Benefit and pension planning based on the (inhomogeneous) preferences of the policyholder

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Preface

He who would learn to fly one day must first learn to stand and walk and run and climb and dance; one cannot fly into flying.
— Friedrich Nietzsche, *Thus Spoke Zarathustra*

This thesis has been prepared in fulfillment of the requirements for the PhD degree at the Department of Mathematical Sciences, Faculty of Science, University of Copenhagen. The work has been carried out under the supervision of Professor Mogens Steffensen from the University of Copenhagen and fully funded and conducted in cooperation with Mancofi A/S from May 2022 to April 2025.

The core of this thesis consists of five distinct manuscripts, each serving as an independent academic contribution on related topics. While the manuscripts are self-contained and can be read individually, minor overlaps and variations in notation occur across the chapters. The manuscripts are divided into two categories: the first exclusively addresses time-additive utility functions, while the second also explores time-inconsistent preferences. Chapter 1 introduces the first three manuscripts, which belong to the first category, whereas Chapter 5 introduces the final two manuscripts, which belong to the second category.

Acknowledgments

This work and my years of research are made possible thanks to the funding and trust Mancofi placed in me, for which I am eternally grateful. This includes a large degree of freedom and faith placed in me by, among others, Line and Allan, who have given me all the opportunities I needed to succeed. I hope our collaborations continue.

I want to thank my supervisor, Mogens Steffensen, who took a chance on me as a PhD student, guided me in a new world of academia, and gave me a thicker skin that will undoubtedly help me face challenges in the future.

During my PhD, I worked partly in the offices of Mancofi and partly present at the University of Copenhagen, not forgetting my immense travels and visits to several departments. I thank all who have hosted me more or less permanently in their office space, encouraged me, and helped me on my journey. Building on that, I want to thank the Universidad Nacional de Colombia Sede Manizales for the several months I spent there. It was a genuinely great and insightful experience. Also, I sincerely thank Università Cattolica del Sacro Cuore and Francesco for the opportunity to lecture on multi-state Markov processes and present my research. Further, thanks to all my colleagues at Mancofi for their great company and insightful discussions. I want to thank my numerous office mates at UCPH for their overbearing with my frustrating or happy outbursts and unconstrained comments on most things. It has always been a pleasure to work in your presence.

During this work, two people have continuously listened to my frustration and provided me with honest advice. I would like to thank Debbie for always being ready with an open-minded and nonjudgmental ear when I am like Bambi on thin ice. In the last part of my PhD, a significant thanks goes to Christian, who has acted as guidance and a friend in times when an open door was much needed.

I thank my parents and brother for believing in me and their eternal support, regardless of the distance between us. Naomi, thank you for your kindness and everlasting memories you have given me, which have, time after time, reminded me of the importance of life outside of the actuarial and academic bobble.

Finally, I extend my heartfelt gratitude to my partner Alaric, whose unwavering belief in me inspires me to strive for greater heights. Thank you for encouraging me to follow my dreams wherever they may bring us and enabling us to dream bigger together. You give me the best reason not to work.

Julie Bjørner Søe
Copenhagen, April 2025

List of papers

Besides chapter 1 and 5 that has been prepared specifically for this thesis, each of the five remaining chapters presents a self-contained, working, submitted or already published manuscript according to the following scheme:

Chapter 2:
Mogens Steffensen & Julie Bjørner Søe (2023) Optimal consumption, investment, and insurance under state-dependent risk aversion. *ASTIN Bulletin*, 53(1), 104–128. doi:10.1017/asb.2022.25

Chapter 3:
Jaime A. Londoño & Julie Bjørner Søe (2025) A State-Dependent Approach to Optimal Consumption, Investment, and Life Insurance by Risk-Adjusted Utilities. *To be submitted.*

Chapter 4:
Julie Bjørner Søe (2025) Implicit Prioritization of Life Insurance Coverage: A Study of Policyholders Preferences in PFA Pension. *Submitted for publication.*

Chapter 6:
Mogens Steffensen & Julie Bjørner Søe (2024) What is the value of the annuity market?. *Decisions Econ Finan.* https://doi.org/10.1007/s10203-023-00411-3

Chapter 7:
Mogens Steffensen & Julie Bjørner Søe (2025) Optimal Equilibrium Investment and Insurance with State-Dependent Risk Aversion. *Working paper.*

Abstract

This thesis consists of independent investigations related to benefit and pension planning based on policyholders' inhomogeneous preferences. The first four chapters are centered around the assumption of time-additive preferences, and the last three chapters include the concept of time inconsistency in optimization problems. The chapters are linked by the shared objective of understanding policyholders' decisions and preferences to better design, manage, and advise on life insurance and savings products.

Chapter 1 provides the reader with background while easing into the general topic of the thesis and provides the main contributions for Chapters 2, 3, and 4.

In Chapter 2, containing the manuscript *Optimal consumption, investment, and insurance under state-dependent risk aversion*, we formalize and solve the problem of optimal consumption, investment, and life insurance decisions with state-dependent risk aversion, where the states can represent the biometric states of a policyholder. Using a multi-state Markov chain model to describe the states, we construct the value function by aggregating value functions of underlying sub-problems, where each sub-problem represents a state. We illustrate numerically the optimal decisions in a three-state disability model of a single policyholder, demonstrating the effect on optimal controls by changes in preferences.

Chapter 3, investigates the possibility of handling state-dependent utility, where the state is the underlying state of the market, by adjusting for risk we measure at time zero the utility of a financial value of a payment at a future time t. Consequently, we acknowledge that the preferences of policyholders may evolve in response to changes in the underlying market conditions. We introduce a deflator that accounts for both financial and actuarial risks. We solve the problem of optimal consumption, investment, and life insurance in this deflated setting by using dynamic programming and providing explicit solutions for CRRA utility functions.

In Chapter 4, we establish a link to the practical usage of optimal controls in the industry by evaluating policyholders' perceptions of their life insurance coverages. In *Implicit Prioritization of Life Insurance Coverage: A Study of Customer Preferences in PFA Pension* we compare the choices and preferences for an entire portfolio with data supplied by PFA Pension.

In the last three chapters, we study two seemingly different utility optimization problems, both in the context of time inconsistency issues handled using equilibrium theory. Chapter 5 introduces and eases the reader into the context of time-inconsistent preferences and provides the main contributions from Chapters 6 and 7. In Chapter 6, we examine the value of the annuity market in the decumulation phase of a pension plan for an investor who either experiences time-additive preferences or separates the preferences of risk aversion and elasticity of inter-temporal substitution. *What is the value of the annuity market?* characterizes the relative loss of wealth by losing access to annuities, and numerical examples illustrate the results.

In the last chapter, *Optimal Equilibrium Investment and Insurance with State-Dependent Risk Aversion*, we investigate the optimal equilibrium investment and insurance strategies, with state-dependent risk aversion as in Chapter 2, but incorporating uncertainty and randomness in the preferences and evaluating certainty equivalents resulting in a time-inconsistent optimization problem. We handle this by applying the equilibrium approach, presenting a verification theorem, proof, and solutions for different utility functions.

Resumé

Denne afhandling består af uafhængige undersøgelser relateret til ydelses- og pensionsplanlægning baseret på forsikringstagernes heterogene præferencer. De første fire kapitler er centreret omkring antagelsen om tidsadditive præferencer, mens de sidste tre kapitler inkluderer begrebet tidsinkonsistens i nytteoptimeringsproblemer. Kapitlerne er forbundet af det fælles mål at forstå forsikringstagernes beslutninger og præferencer for bedre at kunne designe, administrere og rådgive om livsforsikrings- og opsparingsprodukter.

Kapitel 1 giver læseren en baggrund og introducerer det generelle emne for afhandlingen samt de vigtigste bidrag fra kapitlerne 2, 3 og 4.

I kapitel 2, som indeholder manuskriptet *Optimal consumption, investment, and insurance under state-dependent risk aversion*, formaliserer og løser vi problemet med optimale forbrugs-, investerings- og livsforsikringsbeslutninger med tilstandsafhængig risikovillighed, hvor tilstandene kan repræsentere de biometriske tilstande for en forsikringstager. Ved hjælp af en Markov-flertilstandsmodel til at beskrive tilstandene konstruerer vi værdifunktionen ved at aggregere værdifunktioner af underliggende delproblemer, hvor hvert delproblem repræsenterer en tilstand. Vi illustrerer numerisk de optimale beslutninger i en tre-tilstands invaliditetsmodel for en enkelt forsikringstager og demonstrerer effekten på optimale kontroller ved ændringer i præferencer.

Kapitel 3, undersøger muligheden for at håndtere tilstandsafhængig nytte, hvor tilstanden er den underliggende tilstand af markedet, ved at justere for risikoen og dermed evaluere nytten af købekraft. Hermed anerkender vi, at forsikringstagernes præferencer kan udvikle sig som reaktion på ændringer i de underliggende markedsforhold. Vi introducerer en deflator, der tager højde for både finansielle og aktuarielle risici. Vi løser problemet med optimal forbrug, investering og livsforsikring i denne deflaterede indstilling ved hjælp af dynamisk programmering og giver eksplicitte løsninger for CRRA-nyttefunktioner.

I kapitel 4 etablerer vi en forbindelse til den praktiske anvendelse af de optimale kontroller i branchen ved at evaluere forsikringstagernes opfattelser af deres livsforsikringsdækninger. I *Implicit Prioritization of Life Insurance Coverage: A Study of*

Customer Preferences in PFA Pension sammenligner vi valg og præferencer for en hel portefølje med data leveret af PFA Pension.

I de sidste tre kapitler studerer vi to tilsyneladende forskellige nytteoptimeringsproblemer, begge i konteksten af tidsinkonsistensproblemer håndteret ved hjælp af ligevægtsteori. Kapitel 5 introducerer og fører læseren ind i konteksten af tidsinkonsistente præferencer og præsenterer de vigtigste bidrag fra kapitlerne 6 og 7. I kapitel 6 undersøger vi værdien af annuitetsmarkedet i dekumulationsfasen af en pensionsplan for en investor, der enten oplever tidsadditive præferencer eller adskiller præferencerne for risikovillighed og elasticitet af intertemporal substitution. *What is the value of the annuity market?* karakteriserer det relative tab af formue ved at miste adgangen til annuiteter, og numeriske eksempler illustrerer resultaterne.

I det sidste kapitel, *Optimal Equilibrium Investment and Insurance with State-Dependent Risk Aversion*, undersøger vi de optimale ligevægtsinvesterings- og forsikringsstrategier med tilstandsafhængig risikovillighed som i kapitel 2, men inkorporerer usikkerhed og tilfældighed i præferencerne og evaluerer sikkerhedsekvivalenter, hvilket resulterer i et tidsinkonsistent optimeringsproblem. Vi håndterer dette ved at anvende ligevægtstilgangen, præsenterer et verifikationsteorem, beviser og løsninger for forskellige nyttefunktioner.

Contents

Chapter 1

Introduction to decision making in life insurance

This introduction provides an intuitive background to comprehend the basis for the forthcoming three chapters and outlines the specific contributions from these chapters. Each chapter is an independent manuscript and thus contains an introduction that includes the relevant references to related literature. Hence, this background introduction holds no references and gives a broader background and ease into the relevant subject. First, we introduce the overall topic of classical decision-making in multi-state life insurance and the contribution of Chapter 2, then we introduce a new concept of risk adjusted utilities and the corresponding contributions of Chapter 3 and lastly we explain how Chapter 4 contributes to the practical analysis and understanding of application of these themes in the industry.

1.1 From classical decision making to multi-state heterogeneity

When planning for the future, both as an individual and as a society, pensions, and life insurance are essential tools to ensure peace of mind and provide a sense of security by managing risks associated with biometric events such as disability, unforeseen death, or ensuring stability while retired. Actuaries play an essential role in designing and managing life insurance and pension products. Our knowledge brings on responsibilities; some might even say obligations, such as safeguarding the policyholder's best interests and other ethical aspects. Importantly, pension and life insurance providers' interest in financial growth and stability often aligns with that of the policyholder.

