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# Two interactions between stable homotopy theory and geometry

Florian Riedel

**Advisors:** Jesper Grodal and Robert Burklund

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Florian Riedel
florian.riedel@pm.me
Department of Mathematical Sciences
Universitetsparken 5
2100 Copenhagen Ø
Office 04.4.03

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| **Thesis Title** | Two interactions between stable homotopy theory and geometry |
| **Advisors** | Jesper Grodal (principal)<br>Robert Burklund |
| **Assessment Committee** | Nathalie Wahl (chair)<br>*University of Copenhagen*<br>Achim Krause<br>*University of Oslo*<br>Elden Elmanto<br>*Universit of Toronto* |
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# Summary

This thesis consists of two papers, both exploring different aspects of the interaction of stable homotopy theory and geometry.

The first is about spectral algebraic geometry in positive characteristic. We discuss different notions of points of $\mathbb{E}_\infty$-rings over a finite field and find that nilpotent classes play a fundamentally different role, depending on whether the characteristic is odd or even.

The second studies how the motivic $\mathbb{E}_\infty$-ring representing algebraic $K$-theory of smooth schemes. We construct a kind of Fourier transform for this ring, which in particular witnesses that the category of representations of an algebraic torus is semi-simple. Using this, we give a formula for the $K$-theory of an algebraic torus as the equivariant homology of its topological mirror.

# Sammenfatning

Denne afhandling består af to artikler. De to artikler udforsker forskellige aspekter af forholdet mellem stabil homotopi teori og geometri.

Den første omhandler spectral algebraisk geometri i positiv karakteristik. Vi diskuterer forskellige eksempler på punkter af $\mathbb{E}_\infty$-ringe over endelige legemer og viser at nilpotente klasser spiller fundamentalt forskellige roller, afhængigt af hvorvidt karakteristikken er ulige eller lige.

Den anden artikel studerer vi den motiviske $\mathbb{E}_\infty$-ring som repræsenterer algebraisk $K$-teori af glatte skemaer. Vi konstruer en form for Fourier transformation for denne ring, som blandt andet er vidne til at kategorien af repræsentationer af en algebraisk torus er semi simpel. Ved hjælp af dette giver vi en formal for $K$-teorien af en algebraisk torus som den equivariante homologi af dens topologisk spejlbillede.

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# Chapter 1

# Introduction

The fundamental idea of algebraic topology is that, to understand a geometric object $X$, we should associate to it an *algebraic invariant* $E^*(X)$, as algebra is easier than geometry. More precisely, we think of $X$ as living in some "geometric" category $\mathcal{B}$ and would like a *cohomology theory* given by functor $E^*(-)$ from $\mathcal{B}^{\mathrm{op}}$ to the category of graded abelian groups, such that pulling back along the diagonal map $\Delta\colon X \to X \times X$ induces a cup product in cohomology $E^*(X) \otimes E^*(X) \to E^*(X)$.

Stable homotopy theory is the study of such invariants. It tells us that the cohomology theories defined on $\mathcal{B}$ in of themselves ought to form a reasonable category of *spectra* $\mathrm{SH}(\mathcal{B})$ that comes equipped with a tensor product akin to the (derived) tensor product of abelian groups. Moreover, this tensor product is set up such that the multiplication on $E^*(X)$ as $X$ varies produces a commutative ring object $E \in \mathrm{CAlg}(\mathrm{SH}(\mathcal{B}))$.

Waldhausen's "Brave New Algebra" posits that, we should treat these ring objects in $\mathrm{SH}(\mathcal{B})$ as generalizations of ordinary commutative rings. Indeed, any commutative ring $R$ gives rise to a ring object $R_{\mathcal{B}} \in \mathrm{CAlg}(\mathrm{SH}(\mathcal{B}))$ playing the role of cohomology with constant $R$-coefficients. Naturally, the theory of these higher commutative rings is much richer and more subtle than ordinary[1] algebra, which presents much of the allure of the field.

In turn, the modern perspective on algebraic geometry tells us that any notion of ring should give rise to a notion of geometry, glued together from affine spaces, determined by their rings of functions. In this way, studying the "ordinary" geometric objects $X \in \mathcal{B}$ naturally leads us to also study the "spectral" geometry of commutative rings in in the category of cohomology theories $\mathrm{SH}(\mathcal{B})$. What's more, these two co-morbid notions of geometry inform and illuminate one another.

The two projects that make up this thesis are at their core an exploration of this network of ideas.

Chapter 2 explores different generalizations of the classical notion of points in ordinary algebraic geometry to the spectral world, which turn out to behave rather strangely over a field of positive characteristic.

Conversely, Chapter 3 uses the methods of higher algebra to say something about an invariant attached to a more classical object. Namely, we give a presentation of the algebraic $K$-theory of a not-necessarily split algebraic torus as an $\mathbb{E}_\infty$-ring in the stable motivic category.

[1] cf. cowardly old

# Chapter 2

# Reduced points of $\mathbb{E}_\infty$-rings in positive characteristic

Florian Riedel

We investigate whether an arbitrary non-zero $\mathbb{E}_\infty$-ring $A$ admits a reduced point, meaning an $\mathbb{E}_\infty$-map $A \to T$ such that $\pi_* T$ is a graded field. We show that if $2 \in \pi_0 A$ is not invertible, then $A$ admits a reduced point and as an application deduce that a free $A$-module on $n$ generators cannot be built from $n-1$ many cells. Perhaps surprisingly, the existence of reduced points completely fails at odd primes. More precisely, for any prime $p > 2$, we construct a non-zero $\mathbb{E}_\infty$-ring over $\mathbb{F}_p$ which admits no map to an $\mathbb{E}_2$-algebra $T$ such that $\pi_0 T$ is a field.

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## 2.1 Introduction

### Overview

If $X$ is a CW-complex built from $n-1$ many cells, then $X$ cannot admit a wedge of $n$ many spheres $S^{t_1} \vee \cdots \vee S^{t_n}$ as a retract. Indeed, by taking rational homology, this reduces to the fact that a finite dimensional vector space cannot be a retract of a vector space of strictly smaller dimension.

More generally, say that an $\mathbb{E}_\infty$-ring $A$ has *invariant cell numbers* if, for all $1 \leq k < n$, any free $A$-module on $n$ generators cannot be written as retract of a stable, $A$-linear cell complex consisting of $k$-many cells. This raises the natural question:

**Question 2.1.1.** Does any non-zero $\mathbb{E}_\infty$-ring $A$ have invariant cell numbers?

The statement above about CW-complexes is, up to stabilization, equivalent to the fact that the sphere spectrum $\mathbb{S}$ has invariant cell numbers. We give a partial answer to Question 2.1.1 by showing that this also holds for $\mathbb{E}_\infty$-rings whose $\mathbb{F}_2$-homology is non-zero.

**Theorem A** (Corollary 2.4.18)**.** If $A$ is a non-zero $\mathbb{E}_\infty$-ring such that $2 \in \pi_0 A$ is not invertible, then $A$ has invariant cell numbers.

This will be an immediate consequence of our more technical Theorem B. Indeed, for $A = \mathbb{S}$ the trick was to base change to the Eilenberg–MacLane spectrum $\mathbb{Q}$, over which any module is free, thus reducing the claim to linear algebra. Our approach to proving Theorem A is to construct a suitable replacement for the rationalization map $\mathbb{S} \to \mathbb{Q}$, which will allows us to run the same argument.

**Definition 2.1.2.** An $\mathbb{E}_\infty$-ring $T \in \mathrm{CAlg}_A$ is called a *reduced point* of $A$ if $\pi_* T$ is a graded field.[1]

It is straightforward to show that any $\mathbb{E}_\infty$-ring whose homotopy groups form a graded field has invariant cell numbers. Since cell-structures and retracts are preserved by base change, it follows that the same is true for any $\mathbb{E}_\infty$-ring which admits a reduced point. Let us mention some classes of $\mathbb{E}_\infty$-rings where reduced points are known to exist.

**Example 2.1.3.** For any non-zero ordinary commutative ring, the existence of a reduced point is immediate by choosing a maximal ideal. More generally if $A$ is connective, then it admits a reduced point as it maps to the ordinary commutative ring given by the 0-truncation $A \to \tau_{\leq 0} A$.

Moreover, if for any $n \geq 0$ the telescopic localization $A \to L_{T(n)} A$ is non-zero, then by the Chromatic Nullstellensatz [BSY22, Theorem D] the ring $A$ admits a map $A \to E_n \otimes \mathbb{Q}$ where $E_n$ is a height $n$ Lubin–Tate theory. Thus, if a ring is chromatically supported at some finite height, it admits a reduced point and thus has invariant cell numbers.

For any prime $p$, we investigate the "infinite height" case of $\mathbb{E}_\infty$-rings over the finite field $\mathbb{F}_p$, where the situation turns out to differ drastically depending on whether $p$ is odd or even. For $p = 2$ our main result is the following:

**Theorem B** (Theorem 2.4.17)**.** If $A$ is an $\mathbb{E}_\infty$-ring over $\mathbb{F}_2$, then $A$ admits a reduced point $T \in \mathrm{CAlg}_A$ such that $T$ is 1-periodic and $\pi_0 T$ is separably closed.

This raises the question whether non-zero $\mathbb{E}_\infty$-rings always admit reduced points. Somewhat surprisingly, the answer is *no*. Indeed, for any odd prime $p > 2$, we construct an $\mathbb{E}_\infty$-ring over $\mathbb{F}_p$ which admits no non-zero maps out to a reduced ring. Denote by $\mathbb{F}_p\{x\}$ the free $\mathbb{E}_\infty$-algebra on a generator $x$ in topological degree 0.

**Theorem C** (Theorem 2.4.23)**.** For any odd prime $p$, there exists a class $\theta(x) \in \pi_* \mathbb{F}_p\{x\}$, such that the image of $\theta(x)$ under the $\mathbb{E}_\infty$-cofiber $\mathbb{F}_p\{x\} \to \mathbb{F}_p\{x\} /\!\!/^\infty x^p$ is not nilpotent. In particular, the localization $(\mathbb{F}_p\{x\} /\!\!/^\infty x^p)[\theta(x)^{-1}]$ is a non-zero $\mathbb{E}_\infty$-ring which admits no reduced points.

Indeed, any $\mathbb{E}_\infty$-map to a ring $T$ such that $\pi_0 T$ is reduced takes the nilpotent class $x$ to zero. In particular, it must factor through the $\mathbb{E}_\infty$-cofiber by $x$, which kills the invertible class $\theta(x)$, forcing $T \simeq 0$. In fact, as explained in Remark 2.4.24, the class $\theta(x)$ is already killed by the $\mathbb{E}_2$-cofiber, and so the same conclusion holds if $T$ is only an $\mathbb{E}_2$-algebra.

[1] Meaning $\pi_0 T$ is a field and $\pi_* T$ is either concentrated in degree 0 or of the form $\pi_* T \simeq \pi_0 T[u^\pm]$.

We do not know if the ring of Theorem C has invariant cell numbers. Approaching the problem from the other end, one may ask what kind of $\mathbb{E}_\infty$-ring we can always map out to. Indeed, as a consequence of [BSY22, Theorem A.3] *any* non-zero $\mathbb{E}_\infty$-ring $A$ admits an $\mathbb{E}_\infty$-map $A \to T$ such that $T \in \mathrm{CAlg}_A$ is *nullstellensatzian*[2]. Over $\mathbb{F}_2$ the methods proving Theorem B imply that any nullstellensatzian $\mathbb{E}_\infty$-ring is a 1-periodic, separably closed field which is not perfect[3]. For odd primes, we give the following description:

**Proposition 2.1.4** (Proposition 2.4.27)**.** Let $p > 2$ and $T$ be a nullstellensatzian $\mathbb{F}_p$-algebra. Then the graded ring $\pi_* T$ has the following properties:

1. $\pi_* T$ is 2-periodic and $\pi_0 T$ is a local ring of Krull dimension zero.
2. $\pi_1 T \neq 0$
3. The Nilradical $\mathrm{Nil}(\pi_0 T)$ is not a nilpotent ideal. In particular, $\pi_0 T$ is not reduced.
4. Any $v \in \mathrm{Nil}(\pi_0 T)$ satisfies $v^p = 0$.

As the first point is essentially automatic for any nullstellensatzian algebra over a discrete ring, this is in some sense a worst case scenario. In particular $\pi_0 T$ is not noetherian, and we are unable to extract further insight into Question 2.1.1 for $\mathbb{E}_\infty$-rings over $\mathbb{F}_p$.

As any connective $\mathbb{E}_\infty$-ring $A$ is a limit of square zero extensions of $\pi_0 A$, connective spectral algebraic geometry is commonly thought of as a deformation of ordinary algebraic geometry. The non-connective world is different, and Theorem C together with Proposition 2.1.4 provide concrete examples of a phenomena that are unique to this setting. Arbitrary $\mathbb{E}_\infty$-rings simply do not map out to rings which are reasonable from the point of view of ordinary commutative algebra.

## Strategy and methods

The key to proving Theorem B and Theorem C is to control what happens when we take the $\mathbb{E}_\infty$-cofiber by a nilpotent class.

Let us again begin with discussing the $p = 2$ case. Given a non-zero $A \in \mathrm{CAlg}_{\mathbb{F}_2}$ and a nilpotent class $v \in \pi_* A$, we want to get rid of $v$ by taking the $\mathbb{E}_\infty$-cofiber $A \to A/\!\!/^\infty v$, while ensuring that this does not accidentally produce the zero ring. We do this by showing something stronger, namely that nilpotent classes die in a nilpotence detecting way. Say that a map of $\mathbb{E}_\infty$-rings $f\colon A \to B$ *detects nilpotence* if, for any $A$-linear map $v\colon \mathcal{I} \to A$ with $\mathcal{I} \in \mathrm{Mod}_A^\omega$ compact, the composite $f(v)$ being null implies that $v^{\otimes n} \simeq 0$ for some $n > 0$. The following proposition is the main technical result over $\mathbb{F}_2$.

**Proposition 2.1.5** (Proposition 2.4.14)**.** For any $A \in \mathrm{CAlg}_{\mathbb{F}_2}$ and any nilpotent class $v \in \pi_* A$, the $\mathbb{E}_\infty$-cofiber $A \to A/\!\!/^\infty v$ detects nilpotence.

The straightforward reason the map $A \to A/\!\!/^\infty v$ might *fail* to detect nilpotence is that it not only kills the class $v$, but also the extended powers $Q_i(v)$ and their iterates. While a simple computation with the Cartan formula, see Lemma 2.4.12, shows that these classes are all again nilpotent, we need to quantify the idea that these classes dying is, in some sense, the only thing that happens in the cofiber.

By work of Araki–Kudo [KA56] and Browder [Bro58], we know that for any $0 \leq n \leq \infty$ the free $\mathbb{E}_n$-algebra on a generator $\mathbb{F}_2\{x\}_n$ has homotopy groups given by an infinite polynomial ring on the (iterated) extended powers $Q_1^{i_1} \cdots Q_{n-1}^{i_{n-1}}(x)$. By constructing a highly structured comparison map, we deduce that, for any $k \leq n$, the underlying $\mathbb{E}_k$-algebra of $\mathbb{F}_2\{x\}_n$ is equivalent

[2] Meaning every compact algebra $B \in \mathrm{CAlg}_T^\omega$ admits a $T$-algebra map $B \to T$.

[3] This follows from the relation $Q_1(x^2) = 0$, where $Q_1$ is the power operation induced by the two-cell of $\mathbb{R}P^\infty$, see also Proposition 2.4.16.

to an infinite tensor product of free $\mathbb{E}_k$-algebras, cf. Proposition 2.3.6. From this, we obtain the following decomposition of the filtered $\mathbb{E}_n$-cofiber by a class in filtration degree one.

**Lemma 2.1.6** (Corollary 2.3.8)**.** Let $A \in \mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathrm{Mod}_{\mathbb{F}_2})$ for some $n \geq 0$ and let $v \in \pi_* A$ be some class. For any $k \geq n$, we have an equivalence of filtered $\mathbb{E}_{k-1}$-algebras

$$\bigotimes_{(i_{k+1},\dots,i_n)} A/\!\!/^{k-1} Q_{k+1}^{i_{k+1}} \cdots Q_n^{i_n}(\tau v) \xrightarrow{\sim} A/\!\!/^n \tau v\,,$$

where the infinite tensor product runs over all sequences of non-negative integers $(i_{k+1},\dots,i_n)$.

In particular, for $k = n = 1$, we learn that the $\mathbb{E}_1$-cofiber $A/\!\!/^1 v$ of a class in an $\mathbb{E}_2$-algebra $A$ has a natural multiplicative filtration whose $(2^{i+1}-1)$ stage is given by the $A$-module $A/(v, Q_1(v), \dots, Q_1^i(v))$ cf. Lemma 2.4.13. This analysis gives us enough control to deduce Proposition 2.1.5 from the fact that $v$ being nilpotent forces the extended powers to be nilpotent, and Theorem B is a straightforward consequence.

For the case of an odd prime $p > 2$, we want to employ the same methods to arrive at the opposite conclusion. The name of the game is again to understand the free algebra and the $\mathbb{E}_\infty$-cofiber.

By work of Dyer–Lashof [DL62] and Cohen [Coh76], we know that the homotopy groups of the free $\mathbb{E}_\infty$-algebra $\mathbb{F}_p\{x\}$ form a free *graded commutative*[4] $\mathbb{F}_p$-algebra on certain composites of the extended $p$-th powers $P_i(x), \beta P_i(x)$[5]. In particular, any even degree class is not nilpotent in $\pi_*\mathbb{F}_p\{x\}$. Moreover, in Proposition 2.4.22 we observe that $P_I(x^p) \simeq 0$ in $\mathbb{F}_p\{x\}$ for any iterated operation $P_I$ containing at least one Bockstein. Further analyzing the Cartan formula, we learn that the $p$-th power map

$$\mathbb{F}_p\{x\} \longrightarrow \mathbb{F}_p\{x\}, \quad x \mapsto x^p$$

does not hit any iterated operation $P_I(x)$ in even degree containing a Bockstein and thus the class

$$\theta(x) := \beta P_{\frac{1}{2}} \beta P_1(x) \in \pi_{2(p^2+p-1)}\mathbb{F}_p\{x\}$$

has no a priori reason to become nilpotent in the $\mathbb{E}_\infty$-cofiber by $x^p$.

To ensure that nothing unexpected happens in the Tor spectral sequence, we first use the computations of [DL62] to deduce that, the underlying $\mathbb{E}_1$-algebra of $\mathbb{F}_p\{x\}$ is naturally equivalent to an infinite tensor product of free $\mathbb{E}_1$-algebras on even degree classes and free $\mathbb{E}_2$-algebras on odd degree classes, see Proposition 2.3.14. From this, we get the following:

**Lemma 2.1.7** (Corollary 2.3.15)**.** Let $A \in \mathrm{CAlg}_{\mathbb{F}_2}$ and let $v \in \pi_* A$ be a class in even topological degree. Then the $\mathbb{E}_\infty$-cofiber $A/\!\!/^\infty v$ is, as an $\mathbb{E}_0$-algebra over $A$, equivalent to an infinite tensor product

$$\bigotimes_I A/P_I(v) \otimes \bigotimes_J A/\!\!/^1 P_J(v) \xrightarrow{\sim} A/\!\!/^\infty v$$

of $\mathbb{E}_0$- and $\mathbb{E}_1$-cofibers, where the $P_I(v)$ sit in even and the $P_J(v)$ in odd topological degree.

Applying this to the ring $A = \mathbb{F}_p\{x\}$ and $v = x^p$, we see that most of the terms appearing are taking a cofiber by a class that is already null, making the $A$-module structure very simple. Moreover, we have enough structure to conclude that the class $\theta(x)$ does not act nilpotently on the cofiber $\mathbb{F}_p\{x\}/\!\!/^\infty x^p$, which is precisely the content of Theorem C.

[4] Meaning in particular that classes in odd topological degree square to zero.

[5] See Construction 2.3.9 for a precise definition and our indexing conventions.

### Outline

We begin in Section 2.2 by reviewing some facts about locally graded categories and the shearing construction. Moreover, we discuss free $\mathbb{E}_n$-algebras an $\mathbb{E}_{n-1}$-cofibers. In Section 2.3, we recall the construction of Dyer–Lashof operations in arbitrary $\mathbb{F}_p$-linear, $\mathbb{E}_n$-monoidal categories and deduce the decompositions of free algebras, as well as of the filtered $\mathbb{E}_{n-1}$-cofiber by a class in filtration degree one. Lastly in Section 2.4, we first review some material on nullstellensatzian rings and detecting nilpotence, before moving on to prove Theorem B and Theorem C.

### Conventions

1. We freely use the theory of $(\infty, 1)$-categories as developed in [HTT] and [HA] and henceforth refer to $(\infty, 1)$-categories simply as categories.

2. We denote by $\mathcal{S}$ the category of spaces and by Sp the category of spectra.

3. We write $\mathrm{Pr}^{\mathrm{L}}$ for the category of presentable categories equipped with the Lurie tensor product.

4. By commutative ring we will always mean $\mathbb{E}_\infty$-ring and call the objects of $\mathrm{CAlg}(\mathrm{Ab})$ discrete commutative rings.

5. If $A$ is an $\mathbb{E}_n$-algebra for some $1 \leq n \leq \infty$, we write $\mathrm{Mod}_A$ for the category of left $A$-modules.

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## 2.2 Graded $\mathbb{E}_n$-algebras and shearing

The main purpose of this section is to alleviate the authors own confusion about $\mathbb{E}_n$-algebras and locally graded categories, and as such is largely expository.

Firstly, in Section 2.2.1 we review some well known facts about $\mathbb{E}_n$-algebras which we extract from [HA]. Concretely, we need the fact that $\mathbb{E}_{n-1}$-cofibers can be computed as a Bar construction, which is Lemma 2.2.2, see also [HL21] for a comprehensive treatment. Moreover, we recall the description of the free $\mathbb{E}_n$-algebra as the homology of iterated loop spaces in Proposition 2.2.5.

Secondly, we discuss graded and filtered categories, following the ideas laid out in [Lur14]. Moreover, we discuss how strict Picard elements lead to a shearing equivalence Lemma 2.2.10, which substantially simplifies the computations of the free algebras in Section 2.3 and allows us to construct the Dyer–Lashof operations efficiently via Construction 2.2.11.

### 2.2.1 $\mathbb{E}_n$-cofibers and iterated loop spaces

Let $\mathcal{C}$ be a presentably $\mathbb{E}_n$-monoidal category for some $0 \leq n \leq \infty$. By [HA, 3.2.4.3] combined with the Dunn-additivity theorem [HA, 5.1.2.2], we know that the category $\mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$ inherits a natural $\mathbb{E}_{n-k}$-monoidal structure, together with an $\mathbb{E}_{n-k}$-monoidal enhancement of the forgetful functor $U_{\mathcal{C}}\colon \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}) \to \mathcal{C}$. By [HA, 3.1.3.5], the functor $U_{\mathcal{C}}$ admits a left adjoint

$$\mathbb{1}_{\mathcal{C}}\{-\}_k\colon \mathcal{C} \longrightarrow \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})\,.$$

For any $\mathcal{I} \in \mathcal{C}$ we call $\mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_k$ the *free $\mathbb{E}_k$-algebra* on $\mathcal{I}$. If $\mathcal{I} = \Sigma^t \mathbb{1}_{\mathcal{C}}$ is suspension of the unit for some integer $t$, we also refer to $\mathbb{1}_{\mathcal{C}}\{\Sigma^t \mathbb{1}_{\mathcal{C}}\}$ as the free algebra on a generator in topological degree $t$.

By [HA, 3.4.4.6], we know that for any $A \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ the category of (left)-modules $\mathrm{Mod}_A(\mathcal{C})$ naturally inherits the structure of an $\mathbb{E}_{n-1}$-monoidal category via the Bar construction

$$M \otimes_A N = \mathrm{colim}(M \otimes A^{\otimes \bullet} \otimes N), \quad M, N \in \mathcal{C}$$

such that the colimit preserving base change

$$- \otimes A\colon \mathcal{C} \longrightarrow \mathrm{Mod}_A(\mathcal{C})$$

naturally refines to a $\mathbb{E}_{n-1}$-monoidal functor. In particular, it induces a colimit preserving functor $\mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Mod}_A(\mathcal{C}))$ which we will use implicitly.

**Definition 2.2.1.** Let $\mathcal{C}$ be a stable, presentably $\mathbb{E}_n$-monoidal category and let $A \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$. Given a map $v\colon \mathcal{I} \to A$ in $\mathcal{C}$, we denote by

$$A \longrightarrow A/\!\!/^{n-1} v \in \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Mod}_A(\mathcal{C})).$$

the *$\mathbb{E}_{n-1}$-cofiber* of $v$, i.e. the map of $\mathbb{E}_{n-1}$-algebras obtained by the pushout

$$\begin{tikzcd}
A \otimes \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_{n-1} \arrow[r, "A\otimes \mathbb{1}_{\mathcal{C}}\{v\}_{n-1}"] \arrow[d, "A\otimes \mathbb{1}_{\mathcal{C}}\{0\}_{n-1}"'] & A \arrow[d] \\
A \arrow[r] & A/\!\!/^{n-1} v\,.
\end{tikzcd}$$

in the category $\mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Mod}_A(\mathcal{C}))$.

These cofibers admit a more explicit presentation using the Bar construction.

**Lemma 2.2.2.** Let $1 \leq n \leq \infty$ and let $\mathcal{C}$ be a presentably $\mathbb{E}_n$-monoidal category. Suppose we are given $A \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ and a map $v\colon \mathcal{I} \to A \in \mathcal{C}$. Then, the $\mathbb{E}_{n-1}$-cofiber by $v$ is computed by the Bar construction

$$A/\!\!/^{n-1} v \xrightarrow{\sim} A \otimes_{\mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n} \mathbb{1}_{\mathcal{C}} \in \mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Mod}_A)\,.$$

*Proof.* The case $n = \infty$ is an easy consequence of [HA, 3.2.4.7]. Moreover, for finite $n$, we know by [HA, 5.3.1.16] that the $\mathbb{E}_1$-monoidal structure on $\mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathcal{C})$ preserves colimits of contractible shape in each variable. Consequently, by [HA, 5.2.2.12], the unit $\mathbb{1}_{\mathcal{C}}$ considered as an object of $\mathrm{Alg}_{\mathbb{E}_{n-1}}(\mathrm{Mod}_{\mathbb{1}\{\mathcal{I}\}_n})$ via the 0-augmentation is given by the pushout

$$\begin{tikzcd}
\mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_{n-1} \otimes \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n \arrow[r, "m"] \arrow[d, "e"'] & \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n \arrow[d] \\
\mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n \arrow[r] & \mathbb{1}_{\mathcal{C}}\,.
\end{tikzcd}$$

Here, $e$ is induced by the 0 augmentation of $\mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_{n-1}$ and $m$ is induced by the natural map $\mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_{n-1} \to \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n$. Base changing along the map of $\mathbb{E}_n$-algebra $\mathbb{1}_{\mathcal{C}}\{v\}\colon \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n \to A$ and using that the base change preserves colimits, we learn that we have an equivalence

$$\begin{aligned}
A \otimes_{\mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n} \mathbb{1}_{\mathcal{C}} &\simeq A \otimes_{\mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n} (\mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n \amalg_{\mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n \otimes \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_{n-1}} \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n) \\
&\simeq A \amalg_{A \otimes \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_{n-1}} A
\end{aligned}$$

of $\mathbb{E}_{n-1}$-algebra in $\mathrm{Mod}_A$. $\square$

**Remark 2.2.3.** Let $\mathcal{C}$ be a stable, presentably $\mathbb{E}_n$-monoidal category and $Q\colon \mathcal{J} \to \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_n$ be a class in the free $\mathbb{E}_n$-algebra. Then for any $A \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ and any class $v\colon \mathcal{I} \to A$, the presentation of the $\mathbb{E}_{n-1}$-cofiber in Lemma 2.2.2 provides us with a nullhomotopy of the composite map

$$\mathcal{J} \xrightarrow{Q(v)} A \longrightarrow A/\!\!/^{n-1}v\,.$$

In particular, this tells us that any $\mathbb{E}_{n-1}$-algebra map $A \to B$ that kills $v$ also kills the class $Q(v)$, even though the operation $Q$ may not be defined on the $\mathbb{E}_{n-1}$-algebra $B$.

**Construction 2.2.4.** Let $\mathcal{C}$ be a presentably $\mathbb{E}_{n+1}$-monoidal category. Given two augmented $\mathbb{E}_n$-algebras $A, B \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})_{/\mathbb{1}_{\mathcal{C}}}$ we can form their tensor product to obtain an augmented $\mathbb{E}_n$-algebra $C = A \otimes B$. Using the projection formula, we obtain an equivalence

$$\begin{aligned}(\mathbb{1}_{\mathcal{C}} \otimes_A C) \otimes_C (C \otimes_B \mathbb{1}_{\mathcal{C}}) &\simeq (C \otimes_A \mathbb{1}_{\mathcal{C}}) \otimes_B \mathbb{1}_{\mathcal{C}} \\ &\simeq ((A \otimes B) \otimes_A \mathbb{1}_{\mathcal{C}}) \otimes_B \mathbb{1}_{\mathcal{C}} \\ &\simeq B \otimes_B \mathbb{1}_{\mathcal{C}} \\ &\simeq \mathbb{1}_{\mathcal{C}}\,,\end{aligned}$$

of $\mathbb{E}_{n-1}$-algebras in $\mathrm{Mod}_C(\mathcal{C})$. We call this the *unit decomposition* associated to the decomposition of augmented algebras $C \simeq A \otimes B$.

In Section 2.3 we will utilize Construction 2.2.4 combined with Lemma 2.2.2 to deduce from decompositions of free $\mathbb{E}_n$-algebras, associated decompositions of $\mathbb{E}_{n-1}$-cofibers. The main tool for computing free $\mathbb{E}_n$-algebras in the first place is the following story.

The free $\mathbb{E}_k$-algebra $\mathbb{1}_{\mathcal{C}}\{\Sigma^t \mathbb{1}_{\mathcal{C}}\}_k$ on a generator in positive topological degree $t$ admits a well known presentation in terms of the homology of loop spaces, which we want to record here. Since $\mathcal{C}$ is presentably $\mathbb{E}_n$-monoidal, it comes equipped with a $\mathbb{E}_n$-monoidal functor

$$\mathbb{1}_{\mathcal{C}}[-]\colon \mathcal{S} \longrightarrow \mathcal{C}\,,$$

which takes a space $X \in \mathcal{S}$ to the constant colimit $\mathbb{1}_{\mathcal{C}}[X] = \mathrm{colim}_X\, \mathbb{1}_{\mathcal{C}}$, which we think of the homology of $X$ with coefficients in $\mathcal{C}$. If $\mathcal{C}$ is moreover stable, then for any pointed space $(y\colon * \to Y) \in \mathcal{S}_*$ write

$$\mathbb{1}_{\mathcal{C}}^{\mathrm{red}}[Y] \coloneqq \mathrm{cofib}(\mathbb{1}_{\mathcal{C}} \xrightarrow{\mathbb{1}_{\mathcal{C}}[y]} \mathbb{1}_{\mathcal{C}}[Y])$$

for the reduced homology of $Y$.

This allows for the following description of the free $\mathbb{E}_n$-algebra on a reduced homology object.

**Proposition 2.2.5.** Let $0 \leq n \leq \infty$ and let $\mathcal{C} \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Pr}^{\mathrm{L,St}})$. Then for any pointed, connected space $X$, we have a natural equivalence

$$\mathbb{1}_{\mathcal{C}}\{\mathbb{1}_{\mathcal{C}}^{\mathrm{red}}[X]\}_n \simeq \mathbb{1}_{\mathcal{C}}[\Omega^n \Sigma^n X] \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C})$$

of $\mathbb{E}_n$-algebras in $\mathcal{C}$. In particular, taking $X = S^t$ for any $t \geq 1$, we get an equivalence

$$\mathbb{1}_{\mathcal{C}}\{\Sigma^t \mathbb{1}_{\mathcal{C}}\}_n \simeq \mathbb{1}_{\mathcal{C}}[\Omega^n S^{n+t}] \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}).$$

*Proof.* Considering the commutative diagram in $\mathrm{Pr}^{\mathrm{R}}$ given by the forgetful functors

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S}) \arrow[d, "U_{\mathcal{S}_*}"'] & \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \arrow[l, "\Omega^\infty"'] \arrow[d, "U_{\mathcal{C}}"] \\ \mathcal{S}_* & \mathcal{C}\,, \arrow[l, "\Omega^\infty(0\to(-))"] \end{tikzcd}$$

we may pass to the left adjoints to obtain a diagram in $\mathrm{Pr}^{\mathrm{L}}$ of the form

$$\begin{tikzcd} \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S}) \arrow[r, "\mathbb{1}[-]"] & \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \\ \mathcal{S}_* \arrow[u, "F_k^{\mathrm{red}}(-)"] \arrow[r, "\mathbb{1}^{\mathrm{red}}[-]"'] & \mathcal{C}\,. \arrow[u, "\mathbb{1}\{-\}_k"'] \end{tikzcd}$$

It thus suffices to show that, for any connected $X \in \mathcal{S}_*$, we have $F_k^{\mathrm{red}}(X) \simeq \Omega^n\Sigma^n X$. For $n < \infty$ this is an easy consequence of the recognition theorem [HA, 5.2.6.10] combined with Dunn-additivity [HA, 5.1.2.2]. The $n = \infty$ case then follows readily by taking the filtered colimit and using [HA, 5.1.1.5]. □

### 2.2.2 Recollections on locally graded categories

Let $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ be a presentably symmetric monoidal category and write $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}} = \mathrm{Mod}_{\mathcal{V}}(\mathrm{Pr}^{\mathrm{L}})$ for the category of $\mathcal{V}$-linear categories. Throughout this section, we fix some $\mathcal{V} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ unless otherwise specified.

The unique, colimit preserving, symmetric monoidal functor $\mathcal{S} \to \mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$ taking the point to $\mathcal{V}$ induces a colimit preserving functor

$$\mathcal{V}[-] \colon \mathrm{Sp}^{\mathrm{cn}} \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}})\,,$$

whose right adjoint takes a $\mathcal{V}$-linear, presentably symmetric monoidal category $\mathcal{C}$ to the connective spectrum $\mathrm{pic}(\mathcal{C}) \in \mathrm{Sp}^{\mathrm{cn}}$ of $\otimes$-invertible elements in $\mathcal{C}^{\simeq}$ called the *Picard spectrum*.

**Definition 2.2.6.** For any $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$ we call

$$\mathcal{C}^{\mathrm{gr}} := \mathcal{C} \otimes_{\mathcal{V}} \mathcal{V}[\mathbb{Z}]$$

the category of *$\mathbb{Z}$-graded objects* in $\mathcal{C}$. Moreover, we refer to $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}^{\mathrm{gr}}}$ as the category of *locally $\mathbb{Z}$-graded*, $\mathcal{V}$-linear categories.

Since we only talk about $\mathbb{Z}$-gradings, we often leave the $\mathbb{Z}$ implicit and refer only to graded objects and locally graded categories.

**Remark 2.2.7.** Let us make some observations about categories of graded objects.

1. The symmetric monoidal category $\mathcal{V}^{\mathrm{gr}}$ is equivalent to the functor category $\mathrm{Fun}(\mathbb{Z}, \mathcal{V})$ carrying the Day-convolution monoidal structure. Thus, we think of an object of $\mathcal{I}_* \in \mathcal{V}^{\mathrm{gr}}$ as an integer indexed sequence $\{\mathcal{I}_n\}_{n\in\mathbb{Z}}$ of objects $\mathcal{I}_n \in \mathcal{V}$.

2. The inclusion $0 \to \mathbb{Z}$ induces a fully faithful, symmetric monoidal functor

$$\mathcal{V} \longrightarrow \mathcal{V}^{\mathrm{gr}}, \quad \mathcal{I} \mapsto \mathcal{I}(0)$$

which takes an object $\mathcal{I} \in \mathcal{V}$ to the graded object $\mathcal{I}(0)$ concentrated in grading 0. We think of $\mathcal{V} \subseteq \mathcal{V}^{\mathrm{gr}}$ as a full subcategory via this functor and omit the (0) whenever it is clear from context.

3. Dually, the terminal map $\mathbb{Z} \to 0$ induces a symmetric monoidal functor

$$\mathcal{V}^{\mathrm{gr}} \longrightarrow \mathcal{V}, \quad \mathcal{I}_* \mapsto \coprod_{n\in\mathbb{Z}} \mathcal{I}_n$$

which we think of taking a graded object to its "underlying" object. Moreover, this equips $\mathcal{V}$ with the structure of a $\mathcal{V}^{\mathrm{gr}}$-algebra, i.e. a locally graded category via the trivial grading.

4. Barring the symmetric monoidality, the above points apply to any $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$ and we make the same notational conventions and identifications.

For any $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}})$, giving a local grading on $\mathcal{C}$ that is compatible with the symmetric monoidal structure is equivalent to a map of commutative $\mathcal{V}$-algebras $\mathcal{V}[\mathbb{Z}] = \mathcal{V}^{\mathrm{gr}} \to \mathcal{C}$. By adjunction, this is the same datum as a map of connective spectra $\mathbb{Z} \to \mathrm{pic}(\mathcal{C})$. This description of the moduli spaces of local gradings gets a name.

**Definition 2.2.8.** For any $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}})$, following [Car22] we call the connective spectrum

$$\mathrm{pic}_{\mathbb{Z}}(\mathcal{C}) \coloneqq \tau_{\geq 0} \operatorname{map}_{\mathrm{Sp}}(\mathbb{Z}, \mathrm{pic}(\mathcal{C})) \in \mathrm{Sp}^{\mathrm{cn}}$$

the *strict Picard spectrum* of $\mathcal{C}$.

Restriction along the map $\mathbb{S} \to \mathbb{Z}$ induces a map $\mathrm{pic}_{\mathbb{Z}}(\mathcal{C}) \to \mathrm{pic}(\mathcal{C})$ and so any strict Picard element $\mathcal{L} \colon \mathbb{Z} \to \mathrm{pic}(\mathcal{C})$ has an underlying $\otimes$-invertible object, which we abusively[6] denote by the same name. Unwinding the definitions we see that, given a strict Picard element $\mathcal{L} \in \mathrm{pic}_{\mathbb{Z}}(\mathcal{C})$, the induced grading on $\mathcal{C}$ is a symmetric monoidal, $\mathcal{V}$-linear functor, which on objects is given by the formula

$$\mathcal{L}^{\otimes} \colon \mathcal{V}^{\mathrm{gr}} \longrightarrow \mathcal{C}, \quad \mathcal{I}_* \mapsto \coprod_{n \in \mathbb{Z}} \mathcal{I}_n \otimes \mathcal{L}^{\otimes n} .$$

**Notation 2.2.9.** The category $\mathcal{V}^{\mathrm{gr}}$ comes equipped with a "tautological" strict Picard element[7] which we denote by $\mathbb{1}_{\mathcal{V}}(1)$. Moreover, for any locally graded $\mathcal{V}$-linear category $\mathcal{C}$ and an integer $d \in \mathbb{Z}$, we write $\mathcal{I}(d) \coloneqq \mathcal{I} \otimes \mathbb{1}_{\mathcal{V}}(d)$ and refer to this as the $d$-fold *Serre-twist* of $\mathcal{I} \in \mathcal{C}$. Finally, if $\mathcal{V}$ is stable, we write

$$\pi_{t,d}(\mathcal{I}) \coloneqq \pi_0 \operatorname{Map}(\Sigma^t \mathbb{1}_{\mathcal{V}}(d), \mathcal{I}) \in \mathrm{Ab}$$

the graded homotopy groups of any $\mathcal{I} \in \mathcal{C}$. Note that, if the local grading on $\mathcal{C}$ is induced by a strict Picard element $\mathcal{L} \in \mathrm{pic}_{\mathbb{Z}}(\mathcal{C})$, then we have $\mathcal{I}(d) \simeq \mathcal{I} \otimes \mathcal{L}^{\otimes d}$ for all $\mathcal{I} \in \mathcal{C}$.

