0 \le t \le \frac{1}{4}; \\ \phi^1(x_1) & \text{if } \frac{1}{4} \le t \le \frac{3}{4}; \\ \phi^{\chi(2t-1)}(\gamma(2t-1)) & \text{if } \frac{3}{4} \le t \le 1; \end{cases}\right)\right] \\ &\simeq \left[\gamma \mapsto \left(t \mapsto \begin{cases} \phi^0(\gamma(2t)) & \text{if } 0 \le t \le \frac{1}{4}; \\ \phi^{4t-1}(x_1) & \text{if } \frac{1}{4} \le t \le \frac{1}{2}; \\ \phi^{3-4t}(x_1) & \text{if } \frac{1}{2} \le t \le \frac{3}{4}; \\ \phi^0(\gamma(2t-1)) & \text{if } \frac{3}{4} \le t \le 1; \end{cases}\right)\right] \\ &= [\gamma \mapsto \gamma|_{[0,\frac{1}{2}]} \cdot \tau \cdot \overline{\tau} \cdot \gamma].\end{aligned}$$

Therefore, $c_{x_2} \circ \Phi$ is homotopic to $(r_\tau \times l_{\overline{\tau}}) \circ c_{x_1}$ and, on homology, we have

$$(r_\tau \times l_{\overline{\tau}}) \circ \vee^{x_1}_{\mathrm{triv}} = \vee_{\sigma_2, x_2} \circ \Phi_*.$$

□

3.2. The avoiding-stick pre-coproduct. In this subsection, we define and study an analogous construction to the avoiding-stick coproduct where we cut at different points. We relate the maps that arise from cutting at different points to each other. Using this, we can prove a π_1-invariance result in Corollary 3.11 and prove the Sullivan relation in Proposition 3.13 between the avoiding-stick coproduct and the Pontryagin product.

Analogous to Lemma 2.9, one proves the following lemma:

LEMMA 3.4. *Let σ be a stick avoiding x_0, x. The inclusion*

$$C_*(r((\varepsilon, 1-\varepsilon) \times \Omega^\sigma M), \mathcal{F}_x^c) \to C_*(r(I \times \Omega^\sigma M), \mathcal{F}_x^c \cup \mathcal{H})$$

is a homotopy equivalence by excision.

We can thus fix a homotopy inverse

$$\rho_\sigma^x \colon C_*(r(I \times \Omega^\sigma M), \mathcal{F}_x^c \cup \mathcal{H}) \to C_*(r((\varepsilon, 1-\varepsilon) \times \Omega^\sigma M), \mathcal{F}_x^c).$$

DEFINITION 3.5. Let σ be a stick avoiding x_0, x. The *avoiding-stick pre-coproduct at x* is defined as the composition

$$\begin{aligned} \vee_\sigma^x \colon C_*(\Omega M) &\overset{(\pi_\sigma)_*}{\to} C_*(\Omega^\sigma M) \overset{I\times}{\to} C_{1+*}(I \times \Omega^\sigma M, \partial I \times \Omega^\sigma M) \\ &\overset{r_*}{\to} C_{1+*}(r(I \times \Omega^\sigma M), \mathcal{F}_x^c \cup \mathcal{H}) \\ &\overset{\rho_\sigma^x}{\to} C_{1+*}(r((\varepsilon, 1-\varepsilon) \times \Omega^\sigma M), \mathcal{F}_x^c) \\ &\overset{\vee_{\mathrm{triv}}^x}{\to} C_{1-n+*}(\mathcal{P}_{x_0 \to x} \times \mathcal{P}_{x \to x_0}). \end{aligned}$$

REMARK 3.6. The avoiding-stick pre-coproduct at x_0 is the same as the avoiding-stick coproduct as in Definition 2.10.

PROPOSITION 3.7. *Let $x_1, x_2 \in M$ and σ_1 be a stick avoiding x_0, x_1. Let $\phi^s \colon M \to M$ be a 1-parameter family of homeomorphisms (see Figure 22) such that*

(i) $\phi^0 = \mathrm{id}$,
(ii) $\phi^1(x_1) = x_2$,
(iii) $\sigma_2(t) := \phi^{\chi(t)} \circ \sigma_1(t) \neq x_2$ *is a stick avoiding* x_0, x_2.

Let τ be the path $\tau(t) = \phi^t(x_1)$, then in homology the following formula holds

$$(r_\tau \times l_{\overline{\tau}}) \circ \vee_{\sigma_1}^{x_1} = \vee_{\sigma_2}^{x_2}$$

as maps from $H_*(\Omega M) \to H_*(\mathcal{P}_{x_0 \to x_2} \times \mathcal{P}_{x_2 \to x_0})$.

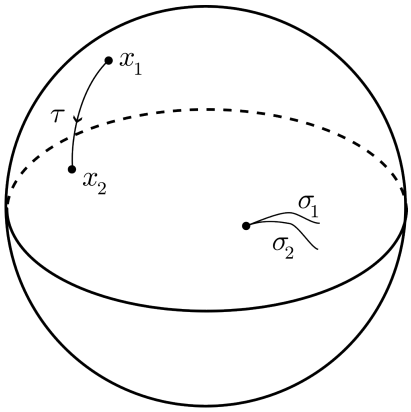

FIGURE 22. The set up of Proposition 3.7 for $M = S^2$ where the 1-parameter family is given by a flow from the north pole to the south pole. The path σ_2 is then given by a "bent version" of σ_2.

PROOF. We again denote the map

$$\Phi \colon \Omega M \to \Omega M,$$
$$\gamma \mapsto (t \mapsto \phi^{\chi(t)} \circ \gamma(t)).$$

By Lemma 3.3, we have $\vee^{x_2} \circ \Phi_* = (r_\tau \times l_{\overline{\tau}})_* \circ \vee^{x_1}_{\mathrm{triv}}$. We therefore need to show that Φ_* intertwines the other maps in Definition 3.5.

We note that $\mathrm{id} \times \Phi$ sends $(I \times \Omega^{\sigma_1} M, \partial I \times \Omega^{\sigma_1} M)$ to $(I \times \Omega^{\sigma_2} M, \partial I \times \Omega^{\sigma_2} M)$. In fact, we have $\Phi \circ \pi_{\sigma_1} = \pi_{\sigma_2} \circ \Phi$.

It thus remains to show that on homology it holds:

$$\Phi_* \circ \rho_{\sigma_1} \circ r_* = \rho_{\sigma_2} \circ r_* \circ (\mathrm{id} \times \Phi)_* \tag{22}$$

as maps $H_*(I \times \Omega^{\sigma_1} M, \partial I \times \Omega^{\sigma_1}) \to H_*(\Omega^{\sigma_2} M, e^{-1}(x_2))$.

In the proof of Lemma 2.9, we showed that r is a homeomorphism onto its image. We define ${}^r\Phi := r \circ (\mathrm{id} \times \Phi) \circ r^{-1}$. By construction, we thus have $r \circ (\mathrm{id} \times \Phi) = {}^r\Phi \circ r$. It remains to show that $\Phi_* \circ \rho_{x_1} = \rho_{x_2} \circ {}^r\Phi_*$ on homology. The maps ρ_{σ_i} were constructed as inverses to the inclusions

$$C_*(r((\varepsilon, 1-\varepsilon) \times \Omega^{\sigma_i} M), \mathcal{F}^c_{x_i}) \to C_*(r(I \times \Omega^{\sigma_i} M), \mathcal{F}^c_{x_i} \cup \mathcal{H})$$

These inclusions are intertwined by ${}^r\Phi_*$ and thus so are ρ_{σ_i} on homology.

Finally, the square

$$\begin{array}{ccc} C_*(r((\varepsilon, 1-\varepsilon) \times \Omega^{\sigma_1} M), \mathcal{F}^c_{x_1}) & \xrightarrow{({}^r\Phi)_*} & C_*(r((\varepsilon, 1-\varepsilon) \times \Omega^{\sigma_2} M), \mathcal{F}^c_{x_2}) \\ \downarrow & & \downarrow \\ C_*(\Omega M, \mathcal{F}^c_{x_1}) & \xrightarrow{\Phi_*} & C_*(\Omega M, \mathcal{F}^c_{x_2}) \end{array}$$

commutes in homology because ${}^r\Phi(\gamma)$ and $\Phi(\gamma)$ are the same loop up to reparametrization.

This shows (22) and thus concludes the proof. $\square$

This result proves to be very powerful as we can often find a 1-parameter family ϕ^s satisfying the assumptions and achieving some other conditions.

Corollary 3.8. *Let σ_1, σ_2 be two paths avoiding x_0, x_1 in a manifold of dimension $n > 1$. The two pre-coproducts $\vee_{\sigma_1,x_1}$ and $\vee_{\sigma_2,x_1}$ are the same in homology.*

In particular for $x_0 = x_1$, we have that $\vee_{\sigma_1} = \vee_{\sigma_2}$ in homology, that is that the avoiding-stick coproduct is independent of the choice of the avoiding-stick in homology.

Proof. Our strategy to prove this corollary is to find a 1-parameter family ϕ^t that satisfies the assumptions for Proposition 3.7 such that $\tau(t) = \phi^t(x_1)$ is a contractible loop based at x_1.

By assumption, we can write, for $i = 1, 2$, $\sigma_i(t) = \varphi_i(t, 0, \dots, 0)$ for charts $\varphi_i \colon \mathbb{R}^n \cong U_i \subseteq M$.

If $x_1 \neq x_0$, we have that $x_1 \notin U_1, U_2$. Moreover, by applying Proposition 3.7 to σ_2 and a 1-parameter family that shrinks U_2, we can assume that $U_2 \subseteq U_1$.

We consider $\sigma_1(s\varepsilon)$ as an isotopy of topological embeddings (historically called locally flat maps) of the point x_0 into M. By the isotopy extension theorem for topological manifolds [EK71, Corollary 1.4], we can find an isotopy $H_1^s \colon M \to M$ such that $H_1^s(x_0) = \sigma_1(s\varepsilon)$ that is supported in U_1.

Because U_1 is contractible, we can find a path ρ between $y := H_1^{-1}(x_0)$ and x_0. We then can extend the isotopy

$$\begin{aligned} h_2^s \colon \{x_0, y\} &\to M, \\ x_0 &\mapsto \sigma_2(\varepsilon s), \\ y &\mapsto \rho(s) \end{aligned}$$

to a isotopy $H_2^s \colon M \to M$.

The 1-parameter family $\phi^s := H_2^s \circ H_1^{-s}$ satisfies

(i) $\phi^0 = H_2^0 \circ H_1^0 = \mathrm{id}$;

(ii) $\phi^1(x_1) = H_2^1 \circ H_1^{-1}(x_1) = x_1$, if $x_1 \neq x_0$, or

$$\phi^1(x_1) = \phi^1(x_0) = H_2^1(H_1^{-1}(x_0)) = H_2^1(y) = \rho(1) = x_0 = x_1,$$

if $x_1 = x_0$;

(iii) for $0 \leq t \leq \varepsilon$, we set $s := t/\varepsilon \in I$ and compute

$$\phi^{\chi(t)}(\sigma_1(t)) = \phi^s(\sigma_1(s\varepsilon)) = H_2^s(H_1^{-s}(\sigma_1(s\varepsilon)) = H_2^s(x_0) = \sigma_2(s\varepsilon) = \sigma_2(t).$$

We therefore can apply Proposition 3.7 and find in homology that $(r_\tau \times l_{\overline{\tau}}) \circ \vee_{\sigma_1}^{x_1} = \vee_{\sigma_2}^{x_1}$ for $\tau(t) = \phi^t(x_1)$.

If $x_1 \neq x_0$, we have $\phi^t(x_1) = H_2^t(H_1^{-t}(x_1)) \equiv x_1$. If $x_1 = x_0$, then $\phi^t(x_1)$ is a loop in U_1 and thus contractible. In both cases, we deduce $(r_\tau \times l_{\overline{\tau}}) = \mathrm{id}$ in homology which shows the Corollary. $\square$

This in particular answers Question 2.13 whether or not the avoiding-stick coproduct depends on the stick σ. We are therefore justified to use the following notation:

Notation 3.9. For $n > 1$, we denote the avoiding-stick coproduct

$$\vee_\sigma \colon H_*(\Omega M) \to H_{1-n+*}(\Omega M \times \Omega M)$$

by $\vee$ or $\vee_{\Omega M}$. Corollary 3.8 for $x_0 = x_1$ shows that this is well-defined.

Similarly, we denote for $x \in M$ and σ a stick avoiding x_0, x, $\vee^x := \vee_\sigma^x$.

Corollary 3.10. *Assume that the dimension is $n > 1$. For any path τ from x_1 to x_2, it holds in homology that*

$$(r_\tau \times l_{\overline{\tau}}) \circ \vee^{x_1} = \vee^{x_2}.$$

Proof. We again want to apply Proposition 3.7.

We note that for τ_1 a path from x_1 and x_2 and τ_2 a path from x_2 to x_3, the statement for τ_1 and τ_2 implies it for $\tau_1 \cdot \tau_2$. Therefore, we can decompose τ and thus assume that τ is contained in a contractible neighbourhood. Moreover, we note that the statement only depends on the homotopy type (relative to ∂I) of τ. We can thus assume that τ is the straight line from $(0,0,\dots,0)$ to $(0,1,0,\dots,0)$ in some $\varphi\colon \mathbb{R}^n \cong U \subseteq M$.

If $x_0 \neq x_1, x_2$, we can assume that $x_0 \notin U$, we take σ_1 a stick avoiding x_0, x_1, x_2 in $M \setminus U$. If $x_0 = x_1$ or $x_0 = x_2$, we can first assume that $x_0 = x_1 = \varphi(0)$ and we fix $\sigma_1(t) := \varphi(t, 0, \dots, 0)$ (see Figure 23).

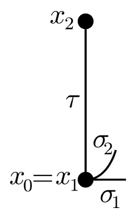

FIGURE 23. In the case $x_0 = x_1$, the stick σ_1 is going to the right and the path τ and the 1-parameter family is moving upwards in a local chart. Here we use the assumption that $n > 1$.

We then fix a 1-parameter family supported in U with $\phi^0 = \mathrm{id}$ and that, in the ball $\varphi(B_2(0))$, is given by

$$\phi^s(\varphi(x_1, \dots, x_n)) = \varphi(x_1, x_2 + s, \dots, x_n).$$

We then have that $\sigma_2(t) := \phi^{\chi(t)}(\sigma_1(t))$ is a stick avoiding x_0 and x_2. Therefore we can apply Proposition 3.7 and find

$$(r_\tau \times l_{\overline{\tau}}) \circ \vee^{x_1}_{\sigma_1} = \vee^{x_2}_{\sigma_2}.$$

After dropping the dependence on σ_1 and σ_2, this gives the corollary. $\square$

COROLLARY 3.11. *Assume that the dimension is $n > 1$. The avoiding-stick coproduct $\vee$ factors through the fixed-points $H_*(\Omega M \times \Omega M)^{\pi_1(M,x_0)}$ where $\pi_1(M, x_0)$ acts on $\Omega M \times \Omega M$ via*

$$[\tau] \cdot (\gamma, \gamma') := (\gamma \cdot \tau, \tau^{-1} \cdot \gamma').$$

PROOF. This is the previous Corollary for the case $x_0 = x_1 = x_2$. $\square$

COROLLARY 3.12. *Let M be a manifold with infinite fundamental group of dimension $n > 1$. The avoiding-stick coproduct and the Goresky-Hingston coproduct and the avoiding stick coproduct are zero on $H_*(\Omega M)$.*

PROOF. By Proposition 2.12, it suffices to show that the avoiding-stick coproduct is zero.

For all manifolds, it holds

$$H_*(\Omega M) = \bigoplus_{\gamma \in \pi_1(M,x_0)} H_*(\Omega_\gamma M),$$

$$H_*(\Omega M \times \Omega M) = \bigoplus_{\gamma,\delta \in \pi_1(M,x_0)} H_*(\Omega_\gamma M \times \Omega_\delta M)$$

where $\Omega_\gamma M$ is the connected component of ΩM of all loops that represent $\gamma \in \pi_1(M, x_0)$. The actions $r_\tau \times l_{\tau^{-1}}$ permutes the summands where $\gamma\delta$ are equal. Because $\pi_1(M, x_0)$ is infinite, there are infinite pairs in each orbit. Therefore the action has no fixed points except 0. Corollary 3.11 now shows that the coproduct is zero. $\square$

PROPOSITION 3.13. *Let σ be a stick avoiding x_0. The Pontryagin product $\wedge$ and the avoiding-stick coproduct $\vee_\sigma$ satisfy the Sullivan relation:*

$$\vee_\sigma \circ \wedge = (\mathrm{id} \times \wedge) \circ (\vee_\sigma \times \mathrm{id}) + (\wedge \times \mathrm{id}) \circ (\mathrm{id} \times \vee_\sigma).$$

PROOF. The strategy is to compute $\vee^x_{\widetilde{\sigma}} \circ \wedge(\alpha \times \beta)$ for a point $x \neq x_0$ and then use Proposition 3.7.

We fix a 1-parameter family of homeomorphisms ϕ^s such that

$$\widetilde{\sigma} \colon [0, \varepsilon] \to M, \qquad s \mapsto \phi^s \circ \sigma(s)$$

is a stick avoiding x_0 and $x := (\phi^1)^{-1}(x_0) \neq x_0$. Moreover, we fix a contractible neighbourhood U_x such that $\widetilde{\sigma}(t) \notin U_x$ for all $t \in [0, \varepsilon]$. We set $\tau(t) := \phi^t(x)$. By Proposition 3.7, we have

$$(r_\tau \times l_{\overline{\tau}}) \circ \vee^x_{\widetilde{\sigma}} = \vee_\sigma. \tag{23}$$

We define $\widetilde{\sigma}'(t) := \widetilde{\sigma}(2t)$ for $t \in [0, \frac{\varepsilon}{2}]$. We then have

$$\wedge \circ (\pi_{\widetilde{\sigma}} \times \pi_{\widetilde{\sigma}}) = \pi_{\widetilde{\sigma}'} \circ \wedge$$

as maps $\Omega M \times \Omega M \to \Omega^{\widetilde{\sigma}'} M$. Moreover, the map

$$r' := r \circ (\mathrm{id} \times \wedge) \colon I \times \Omega^{\widetilde{\sigma}} M \times \Omega^{\widetilde{\sigma}} M \to \Omega M$$

sends $([0, \frac{\varepsilon}{2}] \cup [1 - \frac{\varepsilon}{2}, 1]) \times \Omega^{\widetilde{\sigma}} M \times \Omega^{\widetilde{\sigma}} M$ to $\mathcal{F}^c_x$ because $\widetilde{\sigma}'$ avoids x. But because $\widetilde{\sigma}$ avoids x, r' sends also $([\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}]) \times \Omega^{\widetilde{\sigma}} M \times \Omega^{\widetilde{\sigma}} M$ to $\mathcal{F}^c_x$.

We therefore find that the inclusion

$$C_*\left(r'\left(((\tfrac{\varepsilon}{2}, \tfrac{1-\varepsilon}{2}) \cup (\tfrac{1+\varepsilon}{2}, 1 - \tfrac{\varepsilon}{2})) \times \Omega^{\widetilde{\sigma}} M \times \Omega^{\widetilde{\sigma}} M\right), \mathcal{F}^c_x\right) \to C_*\left(r'\left(I \times \Omega^{\widetilde{\sigma}} M \times \Omega^{\widetilde{\sigma}} M\right), \mathcal{F}^c_x\right)$$

is a homotopy equivalence by excision. We fix a homotopy inverse ρ''.

Because r' is the composition of two injective maps, it is injective and thus

$$\begin{aligned} &C_*\left(r'\left(((\tfrac{\varepsilon}{2}, \tfrac{1-\varepsilon}{2}) \cup (\tfrac{1+\varepsilon}{2}, 1 - \tfrac{\varepsilon}{2})) \times \Omega^{\widetilde{\sigma}} M \times \Omega^{\widetilde{\sigma}} M\right), \mathcal{F}_x\right) \\ &= C_*\left(r'\left((\tfrac{\varepsilon}{2}, \tfrac{1-\varepsilon}{2}) \times \Omega^{\widetilde{\sigma}} M \times \Omega^{\widetilde{\sigma}} M\right), \mathcal{F}_x\right) \\ &\quad \oplus C_*\left(r'\left((\tfrac{1+\varepsilon}{2}, 1 - \tfrac{\varepsilon}{2}) \times \Omega^{\widetilde{\sigma}} M \times \Omega^{\widetilde{\sigma}} M\right), \mathcal{F}_x\right). \end{aligned}$$

Our computations so far show that $\vee^x_{\widetilde{\sigma}'} \circ \wedge$ is computed via the composition

$$\begin{aligned} C_*(\Omega M \times \Omega M) &\overset{(\pi_{\widetilde{\sigma}} \times \pi_{\widetilde{\sigma}})_*}{\to} C_*\left(\Omega^\sigma M \times \Omega^\sigma M\right) \overset{I \times}{\to} C_{1+*}(I \times \Omega^\sigma M \times \Omega^\sigma M, \partial I \times \Omega^\sigma M \times \Omega^\sigma M) \\ &\overset{r'_*}{\to} C_{1+*}\left(r'\left(I \times \Omega^{\widetilde{\sigma}} M \times \Omega^{\widetilde{\sigma}} M\right), \mathcal{F}^c_x\right) \\ &\overset{\rho''}{\to} C_{1+*}\left(r'\left((\tfrac{\varepsilon}{2}, \tfrac{1-\varepsilon}{2}) \times \Omega^{\widetilde{\sigma}} M \times \Omega^{\widetilde{\sigma}} M\right), \mathcal{F}^c_x\right) \\ &\qquad \oplus C_{1+*}\left(r'\left((\tfrac{1+\varepsilon}{2}, 1 - \tfrac{\varepsilon}{2}) \times \Omega^{\widetilde{\sigma}} M \times \Omega^{\widetilde{\sigma}} M\right), \mathcal{F}^c_x\right) \\ &\overset{\vee^x_{\mathrm{triv}}}{\to} C_{1-n+*}(\mathcal{P}_{x_0 \to x} \times \mathcal{P}_{x \to x_0}). \end{aligned}$$

The injectivity of r' also shows that for $\gamma_1, \gamma_2 \in H_*(\Omega^\sigma M)$, we can represent $I \times \gamma_1 \times \gamma_2 \in H_{1+*}(I \times \Omega^\sigma M \times \Omega^\sigma M, \partial I \times \Omega^\sigma M \times \Omega^\sigma M)$ by

$$[\tfrac{\varepsilon}{2}, \tfrac{1-\varepsilon}{2}] \times \gamma_1 \times \gamma_2 + [\tfrac{1+\varepsilon}{2}, 1 - \tfrac{\varepsilon}{2}] \times \gamma_1 \times \gamma_2 + \alpha$$

for some

$$\alpha \in C_*((r')^{-1}(\mathcal{F}_x^c)).$$

Because definition of $\vee^x_{\mathrm{triv}}$ is zero on $C_*(\mathcal{F}_x^c)$, we find $\vee^x_{\mathrm{triv}}(r'(\alpha)) = 0$.

We also have that

$$\vee^x_{\mathrm{triv}}(r'_*([\tfrac{\varepsilon}{2}, \tfrac{1-\varepsilon}{2}] \times \gamma_1 \times \gamma_2)) = (\mathrm{id} \times \wedge)(\vee^x_{\widetilde{\sigma}}(\gamma_1) \times \gamma_2).$$

Indeed after reparametrisation, we find

$$\vee^x_{\mathrm{triv}}(r'_*([\tfrac{\varepsilon}{2}, \tfrac{1-\varepsilon}{2}] \times \gamma_1 \times \gamma_2)) = (\mathrm{id} \times \wedge)(\vee^x_{\mathrm{triv}}(r_*([\varepsilon, 1-\varepsilon] \times \gamma_1) \times \gamma_2))$$

and $\vee^x_{\mathrm{triv}}(r_*([\varepsilon, 1-\varepsilon] \times \gamma_1))$ computes $\vee^x_{\widetilde{\sigma}}(\gamma_1)$.

Similarly, we find

$$\vee^x_{\mathrm{triv}}(r'_*([\tfrac{1+\varepsilon}{2}, 1 - \tfrac{\varepsilon}{2}] \times \gamma_1 \times \gamma_2)) = (-1)^{(n-1)|\gamma_1|}(\wedge \times \mathrm{id})(\gamma_1 \times \vee^x_{\widetilde{\sigma}}(\gamma_2))$$

where the sign comes from the Koszul sign of moving the degree $-n$ map $\vee^x_{\mathrm{triv}}$ and the interval past γ_1.

We thus have shown:

$$\vee^x_{\widetilde{\sigma}'} \circ \wedge(\gamma_1 \times \gamma_2) = (\mathrm{id} \times \wedge)(\vee^x_{\widetilde{\sigma}}(\gamma_1) \times \gamma_2) + (-1)^{(n-1)|\gamma_1|}(\wedge \times \mathrm{id})(\gamma_1 \times \vee^x_{\widetilde{\sigma}}(\gamma_2)).$$

After postcomposing with $r_\tau \times l_{\overline{\tau}}$, we find by (23)

$$\vee_\sigma \circ \wedge(\gamma_1 \times \gamma_2) = (\mathrm{id} \times \wedge)(\vee_\sigma(\gamma_1) \times \gamma_2) + (-1)^{(n-1)|\gamma_1|}(\wedge \times \mathrm{id})(\gamma_1 \times \vee_\sigma(\gamma_2)).$$

This is the Sullivan relation evaluated on $\gamma_1 \times \gamma_2$. $\square$

3.3. Higher dimensional spheres. In this subsection, we consider the example of spheres of dimension $n > 1$.

Example 3.14 (Spheres). We recall that

$$H_*(\Omega S^n) \cong \mathbb{Z}[u]$$

with u in degree $n-1$. The right hand side is a free *associative* algebras. In particular, the Pontryagin product is not (graded) commutative for n even. The class u corresponds to the class au in [CJY04, Theorem 2 (2)] and AU in [CHO23, Subsection 8.2] for n odd.

The class u can be characterised as follows: we fix a vector $v_0 \in UT_{x_0}S^n$. We have

$$D^{n-1} \cong K^{n-1} := \{v \in UT_{x_0}\mathbb{R}^{n+1} \cong S^n \mid \langle v_0, v\rangle = 0, \langle v, x_0\rangle \leq 0\}.$$

Here we interpret $UT_{x_0}\mathbb{R}^{n+1} \cong S^n$ via the canonical embedding $S^n \to \mathbb{R}^{n+1}$. This is the space of all vectors pointing into the sphere which are perpendicular to v_0.

A vector in $v \in K^{n-1}$ and v_0 define a plane P_v containing x_0. There is (up to parametrization) a unique loop $\gamma_v \in \Omega S^n$ in $P_v \cap S^n$ with start direction v_0. For $v \in \partial K^{n-1}$, we have that P_v is tangential to S^n and thus $P_v \cap S^n = x_0$. We then define γ_v as the constant loop at x_0. See Figure 24.

The class u is represented by[1]:

$$\begin{aligned} S^{n-1} \cong K^{n-1}/\partial K^{n-1} &\to \Omega S^n, \\ v &\mapsto \gamma_v. \end{aligned}$$

[1]The orientation on S^n induces an orientation on $K^{n-1}/\partial K^{n-1}$: we have natural homeomorphisms $\langle v_0\rangle^\perp \cap S^n \cong S^{n-1} \cong K^{n-1}/\partial K^{n-1}$

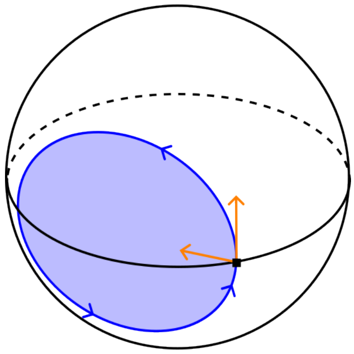

FIGURE 24. The vector v_0 pointing upward and a second perpendicular vector v define a plane P_v in bright blue. The plane P_v intersects S^n in one circle γ_v in dark blue.

For our computations, we fix $x_0 = (1, 0, \dots, 0), x_1 = (-1, \dots, 0, 0) \in S^n$. We choose the stick $\sigma(t) = (t, 1 - t^2, 0, \dots, 0)$ avoiding x_0 and x_1. Moreover, we fix $v_0 = (0, \dots, 0, 1) \in UT_{x_0}S^n$. Then K^{n-1} is given by

$$\{(v_0, \dots, v_{n-1}, 0) \in S^n \mid v_0 \leq 0\}.$$

For a $v = (v_0, \dots, v_{n-1}, 0) \in K^{n-1}$, we have

$$P_v := \{(1, 0, \dots, \lambda) + \mu(v_0, \dots, v_{n-1}, 0) \mid \lambda, \mu \in \mathbb{R}\}.$$

In particular, solving

$$x_1 = (-1, \dots, 0, 0) = (1, 0, \dots, \lambda) + \mu(v_0, \dots, v_{n-1}, 0)$$

we find $\lambda = 0$, $\mu = 2$ and $v = (-1, 0, \dots, 0)$ as the only solution. In other words, the only $(t, v) \in I \times K^{n-1}$ such that $e(r(t, \sigma\overline{\sigma}\gamma_v\sigma\overline{\sigma})) = x_1$ are $t = \frac{1}{2}$ and $v = (-1, 0, \dots, 0)$.

Moreover, this intersection is transversal because varying t and varying v gives the tangent space $T_{x_1}S^n$. Additionally, the local degree of these intersection is 1.

Similarly, the only points $(t, v^{(1)}, \dots, v^{(k)}) \in I \times (K^{n-1})^k$ with $e(r(t, (\sigma\overline{\sigma}(\gamma_{v^{(1)}} \cdots \gamma_{v^{(k)}})\sigma\overline{\sigma})))$ are

$$t_i = \frac{i - 1 + \frac{1}{2}}{k}(1 - 4\varepsilon) + 2\varepsilon, \text{ where } v^{(i)} = (-1, 0, \dots, 0)$$

for $i = 1, \dots, k$.

We thus computed

$$\vee_{x_1}(u^k) = \sum_{i=1}^{k} u^{i-1} \cdot \gamma_{(-1,0,\dots,0)}|_{[0,\frac{1}{2}]} \otimes \gamma_{(-1,0,\dots,0)}|_{[\frac{1}{2},1]} \cdot u^{k-i} \in H_*(\mathcal{P}_{x_0 \to x_1}) \otimes H_*(\mathcal{P}_{x_1 \to x_0}).$$

We fix a path τ from x_1 to x_0. By Corollary 3.10, we find

$$\vee_{x_0}(u^k) = (r_\tau \otimes l_{\tau^{-1}}) \circ \vee_{x_1}(u^k) = \sum_{i=1}^{k} u^{i-1} \cdot \gamma_{(-1,0,\dots,0)}|_{[0,\frac{1}{2}]} \cdot \tau \otimes \tau^{-1} \cdot \gamma_{(-1,0,\dots,0)}|_{[\frac{1}{2},1]} \cdot u^{k-i}.$$

Because all loops in S^n are contractible, so are the loops $\gamma_{(-1,0,\ldots,0)}|_{[0,\frac{1}{2}]}\cdot\tau$ and $\tau^{-1}\cdot\gamma_{(-1,0,\ldots,0)}|_{[\frac{1}{2},1]}$. Therefore we have

$$\vee_{x_0}(u^k)=\sum_{i=1}^{k}u^{i-1}\otimes u^{k-i}.$$

4. Maps and the coproduct

In this section, we consider how our different coproducts interact with maps of manifolds. In Subsection 4.1, we apply our results to the case of the universal covering map.

We start with the trivial coproduct and give a formula for it on chains. This lets us produce more interesting formulas for the avoiding-stick coproduct in Proposition 4.2 and Corollaries 4.3, 4.4, 4.5.

LEMMA 4.1. *Let $f\colon (M,x_0)\to(N,y_0)$ be a map of closed, oriented, pointed manifolds. Assume that y_0 is a regular value with $f^{-1}(y_0)=\{x_0,\ldots,x_k\}$. There exist chain level coproducts $\vee_{\mathrm{triv},\Omega N}\colon C_*(\Omega N)\to C_*(\Omega N\times\Omega N)$ and $\vee^{x_i}_{\mathrm{triv}}\colon C_*(\Omega M)\to C_{-n+*}(\mathcal{P}_{x_0\to x_i}\times\mathcal{P}_{x_i\to x_0})$ for $i=0,\ldots,k$ such that the following formula holds*

$$\vee_{\mathrm{triv},\Omega N}\circ f_*=\sum_{i=0}^{k}\deg(f)_{x_i}(f\times f)_*\circ\vee^{x_i}_{\mathrm{triv}}$$

as maps $C_*(\Omega M)\to C_{-n+*}(\Omega N\times\Omega N)$.

PROOF. Because y_0 is a regular value, we can find a contractible neighbourhood $y_0\in V\subseteq N$

$$f^{-1}(V)=\bigsqcup_{i=0}^{k}U_i$$

where

$$f|_{U_i}\colon U_i\to V$$

is a homeomorphism and $x_i\in U_i$ for $i=0,\ldots k$.

We fix a trivial coproduct $\vee_{\mathrm{triv},\Omega N}\colon C_*(\Omega N)\to C_*(\Omega N\times\Omega N)$. This in particular fixes a homotopy retract $R_V\colon I\times V\to V$ fixing y_0, a local orientation $\tau_{y_0}\in C^n(V,\{y_0\}^c)$ and a homotopy equivalence

$$\rho^y\colon C_*(\Omega N,e^{-1}(y_0)^c)\to C_*(e^{-1}(V),e^{-1}(y_0)^c).$$

We have that $R_{U_i}:=(\mathrm{id}\times f|_{U_i}^{-1})\circ R_V\circ(\mathrm{id}\times f)$ defines a homotopy retraction of U_i.