Classical decision-making in life insurance and pensions concerns optimizing spending, insurance, and portfolio management throughout the lifespan of a policyholder. This is traditionally done under the assumption of constant relative risk aversion, which means that the policyholder maintains an unchanging level of risk aversion regardless of their life circumstances. A classical problem would be to optimize utility from consumption (c) while alive, indicated by the function $I = 1 - N$, such that $N = 1$ only upon death of the policyholder, the utility of bequest wealth $(x + b)$ upon death and utility of terminal wealth $(X(T))$ if the policyholder survives until termination time T.

$$\sup_{c,\pi,b} \mathbb{E}\Big[\int_0^T u(t, c(t))I(t)dt + v(t, X^{c,\pi,b}(t) + b(t))dN(t) \\ + I(T)U(X^{c,\pi,b}(T))\Big].$$

Here, we use three different utility functions: u is an instantaneous utility function, v is the utility function for a bequest, and U measures the utility of terminal wealth. These functions characterize the policyholder's preferences regarding risk. In this situation, the policyholder has a wealth which evolvement we describe by the notation of dynamics as long as he is alive as

$$\begin{aligned} dX^{c,\pi,b}(t) &= \Big((r + \pi(t)(\alpha - r)) - c(t) - \mu^*(t)b(t)\Big)X^{c,\pi,b}(t)dt \\ &\quad + \pi(t)X^{c,\pi,b}(t)\sigma dW(t), \\ X^{c,\pi,b}(0) &= x_0. \end{aligned}$$

Where the policyholder invests a proportion of their wealth (π) in a risky asset modeled by a Black-Scholes stock, such that the expected return on the wealth process from the investments are $(r + \pi(t)(\alpha - r))X$. As mentioned, c is the policyholder's consumption rate. The additional insurance sum received if death occurs before termination time T is b, which is priced by a pricing intensity μ^*. Thereby, we allow for the possibility of a premium loading if $\mu(t) \neq \mu^*(t)$, where $\mu(t)$ is the objective transition intensity. This problem is solved by embedding it in a value function as

$$\sup_{c,\pi,b} \mathbb{E}_{t,x}\Big[\int_t^T u(s, c(s))I(s)ds + v(s, X^{c,\pi,b}(s) + b(s))dN(s) \\ + I(T)U(X^{c,\pi,b}(T))\Big],$$

where the expectation is conditionally given $X^{c,\pi,b}(t) = x$, and $I(t) = 1$. With the techniques from dynamic programming, we characterize this function by the Hamilton-Jacobi-Bellman equation, containing a local optimization problem for each point (t, x) and by the favorable properties from the expectation the solution to the controls (c, π, b) to the continuum of local optimization problems is also a solution to the global optimization problem. Due to the pleasant time-additive utility, which we question and discuss in the later chapters. The mentioned utility functions u, v, and U with a constant relative risk aversion can, e.g., be of the traditional power utility family formulated as

$$u(t,x) = \frac{1}{1-\gamma} x^{1-\gamma} g(t)^{\gamma}, \tag{1.1.1}$$

where the time dependence is present in the function g, which could represent the subjective discount factor of the policyholder. In this problem, the attitude to risk is constant, and the "only" heterogeneity considered is from the simplest survival model.

Multi State Life Insurance

Multi-state models are indispensable tools in the mathematics of life insurance and pensions. While the simplest survival model considers only the states of being alive or dead, more complex scenarios require a broader framework. These include models for disability, multi-life policies, couples, multiple causes of death, and health conditions. Typically assumed to be Markovian, multi-state models are the workhorses in theoretical and practical applications.

In life insurance, the position of the insured and their policy can be described by a finite-state, time-inhomogeneous, continuous-time Markov chain, Z, on a state space $\mathcal{J}$. The insurance policy terminates at a specified time T, and the state of the insured at any time $t \in [0,T]$ is denoted by $Z(t)$. This Markov chain Z is assumed to be independent of the Brownian motion W, and both are defined on a measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the natural filtration of (Z,W).

We can treat life insurance policies as standard financial contracts by defining two equivalent probability measures—one for real-world probabilities (objective measure) and one for pricing risks (pricing measure). The objective measure reflects the actual likelihood of events, while the pricing measure values risks in both financial and insurance markets.

To understand how individuals move between different states (such as healthy, disabled, or deceased for an individual), we use a process that tracks these transitions. This process has specific rates at which transitions occur, determined by the objective measure. If some transitions are impossible and does not happen the rate is zero.

Under the pricing measure, we have similar transition rates, but they are adjusted to reflect the cost of insuring these risks. The difference between the two sets of rates represents an additional expense by transferring the risks to an insurance or pension company. Usually, adding premium loading (also denoted risk loading) adds complexity to notation and mathematical solutions. Finally, we assume that the process remains Markovian under the pricing measure, meaning that the future state depends only on the current state and not on the past. This assumption ensures that the independence between financial market risks and insurance risks holds under both measures.

Multi State heterogeneity in decision making

Aristotle wrote in his work "Nicomachean Ethics" a long time ago

There is nothing more unequal than the equal treatment of unequal people

the message extends to the equal treatment of people during their lifespan regardless of the state of their lives. Thus, the assumption that the policyholder has a constant attitude toward risk over the whole life cycle, regardless of the state of her life, is, to say the least, limiting. Heterogeneous preferences across states are relevant to single-agent models with health states and multi-generation and multi-agent models. Previously, the heterogeneity in the biometric state of the insured is incorporated by allowing the time-dependent function in the utility function, denoted as $g(t)$, to depend upon the state, such that $g^j(t)$ applies for all $j \in \mathcal{J}$.

The contributions from chapter 2

In Chapter 2, we present a new method to integrate multi-state Markov chain models with decision-making processes, allowing relative risk aversion to vary depending on the state of the Markov chain, meaning that the preference parameter in the utility function (1.1.1) γ now is denoted γ^j for all $j \in \mathcal{J}$ alongside with the time-dependent function $g^j(t)$ and thereby the risk aversion while active on the labor market might, very realisticly, differ from the risk aversion while disabled.

We construct a value function by breaking down the problem into several sub-problems, each representing the utility from consumption in a specific state j within a general state space $\mathcal{J}$. Further, as part of the construction, we allocate the initial wealth to the different sub-problems determined so the marginal indirect utility from the different sub-problems coincide. Thereby we ensure an individual would not gain from moving wealth from one sub-problem to another. We form the candidate value function of the original problem by aggregation of the candidate value functions of the sub-problems. We verify that this candidate function satisfies the Hamilton-Jacobi-Bellman equation and compare the structure with cases earlier studied in the literature.

Further, we numerically illustrate the implicit solution using a three-state disability model of a single policyholder, where her risk aversion changes based on whether she is active or disabled — demonstrating the effect on optimal controls by changes in preferences and addressing the limitations by maintaining them constant. Understanding the effect of shifts in risk aversion in different biometric states on financial decisions can enable more rigorous reflection when designing products that better encompass policyholders' complexities.

1.2 Risk-adjusted utilities

When addressing the influence of real-life complex aspects on the preferences of policyholders and individuals in general, it is not only the biometric states that are of consequence; traditionally and generally, the state of the underlying financial market has a considerable impact —individuals' valuation of money and preferences for stability change responding to the changing market. We shift our attention to another concept of state dependence, where the state is that of the underlying market, hereby encapsulating the interplay between preferences and market conditions and the reality that the decisions of policyholders are influenced by the economic situation.

We can see this, for example, in an everyday routine; imagine someone who stops at a local coffee shop on the way to work every morning. In a situation with a strong and sturdy economy, they would treat themselves to a grande cappuccino and might even get a pastry. Nevertheless, if inflation and prices are climbing, and there might be a recession on the horizon if they even still go to the coffee shop every morning, they might merely get a drip coffee, as saving is a higher priority in this financial scenario. The change in behavioral patterns is a response to an underlying economic situation, which we ought to consider when discussing policyholders' decisions.

The idea is to take cues from derivative pricing techniques and compare by price or financial value of money instead of the moral value. We know that the time 0 financial value of a payment $c'(t)$ at time t is given by $\mathbb{E}[\Lambda(t)c'(t)]$ with a deflator

$$d\Lambda(t) = -\Lambda(t)\big(rdt + \frac{\alpha - r}{\sigma}dW(t)\big), \quad \Lambda(0) = 1.$$

Assumed $\frac{\alpha-r}{\sigma} \neq 0$. At time 0, the deflator is essential in quantifying and comparing financial values of payments at different points in time. Regaring the utility optimizatin, the idea, to measure at time 0 the moral value of a payment $u'(t)$ at time t by $E(u(\Lambda(t)c'(t)))$, where u is a utility function. Accounting for the market's attitude for risk before individual decisions.

As explained previously, a premium loading can appear in life insurance decision-making due to the possible difference between the objective and pricing mortality rate. This can lead to a possible deviation between how insurance is priced and how it is perceived and further introduce a risk beyond the financial market. The uncertainty related to the actuarial risk should be considered in addition to the financial risk.

The contributions from chapter 3

In Chapter 3, we marry the concept of risk-adjusted utilities, from a financial point of view, with the framework of jump-diffusion processes, which arises when introducing a multi-state Markov chain model for life insurance. This means adjusting for both financial and actuarial risks before evaluating the utility of purchasing power instead of monetary values. Introducing a stochastic deflator for both financial and actuarial risks allows us to account not only for the financial market conditions but also the actuarial risks before evaluating the optimal decisions a policyholder faces in the midst of these markets. We formulate and solve the optimization problem using dynamic programming and, in the CRRA case, derive closed-form expressions for optimal strategies and further illustrate the numerical effect of these. This chapter addresses the need for a model that includes the effect of not only the underlying financial market but also the actuarial market, which is of great influence when discussing life insurance decisions. Instead of complicating the calculations with additional state variables, it suggests a simplification.

The contributions from chapter 4

We now focus on practical data analysis, with the aim of bridging the gap between theoretical models and a possible application in the life insurance industry. In Chapter 4, we examine data from a pension company to evaluate how policyholders perceive their life insurance coverages. This practical chapter leverages the theoretical foundations discussed earlier to provide insights into business practices and product development.

Data is provided by the Danish pension company PFA Pension, and together, we have selected individual attributes to be analyzed in order to understand their influence on policyholders' evaluation of their life insurance contracts. In this analysis, we compare the implicit priority each policyholder places on their current life insurance coverage by assessing the utility they currently place upon their coverage and comparing it with the utility they would place upon complete coverage. We illustrate the implicit priorities for the entire portfolio and for relevant sub-portfolios in order to compare the implicit priorities of peers. This enables us to identify trends and outliers and, uncover insights, and enhance our understanding of both how the policyholders evaluate their life insurance coverages, moreover to potential optimization that can enable the diversity of policyholders.

As actuaries, we strive to maintain the best interests of the policyholders; this includes considering heterogeneities of choices and policyholders and balancing financial and individual choices throughout the lifespan of a policyholder. The knowledge of actuaries brings the potential to positively affect the financial well-being of policyholders. A better understanding of the mathematical framework that can capture unique needs and preferences fosters a better endeavor to incorporate these diversities into the actuarial framework. A commitment to ethical practice builds trust and ensures that our work contributes positively and will continue to do so.

Chapter 2

Optimal consumption, investment, and insurance under state-dependent risk aversion

Abstract

We formalize a consumption-investment-insurance problem with the distinction of a state-dependent relative risk aversion. The state-dependence refers to the state of the finite state Markov chain that also formalizes insurable risks such as health and lifetime uncertainty. We derive and analyze the implicit solution to the problem, compare it with special cases in the literature, and illustrate the range of results in a disability model where the relative risk aversion is preserved, decreases, or increases upon disability.

2.1 Introduction

We formalize and solve a consumption-investment-insurance problem in a multi-state framework where the risk aversion depends on the state. Heterogeneous preferences across states are relevant in both multi-generation models, multi-agent models, and single-agent models with health states. The solution is here characterized implicitly and numerically illustrated in a three-state (so-called) disability model of a single individual with risk aversion dependent on whether she is active in the labour market or disabled from working.

The academic tradition of considering consumption-investment problems formulated in continuous time dates back to Merton, 1971, 1969. The fundamental, stylized case of an agent seeking to optimize expected utility, with a constant relative risk aversion, of consumption in a Merton market model has been generalized and varied over again and again during the last four decades. The starting point of our work is the generalization of an uncertain lifetime already studied by Richard, 1975 and before by Yaari, 1965 in a simpler setting. The uncertain lifetime is matched by access to life insurance which the agent also optimizes. A simple rationale for our work is the following: Richard, 1975 worked through, explicitly, the case where the utility of both the consumption and the insurance death benefit paid out upon death are based on the same constant relative risk aversion, although these amounts are, clearly to be consumed by different groups of individuals insofar that the decision maker is not present to consume the death benefit. But what happens if the risk aversions are different?

Multi-state models are an inevitable tool in the mathematics of life insurance and pensions. The survival model with alive and dead as the only states is the simplest possible and one can, in that case, easily work without the concept of states, as e.g., Richard, 1975 did. But for generalizations to a disability, multi-life, couples, multiple causes of death, and health models, the multi-state models are the workhorse models in both theory and practice, typically assumed to be Markovian, see e.g., Hoem, 1988. The problem solved by Richard, 1975 was generalized to multi-state models by Kraft and Steffensen, 2008a. They allow both income and consumption to be state-dependent and the optimal risk position now includes optimal insurance against all risks to which one is exposed in a given state. A generalization where market and decision constraints are added to the special case of a disability model is the object of study in Hambel et al., 2017. However, in both Kraft and Steffensen, 2008a and Hambel et al., 2017, the risk aversion is homogeneous across states; thus, none of them helps us answer the question posed at the end of the preceding paragraph.

One generalization of Merton's consumption-investment problem is in the direction of heterogeneous preferences. Heterogeneous preference is a standard topic in multi-agent models where heterogeneity exists across agents. It is less standard in single-agent models where heterogeneity exists across the time and space of the single agent. Heterogeneity in time is studied by Steffensen, 2011 and Aase, 2017. The solution by Steffensen, 2011 is based on an idea of how to construct a value function, presented by Lakner and Nygren, 2006. They solve a problem with different utility functions for (constrained) consumption and (constrained) terminal wealth by introducing the main idea that a candidate value function can be constructed by distributing initial wealth optimally to the consumption and the terminal wealth projects, respectively, and, thereafter, allocating distributed wealth to risky assets marginally for each project. Steffensen, 2011 adopts the idea and constructs on that basis a value function to solve a problem with generally age/time-dependent risk aversion by separating the problem in a continuum of marginal terminal wealth problems terminating at a continuum of time points and an initial wealth distribution problem. Lichtenstern, Shevchenko, and Zagst, 2021 generalize to an age-dependent subsistence level and discuss calibration to observed life-cycle consumption profiles.