We think of a class $\mathcal{L} \in \mathrm{pic}_{\mathbb{Z}}(\mathcal{V})$ as a $\otimes$-invertible element, together with coherent trivializations of the natural $\Sigma_n$-actions on the $\otimes$-powers $\mathcal{L}^{\otimes n}$. More precisely, we have the following interaction of strict Picard elements with free algebras.

**Lemma 2.2.10.** Let $\mathcal{O}$ be an operad and $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}})$. For any strict Picard element $\mathcal{L} \in \mathrm{pic}(\mathcal{V})_{\mathbb{Z}}$, and any $\mathcal{I} \in \mathcal{C}$, we have a natural equivalence

$$\mathcal{S}^{\mathcal{L}} \colon \coprod_{n \geq 0} \big( (\mathcal{I}^{\otimes n} \otimes \mathcal{O}(n))_{h\Sigma_n} \otimes \mathcal{L}^{\otimes n} \big) \xrightarrow{\sim} U_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}\{\mathcal{I} \otimes \mathcal{L}\}_{\mathcal{O}}) .$$

*Proof.* Consider the free graded $\mathcal{O}$-algebra $\mathbb{1}_{\mathcal{C}}\{\mathcal{I}(1)\}_{\mathcal{O}}$ on the Serre-twist of $\mathcal{I}$. By the formula for the free $\mathcal{O}$-algebra [HA, 3.1.3.13] we have an equivalence

$$U_{\mathcal{C}}(\mathbb{1}_{\mathcal{V}}\{\mathcal{I}(1)\}) \simeq \coprod_{n \geq 0} (\mathcal{I}^{\otimes n}(n) \otimes \mathcal{O}(n))_{h\Sigma_n} .$$

Thus, the claim is immediate by applying the symmetric monoidal functor $\mathcal{C}^{\mathrm{gr}} \to \mathcal{C}$ induced by the strict Picard element $\mathcal{L}$. ☐

This equivalence gives rise to the *shearing construction* on classes in the free algebra, which we want to spell out more explicitly.

[6]Beware that, in general lifting a Picard element to a strict Picard element is additional data.
[7]induced by the identity functor $\mathcal{V}^{\mathrm{gr}} \to \mathcal{V}^{\mathrm{gr}}$.

**Construction 2.2.11.** Given a strict Picard element $\mathcal{L} \in \mathrm{pic}_{\mathbb{Z}}(\mathcal{V})$ and any class in the free algebra

$$Q\colon \mathcal{J}(n) \longrightarrow \mathbb{1}_{\mathcal{V}}\{\mathcal{I}(1)\}_{\mathcal{O}} \in \mathcal{V}^{\mathrm{gr}}$$

in grading $n$, we may use the equivalence of Lemma 2.2.10 to obtain a class

$$\mathcal{S}^{\mathcal{L}}(Q)\colon \mathcal{J}(n) \otimes \mathcal{L}^{\otimes n} \longrightarrow \mathbb{1}_{\mathcal{V}}\{\mathcal{I}(1) \otimes \mathcal{L}\} \in \mathcal{V}^{\mathrm{gr}}$$

which we refer to as the *shearing* of $Q$ by $\mathcal{L}$. Moreover, by using the inverse $\mathcal{L}^{-1}$, we see that this defines a bijection between $\mathcal{J}$-based classes in grading $n$ in the free algebra on $\mathcal{I}$ and $\mathcal{J} \otimes \mathcal{L}^{\otimes n}$-based classes in grading $n$ in the free algebra on $\mathcal{I} \otimes \mathcal{L}$.

The situation we are most interested in comes with an abundance of strict units, which is the following lemma.

**Lemma 2.2.12.** For any field $k$, the 2-fold suspension $\Sigma^2 k \in \mathrm{Mod}_k$ admits a refinement to strict Picard element, which is unique up to a choice of unit $u \in k^\times$ Moreover, if $\mathrm{char}(k) = 2$, the 1-fold suspension $\Sigma k \in \mathrm{Mod}_k$ refines to a strict Picard element, unique up to a unit of $k$.

*Proof.* Since $k$ is a field, the Picard spectrum has homotopy groups concentrated in two degrees, namely

$$\pi_0 \,\mathrm{pic}(\mathrm{Mod}_k) = \mathbb{Z} \quad \text{and} \quad \pi_1 \,\mathrm{pic}(\mathrm{Mod}_k) = k^\times .$$

By [Car22, Proposition 3.8] combined with [Car22, Proposition 3.2], we have $\pi_1 \,\mathrm{pic}_{\mathbb{Z}}(\mathrm{Mod}_k) = k^\times$. Moreover, $\pi_0$ is given by the kernel of the symmetric monoidal dimension map

$$\dim_k\colon \pi_0 \,\mathrm{pic}(\mathrm{Mod}_k) \to k^\times ,$$

so the claim follows. □

Consider the map of commutative monoids $\tau\colon \mathbb{N} \to \Omega^\infty \mathrm{pic}_{\mathbb{Z}}(\mathcal{V}^{\mathrm{gr}})$ picking out the tautological strict unit $\mathbb{1}_{\mathcal{V}}(1)$. Taking the Thom-construction, we obtain a commutative algebra in $\mathcal{V}^{\mathrm{gr}}$, which we denote by

$$\mathbb{1}_{\mathcal{V}}[\tau] \coloneqq \operatorname*{colim}_{n\in\mathbb{Z}} \mathbb{1}_{\mathcal{V}}(1)^{\otimes n} \in \mathrm{CAlg}(\mathcal{V}^{\mathrm{gr}}).$$

**Definition 2.2.13.** For any $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$ the category of *filtered objects* in $\mathcal{C}$ is defined to be the relative tensor product

$$\mathcal{C}^{\mathrm{fil}} \coloneqq \mathrm{Mod}_{\mathbb{1}_{\mathcal{V}}[\tau]}(\mathcal{V}^{\mathrm{gr}}) \otimes_{\mathcal{V}^{\mathrm{gr}}} \mathcal{C}^{\mathrm{gr}} \in \mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}^{\mathrm{gr}}} .$$

By construction, $\mathcal{V}^{\mathrm{fil}}$ is a presentably symmetric monoidal, $\mathcal{V}$-linear category equipped with a natural local grading given by the base change functor

$$- \otimes \mathbb{1}_{\mathcal{V}}[\tau]\colon \mathcal{V}^{\mathrm{gr}} \longrightarrow \mathcal{V}^{\mathrm{fil}} ,$$

which we call the *$\tau$-grading* on $\mathcal{V}^{\mathrm{fil}}$. Unwinding the definitions, we extract from the $\mathbb{1}_{\mathcal{V}}[\tau]$-action on any $\mathcal{I} \in \mathcal{C}^{\mathrm{fil}}$ a natural map

$$\tau\colon \mathcal{I}(1) \longrightarrow \mathcal{I} ,$$

i.e. a degree one map of the underlying graded objects. Accordingly, for any $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}_{\mathcal{V}}$, we depict the objects of $\mathcal{C}^{\mathrm{fil}}$ as integer indexed sequences

$$\cdots \to \mathcal{I}_{-2} \to \mathcal{I}_{-1} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \mathcal{I}_2 \to \cdots$$

where $\mathcal{I}_i \in \mathcal{C}$.

**Remark 2.2.14.** By [Lur14, Proposition 3.16], inverting the class $\tau\colon \mathbb{1}_{\mathcal{V}^{\mathrm{fil}}}(1) \to \mathbb{1}_{\mathcal{V}^{\mathrm{fil}}}$ gives us a commutative algebra $\mathbb{1}_{\mathcal{V}^{\mathrm{fil}}}[\tau^{-1}] \in \mathrm{CAlg}(\mathcal{V}^{\mathrm{fil}})$ such that the composite functor

$$\mathcal{V} \longrightarrow \mathcal{V}^{\mathrm{fil}} \xrightarrow{\tau^{-1}} \mathrm{Mod}_{\mathbb{1}_{\mathcal{V}^{\mathrm{fil}}}[\tau^{-1}]}(\mathcal{V}^{\mathrm{fil}})\,.$$

is an equivalence of categories. Under this identification, for any $\mathcal{I} \in \mathcal{V}^{\mathrm{fil}}$ the $\tau$-localization is given by taking the colimit along the filtration

$$\tau^{-1}\mathcal{I} = \varinjlim(\cdots \to \mathcal{I}_{-1} \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots) \in \mathcal{V}\,.$$

## 2.3 Power operations and decomposing free $\mathbb{E}_n$-algebras

Let $\mathcal{C}$ be a presentably monoidal category and let $\mathcal{O}$ be an operad. Given $\mathcal{I}, \mathcal{J} \in \mathcal{C}$, some class in the free $\mathcal{O}$-algebra

$$Q\colon \mathcal{J} \to \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_{\mathcal{O}}$$

and an $A \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, we get a natural map of sets

$$Q(-)\colon \pi_0 \mathrm{Map}_{\mathcal{C}}(\mathcal{I}, A) \to \pi_0 \mathrm{Map}_{\mathcal{C}}(\mathcal{J}, A)$$

defined by taking a class $v\colon \mathcal{I} \to A$ to the composite map

$$Q(v)\colon \mathcal{J} \xrightarrow{Q} \mathbb{1}_{\mathcal{C}}\{\mathcal{I}\}_{\mathcal{O}} \xrightarrow{\mathbb{1}_{\mathcal{C}}\{v\}_{\mathcal{O}}} A\,.$$

On typically considers the case where $\mathcal{I}, \mathcal{J}$ are shifts of a compact generator. Moreover, as we used in Lemma 2.2.10, there is a natural filtration on the free algebra given by the *arity* of the classes. Let us be precise about what we mean.

**Definition 2.3.1.** Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ and let $\mathcal{O}$ be an operad. An $\mathcal{O}$-algebra *power operation* in $\mathcal{C}$ of *type* $(t,s)$ and *arity* $d$ is a map of graded objects

$$Q\colon \Sigma^s \mathbb{1}_{\mathcal{C}}(d) \longrightarrow \mathbb{1}\{\Sigma^t \mathbb{1}_{\mathcal{C}}(1)\}_{\mathcal{O}} \in \mathcal{C}^{\mathrm{gr}}\,.$$

We also refer to $s - t$ as the *total degree* of $Q$.

For a prime $p$ and some $1 \le n \le \infty$, we discuss $\mathbb{E}_n$-power operations in $\mathbb{F}_p$-linear categories for $p$ a prime.

### 2.3.1 Operations and $\mathbb{E}_k$-decompositions for $p = 2$

We begin by describing the $\mathbb{E}_n$-power operations in $\mathrm{Mod}_{\mathbb{F}_2}$ for any $0 \le n \le \infty$. The construction is originally Kudo and Araki [KA56] and Browder [Bro58]. We briefly review it in our language, while also keeping track of the arity filtration by working with graded algebras.

Recall from Lemma 2.2.12 that for each $t \in \mathbb{Z}$ the suspension $\Sigma^t \mathbb{F}_2$ canonically refines to a strict Picard element. We denote the associated shearing map of Construction 2.2.11 simply by $\mathcal{S}^{(t)}$.

**Construction 2.3.2.** Let $1 \le n \le \infty$, then the space $\mathbb{E}_n(2)$ of configurations of two points in $\mathbb{R}^n$ is $C_2$-equivariantly homotopic to $S^{n-1}$ with the flip action. In particular, the $\mathbb{E}_n$-operations of arity 2 on a class in topological degree 0 and grading $d$ are parametrized by the homology of real projective space

$$(\mathbb{F}_2(d)^{\otimes 2} \otimes \mathbb{E}_n(2))_{hC_2} \simeq \mathbb{F}_2(2d) \otimes \mathbb{R}P^{n-1} \simeq \bigoplus_{0 \le i \le n-1} \Sigma^i \mathbb{F}_2(2d) \in \mathrm{Mod}^{\mathrm{gr}}_{\mathbb{F}_2}\,.$$

Thus, for each $0 \leq i \leq n-1$ we obtain a canonical arity 2 operation of type $(0, i)$ which we denote by

$$Q_i \colon \Sigma^i \mathbb{F}_2(2d) \longrightarrow \mathbb{F}_2\{\mathbb{F}_2(d)\}_n.$$

Thus, for any $t \in \mathbb{Z}$, we obtain a canonical arity 2 operation $\mathcal{S}^{(t)} Q_i$ of type $(t, i+2t)$. We often suppress the shearing in our notation, meaning for an element $v \in \pi_* A$ of some $A \in \mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}_{\mathbb{F}_2})$, we define

$$Q_i(v) := \mathcal{S}^{(|v|)} Q_i(v).$$

If $\mathcal{C}$ is any presentably $\mathbb{E}_n$-monoidal, locally graded, $\mathbb{F}_2$-linear category, we obtain corresponding classes $\mathcal{S}^{(t)} Q_i$ in the free algebra $\mathbb{1}_{\mathcal{C}}\{\Sigma^t \mathbb{1}_{\mathcal{C}}(d)\}_n$ via the $\mathbb{E}_n$-monoidal unit functor $\mathrm{Mod}_{\mathbb{F}_2}^{\mathrm{gr}} \to \mathcal{C}$, for which we use the same notation.

**Remark 2.3.3.** This is the *lower indexing* convention for the even primary power operations. The *upper indexing* is obtained by setting $Q^i(v) = Q_{i-|v|}(v)$, with the convention that this vanishes whenever the right hand side is not defined. This has the advantage that $Q^i$ always has total degree $i$, but for us the lower indexing will be more convenient.

**Proposition 2.3.4.** Let $\mathcal{C}$ be a presentably $\mathbb{E}_n$-monoidal, locally graded, $\mathbb{F}_2$-linear category and let $A \in \mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$. For any $i < n-1$, the operations

$$Q_i \colon \pi_{*,*} A \longrightarrow \pi_{2*+i, 2*} A,$$

of Construction 2.3.2 satisfy the following identities:

1. Frobenius: $Q_0(v) = v^2$ for all $v \in \pi_{*,*} A$
2. Unitality: $Q_i(1) = 0$ for $i > 0$
3. Additivity: $Q_i$ is an $\mathbb{F}_2$-linear map
4. Cartan formula: For all $v, w \in \pi_{*,*} A$ we have that

$$Q_i(vw) = \sum_{k+l=i} Q_k(v) Q_l(w)$$

*Proof.* The Frobenius and Unitality relations are straightforward from the definitions. The additivity and Cartan formula can be checked on the free algebra by analyzing the map

$$\mathbb{1}_{\mathcal{C}}\{u\}_n \longrightarrow \mathbb{1}_{\mathcal{C}}\{v, w\}_n, \quad u \mapsto v + w$$

classifying addition, as well as the map

$$\mathbb{1}_{\mathcal{C}}\{u\}_n \longrightarrow \mathbb{1}_{\mathcal{C}}\{v, w\}_n, \quad u \mapsto vw$$

classifying multiplication, respectively. Since these relations are preserved by the unit functor $\mathrm{Mod}_{\mathbb{F}_2}^{\mathrm{gr}} \to \mathcal{C}$, we can further reduce to the case $\mathcal{C} = \mathrm{Mod}_{\mathbb{F}_2}^{\mathrm{gr}}$, which is entirely classical, see for example [BMMS86, III. Theorem 3.1]. □

We want to give a coherent decomposition of the free graded $\mathbb{E}_n$-algebra in any $\mathbb{F}_2$-linear category. To do this, we fix the following setup for the remainder of this section.

**Setting 2.3.5.** Fix $0 \leq n \leq \infty$ and let $\mathcal{C}$ be a presentably $\mathbb{E}_{n+1}$-monoidal, locally graded, $\mathbb{F}_2$-linear category. Let $t$ and $d$ be integers and let $x$ be a generator in topological degree $t$ and grading $d$.

The computation is essentially due to Araki–Kudo in [KA56] where they give a description of the homology $\mathbb{F}_2[\Omega^n\Sigma^n S^t]$ as a polynomial algebra on the iterated operations $Q_i$. We observe that the proof given by Browder [Bro58] easily upgrades to a highly coherent statement.

For integers $0 \leq a \leq b < \infty$ and a sequence of non-negative integers $I = (i_a, i_{a+1} \ldots, i_b)$, we write

$$Q_I \coloneqq Q_a^{i_a} Q_{a+1}^{i_{a+1}} \cdots Q_b^{i_b}$$

for the composite operation, where by definition $Q_k^0 = \mathrm{id}$ for any $k$.

**Proposition 2.3.6.** In Setting 2.3.5 we have for any $0 \leq k < n+1$ a natural equivalence

$$\bigotimes_I \mathbb{1}_{\mathcal{C}}\{Q_I(x)\}_k \xrightarrow{\sim} \mathbb{1}_{\mathcal{C}}\{x\}_{n+1} \in \mathrm{Alg}_{\mathbb{E}_k}(\mathcal{C}^{\mathrm{gr}}),$$

of $\mathbb{E}_k$-algebras in $\mathcal{C}^{\mathrm{gr}}$, where the infinite tensor product is indexed over all sequences of non-negative integers $I = (i_{k+1}, \ldots, i_n)$.

*Proof.* Since $\mathcal{C}$ comes equipped with a colimit preserving $\mathbb{E}_{n+1}$-monoidal functor $\mathrm{Mod}_{\mathbb{F}_2} \to \mathcal{C}$, it suffices to show the claim in the universal case $\mathcal{C} = \mathrm{Mod}_{\mathbb{F}_2}$. The $n = \infty$ case follows readily by taking the filtered colimit over the finite cases, so we may assume $n < \infty$.

We argue using a simultaneous induction on $n$ and $k$. First, note that the case $k = 0$ and $n = 1$ is immediate from the fact that for any $\mathcal{I} \in \mathrm{Mod}_{\mathbb{F}_2}^{\mathrm{gr}}$ we have the formula

$$\mathbb{F}_2\{\mathcal{I}\}_1 \simeq \bigoplus_{m\geq 0} \mathcal{I}^{\otimes m}$$

for the underlying module of a free $\mathbb{E}_1$-algebra, i.e. the homotopy groups are polynomial on any $\mathbb{F}_2$-linear basis of $\mathcal{I}$.

Thus, we assume that $k = n$ and that we have shown the claim for all pairs $(k', n')$ with $k' \leq n' < n$. In this case, the only operations in the allowed range are the iterates $Q_n^i$ for $i \geq 0$. By construction of the $Q_n^i$, we have natural maps of graded $\mathbb{E}_n$-algebras

$$q^i \colon \mathbb{F}_2\{Q_n^i(x)\}_n \to \mathbb{F}_2\{x\}_{n+1}.$$

By Dunn-additivity [HA, 5.1.2.2], we know that for any $j \geq 0$ the multiplication map

$$\mu_j \colon \mathbb{F}_2\{x\}_n^{\otimes j} \longrightarrow \mathbb{F}_2\{x\}_n$$

is a map of $\mathbb{E}_n$-algebras and so the composite

$$A_j \coloneqq \bigotimes_{0\leq i\leq j} \mathbb{F}_2\{Q_n^i(x)\}_n \xrightarrow{\otimes q^i} \bigotimes_{0\leq i\leq j} \mathbb{F}_2\{x\}_n \xrightarrow{\mu_j} \mathbb{F}_2\{x\}_{n+1},$$

defines a map of $\mathbb{E}_n$-algebras in $\mathrm{Mod}_{\mathbb{F}_2}^{\mathrm{gr}}$ as well. Taking the filtered colimit over $j$, we obtain the desired comparison map $q \colon A = \mathrm{colim}_j A_j \to \mathbb{F}_2\{x\}_{n+1}$. Since the construction of the map $q$ is compatible with the shearing functor of Lemma 2.2.10, we may assume that $x$ is in positive topological degree by Lemma 2.2.12. Moreover, the functor that forgets the grading and taking homotopy groups are both conservative, so it suffices to show that $q$ induces an equivalence on (un-graded) homotopy groups. Using the identification of the free $\mathbb{E}_m$-algebra on class in positive degree $t$ with the homology of $\Omega^m\Sigma^m S^t$ of Proposition 2.2.5, we obtain a comparison map of homologies

$$\pi_* q \colon \bigotimes_{i\geq 0} H_*(\Omega^n\Sigma^n S^{|Q_n^i(x)|}; \mathbb{F}_2) \longrightarrow H_*(\Omega^{n+1}\Sigma^{n+1}S^t; \mathbb{F}_2).$$

By the inductive assumption, the left hand side is given by an infinite polynomial algebra on classes of the form $Q_1^{i_1} \ldots Q_{n-1}^{i_{n-1}}(x)$ and hence the map is an equivalence, for example by the computation of the right hand side in [Bro58, Theorem 3]. $\square$

As a consequence, we get the following decomposition of the unit in any $\mathbb{E}_{n+1}$-monoidal $\mathbb{F}_2$-linear category.

**Corollary 2.3.7.** In Setting 2.3.5 with $1 \le k \le n$, the augmentation map $\mathbb{1}_\mathcal{C}\{x\}_{n+1} \to \mathbb{1}_\mathcal{C}$ taking $x$ to 0 induces an equivalence of $\mathbb{E}_{k-1}$-algebras

$$\bigotimes_I (\mathbb{1}_\mathcal{C}\{x\}_{n+1} \otimes_{\mathbb{1}_\mathcal{C}\{Q_I(x)\}_k} \mathbb{1}_\mathcal{C}) \xrightarrow{\sim} \mathbb{1}_\mathcal{C} \in \mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathrm{Mod}_{\mathbb{1}_\mathcal{C}\{x\}_{n+1}}(\mathcal{C}^{\mathrm{gr}}))\,,$$

where $I$ runs over all sequences of non-negative integers $I = (i_{k+1}, \dots, i_n)$.

*Proof.* Observe that, by the relation $Q_I(0) = 0$ of Proposition 2.3.4 the comparison map of Proposition 2.3.6 upgrades to an equivalence of augmented algebras if we equip all free algebras in sight with the 0-augmentation. Thus, we may iteratively apply Construction 2.2.4 to the finite stages of the decomposition in Proposition 2.3.6 and take the filtered colimit to get the desired equivalence. □

With this in hand, we may apply Lemma 2.2.2 to obtain a filtered decomposition of $\mathbb{E}_n$-cofibers.

**Corollary 2.3.8.** In Setting 2.3.5 with $1 \le k \le n$, for any $A \in \mathrm{Alg}_{\mathbb{E}_{n+1}}(\mathcal{C})$ and any class $v \colon \Sigma^t \mathbb{1}_\mathcal{C} \to A$ we have an equivalence of filtered $\mathbb{E}_{k-1}$-algebras

$$\bigotimes_I A /\!\!/^{k-1} Q_I(\tau v) \xrightarrow{\sim} A /\!\!/^{n} \tau v \in \mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}^{\mathrm{fil}})$$

where $I$ runs over all sequences of non-negative integers $I = (i_{k+1}, \dots, i_n)$.

*Proof.* Applying Corollary 2.3.7 to the category $\mathcal{C}^{\mathrm{fil}}$ with the $\tau$-grading, we get an equivalence

$$\mathbb{1}_\mathcal{C} \simeq \bigotimes_I (\mathbb{1}_\mathcal{C} \otimes_{\mathbb{1}_\mathcal{C}\{Q_I(\tau v)\}_k} \mathbb{1}_\mathcal{C}\{\tau v\}_{n+1})$$

in the category $\mathrm{Mod}_{\mathbb{1}_\mathcal{C}\{\tau v\}_n}(\mathrm{Alg}_{\mathbb{E}_{k-1}}(\mathcal{C}^{\mathrm{fil}}))$. Base changing along the map $\mathbb{1}_\mathcal{C}\{\tau v\}_{n+1} \to A$ induced by $v$, we learn that we have an equivalence of filtered $\mathbb{E}_{k-1}$-algebras

$$\mathbb{1}_\mathcal{C} \otimes_{\mathbb{1}_\mathcal{C}\{\tau v\}_{n+1}} A \simeq \bigotimes_I (\mathbb{1}_\mathcal{C} \otimes_{\mathbb{1}_\mathcal{C}\{Q_I(\tau v)\}_k} A)\,.$$

Thus, we may apply Lemma 2.2.2 to identify the relative tensor products with $\mathbb{E}_n$ and $\mathbb{E}_{k-1}$ cofibers respectively, which yields our claim. □

### 2.3.2 Operations and $\mathbb{E}_k$-decompositions for $p > 2$

For odd primes $p > 2$, the situation is more subtle, but still uniform. The following operations were originally constructed by Dyer and Lashof in [DL62]. We briefly review them, roughly following the later definition of Cohen in [Coh76], while keeping track of the grading by arity.

Throughout this section, we consider the refinement of $\Sigma^t \mathbb{F}_p$ to a strict Picard element for any even $t \in \mathbb{Z}$, as induced by the unit $1 \in \mathbb{F}_p^\times$ via Lemma 2.2.12. We denote the associated shearing map of Construction 2.2.11 simply by $\mathcal{S}^{(t)}$.

**Construction 2.3.9.** Let $2 \le n \le \infty$ and $p$ be an odd prime. Then, for a generator in topological degree 0 and grading $d$, by the computation of $\mathbb{E}_n(p) \otimes \mathbb{F}_p$ in [Coh76, Theorem 5.2] there are classes in grading $pd$

$$\beta^\varepsilon P_i \colon \Sigma^{2i(p-1)-\varepsilon} \mathbb{F}_p(pd) \longrightarrow \mathbb{F}_p\{\mathbb{F}_p(d)\}_n$$

for any integer $1 \leq i \leq \frac{n-1}{2}$ and $\varepsilon \in \{0,1\}$ as well as the degree zero operation given by the $p$-th power $P_0 = (-)^p$. Thus, we obtain sheared operations $\mathcal{S}^{(t)}(\beta^\varepsilon P_i)$ of type $(t, pt + 2i(p-1) - \varepsilon)$ for any even integer $t$.

Similarly, for a generator in topological degree 1 we have classes

$$\beta^\varepsilon P_{j-\frac{1}{2}} \colon \Sigma^{p+(2j-1)(p-1)-\varepsilon}\mathbb{F}_p(pd) \longrightarrow \mathbb{F}_p\{\Sigma\mathbb{F}_p(d)\}_n$$

for any integer $1 \leq j \leq \frac{n-1}{2}$ and thus obtain sheared arity $p$ operations $\mathcal{S}^{(t)}(\beta^\varepsilon P_{j-\frac{1}{2}})$ of type $(t+1, p(t+1) + (2j-1)(p-1))$ for any even integer $t$.

We again suppress the shearing from the notation and define operations acting on *even* classes

$$\beta^\varepsilon P_i(v) \coloneqq \mathcal{S}^{(|v|)}\beta^\varepsilon P_i(v), \quad |v| \text{ even}$$

as well as operations acting on *odd* classes

$$\beta^\varepsilon P_{j-\frac{1}{2}}(w) \coloneqq \mathcal{S}^{(|w|-1)}\beta^\varepsilon P_{j-\frac{1}{2}}(w), \quad |w| \text{ odd}\,,$$

with the convention that the operations act by 0 whenever the parities do not match.

As in the $p = 2$ case of Construction 2.3.2, these operations are defined in any presentably $\mathbb{E}_n$-monoidal, locally graded, $\mathbb{F}_p$-linear category.

**Remark 2.3.10.** The purpose of the half-integer notation is two-fold. First, as in the $p = 2$ case, there also exists an upper indexing convention, defined such that $\beta^\varepsilon P^i$ has total degree $2i(p-1) - \varepsilon$ which is given by setting

$$\beta^\varepsilon P^i(v) \coloneqq \beta^\varepsilon P_{i-\frac{|v|}{2}}(v)$$

Second, writing $\frac{1}{2}\mathbb{Z} \subseteq \mathbb{Q}$ for the additive subgroup of half-integers, for any $i \in \frac{1}{2}\mathbb{Z}$ the operation $\beta^\varepsilon P_i$ acting on a class in (not-necessarily even) topological degree $t$ has type $(t, pt + 2i(p-1) - \varepsilon)$, so we do not need to explicitly differentiate between the action on odd or even degree classes.

**Proposition 2.3.11.** Let $\mathcal{C}$ be a presentably $\mathbb{E}_n$-monoidal, locally graded, $\mathbb{F}_p$-linear category. For any $A \in \mathrm{CAlg}(\mathcal{C})$ and any pair $(i, \varepsilon)$ where $i$ is half integer such that $0 \leq i < \frac{n-1}{2}$ and $\varepsilon \in \{0,1\}$, the operations

$$\beta^\varepsilon P_i \colon \pi_{*,*}A \longrightarrow \pi_{p*+2i(p-1)-\varepsilon, p*}A$$

defined in Construction 2.3.9 satisfy the following relations:

1. Frobenius: $P_0(v) = v^p$ for all $v \in \pi_{*,*}A$
2. Unitality: $\beta^\varepsilon P_i(1) = 0$ for all $i > 0$.
3. Additivity: $\beta^\varepsilon P_i(v + w) = \beta^\varepsilon P_i(v) + \beta^\varepsilon P_i(w)$ for all $v, w \in \pi_{*,*}A$.
4. Cartan Formula:
$$P_i(vw) = \sum_{k+l=i} P_k(v)P_l(w)$$
$$\beta P_i(vw) = \sum_{k+l=i} \Big(\beta P_k(v)P_l(w) + (-1)^{|x|}P_k(v)\beta P_l(w)\Big)$$
for all $v, w \in \pi_{*,*}A$.

*Proof.* The Frobenius and Unitality relations are straightforward from the definitions. For the remaining two, we may argue as in Proposition 2.3.4 to reduce to the universal case of $\mathcal{C} = \mathrm{Mod}^{\mathrm{gr}}_{\mathbb{F}_p}$. In this case, the claims are shown in Theorem 2.2 and Theorem 2.3 in [DL62] respectively. □

We also adopt and expand the following terminology of [DL62] to our indexing convention.

**Definition 2.3.12.** A *Dyer–Lashof* sequence $I = (\varepsilon_n, i_n, \dots, \varepsilon_1, i_1)$ is a sequence of half-integers with $i_m \in \frac{1}{2}\mathbb{Z}_{\geq 1}$ and $\varepsilon_m \in \{0,1\}$ for all $m$. For any $1 \leq k \leq \infty$, the composite of Dyer–Lashof operations $P_I(x) = \beta^{\varepsilon_n} P_{i_n} \cdots \beta^{\varepsilon_1} P_{i_1}(x)$ applied to a class $x$ in topological degree $t$ is called *$k$-allowable* if the following hold:

1. $i_1$ is an integer if and only if $t$ is even.
2. We have $i_{m+1} \leq i_m$ for all $m$.
3. If $\varepsilon_m = 1$ for any $k$, then $i_m - i_{m+1}$ is a half integer.
4. We have $i_m \leq \frac{k-1}{2}$ for all $m$.

For any $n \leq k$, we say that a $k$-allowable sequence $P_I(x)$ is *$n$-bounded* if $i_m > \frac{n-1}{2}$ for all $m$ and write $I \gtrsim n$ in this case.

The definition above precisely excludes composites that are automatically zero or to which an Adem relation could be applied. For example, the sequences $P_1P_1, P_{\frac{1}{2}}\beta P_1$ and $\beta P_{\frac{1}{2}}\beta P_1$ are allowable. With this terminology in hand, we get a description of the homotopy groups of the free $\mathbb{E}_k$-algebra from the computations of [DL62].

**Lemma 2.3.13.** Let $x$ be a generator in topological degree $t \in \mathbb{Z}$. For any $k \geq 0$ such that $t$ and $k$ have different parity, $\pi_*\mathbb{F}_p\{x\}_k$ is freely generated as a discrete, graded commutative $\mathbb{F}_p$-algebra by the classes $P_I(x)$ given by all $k$-allowable composite operations.

*Proof.* Since the two-fold suspension $\Sigma^2\mathbb{F}_p$ refines to a strict Picard element by Lemma 2.2.12, the shearing equivalence of Lemma 2.2.10 combined with the construction of the operations in Construction 2.3.9 tells us that it suffices to prove the claim for $t > 0$. By Proposition 2.2.5 the statement reduces to the computation of the homology $\pi_*\mathbb{F}_p[\Omega^k S^{k+t}] = H_*(\Omega^k S^{k+t}; \mathbb{F}_p)$ done in [DL62, Theorem 5.2]. $\square$

We are now ready to state an odd primary analogue of Proposition 2.3.6. For simplicity and since it as all we will need, we restrict ourselves to the decomposition of a free $\mathbb{E}_\infty$-algebra.

**Proposition 2.3.14.** Let $\mathcal{C}$ be a presentably symmetric monoidal, locally graded, $\mathbb{F}_p$-linear category and $x$ a generator in topological degree $t$ and grading $d$. For any $k \geq 0$ such that $t$ and $k$ have different parity, we have a natural equivalence of graded $\mathbb{E}_k$-algebras

$$\left(\bigotimes_{I \gtrsim k} \mathbb{1}_{\mathcal{C}}\{P_I(x)\}_k\right) \otimes \left(\bigotimes_{J \gtrsim k+1} \mathbb{1}_{\mathcal{C}}\{P_J(x)\}_{k+1}\right) \xrightarrow{\sim} \mathbb{1}_{\mathcal{C}}\{x\}_\infty\,,$$

where $P_I(x)$ runs over all $\infty$-allowable, $k$-bounded composites which preserve parity and $P_J(x)$ runs over all $\infty$-allowable, $k+1$-bounded composites which flip parity.

*Proof.* We may construct a natural comparison map of graded $\mathbb{E}_k$-algebras as in the proof of Proposition 2.3.6 and only consider the universal case $\mathcal{C} = \mathrm{Mod}^{\mathrm{gr}}_{\mathbb{F}_p}$. Since the functor which forgets the grading is conservative, we further reduce to the ungraded statement. Moreover, the shearing equivalence of Lemma 2.2.10 combined with the strictness of even suspensions Lemma 2.2.12 tells us that we can assume $t > 0$. Using the presentation of the free $\mathbb{E}_n$-algebra via loop spaces discussed in Proposition 2.2.5 and the fact we can check equivalences on homotopy groups, the map being an equivalences becomes precisely the statement of [DL62, Theorem 5.1] computing the homology $H_*(\Omega^\infty\Sigma^\infty S^t; \mathbb{F}_p)$ combined with Lemma 2.3.13. $\square$

**Corollary 2.3.15.** Let $\mathcal{C}$ be a presentably symmetric monoidal, $\mathbb{F}_p$-linear category and let $A \in \mathrm{CAlg}(\mathcal{C})$. For any $t \in \mathbb{Z}$ such that $t$ and $k$ have different parity and a class $v\colon \Sigma^t \mathbb{1}_{\mathcal{C}} \to A$, we have a natural equivalence

$$\bigotimes_{I \gtrsim k} A/\!\!/^k P_I(\tau v) \otimes \bigotimes_{J \gtrsim k+1} A/\!\!/^{k+1} P_J(\tau v) \xrightarrow{\sim} A/\!\!/^\infty \tau v$$

of filtered $\mathbb{E}_{k-1}$-algebras in $\mathcal{C}$.

*Proof.* Argue exactly as in the proof of Corollary 2.3.8 using Proposition 2.3.14. □

## 2.4 Points in positive characteristic

Having done our homework on Dyer–Lashof operations, we can now turn towards discussing whether $\mathbb{E}_\infty$-rings admit reduced points.

We begin by reviewing the machinery of nullstellensatzian rings and nilpotence detecting maps of [BSY22], which forms the basis of our arguments.

At the prime $p = 2$, we show that the universal map killing an element which squares to 0 detects nilpotence in Proposition 2.4.14. From this, we deduce that all nullstellensatzian $\mathbb{F}_2$-algebras must be separably closed, graded fields and so our main theorem in the form of Theorem 2.4.17 follows.

For $p > 2$, we show in Proposition 2.4.22 that the free commutative $\mathbb{F}_p$-algebra on a $p$-nilpotent generator $\mathbb{F}_p\{x\}/\!\!/^\infty x^p$ contains non-nilpotent classes, given by certain parity preserving composite operations $P_I(x)$. As am immediate consequence in Theorem 2.4.23, we obtain a commutative ring which admits no reduced points in the sense of Definition 2.1.2. By further analyzing the interaction of nilpotent classes with the Dyer–Lashof operations, we also deduce some properties of nullstellensatzian $\mathbb{F}_p$-algebras in Proposition 2.4.27.

### 2.4.1 Geometric points and detecting nilpotence

Let us begin by recalling the definition of nullstellensatzian objects of [BSY22] in an arbitrary category.

**Definition 2.4.1.** Let $\mathcal{C}$ be a category and $\alpha$ be a regular cardinal. An object $X \in \mathcal{C}$ is called *$\alpha$-nullstellensatzian* if every $\alpha$-compact object $(f\colon X \to Y) \in (\mathcal{C}_{X/})^\alpha$ admits a section $Y \to X$. For $\alpha = \omega_0$ we call $X$ just *nullstellensatzian*.

By the weak form of Hilbert's Nullstellensatz, the nullstellensatzian objects in the 1-category of ordinary commutative rings $\mathrm{CAlg}(\mathrm{Ab})$ are precisely the algebraically closed fields, which is the reason for the terminology.

A higher algebra variant of the Nullstellensatz was proven by Burklund–Schlank–Yuan for the monochromatic category. More precisely [BSY22, Theorem A] shows that, in the category $T(n)$-local $\mathbb{E}_\infty$-rings for some $n \geq 0$, the nullstellensatzian objects are precisely those given by the Lubin–Tate theory $E_n(F)$ attached to an algebraically closed field $F$.

Moreover, they show that nullstellensatzian objects exist in any well behaved category of commutative algebras. We are in an even nicer situation, namely our categories are rigid in the following sense.

**Definition 2.4.2.** A compactly generated category $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ is called *rigid* if every compact object is dualizable and the monoidal unit $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$ is a compact object.

Note that, $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}$ being compact implies that *every* dualizable object of $\mathcal{C}$ is compact. One can define rigid categories in greater generality, but for us rigid will always mean compactly generated.

**Example 2.4.3.** The category of spectra Sp is rigid. Moreover, for any rigid category $\mathcal{C}$ and any $A \in \mathrm{CAlg}(\mathcal{C})$, the category $\mathrm{Mod}_A(\mathcal{C})$ is again rigid.

The nullstellensatzian objects in a category of algebras get a less cumbersome name.

**Terminology 2.4.4.** Let $\mathcal{C}$ be a symmetric monoidal category and $A \in \mathrm{CAlg}(\mathcal{C})$. A *geometric point* of $A$ is a nullstellensatzian object $T \in \mathrm{CAlg}_A(\mathcal{C})$.

Note that we slightly deviate from the terminology of [BSY22], where a geometric point is an equivalence class of nullstellensatzians in $\mathrm{CAlg}_A(\mathcal{C})$ under the relation that $T_1 \sim T_2$ if $T_1 \otimes T_2$ is non-terminal. We then have the following very general existence result.