We note that

$$f^{-1}(e^{-1}(V),e^{-1}(y_0)^c)=(e^{-1}(f^{-1}(V)),e^{-1}(f^{-1}(y_0))^c)=\bigsqcup_{i=0}^{k}(e^{-1}(U_i),e^{-1}(x_i)^c).$$

We have

$$f^*\tau_{y_0}=\sum_{i=0}^{k}\deg(f)_{x_i}\tau_{x_i}$$

for some local orientations $\tau_{x_i} \in C^n(U_i, \{x_i\}^c)$. This implies that

$$f^* e^* \tau_{y_0} = \sum_{i=0}^{k} \deg(f)_{x_i} e^* \tau_{x_i}. \tag{24}$$

Moreover, we can find homotopy equivalences $\rho^{x_i} \colon C_*(\Omega M, e^{-1}(x_i)^c) \to C_*(e^{-1}(U_i), e^{-1}(x_i)^c)$ such that $f_* \circ \rho^{x_i} = \rho^{y_0} \circ f_*$.

Therefore the following diagram commutes:

$$\begin{tikzcd}
C_*(\Omega M) \arrow[r, "f_*"] \arrow[d] & C_*(\Omega N) \arrow[d] \\
\bigoplus_{i=0}^k C_*(\Omega M, e^{-1}(x_i)^c) \arrow[r, "f_*"] \arrow[d, "\sum \rho^{x_i}"] & C_*(\Omega N, e^{-1}(y_0)^c) \arrow[d, "\rho^{y_0}"] \\
\bigoplus_{i=0}^k C_*(e^{-1}(U_i), e^{-1}(x_i)^c) \arrow[r, "f_*"] \arrow[d, "\sum \deg(f)_{x_i} e^* \tau_{x_i} \cap"] & C_*(e^{-1}(V), e^{-1}(y_0)^c) \arrow[d, "e^* \tau_{y_0} \cap"] \\
\bigoplus_{i=0}^k C_{-n+*}(e^{-1}(U_i)) \arrow[r, "f_*"] \arrow[d, "R_*"] & C_{-n+*}(e^{-1}(V)) \arrow[d, "R_*"] \\
\bigoplus_{i=0}^k C_{-n+*}(\mathcal{F}_{x_i} M) \arrow[r, "f_*"] \arrow[d, "c_*"] & C_{-n+*}(\mathcal{F} N) \arrow[d, "c_*"] \\
\bigoplus_{i=0}^k C_{-n+*}(\mathcal{P}_{x_0 \to x_i} \times \mathcal{P}_{x_i \to x_0}) \arrow[r, "(f \times f)_*"] & C_{-n+*}(\Omega N \times \Omega N).
\end{tikzcd}$$

The middle square commutes by naturality of the cap product and (24).

This square proves the statement because the composition along the top right computes $\vee_{\mathrm{triv}, \Omega N} \circ f_*$ while composition along the bottom left computes $\sum_{i=0}^k \deg(f)_{x_i} (f \times f)_* \circ \vee_{\mathrm{triv}}^{x_i}$. $\square$

We next prove a formula for the avoiding-stick coproduct. This is a generalization of [Hin10, Lemma 3].

Proposition 4.2. *Let $f \colon (M, x_0) \to (N, y_0)$ be a map of closed, oriented, pointed manifolds of dimension $n > 1$. Assume that y_0 is a regular value with $f^{-1}(y_0) = \{x_0, \dots, x_k\}$. The following formula holds*

$$\vee_{\Omega N} \circ f_* = \sum_{i=0}^{k} \deg(f)_{x_i} (f \times f)_* \circ \vee^{x_i}$$

as maps $H_*(\Omega M) \to H_{1-n+*}(\Omega N \times \Omega N)$.

Proof. We can fix σ a stick avoiding $x_0, \dots, x_k$ such that $f \circ \sigma$ is a stick avoiding y_0.

We have that $f \circ \pi_\sigma = \pi_{f \circ \sigma} \circ f$ as maps $\Omega M \to \Omega^{f \circ \sigma} N$. Moreover, f commutes with the map $I \times$ and intertwines the maps

$$r \colon (I \times \Omega^\sigma M, \partial I \times \Omega^\sigma M) \to (r(I \times \Omega^\sigma M), e^{-1}(\{x_0, \dots, x_k\})^c \cup \mathcal{H})$$

and

$$r \colon (I \times \Omega^{f \circ \sigma} N, \partial I \times \Omega^{f \circ \sigma} N) \to (r(I \times \Omega^{f \circ \sigma} N), e^{-1}(y_0)^c \cup \mathcal{H}).$$

Similarly, f_* intertwines the map

$$C_*(r((\varepsilon, 1-\varepsilon)\times \Omega^\sigma M), e^{-1}(\{x_0,\dots,x_k\})^c) \to C_*(r(I\times \Omega^\sigma M), e^{-1}(\{x_0,\dots,x_k\})^c \cup \mathcal{H})$$

and

$$C_*(r((\varepsilon, 1-\varepsilon)\times \Omega^{f\circ\sigma} N), e^{-1}(y_0)^c) \to C_*(r(I\times \Omega^{f\circ\sigma} M), e^{-1}(y_0)^c \cup \mathcal{H}).$$

Therefore their inverses in homology $\sum_i \rho_\sigma^{x_i}$ and $\rho_{f\circ\sigma}$ are intertwined by f_* in homology.

We thus have shown that

$$\begin{aligned}\vee_{f\circ\sigma}\circ f_* &= \vee_{\mathrm{triv}}\circ \rho_{f\circ\sigma}\circ r_*\circ (I\times)\circ (\pi_{f\circ\sigma})_*\circ f_*\\ &= \vee_{\mathrm{triv}}\circ f_*\circ \left(\sum_{i=0}^k \rho_\sigma^{x_i}\right)\circ r_*\circ (I\times)\circ(\pi_\sigma)_*.\end{aligned}$$

Thus by Lemma 4.1, we find

$$\vee_{f\circ\sigma}\circ f_* = \left(\sum_{i=0}^k \deg(f)_{x_i}(f\times f)_*\circ \vee_{\mathrm{triv}}^{x_i}\right)\circ\left(\sum_{i=0}^k \rho_\sigma^{x_i}\right)\circ r_*\circ (I\times)\circ(\pi_\sigma)_*.$$

Finally, we observe that, if $i\neq j$, $\vee^{x_i}_{\mathrm{triv}}$ is zero on $C_*(r((\varepsilon,1-\varepsilon)\times\Omega^\sigma M), e^{-1}(x_j)^c)$ and thus $\vee^{x_i}_{\mathrm{triv}}\circ\rho_\sigma^{x_j} = 0$. This lets us compute:

$$\begin{aligned}\vee_{f\circ\sigma}\circ f_* &= \left(\sum_{i=0}^k \deg(f)_{x_i}(f\times f)_*\circ \vee_{\mathrm{triv}}^{x_i}\circ\rho_\sigma^{x_i}\right)\circ r_*\circ (I\times)\circ(\pi_\sigma)_*\\ &= \sum_{i=0}^k \deg(f)_{x_i}(f\times f)_*\circ\vee_\sigma^{x_i}.\end{aligned}$$

Because we work in homology and dimension $n>1$, we can drop the dependence on the stick which gives the desired formula (see Corollary 3.8). $\square$

Using the fact that we can relate pre-coproducts to the usual coproduct, we can find the following clean formula.

Corollary 4.3. *Let $f\colon (M,x_0)\to(N,y_0)$ be a map of closed, oriented, pointed manifolds of dimension $n>1$. Assume that y_0 is a regular value with $f^{-1}(y_0)=\{x_0,\dots,x_k\}$. Let τ_i be paths from x_i to x_0. For the avoiding-stick coproducts, it holds*

$$\vee_{\Omega N}\circ f_* = \sum_{i=0}^k \deg(f)_{x_i}(r_{[f_*(\tau_i)]}\times l_{[f_*(\tau_i)]^{-1}})\circ(f\times f)_*\circ\vee_{\Omega M}$$

as maps $H_*(\Omega M)\to H_{1-n+*}(\Omega N\times\Omega N)$

Proof. By Proposition 4.2, we have

$$\vee_{\Omega N}\circ f_* = \sum_{i=0}^k \deg(f)_{x_i}(f\times f)_*\circ\vee^{x_i}.$$

By Corollary 3.10, we have

$$\vee^{x_i} = (r_{\tau_i}\times l_{\overline{\tau_i}})\circ\vee^{x_0} = (r_{\tau_i}\times l_{\overline{\tau_i}})\circ\vee_{\Omega M}.$$

We further note that

$$(f_*\times f_*)\circ(r_{\tau_i}\times l_{\overline{\tau_i}}) = (r_{f\circ\tau_i}\times l_{\overline{f\circ\tau_i}})\circ(f_*\times f_*).$$

Here $f \circ \tau_i$ is a loop and it thus represents an element in $\pi_1(N, y_0)$. In this group, it holds $[\overline{f \circ \tau_i}] = [f \circ \tau_i]^{-1}$. Therefore, we have

$$\begin{aligned} \vee_{\Omega N} \circ f_* &= \sum_{i=0}^{k} \deg(f)_{x_i} (f \times f)_* \circ \vee^{x_i} \\ &= \sum_{i=0}^{k} \deg(f)_{x_i} (f_* \times f_*) \circ (r_{\tau_i} \times l_{\overline{\tau_i}}) \circ \vee_{\Omega M} \\ &= \sum_{i=0}^{k} \deg(f)_{x_i} (r_{[f \circ \tau_i]} \times l_{[f \circ \tau_i]^{-1}}) \circ (f_* \times f_*) \circ \vee_{\Omega M}. \end{aligned}$$

□

In particular, for degree 1 maps we get that the Goresky-Hingston coproduct commutes with them:

Corollary 4.4. *Let $f\colon (M, x_0) \to (N, y_0)$ be a degree 1 map of closed, oriented, pointed manifolds which preserves orientation. Then $(f \times f)_* \circ \vee_{\Omega M} = \vee_{\Omega N} \circ f_*$ holds.*

Proof. If $n = 1$, then f is homotopic to a homeomorphism and we are done.

If $n > 1$, we can replace f with a homotopic map g such that $g^{-1}(y_0) = \{x_0\}$ (see [Hop30] for the case $n > 2$ and [Kne28] for $n = 2$). We thus have $\deg(g)_{x_0} = 1$. We then choose τ_0 the constant path at x_0 and apply Corollary 4.3 which concludes the proof. □

Corollary 4.5. *Let (M, x_0) and (N, y_0) be closed, oriented, pointed manifolds of dimension $n > 1$ and N simply connected. Let $f\colon (M, x_0) \to (N, y_0)$ be a map of degree d. It holds $\vee_{\Omega N} \circ f_* = d \cdot (f \times f)_* \circ \vee_{\Omega M}$.*

Proof. Because N is simply connected, we have that $f_*(\tau_i) \in \pi_1(N, y_0)$ is trivial for all choices τ_i. We apply Corollary 4.3 and find

$$\vee_{\Omega N} \circ f_* = \left(\sum_{i=0}^{k} \deg(f)_{x_i} \right) (f \times f)_* \circ \vee_{\Omega N} = d \cdot (f \times f)_* \circ \vee_{\Omega N}.$$

□

4.1. Universal covering. In this subsection, we study the case where the map $p\colon \widetilde{M} \to M$ we consider is the universal covering. In this special case, we can describe the coproduct on M in terms of the coproduct of $\widetilde{M}$ and vice-versa.

We first recall that, as chain complexes, the chains on the based loop space of a space is described by the chains of the based loop space of the universal covering.

Lemma 4.6. *Let (X, x_0) be a pointed, connected CW complex with universal covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$. The chains of the based loop space of X has the following description:*

$$C_*(\Omega_{x_0} X) \simeq \bigoplus_{l \in \pi(X, x_0)} C_*(\Omega_{\widetilde{x}_0} \widetilde{X}).$$

Proof. We know that ΩX has connected components corresponding to $l \in \pi_1(X)$:

$$\Omega_l X := \{\gamma \in \Omega X \mid [\gamma] = l \in \pi_1(X)\}.$$

Because ΩX is group-like, we have that $\Omega_l X \simeq \Omega_1 X$ for $1 \in \pi_1(X)$ the neutral element.

On the other hand, we know that the map $p_*\colon \pi_k(\widetilde{X}) \to \pi_k(X)$ is an isomorphism for $k \geq 2$. Thus for $k \geq 1$, the map that p induces on $\pi_k(\Omega\widetilde{X})$ is given by the isomorphism

$$\pi_k(\Omega\widetilde{X}, \widetilde{x}_0) \cong \pi_{k+1}(\widetilde{X}, \widetilde{x}_0) \overset{\pi_{k+1}(p)}{\cong} \pi_{k+1}(X, x_0) \cong \pi_k(\Omega X, x_0) \cong \pi_k(\Omega_1 X).$$

Therefore p induces a homotopy equivalence $\Omega\widetilde{X} \cong \Omega_1 X$. This concludes the proof. $\square$

REMARK 4.7. The proof of Lemma 4.6 gives a description of $H_*(\Omega X)$ in the above setting: any element $\alpha \in H_*(\Omega X)$ can be uniquely written as

$$\alpha = \sum_{g \in \pi_1 X} g \cdot p_*(\alpha_g)$$

for some $\alpha_g \in H_*(\Omega\widetilde{X})$ only finitely many non-zero.

We now turn our attention back to $f\colon \widetilde{M} \to M$, the universal covering of a closed manifold. We want to understand the Goresky-Hingston coproduct on ΩM in terms of the coproduct on $\Omega\widetilde{M}$. We recall that by Corollary 3.12 the coproduct is trivial if $\pi_1(M, x_0)$ is infinite. We can thus assume that $\pi_1(M, x_0)$ is finite. Therefore the covering space $\widetilde{M}$ is itself a closed manifold and has indeed a Goresky-Hingston coproduct on its based loop space. We get the following description of the coproduct on ΩM:

PROPOSITION 4.8. *Let (M, x_0) be a pointed, closed, oriented manifold with finite fundamental group and universal covering $p\colon (\widetilde{M}, y_0) \to (M, x_0)$. Let $\alpha = g \cdot p_*(\beta)$ for some $g \in \pi_1(M, x_0)$ and $\beta \in H_*(\Omega\widetilde{M})$ with avoiding-stick coproduct:*

$$\vee_{\Omega\widetilde{M}}(\beta) = \sum_i \gamma_i \times \delta_i$$

The avoiding-stick coproduct on α is computed as

$$\vee_{\Omega M}(\alpha) = \sum_{h \in \pi_1(M, x_0)} \sum_i g \cdot p_*(\gamma_i) \cdot h \times h^{-1} \cdot p_*(\delta_i).$$

PROOF. By the Sullivan relation as in Proposition 3.13, we have

$$\vee_{\Omega M}(g \cdot p_*(\beta)) = \vee_{\Omega M}(g) \cdot p_*(\beta) + g \cdot \vee_{\Omega M}(p_*(\beta)) = 0 + g \cdot \vee_{\Omega M}(p_*(\beta)).$$

We now apply the formula from Corollary 4.3 to compute $\vee_{\Omega M}(p_*(\beta))$:

$$\vee_{\Omega M} \circ p_* = \sum_{x_i \in p^{-1}(y_0)} \deg(p)_{x_i} (r_{[p_*(\tau_i)]} \times l_{[p_*(\tau_i)]^{-1}}) \circ (p \times p)_* \circ \vee_{\Omega\widetilde{M}},$$

where τ_i are paths connecting x_0 and x_i. We note that in our case all x_i have local degree $+1$ and the loops $[p_*(\tau_i)]$ correspond to all elements $h \in \pi_1(M)$. Therefore, we find

$$\vee_{\Omega M}(p_*(\beta)) = \sum_{h \in \pi_1(M)} \sum_i p_*(\gamma_i) \cdot h \times h^{-1} \cdot p_*(\delta_i)$$

which concludes the proof. $\square$

5. Spheres modulo a group action

In this section, we apply our methods to $M = S^n/G$ for a finite group G. We can fully describe the Goresky-Hingston coproduct on ΩM. This also helps us describe the homology of the free loop space because the coproduct gives non-vanishing results on the relevant Leray-Serre spectral sequence.

Let G be a non-trivial, finite group that acts freely and orientation-preserving on S^n for some $n \geq 2$. We note that $p\colon S^n \to S^n/G$ is the universal covering of the oriented manifold $M := S^n/G$.

We have the following description of the Pontryagin ring:

Proposition 5.1. *Let G be a non-trivial, finite group that acts on S^n freely and orientation-preserving for some $n \geq 2$ and let $M = S^n/G$. The Pontryagin ring on ΩM is given by*

$$H_*(\Omega M) \cong \mathbb{Z}[G][x]$$

where $\mathbb{Z}[G]$ is the group ring and x is a central element of degree $n-1$.

Proof. By Lemma 4.6, we find that as abelian groups

$$H_*(\Omega M) \cong \bigoplus_G H_*(S^n).$$

We recall that $H_*(S^n) \cong \mathbb{Z}[u]$ for u a class of dimension $n-1$. We thus set $x := p_*(u) \in H_{n-1}(\Omega_1 M)$ and find

$$H_*(\Omega M) \cong \mathbb{Z}[G][x]$$

as abelian groups.

Because n is odd, $u \in H_*(S^n)$ commutes with itself and therefore so does x. It thus remains to show that x and $g \in G$ commute. Because conjugation by g is a G-action on $H_{n-1}(\Omega_1 M) \cong \mathbb{Z}$, we have $g^{-1}xg = \pm x$.

We thus have to show that the sign is $+1$. Morally this is due to the fact that the class u comes from the orientation of S^n and g acts trivially on the orientation. We spell this argument out in more detail.

We have that

$$H_{n-1}(\Omega_{x_0} S^n) \cong H_n(\Sigma\Omega_{x_0} S^n) \xrightarrow{\text{ev}} H_n(S^n)$$

sends u to the orientation $[S^n] \in H_n(S^n)$.

The diagram

$$\begin{tikzcd} \Sigma\Omega_{x_0} S^n \arrow[r, "\text{ev}"] \arrow[d, "p"] & S^n \arrow[d, "p"] \\ \Sigma\Omega M \arrow[r, "\text{ev}"] & M \end{tikzcd}$$

commutes. Because we have $p_*(S^n) = |G| \cdot [M]$, we find $\text{ev}_*(x) = |G| \cdot [M]$. It thus suffice to show that $\text{ev}_*(g^{-1}xg) = |G| \cdot [M]$.

The commuting diagram

$$\begin{tikzcd} \Sigma\Omega_{x_0} S^n \arrow[r] \arrow[d, "g"] & S^n \arrow[d, "g"] \\ \Sigma\Omega_{g \cdot x_0} S^n \arrow[r] & S^n \end{tikzcd}$$

shows that ev_* sends $g_*(u) \in H_{n-1}(\Omega_{g\cdot x_0} S^n)$ to $(\varphi_g)_*([S^n]) = [S^n]$.

We now give a different description of $g_*(u)$. Let $\alpha \in C_{n-1}(\Omega_{x_0} S^n)$ be a representative of u and let τ_g be a path from x_0 to $g \cdot x_0$. Then $\tau_g^{-1} \cdot \alpha \cdot \tau_g$ represents $g_*(u)$. We thus have

$$p_*(g_*(u)) = p_*(\tau_g^{-1}) \cdot p_*([\alpha]) \cdot p_*(\tau_g) = |G| g^{-1} \cdot x \cdot g.$$

Now the commuting diagram

$$\begin{tikzcd} \Sigma\Omega_{g\cdot x_0} S^n \arrow[r, "\mathrm{ev}"] \arrow[d, "p"] & S^n \arrow[d, "p"] \\ \Sigma\Omega M \arrow[r, "\mathrm{ev}"] & M \end{tikzcd}$$

shows that

$$\mathrm{ev}_*(g^{-1} u g) = \mathrm{ev}_*(p_*(g_*(u))) = p_*(\mathrm{ev}_*(g_*(u))) = p_*([S^n]) = |G| \cdot [M].$$

This in particular shows that $g^{-1} x g = x$ and thus concludes the proof. $\square$

Corollary 5.2. *Let G be a non-trivial, finite group that acts on S^n freely and orientation-preserving for some $n \geq 2$ and let $M = S^n/G$. Let $gx^k \in H_*(\Omega M)$. The avoiding-stick coproduct extends to a map $H_{n-1+*}(\Omega M) \to H_*(\Omega M) \otimes H_*(\Omega M)$ which is computed as*

$$\vee(gx^k) = \sum_{i+j=k-1} \sum_{h \in G} gh^{-1} x^i \otimes h x^j.$$

Proof. Because all homology groups are free, the Künneth map $H_*(\Omega M) \otimes H_*(\Omega M) \to H_*(\Omega M \times \Omega M)$ is an isomorphism and we can thus postcompose the avoiding-stick coproduct $H_{n-1+*}(\Omega M) \to H_*(\Omega M \times \Omega M)$ with the inverse of the Künneth map and get a map $H_{n-1+*}(\Omega M) \to H_*(\Omega M) \otimes H_*(\Omega M)$.

The formula then follows from the coproduct on $H_*(\Omega S^n)$ as described in Example 3.14 transported to $H_*(\Omega M)$ using Proposition 4.8 and Proposition 5.1. $\square$

5.1. Homology of the free loop space. We now turn our attention to the homology of the free loop space ΛM. We show that the Leray-Serre spectral sequence associated to the fibration $\mathrm{ev}_0 \colon \Lambda M \to M$ is such that the E^2-page corresponds to the E^∞-page. We do this using the description of the coproduct on $H_*(\Omega M)$ as in Corollary 5.2.

We first compute $H_*(M)$: we consider the fibration

$$\begin{tikzcd} S^n \arrow[r] & M \arrow[d] \\ & BG. \end{tikzcd}$$

The associated Leray-Serre spectral sequence collapses to the long exact sequence:

$$\ldots \longrightarrow H_{i+1}(G) \longrightarrow H_{i+1-n}(G) \longrightarrow H_i(M) \longrightarrow H_i(G) \longrightarrow H_{i-n}(G) \longrightarrow \ldots$$

We know that $H_0(M) = H_n(M) = \mathbb{Z}$. The above long exact sequence then additionally implies $H_i(M) \cong H_i(G)$ for $1 \leq i \leq n-1$.

Now we can study the space ΛM. It has connected components corresponding to conjugacy classes in G. For $[\gamma] \in \mathrm{ccl}(G)$, we denote $\Lambda_{[g]} M \subseteq \Lambda M$ for the corresponding connected component.

It has associated the following fibration:

$$\begin{tikzcd} \coprod_{h\in[g]} \Omega_h M \arrow[r] & \Lambda_{[g]}M \arrow[d, "\mathrm{ev}_0"] \\ & M. \end{tikzcd} \tag{25}$$

We note that as G-modules, we have

$$H_*\left(\coprod_{h\in[g]} \Omega_h M\right) \cong \bigoplus_{h\in[g]} H_*(\Omega_h M) \cong \begin{cases} \mathbb{Z}[G/C_G(g)] & \text{if } * = m(n-1),\ m \geq 0; \\ 0 & \text{else.} \end{cases}$$

Therefore, the E^2-page of the associated Leray-Serre spectral sequence thus reads as

$$E^2_{p,q}([g]) := \begin{cases} H_p(M; \mathbb{Z}[G/C_G(g)]) & \text{if } q = m(n-1),\ m \geq 0; \\ 0 & \text{else.} \end{cases}$$

We denote the corresponding higher pages of the spectral sequence by $E^r_{p,q}([g])$.

By [Hat02, Example 3H.2] and our previous computations, we find

$$H_p(M; \mathbb{Z}[G/C_G(g)]) \cong H_p(S^n/C_G(g)) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0, n; \\ H_p(C_G(g)) & \text{if } 1 \leq p \leq n-1; \\ 0 & \text{else.} \end{cases}$$

We thus have found that the E^2-page of Leray-Serre of $E^r_{p,q}([g])$ looks like this.

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$\mathbb{Z}$	$H_1(C_G(g))$	$\cdots$	$H_{n-1}(C_G(g))$	$\mathbb{Z}$	0
$n-2$	0	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	0	0	0	0	0
0	$\mathbb{Z}$	$H_1(C_G(g))$	$\cdots$	$H_{n-1}(C_G(g))$	$\mathbb{Z}$	0
	0	1	$\ldots$	$n-1$	n	$>n$

PROPOSITION 5.3. *For all $g \in G$ it holds that $E^2_{*,*}([g]) \cong E^\infty_{*,*}([g])$.*

PROOF. For degree reasons, the only differential that may be non-zero is $d^n \colon E^n_{n,2k}([g]) \to E^n_{0,(n-1)k}([g])$. Thus it suffices to show that $E^\infty_{0,(n-1)k}([g]) \cong E^2_{0,(n-1)k}([g])$ for $k > 0$. By our computations, we have $E^2_{0,(n-1)k}([g]) \cong \mathbb{Z}$. Moreover because all other groups of total degree of $(n-1)k$ are torsion, it suffices to show that $\iota_* \colon H_{(n-1)k}(\Omega_g M) \to H_{(n-1)k}(\Lambda_{[g]}M, M)$ is injective for $g \neq 1$ or $k > 0$.

We do this by induction on k. The statement holds for $k = 0$ and $g \neq 1$ because $H_0(\Lambda_{[g]}M) \cong \mathbb{Z}$.

For the induction step, we use the Goresky-Hingston coproduct on $H_*(\Lambda M, M)$ and the avoiding-stick coproduct on $H_*(\Omega M)$. The avoiding-stick coproduct is a lift of the Goresky-Hingston coproduct on the based loop space, which is a restriction of the Goresky-Hingston coproduct on the free loop space. We thus have the commuting diagram

$$\begin{tikzcd}
H_{k(n-1)}(\Omega_g M) \arrow[r, "\iota_*"] \arrow[d, "\vee_\Omega"] & H_{k(n-1)}(\Lambda_{[g]}M, M) \arrow[d, "\vee_{\mathrm{GH}}"] \\
H_{(k-1)(n-1)}(\Omega M \times \Omega M) \arrow[r, "(\iota\times\iota)_*"] & H_{(k-1)(n-1)}((\Lambda M, M) \times (\Lambda M, M)).
\end{tikzcd}$$

Our strategy is to find an element in $H_{k(n-1)}(\Omega_g M)$ that gets send to a non-torsion element in $H_{(k-1)(n-1)}((\Lambda M, M) \times (\Lambda M, M))$ under composition on the bottom left. This then shows that there exists a non-torsion element in the image of the top map which is what we want to show.

By induction, we can assume that, for $k_1 + k_2 = k - 1$, the map

$$H_{k_1(n-1)}(\Omega_{g_1} M) \otimes H_{k_2(n-1)}(\Omega_{g_2} M) \to H_{(k-1)(n-1)}((\Lambda M, M) \times (\Lambda M, M))$$

is injective if $k_j \geq 1$ or $g_j \neq 1$ for $j = 1, 2$.

Corollary 5.2 then gives on $gx^k \in H_{k(n-1)}(\Omega_g M)$

$$(\iota \times \iota)_* \circ \vee_{\Omega M}(gx^k) = \sum_{g_1 \cdot g_2 = g} \sum_{k_1 + k_2 = k-1} \iota_*(g_1 x^{k_1 - 1}) \times \iota_*(g_2 x^{k_2 - 1})$$

which is non-zero if there exists $g_1, g_2 \neq 1$ with $g_1 g_2 = g$ or $k_1, k_2 > 1$. This is satisfied except in the case where $G = \{\pm 1\}$, $g = -1$ and $k = 1$.

The remaining case is for the manifold $M = \mathbb{R}P^n$. We thus need to show that $H_{n-1}(\Lambda_{[-1]}\mathbb{R}P^n, \mathbb{R}^n)$ has a non-torsion summand. This can be seen from the fact that the Chas-Sullivan product with x gives an isomorphism $H_*(\Lambda\mathbb{R}P^n, \mathbb{R}P^n) \cong H_{n-1+*}(\Lambda\mathbb{R}P^n, \mathbb{R}P^n)$ [GH09, p. 1.11] and our computation that shows that $H_{2(n-1)}(\Lambda_{[-1]}\mathbb{R}P^n, \mathbb{R}^n)$ has a non-torsion summand. $\square$

Corollary 5.4. *The homology of the free loop space of $M = S^n/G$ fits in the following short exact sequence:*

$$0 \longrightarrow H_{i-n-1}(S^n/C_G(g)) \longrightarrow H_{i+k(n-1)}(\Lambda_{[g]}M) \longrightarrow H_i(S^n/C_G(g)) \longrightarrow 0$$

for $k \geq 0$ and $1 \leq i \leq n$. This short exact sequence splits for all $i \neq n - 1$.

Proof. The short exact sequence comes immediately from the description $E^\infty_{*,*}([g])$ as in Proposition 5.3. Moreover, for $i < n - 1$ we have $H_{i-n-1}(S^n/C_G(g)) \cong 0$. For $i = n$, we have $H_i(S^n/C_G(g)) \cong \mathbb{Z}$ is projective and thus the short exact sequence splits. $\square$

5.2. Coproduct on the free loop space. We conclude this article by computing the Goresky-Hingston coproduct on the free loop space with rational coefficients for $M = S^n/G$. This can be done because the coproduct has a description on the Leray-Serre spectral sequence and the rational homology of $H_*(\Lambda M)$ is sparse enough such that there are no extension issues.

We use the following description of the Goresky-Hingston coproduct on the Leray-Serre spectral sequence.

Proposition 5.5 ([Mei11, Theorem 4.8]). *Let F be a field and M be a F-oriented manifold. The Goresky-Hingston coproduct gives a map of spectral sequences with F-coefficients. Moreover, the coproduct is computed on the E^2-page $H_*(M; H_*(\Omega M, x_0; F))$ as $\Delta_* \circ \nu_\Omega$ where ν_Ω denotes the Goresky-Hingston coproduct on the coefficients.*

Remark 5.6. Because the statment uses field coefficients, $H_*(\Lambda M, M) := H_*(\Lambda M, M; k)$ is determined by its dimension. In particular, we get a non-canonical isomorphism:

$$H_*(\Lambda M, M) \cong \bigoplus_{p+q=*} E^\infty_{p,q}$$

where $E^{\infty}_{p,q}$ is the E^{∞}-page of the Leray-Serre spectral with E^2-page $H_p(M, H_q(\Omega M, x_0; k))$.

In particular, we get a filtration

$$F^p_* = \bigoplus_{\substack{i+j=*\\ j\leq p}} E^{\infty}_{i,j} \tag{26}$$

of $H_*(\Lambda M, M)$.

Proposition 5.5 implies that the Goresky-Hingston coproduct preserves the filtration (26). In particular, the coproduct on $E^{\infty}_{p,q}$ lands in

$$\bigoplus_{\substack{p_1+p_2+q_1+q_2=p+q+1-n\\ p_1+p_2\leq p}} E^{\infty}_{p_1,q_1} \otimes E^{\infty}_{p_2,q_2}.$$

The description of the coproduct on the spectral sequence as in Proposition 5.5 only lets us compute the image of the coproduct in

$$\bigoplus_{\substack{p_1+p_2+q_1+q_2=p+q+1-n\\ p_1+p_2=p}} E^{\infty}_{p_1,q_1} \otimes E^{\infty}_{p_2,q_2}.$$

In the case where $M = S^n/G$, this is enough (see Theorem 5.8). In other cases such as Example 5.7, this description is missing all information.

EXAMPLE 5.7. We consider the case for $M = \Sigma_g$ of a surface of genus $g \geq 2$.

Because M has contractible universal covering, we have $\pi_*(\Omega M) \cong \pi_{1+*}(M) \cong 0$ for all $* \geq 1$. Therefore ΩM has contractible path components and $H_*(\Omega M)$ is concentrated in degree 0. In particular, the Goresky-Hingston coproduct on $H_*(\Omega M, x_0)$ is of degree $n-1 = 1$ and thus zero.

Therefore Proposition 5.5 shows that the coproduct is zero on the Leray-Serre spectral sequence. This is in contrast to the fact that the coproduct on $H_*(\Lambda M, M)$ is not zero (see [HS25]).

THEOREM 5.8. *Let G be a non-trivial, finite group acting orientation-preserving on S^n for $n > 1$. The homology of the free loop space of $M = S^n/G$ with rational coefficients is given by*

$$H_*(\Lambda M; \mathbb{Q}) \cong \bigoplus_{[g]\in \mathrm{ccl}(G)} \bigoplus_{k\geq 0} \mathbb{Q}x_{[g],k} \oplus \mathbb{Q}y_{[g],k}$$

with $|x_{[g],k}| = k(n-1)$ and $|y_{[g],k}| = k(n-1)+n$. The relative homology $H_*(\Lambda M, M; \mathbb{Q})$ has the same description except missing the $[g] = [1]$ and $k = 0$ summand.

The Goresky-Hingston coproduct lifts to a map

$$\widehat{\vee}\colon H_*(\Lambda M; \mathbb{Q}) \to H_*(\Lambda M; \mathbb{Q}) \otimes H_*(\Lambda M; \mathbb{Q})$$

computed by

$$\widehat{\vee}(x_{[g],k}) = \sum_{i+j=k-1} \sum_{h\in G} x_{[gh^{-1}],i} \otimes x_{[h],j}$$

and

$$\widehat{\vee}(y_{[g],k}) = \sum_{i+j=k-1} \sum_{h\in G} (x_{[gh^{-1}],i} \otimes y_{[h],j} + y_{[gh^{-1}],i} \otimes x_{[h],j}).$$

PROOF. Considering the spectral sequence associated to the fibration

$$\begin{tikzcd} S^n \arrow[r] & S^n/C_G(g) \arrow[d] \\ & B(C_G(g)), \end{tikzcd}$$

we note that $H_i(S^n/C_G(g)) \cong H_i(C_G(g))$ for all $0 < i < n$. Thus the groups $H_i(S^n/C_G(g))$ for $0 < i < n$, appearing in Corollary 5.4, are finite groups and thus disappear after we tensor with $\mathbb{Q}$. This shows the description as vector space.