The basic idea in the present paper is to adopt and adapt that technique to state-dependent utility in a multi-state model to cope with state heterogeneity. This allows us to study a single agent who changes risk aversion if e.g., she becomes disabled or unemployed or whatever the (insurable) risky event, the state transition represents. We believe that our work contributes to state-dependence in the health dimension for which demand was expressed already by Karni, 1983. Several authors have worked with the state-dependent utility since then but most often with either a multiplicative state effect or with the state being the financial state rather than some orthogonal state stemming from, e.g., health. See Jarrow and Li, 2021 for recent theoretical results on the notion of state-dependent utility.

Our setup also allows us to study household problems where the preferences of the household planner change across states - presumably because the household itself changes - e.g. if someone in the family dies. A similar type of heterogeneity is studied by Kwak, Shin, and Choi, 2011 who work in a three-state model where each state represents the number out of two generations in a household that is alive (2, 1 or 0). What differs from our work is that in Kwak, Shin, and Choi, 2011, each generation has its risk aversion such that, initially, when both generations are alive, utility from the consumption of each generation is simply added up, tacitly assuming generation-additivity of utility. We, instead, assume that the household has a single representative risk aversion in each state, and we solve for a general J-state model. Also, Choi and Koo, 2005 solves a related problem where; however, the event upon which the preferences change is an optimal stopping time, with the retirement time as the most immediate application in mind.

We address the generalization of Kraft and Steffensen, 2008a to include state-dependent utility. Other recent contributions and generalizations in the area include Wei et al., 2020, who consider optimal life insurance in a household with correlated lifetimes; Wang et al., 2021, who allow income to increase in a random and non-hedgeable way and allow for market ambiguity; Wang et al., 2019, who generalize the financial market to a continuous-time, finite-state self-exciting threshold model; and Doctor, 2021, who also generalize the financial market and include inflation risk. Common for the recent literature is that the contributions are driven by generalized financial markets or general insurance risk models whereas our contribution is in the direction of generalized preferences.

Adding the separation idea of Lakner and Nygren, 2006 to the already involved solution to the problem with state-dependence studied by Kraft and Steffensen, 2008a, such that also risk aversion can be state-dependent, is the key contribution of the present paper. This makes the key difference to Kraft and Steffensen, 2008a and this is the key difficulty in the sections ahead. We show that the mathematical structure of the solution is similar to the one obtained by Kraft and Steffensen, 2008a together with an initial optimal allocation of capital. Further, we exemplify the steps concretely to show the impact of state-varying risk aversion.

The outline of the paper is as follows: In Section 2 we present the problem and the Hamilton-Jacobi-Bellman Equation characterizing its solution and formulate a verification theorem. In Section 3 a candidate for the value function is established. In Section 4 we verify that the candidate function fulfills the Hamilton-Jacobi-Bellman equation and compare the structure with special cases earlier studied in the literature. We illustrate in Section 5 how our setup impacts consumption and wealth dynamics for an individual with a risk aversion that depends on his state of health.

2.2 The Problem and characterization of its solution

We consider an individual, henceforth called the insured, who makes decisions in a standard Black-Scholes market consisting of a risk-free asset, the bond, and a risky asset, the stock, such that the price dynamics are given by

$$\begin{aligned} dB(t) &= rB(t)dt, & B(0) &= 1, \\ dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t), & S(0) &= s_0, \end{aligned}$$

where r, α and σ are constants, and $W(t)$ a standard Brownian motion.
We consider a situation where the position of the insured and his insurance policy is described by a finite-state time-inhomogeneous continuous-time Markov chain, Z, on a state space $\mathcal{J}$. The insurance policy terminates at time T and we denote the Markov chain state of the insured at time $t \in [0, T]$ by $Z(t)$. The Brownian motion W and the Markov chain Z are assumed to be independent and defined on the measurable space $(\Omega, \mathcal{F})$, here $\mathcal{F}$ is the natural filtration of (Z, W).

We define two equivalent probability measures on the measurable space $(\Omega, \mathcal{F})$. First, the objective measure is denoted by $\mathbb{P}$ and second, the pricing measure is denoted by $\mathbb{P}^*$ used for pricing both financial market risk (W) and insurance market risk (Z). Thus, we consider life insurance policies as standard marketed contracts, as Richard, 1975 and Kraft and Steffensen, 2008a. Further, we assume the pricing measure exists and is unique such that prices are linear and unique but allow for the possibility that the pricing measure is equal to the objective measure.

Let N^{jk} denote the counting process counting the number of transitions from the jth state to the kth such that $N^{jk}(t)$ equals the number of transitions made until time t. The process Z has deterministic objective transition intensities μ^{jk} for any transition, $j \neq k$, under the objective measure, and we assume that all positive transition intensities are bounded away from both zero and infinity, but some transition intensities may be equal to zero, that is, no transition is possible. The relation between the counting processes and the transition intensities is, formally, $E[N^{jk}(t) - N^{jk}(t-h)|Z(t-h) = j] = \mu^{jk}(t)h + o(h)$.

The corresponding transition intensities under the pricing measure are denoted by μ^{*jk} with the same properties, formally defined by $E^*[N^{jk}(t) - N^{jk}(t-h)|Z(t-h) = j] = \mu^{*jk}(t)h + o(h)$, where E^* is the expectation under then pricing measure $\mathbb{P}^*$. If positive intensities are bounded away from zero and infinity, assuming that the measures ($\mathbb{P}^*$ and $\mathbb{P}$) are equivalent corresponds to the assumption that μ^{*jk} is zero if and only if μ^{jk} is zero. The difference between the two transition intensities represents an insurance risk loading. We consider later the special case of zero risk loading as this gives access to particularly simple, but not trivial, results. Finally, we assume that Z is Markovian also under the pricing measure which essentially means that also the pricing transition intensities are deterministic. One convenient consequence of that assumption is that the assumed independence between W and Z holds under both the objective and the pricing measure, see Dhaene et al., 2013 for that topic on a more general level. Then, for all possible states $j \in \mathcal{J}$, the wealth of the insured evolves according to the jump-diffusion process, with initial wealth x_0. The wealth we present here is financial wealth, but is for simplicity referred to as the wealth.

$$\begin{aligned} dX(t) = \Big(& rX(t) + \sum_{j\in\mathcal{J}} \big(\pi^j(t)X(t)(\alpha - r) + Y^j(t) - c^j(t) \\ & - \sum_{k:k\neq j} \mu^{*jk}(t)b^{jk}(t)\big)\mathbb{1}_{\{Z(t)=j\}}\Big)dt \\ & + \sum_{j\in\mathcal{J}}\sum_{k:k\neq j} b^{jk}(t)dN^{jk}(t) + \sigma\sum_{j\in\mathcal{J}} \pi^j(t)X(t)dW(t)\mathbb{1}_{\{\{Z(t)=j\}\}}, \quad t\in[0,T], \\ X(0) =& \, x_0, \end{aligned} \tag{2.2.1}$$

where π^j describes the proportion of wealth invested in stocks in the jth state, Y^j is a deterministic function formalizing the income rate in the jth state, $c^j(t)$ the consumption rate at time t in the jth state, and $b^{jk}(t)$ is the insurance benefit received upon transition from state j to k at time t.

We take the investment proportion, the consumption rates, and the insurance benefits to be the control process. Whereas investment and consumption are standard control processes, the insurance benefit is less standard. When Z is in state j, the policyholder is exposed to making a transition from j to k for all $k : k \neq j$ with the intensity $\mu^{jk}(t)$. The policyholder buys insurance protection that pays out the lump sum $b^{jk}(t)$ if a transition takes place and for that, the policyholder pays a premium at the rate $\mu^{*jk}(t)b^{jk}(t)$. A controllable insurance sum means that the sum paid out upon any insurance risk can be continuously adjusted. When the policyholder is in state j, she decides on all insurance sums $b^{jk}, k : k \neq j$ where the transition rate μ^{jk} is positive. All other insurance sums play no role as long as the policyholder is in state j. They simply do not appear in the dynamics of X. The second line in the dynamics of X is the insurance premium payment paid out of the wealth for the benefits $b^{jk}(t), k : k \neq j$. That line shows how the pricing transition intensities are used to calculate the premium for the insurance benefits.
We introduce the notion of human capital - and the notation a for it - which is the financial value of future labour income. We speak of the sum of the financial wealth and the human capital as the total wealth of an investor. The human capital is represented in (2.2.1) by the labor income process Y^j. The labour income rate that drives a is stochastic since it depends on Z; therefore, a itself becomes stochastic. However, by access to the insurance market, the individual can hedge their future income. In that sense, access to the insurance market makes the individual face a complete market. The human capital is, formally, the unique value of the future income hedging portfolio.

With the set-up established we consider the objective to maximize the expected utility of consumption until termination T, i.e.

$$\sup_{c,\pi,b\in\mathcal{A}} E\left[\int_0^T \sum_{j\in\mathcal{J}} u^j(t, c^j(t))\mathbb{1}_{\{Z(t)=j\}}dt\right].$$

The supremum is taken over consumption, investment, and insurance processes in the set $\mathcal{A}$ of admissible controls and the utility functions are specified as

$$u^j(t, c^j(t)) = \frac{1}{1-\gamma_j}(g^j(t))^{\gamma_j}(c^j(t))^{1-\gamma_j}. \tag{2.2.2}$$

Here, the time-dependence of the utility function appears through g^j, a deterministic positive time-weight function, taken to the power of γ_j for mathematical convenience. The standard subjective utility discounting is included in the setup by a specific exponential choice of g^j. In the numerical calculations in Section 5, we assume such an exponential discounting of utility. We note here that γ is decorated with j which formalizes the essential contribution of this paper compared to Kraft and Steffensen, 2008a. The value function is, correspondingly, defined as

$$V^j(t,x) = \sup_{c,\pi,b\in\mathcal{A}} E_{t,x,j}\left[\int_t^T \sum_{k\in\mathcal{J}} u^k(s,c^k(s))\mathbb{1}_{\{Z(s)=k\}}ds\right], \tag{2.2.3}$$

where $E_{t,x,j}$ denotes conditional expectation, given that $X(t) = x$ and $Z(t) = j$.

We say that the controls, (c^j, π^j, b^{jk}) for all $j, k \in \mathcal{J}$ are admissible if they meet the following requirements: First, the insured cannot have a negative total capital in the sense of wealth including human capital, that is, $X(t) + a(t) \geq 0$ for all $t \in [0, T]$. Second, (2.2.1) has a unique solution. Third, the expectation in (2.2.3), based on that specific strategy, is well-defined. Finally, we have that

$$E\left[\int_0^T \sigma\pi^j(t)X(t)dW(t)\mathbb{1}_{\{Z(t)=j\}}\right] = 0,$$
$$E\left[\int_0^T b^{jk}(t)\left(dN^{jk}(t) - \mu^{jk}(t)\mathbb{1}_{\{Z(t)=j\}}dt\right)\right] = 0.$$

We denote by $\mathcal{A}$ the set of admissible controls. It is important to note that the rate of the compensator for the jump process $N^{jk}(t)$ is actually $\mu^{jk}(t)\mathbb{1}_{\{Z(t)=j\}}$, unlike a standard Poisson process that has a deterministic compensator.

Theorem 2.2.1 (Verification theorem)**.** *Assume that there exists a system of sufficiently differentiable functions $U^j(t,x)$, $j \in \mathcal{J}$, and admissible controls $(c^j, \pi^j, b^{jk}) \in \mathcal{A}$ such that $U^j(t,x)$ solves the equation*

$$\begin{aligned}
0 = \sup_{c^j,\pi^j,b^{jk}\in\mathcal{A}} \Big\{ \quad & (rx + \pi^j x(\alpha - r) + \textstyle\sum_{k\in\mathcal{J}}(Y^k(t) - c^k)\mathbb{1}_{\{k=j\}} \\
& - \textstyle\sum_{k:k\neq j} \mu^{*jk}(t)b^{jk})\frac{\partial}{\partial x}U^j(t,x) \\
& + \tfrac{1}{2}(\pi^j)^2 x^2\sigma^2 \frac{\partial^2}{\partial x\partial x}U^j(t,x) + \textstyle\sum_{k\in\mathcal{J}} u^k(t,c^k)\mathbb{1}_{\{k=j\}} \\
& + \textstyle\sum_{k:k\neq j} \mu^{jk}(t)\Big(U^k(t,x+b^{jk}) - U^j(t,x)\Big)\Big\} \\
& + \tfrac{\partial}{\partial t}U^j(t,x), \\
0 = U^j(T,x), \quad &
\end{aligned}$$

and such that

$$\begin{aligned}
\arg\sup\nolimits_{c^j,\pi^j,b^{jk}\in\mathcal{A}} \Bigg\{ & (rx + \pi^j x(\alpha - r) + \sum_{k\in\mathcal{J}}(Y^k(t) - c^k)\mathbb{1}_{\{k=j\}} \\
& - \sum_{k:k\neq j} \mu^{*jk}(t)b^{jk})\frac{\partial}{\partial x}U^j(t,x) \\
& + \frac{1}{2}(\pi^j)^2 x^2\sigma^2 \frac{\partial^2}{\partial x\partial x}U^j(t,x) + \sum_{k\in\mathcal{J}} u^k(t,c^k)\mathbb{1}_{\{k=j\}} \\
& + \sum_{k:k\neq j} \mu^{jk}(t)\Big(U^k(t,x+b^{jk}) - U^j(t,x)\Big)\Bigg\},
\end{aligned}$$

exists and constitutes $(c^{*j}, \pi^{*j}, b^{*jk}) \in \mathcal{A}$.
Then the optimal value function V^j *to the control problem is given by*

$$V^j(t,x) = U^j(t,x),$$

and $(c^{*j}, \pi^{*j}, b^{*jk})$ *is the optimal control function.*

Remark 2.2.2. Note that the above system is deterministic and holds for all states of Z and X. Specifically it holds for $Z(t) = j$ and $X(t) = x$ we can; therefore, leave out the indicator $\mathbb{1}_{\{Z(t)=j\}}$. Clearly, in the above equation we could write $\sum_{k\in\mathcal{J}}(Y^k(t) - c^k)\mathbb{1}_{\{k=j\}}$ simpler as $Y^j(t) - c^j$ but we insist on the cumbersome version as it will make certain operations later on easier to follow.