**Proposition 2.4.5** ([BSY22, Proposition A.17])**.** Let $\mathcal{C}$ be a stable rigid category. For any regular cardinal $\alpha$ and any non-zero $A \in \mathrm{CAlg}(\mathcal{C})$, there exists an $\alpha$-nullstellensatzian object $T \in \mathrm{CAlg}_A(\mathcal{C})$. In particular, $A$ admits a geometric point.

Since we assume compact generation, we immediately learn the following about detecting non-nilpotent classes.

**Corollary 2.4.6.** Let $\mathcal{C}$ be a stable, rigid category and $A \in \mathrm{CAlg}(\mathcal{C})$. For any non-nilpotent class $v\colon \Sigma^t \mathbb{1}_{\mathcal{C}} \to A$ there exists a geometric point $f\colon A \to T$ such that $f(v)$ is invertible.

*Proof.* Indeed, since $v$ is not nilpotent and $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}^\omega$ by assumption, the localization $\mathbb{1}_{\mathcal{C}} \to A[v^{-1}]$ is non-zero, and hence admits a geometric point $T$, so the claim follows. □

In particular, we learn that, all non-nilpotent classes of a nullstellensatzian commutative ring are already invertible.[8] Hence, what remains is the much more subtle task of analyzing the behavior of nilpotent elements.

To do this, we recall some definitions and useful facts discussed in [BSY22, Section 4], and prove a closure property of nilpotence detecting $\mathbb{E}_0$-algebras. For any $A \in \mathcal{C}$, we say that a map $v\colon \mathcal{I} \to \mathcal{J} \in \mathcal{C}$ is *$\otimes$-nilpotent at $A$* if $v^{\otimes n} \otimes A \simeq 0$ for some $n \geq 1$. If $A = \mathbb{1}_{\mathcal{C}}$, we simply say that $v$ is $\otimes$-nilpotent.

**Definition 2.4.7.** An object $A \in \mathcal{C}$ *detects nilpotence* if any map $v\colon \mathcal{I} \to \mathcal{J}$ in $\mathcal{C}$ with compact source $\mathcal{I} \in \mathcal{C}^\omega$ is $\otimes$-nilpotent at $A$ if and only if it is $\otimes$-nilpotent. Moreover, we say that a map $f\colon A \to B \in \mathrm{CAlg}(\mathcal{C})$ *detects nilpotence* if $B \in \mathrm{Mod}_A(\mathcal{C})$ detects nilpotence.

**Example 2.4.8.** Let $A \in \mathcal{C}$ such that the functor $-\otimes A\colon \mathcal{C} \to \mathcal{C}$ is conservative. Then by [BSY22, Lemma 4.18] the object $A \in \mathcal{C}$ detects nilpotence. As consequence, see [BSY22, Lemma 4.21], we learn that if $v\colon \mathcal{I} \to \mathbb{1}_{\mathcal{C}}$ is $\otimes$-nilpotent, the cofiber $\mathbb{1}_{\mathcal{C}}/v \in \mathcal{C}$ detects nilpotence.

As we assume that $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}^\omega$, any nilpotence detecting map of commutative algebras $f\colon A \to B$ has the property that, the induced map of graded rings $\pi_* A \to \pi_* B$ contains only nilpotent classes in its kernel. One may think of the notion of Definition 2.4.7 as a highly coherent version of this $\pi_*$ condition. Crucially, nilpotence detecting maps of $\mathbb{E}_\infty$-rings are closed under base change and composition.

**Proposition 2.4.9.** Let $f\colon A \to B$, $g\colon B \to C$ and $h\colon A \to D$ be maps in $\mathrm{CAlg}(\mathcal{C})$ and suppose $f$ and $g$ detect nilpotence. Then the composite $gf\colon A \to C$ and the base change $f \otimes_A h\colon D \to D \otimes_A B$ detect nilpotence.

*Proof.* The first claim is [BSY22, Lemma 4.30] while the second is [BSY22, Proposition 4.24]. □

[8] Beware that, if one does not assume $\mathbb{1}_{\mathcal{C}} \in \mathcal{C}^\omega$, the notion of a class being nilpotent is dependent on the ambient category. For example, element $p \in \pi_0 \mathbb{S}_p^\wedge$ is not nilpotent in the $\mathbb{E}_\infty$-ring $\mathbb{S}_p^\wedge$, but the localization at $p$ is zero in the category of $p$-complete spectra.

We call an $\mathbb{E}_0$-algebra $(f\colon \mathbb{1}_{\mathcal{C}} \to A) \in \mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$ a *weak ring* if the map $A \otimes f\colon A \to A \otimes A$ admits a retraction $\mu\colon A \otimes A \to A$. Note that we do not take $\mu$ to be part of the datum of $A$, so that being a weak ring is a *property* of an $\mathbb{E}_0$-algebra.

**Lemma 2.4.10.** Let $f\colon \mathbb{1}_{\mathcal{C}} \to A$ be a weak ring such that, for any map $v\colon \mathcal{I} \to \mathbb{1}_{\mathcal{C}}$ with $\mathcal{I} \in \mathcal{C}^{\omega}$, the composite $f(v)\colon \mathcal{I} \to A$ being null implies that $v$ is $\otimes$-nilpotent. Then $A$ detects nilpotence.

*Proof.* Let $w\colon \mathcal{I} \to \mathcal{J}$ be a map with $\mathcal{I} \in \mathcal{C}^{\omega}$ that is $\otimes$-nilpotent at $A$. Since $\mathcal{C}$ is compactly generated and rigid, to check that the weak ring $A$ detects nilpotence, we can reduce to the case where $\mathcal{J}$ is compact by [BSY22, Lemma 4.15] and thus dualizable. Given a nullhomotopy of the map $w^{\otimes n} \otimes A\colon \mathcal{I}^{\otimes n} \otimes A \to \mathcal{J}^{\otimes n} \otimes A$, we may pre-compose with the unit $f\colon \mathbb{1}_{\mathcal{C}} \to A$ and pass to the mate to obtain a nullhomotopy of the composite

$$f(w^{\otimes n})^{\natural}\colon \mathcal{I}^{\otimes n} \otimes (\mathcal{J}^{\otimes n})^{\vee} \to \mathbb{1}_{\mathcal{C}} \to A\,.$$

Since taking the mate is symmetric monoidal i.e. $(w^{\otimes n})^{\natural} \simeq (w^{\natural})^{\otimes n}$, the assumption on $f\colon \mathbb{1}_{\mathcal{C}} \to A$ implies that $w^{\natural}\colon \mathcal{I} \otimes \mathcal{J}^{\vee} \to \mathbb{1}_{\mathcal{C}}$ is $\otimes$-nilpotent. Thus, by the same reasoning, $w$ is $\otimes$-nilpotent and we conclude. □

We have the following closure property of filtered colimits of $\mathbb{E}_0$-algebras with nilpotent fiber.

**Lemma 2.4.11.** Let $f_i\colon \mathbb{1}_{\mathcal{C}} \to A_i$ for $i \in I$ be a filtered diagram of $\mathbb{E}_0$-algebras in $\mathcal{C}$ and write $f\colon \mathbb{1}_{\mathcal{C}} \to A = \varinjlim A_i \in \mathcal{C}_{\mathbb{1}_{\mathcal{C}}/}$ for the colimit. If each $f_i\colon \mathbb{1}_{\mathcal{C}} \to A_i$ has $\otimes$-nilpotent fiber and $A$ is a weak ring, then $A$ detects nilpotence.

*Proof.* Let $v\colon \mathcal{I} \to \mathbb{1}_{\mathcal{C}}$ be a map with $\mathcal{I} \in \mathcal{C}^{\omega}$ such that the composite $f(v)\colon \mathcal{I} \to \mathbb{1}_{\mathcal{C}} \to A$ is null. Since $A$ is a weak ring, Lemma 2.4.10 tells us that it suffices to show that $v$ is $\otimes$-nilpotent.

As $\mathcal{I}$ is compact, the nullhomotopy of $f(v)$ factors through a finite stage of the filtered colimit, i.e. we obtain a nullhomotopy of $f_i(v)\colon \mathcal{I} \to \mathbb{1}_{\mathcal{C}} \to A_i$. for some $i$. In particular, $v$ factors through the fiber of $f_i\colon \mathbb{1}_{\mathcal{C}} \to A_i$, which was $\otimes$-nilpotent by assumption, and so $v$ is $\otimes$-nilpotent. □

### 2.4.2 Reduced and geometric points over $\mathbb{F}_2$

We now work with the stable rigid category $\mathrm{Mod}_{\mathbb{F}_2}(\mathrm{Sp})$ of modules over the Eilenberg–MacLane spectrum $\mathbb{F}_2$. Given a commutative ring $A \in \mathrm{CAlg}_{\mathbb{F}_2}$, we want to construct a ring map $A \to T$ such that $\pi_*T$ is a graded field. The procedure is in some sense straightforward, given a nilpotent element $v \in \pi_*A$, we take the $\mathbb{E}_\infty$-cofiber

$$A \longrightarrow A/\!\!/^{\infty} v \in \mathrm{CAlg}_{\mathbb{F}_2} \tag{2.1}$$

to kill $v$. Iterating this and taking the filtered colimit, we arrive at a ring $T'$ which has no non-trivial nilpotent elements in $\pi_0 T'$. Since we want to further map to a field however, we need to show that $T'$ is *not* the zero ring and this is the difficult part. We ensure this via Proposition 2.4.14 by showing that the map (2.1) *detects nilpotence* in the sense of Definition 2.4.7. In particular, this implies that at no stage in the construction of $T'$ we ended up with the zero ring, and our main result for $p = 2$ in the form of Theorem 2.4.17 follows.

The natural reason why the map (2.1) might *fail* to detect nilpotence, is that it also kills all the power operations $Q_i(v)$ of Construction 2.3.2 and their iterates. However, these obstructions are themselves nilpotent on nilpotent classes, as the following lemma shows.

**Lemma 2.4.12.** Let $A \in \mathrm{CAlg}_{\mathbb{F}_2}$ and let $v \in \pi_*A$ be some class. Then, for any composite operation $Q_a^{i_a} \dots Q_b^{i_b}$ with $a \leq b$ we have that

$$Q_{2^n a}^{i_a} \cdots Q_{2^n b}^{i_b}(v^{2^n}) = Q_a^{i_a} \cdots Q_b^{i_b}(v)^{2^n} \in \pi_*A.$$

In particular, if $v$ is nilpotent of exponent $m \leq 2^n$, then so is each iterated operation $Q_I(v)$.

*Proof.* By the Cartan formula of Proposition 2.3.4 we know that for any $i \geq 0$ we have

$$Q_i(v^2) = \sum_{k+l=i} Q_k(v)Q_l(v)\,.$$

Observe that, every decomposition $i = k + l$ appears exactly twice, unless $l = k$ which only appears if $i$ is even. Since $2 = 0 \in \mathbb{F}_2$, this means that $Q_i(v^2)$ vanishes, unless every $i$ is divisible by 2, in which case the only remain term is

$$Q_i(v^2) = Q_{i/2}(v)^2.$$

The claim then follows easily by iterating this argument. □

In Corollary 2.3.8 we used the description of the free algebra obtained in Proposition 2.3.6 to obtain a decomposition of the $\mathbb{E}_n$-cofiber by some class. We now analyze this decomposition more closely in the case $n = 1$, which provide us with the minimal amount of structure we need to deduce detecting nilpotence.

**Lemma 2.4.13.** Let $A$ be an $\mathbb{E}_2$-algebra over $\mathbb{F}_2$ and let $v\colon \Sigma^t\mathbb{F}_2 \to A$ be some class. Then the filtered $\mathbb{E}_1$-cofiber

$$A \longrightarrow A/\!\!/^1 v\tau \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}^{\mathrm{fil}}_{\mathbb{F}_2})$$

has filtration stage $(2^{i+1} - 1)$ given by the $A$-module

$$A/(v, Q_1(v), \ldots, Q_1^i(v))\,.$$

*Proof.* By Corollary 2.3.8 we have a filtered colimit description of the $\mathbb{E}_1$-cofiber of the form

$$A/\!\!/^1\tau v \simeq \underset{i}{\underrightarrow{\mathrm{colim}}} \left(A/\tau v \otimes_A A/Q_1(\tau v) \otimes_A \cdots \otimes_A A/Q_1^i(\tau v)\right) \in \mathrm{Mod}_A^{\mathrm{fil}}\,.$$

The operation $Q_1$ of Construction 2.3.2 acting on $\mathrm{Mod}^{\mathrm{fil}}_{\mathbb{F}_2}$ equipped with the $\tau$-grading has arity 2 and hence $Q_1^i(v\tau) \simeq \tau^{2^i} Q_1^{2^i}(v)$ is a map of filtered modules

$$Q_1^i(v\tau)\colon \Sigma^{2^i(t+1)-1}A(2^i) \longrightarrow A(0)\,.$$

Thus, the cofiber is given by the filtered $A$-module

$$\cdots \to 0 \to A \to \cdots \to A \to A/Q_1^i(v) \to A/Q_1^i(v) \to \ldots$$

where the first $A/Q_1^i(v)$-term sits in filtration degree $2^i$. Since the monoidal structure on $\mathrm{Mod}_A^{\mathrm{fil}}$ is given by Day-convolution, we may argue inductively to learn that the filtration on each finite tensor product

$$A/\tau v \otimes_A A/Q_1(\tau v) \otimes_A \cdots \otimes_A A/Q_1^i(\tau v)$$

stabilizes at stage $2^{i+1} - 1$. Moreover, tensoring with the further cofiber $A/Q_1^{i+1}(\tau v)$ does not change the terms in filtration degree $\leq 2^{i+1} - 1$ and hence the $2^{i+1} - 1$ stage of the filtered $\mathbb{E}_1$-ring $A/\!\!/^1\tau v$ is given by

$$\tau^{-1}\left(A/\tau v \otimes_A A/Q_1(\tau v) \otimes_A \cdots \otimes_A A/Q_1^i(\tau v)\right) \simeq A/(v, Q_1(v), \ldots, Q_1^i(v))$$

as claimed. □

We now use the decomposition of Proposition 2.3.6 to upgrade Lemma 2.4.12 to a highly coherent statement about killing nilpotent classes.

**Proposition 2.4.14.** For any $A \in \mathrm{CAlg}_{\mathbb{F}_2}$ and any nilpotent class $v\colon \Sigma^t\mathbb{F}_2 \to A$, the $\mathbb{E}_\infty$-cofiber $A \to A/\!\!/^\infty v$ detects nilpotence.

*Proof.* Since detecting nilpotence is stable under composition and base change by Proposition 2.4.9, and we can kill $v$ by iteratively by killing the smallest non-zero power of $v$ which squares to zero, we may assume that $v^2 = 0$. By inverting the filtration parameter $\tau$ in the $\mathbb{E}_1$-decomposition of the $\mathbb{E}_\infty$-cofiber obtained in Corollary 2.3.8, we learn that we have a description of $A/\!\!/^\infty v$ as an infinite tensor product of $\mathbb{E}_1$-cofibers

$$A/\!\!/^\infty v \simeq \bigotimes_I (A/\!\!/^1 Q_I(v)) \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}_A)\,, \tag{2.2}$$

where $I$ runs over all sequences of non-negative integers $I = (i_1, \ldots, i_n)$ of varying length $n$ and $Q_I(v) = Q_1^{i_1} \cdots Q_n^{i_n}(v)$. Since nilpotence detecting $\mathbb{E}_1$-algebras are closed under tensor products by [BSY22, Lemma 4.14] and filtered colimits by [BSY22, Lemma 4.25], it suffices to show that, for each $I$, the $\mathbb{E}_1$-cofiber

$$A \longrightarrow A/\!\!/^1 Q_I(v)$$

detects nilpotence.

Write $w_0 = Q_I(v)$ and $w_i = Q_1^i(w_0)$, then the analysis of Lemma 2.4.13, the filtration on this $\mathbb{E}_1$-cofiber obtained in Corollary 2.3.8 admits a cofinal system of the form

$$A \to A/w_0 \to A/(w_0, w_1) \to \ldots .$$

Moreover, writing $A_i = A/(w_0, \ldots, w_i)$ and $\varphi_{i,j}\colon A_i \to A_j$ for the structure maps, the $\mathbb{E}_1$-structure on the filtered object $A/\!\!/^1 \tau w_0$ provides us with multiplication maps $\mu_i$, together with a commutative diagram of $A$-linear maps

$$\begin{tikzcd} A_i \otimes A_i \arrow[r, "\mu_i"] & A_{i+1} \\ A_i \arrow[u, "A_i \otimes \varphi_{0,i}"] \arrow[ur, "\varphi_{i,i+1}"'] & \end{tikzcd}$$

We know from Lemma 2.4.12 that $w_i^2 = 0$ in $A$, so the maps $\varphi_{i,i+1}\colon A_i \to A_{i+1} = A_i/w_{i+1}$ and $A \to A/w_j$ have $\otimes$-nilpotent fiber for all $i$ and $j$. In particular, each $A/w_j$ is a conservative $A$-module and so the $A_i$ are conservative as well, as conservative objects are closed under tensor products. For any $i$, consider the fiber sequence of $A$-modules

$$\mathcal{I}_i \xrightarrow{\alpha_i} A \xrightarrow{\varphi_{0,i}} A_i\,.$$

Our goal now is to show that the map $\alpha_i$ is $\otimes$-nilpotent for each $i$. Tensoring up with $A_i$ we obtain another fiber sequence

$$\mathcal{I}_i \otimes A_i \xrightarrow{\alpha_i \otimes A_i} A_i \to A_i \otimes A_i\,,$$

which we may compose with the multiplication $\mu_i$, to obtain a nullhomotopy of the composite

$$\mathcal{I}_i \otimes A_i \xrightarrow{\alpha_i \otimes A_i} A_i \xrightarrow{\varphi_{i,i+1}} A_{i+1}\,.$$

In particular, the map $\alpha_i \otimes A_i$ factors through the fiber of $\varphi_{i,i+1}$ and is thus $\otimes$-nilpotent, meaning there exists some $N \geq 1$ such that that $\alpha_i^{\otimes N} \otimes A_i^{\otimes N} \simeq 0$. As the $A$-module $A_i$ is conservative, the tensor product $A_i^{\otimes N}$ is conservative as well and thus detects nilpotence by Example 2.4.8. In particular, each $\alpha_i$ is $\otimes$-nilpotent, as claimed.

Thus, we have shown that the $\mathbb{E}_1$-cofiber $A \to A/\!\!/^1 Q_I(v)$ can be written as filtered colimit of $\mathbb{E}_0$-algebras $A \to A_i$, each of which has nilpotent fiber. Thus, we are in the situation of Lemma 2.4.11 and learn that $A \to A/\!\!/^1 Q_I(v)$ detects nilpotence, so we conclude. $\square$

As a straightforward consequence, we now get the following:

**Corollary 2.4.15.** Any $A \in \mathrm{CAlg}_{\mathbb{F}_2}$ admits a nilpotence detecting $\mathbb{E}_\infty$-map $A \to T$ such that $T$ is 1-periodic and $\pi_0 T$ is a reduced ring.

*Proof.* Let $F = \mathbb{F}_2[u^{\pm}]$ where $u$ is a strict invertible class in degree one, set $A_0 := A \otimes F$ and denote by $\mathcal{I}_0 \subseteq \pi_0 A$ the ideal of all nilpotent elements. Note that, since $\mathbb{F}_2 \to F$ admits a retract, so does the map $A \to A_0$. Hence, $A_0$ is a conservative $A$-module and thus detects nilpotence by Example 2.4.8.

By Proposition 2.4.14, we know that for each $v \in \mathcal{I}_0$ the $\mathbb{E}_\infty$-map $A_0 \to A_0 /\!\!/^\infty v$ detects nilpotence. In particular, since the $\mathbb{E}_\infty$-cofiber

$$A_0 \longrightarrow A_0 /\!\!/^\infty \mathcal{I}_0$$

is a filtered colimit of tensor products of nilpotence detecting $\mathbb{E}_\infty$-maps, it detects nilpotence by Proposition 2.4.9 combined with [BSY22, Lemma 4.25].

Inductively setting

$$A_i := (A_{i-1} /\!\!/^\infty \mathcal{I}_{i-1}) \otimes F \in \mathrm{CAlg}_A\,,$$

where $\mathcal{I}_{i-1} \subseteq \pi_0 A_{i-1}$ is the ideal of nilpotent elements, we obtain a filtered diagram of $\mathbb{E}_\infty$-maps

$$A_0 \to A_1 \to A_2 \to \dots$$

each of which detects nilpotence. Setting $A_\infty := (\varinjlim A_i) \otimes F$ and again using Proposition 2.4.9, we deduce that the map $A \to A_\infty$ detects nilpotence. Finally, since $\mathbb{F}_2 \in \mathrm{Mod}_{\mathbb{F}_2}$ is compact, any non-zero class $v\colon \Sigma^t \mathbb{F}_2 \to A_\infty$ must be detected in some finite stage and hence cannot be nilpotent by construction. We conclude that $A_\infty$ is reduced and 1-periodic and so we have constructed the desired map. □

Moreover, we now learn the following about the geometric points of $\mathbb{F}_2$-algebras.

**Proposition 2.4.16.** Let $A \in \mathrm{CAlg}_{\mathbb{F}_2}$ and let $A \to T$ be a geometric point of $A$. Then $T$ is 1-periodic and $\pi_0 T$ is a separably closed field that is not perfect.

*Proof.* Observe that the localization

$$\mathbb{F}_2 \longrightarrow \mathbb{F}_2\{u^{\pm}\} = \mathbb{F}_2\{u\}[u^{-1}] \in \mathrm{CAlg}_{\mathbb{F}_2},$$

where $u$ is an class in topological degree 1, is a conservative $\mathbb{F}_2$-module and a compact $\mathbb{F}_2$-algebra. Hence, the base change $T \to T\{u^{\pm}\}$ is non-zero and admits a retract, meaning $T$ is 1-periodic with respect to some invertible class $u\colon \Sigma\mathbb{F}_2 \to T$.

Moreover, if $v \in \pi_0 T$ is nilpotent, then by Proposition 2.4.14 the map $T \to T /\!\!/^\infty v$ detects nilpotence. In particular, $T \to T /\!\!/^\infty v$ is non-zero and thus admits a retraction i.e. we have a nullhomotopy $v \simeq 0 \in \pi_0 T$. Since $T$ is nullstellensatzian, any non-nilpotent class in $\pi_0 T$ is invertible and so $\pi_0 T$ is a field. Now, if $f \in \pi_0 T[x]$ is any separable polynomial, the separable algebra $\pi_0 T \to \pi_0 T[x]/f = \pi_0 T[f]$ admits a unique lift to a separable $\mathbb{E}_\infty$-algebra $T \to T[f]$, for example by [Ram23, Theorem B]. In particular map $T \to T[f]$ is non-zero and hence admits a retraction, showing that $\pi_0 T$ is separably closed.

Lastly, since the $\mathbb{F}_2$-module $\mathbb{F}_2\{x\}[Q_1(x)^{-1}]$ is conservative, it is non-zero after base changing to $T$, and so there exists an element $v \in \pi_0 T$ such that $Q_1(v) \neq 0$. However, since $Q_1(w^2) \simeq 0$, for all $w \in \pi_0 T$ for example by Lemma 2.4.12, we learn that $v$ does not admit a square root, so $\pi_0 T$ is not perfect. □

In particular, this gives our first main theorem.

**Theorem 2.4.17.** Let $A \in \mathrm{CAlg}_{\mathbb{F}_2}$ and $v \in \pi_* A$ be non-nilpotent. Then there exists a map of $\mathbb{E}_\infty$-rings $f\colon A \to T$ such that $\pi_* T$ is a 1-periodic, separably closed, graded field and $f(v)$ is invertible. In particular, if $A$ is non-zero, it admits a reduced point.

*Proof.* By Corollary 2.4.6, there exists a geometric point $f\colon A \to T$ such that $f(v)$ is a unit. By Proposition 2.4.16, the ring $\pi_* T$ is a 1-periodic, separably closed, graded field. The last claim is clear by taking the non-nilpotent element $1 \in \pi_0 A$ of a non-zero ring $A$. $\square$

As an application, we can now verify that any $\mathbb{E}_\infty$-ring with non-zero $\mathbb{F}_2$-homology has invariant cell numbers.

**Corollary 2.4.18.** Let $A \in \mathrm{CAlg}(\mathrm{Sp})$ be a non-zero $\mathbb{E}_\infty$-ring such that $2 \in \pi_0 A$ is not invertible and suppose $\mathcal{I} \in \mathrm{Mod}_A^\omega$ admits a cell structure with $k$ many cells for some $k \geq 1$. Then, for any $n > k$, the module $\mathcal{I}$ does not admit a free module with $n$ many cells as a retract.

*Proof.* By assumption on $A$, the base change $A \otimes \mathbb{F}_2$ is non-zero and thus admits a reduced point $A \otimes \mathbb{F}_2 \to T$ by Theorem 2.4.17. Now consider some $\mathcal{I} \in \mathrm{Mod}_A^\omega$ which admits a finite filtration $\mathcal{I}_0 \to \mathcal{I}_1 \to \cdots \to \mathcal{I}_\ell = \mathcal{I}$ whose associated graded is free on $k$ many cells and suppose we have maps $i\colon \mathcal{F} \rightleftarrows \mathcal{I} : r$ such that $ri \simeq \mathrm{id}$ and $\mathcal{F}$ is free on $n$ many cells. Since everything in sight is preserved under base change, we can assume $A = T$. Since over a graded field, every module is free and so there are no non-trivial extension, we further reduce to the case $\mathcal{I} \simeq A^k$ and $\mathcal{F} \simeq A^n$. In particular, taking $\pi_0$ we learn that the vector space $\pi_0 A^n$ is a retract of $\pi_0 A^k$ and so $n \leq k$ as claimed. $\square$

**Remark 2.4.19.** Since the main technical input was the decomposition of Corollary 2.3.8, the results of this section generalize in a straightforward manner to $\mathbb{E}_\infty$-algebras in any rigid $\mathbb{F}_2$-linear category.

### 2.4.3 A lack of reduced points over $\mathbb{F}_p$ for $p > 2$

Throughout this section, fix a prime $p > 2$ and write $\mathbb{F}_p\{x\} \in \mathrm{CAlg}_{\mathbb{F}_p}$ for the free $\mathbb{E}_\infty$-algebra on a generator $x$.

Our goal is to show that there exist non-zero $\mathbb{E}_\infty$-rings that admit *no* non-zero maps to a ring $T$ such that $\pi_0 T$ is reduced. Concretely, we want to find an $A \in \mathrm{CAlg}_{\mathbb{F}_p}$ together with a nilpotent class $v \in \pi_0 A$ and an (iterated) operation $P_I(v)$ as in Construction 2.3.9 that is not nilpotent. Then, we can invert $P_I(v)$ and obtain a non-zero ring $A[P_I(v)^{-1}]$. As any map $A \to T$ which kills $v$ must factor through the $\mathbb{E}_\infty$-cofiber by $v$, which in turn kills the invertible class $P_I(v)$, the map is forced to be zero and we conclude.

We do this by considering the *universal* case, i.e. by using the decompositions of Proposition 2.3.14 and Corollary 2.3.15 to analyze the ring structure of the free algebra $\mathbb{F}_p\{x\}/\!\!/^\infty x^p$ on a generator whose $p$-th power is zero.

We use the following terminology for book-keeping the iterated Dyer–Lashof operations.

**Definition 2.4.20.** Let $I = (\varepsilon_n, i_n, \ldots, \varepsilon_1, i_1)$ be a Dyer–Lashof sequence that is $\infty$-allowable on even classes in the sense of Definition 2.3.12. We say that $I$ is

1. *purely bosonic* if $P_I$ preserves parity and $\varepsilon_k = 0$ for all $k$.
2. *mixed bosonic* if $P_I$ preserves parity and contains a Bockstein.
3. *fermionic* if $P_I$ does not preserve parity.

If $I$ is purely bosonic and $n \geq 0$ is an integer, we also write $nI$ for the purely bosonic sequence which is entry-wise multiplied by $n$.

The examples to keep in mind are $P_1P_1$ for purely bosonic, $\beta P_1$ for fermionic and $\beta P_{1/2}\beta P_1$ for mixed bosonic operations.

**Lemma 2.4.21.** Let $A \in \mathrm{CAlg}_{\mathbb{F}_p}$ and let $v \in \pi_0 A$ be some class. If $I$ is a Dyer–Lashof sequence that is not purely bosonic or not divisible by $p$, then $P_I(v^p) \simeq 0$. Moreover, for any purely bosonic Dyer–Lashof sequence $J$ we have

$$P_{pJ}(v^p) = P_J(v)^p.$$

In particular, if $v$ is nilpotent, then so is $P_J(v)$.

*Proof.* The Cartan relations of Proposition 2.3.11 imply that the total operation

$$\varphi\colon \pi_0 A \longrightarrow \pi_* A[t^{\pm}] \otimes \Lambda(z), \quad v \mapsto \sum_{i\geq 0} t^i (P_i(v) + \beta P_i(v) z)$$

is a map of graded rings, where $|t| = -2(p-1)$ and $|z| = -1$. Plugging in a $p$-th power $v^p$ and using that the $p$-th power is additive on homogeneous elements, we see that

$$\sum_i P_i(v)^p t^{ip} = \varphi(v)^p = \varphi(v^p) = \sum_i t^i (P_i(v^p) + \beta P_i(v^p) z).$$

Comparing coefficients, we learn that $\beta P_i(v^p) = 0$ for all $i$ and $P_i(v^p) = 0$ unless $p$ divides $i$, in which case $P_i(v^p) = P_{i/p}(v)^p$. The general claim then follows by replacing with $P_{i/p}(v)$ and iterating the argument. □

Thus, the theory immediately diverges from the $p = 2$ situation even on the "algebraic" level. With this in hand, we are ready to show that an even operation on a nilpotent even class need not be nilpotent.

**Proposition 2.4.22.** Let $I$ be a mixed bosonic Dyer–Lashof sequence. Then, for a generator $x$ in topological degree 0, the class

$$P_I(x) \in \pi_* \mathbb{F}_p\{x\} /\!\!/^{\infty} x^p$$

is not nilpotent.

*Proof.* Let $A \in \mathrm{CAlg}_{\mathbb{F}_p}$ take a class in even topological degree $v \in \pi_t A$ and consider the $\mathbb{E}_\infty$-cofiber $A \to A /\!\!/^{\infty} v^p$. Inverting the filtration parameter in the decomposition of Corollary 2.3.15, we learn that we have a $A$-linear decomposition of $A /\!\!/^{\infty} v^p$ as a filtered colimit of tensor products of $\mathbb{E}_0$-algebras of the form

$$A \longrightarrow A /\!\!/^{0} P_I(v^p) = A/P_I(v^p) \quad \text{and} \quad A \longrightarrow A /\!\!/^{1} P_J(v^p)\,,$$

where the $P_I$ are bosonic and the $P_J$ are fermionic. By Lemma 2.4.21, we know that $P_K(v^p) \simeq 0$ unless $K$ is purely bosonic and each $i_k \in K$ is divisible by $p$, in which case $P_K(v^p) \simeq P_{K/p}(v)^p$.

Thus, the decomposition is given by a filtered colimit of tensor products of $\mathbb{E}_0$-algebras of the form

$$A \longrightarrow A\{a_I\}_0, \quad A \longrightarrow A\{z_J\}_1, \quad A \longrightarrow A/P_K(v)^p\,,$$

where $K$ runs over all purely bosonic Dyer–Lashof sequences.

Now take $A = \mathbb{F}_p\{v\}$ to be the free $\mathbb{E}_\infty$-algebra on a generator $v$, then we know from Lemma 2.3.13 that the ring $\pi_* \mathbb{F}_p\{v\}$ is free as a graded commutative $\mathbb{F}_p$-algebra on the $\infty$-allowable operations. In particular, if $I$ is mixed bosonic, no power of $P_I(v)$ is divisible by a purely bosonic $P_K(v)$. Thus, $P_I(v)$ does not act nilpotently on any finite stage of the decomposition. Since $A \in \mathrm{Mod}_A$ is compact, nilpotence would be detected at a finite stage of the filtered colimit, and so the claim follows. □

This now allows us to construct $\mathbb{E}_\infty$-rings which admit no reduced points.

**Theorem 2.4.23.** Let $x$ be a class in topological degree 0 consider the ring $A = \mathbb{F}_p\{x\}/\!\!/^\infty x^p$. For any mixed bosonic Dyer–Lashof sequence $I$ in the sense of Definition 2.4.20, the localization $A[P_I(x)^{-1}]$ is a non-zero $\mathbb{E}_\infty$-ring which admits no non-zero $\mathbb{E}_\infty$-map to any commutative ring $T$ such that $T$ is even or $\pi_0 T$ is reduced.

*Proof.* By Proposition 2.4.22 the element $P_I(x)$ is not nilpotent and hence $A \to A[P_I(x)^{-1}]$ is not the zero map, as $\mathrm{Mod}_A$ is rigid. However, any map of commutative rings $f\colon A[P_I(x)^{-1}] \to B$ with $B$ reduced must take the nilpotent class $x$ to 0 and hence also take $P_I(x)$ to 0, forcing $f = 0$ as claimed. Moreover, since $I$ is mixed bosonic, the composite

$$P_I(x) = \beta^{\epsilon_{i_1}} P_{i_1} \cdots \beta^{\epsilon_{i_n}} P_{i_n}$$

contains at least *two* Bocksteins. Let $k \leq n$ be the largest integer such that $\varepsilon_{i_k} = 1$, meaning that the class

$$\gamma(x) = \beta P_{i_k} P_{i_{k+1}} \cdots P_{i_n}(x)$$

is necessarily in odd degree. If $B$ is even, the map $f$ must take $\gamma(x)$ to 0 and thus also $P_I(x)$, again forcing $f = 0$. $\square$

**Remark 2.4.24.** In fact, combining Remark 2.2.3 with the description of the free $\mathbb{E}_3$-algebra from Lemma 2.3.13, we learn that inverting the mixed bosonic class $\beta P_{\frac{1}{2}} \beta P_1(x)$ in $\mathbb{F}_p\{x\}/\!\!/^\infty x^p$ produces a non-zero $\mathbb{E}_\infty$ ring which admits no $\mathbb{E}_2$-maps to a ring that is even or reduced.

We have identified the mixed bosonic operations $P_I$ an obstruction towards killing nilpotent elements. However, by Proposition 2.4.22 these obstructions vanish on classes which admit a $p$-th root. To utilize this, let us first recall the following well known obstruction theoretic fact.

**Lemma 2.4.25.** Let $A \in \mathrm{CAlg}_{\mathbb{F}_p}$ and let $v \in \pi_* A$ be a class in even topological degree. Then, the cofiber $A \to A/v$ is a weak ring.

*Proof.* Since we are in characteristic $p > 2$, the 2-torsion operation $Q_1$ induced by the 2-cell of $\mathbb{R}P^\infty$ vanishes. In particular, the natural cell structure on the tensor product $A/v \otimes_A A/v$ splits, allowing us to construct a retraction $\mu\colon A/v \otimes_A A/v \to A$. $\square$

This lets us show that killing nilpotent elements which admit a $p$-th root detects nilpotence.

**Proposition 2.4.26.** Let $A \in \mathrm{CAlg}_{\mathbb{F}_p}$ and suppose we have $v \in \pi_0 A$ such that $v$ is nilpotent and admits a $p$-th root. Then the $\mathbb{E}_\infty$-cofiber $A \to A/\!\!/^\infty v$ detects nilpotence.

*Proof.* Arguing as in the proof of Proposition 2.4.14 we can reduce to the case $v^p \simeq 0$. By Corollary 2.3.15, we have a description of $A/\!\!/^\infty v$ as an $\mathbb{E}_0$-algebra in $\mathrm{Mod}_A$ as a filtered colimit of tensor products of terms of the form

$$A \longrightarrow A/P_I(v) \quad \text{and} \quad A \longrightarrow A/\!\!/^1 P_J(v)\,,$$

where the $P_I$ are bosonic and the $P_J$ are fermionic. As the map $A \to A/\!\!/^\infty v$ is $\mathbb{E}_\infty$, combining [BSY22, Lemma 4.14] with [BSY22, Lemma 4.25] tells us that it suffices to prove that each of the terms are nilpotence detecting weak rings.

Let $w \in \pi_0 A$ with $w^p \simeq v$ be a $p$-th root of $v$. Then, if $I$ is mixed bosonic, Proposition 2.4.22 implies that $P_I(v) \simeq P_I(w^p) \simeq 0$ and hence we have

$$A \longrightarrow A/P_I(v) \simeq A\{\sigma(P_I(v))\}_0 = A \oplus \Sigma^{|P_I(v)|+1} A \in \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Mod}_A),$$

which admits an $A$-linear retract and is thus both conservative and a weak ring. In particular, it detects nilpotence by Example 2.4.8. Moreover, if $P_I$ is purely bosonic, we know that $P_I(v)$ is nilpotent by Proposition 2.4.22. Thus, the cofiber

$$A \longrightarrow A/P_I(v)$$

has $\otimes$-nilpotent fiber and consequently is a nilpotence detecting weak ring by Example 2.4.8 combined with Lemma 2.4.25.

Finally, if $P_J$ is fermionic, then it contains a Bockstein and so we know by Proposition 2.4.22 that

$$P_J(v) \simeq P_J(w^p) \simeq 0\,,$$

meaning we get an equivalence of $\mathbb{E}_0$-algebras

$$A \longrightarrow A /\!\!/^1 P_J(v) \simeq A\{\sigma(P_J(v))\}_1 \in \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Mod}_R).$$

In particular, $A/\!\!/^1 P_J(v)$ is a weak ring and admits an $A$-linear retraction, meaning it is conservative and thus detects nilpotence by Example 2.4.8.

Thus, all the terms in the filtered colimit decomposition of $A \to A/\!\!/^\infty v$ are nilpotence detecting weak rings and we conclude. □

**Proposition 2.4.27.** Let $T \in \mathrm{CAlg}_{\mathbb{F}_p}$ be nullstellensatzian. Then $T$ is 2-periodic, has non-zero odd classes and $\pi_0 T$ is not a reduced ring. Moreover, $\pi_0 T$ is a local ring of dimension zero, each $v \in \mathrm{Nil}(\pi_0 T)$ satisfies $v^p \simeq 0$, but the nilradical $\mathrm{Nil}(\pi_0 T)$ is not a nilpotent ideal.

*Proof.* For a generator $u$ in topological degree 2, the localization $\mathbb{F}_p\{u^\pm\} \in \mathrm{CAlg}^\omega_{\mathbb{F}_p}$ is non-zero and conservative, i.e. the map $T \to T\{u^\pm\}$ admits a retraction since $T$ is nullstellensatzian, and so $T$ is 2-periodic. Similarly, let $A = (\mathbb{F}_p\{x\}/\!\!/^\infty x^p)[\alpha(x)^{-1}]$ where $\alpha(x) = \beta P_{\frac{1}{2}} \beta P_1(x)$ and $x$ is a generator in topological degree 0. As the map $\mathbb{F}_p \to A$ admits an $\mathbb{F}_p$-linear retraction, $A$ is conservative and so the base change $A \otimes T$ is non-zero. Moreover, $A \in \mathrm{CAlg}^\omega_{\mathbb{F}_p}$ and so the map $T \to A \otimes T$ admits a retraction. Considering the composite $A \to A \otimes T \to T$, the claim follows immediately from Theorem 2.4.23.