For the coproduct, we note that $H_{n(k-1)}(\Lambda_{[g]}M) \cong E^\infty_{0,n(k-1)}$. By Remark 5.6 the image of $E^\infty_{0,n(k-1)}$ under $\widehat{\vee}$ lands in

$$\bigoplus_{\substack{p_1+p_2+q_1+q_2=k(n-1)+1-n \\ p_1+p_2\leq 0}} E^\infty_{p_1,q_1} \otimes E^\infty_{p_2,q_2}.$$

Because $E^\infty_{p,q} \cong 0$ for $p < 0$, we have that the image of $E^\infty_{0,n(k-1)}$ under $\widehat{\vee}$ lands in

$$\bigoplus_{\substack{p_1+p_2+q_1+q_2=k(n-1)+1-n \\ p_1+p_2=0}} E^\infty_{p_1,q_1} \otimes E^\infty_{p_2,q_2}$$

and as argued in Remark 5.6 is determined by the map it induces on the E^∞-page.

Similarly, $H_{k(n-1)+n}(\Lambda M) \cong E^\infty_{n,k(n-1)}$ lands in

$$\bigoplus_{\substack{p_1+p_2+q_1+q_2=k(n-1)+n+1-n \\ p_1+p_2\leq n}} E^\infty_{p_1,q_1} \otimes E^\infty_{p_2,q_2}. \tag{27}$$

But $H_*(\Lambda M)$ is concentrated in degree $* \equiv 0, 1 \pmod{n-1}$. Therefore (27) is concentrated in degrees where $p_1 + q_1 \equiv 1 \pmod{n-1}$ and $p_2 + q_2 \equiv 0 \pmod{n-1}$ or vice-versa. But $E^\infty_{p,q}$ is concentrated in degrees where $q \equiv 0 \pmod{n-1}$. This shows that $p_1 + p_2 \equiv 1 \pmod{n-1}$ which in turn shows $p_1 + p_2 = n$.

This finally shows that $\widehat{\vee}$ sends $H_{k(n-1)+n}(\Lambda M)$ to

$$\bigoplus_{\substack{p_1+p_2+q_1+q_2=k(n-1)+n+1-n \\ p_1+p_2=n}} E^\infty_{p_1,q_1} \otimes E^\infty_{p_2,q_2}.$$

and as argued in Remark 5.6 is determined by the map it induces on the E^∞-page.

We now compute what $\widehat{\vee} := \Delta_* \circ \widehat{\nu}$ does on $E^\infty_{p,q} \cong E^2_{p,q}$. Here $\widehat{\nu}$ denotes the avoiding stick coproduct on the coefficients $H_*(\Omega M)$. On the E^2-page we have that $E^2_{0,k(n-1)}([g])$ is generated by $x_0 \otimes g \cdot x^k \in H_0(M; H_{k(n-1)}(\Omega M))$ and $E^2_{n,k(n-1)}([g])$ is generated by $[M] \otimes g \cdot x^k \in H_n(M; H_{k(n-1)}(\Omega M))$. We call the corresponding elements in $H_*(\Lambda M)$ $x_{[g],k}$ and $y_{[g],k}$, respectively.

Using Corollary 5.2, $\Delta_*(x_0) = x_0 \otimes x_0$ and $\Delta_*([M]) = x_0 \otimes [M] + [M] \otimes x_0$, we compute

$$\begin{aligned}
\widehat{\vee}(x_{[g],k}) = \Delta_*(x_0 \otimes \vee_\Omega(gt^k)) &= \Delta_*\left(x_0 \otimes \left(\sum_{i+j=k-1} \sum_{h\in H} gh^{-1}x^i \otimes hx^j\right)\right) \\
&= \sum_{i+j=k-1} \sum_{h\in H} (x_0 \otimes gh^{-1}x^i \otimes) \otimes (x_0 \otimes hx^j) \\
&= \sum_{i+j=k-1} \sum_{h\in G} x_{[gh^{-1}],i} \otimes x_{[h],j}
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\vee}(y_{[g],k}) &= \Delta_*([M] \otimes \vee_\Omega(gx^k)) \\
&= \Delta_* \left([M] \otimes \left(\sum_{i+j=k-1} \sum_{h \in H} gh^{-1}x^i \otimes hx^j \right) \right) \\
&= \sum_{i+j=k-1} \sum_{h \in H} \big((x_0 \otimes gh^{-1}x^i) \otimes ([M] \otimes ht^j) + ([M] \otimes gh^{-1}x^i) \otimes (x_0 \otimes ht^j) \big) \\
&= \sum_{i+j=k-1} \sum_{h \in G} (x_{[gh^{-1}],i} \otimes y_{[h],j} + y_{[gh^{-1}],i} \otimes x_{[h],j}).
\end{aligned}$$

$\square$

Part III

The Goresky-Hingston coproduct in Morse homology with DG coefficients

Abstract. We describe the Goresky-Hingston coproduct on the free loop space via Morse homology with DG coefficients. To this end, we study the Goresky-Hingston coproduct on the based loop space and its invariance under changing the basepoint.

Contents

1. Introduction

In this article, we give a description of the Goresky-Hingston coproduct

$$\nu_\Lambda\colon H_*(\Lambda M, M) \to H_{1-n+*}((\Lambda M, M)^2)$$

via the framework of Morse homology with differential graded coefficients as defined in [Bar+23]. Here, ΛM denotes the free loop space of a smooth manifold M of dimension n. It contains M as the constant loops.

The Goresky-Hingston coproduct or string topology coproduct is a coproduct on $H_*(\Lambda M, M)$ of degree $1-n$, introduced by Goresky and Hingston in [GH09] and Sullivan in [Sul04].

Morse homology with DG coefficients gives a quasi-isomorphism

$$C_*(M, C_*(\Omega M, x_0)) \simeq C_*(\Lambda M, M),$$

where $C_*(M, C_*(\Omega M, x_0))$ denotes Morse homology with DG coefficients $C_*(\Omega M, x_0)$ chains of the based loop space. It is generated by chains of the form $\alpha \otimes x$ where x is a critical point of a fixed Morse function and $\alpha \in C_*(\Omega M, x_0)$. It has a differential that encodes how we move a loop from one critical point to another critical point via conjugation with paths.

The Goresky-Hingston coproduct restricts to an operation on the based loop space

$$C_*(\Omega M, x_0) \to C_{1-n+*}((\Omega M, x_0)^2).$$

We show that in Morse homology with DG coefficients the coproduct ν_Λ can be expressed in terms of the coproduct ν_Ω.

THEOREM A (Theorem 7.14). *Let A be a ring and let (M, x_0) be a smooth, pointed manifold of dimension n. Assume that M is parallelizable or $\mathbb{R} \subseteq A$. For any class $[\alpha] \in H_*(\Lambda M, M; A)$, there exists a representative*

$$\sum_x \alpha_x \otimes x \in C_*(M, C_*(\Omega M, x_0; A))$$

and a chain-level model $\vee_\Omega \colon C_*(\Omega M, x_0; A) \to C_{1-n+*}((\Omega M, x_0)^2; A)$ of the Goresky-Hingston coproduct on the based loop space ν_Ω such that

$$\sum_x \vee_\Omega(\alpha_x) \otimes x$$

computes the Goresky-Hingston coproduct.

The chain-level model $\vee_\Omega$ in the above statement depends on the chain representative chosen. The underlying reason for this is that we give a filtration of $C_*(M, C_*(\Omega M))$. This filtration is exhaustive on homology and, on each level of the filtration, there exists a different chain-level model that does the trick. We sketch in §1.2 how the different stages of the filtration arise.

To give a description of ν_Λ in Morse homology with DG coefficients that works for all chains, we use the extension of the Morse homology framework by Riegel in [Rie24] that uses morphisms of $\mathcal{A}_\infty$-modules.

This allows us to prove the following:

THEOREM B (Theorem 7.3). *Let A be a field and let (M, x_0) be a smooth, pointed manifold of dimension n. If M is parallelizable or $\mathbb{R} \subseteq A$, there exists a morphism of $\mathcal{A}_\infty$-modules over $C_*(\Omega M)$*

$$(\nu_k)_{k\geq 1} \colon C_*(\Omega M, x_0; A) \to C_{1-n+*}((\Omega M, x_0)^2; A),$$

where ν_1 computes the Goresky-Hingston coproduct on the based loop space.

It holds moreover that

$$\begin{aligned} C_*(\Lambda M, M; A) \simeq C_*(M, C_*(\Omega M, x_0; A)) &\xrightarrow{\widetilde{\nu}} C_*(M, C_{1-n+*}((\Omega M, x_0)^2; A)) \\ &\xrightarrow{\Delta_*} C_*(M \times M, C_{1-n+*}((\Omega M, x_0)^2; A)) \\ &\simeq C_{1-n+*}((\Lambda M, M)^2; A) \end{aligned}$$

computes ν_Λ, the Goresky-Hingston coproduct on homology of the free loop space. Here $\widetilde{\nu}$ denotes the morphism in Morse homology with DG coefficients induced by $(\nu_k)_{k\geq 1}$ by the work of Riegel [Rie24].

Morally, both statements work by showing that the Goresky-Hingston coproduct on the based loop space ν_Ω has a chain-level model that is equivariant under the ΩM-action given by conjugation. In the following, we sketch how we construct such an equivariant chain-level model.

The Goresky-Hingston coproduct on a class $\alpha \in H_*(\Omega M, x_0)$ is computed as follows:

(I) multiply with the unit interval $I \coloneqq [0, 1]$ and obtain a class $I \times \alpha \in H_*(I \times \Omega, I \times \{x_0\} \cup \partial I \times \Omega)$;

(II) "intersect" $I \times \alpha$ with the subspace of loops with self-intersections:

$$\mathcal{F}\Omega M \coloneqq \{(t, \gamma) \in I \times \Omega M \mid \gamma(t) = \gamma(0)\};$$

(III) cut using the map $\mathcal{F}\Omega M \to \Omega M \times \Omega M, (t, \gamma) \mapsto (\gamma|_{[0,t]}, \gamma|_{[t,1]})$.

While it is straightforward to give an equivariant model for steps (I) and (III), step (II) needs more work.

We follow the description in [HW17]. In this setup, Step (II) is done by fixing open balls

$$x_0 \in V_{x_0} \coloneqq \{x \in M \mid d_M(x, x_0) < \tfrac{\varepsilon}{2}\} \subsetneq U_{x_0} \coloneqq \{x \in M \mid d_M(x, x_0) < \varepsilon\} \subseteq M$$

Picking a homotopy inverse to excision gives a map

$$C_*(I \times \Omega M, \partial I \times \Omega M) \xrightarrow{\simeq} C_*(e_t^{-1}(U_{x_0}), I \times \{x_0\} \cup \partial I \times \Omega M \cup e_t^{-1}(V_{x_0}^c)). \tag{28}$$

Here $e_t \colon I \times \Omega M \to M$ denotes the evaluation at time t. The "intersection" is done by the cap product

(29)

$$e_t^* \tau_{x_0} \cap \colon C_*(e_t^{-1}(U_{x_0}), I \times \{x_0\} \cup \partial I \times \Omega M \cup e_t^{-1}(V_{x_0}^c)) \to C_{-n+*}(e_t^{-1}(U_{x_0}), I \times \{x_0\} \cup \partial I \times \Omega M),$$

where $\tau_{x_0} \in C^n(U_{x_0}, V_{x_0}^c)$ is a local orientation or Thom class at x_0. We explain this in further detail in Subsection 2.2.

The space $e_t^{-1}(U_{x_0})$ can be described as all times t and loops γ such that $\gamma(t)$ is close to being a self-intersection. In particular, we have a homotopy equivalence $R \colon e_t^{-1}(U_{x_0}) \simeq \mathcal{F}\Omega M$.

Step (II) is thus given by the excision (28), the cap product (29) and the homotopy equivalence R.

1.1. Equivariance of the cap product. The question that occupies us for the most part in this article is the following:

Question 1.1. Is the cap product in (29) ΩM-equivariant?

To make sense of this, we need to specify an ΩM-action on all the spaces appearing in (29). For this, we construct an action of ΩM on itself that is homotopic to conjugation and such that, when we extend it trivially to an action on $I \times \Omega M$, it sends the subspaces $e_t^{-1}(U_{x_0})$ and $e_t^{-1}(V_{x_0}^c)$ to themselves.

To construct such an action, we introduce the notion of a *transitive isotopy extension* in Section 3. The idea is that we define for any path $\lambda \colon [0, a] \to M$ a diffeomorphism

$$\Theta(\lambda) \colon M \cong M$$

that

(i) is compatible with composition of paths, that is, for composable paths λ, μ, we have $\Theta(\lambda \cdot \mu) = \Theta(\mu) \circ \Theta(\lambda)$;
(ii) for $\lambda \in \Omega M$, the diffeomorphism $\Theta(\lambda)$ acts via parallel transport on U_{x_0} via the isomorphism $\exp_{x_0} \colon \{v \in T_{x_0} M \mid \|v\| < \varepsilon\} \cong U_{x_0}$.

This defines via

$$\begin{aligned} \Phi \colon \Omega M \times \Omega M &\to \Omega M, \\ (\gamma, \lambda) &\mapsto \Theta(\lambda)_*(\gamma), \end{aligned}$$

the desired ΩM-action on itself making Question 1.1 well-posed.

1.1.1. *Real coefficients.* In Section 5, we construct a local orientation $\tau_{x_0} \in C^n(U_{x_0}, V_{x_0}^c; \mathbb{R})$ that is invariant under the $\mathrm{SO}(n)$-action on the n-dimensional ball U_{x_0}. This $\mathrm{SO}(n)$-invariance comes from an explicit class in de Rham cohomology and thus requires real coefficients.

The transitive isotopy extension $\Theta(\lambda)|_{U_{x_0}}$ acts via parallel transport and hence defines an element in $\mathrm{SO}(n)$. Therefore, the Thom class is invariant under the ΩM-action. This in turn shows that the cap product (29) is ΩM-equivariant (see Proposition 4.6). This gives a positive answer to Question 1.1 for $\mathbb{R} \subseteq A$.

1.1.2. *Parallelizable manifolds.* In the case where M is parallelizable, we have that parallel transport $T_{x_0}M \to T_{x_0}M$ along a loop is trivial. Hence, any Thom class $\tau_{x_0} \in C^n(U_{x_0}, V_{x_0}^c)$ is invariant under the action given by $\Theta(\lambda)|_{U_{x_0}}$.

This in particular shows that the cap product (29) is ΩM-equivariant for any Thom class if M is parallelizable.

1.2. Equivariance of the excision map. The remaining question is whether the inverse to the excision map (28) can be chosen to be ΩM-equivariant. We note that the wrong-way map

$$C_*(e_t^{-1}(U_{x_0}), I \times \{x_0\} \cup \partial I \times \Omega M \cup e_t^{-1}(V_{x_0}^c)) \to C_*(I \times \Omega M, \partial I \times \Omega M \cup e_t^{-1}(V_{x_0}^c)) \tag{30}$$

is ΩM-equivariant as the actions are given by restricting the action on $I \times \Omega M$. Hence, it is an issue of inverting an ΩM-equivariant map.

We propose two solutions for establishing equivariance for (28). Each solution corresponds to one of the two Theorems A and B.

The first solution is to work with small simplices, which gives us the explicit formula in Theorem A. The strategy is to model a class in $H_*(\Lambda M, M)$ via chains of loops that move slowly enough close to the base point and then multiply these chains by a cut-up interval $\sum_{i=0}[\frac{i}{k+1}, \frac{i+1}{k+1}]$.

For any class in $H_*(\Lambda M, M)$, there exists a k such that the above strategy gives a chain representative in $C_*(I \times \Omega M, \partial I \times \Omega M \cup e_t^{-1}(V_{x_0}^c))$ that is in the image of the wrong-way excision map (30). But there is no k, that works for all classes in $H_*(\Lambda M, M)$ at once. This explains the dependence of the chain-level model on the homology class in Theorem A.

The second solution is to find an inverse of the equivariant map (30) via $\mathcal{A}_\infty$-theory. This requires field coefficients and does not produce a strictly equivariant map but only a morphism of $\mathcal{A}_\infty$-modules. Hence, we use $\mathcal{A}_\infty$-modules in Theorem B.

Organisation of the paper. In Section 2, we recall the necessary background on Morse homology with DG coefficients, fix the definition of the Goresky-Hingston coproduct of this article and specify our sign conventions. In Section 3, we define and construct transitive isotopy extensions. In Section 4, we exhibit the Goresky-Hingston coproduct as a morphism of $\mathcal{A}_\infty$-modules. In Section 5, we explain how to construct the invariant Thom classes that we need. In Section 6, we define and study coherent chain homotopies. They are an algebraic gadget that lets us prove that the map $\widetilde{\nu}$ in Morse homology with DG coefficient indeed models the desired map. In Section 7, we prove our main theorems. Finally, the appendix contains a section about inverting morphisms of $\mathcal{A}_\infty$-modules.

Conventions. Throughout this article M denotes a smooth, closed, oriented manifold of dimension n with a Riemannian metric. The Riemannian metric induces the Riemannian distance, a metric on M. We denote it by $d_M\colon M \times M \to \mathbb{R}$. We fix a basepoint $x_0 \in M$.

For A, B smooth manifolds, we denote by $\mathrm{Emb}(A, B)$ the space of smooth embeddings.

The free loop space on M is defined as

$$\Lambda \coloneqq \Lambda M \coloneqq \{\gamma\colon [0,a] \to M \mid a \geq 0, \gamma(0) = \gamma(a), \gamma \text{ is piecewise smooth}\}$$

and the based loop space is defined as

$$\Omega \coloneqq \Omega M \coloneqq \{\gamma\colon [0,a] \to M \mid a \geq 0, \gamma(0) = \gamma(a) = x_0, \gamma \text{ is piecewise smooth}\}.$$

We denote $\mu\colon \Omega M \times \Omega M \to \Omega M, (\lambda, \nu) \mapsto \lambda \cdot \nu$ for the concatenation.

The path space on a smooth manifold X is denoted by

$$\mathcal{P}X \coloneqq \{\gamma\colon [0,a] \to X \mid \gamma \text{ is piecewise smooth}\}$$

and for subspaces $A, B \subseteq X$, we denote

$$\mathcal{P}_{A \to B}X \coloneqq \{(\gamma\colon [0,a] \to X) \in \mathcal{P}X \mid \gamma(0) \in A, \gamma(a) \in B\}.$$

In this article, we work with cubical chains. A cubical chain $\sigma\colon I^k \to X$ is degenerate if there exists $1 \le i \le k$ such that

$$\sigma(v_1, \dots, v_i, \dots, v_k) = \sigma(v_1, \dots, v_i', \dots, v_k)$$

for all $v_1, \dots, v_k, v_i' \in I$.

We denote by $C_k(X)$ the cubical chains on a topological space. It is the free group generated by continuous maps $\sigma\colon I^k \to X$ relative to the subspace generated by all degenerate chains. For a ring A, we denote $C_k(X; A) \coloneqq C_k(X) \otimes A$.

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2. Background

In this section, we recall the machinery of Morse homology with DG coefficients as introduced in [Bar+23]. Moreover, we recall the definition of the Goresky-Hingston coproduct on the free loop space and based loop space of a manifold as first introduced in [GH09] and [Sul04]. Finally in this section, we fix the sign conventions we use in this article for maps of chain complexes which do not preserve degree.

2.1. Morse homology with DG coefficients. In [Bar+23], Barraud, Damian, Humilière and Oancea describe the framework of Morse homology with differential graded coefficients. We recall in this subsection the parts we need in this article. In particular for a fibration

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & M, \end{tikzcd}$$

we construct a complex $C_*(M, C_*(F))$ with a quasi-isomorphism

$$\Psi\colon C_*(M, C_*(F)) \to C_*(E).$$

2.1.1. *Data and choices.* For this, we choose the following data:

- a basepoint $x_0 \in M$;
- a Morse-Smale pair (f, ξ) on M;
- an orientation on the unstable manifolds $W^u(x)$ of (f, ξ);
- a tree $T \subseteq M$ containing x_0 and the critical points of f;
- a homotopy inverse θ to the projection $p\colon M \to M/T$;
- a representing chain system $\{s_{x,y}\}$ (see Definition 2.1);
- a compatible representing system $\{s_x\}$ (see Definition 2.4);
- a transitive lifting function $\Phi\colon E\,{}_\pi\!\times_{e_0} \mathcal{P}M \to E$ (see Definition 2.5).

For any pair of critical points $x, y \in \mathrm{Crit}(f)$, we denote the space of Morse trajectories of ξ between x and y by $\mathcal{L}(x,y)$ and define the *moduli space of broken Morse trajectories*

$$\overline{\mathcal{L}}(x,y) := \mathcal{L}(x,y) \cup \bigcup_{z_1,\dots,z_k \in \mathrm{Crit}(f)} \mathcal{L}(x,z_1) \times \mathcal{L}(z_1,z_2) \times \cdots \times \mathcal{L}(z_k,y).$$

The space $\overline{\mathcal{L}}(x,y)$ admits the structure of a manifold with corners of dimension $|x|-|y|-1$ with interior $\mathcal{L}(x,y)$ (see e.g. [Lat94, Proposition 2.15]). Moreover, the orientations on the unstable manifolds induce an orientation $[\overline{\mathcal{L}}(x,y)] \in H_{|x|-|y|-1}(\overline{\mathcal{L}}(x,y), \partial\overline{\mathcal{L}}(x,y))$.

Definition 2.1. A *representing chain system* of the moduli space of broken Morse trajectories is a collection $\{s_{x,y} \in C_{|x|-|y|-1}(\overline{\mathcal{L}}(x,y)) \mid x, y \in \mathrm{Crit}(f)\}$ such that

(i) $s_{x,y}$ represents a cycle in $C_{|x|-|y|-1}(\overline{\mathcal{L}}(x,y), \partial\overline{\mathcal{L}}(x,y))$ and $[s_{x,y}] = [\overline{\mathcal{L}}(x,y)]$ in homology;
(ii) the following equation holds

$$ds_{x,y} = \sum_{z \in \mathrm{Crit}(f)} (-1)^{|x|-|z|} s_{x,z} \times s_{z,y}.$$

A representing chain system always exists (see [Bar+23, Proposition 5.2.6]).

Following [Bar+23, Lemma 5.2.10] we denote $\Gamma\colon \overline{\mathcal{L}}(x,y) \to \mathcal{P}_{x\to y}M$ the map that sends a trajectory to the path from x to y parametrised by f. We note that this is well-defined as this parametrisation produces smooth paths. We set

$$m_{x,y} := (\theta \circ p)_* \circ \Gamma_*(s_{x,y}) \in C_{|x|-|y|-1}(\Omega M).$$

Remark 2.2. This assumes that $\theta \circ p$ sends piecewise smooth paths to piecewise smooth paths. For simplicity, we may assume that $\theta \circ p$ is smooth. This can be done by a standard smoothing argument. Indeed for any θ', the map $\theta' \circ p|_T$ is constant and hence smooth. Hence, there exists a smooth map $\widetilde{\theta' \circ p}$ homotopic to $\theta' \circ p$ that extends $\theta' \circ p|_T$. This map is constant on T and thus factors as $\widetilde{\theta' \circ p} = \theta \circ p$ for some θ.

It is convenient in the proof of Proposition 6.2 to assume that θ is also a homeomorphism. This is possible to assume but requires more work that we omit here.

By [Bar+23, Definition 5.2.11], the collection $\{m_{x,y} \mid x, y \in \mathrm{Crit}(f)\}$ is a *twisting cocycle* meaning that they satisfy the equation

$$d(m_{x,y}) = \sum_{z \in \mathrm{Crit}(f)} (-1)^{|x|-|z|} \mu(m_{x,z} \otimes m_{z,y}).$$

Definition 2.3. For $x \in \mathrm{Crit}(f)$, the *Latour cell* is defined as

$$\overline{W^u}(x) := W^u(x) \cup \bigcup_{y \in \mathrm{Crit}(f)} \overline{\mathcal{L}}(x,y) \times W^u(y).$$

The natural topology on the Latour cell makes it into a compact manifold with corners which is homeomorphic to the disk $D^{|x|}$ (see [Lat94, Proposition 2.11]).

Definition 2.4. A *compatible representing chain system* is a collection $\{s_x \in C_{|x|}(\overline{W^u}(x)) \mid x \in \mathrm{Crit}(f)\}$ such that

(i) s_x represents a cycle in $C_{|x|}(\overline{W^u}(x), \partial\overline{W^u}(x))$ and $[s_x] = [\overline{W^u}(x)]$;
(ii) the following equation holds

$$ds_x = \sum_{y \in \mathrm{Crit}(f)} s_x \times s_{x,y}.$$

A compatible representing chain system always exists [Bar+23, Lemma 7.3.2]. We denote $m_x \coloneqq \theta_* \circ p_* \circ \Gamma_*(s_x) \in C_{|x|}(\mathcal{P}_{x_0 \to M} M)$.

Definition 2.5. A *transitive lifting function* is a map

$$\Phi\colon E\,{}_\pi\!\times_{e_0} \mathcal{P}M \to E$$

such that

(i) $\pi \circ \Phi = e_1 \circ \mathrm{pr}_2$;
(ii) for all $e \in E$, we have $\Phi(e, c_{\pi(e)}) = e$, where $c_{\pi(e)} \in \mathcal{P}M$ is the constant path at $\pi(e) \in M$;
(iii) for all $e \in E$ and composable $\gamma, \delta \in \mathcal{P}M$ with $\gamma(0) = \pi(e)$, we have

$$\Phi(\Phi(e,\gamma),\delta) = \Phi(e, \gamma \cdot \delta).$$

Any fibration is fibre homotopy equivalent to a fibration that admits a transitive lifting function [DK69, Proposition 5.5].

A transitive lifting function in particular defines a right ΩM-action on F.

2.1.2. *Construction.* We now describe the complex $C_*(M, C_*(F))$ and the quasi-isomorphism $\Psi\colon C_*(M, C_*(F)) \to C_*(E)$: the complex $C_*(M, C_*(F))$ has underlying graded abelian group

$$C_*(F) \otimes \mathbb{Z}\mathrm{Crit}(f)$$

with differential given by

$$d_F(\alpha \otimes x) = d\alpha + (-1)^{|\alpha|} \sum_{y \in \mathrm{Crit}(f)} \Phi_*(\alpha \otimes m_{x,y}) \otimes y.$$

We denote the degree -1 map

$$\begin{aligned} \boldsymbol{m}\colon \mathbb{Z}\mathrm{Crit}(f) &\to C_*(\Omega M) \otimes \mathbb{Z}\mathrm{Crit}(f), \\ x &\mapsto \sum_y m_{x,y} \otimes y. \end{aligned} \tag{31}$$

Then the differential can be written as

$$d_F = d \otimes \mathrm{id} + (\Phi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \boldsymbol{m})$$

and the above sign $(-1)^{|\alpha|}$ comes from a Koszul sign.

For the map $\Psi\colon C_*(M, C_*(F)) \to C_*(E)$, we consider the fibration $\pi'\colon E' \to M/T$ given by the pullback of $\pi\colon E \to M$ along $\theta\colon M/T \to M$. We can assume that θ is a homeomorphism. Thus $\Phi(-, \theta(-))$ defines also a transitive lifting function on $\pi'\colon E' \to M/T$. This also induces a construction for $C_*(M/T, C_*(F))$.

The map Ψ is given by the composition

$$C_*(M, C_*(F)) \to C_*(M/T, C_*(F)) \xrightarrow{\Psi'} C_*(E') \to C_*(E).$$

The first map is an isomorphism. The third map above is given by the homotopy equivalence θ and is thus a quasi-isomorphism. Thus the only map of interest is the middle map Ψ'. This is given by

$$\Psi'(\alpha \otimes x) = \Phi_*(\alpha \otimes m_x).$$

Barraud, Damian, Humilière and Oancea prove that Ψ' is a quasi-isomorphism and therefore so is Ψ [Bar+23, Theorem 7.2.1].

We use an immediate corollary of this statement for relative fibrations:

COROLLARY 2.6. *Let $\pi\colon E' \subseteq E \to M$ be fibrations with fibres $F' \subseteq F$ with Φ a transitive lifting function of $\pi\colon E \to M$ that restricts to a transitive lifting function on E'. The map*

$$\Psi\colon C_*(M, C_*(F, F')) \to C_*(E, E')$$

defines a quasi-isomorphism.

PROOF. The fact that Φ is a transitive lifting function for E and E' means that the inclusion

$$C_*(M, C_*(F')) \to C_*(M, C_*(F))$$

is a chain map. Moreover, the following diagram commutes

$$\begin{tikzcd} C_*(M, C_*(F')) \arrow[r] \arrow[d, "\Psi_{E'}"] & C_*(M, C_*(F)) \arrow[d, "\Psi_E"] \\ C_*(E') \arrow[r] & C_*(E). \end{tikzcd}$$

The five lemma now concludes the proof. $\square$

2.1.3. *Maps.* In this article, we need two different types of maps one can define in a Morse complex with DG coefficients.

In [Bar+23, Section 9], a direct map is defined for a smooth map of based manifolds $\varphi\colon (N, y_0) \to (M, x_0)$ and a fibration $\pi\colon E \to M$

$$C_*(N, \varphi^* C_*(F)) \to C_*(M, C_*(F)).$$

On the other hand, Riegel developed in [Rie24] a theory to construct maps on the Morse complex with DG coefficients for maps $C_*(F) \to C_*(F')$ which are morphisms of $\mathcal{A}_\infty$-modules. We are interested in $\mathcal{A}_\infty$-morphisms between strict $C_*(\Omega M)$-modules. We recall that ΩM denotes (piecewise smooth) Moore loops and thus $C_*(\Omega M)$ is indeed strictly associative.

DEFINITION 2.7. A *morphism of $\mathcal{A}_\infty$-modules* $\boldsymbol{\eta}\colon \mathcal{F}_* \to \mathcal{G}_*$ where $(\mathcal{F}_*, d_\mathcal{F})$ and $(\mathcal{G}_*, d_\mathcal{G})$ are strict right $C_*(\Omega M)$-modules with action $\mu_\mathcal{F}$ and $\mu_\mathcal{G}$ is given by a collection of maps for $k \geq 1$

$$\eta_k\colon \mathcal{F}_* \otimes C_*(\Omega M)^{\otimes k-1} \to \mathcal{G}_{k-1+*}$$

such that for all $N \geq 0$, it holds:

$$\begin{aligned} &\eta_{N+1} \circ d_{\mathcal{F}\otimes C_*(\Omega)^{\otimes N-1}} + (-1)^{N+1} d_\mathcal{G} \circ \eta_{N+1} \\ &\qquad = (-1)^{N+1}\eta_N(\mu_\mathcal{F} \otimes \mathrm{id}^{\otimes N-1}) - \mu_\mathcal{G}(\eta_N \otimes \mathrm{id}) + \sum_{r=1}^{N-1} (-1)^{N+1+r}\eta_N(\mathrm{id}^{\otimes r} \otimes \mu \otimes \mathrm{id}^{\otimes N-1-r}). \end{aligned}$$

We denote $\widetilde{\boldsymbol{m}}$ for the degree -1 map on $T(C_*(\Omega M)) \otimes \mathbb{Z}\mathrm{Crit}(f) \to T(C_*(\Omega M)) \otimes \mathbb{Z}\mathrm{Crit}(f)$ defined on $C_*(\Omega M)^{\otimes k} \otimes \mathbb{Z}\mathrm{Crit}(f)$ by $\mathrm{id}^{\otimes k} \otimes \boldsymbol{m}$ for $\boldsymbol{m}$ as in (31).

PROPOSITION 2.8 ([Rie24, Proposition 5.1]). *Let $\boldsymbol{\eta}\colon C_*(F_1) \to C_*(F_2)$ be a morphism of $\mathcal{A}_\infty$-modules over $C_*(\Omega M)$. The map $\widetilde{\eta}\colon C_*(M, C_*(F_1)) \to C_*(M, C_*(F_2))$ defined by*

$$\sum_{k\geq 0} (\eta_{k+1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \widetilde{\boldsymbol{m}}^k)$$

is a chain map.

REMARK 2.9. The sum in the definition of $\widetilde{\eta}$ is in fact a finite sum because $\widetilde{\boldsymbol{m}}^{n+1} = 0$.

The goal of this article is to apply Proposition 2.8 to a morphism of $\mathcal{A}_\infty$-modules

$$\boldsymbol{\nu}\colon C_*(\Omega M, x_0) \to C_{1-n+*}((\Omega M, x_0)^2)$$

such that ν_1 is the Goresky-Hingston coproduct on the based loop space.

2.2. The Goresky-Hingston coproduct. In this subsection, we recall the definition of the Goresky-Hingston coproduct following [HW17].

We fix $\varepsilon > 0$ such that 2ε is strictly smaller than the injectivity radius of M. We denote the following tubular neighbourhoods of the diagonal $\Delta \subseteq M \times M$

$$U_\Delta \coloneqq \{(x,y) \in M \times M \mid d_M(x,y) < \varepsilon\}, \qquad V_\Delta \coloneqq \{(x,y) \in M \times M \mid d_M(x,y) < \tfrac{\varepsilon}{2}\}.$$

A *Thom class of the diagonal* $\tau_\Delta \in C^n(U_\Delta, V_\Delta^c)$ is a cochain such that

$$\iota^*\tau_\Delta \cap [M \times M] = \Delta_*[M] \in H_*(M \times M),$$

where ι denotes inclusion $U_\Delta \hookrightarrow M \times M$.