The proof of the verification theorem follows standard calculations, exactly as Asmussen and Steffensen, 2020, Theorem 6.1, where the verification theorem for the multi-state problem with constant risk aversion case is proved. The generalization to state-dependent risk aversion does not make the verification theorem any more complicated. However, as we shall see, constructing a candidate for U and verifying that it fulfills the requirements in the verification, is considerably more complicated.

In the following two sections, first we construct and motivate our value function candidate in Section 3. Second, in Section 4 we verify that this candidate value function fulfills the requirements for U in the verification theorem. Once we have verified this in Section 4, we know by the verification theorem that our candidate value function is indeed the value function of the control problem.

2.3 The Value Function

In this section, we develop an intuitive understanding of and construct a candidate value function. The argument for the construction of the candidate value function is informal, whereas the formal verification that it solves the HJB equation, is found in Section 2.4.

A classical guess for the value function in the case with no insurance state risk and with constant risk aversion is based on the separation of the time and wealth variables such that

$$U(t,x) = \frac{1}{1-\gamma} f(t)^{\gamma}(x + a(t))^{1-\gamma},$$

where the function f together with a the human capital, captures the time dependence of the candidate value function, but this candidate does not include any insurance state variation. Consequently, we turn to the construction in Kraft and Steffensen, 2008a, with insurance state risk included but the risk aversion is constant. They obtain

$$U^j(t,x) = \frac{1}{1-\gamma} f^j(t)^{\gamma}(x + a^j(t))^{1-\gamma}. \tag{2.3.1}$$

As seen, now the candidate value function depends on the state, as do also the functions f^j and a^j. This still does not include the state variation of risk aversion, though. Lakner and Nygren, 2006 suggest a way to construct a value function that copes with variation of risk aversion where the variation is with respect to consumption and terminal wealth. This is done by dividing the initial wealth into one part for consumption and another part for terminal wealth. Then it is possible to solve each sub-problem marginally since each sub-problem has no variation in risk aversion. In the end, the optimal allocation of the initial wealth is determined by making sure the marginal indirect utility from the two problems coincide such that the individual does not gain further from moving wealth from one sub-problem to the other. Steffensen, 2011 uses the same method for solving a pure consumption problem with age-dependent risk aversion. This way of constructing a candidate value function gives us the idea of how to include the state-varying relative risk aversion.

We construct our candidate value function by decomposing our problem into several sub-problems such that each sub-problem is formed by its measure of utility from consumption in a particular state in the state space $\mathcal{J}$. Further, as part of the construction, we also allocate the initial wealth to the different sub-problems. We decorate by subscript the sub-problem to which a given quantity belongs. Thus, we have a wealth process, an income process, a consumption rate, an investment proportion in state j, and an insurance lump sum upon transition from j to k, all corresponding to sub-problem l, given as X_l, Y_l^j, c_l^j, π_l^j, and b_l^{jk}, respectively. The sub-problem quantities aggregate to the quantities of the single original problem in the following way, $X = \sum_l X_l$, $Y^j = \sum_l Y_l^j$, $c^j = \sum_l c_l^j$, $\pi^j X = \sum_l \pi_l^j X_l$, and $b^{jk} = \sum_l b_l^{jk}$.

The decomposition into sub-problems introduces a long list of control variables, and one can expect that there are different decompositions where the overall optimal controls are distributed differently to the sub-problems and where the wealth is distributed to the different sub-problems accordingly. These decompositions are all different decompositions of the same overall optimal control and are, thus, all optimal. The ambition here is not to characterize *all* optimal decompositions but to characterize a single one of them. We, therefore, restrict ourselves to the case where consumption in state k only happens in the sub-problem where utility from that consumption is actually measured, i.e. $c_l^j = 0$ for $l \neq j$ such that $c^j = \sum_l c_l^j = c_j^j$, and c_j^j can henceforth simply be denoted by c^j. Similarly, we restrict ourselves to the case where income in the state k is fully allocated to the state where it is received, i.e. $Y_l^j = 0$ for $l \neq j$ such that $Y^j = \sum_l Y_l^j = Y_j^j$, and we can denote Y_j^j by Y^j. We also restrict ourselves to the case where all of the insurance lump sums paid upon jump into state k are allocated to the wealth of the sub-problem related to utility from consumption in state k, i.e. $b_l^{jk} = 0$ for $l \neq k$ such that $b^{jk} = \sum_l b_l^{jk} = b_k^{jk}$, and we can denote b_k^{jk} by b^{jk}. Finally, we also restrict ourselves to the case where the insurance premium is financed by the wealth in the sub-problem where the corresponding insurance benefits are also earned upon transitions. We emphasize that along with all these restrictions follows a specific initial wealth distribution, and the whole set of controls and initial wealth distribution are integrated parts of the guess on the value function below. If we can find a solution to the HJB equation for a given set of restrictions, we certainly have an optimal solution to the problem. We pay no further attention to the idea that the decomposition of the optimal control is not unique.

We mention, en passant, that these restrictions are no different from the restriction that Lakner and Nygren, 2006 make when they allocate all the consumption to their utility of consumption sub-problem. One can easily imagine a different allocation where parts of the consumption are withdrawn from the wealth in the utility of terminal wealth sub-problem, but the utility of this consumption is measured in the utility of consumption sub-problem. That would simply lead to a different distribution of initial wealth but the same overall aggregate optimal solution.

Based on these restrictions we have that the dynamics of X_l are quite similar to the dynamics given by X in (2.2.1). We have with initial wealth $x_{l,0}$,

$$
\begin{aligned}
dX_l(t) &= \Big(rX_l(t) + \sum_{j\in\mathcal{J}} \pi_l^j(t) X_l(t)(\alpha - r) + \sum_{j\in\mathcal{J}} \big(Y_l^j(t) - c_l^j(t) \\
&\quad - \sum_{k:k\neq j} \mu^{*jk}(t) b_l^{jk}(t) \big) \mathbb{1}_{\{Z(t)=j\}} \Big) dt \\
&\quad + \sum_{j\in\mathcal{J}} \sum_{k:k\neq j} b_l^{jk}(t) dN^{jk}(t) + \sigma \sum_{j\in\mathcal{J}} \pi_l^j(t) X_l(t) dW(t), \quad t \in [0,T], \\
X_l(0) &= x_{l,0}.
\end{aligned}
\tag{2.3.2}
$$

We look at the dynamics dX in (2.2.1) to confirm this to be equal to $d(\sum_l X_l)$. We consider (2.3.2) and sum over all the sub-problems

$$
\begin{aligned}
\sum_{l\in\mathcal{J}} dX_l(t) &= \sum_{l\in\mathcal{J}} \Big(rX_l(t) + \sum_{j\in\mathcal{J}} \pi_l^j(t) X_l(t)(\alpha - r) + \sum_{j\in\mathcal{J}} \big(Y_l^j(t) - c_l^j(t) \\
&\quad - \sum_{k:k\neq j} \mu^{*jk}(t) b_l^{jk}(t) \big) \mathbb{1}_{\{Z(t)=j\}} \Big) dt \\
&\quad + \sum_{l\in\mathcal{J}} \Big(\sum_{j\in\mathcal{J}} \sum_{k:k\neq j} b_l^{jk}(t) dN^{jk}(t) + \sigma \sum_{j\in\mathcal{J}} \pi_l^j(t) X_l(t) dW(t) \Big), \quad t \in [0,T], \\
\sum_{l\in\mathcal{J}} X_l(0) &= \sum_{l\in\mathcal{J}} x_{l,0}.
\end{aligned}
$$

Inserting the above-explained notation and moving the sum over the sub-problems l

to the affected parts we get

$$\begin{aligned} d\sum_{l\in\mathcal{J}} X_l(t) &= \Big(r\sum_{l\in\mathcal{J}} X_l(t) + \sum_{j\in\mathcal{J}}\sum_{l\in\mathcal{J}} \pi_l^j(t)X_l(t)(\alpha - r) + \sum_{j\in\mathcal{J}}\big(\sum_{l\in\mathcal{J}} Y_l^j(t) - \sum_{l\in\mathcal{J}} c_l^j(t) \\ &\quad - \sum_{k:k\neq j}\mu^{*jk}(t)\sum_{l\in\mathcal{J}} b_l^{jk}(t)\big)\Big)dt \\ &\quad + \sum_{j\in\mathcal{J}}\sum_{k:k\neq j}\sum_{l\in\mathcal{J}} b_l^{jk}(t)dN^{jk}(t) + \sigma\sum_{j\in\mathcal{J}}\sum_{l\in\mathcal{J}} \pi_l^j(t)X_l(t)dW(t), \quad t\in[0,T], \\ \sum_{l\in\mathcal{J}} X_l(0) &= \sum_{l\in\mathcal{J}} x_{l,0}. \end{aligned}$$

Now, if we notice $\sum_l \pi_l^j X_l = \pi^j X$, $\sum_l Y_l^j = Y^j$, $\sum_l c_l^j = c^j$, and $\sum_l b_l^{jk} = b^{jk}$, we recognize this as (2.2.1) and, thereby, $dX(t) = \sum_l dX_l(t)$.

When we condition on $X(t) = x$ and $X_k(t) = x_k$, and we know that $X = \sum_{k\in\mathcal{J}} X_k$, we are in the subspace where $x = \sum_{k\in\mathcal{J}} x_k$. This relation between x and $x_k, k\in\mathcal{J}$ is used frequently from now.

The sub-problem corresponding to measuring utility from consumption in state $k\in\mathcal{J}$ becomes

$$\sup_{c^k,\pi^k,b^{jk}} E\Big[\int_0^T u^k(t,c^k(t))\mathbb{1}_{\{Z(t)=k\}}dt\Big]. \tag{2.3.3}$$

This draws on our specific decomposition since c^k is not only the total consumption rate spent in state k. It is even equal to the consumption rate subtracted from the wealth belonging to sub-problem k as the consumption from all other sub-problems is zero. Each sub-problem isolates a single risk aversion and, therefore, with reference to Kraft and Steffensen, 2008a, a sub-problem value function candidate is

$$U_k^j(t,x_k) = \frac{1}{1-\gamma_k} f_k^j(t)^{\gamma_k}(x_k + a_k^j(t))^{1-\gamma_k}. \tag{2.3.4}$$

Note again here how all sub-scripts refer to the sub-problem where utility is measured and all topscripts refer to the possible states of Z at time t.
The candidate function of the original problem is now formed by aggregation of the candidate functions of the sub-problems, i.e.

$$U^j(t,x) = \sum_{k\in\mathcal{J}} U_k^j(t,x_k),$$

along with the relation between the arguments mentioned above, $x = \sum_{k\in\mathcal{J}} x_k$.

The decomposition into sub-problems and an initial wealth allocation follows the fundamental idea by Lakner and Nygren, 2006. Lakner and Nygren, 2006 work with such a separation idea under a general utility function. We consider only the case of the power utility, see ((2.2.2)), and we choose the decomposition into sub-problems such that each sub-problem measures the utility from a specific constant relative risk aversion. That makes the candidate value function for each sub-problem further separable in the time and wealth.

We show that the fundamental idea of allocating to sub-problems works well together with state-varying relative risk aversion. Based on the general utility function approach by Lakner and Nygren, 2006, there is all reason to believe that the decomposition into sub-problems works well beyond that case. However, the separability of the sub-problem value function candidates in time and wealth cannot be expected to work outside our case of state-varying relative risk aversion. This separability allows, e.g., a simple explicit calculation of the distribution of initial capital to sub-problems, see below. Applications of the decomposition idea beyond the power utility case are beyond the scope of this paper.