The fact that $\pi_0 T$ is local of dimension zero is clear, since $\mathrm{Mod}_{\mathbb{F}_p}$ is rigid and hence every class in $\pi_*$ of a nullstellensatzian is either invertible or nilpotent. Moreover, if $v \in \pi_0 T$ is nilpotent, Proposition 2.4.26 tells us that the map $T \to T/\!\!/^\infty v^p$ detects nilpotence. In particular, since $T$ is nullstellensatzian, $v^p = 0$ holds in $\pi_0 T$.

Now let $x_1, \ldots, x_n$ be generators in topological degree zero and consider the free algebra

$$B_n = \mathbb{F}_p\{x_1, \ldots, x_n\}/\!\!/^\infty(x_1^p, \ldots, x_n^p) \simeq \mathbb{F}_p\{x_1\}/\!\!/^\infty x_1 \otimes \cdots \otimes \mathbb{F}_p\{x_n\}/\!\!/^\infty x_n\,.$$

Let $P_I = \beta P_{i_{2n}} \beta P_{i_{2n-1}} \cdots \beta P_{i_1}$ be a mixed bosonic Dyer–Lashof operation of length $2n$ where every extended power is followed by a Bockstein. Expanding the Cartan formula for $P_I(x_1 \cdots x_n)$ we see that it contains a non-trivial term of the form $P_{J_1}(x_1) \cdots P_{J_n}(x_n)$ where the $k$-th factor is a mixed bosonic operation

$$P_{J_k}(x_k) = P_{j_{k,2n}} P_{j_{k,2n-1}} \cdots \beta P_{j_{k,2k}} \beta P_{j_{k,2k-1}} \cdots P_{j_{k,2}} P_{j_{k,1}}(x_k)$$

with exactly two Bocksteins. By Lemma 2.3.13 combined with the fact that, since $\mathbb{F}_p$ is a field, taking homotopy groups is symmetric monoidal, we know that $\pi_* \mathbb{F}_p\{x_1, \ldots, x_n\}$ is a free graded commutative $\mathbb{F}_p$-algebra on the $\infty$-allowable operations $P_K(x_i)$. Moreover, by the analysis of the $\mathbb{E}_\infty$-cofiber carried out in Proposition 2.4.22, we know that each $P_{J_k}(x_i)$ does not act nilpotently on the ring $B_n$, which implies that $P_I(x_1 \cdots x_n)$ does not act nilpotently either.

Arguing as in the proof of Theorem 2.4.23, we can thus find nilpotent elements $v_1, \ldots, v_n \in \pi_0 T$ such that $P_I(v_1 \cdots v_n)$ is invertible and hence the product $v_1 \cdots v_n \in \pi_0 T$ is non-zero. Since $n$ was arbitrary, this proves the claim. □

# Chapter 3

# On the $K$-theory of algebraic tori

QINGYUAN BAI SHACHAR CARMELI BRANKO JURAN FLORIAN RIEDEL

Given an algebraic torus $T$ over a field $F$, its lattice of characters $\Lambda$ gives rise to a topological torus $\mathfrak{T}(T) = \Lambda_{\mathbb{R}}/\Lambda$ with a continuous action of the absolute Galois group $G$. We construct a natural equivalence between the algebraic $K$-theory $K_*(T)$ and the equivariant homology $H_*^G(\mathfrak{T}(T); K_G(F))$ of the topological torus $\mathfrak{T}(T)$ with coefficients in the $G$-equivariant $K$-theory of $F$. This generalizes a computation of $K_0(T)$ due to Merkurjev and Panin. We obtain this equivalence by analyzing the motive $\mathbb{K}_F^T$ in the stable motivic category $\mathrm{SH}F$ of Voevodsky and Morel, where $\mathbb{K}_F$ is the motivic spectrum representing homotopy $K$-theory. We construct a natural comparison map $\mathfrak{F}\colon \mathbb{K}_F[\mathrm{B}\Lambda] \to \mathbb{K}_F^T$ from the $\mathbb{K}_F$-homology of the étale delooping of $\Lambda$ to $\mathbb{K}_F^T$ as a special case of a motivic Fourier transform and prove that it is an equivalence by using a motivic Eilenberg–Moore formula for classifying spaces of tori.

## Contents

## 3.1 Introduction

The "fundamental theorem" of algebraic $K$-theory, due to Quillen [Qui72], asserts that for a field $F$, the $K$-theory of the multiplicative group $\mathbb{G}_{m,F}$ is given by $K_*(\mathbb{G}_{m,F}) \simeq K_*(F) \oplus K_{*-1}(F)$. Put differently, the $K$-groups are computed by the homology $H_*(S^1; K(F))$ of the circle $S^1$ with coefficients in the $K$-theory spectrum of $F$. More generally, the $K$-groups of the $n$-dimensional algebraic torus $\mathbb{G}_{m,F}^{\times n}$ are given by the homology of the $n$-dimensional topological torus $(S^1)^{\times n}$:

$$K_*(\mathbb{G}_{m,F}^{\times n}) \simeq H_*((S^1)^{\times n}; K(F)) \tag{3.1}$$

We generalize this formula to algebraic tori which are non-split. Before we set up the statement, let us first be explicit about what our objects of study are.

**Terminology 3.1.1.** Let $F$ be a field and $T$ a commutative group scheme over $F$. Then $T$ is called an *algebraic torus* if it is isomorphic to a split torus $T_E \simeq \mathbb{G}_{m,E}^{\times n}$ for some $n \geq 1$ after base change along a finite Galois extension $E/F$. In this situation, $n$ is called the *rank* and the field extension $E/F$ is called a *splitting field* of the torus $T$.

Given a rank $n$ torus $T/F$ with a choice of splitting field $E$ and Galois group $G = \mathrm{Gal}(E/F)$, we can attach an algebraic invariant to $T$ as follows:

**Definition 3.1.2.** The *character lattice* of $T$, with respect to the splitting field $E/F$, is the free abelian group of rank $n$ with $G$-action defined as the set of group homomorphisms

$$\Lambda\left(T\right) \coloneqq \hom(T_E, \mathbb{G}_{m,E})\,,$$

where $G$ acts via conjugation. Moreover, the associated *topological mirror* is the topological space with $G$-action given by the quotient

$$\mathfrak{T}(T) \coloneqq (\Lambda\left(T\right) \otimes_{\mathbb{Z}} \mathbb{R})/\Lambda\left(T\right)\,.$$

**Example 3.1.3.** For $T = \mathbb{G}_m^{\times n}$, the character lattice is given by $\mathbb{Z}^n$ with trivial action and we see that $\mathfrak{T}(T) = (S^1)^{\times n}$. For a non-split example, the torus over $\mathbb{R}$ defined by the equation $x^2 + y^2 = 1$ is a non-split torus of rank one and has topological mirror given by $S^1$ equipped with the $C_2$-action induced by complex conjugation on $S^1 \subseteq \mathbb{C}$.

To extend Quillen's computation (3.1) to non-split tori, we make the following replacements:

1. The role of the space $(S^1)^{\times n}$ will be played by the *topological mirror* $\mathfrak{T}(T)$.
2. The role of the coefficients $K(F)$ will be played by the *$G$-equivariant $K$-theory* of $F$ which we denote by $K_G(F)$.

Here, the equivariant $K$-theory $K_G(F)$ is a genuine $G$-spectrum representing the equivariant homology theory which, for any subgroup $H \subseteq G$, evaluates on the $G$-space $G/H$ as $H_*^G(G/H; K_G(F)) = K_*(E^H)$.[1]

With all players in place, we can now state the most elementary version of our main theorem.

**Theorem D** (Corollary 3.5.11)**.** Let $T$ be an algebraic torus over a field $F$, $E/F$ a choice of splitting field with Galois group $G = \mathrm{Gal}(E/F)$ and associated topological mirror $\mathfrak{T}(T)$. We have a natural equivalence of graded rings

$$H_*^G(\mathfrak{T}(T); K_G(F)) \xrightarrow{\sim} K_*(T)$$

between the $G$-equivariant homology of the topological mirror $\mathfrak{T}(T)$ with coefficients in $K_G(F)$ and the algebraic $K$-theory of $T$.

[1] See [Bar17, Ex. D.24] as well as Definition 3.2.15 for a precise construction of $K_G(F)$.

Here, the ring structure on $H_*^G(\mathfrak{T}(T); K_G(F))$ is given by convolution, i.e. induced by the natural $G$-equivariant Lie group structure on the torus $\mathfrak{T}(T)$.

This provides a method for computing $K_*(T)$ in terms of the $K$-theory of all intermediate extensions $F \subseteq L \subseteq E$ via any $G$-equivariant cell structure on the topological torus $\mathfrak{T}(T)$. Moreover, for $* = 0$, the explicit formula for Bredon homology combined with Hilbert 90, recovers the following result of Merkurjev and Panin [MP97], therein stated in terms of generators and relations.

**Corollary 3.1.4** (Corollary 3.5.14)**.** Let $T$ be an algebraic torus over a field $F$, $E/F$ a choice of splitting field with Galois group $G = \mathrm{Gal}(E/F)$ and associated topological mirror $\mathfrak{T}(T)$. There is natural equivalence

$$H_0^G(\mathfrak{T}(T); \underline{\mathbb{Z}}) \xrightarrow{\sim} K_0(T)$$

between the 0-th integral Bredon homology of $\mathfrak{T}(T)$ and the algebraic $K$-theory group $K_0(T)$.

Although we write it in the equivariant language, the proof of Theorem D does not proceed through equivariant homotopy theory. Instead, we lift everything to the land of *motivic* homotopy theory and prove a more highly structured statement, namely Theorem E. Let us switch gears to motivic homotopy theory and outline what actually goes into the proof.

## The motivic enhancement

For a field $F$, we denote by $\mathrm{SH}(F)$ the stable motivic category of $F$ in the sense of Voevodsky–Morel. This is a stable, symmetric monoidal $\infty$-category, which is constructed such that an object $E \in \mathrm{SH}(F)$ represents cohomology theory $H^*(-; E)$ on the category of smooth $F$-schemes $\mathrm{Sm}_F$, given on some $X \in \mathrm{Sm}_F$ by the (bigraded) homotopy groups of the motive $E^X \in \mathrm{SH}(F)$. The objects appearing in Theorem D translate as follows into this setup:

1. The role of the genuine $G$-spectrum $K_G$ is played by the motivic spectrum $\mathbb{K}_F \in \mathrm{SH}(F)$ representing algebraic $K$-theory of smooth $F$-schemes.

2. For a torus $T/F$, the role of the topological mirror $\mathfrak{T}(T)$ is played by the étale delooping $\mathrm{B}\Lambda\,(T)$ where $\Lambda\,(T)$ is the étale sheaf of (algebraic) group homomorphisms $T \to \mathbb{G}_m$.

Our goal is to produce a natural equivalence

$$H_*(\mathrm{B}\Lambda\,(T); \mathbb{K}_F) \xrightarrow{\sim} H^*(T; \mathbb{K}_F) = K_*(T)$$

between the motivic *homology* of the étale delooping $\mathrm{B}\Lambda\,(T)$ and the algebraic $K$-theory of $T$, which is a *cohomology theory* applied to $T$. The first step then is to construct a highly structured, natural comparison map, which we can manipulate using the machinery of motivic homotopy theory. We interpret it as a kind of *Fourier transform*, in the spirit of the chromatic Fourier transform of Barthel, Schlank, Yanovski, and the second author [BCSY24]. Recall that, if $R$ is an ordinary commutative ring and $G$ a finite abelian group with Pontryagin dual $G^\vee = \hom(G, \mathbb{Q}/\mathbb{Z})$, the classical Fourier transform is a map of commutative rings

$$\mathfrak{F}_\omega \colon R[G] \longrightarrow R^{G^\vee}$$

depending on a choice of orientation $\omega\colon \mathbb{Q}/\mathbb{Z} \to R^\times$. Here, $R[G]$ is the group ring construction and $R^{G^\vee}$ is the ring of functions $G^\vee \to R$. This tells us that, a natural first step towards writing down our comparison map is formulating a notion of *duality* that intertwines a torus with the delooping of its character lattice.

We write $\mathcal{S}_F^{\mathrm{big,ét}} = \mathrm{Shv}_{\mathrm{ét}}(\mathrm{Sch}_F; \mathcal{S})$ for the $\infty$-category of sheaves of spaces on the big étale site of $F$. Then any commutative group scheme $A/F$ as well as its étale classifying stack $\mathrm{B}A$ naturally refine to abelian group objects in the category $\mathcal{S}_F^{\mathrm{big,ét}}$ and we make this identification implicitly from now on.

**Definition 3.1.5.** For any abelian group object $M \in \mathrm{Ab}(\mathcal{S}_S^{\mathrm{big,\acute{e}t}})$ we define the *shifted Cartier dual* of $M$ as the internal mapping object

$$M^\vee := \mathrm{map}(M, \mathrm{B}\mathbb{G}_m) \in \mathrm{Ab}(\mathcal{S}_S^{\mathrm{big,\acute{e}t}}).$$

Moreover, we define the motivic group ring construction

$$\mathbb{K}_F[-]\colon \mathrm{Ab}(\mathcal{S}_S^{\mathrm{big,\acute{e}t}}) \longrightarrow \mathrm{CAlg}_{\mathbb{K}_F}(\mathrm{SH}(F))$$

as the composite of the forgetful functor to Nisnevich sheaves with the left Kan extension of the motivic suspension spectrum functor $\mathbb{K}_F \otimes \Sigma^\infty_+(-)\colon \mathrm{Sm}_F \to \mathrm{Mod}_{\mathbb{K}_F}(\mathrm{SH}(F))$.

If $T$ is a torus over $F$, recalling that $\Lambda(T) = \mathrm{map}(T, \mathbb{G}_m)$ it is straightforward to show that there are natural equivalences of étale sheaves

$$\Lambda(T)^\vee \simeq \mathrm{B}T \quad \text{and} \quad \mathrm{B}\Lambda(T)^\vee \simeq T,$$

i.e. the delooping of the torus is dual to its character lattice, and the torus itself is dual to the delooping of its character lattice. Moreover, we can naturally identify $\mathrm{B}\mathbb{G}_m$ with the sheaf of spaces taking a scheme $X/F$ to the Picard groupoid $\mathrm{pic}(X)$. Under this identification, we get an "orientation" for free via the natural map $\mathrm{pic}(X) \to K(X)^\times$. Building on this observation, we construct the comparison map as a $K$-theoretic delooping of a categorical Fourier–Mukai transform, see Section 3.3 for a detailed discussion. Concretely, we prove the following:

**Proposition 3.1.6** (Construction 3.3.18)**.** Let $M \in \mathrm{Ab}(\mathcal{S}_F^{\mathrm{big,\acute{e}t}})$, such that the shifted Cartier dual $M^\vee$ is representable by a stack [2]. There is a natural map of $\mathbb{E}_\infty$-algebras

$$\mathfrak{F}^M_{\mathrm{mot}}\colon \mathbb{K}_F[M] \longrightarrow \mathbb{K}_F^{M^\vee}$$

in the category $\mathrm{Mod}_{\mathbb{K}_F}(\mathrm{SH}(F))$ which we call the *motivic Fourier transform*.

Here, the right hand side is *not* defined via right Kan-extension from smooth $F$-schemes, but using the global structure of the motivic spectrum $\mathbb{K}_F$, see also the discussion at Warning 3.2.10. Concretely, if $M^\vee = \mathrm{B}\mathbb{G}$ for a well-behaved, smooth, commutative group scheme $\mathbb{G}$, then the motive $\mathbb{K}_F^{\mathrm{B}\mathbb{G}}$ represents $\mathbb{G}$-*equivariant* $K$-theory. In particular, the global sections are given by the spectrum $C^*(F; \mathbb{K}_F^{\mathrm{B}\mathbb{G}}) = K(\mathcal{D}_{\mathrm{perf}}(\mathrm{B}\mathbb{G}))$. Our main theorem in the motivic world goes as follows:

**Theorem E** (Theorem 3.4.7)**.** For any torus $T$ over a field $F$, the motivic Fourier transform

$$\mathfrak{F}^{\mathrm{B}\Lambda(T)}_{\mathrm{mot}}\colon \mathbb{K}_F[\mathrm{B}\Lambda(T)] \xrightarrow{\sim} \mathbb{K}_F^T,$$

is an equivalence of $\mathbb{E}_\infty$-algebras in $\mathrm{Mod}_{\mathbb{K}_F}(\mathrm{SH}(F))$.

The equivariant statement of Theorem D then follows by a standard assembly argument about the realization functor $\mathrm{SH}(F) \to \mathrm{Sp}_G$ where $G$ is the absolute Galois group of $F$. Having constructed the motivic Fourier transform, the proof of Theorem E proceeds by first making an auxiliary computation.

For a torus $T$, we show that the Fourier transform $\mathfrak{F}^{\Lambda(T)}_{\mathrm{mot}}$ witnesses the well-known fact that the $\infty$-category of representations $\mathcal{D}_{\mathrm{perf}}(\mathrm{B}T)$ is *semi-simple*. Concretely, given a splitting field $E/F$ and representatives $\chi\colon T \to E^\times$ of each orbit of the character lattice, we obtain a decomposition

$$\mathcal{D}_{\mathrm{perf}}(\mathrm{B}T) \simeq \bigoplus_{\chi} \mathcal{D}_{\mathrm{perf}}(F_\chi),$$

where $F_\chi$ is the fixed field of the stabilizer of $\chi$. We check that this equivalence is induced by a categorical version of the motivic Fourier transform, see Proposition 3.3.13, and learn the following from an assembly argument:

[2] In the sense of Terminology 3.2.1.

**Proposition 3.1.7** (Proposition 3.3.20)**.** Let $T$ be a torus over $F$ with character lattice $\Lambda(T)$. The motivic Fourier transform

$$\mathfrak{F}_{\mathrm{mot}}^{\Lambda(T)} : \mathbb{K}_F[\Lambda(T)] \xrightarrow{\sim} \mathbb{K}_F^{\mathrm{B}T}$$

is an equivalence of $\mathbb{E}_\infty$-algebras in $\mathrm{Mod}_{\mathbb{K}_F}(\mathrm{SH}(F))$.

Having shown this, we want to use the naturality of the Fourier transform to deduce Theorem 3.4.7 from an Eilenberg–Moore formula. Following ideas of Merkurjev and Panin from [MP97], we do this using the following presentation of an arbitrary torus $T$ in terms of a Weil-restricted torus.

**Definition 3.1.8.** Let $T/F$ be a torus and $E$ a splitting field of $T$. The Weil restriction $W$ of the base change $T_E$ back to $F$ comes equipped with a surjective counit map $W \to T$. Taking the kernel, we get a short exact sequence of tori

$$0 \to A \to W \to T \to 0$$

which we call the *Weil-resolution* of $T$ relative to $E$.

Given a Weil resolution of $T$, we obtain a more "tame" variant of the natural fiber sequence $T \to * \to \mathrm{B}T$, namely we rotate the resolution twice to obtain a pullback diagram of stacks of the form

$$\begin{tikzcd} T \arrow[r] \arrow[d] & * \arrow[d] \\ \mathrm{B}A \arrow[r] & \mathrm{B}W \,. \end{tikzcd}$$

By the naturality of the motivic Fourier transform $\mathfrak{F}_{\mathrm{mot}}$, combined with the auxiliary result of Proposition 3.1.7 the proof of Theorem E reduces to the following claim.

**Proposition 3.1.9** (Proposition 3.4.6)**.** Let $T/F$ be a torus and $E$ a splitting field. Then, the associated Weil-resolution induces an Eilenberg–Moore isomorphism

$$\mathbb{K}_F^{\mathrm{B}A} \otimes_{\mathbb{K}_F^{\mathrm{B}W}} \mathbb{K}_F \xrightarrow{\sim} \mathbb{K}_F^T \in \mathrm{CAlg}_{\mathbb{K}_F}(\mathrm{SH}(F)).$$

An analogues Eilenberg–Moore formula for Chow groups was proven by Ananyevskiy and Geldhauser [AG24, Cor. 2.3.3].

As $* \to \mathrm{B}W$ is in general not proper, we cannot deduce Proposition 3.1.9 from a proper projection formula. To rectify this, we show that to obtain an Eilenberg–Moore formula, it suffices to check that the leg $* \to \mathrm{B}W$ satisfies a certain cellularity condition, namely that its motive in $\mathrm{SH}(\mathrm{B}W)$ is *Artin–Tate* in the sense of Definition 3.4.8.

The final and crucial geometric input is to verify that this condition is satisfied, which uses that we chose $W$ to be Weil-restricted from a split torus. Using an inductive argument, analogous to the one used by Merkurjev and Panin [MP97] in order to prove their variant of Corollary 3.1.4 and the one Ananyevskiy and Geldhauser [AG24] used to prove their variant of Proposition 3.1.9, we show the following:

**Proposition 3.1.10** (Proposition 3.4.20)**.** Let $F \to E$ be a finite separable field extension and let $W$ denote the Weil-restriction of a split torus $\mathbb{G}_m^{\times n}$ along $F \to E$. Then the motive of the 0-section $* \to \mathrm{B}W$ in $\mathrm{SH}(\mathrm{B}W)$ is Artin–Tate.

## Outline

We will start by recalling background material from motivic and equivariant homotopy theory as well as parameterized category theory in Section 3.2. Afterwards, in Section 3.3, we will set up the categorical and the motivic Fourier transform and prove Proposition 3.1.6 and Proposition 3.1.7. Section 3.4 deals with the more geometric aspects: We will prove the Eilenberg–Moore formula of Proposition 3.1.9 as well as the Artin–Tate property of Proposition 3.1.10, resulting in Theorem E. Finally, in Section 3.5, we will discuss the relation to equivariant homotopy theory, introduce the topological mirror and deduce Theorem D.

## Conventions

1. We work with the theory of $(\infty, 1)$-categories as developed by Lurie in [HTT] and [HA]. By a category we always mean an $(\infty, 1)$-category.

2. We fix three nested Grothendieck universes. We call categories defined in the smallest one *small* and categories defined in the largest one *large*.

3. We write $\mathcal{S}$ for the category of (small) spaces, Cat for the category of (small) categories, $\widehat{\mathrm{Cat}}$ for the large category of categories and Sp for the category of spectra.

4. All schemes are quasi-compact and quasi-separated.

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# 3.2 Preliminaries

We will make heavy use of the machinery of the *stable motivic category* $\mathrm{SH}(\mathfrak{X})$ of any (well behaved) stack $\mathfrak{X}$, which we review in Section 3.2.1. In the case of a scheme, this category was first constructed by Vovoedsky and Morel [MV99] and the generalization to the equivariant schemes is due to Hoyois [Hoy17]. We will use the version for stacks due to Khan and Ravi [KR24].

In Section 3.2.3, we discuss the relation between motivic homotopy theory over a field $F$ and $G$-equivariant homotopy theory where $G$ is the absolute Galois group of $F$.

Finally, in Section 3.2.4, we recall some basic results and constructions from parametrized homotopy theory as developed by Martini and Wolf in [Mar22], [MW24] and [MW25].

## 3.2.1 Motivic homotopy theory of stacks

Let us first fix what we actually mean by *stack* throughout this paper.

**Terminology 3.2.1.** Let $S$ be a base scheme. Throughout this paper, a *stack* over $S$ is a sheaf of (1-truncated) spaces on the big étale site of $S$ which is *nicely scalloped* in the sense of [KR24, Def. 2.9] unless otherwise specified.

Having declared this once and for all, we also fix the following pieces of notation.

**Notation 3.2.2.** For a base scheme $S$, we write $\mathrm{Sch}_S$ for the category of qcqs schemes over $S$. Moreover, we denote by $\mathrm{Stk}_S$ the category of stacks over $S$ and given any $\mathfrak{X} \in \mathrm{Stk}_S$, we often leave the base implicit and write $\mathrm{Stk}_{\mathfrak{X}} = (\mathrm{Stk}_S)_{/\mathfrak{X}}$ for the slice category. Moreover, we write $\mathrm{Sm}_{\mathfrak{X}} \subseteq \mathrm{Stk}_{\mathfrak{X}}$ for the full subcategory spanned by the smooth, representable morphisms of finite presentation.

Beware that, the notion of stack of Terminology 3.2.1 is relative to the base $S$. Its primary function is to *exclude* stacks that are badly behaved, such as the classifying stack of a finite group whose order is divisible by a residue field. However, the only stacks that are essential for us are the following, which are well behaved over any base.

**Example 3.2.3.** For any scheme $S$, a *torus* over $S$ is an abelian group scheme $T$, with the property that after a finite étale extension $S' \to S$ it splits as $T_{S'} \simeq \mathbb{G}_m^{\times n}$ for some $n \geq 1$. For any torus $T/S$, the étale classifying stack $\mathrm{B}T$ is nicely scalloped by [KR24, Ex. 2.2]. Moreover, for any injective group homomorphism of tori $W \to T$ the étale delooping $\mathrm{B}W \to \mathrm{B}T$ is smooth, representable and of finite presentation and so defines an object of $\mathrm{Sm}_{\mathrm{B}T}$.

We now outline the construction of the stable motivic category $\mathrm{SH}(\mathfrak{X})$ of a stack $\mathfrak{X}$, in this form due to [KR24].

Fix a base scheme $S$ and for any $\mathfrak{X} \in \mathrm{Stk}_S$ write $\mathcal{S}_{\mathfrak{X}}^{\mathrm{ns}}$ for the category of Nisnevich sheaves of spaces on $\mathrm{Sm}_{\mathfrak{X}}$. The category of *motivic spaces* is defined to be the full subcategory $\mathrm{H}(\mathfrak{X}) \subseteq \mathcal{S}_{\mathfrak{X}}^{\mathrm{ns}}$ spanned by the $\mathbb{A}^1$-invariant sheaves. The fully faithful inclusion $\mathrm{H}(\mathfrak{X}) \hookrightarrow \mathcal{S}_{\mathfrak{X}}^{\mathrm{ns}}$ admits a product preserving left adjoint which we denote by

$$L_{\mathbb{A}^1} \colon \mathcal{S}_S^{\mathrm{ns}} \longrightarrow \mathrm{H}(\mathfrak{X}),$$

see [KR24, Rem. 3.4]. If $\mathcal{E}$ is a vector bundle on $\mathfrak{X}$ with total space $\mathcal{V}(\mathcal{E})$, we include $\mathfrak{X} \hookrightarrow \mathcal{V}(\mathcal{E})$ via the zero section and denote by

$$\mathrm{Th}(\mathcal{E}) \coloneqq \mathrm{cofib}\,(\mathcal{V}(\mathcal{E}) \setminus \mathfrak{X} \to \mathcal{V}(\mathcal{E})) \in \mathrm{H}(\mathfrak{X})_*$$

the cofiber in pointed motivic spaces, which we call the *motivic Thom space* of $\mathcal{V}$.

If the stack $\mathfrak{X}$ is *nicely quasi-fundamental* in the sense of [KR24, Def. 2.9] the category of *motivic spectra* over $\mathfrak{X}$ is defined by formally $\otimes$-inverting all motivic Thom spaces in $\mathrm{H}(\mathfrak{X})_*$ with respect to the smash product, i.e. we have

$$\mathrm{SH}(\mathfrak{X}) \coloneqq \mathrm{H}(\mathfrak{X})_*[\{\mathrm{Th}(\mathcal{E})^{-1}\}_{\mathcal{E}}]$$

where $\mathcal{E}$ runs over all vector bundles on $\mathfrak{X}$. The general case construction is obtained via descent and we refer the reader to the proof of [KR24, Thm. 4.5] for details.

The category $\mathrm{SH}(\mathfrak{X})$ is stable, naturally refines to an object of $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ and we denote the monoidal unit by $\mathbb{S}_{\mathfrak{X}} \in \mathrm{SH}(\mathfrak{X})$, the *motivic sphere spectrum*. By construction, $\mathrm{SH}(\mathfrak{X})$ comes equipped with a suspension spectrum functor $\Sigma^\infty \colon \mathrm{H}(\mathfrak{X})_* \to \mathrm{SH}(\mathfrak{X})$. We often implicitly consider smooth representable stacks $\mathfrak{Z} \in \mathrm{Sm}_{\mathfrak{X}}$ as Nisnevich sheaves of spaces on $\mathrm{Sm}_{\mathfrak{X}}$ via the Yoneda-embedding and denote the composition of $\mathbb{A}^1$-localization and (unpointed) motivic suspension by

$$\mathbb{S}_{\mathfrak{X}}[-] \colon \mathcal{S}_{\mathfrak{X}}^{\mathrm{ns}} \longrightarrow \mathrm{SH}(\mathfrak{X}).$$

This functor is symmetric monoidal, colimit preserving and left Kan extended from the full subcategory $\mathrm{Sm}_{\mathfrak{X}} \subseteq \mathcal{S}_{\mathfrak{X}}^{\mathrm{ns}}$.

If $f \colon \mathfrak{Z} \to \mathfrak{X}$ is any map of stacks, pullback of sheaves induces a symmetric monoidal left adjoint functor

$$f^* \colon \mathrm{SH}(\mathfrak{X}) \longrightarrow \mathrm{SH}(\mathfrak{Z})$$

whose lax symmetric monoidal right adjoint we denote by

$$f_*\colon \mathrm{SH}(\mathfrak{Z}) \longrightarrow \mathrm{SH}(\mathfrak{X}).$$

Moreover, by [KR24, Thm. 4.5] the assignment taking a stack to its category of motivic spectra and taking a map to its $(-)^*$ refines to a functor

$$\mathrm{SH}^*\colon \mathrm{Stk}_S^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$$

which satisfies Nisnevich descent.

**Theorem 3.2.4** ([KR24, Thm. 4.10])**.** Let $f\colon \mathfrak{Z} \to \mathfrak{X}$ be smooth and representable. Then the functor $f^*$ admits a left adjoint

$$f_\sharp\colon \mathrm{SH}(\mathfrak{Z}) \longrightarrow \mathrm{SH}(\mathfrak{X}),$$

which fulfills base change along $(-)^*$ and is $\mathrm{SH}(\mathfrak{X})$-linear, meaning for all $A \in \mathrm{SH}(\mathfrak{X})$ and $B \in \mathrm{SH}(\mathfrak{Z})$, we have a projection formula

$$f_\sharp(f^*A \otimes B) \simeq A \otimes f_\sharp B \in \mathrm{SH}(\mathfrak{X}).$$

Moreover, $f_\sharp$ commutes with suspension, so in particular we have

$$f_\sharp \mathbb{S}_{\mathfrak{Z}} \simeq \mathbb{S}_{\mathfrak{X}}[\mathfrak{Z}] \in \mathrm{SH}(\mathfrak{X}).$$

By general internal adjunction nonsense, the smooth projection formula implies that, for all $A \in \mathrm{SH}(\mathfrak{Z})$ and $B \in \mathrm{SH}(\mathfrak{X})$, we have a natural equivalence

$$\mathrm{map}(f_\sharp A, B) \simeq f_*\,\mathrm{map}(A, f^*B).$$

In particular, taking $B = \mathbb{S}_{\mathfrak{X}}$ we learn that we have a weak duality $(f_\sharp(-))^\vee \simeq f_*\left((-)^\vee\right)$.

**Theorem 3.2.5.** Let $f\colon \mathfrak{Z} \to \mathfrak{X}$ be representable, finite type and proper. Then, the functor

$$f_*\colon \mathrm{SH}(\mathfrak{Z}) \longrightarrow \mathrm{SH}(\mathfrak{X})$$

commutes with colimits and is $\mathrm{SH}(\mathfrak{X})$-linear, meaning for all $A \in \mathrm{SH}(\mathfrak{X})$ and $B \in \mathrm{SH}(\mathfrak{Z})$ we have a projection formula

$$A \otimes f_*B \simeq f_*(f^*A \otimes B).$$

*Proof.* This follows by combining statements (ii) and (iv) in [KR24, Thm. 7.1]. ☐

**Theorem 3.2.6** ([KR24, Thm. 6.1. (ii) & (iii)])**.** Let $f\colon \mathfrak{Z} \to \mathfrak{X}$ be a representable map of stacks and suppose we have a pullback diagram of stacks of the form:

$$\begin{tikzcd} \mathfrak{P} \arrow[r, "g"] \arrow[d, "q"'] & \mathfrak{Y} \arrow[d, "p"] \\ \mathfrak{Z} \arrow[r, "f"'] & \mathfrak{X} \end{tikzcd}$$

If $f$ is proper, the Beck-Chevalley transform

$$p^*f_* \longrightarrow g_*q^*$$

is an equivalence. Moreover, if $p$ is additionally smooth and representable, then we also have a natural equivalence

$$p_\sharp g_* \xrightarrow{\sim} f_\sharp q_*.$$

Let $\mathcal{E}$ be a vector bundle on $\mathfrak{X}$. We denote the motivic suspension spectrum of the Thom-space of $\mathcal{E}$ by

$$\mathbb{S}_{\mathfrak{X}}\langle\mathcal{E}\rangle := \Sigma^\infty \mathrm{Th}(\mathcal{E}) \in \mathrm{SH}(\mathfrak{X}),$$

and refer to $\mathbb{S}_{\mathfrak{X}}\langle\mathcal{E}\rangle$ as the *motivic Thom spectrum* of $\mathcal{E}$. By construction of $\mathrm{SH}(\mathfrak{X})$, all motivic Thom spectra $\mathbb{S}_{\mathfrak{X}}\langle\mathcal{E}\rangle$ are $\otimes$-invertible and hence this construction refines to a functor

$$\mathbb{S}_{\mathfrak{X}}\langle-\rangle\colon K_0(\mathfrak{X}) \longrightarrow \mathrm{Pic}(\mathrm{SH}(\mathfrak{X}))\,.$$

We then have the following important duality statement:

**Theorem 3.2.7** (Atiyah Duality)**.** Let $f\colon \mathfrak{Z} \to \mathfrak{X}$ be a smooth and representable map of stacks and denote by $\mathcal{L}_f$ the cotangent complex of $f$.

1. There exists a natural transformation
$$f_\sharp\langle -\mathcal{L}_f\rangle \longrightarrow f_*,$$
which is an equivalence if $f$ is proper.
2. Let $A \in \mathrm{SH}(\mathfrak{Z})$ be a dualizable object. If $f$ is proper, then $f_\sharp A$ is dualizable as well with dual given by $f_*(A^\vee) \simeq f_\sharp(A^\vee\langle -\mathcal{L}_f\rangle)$.

*Proof.* The first claim is [KR24, Cons. 6.7.] combined with [KR24, Thm. 6.1.(iv)]. For the second item we use the smooth and proper projection formulas of Theorem 3.2.4 and Theorem 3.2.5 to see that, for such $A \in \mathrm{SH}(\mathfrak{Z})$ and all $B, C \in \mathrm{SH}(\mathfrak{X})$ we have natural equivalences

$$\begin{aligned}
\mathrm{Map}(B \otimes f_\sharp A, C) &\simeq \mathrm{Map}(f_\sharp(f^*B \otimes A), C)\\
&\simeq \mathrm{Map}(f^*B \otimes A, f^*C)\\
&\simeq \mathrm{Map}(f^*B, A^\vee \otimes f^*C)\\
&\simeq \mathrm{Map}(B, f_*(A^\vee \otimes f^*C)\\
&\simeq \mathrm{Map}(B, f_*A^\vee \otimes C)
\end{aligned}$$

so the claim follows. □

### 3.2.2 K-theory and motivic Bott periodicity

Let $\mathrm{Cat}^{\mathrm{pf}}$ be the category of small, stable, idempotent complete categories with exact functors between them. We denote the lax symmetric monoidal Bass–Thomason–Trobaugh $K$-theory construction by

$$K\colon \mathrm{Cat}^{\mathrm{pf}} \longrightarrow \mathrm{Sp}\,,$$

which we simply refer to as *algebraic $K$-theory*.

The assignment taking a scheme $X$ to its (derived) category of quasi-coherent sheaves $\mathcal{D}_{\mathrm{qc}}(X)$ refines to a functor from $\mathrm{Sch}^{\mathrm{op}}$ to the category of stable, presentably symmetric monoidal categories $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,St}})$. Moreover, $\mathcal{D}_{\mathrm{qc}}(-)$ satisfies étale descent, so we may Kan extend it to all étale sheaves of spaces and obtain a functor

$$\mathcal{D}_{\mathrm{qc}}(-)\colon\ \mathrm{Shv}_{\text{ét}}(\mathrm{Sch};\mathcal{S})^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,St}})\,.$$

For any arbitrary $A \in \mathrm{Shv}_{\text{ét}}(\mathrm{Sch})$ the category $\mathcal{D}_{\mathrm{qc}}(A)$ thus defined has no reason to be compactly generated. However, we know, for example by [KR24, Thm. 2.24], that any $\mathfrak{X} \in \mathrm{Stk}_S$ is *perfect* i.e. the functor $\mathcal{D}_{\mathrm{qc}}(-)$ factors through the (non-full) subcategory $\mathrm{Pr}^{\mathrm{L,St}}_\omega \subseteq \mathrm{Pr}^{\mathrm{L,St}}$ of compactly generated categories with compact object preserving functors between them. Hence, for any stack $\mathfrak{X} \in \mathrm{Stk}_S$, we define the category of perfect complexes on $\mathfrak{X}$ as the category of compact objects

$$\mathcal{D}_{\mathrm{perf}}(\mathfrak{X}) \coloneqq \mathcal{D}_{\mathrm{qc}}(X)^{\omega} \in \mathrm{CAlg}(\mathrm{Cat}^{\mathrm{pf}}) .$$

This lets us apply the algebraic $K$-theory functor and we write

$$K(\mathfrak{X}) \coloneqq K(\mathcal{D}_{\mathrm{perf}}(\mathfrak{X})) \in \mathrm{CAlg}(\mathrm{Sp})$$

which we simply call the *$K$-theory* spectrum of $\mathfrak{X}$.

By [KR24, Thm. 10.2], generalizing the original result of Thomason and Trobaugh [TT90], the composite functor

$$\mathrm{Stk}_{\mathfrak{X}}^{\mathrm{op}} \longrightarrow \mathrm{Sp}, \quad \mathfrak{Z} \mapsto K(\mathfrak{Z})$$

satisfies Nisnevich descent. Restricting to the smooth locus $\mathrm{Sm}_{\mathfrak{X}}$ we may take its $\mathbb{A}^1$-localization to obtain an $S^1$-spectrum in motivic spaces

$$\mathrm{KH}_{\mathfrak{X}} \coloneqq L_{\mathbb{A}^1} K(-) \in \mathrm{CAlg}(\mathcal{S}_{\mathfrak{X}}^{\mathrm{ns}} \otimes \mathrm{Sp})$$

which we refer to as *homotopy $K$-theory*. As discussed in [KR24, Cons. 10.3], if $\mathfrak{X}$ is regular, then we have $\mathrm{KH}_{\mathfrak{X}}(\mathfrak{X}) \simeq K(\mathfrak{X}) = K(\mathcal{D}_{\mathrm{qc}}(\mathfrak{X}))$. Moreover, by [KR24, Rmk. 10.4], we have for any representable map of stacks $f\colon \mathfrak{Z} \to \mathfrak{X}$ a natural equivalence

$$f^*\mathrm{KH}_{\mathfrak{X}} \simeq \mathrm{KH}_{\mathfrak{Z}} .$$

**Definition 3.2.8.** By [KR24, Thm 10.7] the sheaf of spectra $\mathrm{KH}_{\mathfrak{X}}$ is representable by a motivic $\mathbb{E}_\infty$-ring spectrum

$$\mathbb{K}_{\mathfrak{X}} \in \mathrm{CAlg}(\mathrm{SH}(\mathfrak{X})) ,$$

such that, for any smooth and representable map $f\colon \mathfrak{Z} \to \mathfrak{X} \in \mathrm{Sm}_{\mathfrak{X}}$, we have a natural equivalence $f^*\mathbb{K}_{\mathfrak{X}} \simeq \mathbb{K}_{\mathfrak{Z}}$. We call $\mathbb{K}_{\mathfrak{X}}$ the *motivic $K$-theory spectrum* over $\mathfrak{X}$,

**Notation 3.2.9.** For any map of stacks $f\colon \mathfrak{Z} \to \mathfrak{X}$, we write

$$\mathbb{K}_{\mathfrak{X}}^{\mathfrak{Z}} \coloneqq f_*\mathbb{K}_{\mathfrak{Z}} \in \mathrm{CAlg}_{\mathbb{K}_{\mathfrak{X}}}(\mathrm{SH}(\mathfrak{X}))$$

and refer to this motivic spectrum as the $\mathbb{K}_{\mathfrak{X}}$-cohomology of $\mathfrak{Z}$.