We denote the map

$$\begin{aligned} e_\Lambda\colon I \times \Lambda &\to M \times M, \\ (t, (\gamma\colon [0,a] \to M)) &\mapsto (\gamma(0), \gamma(ta)). \end{aligned}$$

We define the *free figure 8 space* as

$$\mathcal{F}\Lambda \coloneqq \mathcal{F}\Lambda M \coloneqq e_\Lambda^{-1}(\Delta(M)) = \{(t,\gamma) \in I \times \Lambda \mid \gamma(0) = \gamma(ta)\}$$

and define the map

$$R\colon e_\Lambda^{-1}(U_\Delta) \to \mathcal{F}\Lambda$$

sending (t,γ) with $(\gamma(0),\gamma(ta)) = (x,y) \in U_\Delta$ to $(t', \gamma|_{[0,ta]} \cdot \overline{\lambda_{x,y}} \cdot \lambda_{x,y} \cdot \gamma|_{[ta,a]})$, where $\lambda_{x,y}$ is the unit speed, geodesic path from x to y and t' is such that e_Λ evaluates $(t', \gamma|_{[0,ta]} \cdot \overline{\lambda_{x,y}} \cdot \lambda_{x,y} \cdot \gamma|_{[ta,a]})$ at the tip of the two geodesics paths. This map is a homotopy inverse of the inclusion $\mathcal{F}\Lambda \hookrightarrow e_\Lambda^{-1}(U_\Delta)$.

Finally on the figure 8 space, we can define a cut map

$$\begin{aligned} \mathrm{c}\colon \mathcal{F}\Lambda &\to \Lambda \times \Lambda, \\ (t,\gamma) &\mapsto (\gamma|_{[0,ta]}, \gamma|_{[ta,a]}). \end{aligned}$$

Definition 2.10. A chain-level description of the *Goresky-Hingston* ν_Λ coproduct is given by the composition

$$\begin{aligned} C_*(\Lambda, M) &\overset{I\times}{\to} C_{1+*}(I \times \Lambda, \partial I \times \Lambda \cup I \times M) \\ &\to C_{1+*}(I \times \Lambda, \partial I \times \Lambda \cup I \times M \cup e_\Lambda^{-1}(V_\Delta^c)) \\ &\simeq C_{1+*}(e_\Lambda^{-1}(U_\Delta), \partial I \times \Lambda \cup I \times M \cup e_\Lambda^{-1}(V_\Delta^c)) \\ &\overset{e_\Lambda^*\tau_\Delta\cap}{\to} C_{1-n+*}(e_\Lambda^{-1}(U_\Delta), \partial I \times \Lambda \cup I \times M) \\ &\overset{R_*}{\to} C_{1-n+*}(\mathcal{F}\Lambda, \partial I \times \Lambda \cup I \times \Lambda) \\ &\overset{\mathrm{c}_*}{\to} C_{1-n+*}((\Lambda, M)^2). \end{aligned} \tag{32}$$

Remark 2.11. This definition agrees with the definition of the Goresky-Hingston coproduct in [HW17, Definition 1.4] up to the sign $(-1)^k$ on $C_k(\Lambda, M)$. We choose these sign conventions because we have the convention that maps act from the left. In particular, we multiply by the interval I from the left while Hingston and Wahl multiply by I on the right in their definition. Our convention leads to slightly neater signs (see [Cli25a, Section 6.3]).

Moreover, we define it as a map with target $C_{1-n+*}((\Lambda, M)^2)$ rather than $C_{1-n+*}(\Lambda, M)^{\otimes 2}$. These two definitions are equivalent after post-composing our definition with a diagonal approximation. For us, it is however more convenient to consider the above map as the coproduct.

2.2.1. *Based loop space.* The Goresky-Hingston coproduct restricts to a coproduct on $C_*(\Omega, x_0)$. We describe it here explicitly. We denote the following open sets

$$U_{x_0} := \{x \in M \mid d_M(x, x_0) < \varepsilon\}, \qquad V_{x_0} := \{x \in M \mid d_M(x, x_0) < \tfrac{\varepsilon}{2}\}.$$

Because ε is smaller than the injectivity radius, we have

$$\exp_x \colon (T_{x_0}M)^{<\varepsilon} := \{v \in T_{x_0}M \mid \|v\| < \varepsilon\} \cong U_{x_0}.$$

We denote the inclusion

$$\begin{aligned} \iota_{x_0} \colon (U_{x_0}, V_{x_0}) &\to (U_\Delta, V_\Delta), \\ x &\mapsto (x_0, x). \end{aligned}$$

A *Thom class of the basepoint* or simply a local orientation is given by $\tau_{x_0} := \iota_{x_0}^* \tau_\Delta \in C^n(U_{x_0}, V_{x_0}^c)$. We denote the map

$$\begin{aligned} e_t \colon I \times \Omega &\to M, \\ (t, (\gamma \colon [0,a] \to M)) &\mapsto \gamma(ta) \end{aligned}$$

and define the *based figure 8 space* as

$$\mathcal{F}\Omega := \mathcal{F}\Omega M := e_t^{-1}(\{x_0\}) = \{(t, \gamma) \in I \times \Omega \mid \gamma(ta) = x_0\}.$$

A chain model of the Goresky-Hingston coproduct as in (32) can be computed on $C_*(\Omega, x_0)$ by

$$\begin{aligned} C_*(\Omega, x_0) &\xrightarrow{I\times} C_{1+*}(I \times \Omega, \partial I \times \Omega \cup I \times \{x_0\}) \\ &\to C_{1+*}(I \times \Omega, \partial I \times \Omega \cup I \times \{x_0\} \cup e_t^{-1}(V_{x_0}^c)) \\ &\simeq C_{1+*}(e_t^{-1}(U_{x_0}), \partial I \times \Omega \cup I \times \{x_0\} \cup e_t^{-1}(V_{x_0}^c)) \\ &\xrightarrow{e_t^* \tau_{x_0} \cap} C_{1-n+*}(e_t^{-1}(U_{x_0}), \partial I \times \Omega \cup I \times \{x_0\}) \\ &\xrightarrow{R_*} C_{1-n+*}(\mathcal{F}\Omega, \partial I \times \Omega \cup I \times \Omega) \\ &\xrightarrow{c_*} C_{1-n+*}((\Omega, x_0)^2). \end{aligned}$$

2.3. Sign conventions. In this article, we heavily use maps of chain complexes that do not preserve degree. For such maps, the signs are delicate. In this subsection, we specify what convention we use.

Let C_* denote a chain complex with differential d. Then $C_{k+*} := \Sigma^{-k} C_*$ denotes the chain complex C_* with the differential $\Sigma^{-k} d := (-1)^k d$.

Thus a degree-shifting map $f \colon C_* \to D_{|f|+*}$ is a chain map if on $\alpha \in C_*$, we have

$$f(d\alpha) = (-1)^{|f|} df(\alpha).$$

Example 2.12. The chain-level Goresky-Hingston coproduct as in (32) is a chain map with this sign convention. Indeed, in the composition (32) the maps

$$\begin{aligned} C_{1+*}(I \times \Lambda, \partial I \times \Lambda \cup I \times M) &\to C_{1+*}(I \times \Lambda, \partial I \times \Lambda \cup I \times M \cup e_\Lambda^{-1}(V_\Delta^c)) \\ &\simeq C_{1+*}(e_\Lambda^{-1}(U_\Delta), \partial I \times \Lambda \cup I \times M \cup e_\Lambda^{-1}(V_\Delta^c)) \end{aligned}$$

and

$$C_{1-n+*}(e_\Lambda^{-1}(U_\Delta), \partial I \times \Lambda \cup I \times M) \xrightarrow{R_*} C_{1-n+*}(\mathcal{F}\Lambda, \partial I \times \Lambda \cup I \times \Lambda)$$
$$\xrightarrow{c_*} C_{1-n+*}((\Lambda, M)^2)$$

are degree-preserving chain maps.

We compute for $\alpha \in C_*(\Lambda, M)$:

$$d(I \times \alpha) = (dI) \times \alpha - I \times (d\alpha) = 0 - I \times (d\alpha) \tag{33}$$

For $\alpha \in C_{1+*}(e_\Lambda^{-1}(U_\Delta), \partial I \times \Lambda \cup I \times M \cup e_\Lambda^{-1}(V_\Delta^c))$, we compute:[2]

$$d(e_\Lambda^* \tau_\Delta \cap \alpha) = (-1)^n (e_\Lambda^* \tau_\Delta \cap d\alpha - \overbrace{\delta(e_\Lambda^* \tau_\Delta)}^{=0} \cap \alpha) = (-1)^n e_\Lambda^* \tau_\Delta \cap d\alpha. \tag{34}$$

Combining (33) and (34), we find

$$\nu_\Lambda(d\alpha) = (-1)^{1-n} d\nu_\Lambda(\alpha)$$

and thus ν_Λ is a chain map with our sign conventions.

This sign convention has moreover the following consequence for DG coefficients: let F, G be fibres of fibrations over M. We write the differential on $C_*(M, C_*(F))$ and $C_*(M, C_*(G))$ as

$$d_F = d \otimes \mathrm{id} + (\Phi_F \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \boldsymbol{m}),$$
$$d_G = d \otimes \mathrm{id} + (\Phi_G \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \boldsymbol{m}).$$

Let $f\colon C_*(F) \to C_{|f|+*}(G)$ be a chain map which is equivariant under the Ω-action. Then $f \otimes \mathrm{id}$ is a chain map with respect to d_F and $\Sigma^{-|f|} d_G$:

$$\begin{aligned}
(f \otimes \mathrm{id}) \circ d_F &= (f \otimes \mathrm{id}) \circ (d \otimes \mathrm{id}) + (f \otimes \mathrm{id}) \circ (\Phi_F \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \boldsymbol{m}) \\
&= (fd \otimes \mathrm{id}) + (\Phi_G \otimes \mathrm{id}) \circ (f \otimes \boldsymbol{m}) \\
&= (-1)^{|f|} df \otimes \mathrm{id} + (-1)^{|f|} (\Phi_G \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \boldsymbol{m}) \circ (f \otimes \mathrm{id}) \\
&= \Sigma^{-|f|} d_G \circ (f \otimes \mathrm{id}).
\end{aligned}$$

3. Transitive isotopy extensions

In this section, we turn our attention to constructing particularly nice transitive lifting functions on the fibration $e_0\colon \Lambda M \to M$. These transitive lifting functions are constructed such that they send the spaces $e_\Lambda^{-1}(U_\Delta), e_\Lambda^{-1}(V_\Delta^c), \mathcal{F}\Lambda$ to themselves. These are the spaces that appear in a chain-level description of the Goresky-Hingston coproduct as in (32). We construct these better-behaved transitive lifting functions by defining and constructing transitive isotopy extensions.

3.1. Definition.

NOTATION 3.1. For $\lambda\colon [0, a] \to M$ and $t, s \in [0, a]$, we denote by $\Gamma(\lambda)_s^t\colon T_{\lambda(s)}M \to T_{\lambda(t)}M$ the parallel transport along λ with respect to the Levi-Civita connection.

Moreover, we denote $U_x \coloneqq \{\exp_x(v) \mid v \in T_xM, \|v\| < \varepsilon\}$ for all $x \in M$, extending the notation U_{x_0}.

---

[2]The signs in [Bre13, Proposition VI.5.1. (v)] are computed as $d(f \cap c) = \delta(f) \cap c + (-1)^{|f|} f \cap dc$. This is a sign mistake in [Bre13, Page 335, Line -2]. The signs that we use are $d(f \cap c) = (-1)^{|f|}(f \cap dc - \delta(f) \cap c)$.
The signs in [Hat02, Section 3.3] are correct. However, we have the opposite convention that in the cap product the cochain acts from the left on the chain.

Definition 3.2. A *transitive isotopy extension* is a map
$$\Theta\colon \mathcal{P}M \to \mathrm{Diff}(M)$$
such that

(i) for composable $\lambda, \delta \in \mathcal{P}M$, it holds
$$\Theta(\lambda \cdot \delta) = \Theta(\delta) \circ \Theta(\lambda);$$

(ii) for all $x, y \in M$ and $(\lambda\colon [0,a] \to M) \in \mathcal{P}_{x \to y}M$, it holds $\exp_y^{-1} \circ \Theta(\lambda)|_{U_x} \circ \exp_x = \Gamma(\lambda)_0^a$.

We recall that $\exp_x\colon (T_xM)^{<\varepsilon} \cong U_x$ by definition of U_x. The second property thus implies that $\Theta(\lambda)|_{U_x}\colon U_x \cong U_y$ and $\Theta(\lambda)(x) = y$.

Remark 3.3. A transitive isotopy extension is obtained as the endpoint of an isotopy extension. Indeed, Property (ii) means that $\Theta(\lambda)|_{U_x}$ is the endpoint of the isotopy
$$\begin{aligned} [0,a] \times U_x &\to M, \\ (t,z) &\mapsto \exp_{\lambda(t)} \circ \Gamma(\lambda)_0^t \circ \exp_x(z). \end{aligned}$$
This isotopy is extended to all of M by the isotopy
$$\begin{aligned} [0,a] \times M &\to M, \\ (t,z) &\mapsto \Theta(\lambda|_{[0,t]})(z). \end{aligned}$$
Property (i) that Θ works well with composition explains why we call Θ a *transitive* isotopy extension.

Before we prove that such a transitive isotopy extension always exists in Proposition 3.6, we prove that, given such a function, we can construct a particularly nice transitive lifting function.

Definition 3.4. Let $\Theta\colon \mathcal{P}M \to \mathrm{Diff}(M)$ be a transitive isotopy extension. The *associated transitive lifting function* is defined by
$$\begin{aligned} \Phi\colon \Lambda M \times \mathcal{P}M &\to \Lambda M, \\ (\gamma, \lambda) &\mapsto \Theta(\lambda)_*(\gamma). \end{aligned}$$

Lemma 3.5. *Let $\Theta\colon \mathcal{P}M \to \mathrm{Diff}(M)$ be a transitive isotopy extension and $\Phi\colon \Lambda M \times \mathcal{P}M \to \Lambda M$ its associated transitive lifting function. Then the following holds:*

(a) the restriction of Φ to $\Lambda M\ {}_{e_0}\!\times_{e_0} \mathcal{P}M$ and the map
$$\begin{aligned} \Phi\colon (I \times \Lambda M)\ {}_{e_0}\!\times_{e_0} \mathcal{P}M &\to I \times \Lambda M, \\ ((t,\gamma),\lambda) &\mapsto (t, \Phi(\gamma,\lambda)) \end{aligned} \tag{35}$$
are transitive lifting functions;

(b) for $(t, \gamma, \lambda) \in I \times \Lambda M \times \mathcal{P}M \to M$
$$e_t \circ \Phi((t,\gamma),\lambda) = \Theta(\lambda) \circ e_t(t,\gamma);$$

(c) for $\lambda \in \mathcal{P}_{x \to y}M$ and $z \in U_x$, it holds:
$$d_M(x,z) = d_M(y, \Phi(z,\lambda));$$

(d) the map (35) restricts to a transitive lifting function on $e_\Lambda^{-1}(U_\Delta), e_\Lambda^{-1}(V_\Delta^c), \mathcal{F}\Lambda \subseteq I \times \Lambda$.

Proof. We first prove (a) for the fibration $e_0\colon \Lambda M \to M$. Property (i) of Θ implies $\Theta(c_x) = \mathrm{id}$. This in turn ensures that $\Phi(\gamma, c_{\gamma(0)}) = \gamma$. Property (i) also gives that for composable paths $\lambda, \delta \in \mathcal{P}M$, we have
$$\Phi(\gamma, \lambda \cdot \delta) = \Phi(\Phi(\gamma,\lambda),\delta).$$

Finally, the fact that $\Theta(\lambda)$ sends $\lambda(0)$ to $\lambda(a)$ ensures that $e_0 \circ \Phi = e_1 \circ \mathrm{pr}_2$.

It is immediate that for any transitive lifting function $\Phi\colon \Lambda M\ {}_{e_0}\!\times_{e_0} \mathcal{P}M \to \Lambda$ the map $\mathrm{id} \times \Phi\colon (I \times \Lambda M)\ {}_{e_0}\!\times_{e_0} \mathcal{P}M \to I \times \Lambda M$ is a transitive lifting function. This shows (a).

For (b), we compute for $(t,\gamma) \in I \times \Lambda$ and $\lambda \in \mathcal{P}$

$$e_t \circ \Phi((t,\gamma),\lambda) = e_t(t,\Phi(\gamma,\lambda)) = e_t(t,\Theta(\lambda)_*(\gamma)) = \Theta(\lambda)(e_t(t,\gamma)).$$

For (c), we note that $z \in U_x$ implies that $d_M(x,z) < \varepsilon$ is smaller than the injectivity radius. Therefore, there exists a unique smallest vector $v \in T_xM$ with $\exp_x(v) = z$ and $d_M(x,z) = \|v\|$. By Property (ii), we have

$$\Theta(\lambda)(z) = \exp_y(\Gamma(\lambda)_0^a(v)).$$

The parallel transport $\Gamma(\lambda)_0^a$ is an isometry because the Levi-Civita connection is metric. We thus find $\|\Gamma(\lambda)_0^a(v)\| = \|v\| < \varepsilon$ is smaller than the injectivity radius. This shows $d_M(y,\Gamma(\lambda)_0^a(v)) = \|\Gamma(\lambda)_0^a(v)\| = d_M(x,z)$.

For (d), we need to check that, for $((t,\gamma),\lambda) \in (I \times \Lambda M)\ {}_{e_0}\!\times_{e_0} \mathcal{P}M$, it holds

$$\Phi((t,\gamma),\lambda) \in \begin{cases} e_\Lambda^{-1}(U_\Delta) & \text{if } (t,\gamma) \in e_\Lambda^{-1}(U_\Delta), \\ e_\Lambda^{-1}(V_\Delta^c) & \text{if } (t,\gamma) \in e_\Lambda^{-1}(V_\Delta^c), \\ \mathcal{F}\Lambda & \text{if } (t,\gamma) \in \mathcal{F}\Lambda; \end{cases}$$

Let $\lambda \in \mathcal{P}_{x\to y}M$ be a path and $z \in U_x$. If $(t,\gamma) \in e_\Lambda^{-1}(U_\Delta)$, we have $e_t(t,\gamma) \in U_x$. We compute

$$d_M(\Phi(\gamma(0),\lambda), e_t \circ \Phi((t,\gamma),\lambda)) \overset{\text{(b)}}{=} d_M(y,\Theta(\lambda)(e_t(t,\gamma))) \overset{\text{(c)}}{=} d_M(x,e_t(t,\gamma)) = d_M(\gamma(0),e_t(t.\gamma)).$$

All three spaces $e_\Lambda^{-1}(U_\Delta), e_\Lambda^{-1}(V_\Delta^c)$ and $\mathcal{F}\Lambda = e_\Lambda(\Delta(M))$ are characterised by the distance of $e_t(t,\gamma)$ to $\gamma(0)$ which is invariant under $\Phi(-,\lambda)$. This shows (d). □

3.2. Construction. In this subsection, we construct a transitive isotopy extension for M.

As explained in Remark 3.3, we want to extend the isotopy

$$H_\lambda(t,-) \coloneqq \exp_{\lambda(t)} \circ \Gamma(\lambda)_0^t \circ \exp_x^{-1} \tag{36}$$

to an isotopy defined on all of M that works well with concatenation of paths.

We do this via a standard strategy to construct isotopy extension being careful that the constructed isotopy only depends on $\lambda(t)$ and $\dot\lambda(t)$:

(1) we realise $H_\lambda(t,-)$ as the flow of a time-dependent vector field V_λ that is defined for all $(t,z) \in [0,a] \times M$ with $d_M(\lambda(t),z) < \varepsilon$;
(2) we extend the vector field V_λ to a vector field $\widetilde{V_\lambda}$ defined on all of $[0,a] \times M$;
(3) we define $\Theta(\lambda)$ as the flow along $\widetilde{V_\lambda}$.

For $(t,z) \in [0,a] \times U_x$, we denote

$$V_\lambda(t,H_\lambda(t,z)) \coloneqq \left.\frac{d}{ds}\right|_{s=t} H_\lambda(s,z). \tag{37}$$

This is a time-dependent vector field that is defined on all $(t,z') \in [0,a] \times M$ if $d_M(\lambda(t),z') < \varepsilon$. We note that $H_\lambda(t,z)$ is the flow from 0 to t along this vector field.

The formula (36) in fact is well-defined for all $(t,z) \in [0,a] \times B_{2\varepsilon}(x)$ because 2ε is smaller than the injectivity radius. Similarly, the formula (37) is well-defined for all $(t,z) \in [0,a] \times M$ such that $d_M(\lambda(t),z) \le 2\varepsilon$. This induces a time-dependent vector field

$$V_\lambda\colon \{(t,z) \in [0,a] \times M \mid d_M(\lambda(t),z) \le 2\varepsilon\} \to TM.$$

We fix a smooth bump function $\chi\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ such that $\chi|_{[0,\varepsilon]} \equiv 1$ and $\chi_{[2\varepsilon,\infty)} \equiv 0$. We can now define V_λ on all $(t,z) \in [0,a] \times M$:

$$\widetilde{V_\lambda}(t,z) \coloneqq \begin{cases} \chi(d_M(\lambda(t),z)) \cdot V_\lambda(t,z) & \text{if } d_M(\lambda(t),z) \leq 2\varepsilon; \\ 0 & \text{if } d_M(\lambda(t),z) \geq 2\varepsilon. \end{cases}$$

The two cases agree for $d_M(\lambda(t),z) = 2\varepsilon$ because $\chi(2\varepsilon) = 0$.

This vector field is smooth. Therefore, the flow

$$\psi^\lambda_{t_0,t}\colon M \to M$$

is a well-defined diffeomorphism for all $t_0, t \in [0,a]$. We define $\Theta(\lambda) \coloneqq \psi^\lambda_{0,a}$.

For a general piecewise smooth $(\lambda\colon [0,a] \to M) \in \mathcal{P}M$, we fix $0 = a_0 < \cdots < a_k = a$ for minimal $k \geq 0$ such that $\lambda|_{[a_i,a_{i+1}]}$ is smooth. We then define

$$\Theta(\lambda) \coloneqq \Theta(\lambda|_{[a_{k-1},a_k]}) \circ \cdots \circ \Theta(\lambda|_{[a_0,a_1]}). \tag{38}$$

Proposition 3.6. *The map $\Theta\colon \mathcal{P}M \to \mathrm{Diff}(M)$ as constructed above is a transitive isotopy extension.*

Proof. Property (i) states that Θ commutes with composition. That means that for $\lambda_1\colon [0,a_1] \to M$ and $\lambda_2\colon [0,a_2]$ composable paths, we have $\Theta(\lambda_1 \cdot \lambda_2) = \Theta(\lambda_2) \circ \Theta(\lambda_1)$. If the composition is not smooth, then this property holds directly from the definition as in (38). If the composition is smooth, it suffices to show that the vector $\widetilde{V_{\lambda_1\cdot\lambda_2}}(t+a_1,-)$ equals $\widetilde{V_{\lambda_2}}(t,-)$ for $t \in [0,a_2]$. We thus compute

$$\begin{aligned} H_{\lambda_1\cdot\lambda_2}(t+a_1,-) &= \exp_{\lambda_2(t)} \circ \Gamma(\lambda_1\cdot\lambda_2)_0^{a_1+t} \circ \exp^{-1}_{\lambda_1(0)} \\ &= \exp_{\lambda_2(t)} \circ \Gamma(\lambda_2)_0^t \circ \Gamma(\lambda_1)_0^{a_1} \circ \exp^{-1}_{\lambda_1(0)} \\ &= \exp_{\lambda_2(t)} \circ \Gamma(\lambda_2)_0^t \circ \exp^{-1}_{\lambda_2(0)} \circ \exp_{\lambda_1(a_1)} \circ \Gamma(\lambda_1)_0^{a_1} \circ \exp^{-1}_{\lambda_1(0)} \\ &= H_{\lambda_2}(t, H_{\lambda_1}(a_1,-)). \end{aligned}$$

We therefore have

$$\begin{aligned} V_{\lambda_1\cdot\lambda_2}(t+a_1, H_{\lambda_2}(t, H_{\lambda_1}(a_1,z))) &= V_{\lambda_1\cdot\lambda_2}(t+a_1, H_{\lambda_1\cdot\lambda_2}(t+a_1,z)) \\ &= \left.\frac{d}{ds}\right|_{s=t+a_1} H_{\lambda_1\cdot\lambda_2}(s,z) \\ &= \left.\frac{d}{ds}\right|_{s=t} H_{\lambda_2}(s, H_{\lambda_1}(a_1,z)) \end{aligned}$$

which we recognize as formula (37) for $\lambda \coloneqq \lambda_2$ and $z \coloneqq H_{\lambda_1}(a_1,z)$. In particular, we have $V_{\lambda_1\cdot\lambda_2}(t+a_1,-) = V_{\lambda_2}(t,-)$ and thus $\widetilde{V_{\lambda_1\cdot\lambda_2}}(t+a_1,-) = \widetilde{V_{\lambda_2}}(t,-)$. This shows that

$$\Theta(\lambda_1\cdot\lambda_2) = \psi^{\lambda_1\cdot\lambda_2}_{0,a_2+a_1} = \psi^{\lambda_2}_{0,a_2} \circ \psi^{\lambda_2}_{0,a_1} = \Theta(\lambda_2) \circ \Theta(\lambda_1)$$

which is Property (i).

Property (ii) follows from the observation that for $z \in U_x$, we have

$$\widetilde{V_\lambda}(t,z) = V_\lambda(t,z)$$

for all $t \in [0,a]$. Therefore, we find that $\Theta(\lambda)(z) = \psi^\lambda_{0,a}(z) = H_\lambda(a,z)$ for H_λ as in (36) which is what we wanted to show. $\square$

4. Goresky-Hingston coproduct as $\mathcal{A}_\infty$-morphism

In this section, we construct an $\mathcal{A}_\infty$-morphism $\boldsymbol{\nu}\colon C_*(\Omega M, x_0) \to C_{1-n+*}((\Omega M, x_0)^2)$. The map ν_1 will be the Goresky-Hingston coproduct on the based loop space. This lets us define a map $\widetilde{\nu}\colon C_*(M, C_*(\Omega M, x_0)) \to C_*(M, C_{1-n+*}((\Omega M, x_0)^2))$ on Morse homology with DG coefficients. Our construction requires a suitably invariant Thom class that only exists for certain manifolds or coefficients (see Section 5).

We fix the following data and notation. Let A be a ring of coefficients. We also fix a transitive isotopy extension $\Theta\colon \mathcal{P}M \to \mathrm{Diff}(M)$ and its associated transitive lifting functions $\Phi\colon \Lambda M \,{}_{e_0}\!\times_{e_0} \mathcal{P}M \to \Lambda M$ and $\Phi\colon (I \times \Lambda M)\,{}_{e_0}\!\times_{e_0} \mathcal{P}M \to I \times \Lambda M$. By Lemma 3.5, this restricts to a right ΩM-action on ΩM, $I \times \Omega M$, $e_t^{-1}(U_{x_0})$, $e_t^{-1}(V_{x_0}^c)$, $\mathcal{F}\Omega$ and $\mathcal{H} \coloneqq \partial I \times \Omega \cup I \times \{x_0\} \subseteq I \times \Omega$. It also acts on $\Omega M \times \Omega M$ via the diagonal action.

The based loop space ΩM thus plays a double role: it is the monoid acting but also the space that is acted on. To minimize the resulting confusion, we use the convention that we denote a loop λ or μ if it is acting and γ or δ if it is acted on.

In this section, we study the (failure of) commutativity of the following diagram

$$
\begin{tikzcd}
C_*(\Omega, x_0; A) \otimes C_*(\Omega) \arrow[r, "\Phi_*"] \arrow[d, "(I\times)\otimes\mathrm{id}"] \arrow[dr, phantom, "(1)"] & C_*(\Omega, x_0; A) \arrow[d, "I\times"] \\
C_{1+*}(I \times \Omega, \mathcal{H}; A) \otimes C_*(\Omega) \arrow[r, "\Phi_*"] \arrow[d] \arrow[dr, phantom, "(2)"] & C_{1+*}(I \times \Omega, \mathcal{H}; A) \arrow[d] \\
C_{1+*}(I \times \Omega, \mathcal{H} \cup e_t^{-1}(V_{x_0}^c); A) \otimes C_*(\Omega) \arrow[r, "\Phi_*"] \arrow[d, "\simeq"] \arrow[dr, phantom, "(3)"] & C_{1+*}(I \times \Omega, \mathcal{H} \cup e_t^{-1}(V_{x_0}^c); A) \arrow[d, "\simeq"] \\
C_{1+*}(e_t^{-1}(U_{x_0}), \mathcal{H} \cup e_t^{-1}(V_{x_0}^c); A) \otimes C_*(\Omega) \arrow[r, "\Phi_*"] \arrow[d, "e_t^*\tau_{x_0}\cap\otimes\mathrm{id}"] \arrow[dr, phantom, "(4)"] & C_{1+*}(e_t^{-1}(U_{x_0}), \mathcal{H} \cup e_t^{-1}(V_{x_0}^c); A) \arrow[d, "e_t^*\tau_{x_0}\cap"] \\
C_{1-n+*}(e_t^{-1}(U_{x_0}), \mathcal{H}; A) \otimes C_*(\Omega) \arrow[r, "\Phi_*"] \arrow[d, "R_*\otimes\mathrm{id}"] \arrow[dr, phantom, "(5)"] & C_{1-n+*}(e_t^{-1}(U_{x_0}), \mathcal{H}; A) \arrow[d, "R_*"] \\
C_{1-n+*}(\mathcal{F}\Omega, \mathcal{H}; A) \otimes C_*(\Omega) \arrow[r, "\Phi_*"] \arrow[d, "\mathrm{c}\otimes\mathrm{id}"] \arrow[dr, phantom, "(6)"] & C_{1-n+*}(\mathcal{F}\Omega, \mathcal{H}; A) \arrow[d, "\mathrm{c}"] \\
C_{1-n+*}((\Omega, x_0)^2; A) \otimes C_*(\Omega) \arrow[r, "\Phi_*"] & C_{1-n+*}((\Omega, x_0)^2; A).
\end{tikzcd}
$$

(1) This square commutes because the transitive lifting function on $I \times \Omega$ is defined by $\mathrm{id} \times \Phi$.
(2) This square commutes because it is given by a map of pairs of topological spaces with a compatible action.
(3) In Lemma 4.1, we prove that there exists a morphism of $\mathcal{A}_\infty$-modules making this square commute for field coefficients.
(4) This square commutes if there exists a suitably invariant Thom class $\tau_{x_0} \in C^n(U_{x_0}, V_{x_0}^c; A)$.
(5) This square commutes by Lemma 4.2.
(6) This square commutes at the level of spaces.

The only two squares that commute only under certain assumptions are (3) and (4). If the conditions are satisfied, we denote their composition by

$$\boldsymbol{\nu}\colon C_*(\Omega, x_0; A) \to C_{1-n+*}((\Omega, x_0)^2; A). \tag{39}$$

This morphism $\boldsymbol{\nu}$ is such that ν_1 computes the Goresky-Hingston coproduct on the based loop space. We prove in Section 7 that the induced morphism on Morse homology with DG coefficients $\widetilde{\nu}\colon C_*(M, C_*(\Omega M, x_0); A) \to C_{1-n+*}(M, C_*((\Omega M, x_0)^2); A)$ actually models the Goresky-Hingston coproduct on $C_*(\Lambda M, M; A)$.

In this section, we construct the remaining morphism of $\mathcal{A}_\infty$-modules for square (3) and show that squares (4) and (5) commute.

Giving a morphism for (3) is done by the following lemma. The proof uses some general $\mathcal{A}_\infty$-theory. It can be found in Appendix A.

Lemma 4.1. *For coefficients in a field A, there exists a morphism of $\mathcal{A}_\infty$-modules*

$$\boldsymbol{\rho}\colon C_*(I \times \Omega, \mathcal{H} \cup e_t^{-1}(V_{x_0}^c); A) \to C_*(e_t^{-1}(U_{x_0}), \mathcal{H} \cup e_t^{-1}(V_{x_0}^c); A)$$

such that ρ_1 is an inverse to the inclusion $C_*(e_t^{-1}(U_{x_0}), \mathcal{H} \cup e_t^{-1}(V_{x_0}^c); A) \to C_*(I \times \Omega, \mathcal{H} \cup e_t^{-1}(V_{x_0}^c); A)$ on homology.

Proof. The inclusion $C_*(e_t^{-1}(U_{x_0}), \mathcal{H} \cup e_t^{-1}(V_{x_0}^c); A) \to C_*(I \times \Omega, \mathcal{H} \cup e_t^{-1}(V_{x_0}^c); A)$ is a quasi-isomorphism by excision. Moreover, it is $C_*(\Omega)$-equivariant. Therefore, the inclusion defines an ∞-quasi-isomorphism (see Definition A.1). In the appendix in Corollary A.6, we show that over a field we can find for ∞-quasi-isomorphism an ∞-quasi-isomorphism which is the inverse in homology. $\square$

To prove that square (5) commutes, we show the following more general lemma that R is $\Phi(-, \lambda)$-equivariant for all $\lambda \in \mathcal{P}M$. Commutativity of square (5) then follows from the case where $\lambda \in \Omega M$. We prove this more general statement because the proof is not harder and we use the more general statement in Section 7.