The initial wealth allocated to each sub-problem is determined through the marginal indirect utility $\frac{\partial}{\partial x_k} U_k^j(t, x_k)$. Following Lakner and Nygren, 2006, we want the marginal indirect utility from the different sub-problems to coincide such that the individual does not gain further from moving wealth from one sub-problem to another. This is a condition on the allocations $x_k, k \in \mathcal{J}$. Further, we want the marginal indirect utility to be independent of the state of Z. Thus, we require from the allocation of wealth that there exists a function ψ such that

$$\psi(t,x) = \frac{\partial}{\partial x_k} U_k^j(t, x_k), \qquad \text{for all } k \in \mathcal{J}. \tag{2.3.5}$$

When noting that the left side depends on only wealth and time, the marginal wealth allocation on the right side must be linked to each other and the wealth through the relation $x = \sum_{k \in \mathcal{J}} x_k$. When taking the derivative with respect to x_k, we mean the derivative with respect to the second coordinate of the value function U_k^j. Thus, a more direct way to write our assumption (2.3.5) is,

$$\psi(t, \sum_{k \in \mathcal{J}} x_k) = \frac{\partial}{\partial x_1} U_1^j(t, x_1) = \frac{\partial}{\partial x_2} U_2^j(t, x_2) = \dots = \frac{\partial}{\partial x_J} U_J^j(t, x_J). \tag{2.3.6}$$

We cannot stress hard enough that, whenever the total wealth realization x is specified below, we always require implicitly that this wealth is allocated optimally such that $x = \sum_{k \in \mathcal{J}} x_k$ and such that (2.3.6) holds.
The function $\psi(t, x)$ representing the marginal indirect utility is the partial derivative of the sub-problem's value function candidate. Thus, it is just an ingredient in our guess on a value function. Only later when we have verified that our candidate solves the HJB equation we know that the allocation of wealth into sub-problems is correct. By the notation and assumption introduced above we can combine (2.3.4) and (2.3.5) to reach the following relation,

$$x_k = f_k^j(t) \psi(t,x)^{-\frac{1}{\gamma_k}} - a_k^j(t). \tag{2.3.7}$$

By plugging this back into the value function, we can write the marginal value functions in terms of the marginal indirect utility as

$$U_k^j(t, x_k) = \frac{1}{1-\gamma_k} f_k^j(t) \psi(t,x)^{\frac{\gamma_k - 1}{\gamma_k}}. \tag{2.3.8}$$

The marginal indirect utility must be determined implicitly as the solution to

$$x = \sum_{k \in \mathcal{J}} x_k = \sum_{k \in \mathcal{J}} \left(f_k^j(t) \psi(t,x)^{-\frac{1}{\gamma_k}} - a_k^j(t) \right). \tag{2.3.9}$$

For numerical illustrations, this implicit equation must be solved numerically.

The combination of (2.3.8) and (2.3.9) forms our candidate value function. This is based on exactly the same idea as the one Lakner and Nygren, 2006 and Steffensen, 2011 use to form a candidate for the value function. But the application of the idea is here clearly much more advanced, and the notation, motivation, and description are correspondingly more involved. We have now established a candidate. In the next section, we verify that our candidate truly solves the HJB equation.

2.4 Verification

Now we are going to verify our candidate solution indeed solves the HJB equation. By inserting (2.3.8) in (2.2.3) our implicit candidate function can be presented as

$$U^j(t,x) = \sum_{k \in \mathcal{J}} \frac{1}{1-\gamma_k} f_k^j(t) \psi(t,x)^{\frac{\gamma_k - 1}{\gamma_k}}, \tag{2.4.1}$$

$$x = \sum_{k \in \mathcal{J}} \left(f_k^j(t) \psi(t,x)^{-\frac{1}{\gamma_k}} - a_k^j(t) \right). \tag{2.4.2}$$

To ease the next computations, we introduce the auxiliary function

$$h^j(t,x) = \sum_{k \in \mathcal{J}} \frac{1}{\gamma_k} f_k^j(t) \psi(t,x)^{-\frac{1}{\gamma_k}}.$$

2.4.1 Partial derivatives

The first step to verifying our candidate value function is to find the partial derivatives. Deriving (2.4.1) with respect to time t gives

$$\begin{aligned} \frac{\partial}{\partial t} U^j(t,x) &= \sum_{k \in \mathcal{J}} \left(\frac{1}{1-\gamma_k} \psi(t,x)^{\frac{\gamma_k - 1}{\gamma_k}} \frac{d}{dt} f_k^j(t) - \frac{1}{\gamma_k} f_k^j(t) \psi(t,x)^{-\frac{1}{\gamma_k}} \frac{\partial}{\partial t} \psi(t,x) \right), \\ &= \sum_{k \in \mathcal{J}} \left(\frac{1}{1-\gamma_k} \psi(t,x)^{\frac{\gamma_k - 1}{\gamma_k}} \frac{d}{dt} f_k^j(t) \right) - h^j(t,x) \frac{\partial}{\partial t} \psi(t,x). \end{aligned} \tag{2.4.3}$$

The partial derivative $\frac{\partial}{\partial t}\psi(t,x)$ is found by differentiating with respect to time t on both sides of the relation (2.4.2)

$$\begin{aligned} 0 &= \sum_{k\in\mathcal{J}} \left(\psi(t,x)^{-\frac{1}{\gamma_k}} \frac{d}{dt} f_k^j(t) - \frac{d}{dt} a_k^j(t) \right) - \frac{h^j(t,x)\frac{\partial}{\partial t}\psi(t,x)}{\psi(t,x)}, \\ \Leftrightarrow \frac{\partial}{\partial t}\psi(t,x) &= \frac{\sum_{k\in\mathcal{J}} \left(\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \frac{d}{dt} f_k^j(t) - \psi(t,x)\frac{d}{dt} a_k^j(t) \right)}{h^j(t,x)}. \end{aligned}$$

Inserting this in (2.4.3) equals

$$\frac{\partial}{\partial t} U^j(t,x) = \sum_{k\in\mathcal{J}} \left(\frac{\gamma_k}{1-\gamma_k} \psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \frac{d}{dt} f_k^j(t) + \psi(t,x) \frac{d}{dt} a_k^j(t) \right). \tag{2.4.4}$$

For the partial derivative with respect to x, (2.4.1) we get

$$\frac{\partial}{\partial x} U^j(t,x) = \sum_{k\in\mathcal{J}} -\frac{1}{\gamma_k} f_k^j(t) \psi(t,x)^{-\frac{1}{\gamma_k}} \frac{\partial}{\partial x}\psi(t,x) = -h^j(t,x) \frac{\partial}{\partial x}\psi(t,x). \tag{2.4.5}$$

Similarly to the above, differentiating on both sides of (2.4.2) with respect to x gives

$$-\frac{\psi(t,x)}{h^j(t,x)} = \frac{\partial}{\partial x}\psi(t,x).$$

Inserting this in (2.4.5) gives

$$\frac{\partial}{\partial x} U^j(t,x) = \psi(t,x), \tag{2.4.6}$$

$$\frac{\partial^2}{\partial x \partial x} U^j(t,x) = \frac{\partial}{\partial x}\psi(t,x). \tag{2.4.7}$$

This result gives an expression for the partial derivatives of the total value function, where the underlying condition for the wealth allocation is crucial as this was assumed to obtain (2.4.2). The partial derivative in (2.4.6) represents variation in x, where $x = \sum_k x_k$, but this variation also includes a variation in the re-allocation of wealth among the sub-problems because $\sum_k x_k$ varies with x. By the inclusion of this re-allocation, we obtain that (2.4.6) does not depend on j. Thus, ψ is not only the marginal indirect utility for each sub-problem k, see (2.3.5), but it actually coincides with the total marginal indirect utility.

2.4.2 Hamilton-Jacobi-Bellman equation

Next, to verify our candidate value function we plug it (and its partial derivatives obtained above) into the Hamilton-Jacobi-Bellman equation, such that

$$
\begin{aligned}
0 =& \sum_{k\in\mathcal{J}} \left(\psi(t,x)\frac{d}{dt}a_k^j(t) + \frac{\gamma_k}{1-\gamma_k}\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}}\frac{d}{dt}f_k^j(t)\right) \qquad (2.4.8)\\
&+ \sup_{c^j,\pi^j,b^{jk}} \left\{ \begin{array}{l} (rx + \pi^j x(\alpha - r) + \sum_{k\in\mathcal{J}}(Y^k(t) - c^k)\mathbb{1}_{\{k=j\}} \\ - \sum_{k\in\mathcal{J}} \mu^{*jk}(t) b^{jk}\mathbb{1}_{\{k\neq j\}})\psi(t,x) \\ +\frac{1}{2}(\pi^j)^2 x^2\sigma^2 \frac{\partial}{\partial x}\psi(t,x) + \sum_{k\in\mathcal{J}} u^k(t,c^k)\mathbb{1}_{\{k=j\}} \\ + \sum_{k\in\mathcal{J}} \mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}\Big(U^k(t,x+b^{jk}) - U^j(t,x)\Big) \end{array}\right\},\\
0 =& U^j(T,x).
\end{aligned}
$$

The optimal controls are found as the controls that attain the supremum, i.e.

$$
\begin{aligned}
\arg\sup\nolimits_{c^j,\pi^j,b^{jk}} \Bigg\{ &(rx + \pi^j x(\alpha - r) + \sum_{k\in\mathcal{J}}(Y^k(t) - c^k)\mathbb{1}_{\{k=j\}} \qquad (2.4.9)\\
&- \sum_{k\in\mathcal{J}} \mu^{*jk}(t) b^{jk}\mathbb{1}_{\{k\neq j\}})\psi(t,x)\\
&+ \frac{1}{2}(\pi^j)^2 x^2\sigma^2 \frac{\partial}{\partial x}\psi(t,x) + \sum_{k\in\mathcal{J}} u^k(t,c^k)\mathbb{1}_{\{k=j\}}\\
&+ \sum_{k\in\mathcal{J}} \mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}\Big(U^k(t,x+b^{jk}) - U^j(t,x)\Big)\Bigg\}.
\end{aligned}
$$

Before solving the equation, we need to investigate and elaborate on some parts. First, we reconsider the allocation of the insurance benefit into sub-problems. As discussed in Section 3, we restrict ourselves to allocating benefits received upon transition into state k, b^{jk}, to the wealth of the kth sub-problem. This means that this lump sum affects sub-problem k only. Looking at the value function $U^k(t,x)$, we can write it as (2.4.1) with the allocated wealth to each sub-problem and include the benefit in the kth sub-problem,

$$U^k(t, x + b^{*jk}) = U_k^k(t, x_k + b^{*jk}) + \sum_{l:l\neq k} U_l^k(t, x_l). \qquad (2.4.10)$$

We remind the reader that this allocation of insurance lump sum payments is a choice we make. One could take a different distribution and, most importantly, a different allocation of initial wealth. However, as mentioned earlier we just need to point at one decomposition and the related optimal solution, not all decompositions. The allocation of the state k-insurance benefits to the state-k sub-problem is mathematically elegant since it relieves us from carrying around with the cumbersome notation

b_l^{jk} (since it is zero for $l \neq k$) and the insurance lump sum b^{jk} then only appears in a single sub-problem value function. Further, we investigate (2.4.9) in order to find an expression for the optimal controls. Deriving wrt. the insurance benefit in order to find optimal control b^{*jk} and using the chosen allocation as described in (2.4.10) we have the first-order condition

$$\mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}\frac{\partial}{\partial x_k}U_k^k(t,x_k+b^{jk}) - \mu^{*jk}(t)\mathbb{1}_{\{k\neq j\}}\psi(t,x) = 0. \tag{2.4.11}$$

The first part could also be seen as $\mu^{jk}(t)\psi(t,x+b^{jk})$ since by (2.3.5) the marginal indirect utility is $\psi(t,x) = \frac{\partial}{\partial x_k}U_k^j(t,x_k)$ for all $k \in \mathcal{J}$. This allocation of the lump sum payment is a choice as previously described, and the above calculation reflects the decision to allocate b^{jk} fully to sub-problem k. We insert the value function from (2.3.4) and derive it with respect to the second parameter

$$\frac{\partial}{\partial x_k}U_k^k(t,x_k+b^{jk}) = f_k^k(t)^{\gamma_k}(x_k+a_k^k(t)+b^{jk})^{-\gamma_k}.$$

Altogether the above calculations result in

$$b^{*jk}(t,x) = \left(\frac{\mu^{*jk}(t)}{\mu^{jk}(t)}\right)^{-\frac{1}{\gamma_k}}\psi(t,x)^{-\frac{1}{\gamma_k}}f_k^k(t) - (x_k+a_k^k(t)), \qquad \text{for all } k\neq j.$$

Remember $x_k = x - \sum_{l:l\neq k}x_l$, we present the optimal controls in (2.4.9) in the following way for all $j \in \mathcal{J}$

$$\begin{aligned}
c^{*j}(t,x) &= g^j(t)\psi(t,x)^{-\frac{1}{\gamma_j}}.\\
\pi^{*j}(t,x) &= -\frac{\alpha-r}{\sigma^2}\frac{\psi(t,x)}{x\frac{\partial}{\partial x}\psi(t,x)}.\\
b^{*jk}(t,x) &= \left(\frac{\mu^{*jk}(t)}{\mu^{jk}(t)}\right)^{-\frac{1}{\gamma_k}}\psi(t,x)^{-\frac{1}{\gamma_k}}f_k^k(t)\\
&\quad -(x+a_k^k(t)-\textstyle\sum_{l:l\neq k}x_l), \qquad \text{for all } k\neq j.
\end{aligned}$$