**Warning 3.2.10.** Unlike for the motivic sphere, if $f\colon \mathfrak{Z} \to \mathfrak{X}$ is not representable, the motive $\mathbb{K}_{\mathfrak{X}}^{\mathfrak{Z}}$ generally does not agree with the naive cohomology $f_*f^*\mathbb{K}_{\mathfrak{X}}$. Indeed, unwinding the definitions and using the smooth base change formula of [KR24, Thm. 4.10.(i).(b)], we learn that $\mathbb{K}_{\mathfrak{X}}^{\mathfrak{Z}}$ has underlying sheaf of spectra on $\mathrm{Sm}_{\mathfrak{X}}$ given by

$$u_{\mathfrak{X}}(\mathbb{K}_{\mathfrak{X}}^{\mathfrak{Z}})(\mathfrak{A}) = \mathrm{KH}(\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{A})$$

Meanwhile, $f_*f^*\mathbb{K}_{\mathfrak{X}}$ is simply the right Kan extension of its restriction to smooth representable stacks over $\mathfrak{X}$ and will act as a suitable completed version of $\mathbb{K}_{\mathfrak{X}}^{\mathfrak{Z}}$, as explained in [EKS25].

The motivic spectrum $\mathbb{K}_{\mathfrak{X}}$ satisfies a much stronger version of Atiyah duality, namely *Bott-periodicity*. Namely, by [KR24, Thm. 10.7], for any virtual vector bundle $\mathcal{E} \in K_0(\mathfrak{X})$, there is a natural equivalence

$$\mathbb{K}_{\mathfrak{X}}\langle \mathcal{E} \rangle \cong \mathbb{K}_{\mathfrak{X}} .$$

This allows us to describe the category $\mathbb{K}_{\mathfrak{X}}$-modules as a localization of the category $\mathrm{KH}_{\mathfrak{X}}$-modules, which will be useful for us in Section 3.3.3. Denote by $\mathrm{Sp}(\mathrm{H}(\mathfrak{X}))$ the category of $S^1$-spectra in motivic spaces, i.e. spectral $\mathbb{A}^1$-invariant Nisnevich sheaves on $\mathrm{Sm}_{\mathfrak{X}}$. The description in this form is due to Hoyois [Hoy20].

**Proposition 3.2.11.** For any stack $\mathfrak{X}$, the forgetful functor

$$\mathrm{Mod}_{\mathbb{K}_{\mathfrak{X}}}(\mathrm{SH}(\mathfrak{X})) \longrightarrow \mathrm{Mod}_{\mathrm{KH}_{\mathfrak{X}}}(\mathrm{Sp}(\mathrm{H}(\mathfrak{X})_*))$$

is fully faithful.

*Proof.* Let us first consider the case where $\mathfrak{X}$ is quasi-fundamental in the sense of [KR24, Def. 2.9], in which case the stable motivic category $\mathrm{SH}(\mathfrak{X})$ is obtained from $\mathrm{Sp}(\mathrm{H}(\mathfrak{X})_*)$ by $\otimes$-inverting Thom spaces.

In this case, we even have a description of the essential image: It is spanned by the Bott-periodic modules. That is, as explained in [Hoy20, Sec. 4], for each vector bundle $\mathcal{E}$ on $\mathfrak{X}$ we may use the projective bundle formula to construct a map of $S^1$-spectra

$$\beta_{\mathcal{E}} \colon \Sigma^{\infty}_{S^1} \mathrm{Th}(\mathcal{E}) \longrightarrow \mathrm{KH}_{\mathfrak{X}}$$

called the Bott element of $\mathcal{E}$. A $\mathrm{KH}_{\mathfrak{X}}$-module is called Bott-periodic if it is local with respect to those morphisms. The fully faithfulness claim then follows from [Hoy20, Prop. 3.2], using the fact that Thom-spectra are invertible in $\mathrm{SH}(\mathfrak{X})$ and so Bott-periodicity for $\mathbb{K}_{\mathfrak{X}}$-modules is automatic.

For the general case, we can choose a Nisnevich cover of any stack by nicely quasi-fundamental stacks. As the functor $u_{\mathfrak{X}}$ commutes with smooth $(-)^*$ (as $(-)_\sharp$ commutes with the adjoint), we can deduce the general statement by Nisnevich descent, using that fully faithful functors are closed under limits. □

In particular, we may identify the category $\mathrm{Mod}_{\mathbb{K}}(\mathrm{SH}(\mathfrak{X}))$ with a Bousfield localization. We denote the symmetric monoidal left adjoint by

$$L_\beta \colon \mathrm{Mod}_{\mathrm{KH}_{\mathfrak{X}}}(\mathrm{Sp}(\mathrm{H}(\mathfrak{X}))) \longrightarrow \mathrm{Mod}_{\mathbb{K}_{\mathfrak{X}}}(\mathrm{SH}(\mathfrak{X}))$$

and refer to it as *Bott-periodization*.

### 3.2.3 Equivariant and motivic homotopy theory

In this section, we import some equivariant technology, as well as a comparison to motivic homotopy theory, of which we give a brief summary. We direct the reader towards [BH21, Sec. 10] for an in-depth discussion.

Let $F$ be a field, choose a separable closure $F \to \bar{F}$ and denote the absolute Galois group by $G$. As $G$ is a pro-finite group, the theory of $G$-equivariant homotopy theory is slightly modified from the case of a finite group.

**Definition 3.2.12.** For $G$ a pro-finite group, we write $\mathrm{Fin}_G$ for the category of finite continuous $G$-sets and $\mathcal{O}_G$ for the subcategory of transitive $G$-sets. We write

$$\mathcal{S}^{BG} = \lim_{N<G \text{ finite index}} \mathcal{S}^{BG/N},$$

where the limit is taking along taking homotopy fixed points, for the category of Borel $G$-spaces and

$$\mathcal{S}_G = \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{S}) \cong \lim_{N<G \text{ finite index}} \mathcal{S}_{G/N}$$

where the limit is taken along taking fixed points for the category of genuine $G$-spaces. The inclusion of Borel-equivariant spaces into genuine spaces, i.e. the right adjoint to restriction, is denoted by

$$\beta \colon \mathcal{S}^{BG} \longrightarrow \mathcal{S}_G .$$

Finally, we write $\mathrm{Sp}_G$ for the category of *genuine $G$-spectra* as defined in [BH21, Sec. 9].

This category of genuine $G$-spectra was first constructed by Barwick [Bar17] and his construction agrees with the one given by Bachmann and Hoyois by [BH21, Prop. 9.11].

We regard $\mathrm{Fin}_G$ as a full subcategory of $\mathrm{Sm}_F$ via the unique coproduct preserving functor determined by the assignment

$$i\colon \mathrm{Fin}_G \longrightarrow \mathrm{Sm}_F, \quad G/H \mapsto \bar{F}^H. \tag{3.2}$$

The subcategory $\mathrm{Fin}_G \subseteq \mathrm{Sm}_F$ is closed under taking both Nisnevich and étale covers, so that $\mathrm{Fin}_G$ defines a sub-site of $\mathrm{Sm}_F$ in either topology. Let $\tau \in \{\mathrm{ns}, \text{ét}\}$, then restriction and left Kan extension along $i$ induces an adjunction

$$c_\tau\colon \mathrm{Shv}_\tau(\mathrm{Fin}_G) \rightleftarrows \mathrm{Shv}_\tau(\mathrm{Sm}_F) : u_\tau.$$

Recall that we had an adjunction

$$\nu_*\colon \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}_F) \rightleftarrows \mathrm{Shv}_{\mathrm{ns}}(\mathrm{Sm}_F) : \nu^*$$

where $\nu_*$ is the fully faithful inclusion and $\nu^*$ is sheafification. Now let $\mathcal{O}_G \subseteq \mathrm{Fin}_G$ denote the full subcategory spanned by the transitive $G$-sets and consider the following diagram of sheaf categories:

$$\begin{tikzcd}
\mathrm{Shv}_{\text{ét}}(\mathrm{Sm}_F) \arrow[r, "\nu_*"] \arrow[d, "u_{\text{ét}}"] & \mathrm{Shv}_{\mathrm{ns}}(\mathrm{Sm}_F) \arrow[d, "u_{\mathrm{ns}}"] \\
\mathrm{Shv}_{\text{ét}}(\mathrm{Fin}_G) \arrow[r, "\nu_*"] \arrow[d, "\simeq"] & \mathrm{Shv}_{\mathrm{ns}}(\mathrm{Fin}_G) \arrow[d, "\simeq"] \\
\mathcal{S}^{BG} \arrow[r, "\beta"] & \mathcal{S}_G
\end{tikzcd}$$

Here, in the upper square the horizontal maps are the canonical inclusions and the vertical maps are given by restriction along the maps of the respective sites. The bottom right vertical equivalence is given by restriction along the transitive, finite $G$-sets $\mathcal{O}_G$. It restricts to the equivalence on the left between Borel $G$-spaces and étale sheaves.

Since everything in sight is (lax) symmetric monoidal, applying $\mathrm{CAlg}(-)$ and restricting to group-like objects, we get a commutative diagram of right adjoints:

$$\begin{tikzcd}
\mathrm{Sp}^{\mathrm{cn}}_{\text{ét}}(F) \arrow[r, "\nu_*"] \arrow[d, "U_{\text{ét}}"'] & \mathrm{Sp}^{\mathrm{cn}}_{\text{ét}}(F) \arrow[d, "U_{\mathrm{ns}}"] \\
\mathrm{Sp}^{\mathrm{cn}}(\mathrm{B}G) \arrow[r, "\beta"'] & \mathrm{Sp}^{\mathrm{cn}}(\mathcal{O}_G^{\mathrm{op}}),
\end{tikzcd}$$

such that $u_\tau \Omega^\infty = \Omega^\infty U_\tau$ where $\tau$ denotes either topology. Accordingly, we obtain adjunctions on sheaves of spectra which we denote

$$C_{\text{ét}}\colon \mathrm{Sp}^{\mathrm{cn}}(\mathrm{B}G) \rightleftarrows \mathrm{Sp}^{\mathrm{cn}}_{\text{ét}}(F) : U_{\text{ét}}$$

and

$$C_{\mathrm{ns}}\colon \mathrm{Sp}^{\mathrm{cn}}(\mathcal{O}_G^{\mathrm{op}}) \rightleftarrows \mathrm{Sp}^{\mathrm{cn}}_{\mathrm{ns}}(F) : U_{\mathrm{ns}}.$$

We also record the following:

**Lemma 3.2.13.** For $\tau \in \{\text{ét}, \mathrm{ns}\}$ the left adjoints $c_\tau, C_\tau$ are fully faithful. Moreover, the right adjoints $u_\tau, U_\tau$ preserve colimits.

*Proof.* We claim that the unit of the adjunction $c_\tau \dashv u_\tau$ is an equivalence. This is true for the adjunction on presheaf categories, as $\mathrm{Fin}_G \subset \mathrm{Sm}_F$ is fully faithful. Moreover, as $\mathrm{Fin}_G$ is closed under all covers, sheafification does not change the value of the left adjoint at elements in $\mathrm{Fin}_G$. The argument for fully faithfulness of $C_\tau$ is exactly the same.

Similarly, we observe that the right adjoints preserve colimits on the level of presheaf categories. As colimits are computed by taking the colimit pointwise and then sheafifying, the claim follows by the same argument as the fully faithfulness. □

Lastly, these adjunctions actually refine to the level of SH, which we record as the following proposition.

**Proposition 3.2.14.** Restriction and Kan-extension along the inclusion $i\colon \mathrm{Fin}_G \to \mathrm{Sm}_F$ induces an adjunction

$$\mathcal{C}\colon \mathrm{Sp}_G \rightleftarrows \mathrm{SH}(F) : \mathcal{U},$$

such that $\mathcal{C}$ is symmetric monoidal and makes the following diagram commute:

$$\begin{tikzcd} \mathrm{Shv}_{\mathrm{ns}}(\mathrm{Fin}_G) \arrow[r, "{\mathbb{S}[-]}"] \arrow[d, "c_{\mathrm{ns}}"] & \mathrm{Sp}_G \arrow[d, "\mathcal{C}"] \\ \mathrm{Shv}_{\mathrm{ns}}(\mathrm{Sm}_F) \arrow[r, "{\mathbb{S}[-]}"] & \mathrm{SH}(F)\,. \end{tikzcd}$$

*Proof.* This is precisely [BH21, Prop. 10.6], noting that, since everything in sight commutes with colimits, it suffices to check commutativity on representables $G/H \in \mathrm{Fin}_G$. □

The right adjoint $\mathcal{U}$ now gives a convenient definition of the $G$-equivariant algebraic $K$-theory spectrum.

**Definition 3.2.15.** For any $X \in \mathrm{Sm}_F$, we define the *$G$-equivariant $K$-theory* of $X$ as the genuine $G$-spectrum

$$K_G(X) := \mathcal{U}(\mathbb{K}_F^X) \in \mathrm{CAlg}(\mathrm{Sp}_G),$$

where $\mathcal{U}\colon \mathrm{SH}(F) \to \mathrm{Sp}_G$ is the right adjoint of Proposition 3.2.14.

**Remark 3.2.16.** Explicitly, we have that, for any open subgroup $H \subseteq G$, the $G$-spectrum $K^G(X)$ has $H$-fixed points given by the formula

$$K_G(X)^H = K(X_{\bar{F}^H}).$$

Note that, since $K$-theory does in general not satisfy étale descent, this $G$-spectrum is typically not Borel.

### 3.2.4 Recollections on parametrized category theory

In this subsection we fix a topos $\mathcal{B}$ throughout. We will remind the reader of some constructions within parametrized category theory, that is, the category theory internal to $\mathcal{B}$, as developed by Martini–Wolf in [Mar22], [MW24] and [MW25].

Among other things, we recall the notions of parametrized Kan extensions, presentability and the Lurie tensor product. Moreover, we discuss some aspects of higher algebra in the $\mathcal{B}$-parametrized setting.

**Definition 3.2.17.** Let $\mathcal{B}$ be a topos. A *$\mathcal{B}$-parametrized category* $\mathcal{C}$ is a limit preserving functor

$$\mathcal{C}\colon \mathcal{B}^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}}\,, \quad U \mapsto \mathcal{C}(U)\,.$$

We denote the large category of $\mathcal{B}$-categories by $\widehat{\mathrm{Cat}(\mathcal{B})} := \mathrm{Fun}^{\lim}(\mathcal{B}^{\mathrm{op}}, \widehat{\mathrm{Cat}})$. A $\mathcal{B}$-category is called small if it factors through $\mathrm{Cat} \subset \widehat{\mathrm{Cat}}$.

Unless otherwise specified, for a $\mathcal{B}$-category $\mathcal{C}$ and a map $f\colon U \to V$ in $\mathcal{B}$, we denote by $f^*\colon \mathcal{C}(V) \to \mathcal{C}(U)$ the "pullback" functor supplied by the data of $\mathcal{C}$. As usual, we also write $f_\sharp \dashv f^* \dashv f_*$ for the adjoints, if they exist.

Evaluating at any object $U \in \mathcal{B}$ defines a sections functor which we denote by

$$\Gamma_U\colon \widehat{\mathrm{Cat}}(\mathcal{B}) \longrightarrow \widehat{\mathrm{Cat}}, \quad \mathcal{C} \mapsto \Gamma_U(\mathcal{C}) = \mathcal{C}(U).$$

For $U = *$ being the terminal object of $\mathcal{B}$, we refer to $\Gamma(\mathcal{C}) := \Gamma_*(\mathcal{C})$ as the global sections of $\mathcal{C}$. For $U \in \mathcal{B}$, an *element of $\mathcal{C}$ in context $U$* is an element of $\mathcal{C}(U)$. If no context is specified, an element of $\mathcal{C}$ is understand to mean an element of the global sections $\Gamma(\mathcal{C})$.

**Example 3.2.18.** The following are the prototypical examples of $\mathcal{B}$-categories.

1. For any topos $\mathcal{B}$, the slice functor

$$\Omega_{\mathcal{B}} : \mathcal{B}^{\mathrm{op}} \longrightarrow \widehat{\mathrm{Cat}} \quad b \mapsto \mathcal{B}_{/b},$$

   defines a $\mathcal{B}$-category that we refer to as the *universe* of $\mathcal{B}$.

2. For any $A \in \mathcal{B}$, the mapping space functor

$$\mathrm{Map}_{\mathcal{B}}(-, A)\colon \mathcal{B}^{\mathrm{op}} \longrightarrow \mathcal{S} \subseteq \mathrm{Cat}$$

   defines a small $\mathcal{B}$-category. In fact, this construction yields an equivalence $\mathrm{Fun}^{\lim}(\mathcal{B}^{\mathrm{op}}, \mathcal{S}) \simeq \mathcal{B}$ and so we freely consider objects of $\mathcal{B}$ as $\mathcal{B}$-categories.

3. Let $\mathcal{B} = \mathrm{Shv}(\mathcal{X}, \mathcal{S})$ be a sheaf topos on some site $\mathcal{X}$. Then, right Kan extension along the fully faithful inclusion $\mathcal{X} \to \mathcal{B}^{\mathrm{op}}$ defines an equivalence of categories

$$\mathrm{Shv}(\mathcal{X}, \widehat{\mathrm{Cat}}) \xrightarrow{\sim} \widehat{\mathrm{Cat}}(\mathcal{B}).$$

**Definition 3.2.19.** By [MW24, Prop. 2.6.4] the category $\mathrm{Cat}(\mathcal{B})$ is cartesian closed. Thus, for any $\mathcal{C}, \mathcal{D} \in \widehat{\mathrm{Cat}}(\mathcal{B})$ we have a $\mathcal{B}$-category $[\mathcal{C}, \mathcal{D}]$ of $\mathcal{B}$-parametrized functors from $\mathcal{C}$ to $\mathcal{D}$. We write

$$\mathrm{Fun}_{\mathcal{B}}(\mathcal{C}, \mathcal{D}) := \Gamma_*[\mathcal{C}, \mathcal{D}] \in \widehat{\mathrm{Cat}}$$

for the category of global sections.

The content of Definition 3.2.19 provides a natural refinement of $\widehat{\mathrm{Cat}}(\mathcal{B})$ to an $(\infty, 2)$-category, and hence one can talk about adjunctions internal to $\widehat{\mathrm{Cat}}(\mathcal{B})$ as is done in [MW24, Def. 3.1.1]. We will not discuss these notions in detail and instead skip ahead to parametrized Kan extensions and then work backwards. Note that, for any $\mathcal{B}$-functor $f\colon \mathcal{C} \to \mathcal{D}$ we have an induced natural transformation $f^*\colon [\mathcal{D}, -] \to [\mathcal{C}, -]$ of functors $\widehat{\mathrm{Cat}}(\mathcal{B}) \to \widehat{\mathrm{Cat}}(\mathcal{B})$. The following definition is precisely[3] [MW24, Def. 6.3.1] and will be important for us.

**Definition 3.2.20.** Let $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{C} \to \mathcal{E}$ be $\mathcal{B}$-functors. A *left Kan extension* of $g$ along $f$ is a $\mathcal{B}$-functor

$$f_\sharp g : \mathcal{D} \to \mathcal{E},$$

together with an equivalence of $\mathcal{B}$-functors $[\mathcal{D}, \mathcal{E}] \to \Omega_{\mathcal{B}}$ of the form

$$\mathrm{Map}_{[\mathcal{D},\mathcal{E}]}(f_\sharp g, -) \simeq \mathrm{Map}_{[\mathcal{C},\mathcal{D}]}(g, f^*(-))\,.$$

Dually, a *right Kan extension* along $f$ is a functor $f_* g : \mathcal{D} \to \mathcal{E}$, together with an equivalence

$$\mathrm{Map}_{[\mathcal{D},\mathcal{E}]}(-, f_* g) \simeq \mathrm{Map}_{[\mathcal{C},\mathcal{E}]}(f^*(-), g).$$

If $f$ is the unique functor to the terminal $\mathcal{B}$-category $\mathcal{C} \to *$ we call $f_\sharp g$ the *parametrized colimit* and $f_* g$ the *parametrized limit* of $g$.

[3] Beware that, we denote left adjoints to pullback generically by $(-)_\sharp$, while Martini–Wolf use $(-)_!$.

**Example 3.2.21.** Let $\mathcal{C}$ be a $\mathcal{B}$-category and let $A \in \mathcal{B}$ with terminal map $p\colon A \to *$. Let $X$ be an object of $\mathcal{C}(A)$, considered as a $\mathcal{B}$-functor $A \xrightarrow{E} \mathcal{C}$. Unwinding the definitions, we see that, if this functor admits a parametrized colimit, there exists an object $p_\sharp E \in \mathcal{C}(*)$, together with an equivalence

$$\eta\colon \operatorname{Map}_{\mathcal{C}(*)}(p_\sharp E, -) \simeq \operatorname{Map}_{\mathcal{C}(A)}(E, p^*(-))\,.$$

In particular, if $\mathcal{C}$ admits all parametrized colimits of this form, the functor $p^*$ necessarily admits a left adjoint $p_\sharp$. Dually, one gets that the parametrized limit is computed via a faux right adjoint $p_*$. We also write $E^A := p_* E \in \mathcal{C}$ and $E[A] := p_\sharp E$, whenever they exist.

**Definition 3.2.22.** A $\mathcal{B}$-category $\mathcal{D}$ is called (co)complete if any functor $f\colon \mathcal{C} \to \mathcal{D}$ with $\mathcal{C}$ small, admits a $\mathcal{B}$-(co)limit.

Moreover, $\mathcal{D}$ is called *presentable* if it is cocomplete and sectionwise presentable. We write $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ for the large category of presentable $\mathcal{B}$-categories and colimit-preserving functors.

**Remark 3.2.23.** It is shown in [MW24, Cor. 5.4.7] that a $\mathcal{B}$-category is cocomplete if and only if

1. it factors through the large category $\widehat{\mathrm{Cat}}^{\mathrm{R}}_{\mathrm{cc}}$ of cocomplete categories with colimit preserving right adjoint functors between them.

2. The adjoints $(-)_\sharp \dashv (-)^*$ satisfy the Beck-Chevalley condition for any pullback square in $\mathcal{B}$.

**Theorem 3.2.24** ([MW25, Prop. 2.6.3.2 & Prop 2.6.3.7])**.** The large category of presentable $\mathcal{B}$-categories $\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$ admits a symmetric monoidal structure with unit being the universe $\Omega_{\mathcal{B}}$. Moreover, the global sections functor

$$\Gamma(-)\colon \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}) \longrightarrow \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}})$$

admits a strong symmetric monoidal, fully faithful and colimit preserving left adjoint

$$-\otimes_{\mathcal{B}} \Omega_{\mathcal{B}}\colon\ \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}) \longrightarrow \mathrm{Pr}^{\mathrm{L}}(\mathcal{B}).$$

**Definition 3.2.25.** Equipped with the symmetric monoidal structure of Theorem 3.2.24, we call $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ the large category of *presentably symmetric monoidal* $\mathcal{B}$-categories. We also denote the composition

$$\mathrm{Pr}^{\mathrm{L}} \xrightarrow{-\otimes\mathcal{B}} \mathrm{Mod}_{\mathcal{B}}(\mathrm{Pr}^{\mathrm{L}}) \xrightarrow{-\otimes_{\mathcal{B}}\Omega_{\mathcal{B}}} \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$$

simply by $-\otimes \Omega_{\mathcal{B}}$. For any presentable category $\mathcal{C} \in \mathrm{Pr}^{\mathrm{L}}$ we call

$$\mathcal{C} \otimes \Omega_{\mathcal{B}} \in \mathrm{Pr}^{\mathrm{L}}(\mathcal{B})$$

the presentable $\mathcal{B}$-category of *$\mathcal{C}$-valued sheaves on $\mathcal{B}$*.

Since the functor $-\otimes \Omega_{\mathcal{B}}$ is symmetric monoidal by Theorem 3.2.24, it carries presentably symmetric monoidal categories to presentably symmetric monoidal $\mathcal{B}$-categories.

**Remark 3.2.26.** In [MW25, Sec. 2.5], Martini–Wolf develop the notion of commutative algebra objects in a presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{C}$. We will use the following facts:

1. For any $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$, there exists a co-complete $\mathcal{B}$-category $\mathrm{CAlg}(\mathcal{C})$ together with, for all $A \in \mathcal{B}$ a natural identification $\mathrm{CAlg}(\mathcal{C})(A) = \mathrm{CAlg}(\mathcal{C}(A))$.

2. For any commutative algebra $R \in \mathrm{CAlg}(\mathcal{C})$, there exists a $\mathcal{B}$-category $\mathrm{Mod}_R(\mathcal{C})$, together with, for any $p\colon A \to * \in \mathcal{B}$ a natural identification $\mathrm{Mod}_R(\mathcal{C})(A) = \mathrm{Mod}_{p^*R}(\mathcal{C}(A))$. Moreover, by [MW25, Prop. 2.5.2.8] the $\mathcal{B}$-category $\mathrm{Mod}_{\mathcal{C}}(R)$ is again presentably symmetric monoidal.

**Construction 3.2.27.** Let $\mathrm{CAlg}^{\mathrm{grp}}(\Omega_{\mathcal{B}}) \subseteq \mathrm{CAlg}(\Omega_{\mathcal{B}})$ denote the full $\mathcal{B}$-subcategory spanned by the *grouplike* commutative monoid objects in $\Omega_{\mathcal{B}}$. The inclusion admits a parametrized right adjoint

$$(-)^{\times}\colon \mathrm{CAlg}(\Omega_{\mathcal{B}}) \longrightarrow \mathrm{CAlg}^{\mathrm{grp}}(\Omega_{\mathcal{B}})$$

sending a commutative algebra to its units. Both, the inclusion and its right adjoint are computed sectionwise, as they preserve limits.

Let $\mathrm{Sp}^{\mathrm{cn}} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ denote the category of connective spectra. In accordance with Definition 3.2.25, we define the $\mathcal{B}$-category of *connective $\mathcal{B}$-spectra* as

$$\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}} := \Omega_{\mathcal{B}} \otimes \mathrm{Sp}^{\mathrm{cn}} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B})).$$

Applying the Boardman–Vogt–May theorem [HA, Thm. 5.2.6.10] sectionwise gives an equivalence of $\mathcal{B}$-parametrized categories

$$\mathrm{CAlg}^{\mathrm{grp}}(\Omega_{\mathcal{B}}) \simeq \mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}}.$$

Under this identification, we have an evident forgetful functor to the universe, which we denote by

$$\Omega^{\infty}\colon \mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}} \longrightarrow \Omega_{\mathcal{B}}\,.$$

**Warning 3.2.28.** Beware that a sheaf of connective spectra is not the same as a sheaf of spectra whose sections happen to be connective, as the sheaf condition is a limit formula. Nonetheless, there is a full faithful functor $\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}} \to \mathrm{Sp}_{\mathcal{B}}$ given sectionwise by including and then sheafifying, with right adjoint given by taking connective covers, see also the $t$-structure discussion in Section 3.4.1.

The sectionwise loop and suspension functors induce an adjunction

$$\Sigma\colon \mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}} \rightleftarrows \mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}} :\Omega$$

where $\Sigma$ is computed by suspending in presheaves and then sheafifying, while $\Omega$ is just computed in presheaves of connective spectra. In particular, we have that $\Omega \circ \Sigma \simeq \mathrm{id}$, i.e. $\Sigma$ is fully faithful.

Since $\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}}$ is presentably symmetric monoidal by construction, so is $\mathrm{Mod}_A(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$ for any $A \in \mathrm{CAlg}(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$ by Remark 3.2.26. In particular, these categories admit an internal mapping object which for $M, N \in \mathrm{Mod}_A(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$ we denote by

$$\mathrm{map}_A(M,N) \in \mathrm{Mod}_A(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}}).$$

## 3.3 The Fourier transform for motivic $K$-theory

Given a ring $R$ and a finite abelian group $G$ with Pontryagin dual $G^{\vee}$, the classical Fourier transform is a natural map of commutative rings:

$$\mathcal{F}_{\omega} : R[G] \longrightarrow R^{G^{\vee}}$$

which depends on the choice of an orientation, i.e. a map of abelian groups

$$\omega : \mathbb{Q}/\mathbb{Z} \longrightarrow R^{\times}.$$

There are numerous generalizations, categorifications and derived enhancements of this construction. The goal of this section is to produce two more. First, a Fourier–Mukai transform for modules over the category of perfect complexes on a scheme. Secondly, we use this to produce a Fourier transform in the category of $\mathbb{K}$-modules in the stable motivic category SH.

We begin in Section 3.3.1 by generalizing the Yoga of [BCSY24] to the $\mathcal{B}$-parametrized setting for any topos $\mathcal{B}$. Concretely, setting up a Fourier transform has two inputs:

1. A pair $(R, I)$ consisting of a base ring $R \in \mathrm{CAlg}(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$ together with a dualizing object $I \in \mathrm{Mod}_R(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$ which we call a *Fourier context*.

2. A presentably symmetric monoidal $\mathcal{B}$-category $\mathcal{C}$, together with a *Fourier orientation*, given by a map of connective $\mathcal{B}$-spectra
$$\omega\colon I \longrightarrow \mathbb{1}_{\mathcal{C}}^{\times}\,.$$

Using the parametrized adjunction

$$\mathbb{1}_{\mathcal{C}}[-]\colon\ \mathrm{Mod}_R(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}}) \rightleftarrows \mathrm{CAlg}(\mathcal{C})\ :(-)^{\times}$$

and writing $I(-) = \mathrm{map}_R(-, I)$ for the internal mapping object, for any $M \in \mathrm{Mod}_R(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$, the data above gives rise a natural map

$$\mathfrak{F}_{\omega}^{M}\colon \mathbb{1}_{\mathcal{C}}[M] \longrightarrow \mathbb{1}_{\mathcal{C}}^{\Omega^{\infty} I(M)} \in \mathrm{CAlg}(\mathcal{C})\,,$$

which we call the *Fourier transform* associated to the datum $(R, I, \mathcal{C}, \omega)$.

In Section 3.3.2, we apply this formalism to construct a compact object preserving Fourier-Mukai transform. Concretely, we work over the topos $\mathcal{S}_S^{\mathrm{big,\acute{e}t}}$ of étale sheaves of spaces on $\mathrm{Sch}_S$ for some base scheme $S$. Our chosen Fourier context is $(\mathbb{Z}_S, \Sigma\mathbb{G}_m)$, where $\mathbb{Z}_S$ is the constant sheaf and $\Sigma\mathbb{G}_m$ is the sheaf of connective $\mathbb{Z}$-modules represented by the étale delooping of the commutative group scheme $\mathbb{G}_m$.

The functor taking a stack $\mathfrak{X} \in \mathrm{Stk}_S$ to its category of perfect complexes $\mathcal{D}_{\mathrm{perf}}(\mathfrak{X})$ defines an étale sheaf on $\mathrm{Sch}_S$ with values in $\mathrm{Cat}^{\mathrm{pf}}$ and we denote the associated $\mathcal{S}_S^{\mathrm{big,\acute{e}t}}$-category by $\mathcal{O}_S^{\mathrm{nc}}$. Then, the parametrized category of modules over $\mathcal{O}_S^{\mathrm{nc}}$ comes equipped with a natural orientation map
$$\omega_{\mathrm{c}}\colon \Sigma\mathbb{G}_m \longrightarrow (\mathcal{O}_S^{\mathrm{nc}})^{\times}$$
induced by the inclusion $\mathcal{D}_{\mathrm{perf}}(S)^{\heartsuit} \to \mathcal{D}_{\mathrm{perf}}(S)$. For any $\mathbb{Z}_S$-module $M$ we write $M^{\vee} = \mathrm{map}_{\mathbb{Z}_S}(M, \Sigma\mathbb{G}_m)$ which we call the *shifted Cartier dual* of $M$. The formalism of Section 3.3.1 then provides us with a natural map

$$\mathfrak{F}_{\omega_{\mathrm{c}}}^{M}\colon \mathcal{O}_S^{\mathrm{nc}}[M] \longrightarrow (\mathcal{O}_S^{\mathrm{nc}})^{\Omega^{\infty} M^{\vee}}\,,$$

of commutative algebra objects in the category $\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}}$ which we call the *categorical Fourier transform*. In particular, on global sections $\mathfrak{F}_{\omega_{\mathrm{c}}}^{M}$ defines an exact, symmetric monoidal, left adjoint functor between small, stable, idempotent complete, symmetric monoidal categories.

In Section 3.3.3, we use the categorical Fourier transform to construct a Fourier transform in the category of modules over the motivic $K$-theory spectrum $\mathbb{K}_S \in \mathrm{SH}(S)$ by applying $K$-theory sectionwise and then localizing into the category of $\mathbb{K}_S$-modules in $\mathrm{SH}(S)$. More precisely, if $M$ is an étale sheaf of connective $\mathbb{Z}$-modules, such that the dual $M^{\vee}$ is representable by a *stack* in the sense of Terminology 3.2.1, we construct natural map of $\mathbb{K}_S$-algebras in $\mathrm{SH}(S)$

$$\mathfrak{F}_{\mathrm{mot}}^{M}\colon \mathbb{K}_S[M] \longrightarrow \mathbb{K}_S^{\Omega^{\infty} M^{\vee}}\,,$$

which we call the *motivic Fourier transform*. Here, the left hand side is the $\mathbb{K}$-homology of $M$ defined via Kan-extension from smooth representables and the right hand side is the $\mathbb{K}$-motive of the stack $\Omega^{\infty} M^{\vee}$, as defined in Notation 3.2.9.

We conclude this section by proving that, for any torus $T/S$, the motivic Fourier transform provides an equivalence
$$\mathbb{K}_S[\Lambda(T)] \xrightarrow{\sim} \mathbb{K}_S^{\mathrm{B}T}\,,$$
where $\Lambda(T) = \mathrm{B}T^{\vee}$ is the sheaf of characters $T \to \mathbb{G}_m$. This will be a crucial input for proving our main theorem in Section 3.4.

### 3.3.1 Parametrized Fourier theory

Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ be a presentably symmetric monoidal $\mathcal{B}$-category. Then $\mathcal{C}$ comes equipped with a unit map

$$\Omega_{\mathcal{B}} \longrightarrow \mathcal{C}$$

which is left Kan extended from the functor $* \xrightarrow{\mathbb{1}_{\mathcal{C}}} \mathcal{C}$. Taking commutative algebras objects, we obtain a map of presentable $\mathcal{B}$-categories

$$\mathbb{1}_{\mathcal{C}}[-]\colon \mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}} \hookrightarrow \mathrm{CAlg}(\Omega_{\mathcal{B}}) \longrightarrow \mathrm{CAlg}(\mathcal{C})$$

which we call the *group algebra* construction. We denote the parametrized right adjoint by

$$(-)^{\times}\colon \mathrm{CAlg}(\mathcal{C}) \longrightarrow \mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}}\,,$$

and refer to it as the *spectrum of units* functor. Explicitly $(-)^{\times}$ is given by the composition of the mapping object $\mathrm{map}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, -)$ with the functor of Construction 3.2.27.

Dually, by parametrized right Kan extension of the map $* \xrightarrow{\mathbb{1}_{\mathcal{C}}} \mathrm{CAlg}(\mathcal{C})$ in $\mathrm{Cat}(\mathcal{B})$, we obtain a right adjoint functor

$$\mathbb{1}_{\mathcal{C}}^{(-)}\colon \Omega_{\mathcal{B}}^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathcal{C})\,.$$

For any $A \in \mathcal{B}$, we think of $\mathbb{1}_{\mathcal{C}}[A]$ as the *homology* and of $\mathbb{1}_{\mathcal{C}}^{A}$ as the *cohomology* of $A$ with coefficients in $\mathcal{C}$. Note that this is consistent with the notation of Example 3.2.21.

**Definition 3.3.1.** A *Fourier context* is a pair $(R, I)$ consisting of a *base ring* $R \in \mathrm{CAlg}(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$ and a *dualizing object* $I \in \mathrm{Mod}_R(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$. For any $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ a *Fourier orientation*[4] of $\mathcal{C}$ with respect to $(R, I)$ is a map of parametrized connective spectra

$$\omega\colon I \longrightarrow \mathbb{1}_{\mathcal{C}}^{\times}.$$

For any $R$-module $M \in \mathrm{Mod}_R(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$ we write

$$I(M) \coloneqq \mathrm{map}_R(M, I) \in \mathrm{Mod}_R(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$$

for the internal mapping object in connective $\mathcal{B}$-spectra.

The following is the parametrized analog of [BCSY24, Prop. 3.10] and the proof is essentially the same.

**Proposition 3.3.2.** Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$ and let $(R, I)$ be a Fourier context. There is a natural equivalence between the space of parametrized natural transformations

$$\mathbb{1}_{\mathcal{C}}[-] \longrightarrow \mathbb{1}_{\mathcal{C}}^{\Omega^{\infty} I(-)},$$

of parametrized functors $\mathrm{Mod}_{\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}}}(R) \to \mathrm{CAlg}(\mathcal{C})$ and the space of Fourier orientations, i.e. the space of maps of parametrized connective spectra

$$\omega\colon I \longrightarrow \mathbb{1}_{\mathcal{C}}^{\times}\,.$$

*Proof.* We claim that the parametrized right Kan extension of the functor $* \xrightarrow{\mathbb{1}_{\mathcal{C}}} \mathrm{CAlg}(\mathcal{C})$ along the map $* \xrightarrow{I} \mathrm{Mod}_R(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$ is given by the functor sending $M$ to $\mathbb{1}_{\mathcal{C}}^{\Omega^{\infty} I(M)}$. As $* \to \Omega_{\mathcal{B}}$ is fully faithful, the latter map factors through the map $\Omega_{\mathcal{B}}^{\mathrm{op}} \to \mathrm{Mod}_R(\mathrm{Sp}_{\mathcal{B}}^{\mathrm{cn}})$ obtained by parametrized right Kan extension. But this map has a parametrized left adjoint, given by

[4] Called a *pre*-orientation in [BCSY24].

sending $M$ to $\Omega^\infty \mathrm{map}_R(M, I)$. The claim follows because parametrized right Kan extension along a parametrized right adjoint is computed by precomposition with the left adjoint.