Lemma 4.2. *Let $x, y \in M$ and $\lambda \in \mathcal{P}_{x \to y}M$. For $(t, \gamma) \in I \times \Lambda M$ with $\gamma(0) = x$ and $e_t(t, \gamma) \in U_x$, it holds:*

$$\Phi(R(t, \gamma), \lambda) = R(\Phi((t, \gamma), \lambda))).$$

Proof. We fix $(t, \gamma\colon [0, a] \to M) \in I \times \Lambda M$ as in the statement. We denote $z \coloneqq e_t(t, \gamma)$ and $\lambda_{x,z}$ for the unit speed, geodesic path from x to y. In particular, we have the formula $\lambda_{xz}(t) = \exp_x(tv)$ where v is the unit tangent vector with $\exp_x(d_M(x, z) \cdot v) = z$.

We recall from Subsection 2.2 the definition of R that

$$R(t, \gamma) = (t', \gamma|_{[0,ta]} \cdot \overline{\lambda_{zx}} \cdot \lambda_{zx} \cdot \gamma|_{[ta,a]})$$

where $t' \coloneqq t'(t, \gamma)$ is such that the evaluation at t' is at the tip of the two geodesics.

Let $z' \coloneqq e_t(\Phi(t, \gamma), \lambda) \in U_y$. The unit speed, geodesic path $\lambda_{yz'}$ from z' to y coincides with $\Phi(-, \lambda)_*(\lambda_{xz})$. Indeed, we compute

$$\Phi(\lambda_{xz}(t), \lambda) = \exp_y \circ \Gamma(\lambda)_0^a \circ \exp_x^{-1}(\lambda_{xz}(t)) = \exp_y \circ \Gamma(\lambda)_0^a(tv) = \exp_y(t\Gamma(\lambda)_0^a(v)).$$

Because v is a unit tangent vector and parallel transport $\Gamma(\lambda)_0^a$ is an isometry, this indeed defines a unit speed geodesic from y to z'. Because U_y has radius smaller than the injectivity radius, this geodesic is unique and shows $\lambda_{yz'} = \Phi(-, \lambda)_*(\lambda_{xz})$.

To show the claim, we note that $t'(t,\gamma)$ only depends on t and $d_M(x,z)$. Both are invariant under $\Phi(-,\lambda)$ and thus $t'(t,\gamma) = t'(\Phi((t,\gamma),\lambda) =: t'$. We now compute

$$\begin{aligned}\Phi(R(t,\gamma),\lambda) &= \Phi((t',\gamma|_{[0,ta]}\cdot\overline{\lambda_{xz}}\cdot\lambda_{xz}\cdot\gamma|_{[ta,a]}),\lambda)\\ &= (t',\Phi(\gamma|_{[0,ta]},\lambda)\cdot\Phi(\overline{\lambda_{xz}},\lambda)\cdot\Phi(\lambda_{xz},\lambda)\cdot\Phi(\gamma|_{[ta,a]},\lambda))\\ &= (t',\Phi(\gamma,\lambda)|_{[0,ta]}\cdot\lambda_{yz'}\cdot\lambda_{yz'}\cdot\Phi(\gamma,\lambda)|_{[ta,a]})\\ &= R(\Phi((t,\gamma),\lambda)).\end{aligned}$$

□

It thus remains to show that

$$e_t^*\tau_{x_0}\cap\colon C_*(e_t^{-1}(U_{x_0}),\mathcal{H}\cup e_t^{-1}(V_{x_0}^c);A)\to C_{-n+*}(e_t^{-1}(U_{x_0}),\mathcal{H};A) \tag{40}$$

is ΩM-equivariant.

Notation 4.3. We denote the evaluation

$$\begin{aligned}\eta_\Omega\colon U_{x_0}\times\mathrm{Diff}(U_{x_0})&\to U_{x_0},\\ (z,\varphi)&\mapsto\varphi(z).\end{aligned}$$

A transitive isotopy extension defines an ΩM-action on U_{x_0} by

$$\eta_\Omega\circ(\mathrm{id}\times\Theta)\colon U_{x_0}\times\Omega M\to U_{x_0}.$$

The key input to make (40) ΩM-equivariant is a Thom class that is invariant under the above action:

Definition 4.4. Let M be a manifold with a transitive isotopy extension and let A be a ring. A Thom class $\tau_{x_0}\in C^n(U_{x_0},V_{x_0}^c;A)$ is *Θ-invariant* if it holds as maps $C_*(U_{x_0},V_{x_0}^c;A)\otimes C_*(\Omega)\to A$ that

$$\tau_{x_0}\circ\eta_\Omega\circ(\mathrm{id}\times\Theta)\circ\times=\tau_{x_0}\circ\mathrm{pr}_1\circ\times. \tag{41}$$

Remark 4.5. In Section 5, we give conditions on A and M when a Θ-invariant Thom class exists.

Proposition 4.6. *Let A be a ring. Let $\tau_{x_0}\in C^n(U_{x_0},V_{x_0}^c;A)$ be a Θ-invariant Thom class. The cap product $e_t^*\tau_{x_0}\cap$ is ΩM-invariant, that is, on $\alpha\otimes\lambda\in C_*(e_t^{-1}(U_{x_0}),\mathcal{H}\cup e_t^{-1}(x_0)^c;A)\otimes C_*(\Omega;A)$, it holds*

$$e_t^*\tau_{x_0}\cap(\Phi_*(\alpha\times\lambda))=\Phi_*((e_t^*\tau_{x_0}\cap\alpha)\times\lambda)\in C_{-n+*}(e_t^{-1}(U_{x_0}),\mathcal{H};A).$$

Proof. We recall that the map $e_t^*\tau_{x_0}\cap$ on $\alpha\in C_*(e_t^{-1}(U_{x_0}),\mathcal{H}\cup e_t^{-1}(V_{x_0}^c))$ is defined as the composition

$$((\tau_{x_0}\circ e_t)\otimes\mathrm{id})\circ\Delta_*(\alpha).$$

Here τ_{x_0} is interpreted as a degree $-n$ map $C^n(U_{x_0},V_{x_0}^c;A)\to A$ and Δ_* is a diagonal approximation. We fix the *Serre diagonal* as defined in [Ser51] given by

$$\begin{aligned}\Delta_*\colon C_n(X)&\to C_*(X)\otimes C_*(X),\\ \sigma&\mapsto\sum_{J\sqcup K=\{1,\dots,n\}}\mathrm{sgn}(J,K)\sigma|_{I^J\times\{0\}^K}\otimes\sigma|_{\{1\}^J\times I^K}\end{aligned}$$

for $\operatorname{sgn}(J,K) = (-1)^{|\{j>k|j\in J,k\in K\}|}$. We compute for $\alpha\otimes\lambda \in C_a(e_t^{-1}(U_{x_0}),\mathcal{H}\cup e_t^{-1}(x_0)^c;A)\otimes C_l(\Omega;A)$

$$\begin{aligned} e_t^*\tau_{x_0}\cap(\Phi_*(\alpha\times\lambda)) &= ((\tau_{x_0}\circ e_t)\otimes \mathrm{id})\circ\Delta_*(\Phi_*(\alpha\times\lambda)) \\ &= ((\tau_{x_0}\circ e_t\circ\Phi_*)\otimes\Phi_*)\circ\Delta_*(\alpha\times\lambda). \end{aligned}$$

By Lemma 3.5 (b), we find $e_t\circ\Phi = \eta_\Omega\circ(e_t\times\Theta)$. We thus continue

$$\begin{aligned} e_t^*\tau_{x_0}\cap(\Phi_*(\alpha\times\lambda)) &= ((\tau_{x_0}\circ\eta_\Omega\circ(e_t\times\Theta))\otimes\Phi_*)\circ\Delta_*(\alpha\times\lambda) \\ &\overset{(41)}{=} ((\tau_{x_0}\circ\mathrm{pr}_1\circ(e_t\times\Theta))\otimes\Phi_*)\circ\Delta_*(\alpha\times\lambda) \\ &= ((\tau_{x_0}\circ e_t\circ\mathrm{pr}_1)\otimes\Phi_*)\circ\Delta_*(\alpha\times\lambda) \\ &= \sum_{J\sqcup K=\{1,\dots a+l\}} \operatorname{sgn}(J,K)\tau_{x_0}(e_t(\mathrm{pr}_1((\alpha\times\lambda)|_{I^J\times\{0\}^K})))(\Phi(\alpha\otimes\lambda))|_{\{1\}^J\times I^K}. \end{aligned}$$

We note that $\mathrm{pr}_1((\alpha\times\lambda)|_{I^J\times\{0\}^K})$ is a degenerate chain if $J\cap\{a+1,\dots,a+l\}\neq\varnothing$. We thus have

$$\begin{aligned} e_t^*\tau_{x_0}\cap(\Phi_*(\alpha\times\lambda)) &= \sum_{\substack{J\sqcup K=\{1,\dots a+l\}\\ J\subseteq\{1,\dots,a\}}} \operatorname{sgn}(J,K)\tau_{x_0}(e_t(\mathrm{pr}_1((\alpha\times\lambda)|_{I^J\times\{0\}^K})))(\Phi(\alpha\otimes\lambda))|_{\{1\}^J\times I^K} \\ &= \Phi\left(\left(\sum_{J\sqcup K=\{1,\dots a\}} \operatorname{sgn}(J,K)\tau_{x_0}(e_t(\alpha|_{I^J\times\{0\}^K}))\alpha|_{\{1\}^J\times I^K}\right)\otimes\lambda\right) \\ &= \Phi\circ(((\tau_{x_0}\circ e_t)\otimes\mathrm{id})\Delta_*)\otimes\mathrm{id})(\alpha\otimes\lambda) \\ &= \Phi_*((e_t^*\tau_{x_0}\cap\alpha)\times\lambda) \end{aligned}$$

which is what we wanted to show. □

5. Invariant Thom classes

In this section, we construct explicit Thom classes that are equivariant under the action given by a transitive isotopy extension. This is the key input to make the cap product equivariant. We give two alternative constructions. In Subsection 5.1, we give a Thom class for parallelizable manifolds and any coefficient ring. In Subsection 5.2, we construct a Thom class that works for any manifold but requires real coefficients.

In Section 7, we prove that the cap product on Morse homology with DG coefficients actually computes the cap product on the free loop space. For this, we need a Thom class that is not just invariant under the ΩM-action but equivariant under the $\mathcal{P}M$-action. To write the $\mathcal{P}M$-action, we use the notation

$$\begin{aligned} \eta_\Lambda\colon U_{x_0}\times\mathrm{Diff}(M) &\to M\times M, \\ (z,\varphi) &\mapsto (z,\varphi(z)). \end{aligned}$$

In showing that the cap product is ΩM-invariant, we used Θ-invariant Thom classes (Definition 4.4). The analogous property for paths is the following:

Definition 5.1. Let (M,x_0) be a pointed manifold with transitive isotopy extension Θ and let A be a ring. A Thom class $\tau_\Delta\in C^n(U_\Delta,V_\Delta^c;A)$ is Θ-*equivariant* if it holds that

$$\tau_\Delta\circ\eta_\Lambda\circ(\mathrm{id}\times\Theta)\circ\times = \tau_{x_0}\circ\mathrm{pr}_1\circ\times$$

as maps $C_*(U_{x_0},V_{x_0}^c;A)\otimes C_*(\mathcal{P}_{x_0\to M}M;A)\to A$.

Remark 5.2. For a Θ-equivariant Thom class τ_Δ, the restriction $\iota_{x_0}^*\tau_\Delta =: \tau_{x_0}$ gives a Θ-invariant Thom class.

5.1. Parallelizable manifolds. In this subsection, we restrict our attention to parallelizable manifolds. This setting makes transitive isotopy extensions simpler and a Θ-equivariant Thom class can be produced for any coefficient ring A.

Definition 5.3. A manifold M together with a bundle isomorphism $TM \cong \mathbb{R}^n \times M$ is called a *parallelized* manifold.

A parallelized manifold has a canonical identification $T_xM \cong \mathbb{R}^n \cong T_yM$ for all $x, y \in M$. We henceforth drop this identification in the notation. Moreover, it has a canonical Riemannian metric given by the standard inner product on $\mathbb{R}^n$.

Lemma 5.4. *Let M be a parallelized manifold. The transitive isotopy extension $\Theta\colon \mathcal{P}M \to \mathrm{Diff}(M)$ satisfies that*

$$\Theta(\lambda)|_{U_{\lambda(0)}} = \exp_{\lambda(a)} \circ \exp_{\lambda(0)}^{-1}$$

for all $\lambda\colon [0,a] \to M$.

Proof. The parallel transport along $\lambda \in \mathcal{P}_{x\to y}M$ has the following formula:

$$\begin{aligned} \Gamma(\lambda)_0^a\colon T_xM &\to T_yM, \\ (x,v) &\mapsto (y,v). \end{aligned}$$

We recall that by Property (ii) of the transitive isotopy extension, we have

$$\Theta(\lambda)|_{U_x} = \exp_y \circ \Gamma(\lambda)_0^a \circ \exp_x^{-1}$$

which gives the above formula under the fixed identification $T_xM \cong T_yM$. $\square$

The map

$$\begin{aligned} \exp\colon TM &\to M \times M, \\ (x,v) &\mapsto (x, \exp_x(v)) \end{aligned}$$

identifies $(TM)^{<\varepsilon} := \{(x,v) \in TM \mid \|v\| < \varepsilon\}$ with U_Δ. For a parallelized manifold, we thus inherit an identification

$$\alpha\colon U_\Delta \cong B_\varepsilon(0) \times M \xrightarrow{\exp_{x_0} \times \mathrm{id}} U_{x_0} \times M$$

of bundles over M.

Proposition 5.5. *Let M be a parallelized manifold and $\tau_x \in C^n(U_{x_0}, V_{x_0}^c)$ be a Thom class. The formula $\tau_\Delta := \alpha^*(\tau_{x_0} \times 1) \in C^n(U_\Delta, V_\Delta^c)$ defines a Θ-equivariant Thom class.*

Proof. We note that $\tau_{x_0} \times 1$ defines a Thom class in $C^n(M \times U_{x_0}, M \times \{x_0\}^c)$. Because the map α is an isomorphism of bundles that sends $M \times V_{x_0}^c$ to V_Δ^c, the above formula indeed defines a Thom class for U_Δ.

It remains to check that τ_Δ is Θ-equivariant. We recall that we need to check that

$$\tau_\Delta \circ \eta_\Lambda \circ (\mathrm{id} \times \Theta) \circ \times = \tau_{x_0} \circ \mathrm{pr}_1 \circ \times$$

as maps $C_*(U_{x_0}, \{x_0\}^c) \otimes C_*(\mathcal{P}_{x_0\to M}M) \to \mathbb{Z}$.

For any $\sigma \in C_*(U_{x_0}, \{x_0\}^c)$ and $\lambda \in \mathcal{P}_{x_0\to M}M$:

$$\begin{aligned} \tau_\Delta \circ \eta_\Lambda \circ (\mathrm{id} \times \Theta)(\sigma \times \lambda) &= \tau_\Delta(\Theta(\lambda)(\sigma)) = (\tau_{x_0} \times 1)(\alpha(\Theta(\lambda)(\sigma))) \\ &= (\tau_{x_0} \times 1)(\exp^{-1} \circ \exp_{x_0} \circ \Theta(\lambda)(\sigma)). \end{aligned}$$

By Lemma 5.4, we find that $\exp_{x_0} \circ \Theta = \exp_{\lambda(a)}$. This shows $\exp^{-1} \circ \exp_{x_0} \circ \Theta(\lambda)(\sigma) = (\lambda(a), \sigma)$. Evaluating on $\tau_{x_0} \times 1$, we find:

$$\tau_\Delta \circ \eta_\Lambda \circ (\mathrm{id} \times \Theta)(\sigma \times \lambda) = \tau_{x_0}(\sigma) = \tau_{x_0} \circ \mathrm{pr}_1(\sigma \times \lambda).$$

This shows the for $\sigma_1 \otimes \sigma_2 \in C_*(U_{x_0}, \{x_0\}^c) \otimes C_*(\mathcal{P}_{x_0 \to M} M)$ for $|\sigma_2| = 0$. If $|\sigma_2| > 0$, the above computation shows that both sides evaluate to a degenerate chain and thus both sides are zero. $\square$

5.2. Real coefficients. In this subsection we construct an explicit Θ-equivariant Thom class for real coefficients and any manifold. We review the construction of the Mathai-Quillen Thom class in §5.2.1. This gives a Thom class in de Rham cohomology. In §5.2.2, we discuss how to smooth singular chains such that evaluating the Mathai-Quillen Thom class is well-defined and satisfies the desired equivariance.

A Θ-invariant Thom class for a general manifold is given by a Thom class $\tau_{x_0} \in C^n(U_{x_0}, V^c_{x_0})$ that is invariant under the canonical $\mathrm{SO}(n)$-action on U_{x_0}. To produce a Θ-equivariant Thom class, we first isolate the maps that play an analogous role for the path action $U_{x_0} \times \mathcal{P}_{x_0 \to M} M \to U_\Delta$.

Notation 5.6. For $x \in M$, we recall that $U_x = \{y \in M \mid d_M(x, y) < \varepsilon\}$ and we denote

$$\begin{aligned} \iota_x \colon U_x &\hookrightarrow U_\Delta, \\ y &\mapsto (x, y). \end{aligned}$$

The exponential map $\exp_x$ identifies $(T_x M)^{<\varepsilon}$ with U_x and thus the Riemannian metric induces a scalar product on U_x.

Definition 5.7. An embedding $\varphi \colon U_{x_0} \to U_\Delta$ is an *orthogonal half-constant embedding* if it factors as $\varphi = \iota_x \circ \psi$ and $\psi \colon U_{x_0} \to U_x$ commutes with the scalar product on U_{x_0} and U_x. The space of all orthogonal half-constant embeddings is denoted by $\mathrm{SOEmb}(U_{x_0}, U_\Delta)$.

Extending the previous notations, we denote the map

$$\begin{aligned} \eta_\Lambda \colon (U_{x_0}, V^c_{x_0}) \times \mathrm{SOEmb}(U_{x_0}, U_\Delta) &\to (U_\Delta, V^c_\Delta), \\ (x, \varphi) &\mapsto \varphi(x). \end{aligned}$$

The goal of this subsection is to prove the following proposition:

Proposition 5.8. *There exists a Thom class with real coefficients $\tau_\Delta \in C^n(U_\Delta, V^c_\Delta; \mathbb{R})$ such that*

$$\tau_\Delta \circ \eta_\Lambda \circ \times = \tau_\Delta \circ \mathrm{pr}_1 \circ \times$$

holds as maps $\bigoplus_{i+j=n} C_i(U_{x_0}, V_{x_0}; \mathbb{R}) \otimes C_j(\mathrm{SOEmb}(U_{x_0}, U_\Delta); \mathbb{R}) \to \mathbb{R}$.

Remark 5.9. We use de Rham cohomology in the construction of such a Thom class. This is the bottleneck that forces us to work with real coefficients.

In fact if such an invariant Thom class exists, the coefficient ring A must contain $\mathbb{Q}$ as the following proposition shows.

Proposition 5.10. *Let A be a ring and assume that $n > 1$. If $\tau_{x_0} \in C^n(U_{x_0}, V^c_{x_0}; A)$ is a Thom class that satisfies that $\tau_{x_0} \circ \eta_\Lambda \circ \times = \tau_{x_0} \circ \mathrm{pr}_1 \circ \times$ as maps $C_n(U_{x_0}, V^c_{x_0}; A) \otimes C_0(\mathrm{SOEmb}(U_{x_0}, U_\Delta); A) \to A$, then A contains $\mathbb{Q}$.*

Proof. After intersecting U_{x_0} with a plane containing the origin, we can reduce to the statement for $n = 2$. Moreover, we restrict our attention to $\varphi \in \mathrm{SOEmb}(U_{x_0}, U_\Delta)$ that land in U_{x_0}. We can identify such a half-constant orthogonal embedding with an element in $\mathrm{SO}(n)$.

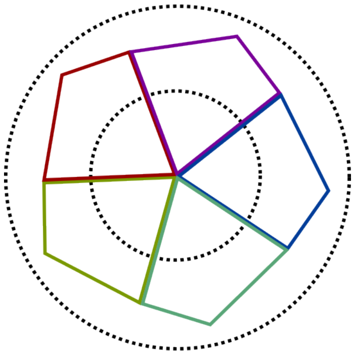

FIGURE 25. A cycle that cuts out a fifth of the V_{x_0}.

In dimension 2, there exists a cycle σ_0 which cuts out $\frac{2\pi}{m}$ of the circle (see Figure 25). Acting on it by $\varphi := \exp(\frac{2\pi i j}{m}) \in \mathrm{SO}(2)$ for $j = 0, \dots, m-1$ closes the circle. Let σ_1 be the chain obtained by gluing together the m rotated copies of σ_0. Then there exists a 3-chain $\widehat{\sigma}$ such that

$$d\widehat{\sigma} = \sigma_1 - \sum_{i=0}^{m-1} \eta_\Lambda(\sigma_0 \times \varphi^i).$$

Moreover, σ_1 is a cycle that generates $H_2(U_{x_0}, V_{x_0}^c; \mathbb{Z})$.

For such $\sigma_0, \sigma_1, \widehat{\sigma}$ and φ, we find

$$1 = \tau_{x_0}(\sigma_1) = \tau_{x_0}(d\widehat{\sigma}) + \sum_{i=0}^{m-1} \tau_{x_0}(\eta_\Lambda(\sigma_0 \times \varphi^i)) = 0 + \sum_{i=0}^{m-1} \tau_{x_0}(\mathrm{pr}_1(\sigma_0 \times \varphi^i)) = m \cdot \tau_{x_0}(\sigma_0).$$

This shows that the element $m \in A^\times$ and thus $\mathbb{Q} \subseteq A$. $\square$

5.2.1. *Mathai-Quillen Thom class.* We recall the construction of the Mathai-Quillen Thom class. This is a de Rham cocycle $\omega_\Delta \in C^n_{\mathrm{dR}}(U_\Delta, V_\Delta^c)$. This class was originally introduced in [MQ86]. We follow the construction as in [Get91, Section 1.6].

We denote $\pi\colon U_\Delta \to M$ the projection to the first coordinate on $U_\Delta \subseteq M \times M$. This is a vector bundle. We denote for $i, j \geq 0$, the following space of sections

$$\mathcal{A}^{i,j} := \Gamma(U_\Delta, \Lambda^i T^* U_\Delta \otimes \Lambda^j \pi^* U_\Delta).$$

The orientation on M defines an orientation on U_Δ. This thus defines a map

$$B\colon \Lambda^n U_\Delta \to M \times \mathbb{R}$$

called the *Berezin integral*. The Berezin integral can be extended to a function

$$\begin{aligned} B\colon \mathcal{A}^{i,j} &\to \Gamma(U_\Delta, \Lambda^i T^* U_\Delta) = C^i_{\mathrm{dR}}(U_\Delta), \\ \chi \otimes \xi &\mapsto \chi \cdot B(\pi_* \xi). \end{aligned}$$

This map is zero if $j \neq n$.

The Riemannian metric defines a metric on the bundle $\pi\colon U_\Delta \to M$. We fix some connection $\nabla\colon \Gamma(M, \Lambda^i T^* M \otimes U_\Delta) \to \Gamma(M, \Lambda^{i+1} T^* M \otimes U_\Delta)$ that is compatible with the metric. The curvature $R := \nabla^2$ can be interpreted as an element in

$$\Gamma(M, \Lambda^2 T^* M \otimes \Lambda^2 U_\Delta).$$

Therefore π^*R can be interpreted as an element in $A^{2,2}$.

The canonical section

$$\begin{aligned}\zeta\colon U_\Delta &\to \pi_* U_\Delta,\\ x &\mapsto (x,x)\end{aligned}$$

defines an element in $\Gamma(U_\Delta, \Lambda^0 T^*U_\Delta \otimes U_\Delta)$. We can thus interpret $(\pi^*\nabla)(\zeta) \in \Gamma(U_\Delta, \Lambda^1 T^*M \otimes U_\Delta) \subseteq \mathcal{A}^{1,1}$. Finally, we interpret $\|\cdot\|\colon U_\Delta \to \mathbb{R}$ as an element in $\mathcal{A}^{0,0}$.

We define $\omega_0 := \pi^*R + (\pi^*\nabla)(\zeta) + \|\cdot\| \in \mathcal{A}^{2,2} \oplus \mathcal{A}^{1,1} \oplus \mathcal{A}^{0,0}$.

For $f\colon \mathbb{R}\to\mathbb{R}$, we define

$$\begin{aligned}f(\omega_0) &\in \mathcal{A}^{*,*},\\ f(\omega_0)(x) &:= \sum_{k=0}^{n} \frac{f^{(k)}(\|x\|)}{k!}((\pi^*\nabla)(\zeta)_{\pi(x)} + \pi^*R_{\pi(x)})^k.\end{aligned}$$

We fix some $f\colon \mathbb{R}\to\mathbb{R}$ smooth such that $f|_{[\frac{\varepsilon}{2},\infty)} \equiv 0$ and $\int_{\mathbb{R}^n} f^{(n)}(\frac{\|x\|}{2})dx = 1$.

Definition 5.11. The *Mathai-Quillen Thom class* is defined as

$$\omega_\Delta := (-1)^{\frac{n(n+1)}{2}} B(f(\omega_0)) \in C^*_{\mathrm{dR}}(U_\Delta, V^c_\Delta).$$

Proposition 5.12 ([Get91, Proposition 1.3]). *The Mathai-Quillen Thom class ω_Δ satisfies*

(i) $\omega_\Delta \in C^n_{\mathrm{dR}}(U_\Delta, V^c_\Delta)$ is a cocycle;
(ii) for any $x\in M$, we have $\int_{B_\varepsilon(x)} \iota_x^*\omega_\Delta = 1$.

Remark 5.13. On a fibre U_x, the class $\iota_x^*\omega_\Delta$ can be described explicitly. We fix some coordinates on $B_\varepsilon(x) \cong B_\varepsilon(0) \subseteq \mathbb{R}^n$ such that the Riemannian metric is given by the standard scalar product on $\mathbb{R}^n$. Then $\iota_x^*\omega_\Delta$ is given by

$$\iota_x^*\omega_\Delta(v) = f^{(n)}(\|v\|)dv_1\cdots dv_n \tag{42}$$

for $v\in\mathbb{R}^n$.

Indeed the term $\pi^*R_{\pi(x)}$ is zero on T^*U_x. Moreover, we have $(\pi^*\nabla)(\zeta) = \sum_{i=1}^n v_i\otimes dv_i$. We therefore have on U_x that

$$\iota_x\omega_0(v) = \sum_{k=0}^{n} \frac{f^{(k)}(\|v\|)}{k!}\left(\sum_{i=1}^n v_i\otimes dv_i\right)^k.$$

The term $(\sum_{i=1}^m v_i\otimes dv_i)^k$ lives in $\mathcal{A}^{k,k}$. Thus the only term that survives after evaluating it under the Berezin integral B is when $k=n$. We thus have

$$\begin{aligned}\iota_x^*\omega_\Delta &= (-1)^{\frac{n(n+1)}{2}}\frac{f^{(n)}(\|v\|)}{n!}B\left(\left(\sum_{i=1}^m v_i\otimes dv_i\right)^n\right)\\ &= (-1)^{\frac{n(n+1)}{2}}\frac{f^{(n)}(\|v\|)}{n!}B\left((-1)^{\frac{n(n+1)}{2}} n! v_1\wedge\cdots\wedge w_n\otimes dv_1\cdots dv_n\right)\\ &= f^{(n)}(\|v\|)dv_1\cdots dv_n.\end{aligned}$$

A consequence of this description is that the Mathai-Quillen Thom class is invariant under orthogonal half-constant maps $\varphi = \iota_x\circ\psi$. Because $D_{x_0}\varphi$ preserves the Riemannian metric, it follows that

$$\iota_{x_0}^*\omega_\Delta = \psi^*\iota_x^*\omega_\Delta = \varphi^*\omega_\Delta. \tag{43}$$

5.2.2. *Smoothing chains.* The problem with using the Mathai-Quillen Thom class is that we can only evaluate ω_Δ on smooth chains in $C_n(U_\Delta, V_\Delta^c;\mathbb{R})$. To solve this issue we smooth chains in $C_n(U_\Delta, V_\Delta^c;\mathbb{R})$. To keep the invariance of the Mathai-Quillen Thom class, we smooth chains in a suitably equivariant way.

We fix the following terminology:

Definition 5.14. A map $\sigma\colon I^k \to X$ is *independent of* $J \subseteq \{1,\dots,k\}$ if the composition

$$I^k \twoheadrightarrow I^{\{1,\dots,k\}\setminus J} \cong I^{\{1,\dots,k\}\setminus J} \times \{0\}^J \xrightarrow{\sigma} X$$

equals σ.

For $1 \leq i \leq n$ and $v \in I$, we denote

$$\lambda_i^v\colon I^{n-1} \to I^n,$$
$$(v_1,\dots,v_{n-1}) \mapsto (v_1,\dots,v_{i-1},v,v_i,\dots,v_n).$$

We use the notation

$$\mu\colon \mathrm{SOEmb}(U_{x_0},U_{x_0}) \times \mathrm{SOEmb}(U_{x_0},U_\Delta) \to \mathrm{SOEmb}(U_{x_0},U_\Delta),$$
$$(\varphi_1,\varphi_2) \mapsto \varphi_2 \circ \varphi_1.$$

We first smooth the chains on $\mathrm{SOEmb}(U_{x_0},U_\Delta)$. This makes sense because $\mathrm{SOEmb}(U_{x_0},U_\Delta)$ is itself a manifold with a canonical smooth structure. Indeed, $\mathrm{SOEmb}(U_{x_0},U_\Delta)$ is a smooth $\mathrm{SO}(n)$ fibre bundle over M.

Lemma 5.15. *For all continuous maps $\sigma\colon I^k \to \mathrm{SOEmb}(U_{x_0},U_\Delta)$ there exists a homotopy*

$$H_\sigma\colon I \times I^k \to \mathrm{SOEmb}(U_{x_0},U_\Delta)$$

such that

(i) $H_\sigma(0,v) = \sigma(v)$;
(ii) $H_\sigma(1,v) =: s(\sigma)(v)$ *is smooth in v;*
(iii) for $k=0$, it holds $H_\sigma(t) \equiv \sigma$;
(iv) the homotopies commute with face maps, that is, for all $i=1,\dots,k$, it holds $H_\sigma \circ (\mathrm{id}\times\lambda_i^0) = H_{\sigma\circ\lambda_i^0}$ and $H_\sigma \circ (\mathrm{id}\times\lambda_i^1) = H_{\sigma\circ\lambda_i^1}$ as maps from $I \times I^{k-1}$;
(v) if $J \subseteq \{1,\dots,k\}$ is such that for $\sigma(v)$ is independent of J, then $H_\sigma(t,x)$ is independent of J for all $t \in I$;
(vi) for $\sigma\colon I^k \to \mathrm{SOEmb}(U_{x_0},U_{x_0}) \subseteq \mathrm{SOEmb}(U_{x_0},U_\Delta)$, it holds $H_\sigma(t,v) \in \mathrm{SOEmb}(U_{x_0},U_{x_0})$ for all $(t,v) \in I \times I^k$;
(vii) if $\sigma \coloneqq \mu\circ(\varphi\times\sigma_0)$ for some $\sigma_0\colon I^k \to \mathrm{SOEmb}(U_{x_0},U_\Delta)$ and $\varphi \in \mathrm{SOEmb}(U_{x_0},U_{x_0})$, we have $H_\sigma = \mu\circ(\varphi\times H_{\sigma_0})$.

Proof. The proof follows the idea of smoothing singular chains on a smooth manifold in a way that commutes with faces. This is a standard result from smoothing theory (see for example [Lee03, Lemma 18.8]).

We prove this statement by induction over k. For $k=0$, we define $H_\sigma(t,v) \coloneqq \sigma(v)$.

Assume we have constructed H_σ as above for all $\sigma\colon I^{k'} \to X$ for $k' < k$. We fix some $\sigma\colon I^k \to X$. If σ is smooth $H_\sigma(t,v) \coloneqq \sigma(v)$ does the trick.

Let $J \subseteq \{1,\dots,k\}$ be the maximal subset such that σ is independent of J. Up to renaming the coordinates we may assume that $J = \{1,\dots,j\}$ for some $j \leq k$.

If $J \neq \varnothing$, we set $\sigma' \coloneqq \sigma|_{\{0\}^J \times I^{\{j+1,\dots,k\}}} : I^{k-j} \to U$. By induction $H_{\sigma'}$ is defined and

$$H_\sigma(t, v_1, \dots, v_k) = H_{\sigma'}(t, v_{j+1}, \dots, v_k)$$

does the trick.

From now on, we can thus assume that σ is not smooth and is not independent of any $\varnothing \neq J \subseteq \{1, \dots, k\}$.

We note that $\mathrm{SOEmb}(U_{x_0}, U_{x_0})$ acts freely (via μ) on all such σ. We can thus fix a system of representatives $\mathcal{X}$ among all such σ.

If $\sigma \in \mathcal{X}$ and $\sigma(I^k) \subseteq \mathrm{SOEmb}(U_{x_0}, U_{x_0})$, we do a construction of H_σ that commutes with face maps as in [Lee03, Lemma 18.8] for the manifold $\mathrm{SOEmb}(U_{x_0}, U_{x_0})$. This ensures (vi).