The results for c^j and π^j are obtained in a similar way as the optimal insurance payment, and these calculations are standard for consumption-investment problems. To make sure we have an optimum, we take the second derivative of the inner part of (2.4.9) with respect to the controls. The derivation for c and π are standard, but for the insurance sum we find, by differentiating the left-hand side of (2.4.11) and plugging in the candidate control,

$$\begin{aligned}
&\frac{\partial}{\partial b^{jk}}\Big(\mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}\frac{\partial}{\partial x_k}U_k^k(t,x_k+b^{*jk}) - \mu^{*jk}(t)\mathbb{1}_{\{k\neq j\}}\psi(t,x)\Big)\\
&= \frac{\partial}{\partial b^{jk}}\Big(\mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}f_k^k(t)^{\gamma_k}(x_k+b^{*jk}+a_k^k(t))^{-\gamma_k} - \mu^{*jk}(t)\mathbb{1}_{\{k\neq j\}}\psi(t,x)\Big)\\
&= -\gamma_k\mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}f_k^k(t)^{\gamma_k}(x_k+b^{*jk}+a_k^k(t))^{-\gamma_k-1} < 0.
\end{aligned}$$

With a negative second derivate, we have an optimum. Later in this section we elaborate on the optimal control strategies without the marginal indirect utility function $\psi(t,x)$, since it is for intermediate calculations mainly. This means the final results do not depend on $\psi(t,x)$ but until our candidate value function is verified as the optimal value function we continue with $\psi(t,x)$ present. With the optimal controls defined we are almost ready to solve the Hamilton-Jacobi-Bellman equation, but before, we need to rewrite the last term from (2.4.8). Therefore, with the design in (2.4.10) we first see

$$\begin{aligned}
&\sum_{k\in\mathcal{J}} \mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}\Big(U^k(t,x+b^{jk}) - U^j(t,x)\Big)\\
&=\sum_{k\in\mathcal{J}} \mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}\left(U_k^k(t,x_k+b^{jk}) + \sum_{l\in\mathcal{J}} U_l^k(t,x_l)\mathbb{1}_{\{l\neq k\}} - \sum_{l\in\mathcal{J}} U_l^j(t,x_l)\right).
\end{aligned}$$

If we gather the last two terms and, on these terms, interchange the order of summation we get

$$\begin{aligned}
&\sum_{k\in\mathcal{J}} \mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}\left(U_k^k(t,x_k+b^{jk}) + \sum_{l\in\mathcal{J}} U_l^k(t,x_l)\mathbb{1}_{\{l\neq k\}} - \sum_{l\in\mathcal{J}} U_l^j(t,x_l)\right)\\
&=\sum_{k\in\mathcal{J}} \mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}U_k^k(t,x_k+b^{jk})\\
&+\sum_{k\in\mathcal{J}}\sum_{l\in\mathcal{J}} \mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}\Big(U_l^k(t,x_l)\mathbb{1}_{\{l\neq k\}} - U_l^j(t,x_l)\Big)\\
&=\sum_{k\in\mathcal{J}} \mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}U_k^k(t,x_k+b^{jk})\\
&+\sum_{k\in\mathcal{J}}\sum_{l\in\mathcal{J}} \mu^{jl}(t)\mathbb{1}_{\{l\neq j\}}\Big(U_k^l(t,x_k)\mathbb{1}_{\{k\neq l\}} - U_k^j(t,x_k)\Big)\\
&=\sum_{k\in\mathcal{J}} \mu^{jk}(t)\mathbb{1}_{\{k\neq j\}}U_k^k(t,x_k+b^{jk}) \qquad (2.4.12)\\
&+\sum_{k\in\mathcal{J}}\sum_{l\in\mathcal{J}} \mu^{jl}(t)\mathbb{1}_{\{l\neq j\}}\left(\frac{1}{1-\gamma_k}f_k^l(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}}\mathbb{1}_{\{k\neq l\}} - \frac{1}{1-\gamma_k}f_k^j(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}}\right).
\end{aligned}$$

This way of rewriting is possible because the lump sum b^{jk} is fully allocated to the kth sub-problem. Now everything is prepared to solve the Hamilton-Jacobi-Bellman equation. We insert the expression (2.4.12) and the optimal controls into the Hamilton-Jacobi-Bellman equation to get

$$
\begin{aligned}
0 = & \sum_{k\in\mathcal{J}} \left(\frac{\gamma_k}{1-\gamma_k} \psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \frac{d}{dt} f_k^j(t) + \psi(t,x) \frac{d}{dt} a_k^j(t) \right) \\
& + rx\psi(t,x) - \frac{1}{2} \frac{(\alpha-r)^2}{\sigma^2} \frac{\psi(t,x)^2}{\frac{\partial}{\partial x}\psi(t,x)} + \sum_{k\in\mathcal{J}} Y^k(t)\psi(t,x)\mathbb{1}_{\{k=j\}} \\
& + \sum_{k\in\mathcal{J}} \frac{\gamma_k}{1-\gamma_k} g^k(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \mathbb{1}_{\{k=j\}} \\
& - \sum_{k\in\mathcal{J}} \mu^{*jk}(t)^{\frac{\gamma_k}{1-\gamma_k}} \mu^{jk}(t)^{\frac{1}{\gamma_k}} f_k^k(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \mathbb{1}_{\{k\neq j\}} \\
& + \sum_{k\in\mathcal{J}} \mu^{*jk}(t)\mathbb{1}_{\{k\neq j\}}(x + a_k^k(t) - \sum_{l:l\neq k} x_l)\psi(t,x) \\
& + \sum_{k\in\mathcal{J}} \frac{1}{1-\gamma_k} \mu^{*jk}(t)^{\frac{\gamma_k}{1-\gamma_k}} \mu^{jk}(t)^{\frac{1}{\gamma_k}} f_k^k(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \mathbb{1}_{\{k\neq j\}} \\
& + \sum_{k\in\mathcal{J}} \sum_{l\in\mathcal{J}} \mu^{jl}(t)\mathbb{1}_{\{l\neq j\}} \Big(\frac{1}{1-\gamma_k} f_k^l(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \mathbb{1}_{\{k\neq l\}} \\
& \qquad\qquad - \frac{1}{1-\gamma_k} f_k^j(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \Big), \\
0 = & \, U^j(T,x).
\end{aligned}
\tag{2.4.13}
$$

The solution to the equation (2.4.13), can be expressed as the solution to the following equations

$$
\begin{aligned}
\frac{d}{dt} f_i^j(t) &= \tilde{r}_i^j(t) f_i^j(t) - \sum_{l:l\neq j} \tilde{\mu}_i^{jl}(t) \left(f_i^l(t) - f_i^j(t) \right) - g^i(t)\mathbb{1}_{\{i=j\}}, \\
f_i^j(T) &= 0.
\end{aligned}
$$

With $\tilde{\mu}_i^{jl}$ ans $\tilde{r}_i^j$ defined as

$$
\begin{aligned}
\tilde{\mu}_i^{jl}(t) =& \mu^{*jl}(t)^{\frac{\gamma_i-1}{\gamma_i}} \mu^{jl}(t)^{\frac{1}{\gamma_i}} \mathbb{1}_{\{l=i\}} + \frac{1}{\gamma_i} \left((\gamma_i - 1)\mu^{*jl}(t) + \mu^{jl}(t) \right) \mathbb{1}_{\{l\neq i\}}, \\
\tilde{r}_i^j(t) =& \frac{\gamma_i - 1}{\gamma_i}(r + \sum_{l:l\neq j} \mu^{*jl}(t)) + \sum_{l:l\neq j} \mu^{jl}(t)\frac{1}{\gamma_i} - \sum_{l\in\mathcal{J}} \tilde{\mu}_i^{jl}(t) + \left(\frac{\alpha-r}{\sigma}\right)^2 \frac{\gamma_i-1}{2\gamma_i^2}.
\end{aligned}
$$

The derivation is presented in Appendix A. Note here that the indicator function elegantly indicates that the utility weight of state i appears only in the sub-problem i and only in (the differential equation for) the state-wise f corresponding to exactly that same state i. Also, note the interpretation of $\tilde{\mu}$ and $\tilde{r}$. The intensities $\tilde{\mu}$ are formed as sums of a geometric and an arithmetic means of the two intensities μ and μ^*. The geometric mean is taken with respect to the transition into the state to which the sub-problem belongs. The arithmetic mean is taken with respect to all other transitions. The artificial interest rate $\tilde{r}$ contains elements from the financial market plus the impact of the multistate model. The impact from the multistate model is a difference between the arithmetic and the geometric means of the exit intensities $\mu^{j\cdot}$ and $\mu^{*j\cdot}$. The system of differential equations for the human capital a becomes

$$\frac{d}{dt}a_i^j(t) = ra_i^j(t) - \sum_{l:l\neq j} \mu^{*jl}(t)\left(a_i^l(t) - a_i^j(t)\right) - Y^i(t)\mathbb{1}_{\{i=j\}},$$
$$a_i^j(T) = 0.$$

Again, note the indicator function elegantly indicates the income in state i appears only in the sub-problem i and only in (the differential equation for) the state-wise human capital corresponding to exactly that same state i. This follows from the convention that all income in a particular state is earned for the sub-problem of that state exclusively. This concludes our verification. Since these systems of differential equations for both f and a are linear, they have unique solutions and we have, thus, found a solution to the Hamilton-Jacob-Bellman. This means the value function candidate consisting of (2.4.1) and (2.4.2) is indeed the optimal value function and we write

$$U^j(t,x) = V^j(t,x) = \sum_{k\in\mathcal{J}} \frac{1}{1-\gamma_k} f_k^j(t)^{\gamma_k}(x_k + a_k^j(t))^{1-\gamma_k},$$
$$x = \sum_{k\in\mathcal{J}} x_k.$$

Furthermore, the optimal controls formulated in (2.4.9) can as previously mentioned be written without the function ψ. First, turning to the optimal investment strategy from (2.4.5) we find

$$\pi^{*j}(t,x) = -\frac{(\alpha-r)}{\sigma^2}\frac{\psi(t,x)}{x\frac{\partial}{\partial x}\psi(t,x)} = \frac{\alpha-r}{\sigma^2}\frac{h^j(t,x)}{x}.$$

By the definition of $h^j(t,x) = \sum_{k\in\mathcal{J}} \frac{1}{\gamma_k} f_k^j(t)\psi(t,x)^{-\frac{1}{\gamma_k}}$ and by using the definition of $\psi(t,x)$ from (2.3.5) for all $j \in \mathcal{J}$ we get

$$\pi^{*j}(t,x) = \frac{\alpha-r}{\sigma^2}\frac{\sum_{k\in\mathcal{J}}\frac{1}{\gamma_k}(x_k + a_k^j(t))}{x}.$$

Second, for all $j \in \mathcal{J}$ the optimal consumption strategy in state j is by (2.3.5) given as

$$c^{*j}(t,x) = g^j(t)\psi(t,x)^{-\frac{1}{\gamma_j}} = \frac{g^j(t)}{f_j^j(t)}(x_j + a_j^j(t)).$$

Third, for the optimal insurance benefit, we look at a transition from the state j to the state k, for all possible states $k \neq j$ where the insured can jump to we have

$$b^{*jk}(t,x) = \left(\frac{\mu^{*jk}(t)}{\mu^{jk}(t)}\right)^{-\frac{1}{\gamma_k}} \psi(t,x)^{-\frac{1}{\gamma_k}} f_k^k(t) - (x_k + a_k^k(t))$$

Again by (2.3.5) for all $k \neq j \in \mathcal{J}$ we have

$$b^{*jk}(t,x) = \left(\frac{\mu^{*jk}(t)}{\mu^{jk}(t)}\right)^{-\frac{1}{\gamma_k}} \frac{f_k^k(t)}{f_k^j(t)}(x_k + a_k^j(t)) - (x_k + a_k^k(t)).$$

For each sub-problem, the system of functions (a, f) relates to the solution found by Kraft and Steffensen, 2008a as we discuss in the following section. However, solving this system for each sub-problem is in our case followed by an initial allocation of capital to the different sub-problems, and through that part relating the $x_k, k \in \mathcal{J}$, to each other, also these sub-problems become entangled. General analytical results about how the state variation of risk aversion impacts the controls are left for future work, but a concrete example where the impact can be illustrated and intuitively explained is presented in Section 2.5.