Using the universal property of the parametrized right Kan extension, we finally deduce that

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{Mod}_R(\mathrm{Sp}^{\mathrm{cn}}_{\mathcal{B}}),\mathrm{CAlg}(\mathcal{C}))}\left(\mathbb{1}_{\mathcal{C}}[-], \mathbb{1}_{\mathcal{C}}^{\Omega^\infty I(-)}\right) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(\mathbb{1}_{\mathcal{C}}[I], \mathbb{1}_{\mathcal{C}}).$$

The claim follows as the spectrum of units was defined to be the right adjoint of the group algebra functor. □

**Definition 3.3.3.** Let $(R, I)$ be a Fourier context in a parametrized presentably symmetric monoidal category $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{B}))$. By Proposition 3.3.2, any orientation $\omega \colon I \to \mathbb{1}_{\mathcal{C}}^\times$ yields an associated natural transformation

$$\mathfrak{F}_\omega^{(-)} \colon \mathbb{1}_{\mathcal{C}}[-] \longrightarrow \mathbb{1}_{\mathcal{C}}^{\Omega^\infty I(-)}$$

of functors $\mathrm{Mod}_R(\mathrm{Sp}^{\mathrm{cn}}_{\mathcal{B}}) \to \mathrm{CAlg}(\mathcal{C})$. We call $\mathfrak{F}_\omega$ the *Fourier transform* associated to $\omega$.

### 3.3.2 A Fourier–Mukai transform for $\mathcal{D}_{\mathrm{perf}}$-linear categories

Throughout this section we fix a base scheme $S$ and work over the topos of étale sheaves on the category of $S$-schemes $\mathrm{Sch}_S$ which we denote as

$$\mathcal{S}_S^{\mathrm{big,ét}} := \mathrm{Shv}_{\mathrm{ét}}(\mathrm{Sch}_S; \mathcal{S}).$$

Note that, since any stack over $S$ is étale locally representable, we also have a natural equivalence $\mathcal{S}_S^{\mathrm{big,ét}} \simeq \mathrm{Shv}_{\mathrm{ét}}(\mathrm{Stk}_S; \mathcal{S})$ so it does not matter which site we use to produce our base topos.

We now use the machinery developed in the previous section to set up a Fourier–Mukai transform for the category of perfect complexes.

Recall from the discussion in Section 3.2.2 that for any $\mathfrak{X} \in \mathrm{Stk}_S$ we have a well behaved notion of perfect complexes on $\mathfrak{X}$, namely the category $\mathcal{D}_{\mathrm{perf}}(\mathfrak{X}) = \mathcal{D}_{\mathrm{qc}}(\mathfrak{X})^\omega \in \mathrm{Cat}^{\mathrm{pf}}$. By [BGT13, Cor. 4.25] the category $\mathrm{Cat}^{\mathrm{pf}}$ is compactly generated and presentable. Moreover, the inclusion $\mathrm{Cat}^{\mathrm{pf}} \simeq \mathrm{Pr}^{\mathrm{L,St}}_\omega \to \mathrm{Pr}^{\mathrm{L,St}}$ preserves colimits and is closed under tensor products and so $\mathrm{Cat}^{\mathrm{pf}}$ inherits the structure of a presentably symmetric monoidal category and the forgetful functor $\mathrm{Cat}^{\mathrm{pf}} \to \mathrm{Cat}$ preserves limits. In particular, the functor $\mathcal{D}_{\mathrm{perf}}(-)$ refines to an étale sheaf of symmetric monoidal categories, which we record in the following definition.

**Definition 3.3.4.** For any scheme $S$, we write

$$\mathcal{O}_S^{\mathrm{nc}} \in \mathrm{CAlg}(\Omega_{\mathcal{S}_S^{\mathrm{big,ét}}} \otimes \mathrm{Cat}^{\mathrm{pf}})$$

for the sheaf of categories given by the functor

$$\mathcal{O}_S^{\mathrm{nc}} \colon \mathrm{Stk}_S^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathcal{D}_{\mathrm{perf}}), \quad X \mapsto \mathcal{D}_{\mathrm{perf}}(X).$$

Moreover, we denote the $\mathcal{S}_S^{\mathrm{big,ét}}$-parametrized category of modules over $\mathcal{O}_S^{\mathrm{nc}}$ as

$$\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}} := \mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}}(\mathrm{Cat}^{\mathrm{pf}} \otimes \Omega_{\mathcal{S}_S^{\mathrm{big,ét}}}) \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}(\mathcal{S}_S^{\mathrm{big,ét}}))$$

and refer to sections of $\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}}$ as *étale $S$-linear categories*.

Explicitly, the parametrized category $\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}}$ sends $A \in \mathcal{S}_S^{\mathrm{big,ét}}$ to the presentably symmetric monoidal category

$$\mathrm{Mod}_{\mathcal{O}^{\mathrm{nc}}}(A) = \mathrm{Mod}_{\mathcal{O}_A^{\mathrm{nc}}}(\mathcal{S}_A^{\mathrm{big,ét}} \otimes \mathrm{Cat}^{\mathrm{pf}}).$$

Note that, by Remark 3.2.26 the fact that the category $\mathrm{Cat}^{\mathrm{pf}}$ is presentably symmetric monoidal, implies that the category $\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}}$ is *parametrized* presentably symmetric monoidal.

**Warning 3.3.5.** Note that, if $A \in \mathcal{S}_S^{\mathrm{big,\acute{e}t}}$ is not representable by a stack, then the category $\Gamma_A \mathcal{O}_S^{\mathrm{nc}}$ as defined by the Kan extension above need not agree with the category of compact objects of $\mathcal{D}_{\mathrm{qc}}(A)$ as defined via Kan extension of $\mathcal{D}_{\mathrm{qc}}(-)$ as a functor into $\mathrm{Pr}^{\mathrm{L,St}}$.

Let $f\colon A \to S$ be an object of $\mathcal{S}_S^{\mathrm{big,\acute{e}t}}$. Since the category $\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}}$ is parametrized presentable, the pullback functor

$$f^*\colon \mathrm{Mod}_{\mathcal{O}^{\mathrm{nc}}}(S) \longrightarrow \mathrm{Mod}_{\mathcal{O}^{\mathrm{nc}}}(A)$$

admits both adjoints $f_\sharp \dashv f^* \dashv f_*$ which are worth discussing explicitly. First, note that $f^*$ is given by restriction along the pushforward map $(\mathcal{S}_S^{\mathrm{big,\acute{e}t}})_{/A} \longrightarrow \mathcal{S}_S^{\mathrm{big,\acute{e}t}}$. Thus, the functor $f_*$ is computed by right Kan extension along the same functor and is thus given by precomposition with the left adjoint. In particular, for any $\mathcal{D} \in \mathrm{Mod}_{\mathcal{O}^{\mathrm{nc}}}(\mathfrak{Z})$ we have that $f_*\mathcal{D}$ is the sheaf of $\mathcal{O}_S^{\mathrm{nc}}$-modules given by

$$f_*\mathcal{D}(B) = \mathcal{D}(f^*B) = \mathcal{D}(A \times_S B) \in \mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}(B)}(\mathrm{Cat}^{\mathrm{pf}}).$$

The left adjoint $f_\sharp$ is computed by left Kan extension over the slice, which is in general mysterious. However, if $f$ is finite étale, the adjoints will agree in any 0-semiadditive category as the adjoints are locally given by product and coproduct. Let us call a $\mathcal{S}_S^{\mathrm{big,\acute{e}t}}$-category $\mathcal{C}$ *finite étale (co)-complete* if it is (co)-complete with respect to the inductible class of finite étale morphisms in the sense of [CLL25, Def. 2.10].

**Lemma 3.3.6.** Let $\mathcal{C}$ be a finite étale complete and cocomplete $\mathcal{S}_S^{\mathrm{big,\acute{e}t}}$-category that is sectionwise 0-semiadditive. Then for any finite étale map $f\colon X \to S$ we have a natural equivalence $\mathrm{Nm}_f\colon f_\sharp \xrightarrow{\sim} f_*$ of functors $\mathcal{C}(X) \to \mathcal{C}(S)$.

*Proof.* As any $(-1)$-truncated map in $\mathcal{S}_S^{\mathrm{big,\acute{e}t}}$ is either empty or an equivalence and $\mathcal{C}$ is sectionwise pointed, we may construct a comparison map

$$\mathrm{Nm}_f\colon f_\sharp \longrightarrow f_*,$$

as in [CLL25, Cons. 3.3] inductively. As $f$ is finite étale, it is étale locally given by a fold map $\sqcup_n S \to S$ for some finite $n$. Thus, as $\mathcal{C}$ is finite étale complete and cocomplete, we may use the $(-)_*$ and $(-)_\sharp$ Beck–Chevalley isomorphisms to see that $\mathrm{Nm}_f$ is étale locally given by the identity matrix

$$\coprod_n S \longrightarrow \prod_n S\,,$$

and thus an equivalence by the assumption that $\mathcal{C}$ is 0-semiadditive. As we can check equivalences étale locally, the claim follows. ☐

In particular Lemma 3.3.6 applies to $\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}}$ as it is even parametrized presentable and sectionwise 0-semiadditive, as well as any sheaf category $\mathcal{S}_S^{\mathrm{big,\acute{e}t}} \otimes \mathcal{C}$ where $\mathcal{C}$ is presentable and 0-semiadditive.

Let us now define our chosen Fourier context. Recall that we denote by $\mathbb{Z}_S \in \mathrm{Sp}^{\mathrm{cn}}_{\mathcal{S}_S^{\mathrm{big,\acute{e}t}}}$ the étale sheafification of the constant presheaf with value $\mathbb{Z}$. We write

$$\mathrm{Mod}_{\mathbb{Z}_S}^{\mathrm{cn}} := \mathrm{Mod}_{\mathbb{Z}_S}(\mathrm{Sp}^{\mathrm{cn}}_{\mathcal{S}_S^{\mathrm{big,\acute{e}t}}})$$

for the parametrized category of $\mathbb{Z}_S$-modules.

**Notation 3.3.7.** Let $\mathcal{E}$ be a commutative group scheme over $S$. Then, via Yoneda, we can regard $\mathcal{E}$ as an étale sheaf of abelian groups on $\mathrm{Sch}_S$ and thus via the Dold–Kan correspondence as an object of $\mathrm{Mod}_{\mathbb{Z}_S}^{\mathrm{cn}}$. As such, we have a canonical identification

$$\Omega^\infty \Sigma \mathcal{E} \simeq \mathrm{B}\mathcal{E} \in \mathcal{S}_S^{\mathrm{big,\acute{e}t}}$$

where $\mathrm{B}\mathcal{E}$ denotes the étale classifying stack. We make this identification implicitly during the rest of this section.

**Definition 3.3.8.** For any $M \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}_S}$, we call the internal mapping objects

$$\Lambda\left(M\right) \coloneqq \mathrm{map}_{\mathbb{Z}_S}(M, \mathbb{G}_m) \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}_S}$$

$$M^\vee \coloneqq \mathrm{map}_{\mathbb{Z}_S}(M, \Sigma\mathbb{G}_m) \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}_S}$$

the *Cartier dual* and *shifted Cartier dual* of $M$ respectively.

**Remark 3.3.9.** Let us say some words about these two notions of duality.

1. By definition, classes in the Cartier dual $\Lambda(M)$ correspond to characters maps $\chi\colon M \to \mathbb{G}_m$. If $M$ is given by an algebraic torus $T$, then after passing to a finite étale extension of $S$ that splits $T$ the sheaf $\Lambda\left(T\right)$ is constant and given by a free abelian group of finite rank, i.e. a lattice. Thus, in this situation, we also call $\Lambda\left(T\right)$ the *character lattice of $T$*.

2. Using the identification $\Sigma\mathbb{G}_m \simeq \mathrm{pic}_S$ of Lemma 3.3.11, we see that classes of the shifted Cartier dual $M^\vee$ are given by rank 0 line bundles $\mathcal{L} \in \mathrm{pic}_S(M)$. Thus, if $M = \mathcal{E}$ is an abelian variety, the shifted Cartier dual $\mathcal{E}^\vee$ is representable and recovers the dual abelian variety of $\mathcal{E}$.

Since the étale suspension functor is fully faithful, it provides a canonical identification

$$\Lambda\left(M\right) \simeq \mathrm{map}_{\mathbb{Z}_S}(\Sigma M, \Sigma\mathbb{G}_m) = \mathrm{pic}_S(\Sigma M) = (\Sigma M)^\vee \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}_S},$$

which we will use freely. Moreover, for any $M \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}_S}$, the equivalence

$$\Lambda\left(M\right) \simeq \Omega\,\mathrm{map}_{\mathbb{Z}_S}(M, \Sigma\mathbb{G}_m) = \Omega M^\vee,$$

induces a natural comparison map

$$\Sigma\Lambda\left(M\right) \longrightarrow M^\vee.$$

**Lemma 3.3.10.** Let $T$ be a torus over $S$ and consider $T$ as an object of $\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}_S}$. The natural maps

$$\Sigma\Lambda\left(T\right) \longrightarrow T^\vee$$

$$(T^\vee)^\vee \longrightarrow T$$

$$((\Sigma T)^\vee)^\vee \simeq \Lambda\left(T\right)^\vee \longrightarrow \Sigma T,$$

are equivalences of étale connective $\mathbb{Z}_S$-modules.

*Proof.* Since both sides are étale sheaves, we can check this étale locally, so we can assume that $T \simeq \mathbb{G}_m^{\times n}$ is split. Moreover, since everything in sight commutes with products, we can take $n = 1$. The first claim is then that the map

$$\Sigma\mathbb{Z}_S \longrightarrow \mathrm{map}_{\mathbb{Z}_S}(\mathbb{G}_m, \Sigma\mathbb{G}_m) = \mathrm{pic}_S(\mathbb{G}_m)$$

is an equivalence, which follows since every line bundle over $\mathbb{G}_m$ is trivial. Using the first equivalence, the second map is identified with the composite

$$(\Sigma\mathbb{Z}_S)^\vee = \mathrm{map}_{\mathbb{Z}_S}(\Sigma\mathbb{Z}_S, \Sigma\mathbb{G}_m) \simeq \mathrm{map}_{\mathbb{Z}_S}(\mathbb{Z}_S, \mathbb{G}_m) \simeq \mathbb{G}_m,$$

where we have again used fully faithfulness of étale suspension. Finally, the third map simply becomes the canonical equivalence

$$\mathbb{Z}_S^\vee = \mathrm{map}_{\mathbb{Z}_S}(\mathbb{Z}_S, \Sigma\mathbb{G}_m) \simeq \Sigma\mathbb{G}_m$$

and so we are done. ∎

Recall that, since $\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}}$ is presentably symmetric monoidal, we have a group ring functor

$$\mathcal{O}_S^{\mathrm{nc}}[-]\colon \mathrm{Sp}^{\mathrm{cn}}_{\mathcal{S}_S^{\mathrm{big,ét}}} \longrightarrow \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}}),$$

which admits a parametrized right adjoint $(-)^{\times}$. Unwinding the definitions, we see that any $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}})$ has an underlying sheaf with values in $\mathrm{CAlg}(\mathrm{Cat}^{\mathrm{pf}})$ and the sheaf of units $\mathcal{C}^{\times}$ is sectionwise given by the Picard spectrum

$$\mathcal{C}^{\times}(A) \simeq \mathrm{pic}(\mathcal{C}(A)) \in \mathrm{Sp}^{\mathrm{cn}},$$

i.e. the grouplike $\mathbb{E}_\infty$-space of $\otimes$-invertible objects in $\mathcal{C}(A)$. For any $X \in \mathrm{Sch}_S$, equip the category $\mathcal{D}_{\mathrm{perf}}(X)$ with the standard $t$-structure, see also the discussion in Section 3.4.1, such that the heart $\mathcal{D}_{\mathrm{perf}}(X)^{\heartsuit}$ is an abelian 1-category, equipping $\mathrm{pic}(\mathcal{D}_{\mathrm{perf}}(X)^{\heartsuit})$ with a natural $\mathbb{Z}$-linear structure.

We write $\mathrm{pic}_S \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}_S}$ for the étale sheaf which takes $X \in \mathrm{Sch}_S$ to the $\mathbb{Z}$-linear connective spectrum $\mathrm{pic}(\mathcal{D}_{\mathrm{perf}}(X)^{\heartsuit})$. Observe that, for any any $A \in \mathrm{Sch}_S$ we have natural equivalences of abelian groups

$$\Omega\,\mathrm{pic}_S(A) \simeq (\Omega\,\mathcal{D}_{\mathrm{qc}}(A)^{\heartsuit})^{\times} \simeq (\mathcal{O}_A)^{\times} \simeq \mathbb{G}_m(A) \tag{3.3}$$

Moreover, we can actually de-loop this identification.

**Lemma 3.3.11.** The natural map $\Sigma\mathbb{G}_m \longrightarrow \mathrm{pic}_S$ induced by (3.3) is an equivalence of étale $\mathbb{Z}$-linear connective spectra.

*Proof.* It follows from the discussion above that the natural map $\Sigma\mathbb{G}_m \longrightarrow \Sigma\Omega\,\mathrm{pic}_S$ is an equivalence, so it suffices to argue that the counit $\Sigma\Omega\,\mathrm{pic}_S \to \mathrm{pic}_S$ is an equivalence. This is true because every line bundle is étale locally trivial, so that $\mathrm{pic}_S$ is étale locally connected. □

The observations above allows us define the desired orientation of the parametrized category $\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}}$. In particular, we get an associated Fourier transform as in Definition 3.3.3.

**Definition 3.3.12.** The *canonical orientation* of the $\mathcal{S}_S^{\mathrm{big,ét}}$-category $\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}}$ in the Fourier context $(\mathbb{Z}_S, \Sigma\mathbb{G}_m)$ is given by the natural map of étale connective spectra

$$\omega_{\mathrm{c}}\colon \Sigma\mathbb{G}_m \longrightarrow (\mathcal{O}_S^{\mathrm{nc}})^{\times}$$

which sectionwise on $X \in \mathrm{Sm}_S$ is induced by the identification $\Sigma\mathbb{G}_m \simeq \mathrm{pic}_S$ combined with the inclusion of the heart $\mathcal{D}_{\mathrm{perf}}(X)^{\heartsuit} \to \mathcal{D}_{\mathrm{perf}}(X)$. For any $M \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}_S}$, we call the associated Fourier transform

$$\mathfrak{F}^{M}_{\omega_{\mathrm{c}}}\colon \mathcal{O}_S^{\mathrm{nc}}[M] \longrightarrow (\mathcal{O}_S^{\mathrm{nc}})^{\Omega^\infty M^\vee} \in \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}})$$

the *categorical Fourier transform*.

Because the group ring construction $\mathcal{O}_S^{\mathrm{nc}}[-]$ is difficult to identify explicitly, the main purpose of the Fourier transform of Definition 3.3.12 is to help us construct the motivic Fourier transform in Section 3.3.3. The one crucial thing we verify however is that, for a torus $T$ it recovers the decomposition of the category of representations of $T$ into characters. This is the content of the following proposition.

**Proposition 3.3.13.** For any torus $T$ over $S$, the categorical Fourier transform

$$\mathfrak{F}^{\Lambda(T)}_{\omega_{\mathrm{c}}}\colon \mathcal{O}_S^{\mathrm{nc}}[\Lambda(T)] \longrightarrow (\mathcal{O}_S^{\mathrm{nc}})^{\mathrm{B}T} \in \mathrm{CAlg}(\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}})$$

is an equivalence of $\mathcal{S}_S^{\mathrm{big,ét}}$-parametrized categories.

*Proof.* First, note that by Lemma 3.3.10 we have $\Lambda(T)^\vee \simeq \mathrm{B}T$, so the target of the Fourier transform is indeed given by $(\mathcal{O}_S^{\mathrm{nc}})^{\mathrm{B}T}$

Since equivalences can be checked étale locally and both sides take products of étale connective spectra to tensor products, we may reduce to the case $T = \mathbb{G}_m$ and $\Lambda(T) = \mathbb{Z}$. It thus suffices to show that, for any representable $A \in \mathrm{Sch}_S$, the $\mathcal{D}_{\mathrm{perf}}(A)$-linear functor

$$\Gamma_A(\mathfrak{F}^{\mathbb{Z}}_{\omega_c})\colon \Gamma_A(\mathcal{O}_S^{\mathrm{nc}}[\mathbb{Z}]) \longrightarrow \Gamma_A(\mathcal{O}_S^{\mathrm{nc}})^{\mathrm{B}T} = \mathcal{D}_{\mathrm{perf}}(\mathrm{B}\mathbb{G}_{m,A})$$

is an equivalence. Since in $\mathrm{Cat}^{\mathrm{pf}}$ the map from the infinite coproduct to the infinite product is fully faithful and hence a monomorphism, the colimit over the discrete space $\mathbb{Z}$ in the sheaf category $\Gamma_A(\mathrm{Mod}_{\mathcal{O}_S^{\mathrm{nc}}})$ is computed sectionwise. Thus, the sections of $\mathcal{O}_S^{\mathrm{nc}}[\mathbb{Z}]$ are given by the coproduct

$$\Gamma_A(\mathcal{O}_S^{\mathrm{nc}}[\mathbb{Z}]) \simeq \mathcal{D}_{\mathrm{perf}}(A)[\mathbb{Z}] \simeq \coprod_{\mathbb{Z}} \mathcal{D}_{\mathrm{perf}}(A) \in \mathrm{Cat}^{\mathrm{pf}}.$$

Moreover, since the composite $\mathrm{Cat}^{\mathrm{pf}} \simeq \mathrm{Pr}^{\mathrm{L}}_\omega \to \mathrm{Pr}^{\mathrm{L}}$ preserves colimits, the symmetric monoidal category $\mathcal{D}_{\mathrm{perf}}(A)[\mathbb{Z}]$ naturally identifies with compact objects in the category of $\mathbb{Z}$-graded objects $\mathcal{D}_{\mathrm{qc}}(A)^{\mathrm{gr}}$ carrying the Day-convolution symmetric monoidal structure.

Unwinding the definitions, we learn that the Fourier transform $\Gamma_A(\mathfrak{F}^{\mathbb{Z}}_{\omega_c})$ is determined by the map of connective spectra

$$\mathbb{Z} \longrightarrow \mathcal{D}_{\mathrm{perf}}(\mathrm{B}\mathbb{G}_{m,A})^\times$$

picking out the canonical line bundle $\mathcal{O}(1)$. Thus, under the identification of the left hand side described above, the functor

$$\Gamma_A(\mathfrak{F}^{\mathbb{Z}}_{\omega_c})\colon\ \mathcal{D}_{\mathrm{qc}}(A)[\mathbb{Z}] \longrightarrow \mathcal{D}_{\mathrm{qc}}(\mathrm{B}\mathbb{G}_{m,A})$$

takes a graded complex of sheaves $(E_n)_{n\in\mathbb{Z}}$ to the complex $\bigoplus_{n\in\mathbb{Z}} E_n$ with the natural $\mathbb{G}_m$-action. This is well known to be an equivalence of categories, for example by [Mou21, Theorem 4.1], with inverse taking $F \in \mathcal{D}_{\mathrm{perf}}(\mathrm{B}\mathbb{G}_m)$ to the graded complex $(\Gamma(B\mathbb{G}_m, F \otimes \mathcal{O}(n)))_{n\in\mathbb{Z}}$ and so we conclude. □

### 3.3.3 Assembling the motivic Fourier transform

Fix $S$ to be an affine, regular base scheme. We now use the parametrized Fourier transform constructed in Definition 3.3.3 to build a Fourier transform in the category of $\mathbb{K}_S$-modules, which we call the *motivic Fourier transform*.

Let $\mathfrak{X} \in \mathrm{Stk}_S$ be a stack and consider the lax symmetric monoidal functor

$$K_{\mathfrak{X}}\colon\ \mathrm{Mod}_{\mathcal{O}^{\mathrm{nc}}}(\mathfrak{X}) \longrightarrow \mathrm{Shv}_{\text{ét}}(\mathrm{Stk}_{\mathfrak{X}}; \mathrm{Cat}^{\mathrm{pf}}) \longrightarrow \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}_{\mathfrak{X}}; \mathrm{Cat}^{\mathrm{pf}}) \xrightarrow{K} \mathrm{Fun}(\mathrm{Sm}_{\mathfrak{X}}^{\mathrm{op}}, \mathrm{Sp})$$

given by first forgetting to the underlying $\mathrm{Cat}^{\mathrm{pf}}$-valued sheaf, then restricting to smooth stacks over $\mathfrak{X}$, followed by applying $K$-theory sectionwise (which generally might not preserve the sheaf condition). For any $E \in \mathrm{SH}(\mathfrak{X})$ we write $u_{\mathfrak{X}}$ for the sheaf of spectra on $\mathrm{Sm}_{\mathfrak{X}}$ given by the assignment taking $Y \in \mathrm{Sm}_{\mathfrak{X}}$ to the mapping spectrum $\mathrm{map}(\mathbb{S}_{\mathfrak{X}}[Y], E)$. We begin by observing the following.

**Lemma 3.3.14.** For any regular stack $\mathfrak{X}$, there is a natural equivalence

$$K_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}}^{\mathrm{nc}}) \xrightarrow{\sim} u_{\mathfrak{X}}(\mathbb{K}_{\mathfrak{X}})$$

of presheaves of spectra on $\mathrm{Sm}_{\mathfrak{X}}$.

*Proof.* Let $p\colon Y \to \mathfrak{X} \in \mathrm{Sm}_{\mathfrak{X}}$ be smooth representable. By construction, $\mathbb{K}_{\mathfrak{X}}$ has underlying (pre)-sheaf given by

$$u_S(\mathbb{K}_{\mathfrak{X}})(Y) = L_{\mathbb{A}^1} K(\mathcal{D}_{\mathrm{perf}}(Y))\,.$$

Since $\mathfrak{X}$ is regular and $p$ is smooth, $Y$ is regular as well and hence we get that the natural map

$$L_{\mathbb{A}^1} K(\mathcal{D}_{\mathrm{qc}}(\mathfrak{X}_Y)) \xrightarrow{\sim} K(\mathcal{D}_{\mathrm{perf}}(\mathfrak{X}_Y)) = K_{\mathfrak{X}}(\mathcal{O}^{\mathrm{nc}}_{\mathfrak{X}})(Y)$$

is an equivalence, as claimed. □

In particular, we learn that we have $K_{\mathfrak{X}}(\mathcal{O}^{\mathrm{nc}}_{\mathfrak{X}}) \simeq \mathrm{KH}_{\mathfrak{X}}$, so that $K_{\mathfrak{X}}$ naturally lifts to a functor to modules over $\mathrm{KH}_{\mathfrak{X}}$ in presheaves of spectra. We denote by

$$\mathcal{K}_{\mathfrak{X}}\colon \mathrm{Mod}_{\mathcal{O}^{\mathrm{nc}}}(\mathfrak{X}) \longrightarrow \mathrm{Mod}_{\mathbb{K}_{\mathfrak{X}}}(\mathrm{SH}(\mathfrak{X}))$$

the functor given by the localization $L_\beta L_{\mathbb{A}^1} L_{\mathrm{ns}} K_{\mathfrak{X}}$, that is we apply $K$-theory sectionwise and then apply Nisnevich localization, $\mathbb{A}^1$-localization and Bott-periodization to the resulting presheaf. All the functors are compatible with representable $(-)^*$, so that $\mathcal{K}_{\mathfrak{X}}$ is as well. This allows us to define assembly and coassembly maps for representable morphisms. In fact we can use Proposition 3.2.11 to get a coassembly equivalence for all maps of stacks, not just representable ones.

**Lemma 3.3.15.** For any stack $f\colon \mathfrak{X} \to S$, we have a natural equivalence of $\mathbb{K}_S$-algebras

$$\mathcal{K}_S(f_*\mathcal{O}^{\mathrm{nc}}_{\mathfrak{X}}) \xrightarrow{\sim} f_*\mathbb{K}_{\mathfrak{X}}\,.$$

*Proof.* Unwinding the definition of $\mathcal{K}_S$, we see that the left hand side is the Bott-periodization of the motivic $S^1$-spectrum which takes a smooth representable $p\colon Y \to S$ to

$$L_{\mathbb{A}^1} L_{\mathrm{ns}} K(\mathcal{D}_{\mathrm{perf}}(Y \times_S \mathfrak{X})) = \mathrm{KH}_S(Y \times_S \mathfrak{X})\,,$$

which by Warning 3.2.10 is naturally equivalent to the underlying sheaf of spectra represented by $f_*\mathbb{K}_{\mathfrak{X}}$. In particular, it is already a $\mathbb{K}_S$-module and so the claim follows from the fully faithfulness of Proposition 3.2.11. □

The assembly is more mysterious, so we spell it out and upgrade it to a map of group algebras.

**Construction 3.3.16.** Let $f\colon X \to S$ be a smooth $S$-scheme. Applying the functor $\mathcal{K}_X$ to the unit of the $f_\sharp \dashv f^*$ adjunction

$$\epsilon\colon \mathcal{O}^{\mathrm{nc}}_X \longrightarrow f^* f_\sharp \mathcal{O}^{\mathrm{nc}}_X \in \mathrm{Mod}_{\mathcal{O}^{\mathrm{nc}}}(X).$$

and using that $\mathcal{K}_S$ commutes with $f^*$ induces a natural map of $\mathbb{K}_X$-modules

$$\mathcal{K}_X(\epsilon)\colon \mathcal{K}_X(\mathcal{O}^{\mathrm{nc}}_X) \longrightarrow f^*\mathcal{K}_S(f_\sharp\mathcal{O}^{\mathrm{nc}}_X) \in \mathrm{Mod}_{\mathbb{K}_X}(\mathrm{SH}(X))\,.$$

Using that $\mathcal{K}_X(\mathcal{O}^{\mathrm{nc}}_X) \simeq \mathbb{K}_X$ by Lemma 3.3.15 and passing to the adjoint yields a map

$$\mathfrak{a}_X\colon f_\sharp\mathbb{K}_X \longrightarrow \mathcal{K}_S(f_\sharp\mathcal{O}^{\mathrm{nc}}_X) \in \mathrm{Mod}_{\mathbb{K}_S}(\mathrm{SH}(S)).$$

By left Kan extending in the source, we obtain for every $A \in \mathcal{S}^{\mathrm{ns}}_S$ a natural map

$$\mathfrak{a}_A\colon \mathbb{K}_S[A] \longrightarrow \mathcal{K}_S(\mathcal{O}^{\mathrm{nc}}_S[v^*A])$$

which we call the *assembly map* of $A$. Here, $v^*$ denotes étale sheafification. By monoidality of the adjunctions, $\mathfrak{a}_{(-)}$ enhances to a lax symmetric monoidal transformation of lax symmetric monoidal functors and thus refines to a natural transformation

$$\mathfrak{a}_{(-)}\colon \mathbb{K}_S[-] \longrightarrow \mathcal{K}_S(\mathcal{O}^{\mathrm{nc}}_S[v^*(-)]$$

of functor $\mathcal{S}^{\mathrm{ns}}_S \otimes \mathrm{Sp}^{\mathrm{cn}} \to \mathrm{CAlg}_{\mathbb{K}_S}(\mathrm{SH}(S))$.

Since we have little control over the $(-)_\sharp$-functors in $\mathrm{Mod}_{\mathcal{O}^{\mathrm{nc}}}(S)$, it is difficult to say things about the assembly in general. However, we know what happens for finite étale maps, which is just enough for our purposes.

**Lemma 3.3.17.** For any finite étale scheme $f\colon X \to S$ the assembly map

$$\mathfrak{a}_X : f_\sharp \mathbb{K}_X \longrightarrow \mathcal{K}_S(f_\sharp \mathcal{O}_X^{\mathrm{nc}})$$

is an equivalence.

*Proof.* By the naturality of the norm construction, explained in [CLL25, Prop. 3.7], the diagram

$$\begin{array}{ccc} f_\sharp \mathbb{K}_X & \longrightarrow & \mathcal{K}_S(f_\sharp \mathcal{O}_X^{\mathrm{nc}}) \\ \downarrow & & \downarrow \\ f_* \mathbb{K}_X & \longleftarrow & \mathcal{K}_S(f_* \mathcal{O}_X^{\mathrm{nc}}) \end{array}$$

commutes, where the vertical maps are the norm maps and the vertical once are the (co-) assembly maps. The lower map is an equivalence by Lemma 3.3.15, the right hand map is an equivalence by Atiyah duality Theorem 3.2.7 and the right hand map is an equivalence by Lemma 3.3.6, so the claim follows. ☐

**Construction 3.3.18.** Let $M \in \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}_S}$ such that $\Omega^\infty M^\vee$ is representable by a stack. The *Motivic Fourier transform* associated to $M$ is the map of $\mathbb{K}_S$-algebras

$$\mathfrak{F}^M_{\mathrm{mot}} : \mathbb{K}_S[M] \longrightarrow \mathbb{K}_S^{\Omega^\infty M^\vee} \in \mathrm{CAlg}_{\mathbb{K}_S}(\mathrm{SH}(S)),$$

given by the composite map

$$\mathbb{K}_S[M] \xrightarrow{\mathfrak{a}_M} \mathcal{K}_S(\mathcal{O}_S^{\mathrm{nc}}[M]) \xrightarrow{\mathcal{K}_S(\mathfrak{F}^M_{\omega_c})} \mathcal{K}_S((\mathcal{O}_S^{\mathrm{nc}})^{\Omega^\infty M^\vee}) \simeq \mathbb{K}_S^{\Omega^\infty M^\vee}$$

of the assembly of $M \to S$, the image of the categorical Fourier transform of Definition 3.3.12 under the functor $\mathcal{K}_S$ and the equivalence from Lemma 3.3.15.

We are now ready to prove the first major ingredient for our main theorem, namely that the representation theoretic equivalence of Proposition 3.3.13 descends to algebraic $K$-theory. We will first start with the following lemma, giving a more explicit description of the character lattice.

**Construction 3.3.19.** Given a torus $T/S$, let $S' \to S$ be a finite étale extension that splits $T$ with Galois-group $G$ and write $\Lambda = \Lambda(T)_{S'}$ for the underlying set with $G$-action of $\Lambda(T)$. We have a natural $G$-equivariant decomposition of $\Lambda$ into the orbits i.e. an equivalence of $G$-sets

$$\Lambda \simeq \coprod_{[\alpha]\in\Lambda_G} \alpha G\,.$$

Since $\Lambda(T)$ is locally constant, it is left Kan extended from the small étale site and so by Lemma 3.2.13 we obtain an induced decomposition of étale sheaf of sets

$$\Lambda(T) \simeq \coprod_{[\alpha]\in\Lambda_G} X_\alpha \in \mathrm{Set}_S^{\text{ét}},$$

where $X_\alpha \to S$ is the finite étale cover corresponding to the finite, transitive $G$-set $\alpha G$.

With this presentation in hand, we are ready to use the machine of the previous section to show that, for a torus $T$, the motivic Fourier transform witnesses the character decomposition of the category $\mathcal{D}_{\mathrm{perf}}(\mathrm{B}T)$ on the level of motivic spectra.

**Proposition 3.3.20.** For any torus $T$ over $S$ the motivic Fourier transform

$$\mathfrak{F}_{\mathrm{mot}}^{\Lambda(T)} \colon \mathbb{K}_S[\Lambda(T)] \longrightarrow \mathbb{K}_S^{\mathrm{B}T} \in \mathrm{CAlg}_{\mathbb{K}}(\mathrm{SH}(S))$$

is an equivalence of commutative $\mathbb{K}_S$-algebras.

*Proof.* By construction, the map $\mathfrak{F}_{\mathrm{mot}}^{\Lambda(T)}$ is given by the composite

$$\mathbb{K}_S[\Lambda(T)] \xrightarrow{\mathfrak{a}_{\Lambda(T)}} \mathcal{K}_S(\mathcal{O}_S^{\mathrm{nc}}[\Lambda(T)]) \xrightarrow{\mathcal{K}_S(\mathfrak{F}_{\omega_c}^{\Lambda(T)})} \mathcal{K}_S((\mathcal{O}_S^{\mathrm{nc}})^{\mathrm{B}T}).$$

As the categorical Fourier transform $\mathfrak{F}_{\omega_c}^{\Lambda(T)}$ is an equivalence by Proposition 3.3.13, it is enough to show that the assembly map for $\Lambda(T)$ is an equivalence. Recall that the functor $\mathcal{K}_S$ is given by the composite $L_\beta L_{\mathbb{A}^1} L_{\mathrm{ns}} K_S$ where $K_S$ is the functor that applies $K$-theory section-wise. As $K$-theory is an additive invariant by [BGT13, Thm. 1.3] it commutes with infinite coproducts and so do the localization functors as they are left adjoints. Thus, $\mathcal{K}_S$ commutes with infinite coproducts as well. By Construction 3.3.19, we can write $\Lambda(T)$ as an infinite coproduct of finite étale extensions of $S$ and so the claim follows by naturality of the assembly combined with Lemma 3.3.17. □

## 3.4 Motivic Eilenberg–Moore formulas

Let $F$ be a field and write $S = \mathrm{Spec}(F)$ throughout this section.

We saw in Proposition 3.3.20 that for any torus $T/S$, the decomposition of a representation of $T$ into characters as in Proposition 3.3.13, induces an equivalence of $\mathbb{K}_S$-algebras

$$\mathfrak{F}_{\mathrm{mot}}^{\Lambda(T)} \colon \mathbb{K}_S[\Lambda(T)] \xrightarrow{\sim} \mathbb{K}_S^{\mathrm{B}T}.$$

By the computations of Lemma 3.3.10, we know that the motivic Fourier transform also gives a natural map

$$\mathfrak{F}_{\mathrm{mot}}^{\Sigma\Lambda(T)} \colon \mathbb{K}_S[\Sigma\Lambda(T)] \longrightarrow \mathbb{K}_S^{T}.$$

The main result of this section, Theorem 3.4.7, is that this map is in fact an equivalence. The proof proceeds along several steps.

In Section 3.4.1, we discuss how to obtain the second equivalence as a base change of the first one. The idea is to replace the presentation of $T$ via the bar construction, i.e. as the fiber of the canonical map $S \to \mathrm{B}T$, by a more delicate resolution. Namely, we write $T$ as a fiber of a map of deloopings of tori $\mathrm{B}A \to \mathrm{B}W$ where $W$ is Weil restricted from a split torus. This will imply our theorem once we prove an Eilenberg–Moore formula for the stacks involved, see Proposition 3.4.6, which occupies us for the remainder of this chapter.

The main important geometric ingredient is an observation which already appeared in the work of Merkurjev and Panin [MP97], namely that the Weil restricted torus $W$ admits a $W$-equivariant cell decomposition. We will formalize this, by showing that the $W$-equivariant motive of $W$ is *Artin–Tate* and hence *Artin* over $\mathbb{K}_{\mathrm{B}W}$. We will recall those two notions in Section 3.4.2, before turning to the concrete case of a torus in Section 3.4.3.

### 3.4.1 Gluing the motivic Fourier transform

As we will be juggling different (co)-fiber sequences of sheaves of connective spectra, let us recall some facts about the standard $t$-structure on sheaves of spectra. We refer the reader to [SAG, Sec. 1.3.1] as well as [HPT25] for details.