If $\sigma \in \mathcal{X}$ and $\sigma(I^k) \not\subseteq \mathrm{SOEmb}(U_{x_0}, U_{x_0})$, we do a construction of H_σ that commutes with face maps as in [Lee03, Lemma 18.8] for the manifold $\mathrm{SOEmb}(U_{x_0}, U_\Delta)$.

If $\sigma \notin \mathcal{X}$, there exists a unique $\varphi \in \mathrm{SOEmb}(U_{x_0}, U_{x_0})$ and $\sigma' \in \mathcal{X}$ such that $\sigma = \mu(\varphi \times \sigma')$. We then define $H_\sigma \coloneqq \mu(\varphi \times H_{\sigma'})$. This construction thus ensures (vii). ☐

We now turn our attention to smoothing the chains in $C_*(U_\Delta, V_\Delta^c)$. For this, we need the following definition.

Definition 5.16. A chain $\sigma\colon I^k \to U_\Delta$ has a *cutting* at $0 \leq i < k$ if there exist chains $\sigma_0\colon I^i \to U_{x_0}$ and $\varphi_1\colon I^{k-i} \to \mathrm{SOEmb}(U_{x_0}, U_\Delta)$ with $\sigma = \eta_\Lambda \circ (\sigma_0 \times \varphi_1)$.

The following lemma shows that we can exhibit all cuttings of a chain at once:

Lemma 5.17. *Let $\sigma\colon I^k \to U_\Delta$ be a chain with cuttings at $0 \leq i_1 < \dots < i_l \leq k$. There exist $\sigma_0\colon I^{i_1} \to U_{x_0}$, $\varphi_j\colon I^{i_{j+1}-i_j} \to \mathrm{SOEmb}(U_{x_0}, U_{x_0})$ for $1 \leq j < l$ and $\varphi_l\colon I^{k-i_l} \to \mathrm{SOEmb}(U_{x_0}, U_\Delta)$ with*

$$\sigma = \eta_\Lambda \circ (\sigma_0 \times \mu \circ (\varphi_1 \times \mu(\sigma_2 \times \dots \times \mu(\varphi_{l-1} \times \varphi_l) \dots))).$$

Proof. We prove this by induction on l. The statement is the definition of cutting when $l = 1$. We can thus assume that we have constructed $\sigma_0, \varphi_1, \dots, \varphi_{l-1}$ as in the statement for $l-1$. After composing $\varphi_1, \dots, \varphi_{l-1}$, the induction step reduces to the statement for $l = 2$.

We thus assume that $\sigma = \eta_\Lambda(\sigma_0' \times \varphi_1') = \eta_\Lambda(\sigma_0'' \times \varphi_1'')$.

We can thus compute

$$\eta_\Lambda \circ (\sigma_0'(v_1, \dots, v_{i_1}) \times \varphi_1'(v_{i_1+1}, \dots, v_{i_2}, 0, \dots, 0)) = \eta_\Lambda \circ (\sigma_0''(v_1, \dots, v_{i_2}) \times \varphi_1''(0, \dots, 0)).$$

We note that the maps in $\mathrm{SOEmb}(U_{x_0}, U_\Delta)$ have the property that their images agree if and only if their images intersect. This calculation thus shows $\mathrm{im}(\varphi_1''(0, \dots, 0)) = \mathrm{im}(\varphi_1'(v_{i_1+1}, \dots, v_{i_2}, 0, \dots, 0)$ for all $v_{i_1+1}, \dots, v_{i_2}$. Moreover, we have by injectivity of maps in $\mathrm{SOEmb}(U_{x_0}, U_\Delta)$:

$$\sigma_0''(v_1, \dots, v_{i_2}) = \eta_\Lambda(\sigma_0'(v_1, \dots, v_{i_1}) \times (\varphi_1''(0, \dots, 0)^{-1} \circ \varphi_1'(v_{i_1+1}, \dots, v_{i_2}, 0, \dots, 0))).$$

We therefore find that $\sigma_0 \coloneqq \sigma_0'$, $\varphi_1(v_{i_1+1}, \dots, v_{i_2}) \coloneqq \varphi_1'(v_{i_1+1}, \dots, v_{i_2}, 0, \dots, 0)$ and $\varphi_2 \coloneqq \varphi_1''$ does the trick. ☐

We are now in a position to prove and state how we smooth chains in $C_*(U_\Delta, V_\Delta^c)$:

Lemma 5.18. *For all continuous maps $\sigma\colon I^k \to U_\Delta$ there exists a homotopy*

$$H_\sigma\colon I \times I^k \to U_\Delta$$

such that

(i) $H_\sigma(0, v) = \sigma(v)$;

(ii) $H_\sigma(1, v) =: s(\sigma)(v)$ *is smooth in* v;

(iii) *if σ is smooth $H_\sigma(t,v) = \sigma(v)$ for all $(t,v) \in I \times I^k$;*
(iv) *the homotopies commute with face maps, that is, for all $i = 1,\ldots,k$, it holds $H_\sigma \circ (\mathrm{id} \times \lambda_i^0) = H_{\sigma\circ\lambda_i^0}$ and $H_\sigma \circ (\mathrm{id} \times \lambda_i^1) = H_{\sigma\circ\lambda_i^1}$ as maps from $I \times I^{k-1}$;*
(v) *if σ is independent of $J \subseteq \{1,\ldots,k\}$, then $H_\sigma(t,x)$ is independent of J for all $t \in I$;*
(vi) *if $\sigma(I^k) \subseteq V_\Delta^c$, then $H_\sigma(t,v) \in V_\Delta^c$ for all $(t,v) \in I \times I^k$;*
(vii) *if $\sigma = \eta_\Lambda(\sigma_0 \times \varphi)$ for some $\sigma_0\colon I^k \to U_{x_0}$ and $\varphi \in \mathrm{SOEmb}(U_{x_0}, U_\Delta)$, then $H_\sigma = \eta_\Lambda(H_{\sigma_0} \times \varphi)$;*
(viii) *if σ has a cutting at i, then so does $H_\sigma(t,v)$ for all $t \in I$.*

Proof. The proof follows similar lines as the proof of Lemma 5.15. We do induction on k and can thus assume it is proven for all $k' < k$. Moreover we may assume that $\sigma\colon I^k \to U_\Delta$ is not smooth and not independent for any $\varnothing \neq J \subseteq \{1,\ldots,k\}$.

We identify $\alpha\colon U_{x_0} \cong B_\varepsilon(0)$ via orthonormal coordinates.
We make a case distinction, according to the criteria:

(1) whether or not σ has a cutting;
(2) whether or not $\sigma(I^k) \subseteq U_{x_0}$;
(3) whether or not $\sigma(I^k) \subseteq V_\Delta^c$;
(4) if there exists $\sigma_0\colon I^k \to U_{x_0}$ and $\varphi \in \mathrm{SOEmb}(U_{x_0}, U_\Delta)$ with $\sigma = \eta_\Lambda(\sigma_0 \times \varphi)$.

We first consider σ with no cuttings and $\sigma(I^k) \subseteq U_{x_0}$. In particular, there always exist σ_0 and φ as in (4).

We define $T_\sigma \subseteq \mathbb{R}^n$ the linear subspace spanned by the image of $\alpha \circ \sigma$ and set $W_\sigma \coloneqq \alpha^{-1}(T_\sigma)$. We note that $\mathrm{SOEmb}(U_{x_0}, U_{x_0})$ acts on all such σ. We fix $\mathcal{X}$ a representation system under this action. For such a σ, its stabilizer is all $\varphi \in \mathrm{SOEmb}(U_{x_0}, U_{x_0})$ that are the identity on W_σ.

We now assume that additionally $\sigma(I^k) \subseteq V_\Delta^c$. If $\sigma \in \mathcal{X}$, we construct H_σ as in [Lee03, Lemma 18.8] for the manifold $W_\sigma \cap V_{x_0}^c$. We note that if φ is in the stabilizer of σ, then it also leaves H_σ invariant.

If $\sigma \notin \mathcal{X}$, there exists a $\sigma_0 \in \mathcal{X}$ and $\varphi \in \mathrm{SOEmb}(U_{x_0}, U_{x_0})$ such that $\sigma = \eta_\Lambda(\sigma_0 \times \varphi)$. We define $H_\sigma \coloneqq \mu(H_{\sigma_0} \times \varphi)$. While σ_0 is unique, φ is not. In fact two different φ may exist but their difference is in the stabilizer of σ_0 and thus leaves H_{σ_0} invariant. Therefore our definition of H_σ is independent of our choice of φ and (vii) is satisfied. This concludes with the first case.

We now assume that σ has no cuttings and $\sigma(I^k) \subseteq U_{x_0}$, but not $\sigma(I^k) \subseteq V_\Delta^c$. If $\sigma \in \mathcal{X}$, we do a construction for H_σ as in [Lee03, Lemma 18.8] for the manifold W_σ. If $\sigma \in \mathcal{X}$, we find $\sigma_0 \in \mathcal{X}$ and $\varphi \in \mathrm{SOEmb}(U_{x_0}, U_{x_0})$ such that $\eta_\Lambda(\sigma_0 \times \varphi)$. We define $H_\sigma \coloneqq \mu(H_{\sigma_0} \times \varphi)$. As before this is independent of our choices and ensures (vii).

We now assume that σ has no cuttings, $\sigma(I^k) \not\subseteq U_{x_0}$, $\sigma(I^k) \subseteq V_\Delta^c$ and there exists no σ_0 and φ as in (4). We then construct H_σ as in [Lee03, Lemma 18.8] for the manifold V_Δ^c.

We now assume that σ has no cuttings, $\sigma(I^k) \not\subseteq U_{x_0}$, $\sigma(I^k) \not\subseteq V_\Delta^c$ and there exists no σ_0 and φ as in (4). We then construct H_σ as in [Lee03, Lemma 18.8] for the manifold U_Δ.

We now assume that σ has no cuttings and that we can write $\sigma = \eta_\Lambda(\sigma_0 \times \varphi)$ for some $\sigma_0\colon I^k \to U_{x_0}$ and $\varphi \in \mathrm{SOEmb}(U_{x_0}, U_\Delta)$. We note that if $\sigma(I^k) \subseteq V_\Delta^c$, then also $\sigma_0(I^k) \subseteq V_{x_0}^c$. We define $H_\sigma \coloneqq \mu(H_{\sigma_0} \times \varphi)$. By (vii) for chains with image in U_{x_0}, this definition is independent of our choice of σ_0 and φ.

Finally we assume that σ has cuttings. In particular we assume that it has precisely cuttings at $0 \leq i_1 < \cdots < i_l < k$. We then choose σ_0 and φ_j as in Lemma 5.17 for $1 \leq j \leq l$.

We then define

$$H_\sigma := \eta_\Lambda(H_{\sigma_0} \times \mu(H_{\varphi_1} \times \mu(H_{\varphi_2} \times \cdots \times \mu(H_{\varphi_{l-1}} \times H_{\varphi_l}) \dots))),$$

where H_{φ_j} is as in Lemma 5.15.

This ensures (viii) and the Conditions (i)-(vii) are inherited from σ_0 and φ_i by Lemma 5.15. Moreover equivariance for σ_0 and φ_i ensures that this definition is independent of choice. □

We can now define the following Thom class:

Definition 5.19. The class $\tau_\Delta \in C^n(U_\Delta, V_\Delta^c; \mathbb{R})$ is defined as $s^*\omega_\Delta$ and τ_{x_0} is defined as $\iota_{x_0}^*\tau_\Delta$.

This construction gives a Thom class as described in Proposition 5.8:

Proof of Proposition 5.8. We recall that Proposition 5.8 states that τ_{x_0} is a Thom class and

$$\tau_\Delta \circ \eta_\Lambda \circ \times = \tau_{x_0} \circ \mathrm{pr}_1 \circ \times \tag{44}$$

holds as maps $C_i(U_{x_0}, V_{x_0}; \mathbb{R}) \otimes C_j(\mathrm{SOEmb}(U_{x_0}, U_\Delta); \mathbb{R}) \to \mathbb{R}$.

We first proof that τ_{x_0} is a Thom class. We recall that ω_Δ is a Thom class of smooth chains (see e.g. [BGV03, Proposition 1.48]).

Next, we note that the map $s_*\colon C_*(U_\Delta) \to C_*(U_\Delta)$ is a well-defined map as it sends degenerate chains to degenerate chains by Lemma 5.18 (v). Because smoothing commutes with face maps as in Lemma 5.18 (iv), s_* is a chain map and it lands in the image of all smooth chains. Moreover, the map

$$\begin{aligned} h\colon C_*(U_\Delta) &\to C_{1+*}(U_\Delta), \\ \sigma &\mapsto H_\sigma(I \otimes -) \end{aligned}$$

defines a chain homotopy between s_* and id. On the other hand for $\sigma \in C_*^{\mathrm{sm}}(U_\Delta^c)$, we have $s_*(\sigma) = \sigma$ by Lemma 5.18 (iii), where C_*^{sm} denotes the smooth chains. We therefore have $s\colon C_*(U_\Delta) \simeq C_*^{\mathrm{sm}}(U_\Delta)$.

By Lemma 5.18 (vi), the map s sends $C_*(V_\Delta^c)$ to $C_*^{\mathrm{sm}}(V_\Delta^c)$ and it is also a quasi-isomorphism. Therefore, by the five lemma $s\colon C_*(U_\Delta, V_\Delta^c) \to C_*^{\mathrm{sm}}(U_\Delta, V_\Delta^c)$ is also a quasi-isomorphism. This proves that $\tau_\Delta = s^*\omega_\Delta$ is a Thom class.

It remains to check (44). Because $\mathrm{pr}_1 \circ \times$ is degenerate if $j > 0$, we note that (44) entails two statements:

(a) on $C_i(U_{x_0}, V_{x_0}; \mathbb{R}) \otimes C_j(\mathrm{SOEmb}(U_{x_0}, U_\Delta); \mathbb{R})$ with $j > 0$ and $i + j = n$, we have $\tau_\Delta \circ \eta_\Lambda \circ \times = 0$;
(b) for $\sigma \otimes \varphi \in C_n(U_{x_0}, V_{x_0}; \mathbb{R}) \otimes C_0(\mathrm{SOEmb}(U_{x_0}, U_\Delta); \mathbb{R})$, we have $\tau_\Delta(\eta_\Lambda(\sigma \times \varphi)) = \tau_{x_0}(\sigma)$.

For (a) we fix $\sigma_0 \otimes \varphi_1 \in C_i(U_{x_0}, V_{x_0}; \mathbb{R}) \otimes C_j(\mathrm{SOEmb}(U_{x_0}, U_\Delta); \mathbb{R})$. By Definition 5.16, $\eta_\Lambda(\sigma_0 \times \varphi_1)$ has a cutting at $i < n$. By Lemma 5.18 (viii), $s(\eta_\Lambda(\sigma_0 \times \varphi_1))$ has a cutting at $i < n$. That means that there exist $\sigma_0' \otimes \varphi_1' \in C_i^{\mathrm{sm}}(U_{x_0}, V_{x_0}; \mathbb{R}) \otimes C_j^{\mathrm{sm}}(\mathrm{SOEmb}(U_{x_0}, U_\Delta); \mathbb{R})$ with $s(\eta_\Lambda(\sigma_0 \times \varphi_1)) = \eta_\Lambda(\sigma_0' \times \varphi_1')$. We therefore compute:

$$\tau_\Delta \circ \eta_\Lambda(\sigma_0 \times \varphi_1) = \omega_\Delta \circ s \circ \eta_\Lambda(\sigma_0 \times \varphi_1) = \omega_\Delta \circ \eta_\Lambda(\sigma_0' \times \varphi_1') = \omega_\Delta \circ \mathrm{pr}_1(\sigma_0' \times \varphi_1'),$$

where we used that ω_Δ is invariant in the sense of (43). But φ_1' is not a zero chain, therefore $\mathrm{pr}_1(\sigma_0' \times \sigma_1')$ is degenerate and the above equation evaluates to zero. This shows (a).

For (b) we note that, by Lemma 5.18 (vii), we have $s(\eta_\Lambda(\sigma \times \varphi)) = \eta_\Lambda(s(\sigma) \times \varphi)$. Using again the invariance of ω_Δ as in (43), we compute:

$$\tau_\Delta(\eta_\Lambda(\sigma \times \varphi)) = \omega_\Delta \circ s \circ \eta_\Lambda(\sigma \times \varphi) = \omega_\Delta \circ \eta_\Lambda(s(\sigma) \times \varphi) = \varphi^*\omega_\Delta(s(\sigma)) \overset{(43)}{=} \iota_{x_0}^*\omega_\Delta(s(\sigma)) = \tau_{x_0}(\sigma)$$

which is what we wanted to proof. □

6. Coherent chain homotopies

In this section, we give a proof strategy how one can prove that a map $\widetilde{\eta}\colon C_*(M, C_*(F_1, F_1')) \to C_*(M, C_{m+*}(F_2, F_2'))$ models a certain map $\eta\colon C_*(E_1, E_1') \to C_*(E_2, E_2')$. The main application that we have in mind is that the map constructed in Section 4 $\widetilde{\nu}\colon C_*(M, C_*(\Omega M, x_0; A)) \to C_*(M, C_{1-n+*}((\Omega M, x_0)^2); A)$ actually models the coproduct on the free loop space.

We follow the proof strategy from [Rie24, Theorem 5.8]. Riegel constructs a collection of maps called coherent homotopies ([Rie24, Definition 5.10]) that witness that a map $\widetilde{\varphi}\colon C_*(M, C_*(F_1)) \to C_*(M, C_*(F_2))$ coming from a morphism of fibration $\varphi\colon E_1 \to E_2$ models $\varphi_*\colon C_*(E_1) \to C_*(E_2)$. This is in contrast to our case where we are interested in maps that do not come from maps of fibrations. We thus isolate the algebraic part of his argument: we define in Definition 6.1 coherent chain homotopies as an algebraic version of Riegel's coherent homotopies. In Proposition 6.2, we show that given a coherent chain homotopy, $\widetilde{\eta}$ models η on $C_*(M, C_*(F_1, F_1'))$. Finally in Subsection 6.2, we outline how to invert certain coherent chain homotopies. The main application is for inverting the excision map.

6.1. Definition.

Definition 6.1. For $i = 1, 2$, let $\pi_i\colon E_i \to M$ be two fibrations with fibres F_i and transitive lifting functions Φ_i. Let $\eta_1 \coloneqq \eta\colon C_*(E_1) \to C_{m+*}(E_2)$ be a chain map of degree $m \in \mathbb{Z}$.

A *coherent chain homotopy* $\boldsymbol{\eta}\colon C_*(E_1) \to C_{m+*}(E_2)$ *for* η is a collection of maps

$$\eta_k\colon C_*(F_1) \otimes C_*(\Omega)^{\otimes(k-2)} \otimes C_*(\mathcal{P}_{x_0\to M}M) \to C_{k-1+m+*}(E_2)$$

for $k \geq 2$ that satisfy the $\mathcal{A}_\infty$-equations

(45)

$$\eta_{N+1} \circ d + (-1)^{N+1+m} d \circ \eta_{N+1}$$
$$= (-1)^{N+1}\eta_N \circ (\Phi_1 \otimes \mathrm{id}^{\otimes N-1}) - \Phi_2 \circ (\eta_N \otimes \mathrm{id}) + \sum_{r=1}^{N-1} (-1)^{N+1+r}\eta_N \circ (\mathrm{id}^{\otimes r} \otimes \mu \otimes \mathrm{id}^{\otimes N-1-r})$$

for all $N \geq 1$ and restrict to maps

$$\eta_k\colon C_*(F_1) \otimes C_*(\Omega)^{\otimes k-1} \to C_{k-1+m+*}(F_2).$$

The following proposition makes precise that a coherent chain homotopy is a witness that a map $\widetilde{\eta}\colon C_*(M, C_*(F_1)) \to C_*(M, C_{m+*}(F_2))$ models the map $\eta\colon C_*(E_1) \to C_*(E_2)$.

Proposition 6.2. *If there exists a coherent chain homotopy η_k for $\eta\colon C_*(E_1) \to C_{m+*}(E_2)$, the restrictions*

$$\eta_k\colon C_*(F_1) \otimes C_*(\Omega M)^{\otimes(k-1)} \to C_{k-1+m+*}(F_2)$$

define a morphism of $\mathcal{A}_\infty$-modules and the following diagram commutes

$$\begin{tikzcd} H_*(M, C_*(F_1)) \arrow[r, "\widetilde{\eta}"] \arrow[d, "\Psi_1"] & H_*(M, C_{m+*}(F_2)) \arrow[d, "\Psi_2"] \\ H_*(E_1) \arrow[r, "\eta"] & H_*(E_2). \end{tikzcd}$$

Proof. Because the equation for morphisms of $\mathcal{A}_\infty$-modules in Definition 2.7 is simply a restriction of the $\mathcal{A}_\infty$-equation in Definition 6.1, it is immediate that the restrictions

$$\eta_k\colon C_*(F_1) \otimes C_*(\Omega M)^{k-1} \to C_{k-1+m+*}(F_2)$$

define a morphism of $\mathcal{A}_\infty$-modules. It remains to prove that the induced map $\widetilde{\eta}$ models η, that is, the above diagram commutes.

We recall that the quasi-isomorphisms $\Psi_i\colon C_*(M, C_*(F_i)) \to C_*(E_i)$ are given by compositions

$$C_*(M, C_*(F_i)) \to C_*(M/T, C_*(F_i')) \xrightarrow{\Psi_i'} C_*(E_i') \xrightarrow{\theta_*} C_*(E_i),$$

where E_i' is the pullback of E_i along the homotopy equivalence $\theta\colon M/T \to M$ as in Subsection 2.1. We may identify $F_i \cong F_i'$ and then the first map becomes an isomorphism.

We now consider the following diagram:

$$\begin{tikzcd}
C_*(M, C_*(F_1)) \arrow[r] \arrow[d, "\widetilde{\eta}"] & C_*(M/T, C_*(F_1')) \arrow[r, "\Psi_1'"] \arrow[d, "\theta^*\widetilde{\eta}"] & C_*(E_1') \arrow[r, "\theta_*"] \arrow[d, "\theta^*\eta"] & C_*(E_1) \arrow[d, "\eta"] \\
C_*(M, C_{m+*}(F_2)) \arrow[r] & C_*(M/T, C_{m+*}(F_2')) \arrow[r, "\Psi_2'"] & C_{m+*}(E_2') \arrow[r, "\theta_*"] & C_{m+*}(E_2).
\end{tikzcd}$$

The first square commutes as the horizontal maps are given by isomorphisms and the vertical maps commute with these identifications. The third square commutes by definition of $\theta^*\eta$.

It thus remains to give a chain homotopy making the middle square commute. If we assume that θ is a homeomorphism,[3] we may interpret η_k as maps

$$C_*(F_1') \otimes C_*(\Omega(M/T))^{\otimes(k-2)} \otimes C_*(\mathcal{P}_{x_0 \to M/T}(M/T)) \to C_{k-1+m+*}(E_2')$$

and η_1 as a map $C_*(E_1') \to C_*(E_2')$.

We recall some notation from Subsection 2.1. We have for $x, y \in \mathrm{Crit}(f)$ the chains $m_x \in C_*(\mathcal{P}_{x_0 \to M} M)$ and $m_{x,y} \in C_*(\Omega M)$. Moreover, $\boldsymbol{m}$ denotes the degree -1 map

$$\begin{gathered}\mathbb{Z}\mathrm{Crit}(f) \to C_*(\Omega M \otimes \mathbb{Z}\mathrm{Crit}(f), \\ x \mapsto \sum_y m_{x,y} \otimes y\end{gathered}$$

which we extend to a map

$$\widetilde{\boldsymbol{m}}\colon C_*(F_1') \otimes T(C_*(\Omega M) \otimes \mathbb{Z}\mathrm{Crit}(f) \to C_*(F_1') \otimes T(C_*(\Omega M) \otimes \mathbb{Z}\mathrm{Crit}(f)$$

by $\mathrm{id} \otimes \mathrm{id} \otimes \boldsymbol{m}$.

We also denote

$$\begin{gathered}\widetilde{\boldsymbol{l}}\colon C_*(F_1') \otimes T(C_*(\Omega(M/T))) \otimes \mathbb{Z}\mathrm{Crit}(f) \to C_*(F_1') \otimes T(C_*(\Omega(M/T))) \otimes C_*(\mathcal{P}_{x_0 \to M/T}(M/T)), \\ \alpha \otimes \lambda_1 \otimes \cdots \otimes \lambda_k \otimes x \mapsto \alpha \otimes \lambda_1 \otimes \cdots \otimes \lambda_k \otimes m_x.\end{gathered}$$

Then we define

$$v \coloneqq \sum_{N \geq 1} (-1)^{N+1} \eta_{N+1} \circ \widetilde{\boldsymbol{l}} \circ \widetilde{\boldsymbol{m}}^{N-1}\colon C_*(M, C_*(F_1)) \to C_{m+1+*}(E_2).$$

This is well-defined because $\widetilde{\boldsymbol{m}}^{n+1} = 0$. The proposition now follows from the following lemma:

Lemma 6.3. *It holds*

$$vd + (-1)^m dv = (\theta^*\eta) \circ \Psi_1' - \Psi_2' \circ (\theta^*\widetilde{\eta}).$$

[3] The assumption that $\theta\colon M/T \to M$ makes the argument simpler here. But one has to be careful that it does not contradict our running assumption that $\theta \circ p\colon M \to M$ is smooth. We only give the argument under this assumption here.

Proof of Lemma 6.3. We compute by (45)

$$
\begin{aligned}
(-1)^m dv = & \sum_{N\geq 1}(-1)^{N+1+m} d\eta_{N+1}\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1} \\
= & -\sum_{N\geq 1}\eta_{N+1} d\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1} \\
& + \sum_{N\geq 1}(-1)^{N+1}\eta_N(\Phi_1\otimes \mathrm{id}^{\otimes N-1})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1} \\
& - \sum_{N\geq 1}\Phi_2(\eta_N\otimes \mathrm{id})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1} \\
& + \sum_{N\geq 1}\sum_{r=1}^{N-1}(-1)^{N+1+r}\eta_N(\mathrm{id}^{\otimes r}\otimes\mu\otimes \mathrm{id}^{\otimes N-1-r})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1}.
\end{aligned}
$$

We turn our attention to the first term:

$$
\eta_{N+1} d\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1} = \sum_{r=0}^{N}\eta_{N+1}(\mathrm{id}^{\otimes r}\otimes d\otimes \mathrm{id}^{\otimes N-r})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1}.
$$

Because $\widetilde{\boldsymbol{l}}$ and $\widetilde{\boldsymbol{m}}$ are the identity on all components except the last, we have

$$
\eta_{N+1}(d\otimes \mathrm{id}^{\otimes N})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1} = (-1)^{N-1}\eta_{N+1}\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1}(d\otimes \mathrm{id})
$$

and, for $0 < r < N$,

$$
\begin{aligned}
\eta_{N+1}(\mathrm{id}^{\otimes r}\otimes d\otimes \mathrm{id}^{\otimes N-r})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1} &= (-1)^{N-1-r}\eta_{N+1}\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1-r}(\mathrm{id}^{\otimes r}\otimes d\otimes \mathrm{id})\widetilde{\boldsymbol{m}}^{r} \\
&= (-1)^{N+r}\eta_{N+1}\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1-r}(\mathrm{id}^{\otimes r}\otimes \mu\otimes \mathrm{id})\widetilde{\boldsymbol{m}}^{r+1} \\
&= (-1)^{N+r}\eta_{N+1}(\mathrm{id}^{\otimes r}\otimes\mu\otimes \mathrm{id}^{\otimes N-r})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N},
\end{aligned}
$$

where the second equality follows from the compatibility of the $m_{x,y}$:

$$
d(m_{x,y}) = \sum_z (-1)^{|x|-|z|}\mu(m_{x,z}\otimes m_{z,y})
$$

(see [Rie24, Lemma 4.4] for a detailed treatment of the signs). Similarly, we have $d(m_x) = \sum_y \mu(m_{x,y}\otimes m_y)$ and thus

$$
\eta_{N+1}(\mathrm{id}^{\otimes N}\otimes d)\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1} = \eta_{N+1}(\mathrm{id}^{\otimes N}\otimes\mu)\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N}.
$$

We therefore have:

$$
\begin{aligned}
\sum_{N\geq 1}\eta_{N+1} d\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1} = & \sum_{N\geq 1}\eta_{N+1}\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1}(d\otimes \mathrm{id}) \\
& + \sum_{N\geq 1}\sum_{r=1}^{N-1}(-1)^{N+r}\eta_{N+1}(\mathrm{id}^{\otimes r}\otimes\mu\otimes \mathrm{id}^{\otimes N-r})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N} \\
& + \sum_{N\geq 1}\eta_{N+1}(\mathrm{id}^{\otimes N}\otimes\mu)\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N} \\
= & \; v(d\otimes \mathrm{id}) + \sum_{N\geq 1}\sum_{r=1}^{N}(-1)^{N+r}\eta_{N+1}(\mathrm{id}^{\otimes r}\otimes\mu\otimes \mathrm{id}^{\otimes N-r})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N}.
\end{aligned}
$$

We therefore compute

$$\begin{aligned}(-1)^m dv =& - v(d\otimes \mathrm{id})\\ &- \sum_{N\geq 1}\sum_{r=1}^{N}(-1)^{N+r}\eta_{N+1}(\mathrm{id}^{\otimes r}\otimes\mu\otimes\mathrm{id}^{\otimes N-r})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N}\\ &+\sum_{N\geq 1}(-1)^{N+1}\eta_N(\Phi_1\otimes\mathrm{id}^{\otimes N-1})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1}\\ &-\sum_{N\geq 1}\Phi_2(\eta_N\otimes\mathrm{id})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1}\\ &+\sum_{N\geq 1}\sum_{r=1}^{N-1}(-1)^{N+1+r}\eta_N(\mathrm{id}^{\otimes r}\otimes\mu\otimes\mathrm{id}^{\otimes N-1-r})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1}.\end{aligned}$$

The second and last term cancel. Moreover by the fact that $\widetilde{\boldsymbol{l}}$ and $\widetilde{\boldsymbol{m}}$ are the identity on all components except the last, we find that:

$$\begin{aligned}\sum_{N\geq 1}(-1)^{N+1}\eta_N(\Phi_1\otimes\mathrm{id}^{\otimes N-1})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1} &= \eta_1\Phi_1\widetilde{\boldsymbol{l}}+\sum_{N\geq 2}(-1)^{N+1}\eta_N\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-2}(\Phi_1\otimes\mathrm{id})\widetilde{\boldsymbol{m}}\\ &=(\theta^*\eta_1)\Psi_1'-v(\Phi_1\otimes\mathrm{id})\widetilde{\boldsymbol{m}}\end{aligned}$$

and

$$\begin{aligned}\sum_{N\geq 1}\Phi_2(\eta_N\otimes\mathrm{id})\widetilde{\boldsymbol{l}}\widetilde{\boldsymbol{m}}^{N-1} &= \sum_{N\geq 1}\Phi_2\widetilde{\boldsymbol{l}}(\eta_N\otimes\mathrm{id})\widetilde{\boldsymbol{m}}^{N-1}\\ &=\sum_{N\geq 1}\Psi_2(\eta_N\otimes\mathrm{id})\widetilde{\boldsymbol{m}}^{N-1}\\ &=\Psi_2'(\theta^*\widetilde{\eta}).\end{aligned}$$

We thus have

$$\begin{aligned}(-1)^m dv =& - v(d_{F_1}\otimes\mathrm{id})+(\theta^*\eta_1)\Psi_1'-v(\Phi_1\otimes\mathrm{id})\widetilde{\boldsymbol{m}}-\Psi_2(\theta^*\widetilde{\eta})\\ =&\ -vd+(\theta^*\eta_1)\Psi_1'-\Psi_2'(\theta^*\widetilde{\eta}).\end{aligned}$$

This proves Lemma 6.3 and thus Proposition 6.2 ☐

☐

Corollary 6.4. *For $i=1,2$, let $\pi_i\colon E_i'\subseteq E_i\to M$ be fibrations with fibres F_i and F_i'. Let Φ_i be transitive lifting functions on E_i that restrict to transitive lifting functions on E_i'. Let $\eta\colon C_*(E_1)\to C_{m+*}(E_2)$ be a map of degree $m\in\mathbb{Z}$ that restricts to a map $C_*(E_1')\to C_{m+*}(E_2')$.*

If there exists a coherent chain homotopy $\boldsymbol{\eta}\colon C_*(E_1)\to C_{m+*}(E_2)$ for η that restricts to a coherent chain homotopy $\boldsymbol{\eta}\colon C_*(E_1')\to C_{m+*}(E_2')$, then there exists a $\boldsymbol{\eta}\colon C_*(E_1,E_1')\to C_{m+*}(E_2,E_2')$ a relative coherent chain homotopy and the following square commutes

$$\begin{tikzcd} H_*(M,C_*(F_1,F_1')) \arrow[r, "\widetilde{\eta}"] \arrow[d, "\Psi_1"] & H_*(M,C_{m+*}(F_2,F_2')) \arrow[d, "\Psi_2"] \\ H_*(E_1,E_1') \arrow[r, "\eta"] & H_{m+*}(E_2,E_2'). \end{tikzcd}$$

Proof. This follows from Proposition 6.2 and the five lemma. ☐

6.2. Inverting coherent chain homotopies. In this subsection, we construct inverses to coherent chain homotopies when η_1 is an isomorphism or a quasi-isomorphism. This follows an analogous strategy to Appendix A.