2.4.3 Comparison

We finalize this section on the verification by comparing the system of differential equations for f and a with those obtained by Kraft and Steffensen, 2008a and further specified in Kraft and Steffensen, 2008b. They also have functions f and a like ours, but things are simpler since the risk aversion γ does not depend on the state. If we let γ be constant the double-notation becomes redundant, and we are back with one single value function in the form

$$V^j(t,x) = \frac{1}{1-\gamma} f^j(t)^\gamma (x + a^j(t))^{1-\gamma}.$$

In that case, we do not have to divide the problem into sub-problems corresponding to insurance, income, and consumption for every single state. Then we can perform all the calculations above once and reach a single system of differential equations for f and a single system of differential equations for a. They become

$$\begin{aligned} \tfrac{d}{dt} f^j(t) &= \tilde{r}^j(t) f^j(t) - \sum_{l:l\neq j} \tilde{\mu}^{jl}(t)\left(f^l(t) - f^j(t)\right) - g^j(t), \quad && f^j(T) = 0, \\ \tfrac{d}{dt} a^j(t) &= r a^j(t) - \sum_{l:l\neq j} \mu^{*jl}(t)\left(a^l(t) - a^j(t)\right) - Y^j(t), && a^j(T) = 0, \end{aligned}$$

with

$$\begin{aligned} \tilde{\mu}^{jl}(t) =& \mu^{*jl}(t)^{\frac{\gamma-1}{\gamma}} \mu^{jl}(t)^{\frac{1}{\gamma}}, \\ \tilde{r}^j(t) =& \frac{\gamma-1}{\gamma}(r + \sum_{l:l\neq j} \mu^{*jl}(t)) + \sum_{l:l\neq j} \mu^{jl}(t)\frac{1}{\gamma} - \sum_{l\in\mathcal{J}} \tilde{\mu}^{jl}(t) + (\frac{\alpha - r}{\sigma})^2 \frac{\gamma-1}{2\gamma^2}. \end{aligned}$$

This is the same system as the one obtained in Kraft and Steffensen, 2008a. We note as it was done both there and in Kraft and Steffensen, 2008b that we can write f and a as certain conditional expected present values. The function f is the value of future utility weights under a measure with intensities $\tilde{\mu}$ and state-dependent interest rates $\tilde{r}$, and the function a is the financial value of future income, i.e.

$$f^j(t) = \tilde{E}_{x,j}\left[\int_t^n e^{-\int_t^s \tilde{r}(u)du} d\Upsilon(s)\right],$$
$$a^j(t) = E^*_{t,j}\left[\int_t^n e^{-r(t-s)} dA(s)\right].$$

Here Υ is the process of accumulated utility weights and $A(t)$ is the process of accumulated income. Like in the general case, we note the interpretation of the artificial parameters $\tilde{\mu}$ and $\tilde{r}$. The intensity $\tilde{\mu}$ is now a simple geometric mean of μ and μ^*. The part with the arithmetic mean from the general case has vanished because one never jumps to a state which is part of a different sub-problem. Namely, there is only one single problem, and all jumps relate to entrance into that single problem. Further, the artificial interest rate $\tilde{r}$ is again the combination of the (usual) market terms and then the multi-state market impact which is, simply, the difference between the arithmetic and the geometric means of exit transition intensities. We mention here that this interpretation of the artificial assumptions underlying f in terms of geometric and arithmetic means is mentioned by neither Kraft and Steffensen, 2008a nor Kraft and Steffensen, 2008b. Apart from giving insight, it also helps in the implementation phase.

2.5 Numerical Example

In this section, we illustrate with a numerical example the scope of our setup with state-dependent risk aversion. We illustrate the expected development over time of the wealth process and the consumption process for an individual who controls consumption, investment, and disability insurance before and after retirement. The numerical illustrations are based on the three-state model as illustrated in Appendix: 2.5, interpreted as state 0: Active; state 1: Disabled; and state 2: Dead. We assume that $\mu^{10} = 0$ which together with the obvious $\mu^{21} = \mu^{20} = 0$ leads to closed form solutions. The life insurance scheme consists of a disability sum of b^{01}, paid out upon transitioning to the disabled state 1 and then added to the wealth allocated to consumption during the disability. During disability, the individual receives no labour income, and the disability sum serves as insurance to cover the maintenance of the desired level of consumption. There is no life insurance present which is optimal since we assume that there is no utility from the bequest.

The disability intensity is defined as $\mu^{ai}(t) = \mu^{01}(t) = Ae^{(t+z)\log(B)}$, and the non-differential death intensity as $\mu^{ad}(t) = \mu^{id}(t) = \mu^{02}(t) = \mu^{12}(t) = C + 10^{D+E(t+z)-10}$ with constants defined in Table 2.1. Note that we assume so-called non-differential mortality where disability does not accelerate death. The utility functions are defined in (2.2.2), and for numerical calculations, we choose the exponential function as the time-weight function $g(t) = e^{-\rho t}$, where ρ is the utility discount factor relating the utility of payments at different points in time to each other. In all illustrations we consider an insured at age 30 at initialization who retires at age 70. We study the wealth and consumption patterns for the two cases where the individual becomes disabled at age 50 and age 80, respectively. Normally, at least in this case of non-differential mortality, getting disabled after retirement does not change the consumption pattern since the event does not influence the financial situation. However, here consumption changes because the risk aversion changes, and this holds even after retirement.

Table 2.1: *The parameters used in the numerical examples. Note r, α and σ are thought of as corrected for inflation.*

Parameters	Description	Value
z	Age at initialization	30
T	Time of retirement	40
x_0	Initial wealth	400 000
Y	Constant income rate in USD until retirement or disability	45 000
ρ	Impatience factor for all states	0.05
r	The constant drift of the risk-free asset	0.02
α	The constant drift of the risky asset	0.05
σ	The constant volatility of the risky asset	0.2
A	Parameter for mortality intensity	0.0000005
B	Parameter for mortality intensity	1.14
C	Parameter for disability intensity	0.000400
D	Parameter for disability intensity	4.58
E	Parameter for disability intensity	0.051

Figure 2.1 illustrates the benchmark case with the same risk aversion in both states, 0 and 1, namely $\gamma_0 = \gamma_1 = 2$. As in the Appendix: 2.5, the wealth is allocated at initialization for two sub-problems: One for consumption while active and another for consumption as disabled. The allocation is done such that the marginal indirect utility at $t = 0$, is the same in each state. In other words, the insured does not gain further from moving wealth between the sub-problems regarding the state active and the state disabled, respectively. Recall that the function $\psi(t, x)$ represents exactly this marginal indirect utility as

$$\psi(t,x) = \frac{\partial}{\partial x_k} V_k^j(t, x_k) = f_k^j(t)^{\gamma_k}(x_k + a_k^j(t))^{-\gamma_k}, \quad \forall k \in \mathcal{J}.$$

We denote the initial wealth by x^0, deviating from the more usual x_0 since the latter denotes the wealth allocated to sub-problem 0 at an arbitrary time point. We denote the initial allocations to the two sub-problems by x_0 and x_1 such that we have the constraint $x^0 = x_0 + x_1$. Thus, we solve the following equation for x_0

$$f_0^0(0)^{\gamma_0}(x_0 + a_0^0(0))^{-\gamma_0} = f_1^0(0)^{\gamma_1}(x^0 - x_0)^{-\gamma_1}, \qquad (2.5.1)$$

and define $x_1 = x^0 - x_0$, to obtain the optimal allocation at initialization. The figures contain expected wealth where the expectation is taken over financial risk but not over state risk. Thus, the wealth dynamics are state-wise in the state dimension but not in the financial dimension. When disability occurs, the wealth allocated to consumption as active is lost but the wealth allocated to consumption as disabled, together with the disability sum paid out, takes over financing the consumption as disabled (left). Note that, from the onset of disability even after retirement, the wealth is larger for consumption as disabled than for consumption as active although the consumption rates in the two states are the same. This is because the wealth for consumption as active only finances that consumption rate until death or disability (and pays for the disability insurance) whereas the wealth for consumption as disabled finances the same consumption rate until death.

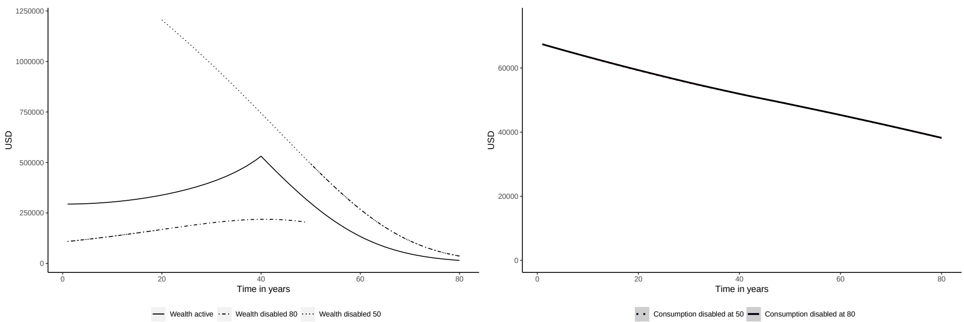

Figure 2.1: *For $\gamma_0 = \gamma_1 = 2$. Left: The expected wealth allocated for the states. Right: The optimal consumption (the two curves overlap).*

Figure 2.1 illustrates (right) that the consumption level is the same regardless of the state, resulting in the same consumption curve independent of the time of disability. The optimal strategies are computed in the Appendix. The wealth allocated to consumption as disabled follows (left) the dotted and dash-dotted lines, respectively. If disability occurs at age 50 the wealth allocated to that problem jumps to the dotted line and that jump is financed by the disability sum paid out. A similar but smaller jump in the disability wealth, also paid by the disability insurance, happens at age 80 if disability occurs then. The allocated wealth at initialization is found by solving (2.5.1), resulting in at time $t = 0$, $x_0 = \$293883.4$ and $x_1 = \$106116.6$. We are now going to investigate how splitting in different risk aversion changes the figures.

We illustrate first a higher risk aversion for the active state than the disabled state, $\gamma_0 > \gamma_1$, in Figure 2.2. The effect on the wealth allocation (left) at initialization is that a higher proportion of initial wealth is allocated to consuming as disabled than in the benchmark case, since solving (2.5.1) results in $x_0 = \$124643.6$ and $x_1 = \$275356.4$. The wealth of the disabled is higher than the wealth of the active individual no matter when disability occurs.

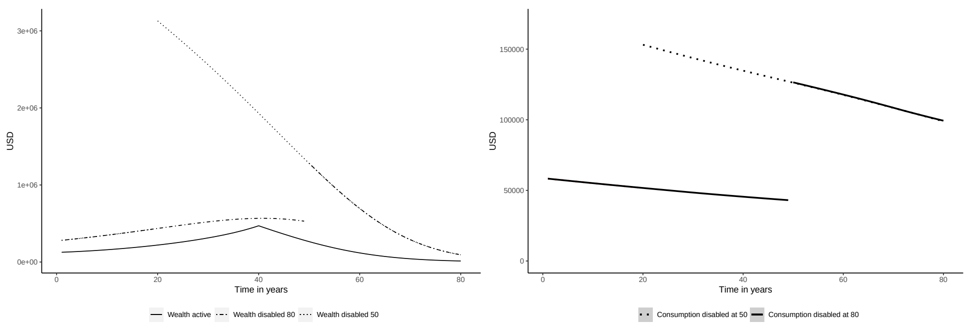

Figure 2.2: *For $\gamma_0 = 2.2$, $\gamma_1 = 2$. Left: The expected allocated wealth. Right: The optimal consumption.*

This conforms with the observation (right) that, upon disability, the consumption jumps upwards to a higher level but with a steeper slope downwards.

The steeper slope shows that the preference for a stable level of consumption is not as high during disability as it is while being active, due to the fall in risk aversion. When the need for stability is lowered upon disability, the impatience factor ρ makes the individual accelerate consumption compared to before the disability occurred. This is the case since we have that $\rho > r$, such that the individual is impatient for consumption relative to the market.

Another factor that affects the slope is that γ actually does not only represent aversion towards risk but also parametrizes the co-called Elasticity of Inter-temporal Substitution (EIS). The risk aversion expresses the willingness to gamble, and an individual with a higher risk aversion is less willing to gamble. The EIS expresses a willingness to substitute consumption over time, and an individual with a higher elasticity is more willing to substitute consumption over time. It is beyond the scope of this paper to enter further into the delicate distinction of these different meanings of the parameter γ.

If the risk aversion in the active state is lower than in the disabled state, $\gamma_0 < \gamma_1$, we find the patterns in Figure 2.3. The wealth allocated (left) for consumption as disabled is now lower than in the benchmark case in Figure 2.1, and at initialization we get from from (2.5.1) that $x_0 = \$357176.8$ and $x_1 = \$42823.2$. We even see that only if disability occurs before age 60 (roughly, when the dotted line crosses the solid line), the disabled individual holds a larger wealth (until around age 60) than the active individual of the same age. The wealth held by the disabled individual is mainly financed by the disability insurance sum paid out.

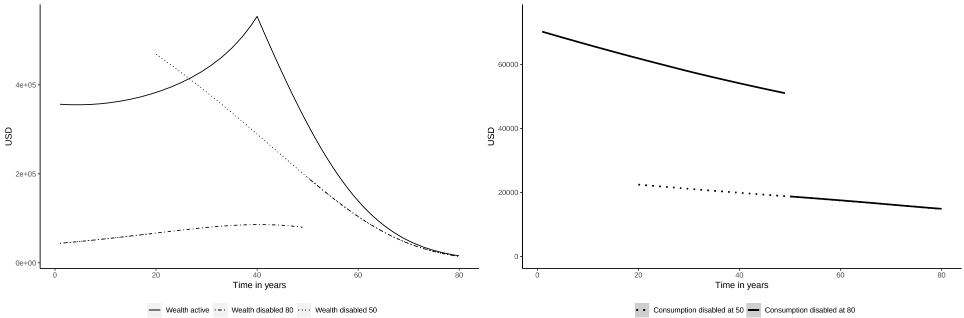

Figure 2.3: *For $\gamma_0 = 2$, $\gamma_1 = 2.2$. Left: The expected allocated wealth. Right: The optimal consumption.*

The consumption (right) begins at a higher level than in the benchmark case but has a steeper downward slope. As was the case for the disabled individual in Figure 2.2 this follows from a smaller need for stability together with impatience for consumption. Upon disability, the consumption jumps to a lower but more stable level because the risk aversion is going up.