For $\sigma \in \{\text{ét}, \text{ns}\}$, we from now on denote by

$$\mathrm{Sp}_\sigma(S) := \mathrm{Shv}_\sigma(\mathrm{Sm}_S; \mathrm{Sp}), \quad \mathrm{Sp}_\sigma^{\mathrm{cn}}(S) := \mathrm{Shv}_\sigma(\mathrm{Sm}_S; \mathrm{Sp}^{\mathrm{cn}})$$

the category of $\sigma$-sheaves of (connective) spectra on $\mathrm{Sm}_S$. By [SAG, Prop. 1.3.2.7] the natural $t$-structure on Sp induces a $t$-structure on $\mathrm{Sp}_\sigma(S)$ called the *standard $t$-structure*. Explicitly, the fully faithful inclusion $i\colon \mathrm{Sp}^{\mathrm{cn}} \subseteq \mathrm{Sp}$ induces an adjunction

$$i^\sigma\colon \mathrm{Sp}_\sigma^{\mathrm{cn}}(S) \rightleftarrows \mathrm{Sp}_\sigma(S) : \tau_{\geq 0}^\sigma,$$

where $i^\sigma$ is again fully faithful and exhibits $\mathrm{Sp}_\sigma^{\mathrm{cn}}(S)$ as the full subcategory

$$\mathrm{Sp}_\sigma^{\mathrm{cn}}(S) \simeq (\mathrm{Sp}_\sigma(S))_{\geq 0} \subseteq \mathrm{Sp}_\sigma$$

of connective objects with respect to the standard $t$-structure. Here, $\tau_{\geq 0}^\sigma$ is computed pointwise by taking connective covers and $i^\sigma$ is computed by applying $i$ pointwise and then sheafifying.

Note that the sheafification built into the functor $i^\sigma$ means that, if $E \in \mathrm{Sp}_\sigma^{\mathrm{cn}}(S)$ is a sheaf of connective spectra, the sheaf $i^\sigma E$ can sectionwise have negative homotopy groups. Indeed, these are the (positive) sheaf cohomology groups, so for any $U \in \mathrm{Sm}_S$ and $E \in \mathrm{Sp}_\sigma^{\mathrm{cn}}(S)$ we write

$$H_\sigma^n(U; E) := \pi_{-n}\Gamma(U; i^\sigma E).$$

This subtlety disappears if we sheafify again. For each $n \in \mathbb{Z}$, we have a functor

$$\pi_n^\sigma\colon \mathrm{Sp}_\sigma(S) \longrightarrow \mathrm{Sp}_\sigma^\heartsuit(S) \simeq \mathrm{Shv}_\sigma(\mathrm{Sm}_S; \mathrm{Ab}),$$

which, for $E \in \mathrm{Sp}_\sigma(S)$, is computed as the sheafification of the presheaf $U \mapsto \pi_n\Gamma(U, E)$. Moreover, $E \in \mathrm{Sp}_\sigma(S)$ is contained in the full subcategory $\mathrm{Sp}_\sigma(S)_{\geq 0}$, i.e. is given by the sheafification of a sheaf of connective spectra, if and only if $\pi_n^\sigma E \simeq 0$ for all $n < 0$.

Before we begin properly, we make a remark about Weil restriction of commutative group schemes.

**Remark 3.4.1.** Let $p\colon S' \to S$ be a finite étale map. The pullback functor

$$p^*\colon \mathrm{Sp}_{\text{ét}}^{\mathrm{cn}}(S) \longrightarrow \mathrm{Sp}_{\text{ét}}^{\mathrm{cn}}(S')$$

admits both adjoints $p_\sharp \dashv p^* \dashv p_*$ which are naturally equivalent by Lemma 3.3.6. In particular, on representable objects, namely commutative group schemes, we have a left adjoint functor $p_\sharp = \mathrm{R}_{S'/S}(-)$ which is underlying given by *Weil restriction* of schemes.

**Construction 3.4.2.** Let $S$ be a scheme and $T$ be a torus over $S$. Given a finite étale map $p\colon S' \to S$ which splits $T$, let $p_*p^*T = W$ denote the Weil restriction of $T_{S'}$ along $p$ and denote by $A \to W$ the kernel of the counit map $W \to T$. Then $A$ is again a torus, and we obtain a short exact sequence

$$0 \longrightarrow A \longrightarrow W \longrightarrow T \longrightarrow 0 \tag{3.4}$$

of algebraic tori over $F$, which we call the *Weil resolution* of $T$ associated to the cover $p\colon S' \to S$. We sometimes leave the choice of a splitting cover implicit and simply refer to a Weil resolution of $T$.

Given a Weil-resolution as in (3.4) upon taking character lattices, see Remark 3.3.9, and restricting to the smooth locus, we get a sequence of étale sheaves of abelian groups on $\mathrm{Sch}_S$ of the form

$$0 \to \Lambda(T) \to \Lambda(W) \to \Lambda(A) \to 0\,, \tag{3.5}$$

which is again exact, as can be seen by passing to the splitting extension $S'$. Since étale covers of smooth schemes are smooth, we may argue as in the proof of Lemma 3.2.13 to see that the functor

$$\mathrm{Shv}_{\text{ét}}(\mathrm{Sch}_S; \mathrm{Sp}^{\mathrm{cn}}) \longrightarrow \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}_S; \mathrm{Sp}^{\mathrm{cn}}) = \mathrm{Sp}^{\mathrm{cn}}_{\text{ét}}(S)$$

induced by restricting along the inclusion $\mathrm{Sm}_S \subseteq \mathrm{Sch}_S$ preserves colimits. Combining this with the fact that the inclusion of the heart takes short exact sequences to cofiber sequences, we learn that we can rotate the sequence (3.5) once to obtain a cofiber sequence in $\mathrm{Sp}^{\mathrm{cn}}_{\text{ét}}(S)$ of the form

$$\Lambda(W) \longrightarrow \Lambda(A) \longrightarrow \Sigma\Lambda(T)\,. \tag{3.6}$$

Our first goal will be showing that this remains a cofiber sequence after forgetting to Nisnevich sheaves of connective spectra.

The inclusion $v\colon \mathrm{Shv}_{\text{ét}}(\mathrm{Sm}_S; \mathcal{S}) \to \mathrm{Shv}_{\mathrm{ns}}(\mathrm{Sm}_S; \mathcal{S})$ induces a commutative diagram of right adjoints

$$\begin{tikzcd}
\mathrm{Sp}_{\mathrm{ns}}(S) \arrow[d, "\tau^{\mathrm{ns}}_{\geq 0}"'] & \mathrm{Sp}_{\text{ét}}(S) \arrow[l, "V_*"'] \arrow[d, "\tau^{\text{ét}}_{\geq 0}"] \\
\mathrm{Sp}^{\mathrm{cn}}_{\mathrm{ns}}(S) & \mathrm{Sp}^{\mathrm{cn}}_{\text{ét}}(S)\,. \arrow[l, "v_*"]
\end{tikzcd}$$

which, upon passing to left adjoints gives a commutative diagram

$$\begin{tikzcd}
\mathrm{Sp}_{\mathrm{ns}}(S) \arrow[r, "V^*"] & \mathrm{Sp}_{\text{ét}}(S) \\
\mathrm{Sp}^{\mathrm{cn}}_{\mathrm{ns}}(S) \arrow[u, "i^{\mathrm{ns}}"] \arrow[r, "v^*"'] & \mathrm{Sp}^{\mathrm{cn}}_{\text{ét}}(S)\,, \arrow[u, "i^{\text{ét}}"']
\end{tikzcd}$$

Note that, while the functor $V_*$ preserves fiber and cofiber sequences, in general it does not preserve the full subcategory of connective objects.

**Lemma 3.4.3.** Let $W = q_*W'$ be the Weil restriction of a split torus $W'$ along a finite étale map $q\colon S' \to S$. Then, we have an equivalence of sheaves

$$\pi^{\mathrm{ns}}_{-1} V_*\Lambda(W) \simeq 0 \in \mathrm{Sp}^{\heartsuit}_{\mathrm{ns}}(S)\,.$$

*Proof.* By definition, $\pi^{\mathrm{ns}}_{-1}(V_*\Lambda(W))$ is the Nisnevich sheafification of the presheaf

$$\mathrm{Sm}^{\mathrm{op}}_S \longrightarrow \mathrm{Ab}, \quad U \mapsto H^1_{\text{ét}}(U; \Lambda(W))\,.$$

We claim that this presheaf is already the zero sheaf. Indeed, we have natural equivalences

$$\Lambda(W) \simeq \mathrm{map}_{\mathbb{Z}_S}(W, \mathbb{G}_m) \simeq \mathrm{map}_{\mathbb{Z}_S}(q_*W', \mathbb{G}_m) \simeq q_*\,\mathrm{map}_{\mathbb{Z}_{S'}}(W', q^*\mathbb{G}_m) \simeq q_*\Lambda(W')\,.$$

As a result, for any $U \in \mathrm{Sm}_S$ we get that

$$H^1_{\text{ét}}(U; \Lambda(W)) \simeq H^1_{\text{ét}}(U_{S'}; \Lambda(W'))\,,$$

where $\Lambda(W') \simeq \mathbb{Z}^d_{S'}$ is a constant sheaf. The right hand side is thus computed by the set of continuous group homomorphisms

$$H^1_{\text{ét}}(U_{S'}; \Lambda(W')) \simeq \mathrm{Hom}_{\mathrm{cont}}(\pi^{\text{ét}}_1 U', \mathbb{Z}^d),$$

where $\mathbb{Z}^d$ carries the discrete topology. Since $S = \mathrm{Spec}(F)$ is a field it is in particular geometrically unibranch. Since $U$ is smooth over $S$ and $S'$ is finite étale over $S$, $U'$ is also geometrically unibranch and thus $\pi_1^{\text{ét}} U'$ is a pro-finite group. In particular there are no non-trivial continuous group homomorphisms $\pi_1^{\text{ét}} U' \to \mathbb{Z}^d$ and so $\pi_{-1}^{\text{ns}} V_* \Lambda(W)$ is the sheafification of the zero presheaf, hence itself zero. $\square$

From this "acyclicity" lemma about Weil restricted tori we learn the following.

**Proposition 3.4.4.** Let $S$ be geometrically unibranch and $T$ a torus over $S$. Given any finite étale extension $S' \to S$ that splits $T$, the sequence of connective Nisnevich spectra

$$v_* \Lambda(W) \longrightarrow v_* \Lambda(A) \longrightarrow v_* \Sigma\Lambda(T)$$

obtained by applying the forgetful functor $v_* \colon \mathrm{Sp}^{\mathrm{cn}}_{\text{ét}}(S) \to \mathrm{Sp}^{\mathrm{cn}}_{\mathrm{ns}}(S)$ to the rotated Weil resolution (3.6), is a cofiber sequence in $\mathrm{Sp}^{\mathrm{cn}}_{\mathrm{ns}}(S)$.

*Proof.* Since the functor $v_*$ is a right adjoint and hence preserves limits, the sequence in the statement is a *fiber* sequence. To show that it is a *cofiber* sequence of sheaves of connective spectra, it suffices to show that boundary map

$$\pi_0^{\mathrm{ns}} V_* \Sigma\Lambda(T) \longrightarrow \pi_{-1}^{\mathrm{ns}} V_* \Lambda(W) \in \mathrm{Sp}^{\heartsuit}_{\mathrm{ns}}(S)$$

of the non-connective restrictions is the zero map. However, this is clear, since the target is 0 by Lemma 3.4.3. $\square$

In summary, we obtain a pushout square of motivic group rings.

**Corollary 3.4.5.** Let $S$ be geometrically unibranch and $T/S$ be a torus. Then for any Weil resolution of $T$, the induced cofiber sequence (3.6) induces a pushout square of motivic group rings

$$\begin{tikzcd} \mathbb{K}_S[\Lambda(W)] \arrow[r] \arrow[d] & \mathbb{K}_S \arrow[d] \\ \mathbb{K}_S[\Lambda(A)] \arrow[r] & \mathbb{K}_S[\Sigma\Lambda(T)]. \end{tikzcd}$$

in the category of commutative $\mathbb{K}_S$-algebras $\mathrm{CAlg}_{\mathbb{K}_S}(\mathrm{SH}(S))$.

*Proof.* The group ring functor $\mathbb{S}[-] \colon \mathrm{Sp}^{\mathrm{cn}}_{\mathrm{ns}}(S) \to \mathrm{CAlg}(\mathrm{SH}(S))$ is colimit preserving by construction and so is the base change to $\mathbb{K}_S$, so the claim follows from Proposition 3.4.4. $\square$

This describes the *source* of the motivic Fourier transform $\mathfrak{F}_{\mathrm{mot}}^{\Sigma\Lambda(T)}$ as a pushout of objects where we know $\mathfrak{F}_{\mathrm{mot}}$ to be an equivalence by Proposition 3.3.20. It remains to analyze the target.

Recall from Definition 3.3.8 that we defined the Cartier dual functor $(-)^\vee$ as the internal mapping object $\mathrm{map}_{\mathbb{Z}_S}(-, \Sigma\mathbb{G}_m)$. In particular $(-)^\vee$ takes cofiber sequences to fiber sequences. Thus, applying $(-)^\vee$ to (3.6) and using the identifications of Lemma 3.3.10, we get a pullback diagram on underlying stacks of the form

$$\begin{tikzcd} T \arrow[r] \arrow[d] & S \arrow[d] \\ \mathrm{B}A \arrow[r] & \mathrm{B}W\,. \end{tikzcd} \tag{3.7}$$

We claim that this pullback square induces a pushout in $\mathbb{K}_S$-cohomology.

**Proposition 3.4.6** (Eilenberg–Moore)**.** Let $S = \mathrm{Spec}(F)$ be a field and $T$ be a torus over $S$. Given any Weil resolution

$$0 \to A \to W \to T \to 0$$

of $T$, the associated pullback square (3.7) induces a pushout diagram

$$\begin{tikzcd}
\mathbb{K}_S^{\mathrm{B}W} \arrow[r] \arrow[d] & \mathbb{K}_S \arrow[d] \\
\mathbb{K}_S^{\mathrm{B}A} \arrow[r] & \mathbb{K}_S^{T}.
\end{tikzcd}$$

in the category of commutative $\mathbb{K}_S$-algebras $\mathrm{CAlg}_{\mathbb{K}_S}(\mathrm{SH}(S))$.

The proof of Proposition 3.4.6 will be the topic of Section 3.4.2. Before we delve into this, we observe that this immediately results in our main theorem.

**Theorem 3.4.7.** Let $F$ be a field and write $S = \mathrm{Spec}(F)$. For any torus $T$ with character lattice $\Lambda(T)$, the motivic Fourier transform of Construction 3.3.18 gives a natural equivalence

$$\mathfrak{F}_{\mathrm{mot}}^{\Sigma\Lambda(T)} \colon \mathbb{K}_S[\Sigma\Lambda(T)] \xrightarrow{\sim} \mathbb{K}_S^{T},$$

of commutative $\mathbb{K}_S$-algebras in $\mathrm{SH}(S)$.

*Proof.* First note that by Lemma 3.3.10 we have $(\Sigma\Lambda(T))^\vee \simeq T$, so the motivic Fourier transform for $\Sigma\Lambda(T)$ is defined and has target $\mathbb{K}_S^T$. For any choice of splitting field $E/F$, the associated Weil resolution combined with the naturality of the Fourier transform provides us with a commutative diagram in the category $\mathrm{CAlg}_{\mathbb{K}}(\mathrm{SH}(S))$ of the form:

$$\begin{tikzcd}
 & & \mathbb{K}_S^{\mathrm{B}W} \arrow[r] \arrow[d] & \mathbb{K}_S \arrow[d] \\
\mathbb{K}_S[\Lambda(W)] \arrow[urr, "\mathfrak{F}_{\mathrm{mot}}^{\Lambda(W)}"] \arrow[r] \arrow[d] & \mathbb{K}_S \arrow[urr, "\mathrm{id}"] \arrow[d] & \mathbb{K}_S^{\mathrm{B}A} \arrow[r] & \mathbb{K}_S^{T} \\
\mathbb{K}_S[\Lambda(A)] \arrow[urr, "\mathfrak{F}_{\mathrm{mot}}^{\Lambda(A)}"] \arrow[r] & \mathbb{K}_S[\Sigma\Lambda(T)] \arrow[urr, "\mathfrak{F}_{\mathrm{mot}}^{\Sigma\Lambda(T)}"'] & &
\end{tikzcd}$$

Now Corollary 3.4.5 and Proposition 3.4.6 tell us that the front and back planar squares are pushouts, respectively. Moreover, by Proposition 3.3.20, we know that the Fourier transforms $\mathfrak{F}_{\mathrm{mot}}^{\Lambda(W)}$ and $\mathfrak{F}_{\mathrm{mot}}^{\Lambda(A)}$ are equivalences and so the lower right map

$$\mathfrak{F}_{\mathrm{mot}}^{\Sigma\Lambda(T)} \colon \mathbb{K}_S[\Sigma\Lambda(T)] \longrightarrow \mathbb{K}_S^{T}$$

is an equivalence as well, as claimed. □

Note that, for a split torus, this recovers the highly structured version of Quillen's computation [Qui73, Theorem 8] of the algebraic $K$-groups of the scheme $\mathbb{G}_m$: Let $F$ be a field and $t \in \mathcal{O}(\mathbb{G}_m)$ be a choice of coordinate Considering $t$ as an automorphism of the trivial line bundle on $\mathbb{G}_m$, we learn that the associated map

$$t \colon S^1 \longrightarrow \mathrm{QCoh}(\mathbb{G}_m)$$

induces an equivalence of commutative $K(F)$-algebras

$$K(F)[S^1] \xrightarrow{\sim} K(\mathbb{G}_m)\,.$$

In Section 3.5, we leverage Theorem 3.4.7 computationally by transferring our result into the world of equivariant homotopy theory and obtain a spectral sequence computing $K(T)$. Before we can reap those rewards, we must prove Proposition 3.4.6. To this end, we need to introduce some further machinery.

### 3.4.2 Artin–Tate motives

The crucial technical input for Proposition 3.4.6 will be that, for a Weil restricted torus $W$, the zero section $S \to \mathrm{B}W$ satisfies a strong cellularity condition. Namely, it is *Artin–Tate* in the sense of [BHS22, Definition 1.5]. We review the notions of Artin(–Tate) motives[5] and discuss in Corollary 3.4.19 how, for an stack $p\colon \mathfrak{X} \to S$ such that $p$ induces an equivalence on $\pi_1^{\text{ét}}$, any pullback diagram of smooth representable stacks over $\mathfrak{X}$ gives an Eilenberg–Moore formula for $\mathbb{K}^{\mathfrak{X}} \in \mathrm{SH}(S)$, if one of the legs is Artin–Tate.

Let $\mathfrak{X}$ be a stack and $R \in \mathrm{CAlg}(\mathrm{SH}(\mathfrak{X}))$ be a motivic commutative ring spectrum. A *virtual vector bundle* on $\mathfrak{X}$ is a class $\mathcal{E} \in \pi_0 K(\mathfrak{X})$. Recall that we denote by

$$R\langle - \rangle\colon \pi_0 K(\mathfrak{X}) \longrightarrow \mathrm{Pic}(\mathrm{Mod}_R(\mathrm{SH}(\mathfrak{X})))$$

the motivic Thom-construction.

**Definition 3.4.8.** Let $R \in \mathrm{CAlg}(\mathrm{SH}(\mathfrak{X}))$ a motivic commutative ring spectrum over $\mathfrak{X}$. We say that an object of $\mathrm{Mod}_R(\mathrm{SH}(\mathfrak{X}))$ is:

1. *Artin–Tate* if it contained in the thick subcategory generated by the objects of the form

   $$\psi_\sharp((\psi^* R)\langle \mathcal{E} \rangle),$$

   where $\psi\colon \mathfrak{Y} \to \mathfrak{X}$ is finite étale and $\mathcal{E}$ is a virtual vector bundle on $\mathfrak{Y}$.

2. *Artin* if it is contained in the thick subcategory generated by objects of the form

   $$\psi_\sharp \psi^* R,$$

   where $\psi\colon \mathfrak{Y} \to \mathfrak{X}$ is finite étale.

Moreover, we say that a smooth representable stack $\varphi\colon \mathfrak{Y} \to \mathfrak{X}$ is *Artin–Tate over $R$* (or *Artin over $R$*, respectively) if $\varphi_\sharp \varphi^* R \in \mathrm{Mod}_R(\mathrm{SH}(\mathfrak{X}))$ is Artin–Tate (or Artin, respectively). If $R$ is the unit, we omit the phrase "over $R$".

**Remark 3.4.9.** Clearly, every Artin motive is Artin–Tate. Moreover, recall that, for a finite étale map $\psi$, we have a natural equivalence $\psi_\sharp \simeq \psi_*$, hence we can freely replace $\psi_\sharp$ by $\psi_*$ when working with Definition 3.4.8.

Our goal is to show that if $W$ is the Weil restriction of a split torus along a finite étale map, then the $W$-equivariant motive of $W$ itself, i.e. the smooth map of stacks $* \longrightarrow \mathrm{B}W$ is Artin–Tate and hence Artin over $\mathbb{K}_{\mathrm{B}W}$ by Lemma 3.4.12. To do this, let us first gather some general facts about Artin and Artin–Tate motives.

**Lemma 3.4.10.** All Artin–Tate motives are dualizable.

*Proof.* Note that Thom spectra are invertible. Since dualizable objects form a thick subcategory closed under tensor products, it suffices to show that, for a finite étale map $\psi$, the functor $\psi_\sharp$ preserves dualizable objects, which is immediate from Atiyah duality Theorem 3.2.7. □

**Lemma 3.4.11.** Let $f\colon R \to S$ be a morphism in $\mathrm{CAlg}(\mathrm{SH}(\mathfrak{X}))$. Then the base change along $f$ preserves the class of Artin–Tate and Artin objects.

*Proof.* Let $\phi\colon \mathfrak{Y} \to \mathfrak{X}$ be finite étale and $\mathcal{E}$ be a virtual vector bundle on $\mathfrak{Y}$. Then, by the projection formula for the adjunction $\phi_\sharp \dashv \phi^*$ of Theorem 3.2.4, we obtain an equivalence in $\mathrm{Mod}_S(\mathrm{SH}(\mathfrak{X}))$ of the form

$$S \otimes_R \phi_\sharp \phi^* R\langle \mathcal{E} \rangle \simeq \phi_\sharp(\phi^* S \otimes_{\phi^* R} \phi^* R\langle \mathcal{E} \rangle) \simeq \phi_\sharp \phi^* S\langle \mathcal{E} \rangle$$

and so $S \otimes_R \phi_\sharp \phi^* R\langle \mathcal{E} \rangle$ is Artin–Tate. Thus, the claim follows, as base change is exact and the respective subcategories are thick. □

[5] Our definition is a small category version of the definition in [BHS22].

**Lemma 3.4.12.** Over the K-theory spectrum $\mathbb{K}$, a motive is Artin if and only if it is Artin–Tate. In particular, if $\mathfrak{Y} \in \mathrm{Sm}_{\mathfrak{X}}$ is Artin–Tate, then it is Artin over $\mathbb{K}_{\mathfrak{X}} \in \mathrm{SH}(\mathfrak{X})$.

*Proof.* By [KR24, Thm. 10.7(ii)] we have, for any virtual vector bundle $\mathcal{E}$ on $\mathfrak{X}$ a canonical Bott-periodicity isomorphism

$$\mathbb{K}_{\mathfrak{X}}\langle \mathcal{E} \rangle \simeq \mathbb{K}_{\mathfrak{X}} \in \mathrm{SH}(\mathfrak{X}),$$

which implies the first statement and the second is immediate by Lemma 3.4.11. □

**Lemma 3.4.13** (Pullback)**.** Let $\phi\colon \mathfrak{X}' \to \mathfrak{X}$ be a morphism of stacks. Then $\phi^*\colon \mathrm{SH}(\mathfrak{X}) \to \mathrm{SH}(\mathfrak{X}')$ preserves the subcategory of Artin–Tate objects. In particular, if $\mathfrak{Z} \in \mathrm{Sm}_{\mathfrak{X}}$ is Artin–Tate, then the pullback $\phi^*\mathfrak{Z} \in \mathrm{Sm}_{\mathfrak{X}'}$ is Artin–Tate.

*Proof.* Let $\psi\colon \mathfrak{Y} \to \mathfrak{X}$ be a finite étale morphism $\mathcal{E}$ be a virtual vector bundle on $\mathfrak{Y}$ and consider the pullback diagram

$$\begin{tikzcd} \mathfrak{Y}' \arrow[r, "\phi'"] \arrow[d, "\psi'"] & \mathfrak{Y} \arrow[d, "\psi"] \\ \mathfrak{X}' \arrow[r, "\phi"] & \mathfrak{X}. \end{tikzcd}$$

Then, by $\sharp$-base change (Theorem 3.2.4), we have equivalences

$$\phi^*\psi_\sharp(\mathbb{S}_{\mathfrak{Y}}\langle \mathcal{E} \rangle) \simeq \psi'_\sharp\phi'^*(\mathbb{S}_{\mathfrak{Y}}\langle \mathcal{E} \rangle) \simeq \psi'_\sharp(\mathbb{S}_{\mathfrak{Y}'}\langle \phi'^*\mathcal{E} \rangle)$$

Since the right hand term is Artin–Tate by definition, the claim follows by virtue of $\phi^*$ being exact and the subcategory of Artin–Tate motives being thick. □

**Lemma 3.4.14** (Pushforward)**.** Let $\phi\colon \mathfrak{X}' \to \mathfrak{X}$ be a finite étale morphism of stacks. Then $\phi_\sharp\colon \mathrm{SH}(\mathfrak{X}') \to \mathrm{SH}(\mathfrak{X})$ preserves the subcategory of Artin–Tate objects.

*Proof.* This follows immediately from the definitions, using that finite étale morphisms are closed under composition. □

Unfortunately, being Artin–Tate is not closed under gluing of smooth $\mathfrak{X}$-stacks, but we have the following partial result which will suffice for our needs.

**Lemma 3.4.15** (Dévissage)**.** Suppose we have a diagram of smooth stacks

$$\begin{tikzcd} \mathfrak{Z} \arrow[r, "i"] \arrow[d, "\alpha"'] & \mathfrak{X} \arrow[d, "\phi"] & \mathfrak{U} \arrow[l, "j"'] \\ \mathfrak{Y}' \arrow[r, "\beta"] & \mathfrak{Y} & \end{tikzcd}$$

in which $\phi$ and $\alpha$ are smooth, $\beta$ is finite étale and $i$ is a closed immersion with complement $j$. Assume that the class of the normal bundle $[\mathcal{N}_{\mathfrak{Z}/\mathfrak{X}}]$ is in the image of the map $\alpha^*\colon K_0(\mathfrak{Y}') \to K_0(\mathfrak{Z})$. Moreover, assume that $\alpha$ is Artin–Tate. Then, $\phi \circ j$ is Artin–Tate if and only if $\phi$ is Artin–Tate.

*Proof.* By [KR24, Remark 5.6] we have a cofiber sequence

$$j_\sharp\mathbb{S}_{\mathfrak{U}} \longrightarrow \mathbb{S}_{\mathfrak{X}} \longrightarrow i_*\mathbb{S}_{\mathfrak{Z}}.$$

Applying $\phi_\sharp$ we thus get a cofiber sequence

$$(\phi\circ j)_\sharp\mathbb{S}_{\mathfrak{U}} \longrightarrow \phi_\sharp\mathbb{S}_{\mathfrak{X}} \longrightarrow \phi_\sharp i_*\mathbb{S}_{\mathfrak{Z}}.$$

As the Artin–Tate subcategory is closed under extensions and fibers, it is enough to show that the right hand term $\phi_\sharp i_*\mathbb{S}_{\mathfrak{Z}}$ is also Artin–Tate. Let $\mathcal{E}\in K(\mathfrak{Y}')$ be a preimage of $\mathcal{N}_{\mathfrak{X}/\mathfrak{Z}}$. By [KR24, Theorem 6.6], we have

$$\phi_\sharp i_*\mathbb{S}_{\mathfrak{Z}} \simeq (\phi\circ i)_\sharp(\mathbb{S}_{\mathfrak{Z}}\langle\mathcal{N}_{\mathfrak{X}/\mathfrak{Z}}\rangle^\vee) \simeq \beta_\sharp\alpha_\sharp(\mathbb{S}_{\mathfrak{Z}}\langle\mathcal{N}^\vee_{\mathfrak{X}/\mathfrak{Z}}\rangle) \simeq \beta_\sharp((\alpha_\sharp\mathbb{S}_{\mathfrak{Z}})\langle\mathcal{E}^\vee\rangle).$$

Since $\alpha_\sharp\mathbb{S}_{\mathfrak{Z}}\in \mathrm{SH}(\mathfrak{Y}')$ is Artin–Tate, the Thom-spectrum $\alpha_\sharp\mathbb{S}_{\mathfrak{Z}}\langle\mathcal{E}^\vee\rangle$ is as well. By Lemma 3.4.14 and the assumption that $\beta$ is finite étale, we deduce that $\beta_\sharp(\alpha_\sharp\mathbb{S}_{\mathfrak{Z}}\langle\mathcal{E}^\vee\rangle)\in\mathrm{SH}(\mathfrak{Y})$ is Artin–Tate and the result follows. □

### Eilenberg–Moore and Künneth

We now deduce formulas for the cohomology of pullback square of stacks where one leg is Artin–Tate.

**Proposition 3.4.16** (Künneth)**.** Let $\mathfrak{X}$ be a stack, $R\in\mathrm{CAlg}(\mathrm{SH}(\mathfrak{X}))$ and suppose we are given two smooth, representable maps $s\colon\mathfrak{A}\to\mathfrak{X}$ and $i\colon\mathfrak{B}\to\mathfrak{X}$. Denote by $\delta\colon\mathfrak{Z}\to\mathfrak{X}$ their fiber product in $\mathrm{Sm}_{\mathfrak{X}}$. The Künneth map

$$\kappa_*\colon s_*s^*R\otimes_R i_*i^*R \longrightarrow \delta_*\delta^*R,$$

is an equivalence if $s$ is Artin–Tate.

*Proof.* Recall from the discussion of Theorem 3.2.4 that, for any smooth representable map of stacks $f$, the functors $f_\sharp$ and $f_*$ are weakly dual to each other, that is we have $(f_\sharp f^*)^\vee\simeq f_*f^*$. Under this identification, we may factor the map $\kappa_*$ as the composite

$$s_*s^*R\otimes_R i_*i^*R\simeq (s_\sharp s^*R)^\vee\otimes_R(i_\sharp i^*R)^\vee\to(s_\sharp s^*R\otimes_R i_\sharp i^*R)^\vee\xrightarrow{\kappa_\sharp^\vee}(\delta_\sharp\delta^*R)^\vee\simeq\delta_*\delta^*R$$

where $\kappa_\sharp$ is the natural map

$$\kappa_\sharp\colon\delta_\sharp\delta^*R\xrightarrow{\sim}s_\sharp s^*R\otimes_R i_\sharp i^*R,$$

which is an equivalence by the definition of the monoidal structure on $\mathrm{Mod}_R(\mathrm{SH}(\mathfrak{X}))$. Thus, $\kappa_*$ is an equivalence if and only if the map

$$(s_\sharp s^*R)^\vee\otimes_R(i_\sharp i^*R)^\vee\to(s_\sharp s^*R\otimes_R i_\sharp i^*R)^\vee$$

is an equivalence. This holds because $s_\sharp s^*R$ is dualizable as an $R$-module by Lemma 3.4.10. □

**Construction 3.4.17.** Let $p\colon\mathfrak{X}\to S$ be a stack and $R\in\mathrm{CAlg}(\mathrm{SH}(\mathfrak{X}))$. For modules $M,N\in\mathrm{Mod}_R(\mathrm{SH}(\mathfrak{X}))$, the lax monoidality of $p_*$ provides us with a comparison map of the relative tensor products

$$\tau_{M,N}: p_*M\otimes_{p_*R}p_*N\longrightarrow p_*\left(M\otimes_R N\right)\in\mathrm{Mod}_{p_*R}(\mathrm{SH}(S))\,. \tag{3.8}$$

For fixed $N$, we denote by

$$\mathcal{C}(N,p)\subseteq\mathrm{Mod}_R(\mathrm{SH}(\mathfrak{X}))$$

the full subcategory spanned by those modules $M$ for which $\tau_{M,N}$ is an equivalence.

Note that, clearly we have $R\in\mathcal{C}(N,p)$. Moreover, the category $\mathcal{C}(N,p)$ is thick, as $p_*$ is exact. Moreover, the following proposition shows that transfers of $R$ along proper maps pulled back from the base, also lie in $\mathcal{C}(N,p)$.

**Proposition 3.4.18.** Let $R \in \mathrm{CAlg}(\mathrm{SH}(\mathfrak{X}))$ and $N \in \mathrm{Mod}_R(\mathrm{SH}(\mathfrak{X}))$. For a smooth and proper map of schemes $f\colon S' \to S$, consider the pullback diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "q"] \arrow[d, "g"] & S' \arrow[d, "f"] \\ \mathfrak{X} \arrow[r, "p"] & S. \end{tikzcd}$$

Suppose we have $M \in \mathrm{Mod}_{g^*R}(\mathrm{SH}(\mathfrak{X}'))$, such that $M \in \mathcal{C}(g^*N, q)$, then also $g_*M \in \mathcal{C}(N, p)$. In particular, we have that $g_*g^*M \in \mathcal{C}(N, p)$.

*Proof.* By assumption, we know that the natural map

$$f_*\left(q_*M \otimes_{q_*g^*R} q_*g^*N\right) \xrightarrow{\sim} f_*q_*\left(M \otimes_{g^*R} g^*N\right) \tag{3.9}$$

is an equivalence. Using the smooth base change formula $q_*g^* \simeq f^*p_*$ and then the proper projection formula for $f^* \dashv f_*$, we obtain a natural equivalence between the left hand side and

$$f_*q_*M \otimes_{p_*R} p_*N \simeq p_*g_*M \otimes_{p_*R} p_*N\,.$$

Moreover, using that $f_*q_* \simeq p_*g_*$ and applying the proper projection formula for $g^* \dashv g_*$, we see that the right hand side is naturally equivalent to

$$p_*\left(g_*M \otimes_R N\right)\,.$$

Under these identifications (3.9) thus becomes an equivalence

$$p_*g_*M \otimes_{p_*R} p_*N \xrightarrow{\sim} p_*\left(g_*M \otimes_R N\right),$$

and we leave it to the reader to verify that this is indeed the natural map $\tau_{g_*M,N}$. □

We say that a stack $p\colon \mathfrak{X} \to S$ is *étale simply connected* if the map $p$ induces an equivalence on étale fundamental groups, i.e. if every finite étale cover $\mathfrak{X}$ is pulled back from $S$.

**Corollary 3.4.19** (Abstract Eilenberg–Moore)**.** Let $p\colon \mathfrak{X} \to S$ be an étale simply connected stack and let $R \in \mathrm{CAlg}(\mathrm{SH}(\mathfrak{X}))$. Then, if $A \in \mathrm{Mod}_R(\mathrm{SH}(\mathfrak{X}))$ is Artin, the natural transformation

$$p_*A \otimes_{p_*R} p_*(-) \longrightarrow p_*\left(A \otimes_R -\right)$$

is an equivalence.

*Proof.* It suffices to show that, for any $M \in \mathrm{Mod}_R(\mathrm{SH}(\mathfrak{X}))$ we have $A \in \mathcal{C}(M, p)$. Since $\mathcal{C}(M, p)$ is thick, and $A$ is Artin, it suffices to prove this for $A = g_\sharp g^*R$ where $g\colon \mathfrak{X}' \to \mathfrak{X}$ is any finite étale map. By assumption on $\mathfrak{X}$, $g$ is pulled back from $S$, so the claim follows from Proposition 3.4.18. □

### 3.4.3 Eilenberg–Moore for algebraic tori

We finally come to the proof of Proposition 3.4.6 and begin with showing that the zero section $S \to \mathrm{B}W$ of a Weil restricted torus $W$ is Artin–Tate. The cell decomposition appearing in the proof was already used by Merkurjev and Panin in their computation of $K_0(T)$ of an algebraic torus $T$ [MP97].

Let $F$ be a field and let $G$ be its absolute Galois group. For a finite set $I$ with an action on $G$ and a variety $X$ over $F$, let $X^I$ be the Weil restriction of $X$ along the map $I \to *$, regarded as morphism of finite $F$-varieties. In other words,

$$X^I \times_F \overline{F} = (X \times_F \overline{F})^I$$

with the action of the Galois group given by

$$\gamma(x)_i = \gamma(x_{\gamma^{-1}(i)}) \quad \forall x = (x_i)_{i\in I} \in (X \times_F \overline{F})^I.$$

**Proposition 3.4.20.** Let $F$ be a field with Galois group $G$ and let $I$ be a finite $G$-set. Then, the motive of the canonical smooth section $e\colon \mathrm{Spec}(F) \to \mathrm{B}\mathbb{G}_m^I$ is Artin–Tate in $\mathrm{SH}(\mathrm{B}\mathbb{G}_m^I)$.

*Proof.* We will prove the result simultaneously over all fields and by induction on $|I|$. For $I = \varnothing$ the map $e$ is the identity and the claim is trivially true.

Now assume we have proven the claim for all fields $F$ and all finite $G$-sets of cardinality $< n$ and let $I$ be a finite $G$-set with $|I| = n$. Consider the $\mathbb{G}_m^I$-equivariant embedding $\mathbb{G}_m^I \hookrightarrow (\mathbb{A}_F^1)^I \simeq \mathbb{A}_F^{|I|}$. For a subset $J \subseteq I$ we let

$$U_J \subseteq (\mathbb{A}_F^1)^I$$

be the $\mathbb{G}_{m,F}^I$-invariant $\overline{F}$-subvariety consisting of tuples $(a_i)_{i\in I} \in (\mathbb{A}_F^1)^I$ for which $a_i \neq 0$ for $i \in J$. Moreover, for any $0 \leq k \leq |I|$, set

$$U_k := \bigcup_{J\subseteq I, |J|=k} U_J \subseteq (\mathbb{A}_F^1)^I,$$

that is, $U_k$ is the open subvariety consisting of tuples with at least $k$ non-zero entries.

We will prove by another induction on $k$ that the smooth $B\mathbb{G}_{m,F}^I$-stack $U_k//\mathbb{G}_{m,F}^I$ is Artin–Tate, which yields our claim for $k = n = |I|$, as we have a canonical $\mathbb{G}_{m,F}^I$-equivariant identification $U_{|I|} = \mathbb{G}_m^I$. For $k = 0$, $U_0 = (\mathbb{A}_F^1)^I$ is an affine space, so that its motive agrees with the unit.

For $k > 0$, consider the $\mathbb{G}_{m,F}^I$-equivariant embedding $U_k \hookrightarrow U_{k-1}$, and denote by $Z_k$ the complement. The group $G$ acts on the set $\binom{I}{k-1}$ of subsets of size $k-1$ of $I$. Let $J_1, \ldots, J_\ell$ be a complete set of representatives for the $G$-orbits on $\binom{I}{k-1}$ and let $G_t$ be the stabilizer of $J_t$. Let $F_t = \bar{F}^{G_t}$ be the fixed field of $G_t$. Then, we have an open and closed subvariety $Z_{J_t} \subseteq Z_k \times_F F_t$ consisting of tuples $(a_i)_{i\in I}$ such that $a_i \neq 0 \iff i \in J_t$. It fits into a diagram

$$\begin{tikzcd}
\coprod_{t=1}^{l} Z_{J_t}//\mathbb{G}_{m,F_t}^I \arrow[r] \arrow[d] & U_{k-1}//\mathbb{G}_{m,F}^I \arrow[d] & U_k//\mathbb{G}_{m,F}^I \arrow[l] \\
\coprod_{t=1}^{l} B\mathbb{G}_{m,F_t}^I \arrow[r] & B\mathbb{G}_{m,F}^I &
\end{tikzcd}$$

in which the upper row is an smooth open-closed decomposition and the lower horizontal arrow is finite étale. Using the induction hypothesis for $k$ and Lemma 3.4.15, we see that $U_k//\mathbb{G}_{m,F}^I \to B\mathbb{G}_{m,F}^I$ is Artin–Tate if the following hold:

1. The map $Z_{J_t}//\mathbb{G}_{m,F_t}^I \to B\mathbb{G}_{m,F_t}^I$ is Artin–Tate.
2. The normal bundle of the map $Z_{J_t}//\mathbb{G}_{m,F_t}^I \to U_{k-1}//\mathbb{G}_{m,F}^I$ is pulled back from $B\mathbb{G}_{m,F_t}^I$.