To achieve this, we replace the topological context of ΩM and $\mathcal{P}_{x_0\to M}M$ with an algebra A_* and a left module P_*:

DEFINITION 6.5. Let (A_*, d_A, μ) be a strict differential graded algebra and (P_*, d_P, μ) a strict left A_*-module and $A_* \subseteq P_*$ as A_*-modules. A *path module over the pair* (A_*, P_*) is given by

- a complex $(\mathcal{E}_*, d_{\mathcal{E}})$;
- a subcomplex $\mathcal{F}_* \subseteq \mathcal{E}_*$;
- a collection of maps
$$m_k^{\mathcal{E}} \colon \mathcal{F}_* \otimes A_*^{\otimes k-2} \otimes \mathcal{P}_* \to \mathcal{E}_{k-2+*},$$
for $k \geq 2$;

such that $m_2^{\mathcal{E}}$ is a chain map and for $N \geq 3$, the following $\mathcal{A}_\infty$-equation is satisfied:
$$\begin{aligned}(-1)^{N+1} m_N^{\mathcal{E}} d + d m_N^{\mathcal{E}} &= \sum_{r=0}^{N-3} (-1)^r m_{N-1}^{\mathcal{E}}(\mathrm{id}^{\otimes r+1} \otimes \mu \otimes \mathrm{id}^{\otimes N-r-3}) \\ &\quad - \sum_{s=2}^{N-1} (-1)^{s(N-s)} m_{N-s+1}^{\mathcal{E}}(m_s^{\mathcal{F}} \otimes \mathrm{id}^{\otimes N-s})\end{aligned}$$
as maps $\mathcal{F}_* \otimes A_*^{\otimes N-2} \otimes P_* \to \mathcal{E}_{N-3+*}$ and $m_k^{\mathcal{E}}$ restricts to maps
$$\mathcal{F}_* \otimes A_*^{\otimes k-1} \to \mathcal{F}_{k-2+*}$$
A path module over (A_*, P_*) is *strict* if $m_k^{\mathcal{E}} = 0$ for $k \geq 3$.

REMARK 6.6. In particular, $m_k^{\mathcal{E}}$ makes $\mathcal{F}_*$ into a right $\mathcal{A}_\infty$-module over A_*.

DEFINITION 6.7. Let $(\mathcal{E}_*^1, \mathcal{F}_*^1, m_k^{\mathcal{E}^1})$ and $(\mathcal{E}_*^2, \mathcal{F}_*^2, m_k^{\mathcal{E}^2})$ be path modules over (A_*, P_*). A *morphism of path modules* $\boldsymbol{\eta} \colon \mathcal{E}_*^1 \to \mathcal{E}_{m+*}^2$ *of degree* $m \in \mathbb{Z}$ is given by a chain map $\eta_1 \colon \mathcal{E}_*^1 \to \mathcal{E}_{m+*}^2$ and
$$\eta_k \colon \mathcal{F}_*^1 \otimes A_*^{\otimes(k-2)} \otimes P_* \to \mathcal{E}_{k-1+m+*}^2,$$
for $k \geq 2$, that satisfy the $\mathcal{A}_\infty$-equations for $N \geq 1$:
$$\begin{aligned}\eta_{N+1} \circ d + (-1)^{N+1+m} d \circ \eta_{N+1} &= \sum_{s=1}^{N} (-1)^{s(N-s)} \eta_{N-s+1}(m_{s+1}^{\mathcal{E}_1} \otimes \mathrm{id}^{N-s}) \\ &\quad + (-1)^N \sum_{k=1}^{N} (-1)^{k(N-k)} m_{k+1}^{\mathcal{E}^2}(\eta_{N-k} \otimes \mathrm{id}^{\otimes k}) \\ &\quad + \sum_{r=1}^{N-1} (-1)^{N+1+r} \eta_N(\mathrm{id}^{\otimes r} \otimes \mu \otimes \mathrm{id}^{\otimes N-1-r})\end{aligned}$$
and restrict to maps
$$\mathcal{F}_*^1 \otimes A_*^{k-1} \to \mathcal{F}_{m+k-1+*}^2.$$
The *identity morphism* is given by $\eta_1 = \mathrm{id}$ and $\eta_k = 0$ for $k \geq 2$.

REMARK 6.8. On $\mathcal{F}_*^1$, the map $\boldsymbol{\eta}$ defines a morphism of $\mathcal{A}_\infty$-modules.

EXAMPLE 6.9. Let $\pi\colon E' \subseteq E \to M$ be fibrations with fibre $F' \subseteq F$ and transitive lifting function Φ. The data $\mathcal{E}_* = C_*(E, E')$, $\mathcal{F}_* := C_*(F, F')$ and $m_2^{\mathcal{E}} := \Phi_*$ defines a strict path module over $(C_*(\Omega), C_*(\mathcal{P}_{x_0 \to M}))$.

A coherent chain homotopy defines a morphism of path modules.

LEMMA 6.10. *For $i = 1, 2, 3$, let $(\mathcal{E}^i_*, \mathcal{F}^i_*, m_k^{\mathcal{E}^i})$ be path modules over (A_*, P_*). Let $\boldsymbol{\eta}\colon \mathcal{E}^1_* \to \mathcal{E}^2_{m_1+*}$ and $\boldsymbol{\eta}'\colon \mathcal{E}^2_* \to \mathcal{E}^3_{m_2+*}$ be morphisms of path modules of degree m_1 and m_2, respectively. The maps $(\eta' \circ \eta)_1 := \eta'_1 \circ \eta_1$ and*

$$(\eta' \circ \eta)_k := \sum_{r=1}^{k} (-1)^{(r-1)(k-r)} \eta'_{k-r+1} \circ (\eta_r \otimes \mathrm{id}^{\otimes k-r}),$$

for $k \geq 2$, define a morphism of path modules $\boldsymbol{\eta}' \circ \boldsymbol{\eta}\colon \mathcal{E}^1_* \to \mathcal{E}^3_{m_1+m_2+*}$ of degree $m_1 + m_2$.

PROOF. This is a computation analogous to the proof that the above formulas defines composition for morphisms of $\mathcal{A}_\infty$-modules. □

REMARK 6.11. By Example 6.9, the previous lemma shows that we can compose coherent chain homotopies.

On this level of algebraic abstraction, we can now invert certain morphisms of path modules.

PROPOSITION 6.12. *For $i = 1, 2$, let $(\mathcal{E}^i_*, \mathcal{F}^i_*, m_k^{\mathcal{E}^i})$ be path modules over (A_*, P_*). Let $\boldsymbol{\eta}\colon \mathcal{E}^1_* \to \mathcal{E}^2_{m+*}$ be a morphism of path modules of degree m such that η_1 is an isomorphism and $\eta_1(\mathcal{F}^1_*) = \mathcal{F}^2_*$. The map $\boldsymbol{\zeta}\colon \mathcal{E}^2_* \to \mathcal{E}^1_*$ given by $\zeta_1 := \eta^{-1}$ and*

$$\zeta_{N+1} := -\eta_1^{-1} \circ \left(\sum_{r=0}^{N-1} (-1)^{r(N-r)} \eta_{N-r+1} \circ \left(\zeta_{r+1} \otimes \mathrm{id}^{\otimes N-r} \right) \right),$$

for $k \geq 2$ is an inverse to $\boldsymbol{\eta}$.

PROOF. By our assumptions, this map is well-defined and restricts to a map

$$\mathcal{F}^2_* \otimes A_*^{k-1} \to \mathcal{F}^1_{-m+k-1+*}.$$

It is a computation analogous to Proposition A.2 to check that ζ_N satisfy the relevant $\mathcal{A}_\infty$-equations and that $\boldsymbol{\eta} \circ \boldsymbol{\zeta} = \mathrm{id}$ and $\boldsymbol{\zeta} \circ \boldsymbol{\eta} = \mathrm{id}$. □

The next step in finding an inverse to morphisms of path modules such that η_1 is a proposition akin to the homotopy transfer theorem.

PROPOSITION 6.13. *For $j = 1, 2$, let $\mathcal{F}^j_* \subseteq \mathcal{E}^j_*$ be chain complexes. Let (i, p, h) exhibit $\mathcal{E}^1_*$ as a homotopy retract of $\mathcal{E}^2_*$ (see Definition A.3) such that their restrictions to $\mathcal{F}^i_*$ exhibit $\mathcal{F}^1_*$ as a homotopy retract of $\mathcal{F}^2_*$. If $\mathcal{E}_1$ is a strict path module over (A_*, P_*), then there exists a path module structure $m_k^{\mathcal{E}^2}$ on $\mathcal{E}^2_*$ such that i and p can be extended to morphisms of path modules.*

PROOF. The path module structure and the higher maps are given analogous to Proposition A.4. The proof is also an analogous computation. □

PROPOSITION 6.14. *For $i = 1, 2$, let $(\mathcal{E}^i_*, \mathcal{F}^i_*, m_k^{\mathcal{E}^i})$ be strict path modules over (A_*, P_*). Let $\boldsymbol{\eta}\colon \mathcal{E}^1_* \to \mathcal{E}^2_*$ be a morphism of path modules such that η_1 is a quasi-isomorphism. If $(i^{\mathcal{E}^i}, p^{\mathcal{E}^i}, h^{\mathcal{E}^i})$ exhibits $(\mathcal{E}^i_*, d_{\mathcal{E}^i})$ and $(\mathcal{F}^i_*, d_{\mathcal{E}^i})$ as a homotopy retract of $(H_*(\mathcal{E}^i), 0)$ and $(H_*(\mathcal{F}^i), 0)$, respectively, then there exists a morphism of path modules $\boldsymbol{\zeta}\colon \mathcal{E}^2_* \to \mathcal{E}^1_*$ such that $H_*(\zeta_1) = H_*(\eta_1)^{-1}$.*

PROOF. By Proposition 6.13, we can define the composition $\boldsymbol{\varepsilon}$ as

$$H_*(\mathcal{E}^1) \xrightarrow{\boldsymbol{i}^1} \mathcal{E}^1_* \xrightarrow{\boldsymbol{\eta}} \mathcal{E}^2_* \xrightarrow{\boldsymbol{p}^2} H_*(\mathcal{E}^2).$$

This is such that $\varepsilon_1 = H_*(\eta_1)$ is an isomorphism. By Proposition 6.12, we can invert $\boldsymbol{\varepsilon}$ and can thus define $\boldsymbol{\zeta}$ as the composition

$$\mathcal{E}^2_* \xrightarrow{\boldsymbol{p}^2} H_*(\mathcal{E}^2) \xrightarrow{\boldsymbol{\varepsilon}^{-1}} H_*(\mathcal{E}^1) \xrightarrow{\boldsymbol{i}^1} \mathcal{E}^1_*.$$

□

This lets us prove the following statement that lets us invert the excision map for field coefficients.

COROLLARY 6.15. *Let A be a field. If there exists a coherent chain homotopy $\boldsymbol{\eta}\colon C_*(E_1, E_1'; A) \to C_*(E_2, E_2'; A)$ for a quasi-isomorphism η, then there exists a coherent chain homotopy*

$$\boldsymbol{\zeta}\colon C_*(E_2, E_2'; A) \to C_*(E_1, E_1'; A)$$

for a map ζ where $H_*(\zeta) = H_*(\eta)^{-1}$.

PROOF. By [LV12, Lemma 9.4.4], we can exhibit $H_*(E_i, E_i'; A)$ as a homotopy retract of the chain complex $C_*(E_i, E_i'; A)$ over field coefficients. Then Example 6.9 and Proposition 6.14 conclude the proof. □

7. Main theorems

In this section, we prove the main theorems of this article. In Subsection 7.1, we show that the map $\widetilde{\nu}\colon C_*(M, C_*(\Omega M, x_0)) \to C_*(M, C_{1-n+*}((\Omega M, x_0)^2))$ from Section 4 indeed models the Goresky-Hingston coproduct. In Subsection 7.2, we show that for explicit classes $\alpha \in H_*(\Lambda M, M; A)$, there exist strict formulas for the Goresky-Hingston coproduct.

The bottle neck for both results is the existence of a Θ-equivariant Thom class which exists if M is parallelizable or if $\mathbb{R} \subseteq A$.

7.1. Chain model.

In this subsection, we construct a coherent chain homotopy $\boldsymbol{\nu}$ for the map $\nu\colon C_*(\Lambda M, M; A) \to C_{1-n+*}(\mathcal{F}\Lambda, \mathcal{H}; A)$ such that $\mathrm{c} \circ \nu\colon C_*(\Lambda M, M; A) \to C_{1-n+*}((\Lambda M, M)^2; A)$ computes the Goresky-Hingston coproduct. This gives an explicit chain model for the Goresky-Hingston coproduct on Morse homology with DG coefficients.

We recall that a coherent chain homotopy $\boldsymbol{\eta}\colon C_*(E_1) \to C_{m+*}(E_2)$ is *trivial* if $\eta_k = 0$ for all $k \geq 2$.

We use the notation that A is a ring, $\mathcal{P} \coloneqq \mathcal{P}_{x_0 \to M} M$, $\mathcal{H}\Omega \coloneqq \partial I \times \Omega \cup I \times \{x_0\}$ and $\mathcal{H}\Lambda \coloneqq \partial I \times \Lambda \cup I \times M$. We consider the following diagram:

$$
\begin{tikzcd}
C_*(\Omega, x_0; A) \otimes C_*(\mathcal{P}) \arrow[r, "\Phi_*"] \arrow[d, "(I\times)\otimes\mathrm{id}"] \arrow[dr, phantom, "(1)"] & C_*(\Lambda, M; A) \arrow[d, "I\times"] \\
C_{1+*}(I \times \Omega, \mathcal{H}\Omega; A) \otimes C_*(\mathcal{P}) \arrow[r, "\Phi_*"] \arrow[d] \arrow[dr, phantom, "(2)"] & C_{1+*}(I \times \Lambda, \mathcal{H}\Lambda; A) \arrow[d] \\
C_{1+*}(I \times \Omega, \mathcal{H}\Omega \cup e_t^{-1}(V_{x_0}^c); A) \otimes C_*(\mathcal{P}) \arrow[r, "\Phi_*"] \arrow[d, "\simeq"] \arrow[dr, phantom, "(3)"] & C_{1+*}(I \times \Lambda, \mathcal{H}\Lambda \cup e_\Lambda^{-1}(V_\Delta^c); A) \arrow[d, "\simeq"] \\
C_{1+*}(e_t^{-1}(U_{x_0}), \mathcal{H}\Omega \cup e_t^{-1}(V_{x_0}^c); A) \otimes C_*(\mathcal{P}) \arrow[r, "\Phi_*"] \arrow[d, "e_t^*\tau_{x_0}\cap\otimes\mathrm{id}"] \arrow[dr, phantom, "(4)"] & C_{1+*}(e_\Lambda^{-1}(U_\Delta), \mathcal{H}\Lambda \cup e_\Lambda^{-1}(V_\Delta^c); A) \arrow[d, "e_\Lambda^*\tau_\Lambda\cap"] \\
C_{1-n+*}(e_t^{-1}(U_{x_0}), \mathcal{H}\Omega; A) \otimes C_*(\mathcal{P}) \arrow[r, "\Phi_*"] \arrow[d, "R_*\otimes\mathrm{id}"] \arrow[dr, phantom, "(5)"] & C_{1-n+*}(e_\Lambda^{-1}(U_{x_0}), \mathcal{H}\Lambda; A) \arrow[d, "R_*"] \\
C_{1-n+*}(\mathcal{F}\Omega, \mathcal{H}\Omega; A) \otimes C_*(\mathcal{P}) \arrow[r, "\Phi_*"] \arrow[d, "\mathrm{c}\otimes\mathrm{id}"] \arrow[dr, phantom, "(6)"] & C_{1-n+*}(\mathcal{F}\Lambda, \mathcal{H}\Lambda; A) \arrow[d, "\mathrm{c}"] \\
C_{1-n+*}((\Omega, x_0)^2; A) \otimes C_*(\mathcal{P}) \arrow[r, "\Phi_*"] & C_{1-n+*}((\Lambda, M)^2; A).
\end{tikzcd}
$$

(1) This square commutes strictly thus the trivial coherent chain homotopy does the trick.
(2) This square commutes strictly thus the trivial coherent chain homotopy does the trick.
(3) A coherent chain homotopy exists for this square because if A is a field: by Corollary 6.15, we can invert the trivial coherent chain homotopy for the inclusion

$$C_*(e_\Lambda^{-1}(U_\Delta), \mathcal{H} \cup e_\Lambda^{-1}(V_\Delta^c); A) \xhookrightarrow{\simeq} C_*(I \times \Lambda, \mathcal{H} \cup e_\Lambda^{-1}(V_\Delta^c); A).$$

(4) This square commutes if there exists a Θ-equivariant Thom class (Proposition 7.1).
(5) This square commutes at the level of topological spaces by Lemma 4.2 and thus the trivial coherent chain homotopy does the trick.
(6) This square commutes strictly thus the trivial coherent chain homotopy does the trick.

The two squares that impose certain conditions are (3) and (4).

It remains to prove that the cap product is $\mathcal{P}M$-equivariant. We recall some notation:

$$\begin{aligned} \eta\colon U_{x_0} \times \mathrm{SOEmb}(U_{x_0}, U_\Delta) &\to U_\Delta, \\ (x, \varphi) &\mapsto \varphi(x) \end{aligned}$$

By Property (ii), we have a map

$$\begin{aligned} \Theta\colon \mathcal{P}_{x_0 \to M} M &\to \mathrm{SOEmb}(U_{x_0}, U_\Delta) \subseteq \mathrm{Emb}(U_{x_0}, U_\Delta), \\ \lambda &\mapsto \Theta(\lambda)|_{U_{x_0}}. \end{aligned}$$

Proposition 7.1. *Let A be a ring. Let $\tau_\Delta \in C^n(U_\Delta, V_\Delta^c; A)$ be a Θ-equivariant Thom class. The cap product $e_t^*\tau_{x_0}\cap$ is $\mathcal{P}M$-equivariant, that is, it holds*

$$e_t^*\tau_{x_0} \cap (\Phi_*(\alpha \times \lambda)) = \Phi_*((e_t^*\tau_{x_0} \cap \alpha) \times \lambda) \in C_{-n+*}(e_t^{-1}(U_{x_0}), \mathcal{H}; A)$$

for $\alpha \otimes \lambda \in C_*(e_t^{-1}(U_{x_0}), \mathcal{H} \cup e_t^{-1}(x_0)^c; A) \otimes C_*(\mathcal{P}_{x_0 \to M} M)$.

Proof. This proof follows from an analogous computation as in the proof of Proposition 4.6. □

We can now compose the coherent chain homotopies of the above squares by Remark 6.11:

Corollary 7.2. *The composition of the coherent chain homotopies of the above squares* (1)-(6) *define a coherent chain homotopy*

$$\boldsymbol{\nu}\colon C_*(\Lambda M, M; A) \to C_{1-n+*}(\mathcal{F}\Lambda, \mathcal{H}\Lambda; A)$$

for a map $\nu\colon C_*(\Lambda M, M) \to C_{1-n+*}(\mathcal{F}\Lambda, \mathcal{H}\Lambda)$ *such that* $\mathrm{c}\circ\nu\colon C_*(\Lambda M, M) \to C_{1-n+*}((\Lambda M, M)^2)$ *computes the Goresky-Hingston coproduct.*

Theorem 7.3. *Let A be a field. Let (M, x_0) be a smooth, pointed manifold of dimension n. If M is parallelizable or $\mathbb{R}\subseteq A$, there exists a morphism of $\mathcal{A}_\infty$-modules*

$$\boldsymbol{\nu}\colon C_*(\Omega M, x_0; A) \to C_{1-n+*}((\Omega M, x_0)^2; A)$$

such that ν_1 computes the Goresky-Hingston coproduct on ΩM and the following diagram commutes

$$\begin{tikzcd}
H_*(M, C_*(\Omega M, x_0; A)) \arrow[r, "\Psi"] \arrow[d, "\widetilde{\nu}"] & H_*(\Lambda M, M; A) \arrow[d] \arrow[dd, bend left=60, "\nu_\Lambda"] \\
H_*(M, \Delta_* C_{1-n+*}((\Omega M, x_0; A)^2) \arrow[r, "\Psi"] \arrow[d, "\Delta_*"] & H_{1-n+*}(\mathcal{F}\Lambda, \mathcal{H}; A) \arrow[d, "\mathrm{c}_*"] \\
H_*(M\times M, C_{1-n+*}((\Omega M, x_0; A)^2) \arrow[r, "\Psi"] & H_{1-n+*}((\Lambda M, M; A)^2).
\end{tikzcd}$$

Proof. By Proposition 5.5 for the case where M is parallelizable and by Proposition 5.8 for the case where $\mathbb{R}\subseteq A$, there exists a transitive isotopy extension Θ and a Θ-equivariant Thom class $\tau_\Delta \in C^n(U_\Delta, V_\Delta^c; A)$. The statement then follows from Corollary 6.4 and the fact that the coherent chain homotopy defined in Corollary 7.2 restricts to the morphism of $\mathcal{A}_\infty$-module defined in (39). $\square$

7.2. Strict formula. We show that for "enough" chains $\alpha\otimes x \in C_*(M; C_*(\Omega M, x_0))$ the Goresky-Hingston coproduct is computed as $\vee(\alpha)\otimes x$ where $\vee$ is a chain model for the Goresky-Hingston coproduct on $C_*(\Omega M, x_0)$. This gives a strict formula to compute the Goresky-Hingston coproduct on the free loop space ΛM.

The construction in the previous subsection does not provide such a strict formula because the inversion of the excision map is not strictly ΩM-equivariant. In this subsection, we restrict our operation to small simplices where the excision map is an isomorphism and thus can be inverted equivariantly.

7.2.1. *Slow loops.*

Definition 7.4. For $k\geq 0$, the *k-slow loops* are given by

$$\Omega M_k \coloneqq \left\{\gamma\in\Omega M \mid \text{if } e_t(t,\gamma)\in V_{x_0},\ \text{then } \forall t'\in I \text{ with } |t-t'|<\tfrac{1}{k}\colon d_M(e_t(t',\gamma), x_0) < \tfrac{3\varepsilon}{4}\right\}.$$

Intuitively speaking, ΩM_k contains all loops γ that move slowly enough around times t where $e_t(t,\gamma)$ is close to x_0.

Remark 7.5. We note that

$$\bigcup_{k\in\mathbb{N}} \Omega M_k = \Omega M.$$

Indeed by uniform continuity, for any loop $\gamma\in\Omega M$, there exists a k such that $|t-t'|<\frac{1}{k}$ implies $d_M(e_t(t,\gamma), e_t(t',\gamma)) < \frac{\varepsilon}{4}$. This in turn implies $\gamma\in\Omega M_k$.

Proposition 7.6. *Let M be a manifold with a transitive isotopy extension $\Theta\colon \mathcal{P}M\to \mathrm{Diff}(M)$. The ΩM-action induced by the transitive isotopy extension sends ΩM_k to itself for all k.*

PROOF. We fix $k \in \mathbb{N}$, $\lambda \in \Omega M$ and $\gamma \in \Omega M_k$. We check that $\Theta(\lambda)(\gamma)$ is also in ΩM_k. For any $t \in I$, we have $d_M(e_t(t,\gamma), x_0) \geq \frac{\varepsilon}{2}$ or $d_M(e_t(t,\gamma), e_t(t',\gamma)) < \frac{\varepsilon}{4}$ for all t' with $|t-t'| < \frac{1}{k}$.

We first assume that $d_M(e_t(t,\gamma), x_0) \geq \frac{\varepsilon}{2}$. If $e_t(t,\gamma) \notin U_{x_0}$, then so is

$$e_t(t, \Phi(\gamma,\lambda)) = e_t(t, \Theta(\lambda)(\gamma)) = \Theta(\lambda)(e_t(t,\gamma)).$$

In particular, we have $d_M(e_t(t,\Phi(\gamma,\lambda)), x_0) \geq \varepsilon$. If $e_t(t,\gamma) \in U_{x_0}$ holds, then Lemma 3.5 (c) shows that

$$d_M(e_t(t,\Phi(\gamma,\lambda)), x_0) = d_M(e_t(t,\gamma), x_0) \geq \tfrac{\varepsilon}{2}.$$

We now assume that $d_M(e_t(t,\gamma), x_0) < \frac{\varepsilon}{2}$. This implies that $d_M(x_0, e_t(t',\gamma)) < \frac{3\varepsilon}{4}$ for all t' with $|t-t'| < \frac{1}{k}$. In particular, we have $e_t(t',\gamma) \in U_{x_0}$. Lemma 3.5 (c) implies that the distance to x_0 is preserved by $\Phi(-,\lambda)$. We thus have by Lemma 3.5 (b)

$$d_M(e_t(t', \Phi(\gamma,\lambda)), x_0) = d_M(\Phi(e_t(t',\gamma),\lambda), x_0) = d_M(\Phi(d_M(e_t(t',\gamma), x_0) < \tfrac{3\varepsilon}{4}$$

which is what we wanted to show. □

NOTATION 7.7. For $k \geq 1$, we denote $\mathfrak{U}_k$ as the covering of ΩM_k consisting of all $U \subseteq \Omega M_k$ satisfying the condition that if $\gamma, \delta \in U$ and $t \in I$ is such that $d_M(e_t(t,\gamma), x_0) < \frac{3\varepsilon}{4}$ then it holds $e_t(t,\delta) \in U_{x_0}$.

We denote $C_*^{\mathfrak{U}_k}(\Omega M_k)$ for the subcomplex of all small simplices $\sigma\colon I^i \to \Omega M_k$ that has its image contained in one $U \in \mathfrak{U}_k$.

LEMMA 7.8. *The inclusion $C_*^{\mathfrak{U}_k}(\Omega M_k) \subseteq C_*(\Omega M_k)$ is a quasi-isomorphism.*

PROOF. This is a standard argument using barycentric subdivision (see for example [Hat02, Proposition 2.21]). We only need to check that the interiors of $U \in \mathfrak{U}_k$ cover ΩM_k. This is true because for any $\gamma \in \Omega M_k$ the following open neighbourhood

$$\left\{ \delta \in \Omega M_k \;\middle|\; \sup_{t \in I} d_M(\gamma(t), \delta(t)) < \tfrac{\varepsilon}{8} \right\}$$

is in $\mathfrak{U}_k$. □

LEMMA 7.9. *Let M be a manifold with a transitive isotopy extension $\Theta\colon \mathcal{P}M \to \mathrm{Diff}(M)$. For any $\sigma \in C_*^{\mathfrak{U}_k}(\Omega M_k)$ and $\lambda \in C_*(\Omega M)$, it holds $\Phi_*(\sigma \times \lambda) \in C_*^{\mathfrak{U}_k}(\Omega M_k)$.*

PROOF. We fix $\sigma\colon (I^i \to \Omega M_k) \in C_*^{\mathfrak{U}_k}(\Omega M_k)$ and $(\lambda\colon I^j \to \Omega M) \in C_*(\Omega M)$. It suffices to check that if $t \in I$ and $u \in I^i$ and $v \in I^j$ are such that

$$d_M(e_t(t, \Phi(\sigma(u), \lambda(v))), x_0) < \tfrac{3\varepsilon}{4} \tag{46}$$

then

$$e_t(t, \Phi(\sigma(u'), \lambda(v'))) \in U_{x_0}$$

for all $u' \in I^i$ and $v' \in I^j$.

By (46) and Lemma 3.5 (b), we have that

$$\Theta(\lambda(v))(e_t(t,\sigma(u))) = e_t(t, \Phi(\sigma(u), \lambda(v))) \in U_{x_0}.$$

By Lemma 3.5 (c), $\Phi(-,\lambda(v))$ preserves the distance to x_0 for points in U_{x_0}. This implies that

$$d_M(e_t(t,\sigma(u)), x_0) < \tfrac{3\varepsilon}{4}. \tag{47}$$

By the fact that σ is contained in a $U \in \mathfrak{U}_k$, it follows for all $u' \in I^i$ that

$$e_t(t, \sigma(u')) \in U_{x_0}. \tag{48}$$

By Lemma 3.5 (d), we thus have $e_t(t, \Phi(\sigma(u'), \lambda(v'))) \in U_{x_0}$ which is what we wanted to show. □

Corollary 7.10. *Let M be a manifold with a transitive isotopy extension $\Theta\colon \mathcal{P}M \to \mathrm{Diff}(M)$. The Morse homology with coefficients in $C_*^{\mathfrak{U}_k}(\Omega M_k, x_0)$ is well-defined. Moreover, the inclusion $C_*(M, C_*^{\mathfrak{U}_k}(\Omega M_k, x_0)) \subseteq C_*(M, C_*(\Omega M, x_0))$ is a map of complexes.*

Proof. By Lemma 7.9, the differential on $C_*(M, C_*^{\mathfrak{U}_k}(\Omega M_k, x_0))$ is well-defined. The inclusion $C_*(M, C_*^{\mathfrak{U}_k}(\Omega M_k, x_0)) \subseteq C_*(M, C_*(\Omega M, x_0))$ is a map of complexes because the differential is given by the same transitive lifting function. □

7.2.2. *Small intervals.*

Notation 7.11. We denote $\mathfrak{U}$ the covering of $I \times \Omega M$ by $e_\Lambda^{-1}(U_{x_0})$ and $e_\Lambda^{-1}(V_{x_0}^c)$. The associated chain complex with small simplices is denoted by $C_*^{\mathfrak{U}}(I \times \Lambda M) \coloneqq C_*(e_\Lambda^{-1}(V_{x_0}^c)) + C_*(e_\Lambda^{-1}(U_{x_0}))$.

The relevant excision map is given by the observing that the inclusion $C_*^{\mathfrak{U}}(I \times \Omega M) \to C_*(I \times \Omega M)$ is a quasi-isomorphism and the fact that $C_*^{\mathfrak{U}}(I \times \Omega M)/(C_*(\partial I \times \Lambda I \times M \cup e_\Lambda^{-1}(V_{x_0}^c))) \cong C_*(e^{-1}(U_{x_0}), \partial I \times \Lambda I \times M \cup e_\Lambda^{-1}(V_{x_0}^c))$.

Notation 7.12. For $k \geq 1$, we denote $I^{(k)} \in C_1(I, \partial I)$ the chain given by

$$\sum_{i=0}^{k} [\tfrac{i}{k+1}, \tfrac{i+1}{k+1}],$$

where $[\frac{i}{k+1}, \frac{i+1}{k+1}]$ is considered as a 1-chain via the unique linear, orientation-preserving homeomorphism $I \to [\frac{i}{k+1}, \frac{i+1}{k+1}]$.

We note that $I^{(k)}$ represents the fundamental class in $H_1(I, \partial I)$.

The relevance of this representative is that $I^{(k)} \times$ sends $C_*^{\mathfrak{U}_k}(\Omega M_k)$ to $C_*^{\mathfrak{U}}(I \times \Omega M, \partial I \times \Omega M)$.

Lemma 7.13. *For any $\alpha \in C_*^{\mathfrak{U}_k}(\Omega M_k)$, it holds that $I^{(k)} \times \alpha$ is in $C_*^{\mathfrak{U}}(I \times \Omega M)$.*

Proof. We fix a simplex $\sigma \in C_*^{\mathfrak{U}_k}(\Lambda M^{<\varepsilon k})$. To show the statement it suffices to check that $[\frac{i}{k+1}, \frac{i+1}{k+1}] \times \sigma$ is in $e_\Lambda^{-1}(U_{x_0})$ or $e_\Lambda^{-1}(V_{x_0}^c)$ for all $0 \leq i \leq k$.

We assume that $[\frac{i}{k+1}, \frac{i+1}{k+1}] \times \sigma$ is not in $e_\Lambda^{-1}(V_\Delta^c)$. We thus need to show that $[\frac{i}{k+1}, \frac{i+1}{k+1}] \times \sigma$ is contained in $e^{-1}(U_{x_0})$. By assumption, there exists a $t \in [\frac{i}{k+1}, \frac{i+1}{k+1}]$ and a γ the image of σ such that $e_\Lambda(t, \gamma) \in V_\Delta$. We want to show that for all $t' \in [\frac{i}{k+1}, \frac{i+1}{k+1}]$ and δ in the image of σ, it holds $e_t(t', \delta) \in U_{x_0}$.

We thus fix $t' \in [\frac{i}{k+1}, \frac{i+1}{k+1}]$ and δ in the image of σ. By assumption, we have $|t - t'| < \frac{1}{k}$. By the definition of ΩM_k, we thus have that

$$d_M(x_0, e_t(t', \gamma)) < \tfrac{3\varepsilon}{4}.$$

Because σ is a small simplex, both γ and δ are in the same $U \in \mathfrak{U}_k$. We thus find:

$$e_t(t', \delta) \in U_{x_0},$$

which is what we wanted to show. □

We are now in a position to prove an explicit formula.

Theorem 7.14. *Let A be a ring. Let (M, x_0) be a smooth, pointed manifold of dimension n. Assume that M is parallelizable or $\mathbb{R} \subseteq A$. For any $k \geq 0$, there exists a chain-level model of the based Goresky-Hingston coproduct $\vee_k\colon C_*^{\mathfrak{U}_k}(\Omega M_k, x_0; A) \to C_{1-n+*}((\Omega M, x_0)^2; A)$ such that*

$$C_*(M, C_*^{\mathfrak{U}_k}(\Omega M_k, x_0); A) \to C_*(M, C_{1-n+*}((\Omega M, x_0)^2); A),$$
$$\alpha \otimes x \mapsto \vee_k(\alpha) \otimes x$$

computes the Goresky-Hingston coproduct.

Moreover, for any homology class in $[\alpha] \in H_*(\Lambda M, M; A)$, there exists a k such that $[\alpha]$ can be represented by a cycle in $C_*(M, C_*^{\mathfrak{U}_k}(\Omega M_k, x_0); A)$.

Proof. By assumption, there exists a Θ-equivariant Thom class $\tau_\Delta \in C^n(U_\Delta, V_\Delta^c; A)$. For convenience, we drop A from the notation throughout the proof.