For (1), note that $Z_{J_t} \cong \mathbb{G}_{m,F_t}^{J_t} \times (\mathbb{A}_{F_t})^{|I\setminus J_t|}$ is acted on by $\mathbb{G}_{m,F_t}^I = \mathbb{G}_{m,F_t}^{J_t} \times \mathbb{G}_{m,F_t}^{I\setminus J_t}$ factor-wise, i.e. $\mathbb{G}_{m,F_t}^{J_t}$ acts on the first factor freely and $\mathbb{G}_{m,F_t}^{I\setminus J_t}$ acts on $(\mathbb{A}_{F_t})^{|I\setminus J_t|}$. By $\mathbb{A}^1$-invariance, the motive is the same as just the first factor, i.e. the motive of $\mathbb{G}_{m,F_t}^{J_t}//\mathbb{G}_{m,F_t} \to B\mathbb{G}_{m,F_t}^{I}$. This morphisms of schemes sits in a pullback

$$\begin{tikzcd} \mathbb{G}_{m,F_t}^{J_t}//\mathbb{G}_{m,F_t}^{I} \arrow[r] \arrow[d] & B\mathbb{G}_{m,F_t}^{I} \arrow[d] \\ \mathrm{Spec}(F_t) \arrow[r] & B\mathbb{G}_{m,F_t}^{J_t} \end{tikzcd}$$

The lower horizontal map is Artin–Tate by the inductive hypothesis, as $|J_t| = k-1 < n = |I|$. Therefore, so is the upper horizontal map by Lemma 3.4.13.

For (2), the normal bundle in question is the restriction of scalars of the (equivariant) normal bundle of $Z_{J_t}$ inside $U_{k-1} \times_F F_t$. Then, since $Z_{J_t}$ is an open subvariety of $\mathbb{A}_{F_t}^{J_t}$ which embeds linearly and equivariantly in $\mathbb{A}_{F_t}^{I}$, the normal bundle of $Z_{J_t}$ identifies with the pullback of the algebraic representation $F_t^{I\setminus J_t}$ of $\mathbb{G}_{m,F_t}^{I\setminus J_t}$, along the map $Z_{J_t}//\mathbb{G}_{m,F_t}^{I} \to B\mathbb{G}_{m,F_t}^{I} \to B\mathbb{G}_{m,F_t}^{I\setminus J_t}$. This yields the result. □

**Corollary 3.4.21.** Let $F$ be a field, write $S = \mathrm{Spec}(F)$ and suppose $W$ is the Weil restriction of a split torus along a finite étale extension of $S$. Then, the section $s\colon S \longrightarrow \mathrm{B}W$ is Artin over $\mathbb{K}_{\mathrm{B}W}$.

*Proof.* By Proposition 3.4.20 the map is Artin–Tate and hence Artin over $\mathbb{K}_{\mathrm{B}W}$ by Lemma 3.4.12. □

Finally, we need the following lemma which will be used in order to verify the condition appearing in Corollary 3.4.19.

**Lemma 3.4.22.** For any torus $T$ over $S = \mathrm{Spec}(F)$, the stack $\mathrm{B}T$ is étale simply connected relative to $S$, namely the structure map $p\colon \mathrm{B}T \to S$ induces an isomorphism of the étale fundamental groups.

*Proof.* For any choice of separably closure $F \to E$ we have a short exact sequence of étale fundamental groups

$$1 \to \pi_1^{\text{ét}}(\mathrm{B}T_E) \to \pi_1^{\text{ét}}(\mathrm{B}T) \to \mathrm{Gal}(E/F) \to 1\,.$$

As $T$ is a connected group scheme and the delooping is in the étale topology, the left hand term vanishes and the claim follows. □

With all this in hand, the proof of the desired Eilenberg–Moore formula is straightforward.

*Proof of Proposition 3.4.6.* Let $q\colon \mathrm{B}A \to S$ and $p\colon \mathrm{B}W \to S$ denote the unique maps to the terminal object. For the pullback diagram of stacks over $S$

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d] \\ \mathrm{B}A \arrow[r] \arrow[dr, "q"'] & \mathrm{B}W \arrow[d, "p"] \\ & S \end{tikzcd}$$

we want to show that the induced comparison map

$$q_*\mathbb{K}_{\mathrm{B}A} \otimes_{p_*\mathbb{K}_{\mathrm{B}W}} \mathbb{K} \longrightarrow \mathbb{K}^T \in \mathrm{CAlg}_{\mathbb{K}}(\mathrm{SH}(S)). \tag{3.10}$$

is an equivalence. First, note that the left hand side is given by

$$q_*\mathbb{K}_{\mathrm{B}A} \otimes_{p_*\mathbb{K}_{\mathrm{B}W}} \mathbb{K} \simeq p_*\mathbb{K}^{\mathrm{B}A}_{\mathrm{B}W} \otimes_{p_*\mathbb{K}_{\mathrm{B}W}} p_*\mathbb{K}^{S}_{\mathrm{B}W},$$

and similarly for the right hand side we have

$$\mathbb{K}^T \simeq p_*\mathbb{K}^T_{\mathrm{B}W}.$$

Under these identifications, the map (3.10) can be written as a composite

$$p_*\mathbb{K}^{\mathrm{B}A}_{\mathrm{B}W} \otimes_{p_*\mathbb{K}_{\mathrm{B}W}} p_*\mathbb{K}^{S}_{\mathrm{B}W} \xrightarrow{\tau} p_*\left(\mathbb{K}^{\mathrm{B}A}_{\mathrm{B}W} \otimes_{\mathbb{K}_{\mathrm{B}W}} \mathbb{K}^{S}_{\mathrm{B}W}\right) \xrightarrow{\mu} p_*(\mathbb{K}^T_{\mathrm{B}W}) \in \mathrm{SH}(S).$$

As $W$ is the Weil restriction of a split torus, we know by Corollary 3.4.21 that the motive $\mathbb{K}^S_{\mathrm{B}W} \in \mathrm{Mod}_{\mathbb{K}_{\mathrm{B}W}}(\mathrm{SH}(\mathrm{B}W))$ is Artin, and so the map $\tau$ is an equivalence by Corollary 3.4.19 and Lemma 3.4.22. Moreover, the morphism $\mu$ is precisely the image under $p_*$ of the Künneth map

$$\kappa_*\colon \mathbb{K}^{\mathrm{B}A}_{\mathrm{B}W} \otimes_{\mathbb{K}_{\mathrm{B}W}} \mathbb{K}^{S}_{\mathrm{B}W} \longrightarrow \mathbb{K}^T_{\mathrm{B}W} \in \mathrm{SH}(\mathrm{B}W)$$

and thus an equivalence by Proposition 3.4.16, so we are done. □

## 3.5 K-Theory of tori via the topological mirror

Let $T$ be an algebraic torus over a field $F$. We want to leverage Theorem 3.4.7 to actually give a presentation of $K$-theory spectrum $K(T)$ in terms of the spectra $K(L)$ where $F \to L \to E$ runs over intermediate extensions. This is done via the motivic–to–equivariant comparison functors which we reviewed in Section 3.2.3. Let $G$ denote the absolute Galois group of $F$.

We start in Section 3.5.1 by showing that any topological torus with an action of $G$ is contained in the subcategory of Borel $G$-spaces and use this to give a topological description of the delooped character lattice $\Sigma\Lambda^*(T)$ in Section 3.5.2

With this in hand, we deduce Theorem D as Corollary 3.5.11, by applying the comparison functor $\mathrm{SH}(F) \to \mathrm{Sp}_G$ to the equivalence of Theorem 3.4.7 and showing that a certainly assembly map is an equivalence of $G$-spectra.

In Section 3.5.3, we record the Atiyah–Hirzebruch spectral sequence associated to the equivalence of Corollary 3.5.11, which gives a tool for actually computing the $K$-*groups* of $T$. As a special case, we recover the formula for $K_0(T)$ of Merkujev–Panin via the generic formula for Bredon homology as a coend. Finally, we illustrate how our formula recovers a computation of the $K$-theory of algebraic tori of rank 1 due to Swan.

### 3.5.1 Topological $G$-tori

The goal of this section is to explain how to associate a topological torus with $G$-action to a lattice with a $G$-action and to prove that the homotopy type of this torus is Borel.

**Construction 3.5.1.** Let **Tori** denote the topological category of *topological tori*, i.e. the category of commutative, connected, compact Lie groups. Moreover, let **Lat** denote the category of *lattices*, i.e. the full subcategory of Ab spanned by the finitely generated free abelian groups. For a lattice $\Lambda \in \mathbf{Lat}$, write $\Lambda_{\mathbb{R}} \coloneqq \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ for the base change considered as a topological space with discrete subspace $\Lambda \subseteq \Lambda_{\mathbb{R}}$. Then there is an equivalence of categories

$$\mathbf{Lat} \xrightarrow{\sim} \mathbf{Tori}$$

given by sending a lattice $\Lambda$ to the topological torus $\Lambda_{\mathbb{R}}/\Lambda$ and inverse given by taking the first homology group $H_1(-,\mathbb{Z})$. In particular, for any profinite group $G$, we get an induced equivalence of $G$-equivariant categories and denote by

$$\mathbb{T}(-)\colon \mathbf{Lat}^{\mathrm{B}G} \xrightarrow{\sim} \mathbf{Tori}^{\mathrm{B}G}.$$

Here we adopt our convention from Section 3.2.3, defining the category of objects with an action of a pro-finite group as the limit of the categories of its finite quotients.

**Proposition 3.5.2.** Let $\Lambda \in \mathbf{Lat}^{\mathrm{B}G}$ be a lattice with $G$-action for $G$ a finite group. Then, the associated torus $\mathbb{T}(\Lambda)$ is Borel, i.e. the natural map

$$\mathbb{T}(\Lambda)^G \longrightarrow \mathbb{T}(\Lambda)^{hG}$$

from the fixed points to the homotopy fixed points is an equivalence.

*Proof.* Both spaces in question are 1-truncated, because $\mathbb{T}(\Lambda)$ is so. Moreover, as $\mathbb{T}(\Lambda)$ is a group-like commutative monoid, so are both fixed points, and it suffices to show that the map between them is an isomorphism on $\pi_0$ and $\pi_1$ at the canonical base point. Consider the exact sequence

$$0 \to \Lambda \to \Lambda_{\mathbb{R}} \to \mathbb{T}(\Lambda) \to 0.$$

Since $H^1(G;\Lambda_{\mathbb{R}}) = 0$ we obtain an exact sequence of abelian Lie groups

$$0 \to \Lambda^G \to \Lambda_{\mathbb{R}}^G \to \mathbb{T}(\Lambda)^G \to H^1(G;\Lambda) \to 0,$$

or equivalently, a short exact sequence

$$0 \to \mathbb{T}(\Lambda^G) \to \mathbb{T}(\Lambda)^G \to H^1(G;\Lambda) \to 0.$$

The Lie group on the right is a finite discrete group and the Lie group on the left is connected. It follows that

$$\pi_0\mathbb{T}(\Lambda)^G \xrightarrow{\simeq} H^1(G;\Lambda) \simeq \pi_0(B\Lambda)^{hG} = \pi_0 T(\Lambda)^{hG}$$

while

$$\pi_1\mathbb{T}(\Lambda)^G \simeq \pi_1\mathbb{T}(\Lambda^G) \simeq \Lambda^G \simeq \pi_1(B\Lambda)^{hG} = \pi_1\mathbb{T}(\Lambda)^{hG}.$$

It is now easy to check that these two equivalences are the ones induced from the map $\mathbb{T}(\Lambda)^G \to \mathbb{T}(\Lambda)^{hG}$, yielding the result. □

We can summarize this result as follows:

**Corollary 3.5.3.** For any profinite group $G$, the following diagram commutes:

$$\begin{tikzcd}
\mathbf{Lat}^{\mathrm{B}G} \arrow[r] \arrow[d, "\mathbb{T}(-)"] & \mathrm{Sp}^{\mathrm{cn}}(\mathrm{B}G) \arrow[r, "\Sigma"] & \mathrm{Sp}^{\mathrm{cn}}(\mathrm{B}G) \arrow[d, "\beta"] \\
\mathbf{Tori}^{\mathrm{B}G} \arrow[rr] & & \mathrm{Sp}^{\mathrm{cn}}(\mathcal{O}_G^{\mathrm{op}})
\end{tikzcd}$$

where the unlabeled horizontal morphisms are forgetful functors, only remembering the underlying (Borel) $G$-space together with its multiplicative structure.

*Proof.* As all categories in question are formally defined as limits of the finite quotients of $G$, we may assume that $G$ is finite. We have to verify that the underlying homotopy type of $\mathbb{T}(\Lambda)$ is Borel, as a connective spectrum. As limits of connective spectra are computed on underlying spaces, it is enough to verify that it is a Borel space. This precisely is the content of Proposition 3.5.2. □

### 3.5.2 Deducing the $G$-equivariant description

Throughout this section, we let $F$ be a field with absolute Galois-group $G$.

Recall from the discussion in Section 3.2.3 that we have adjunctions

$$C_{\text{ét}} \colon \mathrm{Sp}^{\mathrm{cn}}(\mathrm{B}G) \rightleftarrows \mathrm{Sp}^{\mathrm{cn}}_{\text{ét}}(F) : U_{\text{ét}}$$

$$C_{\mathrm{ns}} \colon \mathrm{Sp}^{\mathrm{cn}}(\mathcal{O}_G^{\mathrm{op}}) \rightleftarrows \mathrm{Sp}^{\mathrm{cn}}_{\mathrm{ns}}(F) : U_{\mathrm{ns}}\,,$$

such that the right adjoints commute with the natural inclusions $\beta \colon \mathrm{Sp}^{\mathrm{cn}}(\mathrm{B}G) \to \mathrm{Sp}^{\mathrm{cn}}(\mathcal{O}_G^{\mathrm{op}})$ and $\nu_* \colon \mathrm{Sp}^{\mathrm{cn}}_{\text{ét}}(F) \to \mathrm{Sp}^{\mathrm{cn}}_{\mathrm{ns}}(F)$. For a torus $T$, the considerations of Section 3.5.1 allow us to give a more explicit description of the $G$-space $U_{\mathrm{ns}}(\Sigma\Lambda(T))$ as follows.

**Corollary 3.5.4.** Let $T$ be a torus over $F$ and let $\Lambda = U_{\text{ét}}\Lambda(T) \in \mathbf{Lat}^{\mathrm{B}G}$ denote the underlying lattice with $G$-action of $\Lambda(T)$. Then, we have a natural equivalence of commutative groups in genuine $G$-spaces:

$$U_{\mathrm{ns}}(\Sigma\Lambda(T)) \simeq \mathbb{T}(\Lambda)$$

*Proof.* We know from Lemma 3.2.13 that we have

$$U_{\mathrm{ns}}(\Sigma\Lambda(T)) = \beta U_{\text{ét}}(\Sigma\Lambda(T)) \simeq \beta\Sigma U_{\text{ét}}(\Lambda(T))\,.$$

The right-hand side is by definition the Borel $G$-space associated to $\Sigma U_{\text{ét}}(\Lambda(T))$ which is precisely is the delooping of the character lattice and agrees with $\mathbb{T}(\Lambda)$ by Corollary 3.5.3, so we conclude. □

Put differently, all ways to associate a $G$-equivariant topological torus to an algebraic torus $T$ agree. Accordingly, this construction gets a name.

**Definition 3.5.5.** Let $T/F$ be an algebraic torus and let $\Lambda$ be the underlying lattice with $G$-action of $\Lambda(T)$. We call

$$\mathfrak{T}(T) := \mathbb{T}(\Lambda)$$

the *topological mirror* of $T$. Note that, by Corollary 3.5.4, the underlying $G$-space of $\mathfrak{T}(T)$ is *compact* in $\mathcal{S}_G$.

Recall from Proposition 3.2.14 that the adjunction $C_{\mathrm{ns}} \dashv U_{\mathrm{ns}}$ on the level of sheaves of spectra further refines to an adjunction

$$\mathcal{C} \colon \mathrm{Sp}_G \rightleftarrows \mathrm{SH}(F) \colon \mathcal{U},$$

where $\mathcal{C}$ is symmetric monoidal and $\mathcal{U}$ is lax monoidal.

Our goal is to use this to describe the algebraic $K$-theory of $T$ in terms of the equivariant homology of the mirror $\mathfrak{T}(T)$. To this end, let us first observe the following general fact.

**Lemma 3.5.6.** Let $X \in \mathrm{Sp}_G$ be dualizable and $E \in \mathrm{SH}(F)$ be arbitrary. The natural map

$$\mathcal{U}(E) \otimes X \longrightarrow \mathcal{U}(E \otimes \mathcal{C}(X))$$

is an equivalence of $G$-spectra.

*Proof.* This follows by general nonsense about monoidal adjunctions. Indeed, since $\mathcal{C}$ is symmetric monoidal, we have for any $A \in \mathrm{Sp}_G$ natural equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{Sp}_G}(A, \mathcal{U}(E) \otimes X) &\simeq \mathrm{Map}_{\mathrm{Sp}_G}(A \otimes X^\vee, \mathcal{U}(E)) \\
&\simeq \mathrm{Map}_{\mathrm{SH}(F)}(\mathcal{C}(A) \otimes \mathcal{C}(X)^\vee, E) \\
&\simeq \mathrm{Map}_{\mathrm{SH}(F)}(\mathcal{C}(A), E \otimes \mathcal{C}(X)) \\
&\simeq \mathrm{Map}_{\mathrm{Sp}_G}(A, \mathcal{U}(E \otimes \mathcal{C}(X)))
\end{aligned}$$

and so the claim follows by Yoneda. □

We want to apply this projection formula in the case where $X$ is the $G$-equivariant suspension spectrum $\mathbb{S}_G[\mathfrak{T}(T)] = \mathbb{S}_G[U_{\mathrm{ns}}(\Sigma\Lambda(T))]$. To do this, we compute the following.

**Proposition 3.5.7.** Let $T$ be an algebraic torus over $F$ with character lattice $\Lambda(T)$. The natural comparison map

$$\mathcal{C}(\mathbb{S}_G[U_{\mathrm{ns}}(\Sigma\Lambda(T))]) \longrightarrow \mathbb{S}_F[\Sigma\Lambda(T)] \in \mathrm{SH}(F)$$

is an equivalence of motivic spectra.

*Proof.* The map in question factors as the composite

$$\mathcal{C}(\mathbb{S}_G[U_{\mathrm{ns}}(\Sigma\Lambda(T))]) \xrightarrow{\sim} \mathbb{S}_F[C_{\mathrm{ns}}U_{\mathrm{ns}}(\Sigma\Lambda(T))] \to \mathbb{S}_F[\Sigma\Lambda(T)],$$

where the left hand morphism is an equivalence by Proposition 3.2.14. For the right hand side, we claim that the unit map

$$(C_{\mathrm{ns}} \circ U_{\mathrm{ns}})(\Sigma\Lambda(T)) \longrightarrow \Sigma\Lambda(T)$$

is an equivalence of Nisnevich sheaves valued in connective spectra. Consider the Weil-resolution of $T$ via tori

$$A \to W \to T$$

as in (3.4). Then, by Proposition 3.4.4, we have a cofiber sequence of Nisnevich connective spectra

$$\Lambda(W) \to \Lambda(A) \to \Sigma\Lambda(T).$$

Since the composite $C_{\mathrm{ns}} \circ U_{\mathrm{ns}}$ preserves colimits by Lemma 3.2.13 we obtain a map of cofiber sequences connective Nisnevich spectra of the form

$$\begin{tikzcd}
\Lambda(W) \arrow[r] & \Lambda(A) \arrow[r] & \Sigma\Lambda(T) \\
(C_{\mathrm{ns}} \circ U_{\mathrm{ns}})(\Lambda(W)) \arrow[r] \arrow[u] & (C_{\mathrm{ns}} \circ U_{\mathrm{ns}})(\Lambda(A)) \arrow[r] \arrow[u] & (C_{\mathrm{ns}} \circ U_{\mathrm{ns}})(\Sigma\Lambda(T)) . \arrow[u]
\end{tikzcd}$$

It follows from Construction 3.3.19 (together with Lemma 3.2.13 that the first two vertical maps are equivalences and so the right hand map is an equivalence as well. ☐

This allows us to use Lemma 3.5.6 to import Theorem 3.4.7. Recall from Definition 3.2.15 that for any $X \in \mathrm{Sm}_F$, the $G$-equivariant algebraic $K$-theory spectrum of $X$ is defined as $K_G(X) = \mathcal{U}(\mathbb{K}_F^X) \in \mathrm{CAlg}(\mathrm{Sp}_G)$. As usual, we denote by $K_G(F)[-] = \mathbb{S}_G[-] \otimes K_G(F)$ the symmetric monoidal enhancement of the homology functor.

**Theorem 3.5.8.** Let $T$ be an algebraic torus defined over a field $F$ with absolute Galois group $G$. There is a natural equivalence of commutative $K_G(F)$-algebras

$$K_G(F)[\mathfrak{T}(T)] \xrightarrow{\sim} K_G(T) \in \mathrm{CAlg}_{K_G(F)}(\mathrm{Sp}_G)$$

between the equivariant homology of the topological mirror $\mathfrak{T}(T)$ with coefficients in $K_G(F)$ and the $G$-equivariant algebraic $K$-theory of $T$.

*Proof.* Applying the lax monoidal functor $\mathcal{U}$ to the map of Theorem 3.4.7, yields

$$\mathcal{U}(\mathfrak{F}^{\Sigma\Lambda(T)}) \colon \mathcal{U}(\mathbb{K}_F[\Sigma\Lambda(T)]) \xrightarrow{\sim} \mathcal{U}(\mathbb{K}_F^T) = K_G(T) \in \mathrm{CAlg}_{K_G(F)}(\mathrm{Sp}_G)$$

and so it suffices to show that we have a natural equivalence

$$K_G(F)[\mathfrak{T}(T)] \simeq \mathcal{U}(\mathbb{K}_F[\Sigma\Lambda(T)]) . \tag{3.11}$$

Recall that $\mathfrak{T}(T) \simeq U_{\mathrm{ns}}(\Sigma\Lambda\,(T)) \in \mathcal{S}_G$ is Borel and compact by Corollary 3.5.4, which implies that the spectrum $\mathbb{S}_G[U_{\mathrm{ns}}(\Sigma\Lambda\,(T))] \in \mathrm{Sp}_G$ is dualizable. Thus, we may use the projection formula of Lemma 3.5.6 to see that the left hand side of (3.11) is given by

$$K_G(F)[\mathfrak{T}(T)] = \mathcal{U}(\mathbb{K}) \otimes \mathbb{S}_G[U_{\mathrm{ns}}(\Sigma\Lambda\,(T))] \simeq \mathcal{U}(\mathbb{K}_F \otimes \mathcal{C}(\mathbb{S}_F[U_{\mathrm{ns}}(\Sigma\Lambda\,(T))])).$$

Finally, by Proposition 3.5.7, we have a natural equivalence

$$\mathcal{U}(\mathbb{K}_F \otimes \mathcal{C}(\mathbb{S}_G[U_{\mathrm{ns}}(\Sigma\Lambda\,(T))])) \simeq \mathcal{U}(\mathbb{K}_F \otimes \mathbb{S}_F[\Sigma\Lambda\,(T)]) = \mathcal{U}(\mathbb{K}_F[\Sigma\Lambda\,(T)])$$

so we conclude. □

As any algebraic torus already splits after a finite field extension, the presentation of $K(T)$ via the topological mirror $\mathfrak{T}(T)$ is rather inefficient, as most of the absolute Galois group acts trivially. We now discuss a variant of Theorem 3.5.8 which addresses this inefficiency by incorporating a choice of splitting field. This leads to a smaller and in fact finite Galois group, making the presentation more tractable to computation.

**Definition 3.5.9.** Let $T/F$ be an algebraic torus together with the choice of a splitting field $E/F$ with Galois group $G_E$. We write

$$\Lambda_E(T) = \mathrm{map}(T_E, \mathbb{G}_{m,E})$$

for the character lattice relative to $E$, equipped with a $G_E$-action through conjugation. We write

$$\mathfrak{T}_E(T) = \mathbb{T}(\Lambda_E(T)) \in \mathbf{Tori}^{BG_E}$$

for the *topological mirror* of $T$ relative to $E$.

Let us write $G$ for the absolute Galois group of $F$. As $E$ splits the algebraic torus, the $G$-action on the lattice $\Lambda$ is inflated from $\Lambda_E(T)$ along the surjection $G \to G_E$ classifying the field extension $E$. Therefore, the topological mirror $\mathfrak{T}(T)$ is inflated from $\mathfrak{T}_E(T)$ along the same surjection. We use this observation to deduce the following:

**Corollary 3.5.10.** Let $T$ be an algebraic torus defined over a field $F$ and a choice of a splitting field $E/F$ with Galois group $G_E = \mathrm{Gal}(E/F)$. Let $\mathfrak{T}_E(T)$ denote the topological mirror relative to the field extension $E/F$. There is a natural equivalence of commutative $K_{G_E}(F)$-algebras

$$K_{G_E}(F)[\mathfrak{T}_E(T)] \xrightarrow{\sim} K_{G_E}(T) \in \mathrm{CAlg}_{K_{G_E}(F)}(\mathrm{Sp}_{G_E})$$

between the $G_E$-equivariant homology of the topological mirror $\mathfrak{T}_E(T)$ with coefficients in $K_{G_E}(F)$ and the $G_E$-equivariant algebraic $K$-theory of $T$.

*Proof.* Let $N$ be the normal subgroup classifying the field extension $E$, so that $G_E = G/N$. The result follows from applying genuine $N$-fixed points $(-)^N \colon \mathrm{Sp}_G \to \mathrm{Sp}_{G_E}$ to the equivalence from Theorem 3.5.8. The left hand side becomes the $G_E$-equivariant $K$-theory of $T$. To identity the right hand side, we use the following projection formula from equivariant homotopy theory: Let $K$ be a genuine $G$-spectrum and $X$ a genuine $G/N$-spectrum. Writing $\mathrm{infl}^G_{G/N}$ for the inflation functor from genuine $G/N$-spectra to genuine $G$-spectra, there is a natural equivalence of genuine $G/N$-spectra:

$$K^N \otimes X \simeq (K \otimes \mathrm{infl}^G_{G/N} X)^N.$$

Applying this with $K = K_G$ and $X$ the topological mirror of $T$ relative to $E$, we obtain:

$$(K_G(F) \otimes \mathfrak{T}(T))^N \simeq \Big(K_G(F) \otimes \mathrm{infl}^G_{G/N} \mathfrak{T}_E(T)\Big)^N \simeq K_G(F)^N \otimes \mathfrak{T}_E(T) \simeq K_{G_E}(F) \otimes \mathfrak{T}_E(T)\,,$$

as desired. □

In particular, we immediately get the following by taking $G_E$-fixed points

**Corollary 3.5.11.** Let $T$ be an algebraic torus defined over a field $F$ together with a choice of a splitting field $E/F$ with Galois group $G_E = \mathrm{Gal}(E/F)$. Let $\mathfrak{T}_E(T)$ denote the topological mirror of $T$ relative to the field extension $E/F$. The algebraic $K$-theory spectrum of $T$ is computed by the genuine fixed points

$$C_*^{G_E}(\mathfrak{T}_E(T); K_{G_E}(F)) = (K_{G_E}(F) \otimes \mathfrak{T}(T))^{G_E} \xrightarrow{\sim} K(T) \in \mathrm{CAlg}(\mathrm{Sp}).$$

Taking homotopy groups, we obtain an equivalence of graded rings

$$H_*^{G_E}(\mathfrak{T}_E(T), K_{G_E}(F)) \xrightarrow{\sim} K_*(T)$$

between the equivariant homology of the topological mirror $\mathfrak{T}_E(T)$ with coefficients in $K_{G_E}(F)$ and the algebraic $K$-groups of $T$.

### 3.5.3 Applications

We now illustrate how more careful analysis of the topological mirror can be used to leverage Corollary 3.5.10 to give explicit formulas. We first record that, as a further immediate corollary, we get an Atiyah–Hirzebruch spectral sequence computing the $K$-groups of a torus.

**Corollary 3.5.12.** Let $T$ be an algebraic torus over a field $F$ together with a choice of a splitting field $E/F$ with Galois group $G_E = \mathrm{Gal}(E/F)$. The equivalence of Corollary 3.5.10 induces a multiplicative spectral sequence of the form

$$E^2_{p,q} = H_p^{G_E}(\mathfrak{T}_E(T); \pi_q K_{G_E}(F)) \implies K_{p+q}(T)\,.$$

Any finitely generated projective module over a field is free, so that the rank map $\pi_0 K_G(F) \to \underline{\mathbb{Z}}$ to the constant Mackey functor with value $\mathbb{Z}$ is an equivalence, i.e. $\pi_0^H K_G(F) \simeq \mathbb{Z}$ for all $H < G$ and all the restrictions functors are equivalences and the transfer along an inclusion of subgroups $H' < H$ is given by multiplication with $|H/H'|$. In particular, since the spectral sequence is concentrated in positive degrees, we also learn the following:

**Corollary 3.5.13.** For any algebraic torus $T$ defined over a field $F$ and any choice of splitting field $E$ with Galois group $G_E = \mathrm{Gal}(E/F)$ and associated topological mirror $\mathfrak{T}_E(T)$, we have an equivalence

$$H_0^{G_E}(\mathfrak{T}_E(T); \underline{\mathbb{Z}}) \xrightarrow{\sim} K_0(T), \tag{3.12}$$

where the left hand side is $G_E$-equivariant Bredon homology of the topological mirror $\mathfrak{T}(T)$ with coefficients in the constant Mackey-functor $\underline{\mathbb{Z}}$.

The left hand side of Corollary 3.5.13 now admits an explicit description in terms of generators and relations: For any scheme $X/F$, write $\mathrm{Pic}(X)$ for the Picard group of $X$, i.e. the group of $\otimes$-invertible objects in $\mathrm{QCoh}(X)^{\heartsuit}$. If $F \to E$ is a finite étale extension, then restriction of scalars defines a functor $\mathrm{Res}_{E/F} \colon \mathrm{QCoh}(X_E) \longrightarrow \mathrm{QCoh}(X)$. Restricting to line bundles on $X$ and applying $K_0$ thus gives a map

$$\mathrm{Res}_{E/F} \colon \mathrm{Pic}(X_E) \longrightarrow K_0(X).$$

With this in hand, we recover the following computation of $K_0(T)$ due to Merkurjev and Panin from [MP97].

**Corollary 3.5.14** ([MP97, Thm. 9.1])**.** Let $T$ be a torus over a field $F$ with splitting field $E/F$. The map of abelian groups

$$\mathrm{Res}_{L/F}\colon \bigoplus_{F\subseteq L\subseteq E} \mathbb{Z}[\mathrm{Pic}(T_L)] \longrightarrow K_0(T),$$

is surjective, with kernel generated by all expressions of the form

$$[L' : L](\mathcal{L}) - (\mathcal{L} \otimes_L L'),$$

where $\mathcal{L} \in \mathrm{Pic}(T_L)$ and $L \hookrightarrow L'$ is a $F$-linear extension.[6]

*Proof.* The tensor product of $\mathbb{S}[\mathfrak{T}_E(T)]$ and $K_G(F)$ is computed by the coend formula

$$K_{G_E}(F)[\mathfrak{T}(T)]^G \simeq \int^{G/H\in\mathcal{O}_{G_E}} (\mathfrak{T}(T))^H \otimes K_{G_E}(F)^H.$$

Since $\pi_0$ commutes with colimits and tensor products, Corollary 3.5.13 implies that $K_0(T)$ is given by

$$H_0^{G_E}(\mathfrak{T}_E(T);\underline{\mathbb{Z}}) \simeq \int^{G_E/H\in\mathcal{O}_{G_E}} \mathbb{Z}\otimes\pi_0\mathfrak{T}_E(T)^H \simeq \left(\bigoplus_{G_E/H\in\mathcal{O}_{G_E}} \mathbb{Z}\otimes\pi_0(\mathfrak{T}(T)^H)\right)/\sim .$$

We claim that we have natural equivalence

$$\pi_0(\mathfrak{T}_E(T)^H) \simeq \mathrm{Pic}(T_L)$$

where $L = E^H$. Indeed, since $\mathrm{Pic}(T_E)$ is trivial, we have by Galois descent that

$$\mathrm{Pic}(T_L) \simeq H^1(H, \mathcal{O}_{T_E}^\times),$$

where $\mathcal{O}_{T_E}$ denotes the ring of functions. Then, there is a canonical $G$-equivariant decomposition of the units

$$\mathcal{O}_{T_E}^\times \simeq \Lambda\,(T_E)\times E^\times = \Lambda\times E^\times.$$

By Hilbert 90, we have $H^1(H, E^\times) = 0$. Putting everything together and recalling that $\mathfrak{T}_E(T) \simeq \mathrm{B}\Lambda$, we obtain equivalences

$$\mathrm{Pic}(T_L) \simeq H^1(H,\Lambda) \simeq \pi_0(\mathfrak{T}_E(T)^H)$$

as desired. Thus, we have obtained an equivalence

$$K_0(T) \simeq \left(\bigoplus_{G_E/H\in\mathcal{O}_{G_E}} \mathbb{Z}[\mathrm{Pic}(E^H)]\right)/\sim .$$

Finally, we have to unwind the equivalence relation. The covariant functoriality of $\underline{\mathbb{Z}}$ in $G/H \in \mathcal{O}_{G_E}$ is given by multiplication with the index, whereas the contravariant functoriality of $\mathrm{Pic}(E^H)$ is given by scalar extension.

Given a $G$-equivariant map $G/H \to G/H'$, corresponding to an $F$-linear map $F^{H'} \to F^H$, together with an integer $n \in \mathbb{Z}$ and a class $\mathcal{L} \in \mathrm{Pic}(F^{H'})$, we therefore have to identify the class $n\cdot(L\otimes_{F^{H'}} F^H)$ with $|H'/H|n\cdot\mathcal{L}$, leading to the description in the statement of the proposition. $\square$

[6] not necessarily compatible with the embedding to $E$.

As a further application, we recover a computation of the $K$-theory of tori of rank 1. From now on, assume that $F$ is of characteristic different from 2. Every such torus is of the form

$$T_a := \{(x,y)| : x^2 - ay^2 = 1\}$$

for $a \in F^\times$, and two such tori $T_a$ and $T_{a'}$ are isomorphic if and only if $a \equiv a'(/\!\!/ F^\times)^2$. The character lattice of $T_a$ is then a single copy of $\mathbb{Z}$ with the action of the Galois group being the inflation of the sign representation of $C_2$ along the map $G \to C_2$, $\gamma \mapsto \frac{\gamma(\sqrt{a})}{\sqrt{a}}$. In other words, it is the homomorphism that classifies the quadratic extension $F[\sqrt{a}]/F$.

Since all rank 1 tori over $F$ can be written as affine quadrics, the computation of their $K$-theory is a (very) special case of Swan's computation of the $K$-theory of projective and affine quadratic hypersurfaces [Swa85].

**Corollary 3.5.15** ([Swa85])**.** Let $F$ be a field of characteristic different from 2, $a \in F^\times$ and set

$$T_a \coloneqq \mathrm{Spec}(F[x,y]/(x^2 - ay^2 - 1)).$$

We have an equivalence of spectra

$$K(T_a) \simeq K(F) \oplus \mathrm{cofib}(K(F[\sqrt{a}]) \xrightarrow{\rho} K(F)),$$

where $\rho$ is the map induced by restriction of scalars along the field extension $F \to F[\sqrt{a}]$.

*Proof.* Let $G$ denote the absolute Galois group of $F$. The topological mirror $\mathfrak{T}(T_a)$ is a circle $S^1$, and the action of $G$ on this circle is given by the composition

$$G \to C_2 \to \mathrm{Aut}(S^1)$$

where the first map is the one described above (classifying $F[\sqrt{a}]$) and the second map is the (genuine) action of $C_2$ on $\mathfrak{T}(T_a)$ by reflection along an axis. The space $\mathfrak{T}(T_a)$ has an equivariant cell structure with two 0-cells $x$ and $y$ (the fixed points of the $C_2$-action) and two 1-cells $e_1$ and $e_2$, the upper and lower semicircles, which get switched by the $C_2$-action, see Figure 3.1.

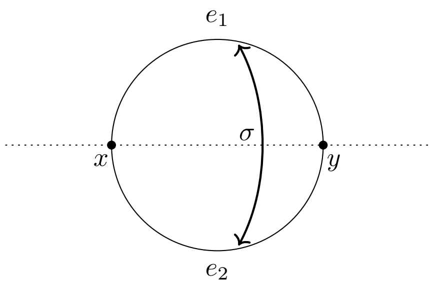


Figure 3.1: The equivariant cell structure of $\mathfrak{T}(T_a)$

We thus have a cell-attachment cofiber sequence

$$G/G_0 \otimes \mathbb{S}_G \longrightarrow \mathbb{S}_G \oplus \mathbb{S}_G \longrightarrow \mathbb{S}_G \otimes \mathfrak{T}(T_a) \in \mathrm{Sp}_G.$$

Tensoring with the equivariant $K$-theory spectrum of $F$ we get a cofiber sequence

$$K_G(F) \otimes G/G_0 \longrightarrow K_G(F) \oplus K_G(F) \longrightarrow K_G(F)[\mathfrak{T}(T-a)] \in \mathrm{Sp}_G.$$

By Theorem 3.5.8, the $G$-fixed points of the last term are given by $K(T_a)$. Since taking fixed points preserves cofiber sequences and

$$(K_G(F) \otimes G/G_0)^G \simeq (K_G(F))^{G_0} \simeq K(F[\sqrt{a}]),$$

we obtain a cofiber sequence of $G$-fixed points spectra

$$K(F[\sqrt{a}]) \longrightarrow K(F) \oplus K(F) \longrightarrow K(T_a).$$

It remains to identify the left hand map.
This map is induced from the attaching map of the 1-cells to the 0-cells. Since the 1-cell is induced from $G_0$, the data of this attaching map is the same as that of the $G_0$-equivariant attaching map of $e_1$ to $\{x\} \sqcup \{y\}$. This is the attaching map of an interval to its endpoints, so the map $\mathbb{S} \to \mathbb{S} \oplus \mathbb{S}$ we get is $(1, -1)$. Passing back to the $G$-equivariant maps, we deduce that the first map is given by

$$(\mathrm{tr}_{G_0}^{G}, -\mathrm{tr}_{G_0}^{G})\colon K(F[\sqrt{a}]) \to K(F),$$

where $\mathrm{tr}_{G}^{G_0}$ is the transfer map from the $G$-fixed points to the $G$-fixed points. By definition of $K^F$, this map is given by restriction of scalars along $F[\sqrt{a}] \to F$. Finally, performing the change of coordinates $(a, b) \mapsto (a, a + b)$ on the middle term, we see that we have a cofiber sequence

$$K(F[\sqrt{a}]) \xrightarrow{(\mathrm{res}_F^{F[\sqrt{a}]},0)} K(F) \oplus K(F) \longrightarrow K(T_a),$$

which proves the claim. $\square$

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