We define $\vee_k$ as the composition

$$\begin{aligned} C_*^{\mathfrak{U}_k}(\Omega M_k, x_0) &\xrightarrow{I^{(k)}} C_{1+*}^{\mathfrak{U}}(I \times \Omega M, \mathcal{H}\Omega) \\ &\to C_{1+*}^{\mathfrak{U}}(I \times \Omega M, \mathcal{H}\Omega)/C_*(e_t^{-1}(V_{x_0}^c)) \\ &\overset{(*)}{\cong} C_{1+*}(e_t^{-1}(U_{x_0}), \mathcal{H}\Omega \cup e_t^{-1}(V_{x_0}^c)) \\ &\xrightarrow{e_t^*\tau_{x_0}\cap} C_{1-n+*}(e_t^{-1}(U_{x_0}), \mathcal{H}) \\ &\xrightarrow{R_*} C_{1-n+*}(\mathcal{F}\Omega, \mathcal{H}\Omega) \\ &\xrightarrow{c_*} C_{1-n+*}((\Omega, x_0)^2). \end{aligned}$$

By Lemma 7.13, the first map is well-defined. Moreover, all maps in the composition are ΩM-equivariant. Indeed, the previous discussion shows that all maps except the excision map $(*)$ are ΩM-equivariant. The excision map in this case is an isomorphism that identifies the two chain complexes. As its inverse is an inclusion and thus ΩM-equivariant, the excision map $(*)$ is also ΩM-equivariant.

This shows that the formula

$$\begin{gathered} C_*(M, C_*^{\mathfrak{U}_k}(\Omega M_k, x_0)) \to C_*(M, C_{1-n+*}((\Omega M, x_0)^2)), \\ \alpha \otimes x \mapsto \vee_k(\alpha) \otimes x \end{gathered}$$

indeed defines a map of chain complexes.

We now want to show that this formula computes the Goresky-Hingston coproduct, that is, we want to show that the following diagram commutes

$$\begin{tikzcd} H_*(M, C_*^{\mathfrak{U}_k}(\Omega M_k, x_0)) \arrow[r] \arrow[d, "\vee_k \otimes \mathrm{id}"] & H_*(M, C_*(\Omega M, x_0)) \arrow[r, "\cong"] & H_*(\Lambda M, M) \arrow[d, "\vee_\Lambda"] \\ H_*(M, C_{1-n+*}((\Omega M, x_0)^2)) \arrow[r] & H_*(M, C_{1-n+*}((\Omega M, x_0)^2)) \arrow[r] & H_{1-n+*}((\Lambda M, M)^2) \end{tikzcd}$$

By Corollary 7.10, the first horizontal map is indeed well-defined.

Both Goresky-Hingston coproducts are given by analogous compositions. We only need to show that the excision map on the left hand side actually models the excision map on the right hand side. The excision map is given on the chain-level by inverting an inclusion which goes in the wrong direction. However in homology, the excision map is invertible thus the fact, that the "wrong-way" diagram

$$\begin{tikzcd} H_*(M, C_*^{\mathfrak{U}}(I \times \Omega M, \mathcal{H}\Omega)/C_*(e_t^{-1}(V_{x_0}^c))) \arrow[r] & H_*(I \times \Lambda M, \mathcal{H}\Lambda \cup e_\Lambda^{-1}(V_\Delta^c)) \\ H_*(M, C_*(e_t^{-1}(U_{x_0}), \mathcal{H}\Omega \cup e_t^{-1}(V_{x_0}^c))) \arrow[r] \arrow[u, "\cong"] & H_*(e_\Lambda^{-1}(U_\Delta), \mathcal{H}\Lambda \cup e_\Lambda^{-1}(V_\Delta^c)) \arrow[u, "\cong"] \end{tikzcd}$$

commutes, shows that the "right-way" diagram commutes.

Finally, we want to show that any homology class in $H_*(\Lambda M, M)$ can be represented by a cycle in $C_*(M, C_*^{\mathfrak{U}_k}(\Omega M_k, x_0))$ for some k. Any homology class in $H_*(\Lambda M, M)$ can be represented by a cycle $\sum_x \alpha_x \otimes x \in C_*(M, C_*(\Omega M, x_0))$ by the relative version of the Fibration Theorem (Corollary 2.6).

By Remark 7.5, any $\alpha \in C_*(\Omega M)$ is contained in $C_*(\Omega M_k)$ for a $k = k(\alpha)$ large enough. We thus can pick k such that all α_x are in $C_*(\Omega M_k)$. We therefore find

$$\sum_x \alpha_x \otimes x \in C_*(M, C_*(\Omega M_k, x_0)).$$

Finally, we note that $C_*(M, C_*^{\mathfrak{U}_k}(\Omega M_k, x_0)) \simeq C_*(M, C_*(\Omega M_k, x_0))$. This can be seen for example by observing that the map on the second page of the associated Leray-Serre spectral sequence

$$H_*(M, H_*(C_*^{\mathfrak{U}_k}(\Omega M_k, x_0))) \to H_*(M, H_*(\Omega M_k, x_0))$$

is an isomorphism by Lemma 7.8. We can thus represent $\sum_x \alpha_x \otimes x$ by some $\sum_x \alpha'_x \otimes x \in C_*(M, C_*^{\mathfrak{U}_k}(\Omega M_k, x_0))$ which is what we wanted to show. $\square$

EXAMPLE 7.15. We consider the example $M = L(p, q)$ for a 3-dimensional lens space. By Stiefel's parallelizability theorem for orientable 3-manifold [Sti35], $L(p, q)$ is parallelizable. We can therefore apply our theorem for $A = \mathbb{Z}$.

We can pick a Morse function on $L(p, q)$ that has four critical points $x_0, \ldots, x_3$ such that x_i has index x_i. Any class in $H_*(\Lambda L(p, q), L(p, q))$ can be represented by a chain

$$\sum_{i=0}^{3} \alpha_i \otimes x_i \in C_*(L(p, q), C_*^{\mathfrak{U}_k}(\Omega L(p, q)_k, x_0))$$

for some k and the coproduct is computed as

$$\sum_{i=0}^{3} \vee_k(\alpha_i) \otimes x_i \in C_*(L(p, q), C_{1-n+*}((\Omega L(p, q), x_0)^2)) \simeq C_{1-n+*}(\mathcal{F}\Lambda L(p, q)).$$

Computing $\vee_k(\alpha_i)$ is often achievable for explicit chains by our work in [Cli25b]. The difficulty comes in finding the explicit chains α_i. This is work in progress.

Appendix

A. Inverting quasi-isomorphisms of $\mathcal{A}_\infty$-Modules

In this section, we prove that, for an ∞-quasi-isomorphism of $\mathcal{A}_\infty$-modules $\boldsymbol{f}\colon \mathcal{M}_* \to \mathcal{N}_*$, we can find a morphism $\boldsymbol{g}\colon \mathcal{N}_* \to \mathcal{M}_*$ where g_1 is an inverse to f_1 on homology. This is a well-known result for $\mathcal{A}_\infty$-algebras (see [Lef03] and [LV12, Theorem 10.4.4] for algebras over any Koszul operad). However, the statement for $\mathcal{A}_\infty$-modules, while known to experts, does not have a proof in the literature to our knowledge.

DEFINITION A.1. Let (A_*, d) be a (strict) differential graded algebra. A morphism of $\mathcal{A}_\infty$-modules $\boldsymbol{f}\colon \mathcal{M}_* \to \mathcal{N}_*$ given by $f_k\colon \mathcal{M}_* \otimes A_*^{\otimes k-1} \to \mathcal{N}_{k-1+*}$ is an *∞-isomorphism* if f_1 is an isomorphism and an *∞-quasi-isomorphism* if f_1 is a quasi-isomorphism.

The goal of this section is to find for an ∞-quasi-isomorphism $\boldsymbol{f}\colon \mathcal{M}_* \to \mathcal{N}_*$ a map $\boldsymbol{g}\colon \mathcal{N}_* \to \mathcal{M}_*$ such that $H_*(g_1)$ is an inverse to $H_*(f_1)$. Our strategy is a classical strategy of $\mathcal{A}_\infty$-morphisms as for example in [LV12, Theorem 10.4.4]:

(1) invert ∞-isomorphisms;
(2) prove a version of the Homotopy Transfer Theorem for $\mathcal{A}_\infty$-modules.

A.1. Inverting ∞-isomorphisms.

PROPOSITION A.2. *Let (A_*, d) be a (strict) differential graded algebra. Let $\boldsymbol{f}\colon \mathcal{M}_* \to \mathcal{N}_*$ be an ∞-isomorphism of A-modules given by*

$$f_k\colon \mathcal{M}_* \otimes A_*^{\otimes k-1} \to \mathcal{N}_{k-1+*}.$$

The map $\boldsymbol{g}\colon \mathcal{N}_* \to \mathcal{M}_*$ given by $g_1 \coloneqq f_1^{-1}$ and for $k \geq 2$:

$$g_{N+1} \coloneqq -f_1^{-1} \circ \left(\sum_{r=0}^{N-1} (-1)^{r(N-r)} f_{N-r+1} \circ \left(g_{r+1} \otimes \mathrm{id}^{\otimes N-r} \right) \right)$$

defines an inverse to $\boldsymbol{f}$. In other words, it holds $\boldsymbol{g} \circ \boldsymbol{f} = \mathrm{id}_{\mathcal{M}_*}$ and $\boldsymbol{f} \circ \boldsymbol{g} = \mathrm{id}_{\mathcal{N}_*}$.

PROOF. We have to check that the above formulas define a morphism of $\mathcal{A}_\infty$-modules and that $\boldsymbol{g} \circ \boldsymbol{f} = \mathrm{id}_{\mathcal{M}_*}$ and $\boldsymbol{f} \circ \boldsymbol{g} = \mathrm{id}_{\mathcal{N}_*}$ holds. We denote $m_k^{\mathcal{M}}$ and $m_k^{\mathcal{N}}$ for the $\mathcal{A}_\infty$-module structure on $\mathcal{M}_*$ and $\mathcal{N}_*$, respectively.

We first check that $\boldsymbol{g} \circ \boldsymbol{f} = \mathrm{id}_{\mathcal{M}_*}$ and $\boldsymbol{f} \circ \boldsymbol{g} = \mathrm{id}_{\mathcal{N}_*}$. We recall that composition on morphisms of $\mathcal{A}_\infty$-modules is defined via

$$(\boldsymbol{\eta} \circ \boldsymbol{\zeta})_{N+1} = \sum_{k=0}^{N} (-1)^{k(N-k)} \eta_{k+1}(\zeta_{N-k+1} \otimes \mathrm{id}^{\otimes k}).$$

We have $g_1 \circ f_1 = \mathrm{id}$ and $f_1 \circ g_1 = \mathrm{id}$ and for $N \geq 1$

$$(\boldsymbol{f} \circ \boldsymbol{g})_{N+1} = f_1 \circ g_{N+1} + \sum_{r=0}^{N-1} (-1)^{r(N-r)} f_{N-r+1} \circ \left(g_{r+1} \otimes \mathrm{id}^{\otimes N-r} \right) = 0$$

by construction. To show that $(\boldsymbol{g} \circ \boldsymbol{f})_{N+1} = 0$, we define for $k_1, \ldots, k_l \geq 1$:

$$\varphi_{k_1,\ldots,k_l} \coloneqq f_1^{-1} f_{k_1+1}(f_1^{-1} f_{k_2+1} \otimes \mathrm{id}^{\otimes k_1}) \ldots (f_1^{-1} f_{k_l+1} \otimes \mathrm{id}^{\otimes k_1+\cdots+k_{l-1}})(f_1^{-1} \otimes \mathrm{id}^{\otimes k_1+\cdots+k_l})$$

as in Figure 26.

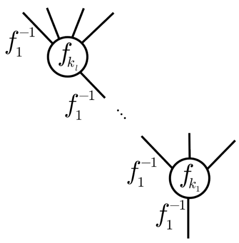

FIGURE 26. The map $\varphi_{k_1,\ldots,k_l}$.

We then find, by induction on N, the formula

$$\begin{aligned} g_{N+1} &= \sum_{k_1+\cdots+k_l=N} (-1)^{l+\sum_{i<j} k_i k_j} \varphi_{k_1,\ldots,k_l} \\ &= \sum_{k_l=1}^{N} (-1)^{k_l(N-k_l)} \sum_{k_1+\cdots+k_{l-1}=N-k_l} (-1)^{l+\sum_{i<j<l} k_i k_j} \varphi_{k_1,\ldots,k_l} \\ &= \left(\sum_{k_l=1}^{N} (-1)^{k_l(N-k_l)} g_{N-k_l+1}(f_{k_l+1} \otimes \mathrm{id}^{\otimes N-k_l}) \right) (f_1^{-1} \otimes \mathrm{id}^{\otimes N}). \end{aligned}$$

From this, it follows that $(\boldsymbol{g} \circ \boldsymbol{f})_{N+1} = 0$.

We now check the $\mathcal{A}_\infty$-equation which reads for $N \geq 0$ as:

$$
\begin{aligned}
dg_{N+1} = & -\sum_{r=1}^{N} (-1)^{(N-r)r} m_{r+1}^{\mathcal{M}} (g_{N-r+1} \otimes \mathrm{id}^{\otimes r}) \\
& + \sum_{r=1}^{N} (-1)^{(N-r)(r+1)} g_{N-r+1} (m_{r+1}^{\mathcal{N}} \otimes \mathrm{id}^{\otimes N-r}) \\
& + (-1)^N g_{N+1} d \\
& + \sum_{r=1}^{N-1} (-1)^r g_N (\mathrm{id}^{\otimes r} \otimes \mu \otimes \mathrm{id}^{\otimes N-1-r}).
\end{aligned}
\tag{49}
$$

We prove it by induction on N. For $N = 0$, the statement is that $g_1 = f_1^{-1}$ is a chain map. This is true because f_1 is a chain map.

Because $\boldsymbol{f}$ is a morphism of $\mathcal{A}_\infty$-modules, we find:

$$
\begin{aligned}
(-f_1^{-1}) df_{N-k+1} = & - m_{N-k+1}^{\mathcal{M}} \\
& + \sum_{r=1}^{N-k} (-1)^{r(N-k-r)} f_1^{-1} m_{r+1}^{\mathcal{N}} (f_{N-k-r+1} \otimes \mathrm{id}^{\otimes r}) \\
& - \sum_{r=1}^{N-k-1} (-1)^{r(N-k-r+1)} f_1^{-1} f_{r+1} (m_{N-k-r+1}^{\mathcal{M}} \otimes \mathrm{id}^{\otimes r}) \\
& - (-1)^{N-k} f_1^{-1} f_{N-k+1} d \\
& - \sum_{r=1}^{N-k-1} (-1)^r f_1^{-1} f_{N-k} (\mathrm{id}^{\otimes r} \otimes \mu \otimes \mathrm{id}^{\otimes N-k-r-1})
\end{aligned}
$$

and thus

(50)

$$
\begin{aligned}
dg_{N+1} = & -\sum_{k=0}^{N-1} (-1)^{k(N-k)} m_{N-k+1}^{\mathcal{M}} (g_{k+1} \otimes \mathrm{id}^{\otimes N-k}) \\
& + \sum_{k=0}^{N-1} (-1)^{k(N-k)} \sum_{r=1}^{N-k} (-1)^{r(N-k-r)} f_1^{-1} m_{r+1}^{\mathcal{N}} (f_{N-k-r+1} \otimes \mathrm{id}^{\otimes r})(g_{k+1} \otimes \mathrm{id}^{\otimes N-k}) \\
& - \sum_{k=0}^{N-1} (-1)^{k(N-k)} \sum_{r=1}^{N-k-1} (-1)^{r(N-k-r+1)} f_1^{-1} f_{r+1} (m_{N-k-r+1}^{\mathcal{M}} \otimes \mathrm{id}^{\otimes r})(g_{k+1} \otimes \mathrm{id}^{\otimes N-k}) \\
& - \sum_{k=0}^{N-1} (-1)^{(k+1)(N-k)} f_1^{-1} f_{N-k+1} (dg_{k+1} \otimes \mathrm{id}^{\otimes N-k}) \\
& + (-1)^N \sum_{k=0}^{N-1} (-1)^{k(N-k)} (-f_1^{-1}) f_{N-k+1} (g_{k+1} \otimes \mathrm{id}^{\otimes N-k})(\mathrm{id}^{\otimes k+1} \otimes d) \\
& + \sum_{r=k+1}^{N-1} (-1)^r g_N (\mathrm{id}^{\otimes r} \otimes \mu \otimes \mathrm{id}^{\otimes N-r-1})
\end{aligned}
$$

The second term can be rewritten as

$$\sum_{r=1}^{N}(-1)^{r(N-r)}f_1^{-1}m^{\mathcal{N}}_{r+1}((\boldsymbol{f}\circ\boldsymbol{g})_{N-r+1}\otimes \mathrm{id}^{\otimes r}) = g_1 m^{\mathcal{N}}_{N+1}$$

because $\boldsymbol{f}\circ\boldsymbol{g}=\mathrm{id}$.

The third and fourth term in (50) evaluate to

$$-\sum_{k=0}^{N-1}(-1)^{(k+1)(N-k)}f_1^{-1}f_{N-k+1}\left(\left(\sum_{r=1}^{k}(-1)^{(k-r)r}m^{\mathcal{M}}_{r+1}(g_{k-r+1}\otimes \mathrm{id}^{\otimes k})+dg_{k+1}\right)\otimes \mathrm{id}^{\otimes N-k}\right).$$

We can apply the induction hypothesis and find:

$$\begin{aligned}
-\sum_{k=0}^{N-1}&(-1)^{(k+1)(N-k)}f_1^{-1}f_{N-k+1}\left(\left(\sum_{r=1}^{k}(-1)^{(k-r)r}m^{\mathcal{M}}_{r+1}(g_{k-r+1}\otimes \mathrm{id}^{\otimes k})+dg_{k+1}\right)\otimes \mathrm{id}^{\otimes N-k}\right)\\
&= \sum_{r=1}^{N-1}(-1)^{N(N-r)}(-f_1^{-1})((\boldsymbol{f}\circ\boldsymbol{g})_{N-r+1}-f_1 g_{N-r+1})\circ(m^{\mathcal{N}}_{r+1}\circ \mathrm{id}^{\otimes N-r})\\
&\quad+(-1)^N\sum_{k=0}^{N-1}(-1)^{k(N-k)}(-f_1^{-1})f_{N-k+1}(g_{k+1}\otimes \mathrm{id}^{\otimes N-k})(d\otimes \mathrm{id}^{\otimes N-k})\\
&\quad+\sum_{r=1}^{N-k-1}(-1)^r g_N(\mathrm{id}^{\otimes r}\otimes\mu\otimes \mathrm{id}^{\otimes N-r-1}).
\end{aligned}$$

Plugging this into (50), we find:

$$\begin{aligned}
dg_{N+1} =& -\sum_{k=0}^{N-1}(-1)^{k(N-k)}m^{\mathcal{M}}_{N-k+1}(g_{k+1}\otimes \mathrm{id}^{\otimes N-k})\\
&+\sum_{r=1}^{N}(-1)^{N(N-r)}g_{N-r+1}\circ(m^{\mathcal{N}}_{r+1}\circ \mathrm{id}^{\otimes N-r})\\
&+(-1)^N\sum_{k=0}^{N-1}(-1)^{k(N-k)}(-f_1^{-1})f_{N-k+1}(g_{k+1}\otimes \mathrm{id}^{\otimes N-k})(d\otimes \mathrm{id}^{\otimes N-k}+\mathrm{id}^{\otimes k+1}\otimes d)\\
&+\sum_{r=1}^{N-k-1}(-1)^r g_N(\mathrm{id}^{\otimes r}\otimes\mu\otimes \mathrm{id}^{\otimes N-r-1})+\sum_{r=k+1}^{N-1}(-1)^r g_N(\mathrm{id}^{\otimes r}\otimes\mu\otimes \mathrm{id}^{\otimes N-r-1})
\end{aligned}$$

which simplifies to (49). □

A.2. Homotopy transfer theorem for $\mathcal{A}_\infty$-modules. In this subsection, we prove that we can transport a (right) module structure along a homotopy retract to an $\mathcal{A}_\infty$-module structure.

Definition A.3. The data (i,p,h) as in

$$\begin{tikzcd} \mathcal{M}_* \arrow[loop left, "h"] \arrow[r, shift left, "p"] & \mathcal{N}_* \arrow[l, shift left, "i"] \end{tikzcd}$$

exhibits $(\mathcal{N}_*, d_{\mathcal{N}})$ as a *homotopy retract* of $(\mathcal{M}_*, d_{\mathcal{M}})$ if i and p are chain maps, i is a quasi-isomorphism and $d_{\mathcal{M}}h+hd_{\mathcal{M}}=\mathrm{id}_{\mathcal{M}}-ip$.

Proposition A.4 (Homotopy Transfer Theorem for $\mathcal{A}_\infty$-modules). *Let (i,p,h) exhibit $(\mathcal{N}_*, d_{\mathcal{N}})$ as a homotopy retract of $(\mathcal{M}_*, d_{\mathcal{M}})$. Let (A_*, d_A, μ_A) be a (strict) differential graded algebra such that $\mu_{\mathcal{M}}\colon \mathcal{M}_*\otimes A_*\to\mathcal{M}_*$ exhibits $\mathcal{M}_*$ as a right A_*-module.*

The maps $m_1 \coloneqq d_{\mathcal{N}}$, $m_2 \coloneqq p \circ \mu_{\mathcal{M}}(i \otimes \mathrm{id})$ *and*

$$m_k \coloneqq (-1)^{\frac{k(k-1)}{2}} p\mu_{\mathcal{M}}(\overbrace{h\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}}^{k-2}(i \otimes \mathrm{id}) \otimes \mathrm{id}) \otimes \dots) \otimes \mathrm{id}),$$

for $k \geq 2$*, define a* $\mathcal{A}_\infty$*-module structure on* $\mathcal{N}_*$*.*

Moreover, the maps $\boldsymbol{p}\colon \mathcal{M}_* \to \mathcal{N}_*$ *and* $\boldsymbol{i}\colon \mathcal{N}_* \to \mathcal{M}_*$ *given by* $p_1 \coloneqq p$*,* $i_1 \coloneqq i$

$$p_k \coloneqq (-1)^{\frac{(k-1)(k-2)}{2}} p\mu_{\mathcal{M}}(\overbrace{h\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(h}^{k-1} \otimes \mathrm{id}) \otimes \mathrm{id}) \otimes \dots) \otimes \mathrm{id}),$$

$$i_k \coloneqq (-1)^{\frac{k(k-1)}{2}} \overbrace{h\mu_{\mathcal{M}}(h\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}}^{k-1}(i \otimes \mathrm{id}) \otimes \mathrm{id}) \otimes \dots) \otimes \mathrm{id}),$$

for $k \geq 2$*, define morphisms of* $\mathcal{A}_\infty$*-modules.*

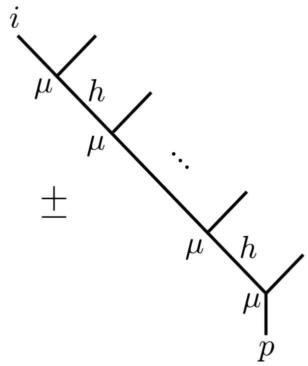

FIGURE 27. The map m_k.

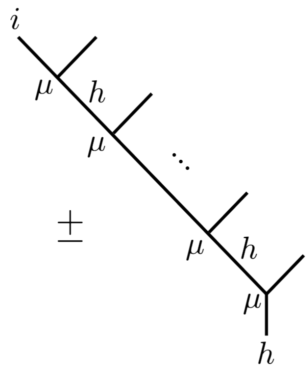

FIGURE 28. The map i_k.

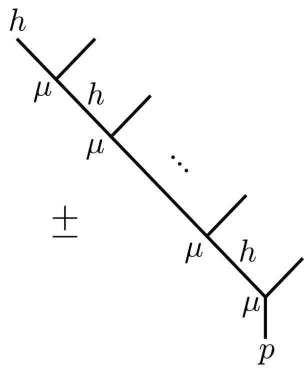

FIGURE 29. The map p_k.

PROOF. We need to check the $\mathcal{A}_\infty$-equation for m_k, which reads as:

$$\begin{aligned} dm_N + (-1)^{N+1} m_N d + \sum_{t=1}^{N-2} (-1)^{(N-t)t} m_{t+1}(m_{N-t} \otimes \mathrm{id}^{\otimes t}) \\ + \sum_{r=1}^{N-2} (-1)^r m_{N-1}(\mathrm{id}^{\otimes r} \otimes \mu \otimes \mathrm{id}^{\otimes N-r-2}) = 0. \end{aligned}$$

Using that $dh = \mathrm{id}_{\mathcal{M}} - ip - hd$, we compute as in Figure 30

$$\begin{aligned} &p\mu_{\mathcal{M}}(h\mu_{\mathcal{M}}(\dots dh\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i \otimes \mathrm{id}) \otimes \mathrm{id}) \cdots \otimes \mathrm{id}) \cdots \otimes \mathrm{id}) \otimes \mathrm{id}) \\ &\quad = p\mu_{\mathcal{M}}(h\mu_{\mathcal{M}}(\dots \mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i \otimes \mathrm{id}) \otimes \mathrm{id}) \cdots \otimes \mathrm{id}) \cdots \otimes \mathrm{id}) \otimes \mathrm{id}) \\ &\qquad - p\mu_{\mathcal{M}}(h\mu_{\mathcal{M}}(\dots ip\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i \otimes \mathrm{id}) \otimes \mathrm{id}) \cdots \otimes \mathrm{id}) \cdots \otimes \mathrm{id}) \otimes \mathrm{id}) \\ &\qquad - p\mu_{\mathcal{M}}(h\mu_{\mathcal{M}}(\dots hd\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i \otimes \mathrm{id}) \otimes \mathrm{id}) \cdots \otimes \mathrm{id}) \cdots \otimes \mathrm{id}) \otimes \mathrm{id}). \end{aligned}$$

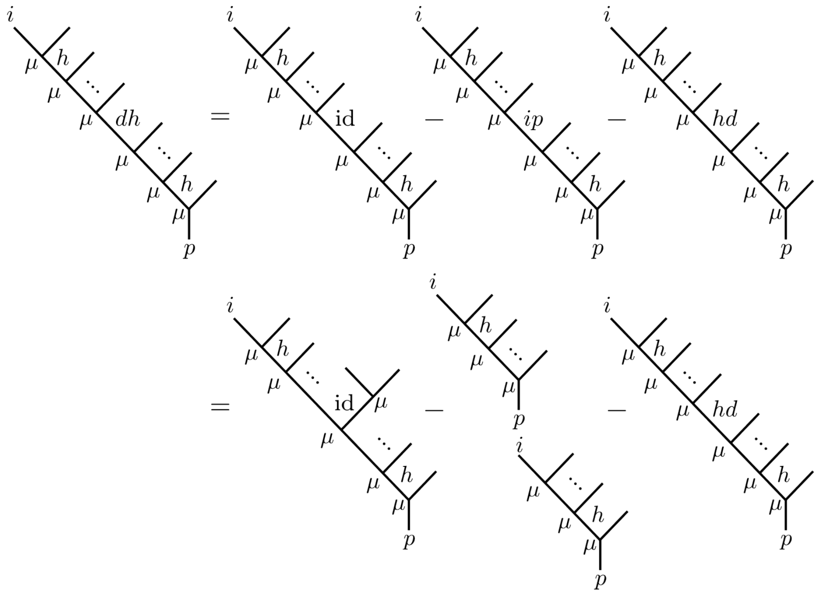

FIGURE 30. Putting a dh on an edge in the tree gives the same tree with $\mathrm{id}_{\mathcal{M}} - ip - hd$ at the edge.

It therefore follows that

$$\begin{aligned}
(-1)^{\frac{N(N-1)}{2}} dm_N &= dp\mu_{\mathcal{M}}(h\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i\otimes \mathrm{id})\otimes \mathrm{id})\cdots\otimes \mathrm{id})\otimes \mathrm{id}) \\
&= p\mu_{\mathcal{M}}(dh\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i\otimes \mathrm{id})\otimes \mathrm{id})\cdots\otimes \mathrm{id})\otimes \mathrm{id}) \\
&\quad + (-1)^{N-2} p\mu_{\mathcal{M}}(h\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i\otimes \mathrm{id})\otimes \mathrm{id})\cdots\otimes \mathrm{id})\otimes d) \\
&= p\mu_{\mathcal{M}}(\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i\otimes \mathrm{id})\otimes \mathrm{id})\cdots\otimes \mathrm{id})\otimes \mathrm{id}) \\
&\quad - p\mu_{\mathcal{M}}(ip\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i\otimes \mathrm{id})\otimes \mathrm{id})\cdots\otimes \mathrm{id})\otimes \mathrm{id}) \\
&\quad - p\mu_{\mathcal{M}}(hd\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i\otimes \mathrm{id})\otimes \mathrm{id})\cdots\otimes \mathrm{id})\otimes \mathrm{id}) \\
&\quad - (-1)^{\frac{N(N-1)}{2}}(-1)^{N+1} m_N(\mathrm{id}^{\otimes N}\otimes d) \\
&= (-1)^{\frac{(N-1)(N-2)}{2}} m_{N-1}(\mathrm{id}^{\otimes N-2}\otimes \mu_{\mathcal{M}}) \\
&\quad - (-1)^{\frac{(N-1)(N-2)}{2}} m_2(m_{N-1}\otimes \mathrm{id}) \\
&\quad - p\mu_{\mathcal{M}}(hd\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i\otimes \mathrm{id})\otimes \mathrm{id})\cdots\otimes \mathrm{id})\otimes \mathrm{id}) \\
&\quad + (-1)^{N-2} p\mu_{\mathcal{M}}(h\mu_{\mathcal{M}}(\dots(h\mu_{\mathcal{M}}(i\otimes \mathrm{id})\otimes \mathrm{id})\cdots\otimes \mathrm{id})\otimes d) \\
&= \dots \\
&= -(-1)^{\frac{N(N-1)}{2}} \sum_{r=1}^{N-2} (-1)^r m_{N-1}(\mathrm{id}^{\otimes r}\otimes \mu_{\mathcal{M}}\otimes \mathrm{id}^{\otimes N-r-2}) \\
&\quad - (-1)^{\frac{N(N-1)}{2}} \sum_{t=1}^{N-2} (-1)^{(N-t)t} m_{t+1}(m_{N-t}\otimes \mathrm{id}^{\otimes t}) \\
&\quad - (-1)^{\frac{N(N-1)}{2}}(-1)^{N+1} m_N d.
\end{aligned}$$

This shows that m_k defines a $\mathcal{A}_\infty$-structure on $\mathcal{N}_*$.

Analogous computations show that the above formulas for p_k and i_k define morphism of $\mathcal{A}_\infty$-modules. □

Proposition A.5. *Let (A_*, d_A) be a (strict) differential graded algebra. Let $(\mathcal{M}_*, d_{\mathcal{M}})$ and $(\mathcal{N}_*, d_{\mathcal{N}})$ be strict right A_*-modules with an ∞-quasi-isomorphism $\boldsymbol{f}\colon \mathcal{M}_* \to \mathcal{N}_*$.*

If $(i^{\mathcal{M}}, p^{\mathcal{M}}, h^{\mathcal{M}})$ and $(i^{\mathcal{N}}, p^{\mathcal{N}}, h^{\mathcal{N}})$ exhibit $(\mathcal{M}_*, d_{\mathcal{M}})$ and $(\mathcal{N}_*, d_{\mathcal{N}})$ as a homotopy retract of $(H_*(\mathcal{M}), 0)$ and $(H_*(\mathcal{N}), 0)$, respectively, then there exists an ∞-quasi-isomorphism $\boldsymbol{g}\colon \mathcal{N}_* \to \mathcal{M}_*$ such that $H_*(g_1) = H_*(f_1)^{-1}$.

Proof. By Proposition A.4, we can define the following composition of morphisms of $\mathcal{A}_\infty$-modules as $\boldsymbol{j}$:

$$H_*(\mathcal{M}) \xrightarrow{\boldsymbol{i}^{\mathcal{M}}} \mathcal{M}_* \xrightarrow{\boldsymbol{f}} \mathcal{N}_* \xrightarrow{\boldsymbol{p}^{\mathcal{N}}} H_*(\mathcal{N}).$$

This morphism is such that $j_1 = H_*(f_1)$ is an isomorphism. Therefore, we can invert it by Proposition A.2. We now define $\boldsymbol{g}$ as the composition

$$\mathcal{N}_* \xrightarrow{\boldsymbol{p}^{\mathcal{N}}} H_*(\mathcal{N}) \xrightarrow{\boldsymbol{j}^{-1}} H_*(\mathcal{M}) \xrightarrow{\boldsymbol{i}^{\mathcal{M}}} \mathcal{M}_*.$$

□

Corollary A.6. *Let (A_*, d_A) be a (strict) differential graded algebra over a field. Let $(\mathcal{M}_*, d_{\mathcal{M}})$ and $(\mathcal{N}_*, d_{\mathcal{N}})$ be strict right A_*-modules with an ∞-quasi-isomorphism $\boldsymbol{f}\colon \mathcal{M}_* \to \mathcal{N}_*$.*

There exists an ∞-quasi-isomorphism $\boldsymbol{g}\colon \mathcal{N}_* \to \mathcal{M}_*$ such that $H_*(g_1) = H_*(f_1)^{-1}$.

Proof. By [LV12, Lemma 9.4.4] any complex over a field admits its homology as a homotopy retract. Together with Proposition A.5, this shows the corollary. □

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