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Appendix A

From equation (2.4.13), we rewrite the term containing $(x + a^k_k(t) - \sum_{l:l\neq k} x_l)$, by changing the order of summation and recalling that $x_k = f^j_k(t)\psi(t,x)^{-\frac{1}{\gamma_k}} - a^j_k(t)$, into

$$
\begin{aligned}
&\sum_{k\in\mathcal{J}} \mu^{*jk}(t)\mathbb{1}_{\{k\neq j\}}(x + a^k_k(t) - \sum_{l:l\neq k} x_l)\psi(t,x) \\
&= \sum_{k\in\mathcal{J}} \mu^{*jk}(t)\mathbb{1}_{\{k\neq j\}} x\psi(t,x) \\
&+ \sum_{k\in\mathcal{J}} \mu^{*jk}(t)\mathbb{1}_{\{k\neq j\}} \Big(a^k_k(t) \\
&\qquad - \sum_{l:l\neq k} (f^k_l(t)\psi(t,x)^{-\frac{1}{\gamma_l}} - a^k_l(t))\Big)\psi(t,x), \\
&= \sum_{k\in\mathcal{J}} \mu^{*jk}(t)\mathbb{1}_{\{k\neq j\}} x\psi(t,x) \\
&+ \sum_{k\in\mathcal{J}}\sum_{l\in\mathcal{J}} \mu^{*jk}(t)\mathbb{1}_{\{k\neq j\}} a^k_l(t)\psi(t,x) \\
&- \sum_{k\in\mathcal{J}}\sum_{l\in\mathcal{J}} \mu^{*jk}(t)\mathbb{1}_{\{k\neq j\}} f^k_l(t)\psi(t,x)^{\frac{\gamma_l-1}{\gamma_l}}, \\
&= \sum_{k\in\mathcal{J}} \mu^{*jk}(t)\mathbb{1}_{\{k\neq j\}} x\psi(t,x) \\
&+ \sum_{k\in\mathcal{J}}\sum_{l\in\mathcal{J}} \mu^{*jl}(t)\mathbb{1}_{\{l\neq j\}} a^l_k(t)\psi(t,x) \\
&- \sum_{k\in\mathcal{J}}\sum_{l\in\mathcal{J}} \mu^{*jl}(t)\mathbb{1}_{\{l\neq j\}} f^l_k(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}}.
\end{aligned}
\tag{2.A.1}
$$

Further, using that $\frac{\psi(t,x)}{\frac{\partial}{\partial x}\psi(t,x)} = -h^j(t,x)$ and that $x\psi(t,x) = \sum_{k\in\mathcal{J}} f_k^j(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} - a_k^j(t)\psi(t,x)$, we find that

$$
\begin{aligned}
0 = &\sum_{k\in\mathcal{J}} \left(\frac{\gamma_k}{1-\gamma_k}\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \frac{d}{dt} f_k^j(t) + \psi(t,x)\frac{d}{dt} a_k^j(t) \right) \\
&+ \sum_{k\in\mathcal{J}} (r + \sum_{l:l\neq j} \mu^{*jl}(t)) \left(f_k^j(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} - a_k^j(t)\psi(t,x) \right) \\
&+ \sum_{k\in\mathcal{J}} \frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2}\frac{1}{\gamma_k} f_k^j(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \\
&+ \sum_{k\in\mathcal{J}} \frac{\gamma_k}{1-\gamma_k} g^k(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \mathbb{1}_{\{k=j\}} + \sum_{k\in\mathcal{J}} Y^k(t)\psi(t,x)\mathbb{1}_{\{k=j\}} \\
&+ \sum_{k\in\mathcal{J}}\sum_{l\in\mathcal{J}} \mu^{*jl}(t)\mathbb{1}_{\{l\neq j\}} a_k^l(t)\psi(t,x) \\
&- \sum_{k\in\mathcal{J}}\sum_{l\in\mathcal{J}} \mu^{*jl}(t)\mathbb{1}_{\{l\neq j\}} f_k^l(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \mathbb{1}_{\{l\neq k\}} \\
&+ \sum_{k\in\mathcal{J}} \frac{\gamma_k}{1-\gamma_k} \mu^{*jk}(t) \Big(\frac{\mu^{*jk}(t)}{\mu^{jk}(t)} \Big)^{-\frac{1}{\gamma_k}} \psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} f_k^k(t)\mathbb{1}_{\{k\neq j\}} \\
&+ \sum_{k\in\mathcal{J}}\sum_{l:l\neq j} \mu^{jl}(t) \Big(\frac{1}{1-\gamma_k} f_k^l(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \mathbb{1}_{\{k\neq l\}} \\
&\qquad - \frac{1}{1-\gamma_k} f_k^j(t)\psi(t,x)^{\frac{\gamma_k-1}{\gamma_k}} \Big), \qquad (2.A.2) \\
0 = &V^j(T,x).
\end{aligned}
$$

We realize that every term in this expression is a sum over $k \in \mathcal{J}$. If we now take each element in the sum to be zero, also the sum of these elements over $k \in \mathcal{J}$ is zero. This creates a separate system of differential equations for f and a, respectively, related to each sub-problem. The system reflects that even for the sub-problem k we have a list of state-wise functions f and a corresponding to the different states, corresponding to the double state-scripting of both f and a. First, we consider the system for f and point out the system for the sub-problem i,

$$\begin{aligned}
\frac{d}{dt}f_i^j(t) = &\Big(\frac{\gamma_i-1}{\gamma_i}(r+\sum_{l:l\neq j}\mu^{*jl}(t))+\frac{(\alpha-r)^2}{\sigma^2}\frac{\gamma_i-1}{2\gamma_i^2}+\sum_{l:l\neq j}\frac{\mu^{jl}(t)}{\gamma_i}\Big)f_i^j(t)\\
&-g^i(t)\mathbb{1}_{\{i=j\}}-\mu^{*ji}(t)^{\frac{\gamma_i-1}{\gamma_i}}\mu^{ji}(t)^{\frac{1}{\gamma_i}}f_i^i(t)\mathbb{1}_{\{i\neq j\}}\\
&-\sum_{l\in\mathcal{J}}\frac{1}{\gamma_i}\left((\gamma_i-1)\mu^{*jl}(t)+\mu^{jl}(t)\right)\mathbb{1}_{\{l\neq j\}}f_i^l(t)\mathbb{1}_{\{l\neq i\}},\\
&=\Bigg(\frac{\gamma_i-1}{\gamma_i}(r+\sum_{l:l\neq j}\mu^{*jl}(t))+\frac{(\alpha-r)^2}{\sigma^2}\frac{\gamma_i-1}{2\gamma_i^2}+\sum_{l:l\neq j}\frac{\mu^{jl}(t)}{\gamma_i}\\
&-\sum_{l:l\neq j}\Big(\mu^{*jl}(t)^{\frac{\gamma_l-1}{\gamma_l}}\mu^{jl}(t)^{\frac{1}{\gamma_l}}\mathbb{1}_{\{l=i\}}\\
&+\frac{1}{\gamma_i}\left((\gamma_i-1)\mu^{*jl}(t)+\mu^{jl}(t)\right)\mathbb{1}_{\{l\neq i\}}\Big)\Bigg)f_i^j(t)\\
&-g^i(t)\mathbb{1}_{\{i=j\}}-\sum_{l:l\neq j}\Big(\mu^{*jl}(t)^{\frac{\gamma_l-1}{\gamma_l}}\mu^{jl}(t)^{\frac{1}{\gamma_l}}\mathbb{1}_{\{l=i\}}\\
&+\frac{1}{\gamma_i}\left((\gamma_i-1)\mu^{*jl}(t)+\mu^{jl}(t)\right)\mathbb{1}_{\{l\neq i\}}\Big)\left(f_i^l(t)-f_i^j(t)\right),\\
f_i^j(T) = &\,0.
\end{aligned}$$

To reach a tighter notation, we define

$$\begin{aligned}
\tilde{\mu}_i^{jl}(t) =&\mu^{*jl}(t)^{\frac{\gamma_i-1}{\gamma_i}}\mu^{jl}(t)^{\frac{1}{\gamma_i}}\mathbb{1}_{\{l=i\}}+\frac{1}{\gamma_i}\left((\gamma_i-1)\mu^{*jl}(t)+\mu^{jl}(t)\right)\mathbb{1}_{\{l\neq i\}},\\
\tilde{r}_i^j(t) =&\frac{\gamma_i-1}{\gamma_i}(r+\sum_{l:l\neq j}\mu^{*jl}(t))+\sum_{l:l\neq j}\mu^{jl}(t)\frac{1}{\gamma_i}-\sum_{l\in\mathcal{J}}\tilde{\mu}_i^{jl}(t)+(\frac{\alpha-r}{\sigma})^2\frac{\gamma_i-1}{2\gamma_i^2}.
\end{aligned}$$

This simplifies the differential equation for f to

$$\begin{aligned}
&\frac{d}{dt}f_i^j(t)=\tilde{r}_i^j(t)f_i^j(t)-\sum_{l:l\neq j}\tilde{\mu}_i^{jl}(t)\left(f_i^l(t)-f_i^j(t)\right)-g^i(t)\mathbb{1}_{\{i=j\}},\\
&f_i^j(T)=0.
\end{aligned}$$

The system of differential equations for a becomes

$$\begin{aligned}
&\frac{d}{dt}a_i^j(t)=ra_i^j(t)-\sum_{l:l\neq j}\mu^{*jl}(t)\left(a_i^l(t)-a_i^j(t)\right)-Y^i(t)\mathbb{1}_{\{i=j\}},\\
&a_i^j(T)=0.
\end{aligned}$$

Appendix B

Example of Calculations

In this section, we focus on the three-state model illustrated in Figure 1. Repeating the core calculations of the verification, in that case, serves two purposes. First, the derivation is fruitful as a confirmation of the more general result because the notation without summation over states can be more reader-friendly. Second, the results of the three-state model are the formulas underlying the numerical illustration in the following Section.

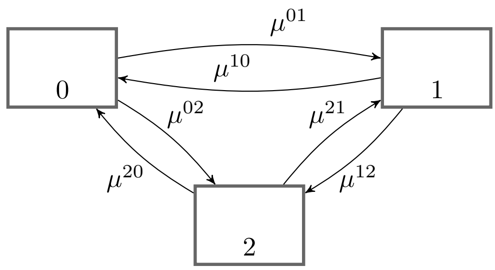

Figure 2.4: *Example of state space with three states, with the true mortality intensities.*

In the mode illustrated in Figure 1, the value function and the implicit function for the marginal indirect utility are described as

$$\begin{aligned}V^{Z(t)}(t,x) &= V_0^{Z(t)}(t,x_0) + V_1^{Z(t)}(t,x_1) + V_2^{Z(t)}(t,x_2),\\ x &= f_0^{Z(t)}(t)\psi(t,x)^{-\frac{1}{\gamma_0}} - a_0^{Z(t)}(t) + f_1^{Z(t)}(t)\psi(t,x)^{-\frac{1}{\gamma_1}} - a_1^{Z(t)}(t)\\ &\quad + f_2^{Z(t)}(t)\psi(t,x)^{-\frac{1}{\gamma_2}} - a_2^{Z(t)}(t). \end{aligned} \tag{2.B.1}$$

The candidate value function is given by $V_i^{Z(t)}(t,x_i) = \frac{1}{1-\gamma_i} f_i^{Z(t)}(t)\psi(t,x)^{\frac{\gamma_i-1}{\gamma_i}}$ for $i = 0,1,2$. The dynamics of the total wealth for $t \in [0,T]$ are given in terms of the stochastic differential equation,

$$\begin{aligned}dX(t) = \Big(& rX(t) + (\pi^0(t)X(t)(\alpha - r) + Y^0(t)\\ &- c^0(t) - \mu^{*01}(t)b^{01} - \mu^{*02}(t)b^{02})\mathbb{1}_{\{Z(t)=0\}}\\ &+ (\pi^1(t)X(t)(\alpha - r) + Y^1(t) - c^1(t) - \mu^{*10}(t)b^{10} - \mu^{*12}(t)b^{12})\mathbb{1}_{\{Z(t)=1\}}\\ &+ (\pi^2(t)X(t)(\alpha - r) + Y^2(t) - c^2(t) - \mu^{*21}(t)b^{21} - \mu^{*20}(t)b^{20})\mathbb{1}_{\{Z(t)=2\}}\Big)dt\\ &+ (b^{01}dN^{01}(t) + b^{02}dN^{02}(t)) + (b^{12}dN^{12}(t) + b^{10}dN^{10}(t))\\ &+ (b^{21}dN^{21}(t) + b^{20}dN^{20}(t)) + \sigma\pi^0(t)X(t)dW(t)\mathbb{1}_{\{Z(t)=0\}}\\ &+ \sigma\pi^1(t)X(t)dW(t)\mathbb{1}_{\{Z(t)=1\}} + \sigma\pi^2(t)X(t)dW(t)\mathbb{1}_{\{Z(t)=2\}}.\end{aligned}$$

We specify the Hamilton-Jacobi-Bellman equation for state 0, i.e., conditional on the policyholder being in state 0. This corresponds to the top-script 0. Note that the subscripts 1 and 2 appear several times, as all three sub-problems corresponding to income, consumption, and insurance of jumps into each state are all relevant to a policyholder in state 0. The Hamilton-Jacobi-Bellman equation is similar if the policyholder is in states 1 and 2, with (some of) the top-scripts replaced accordingly. We have

$$
\begin{aligned}
0 = &\frac{\gamma_0}{1-\gamma_0}\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}}\frac{d}{dt}f_0^0(t) + \psi(t,x)\frac{d}{dt}a_0^0(t) + \frac{\gamma_1}{1-\gamma_1}\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}}\frac{d}{dt}f_1^0(t) \\
&+ \psi(t,x)\frac{d}{dt}a_1^0(t) + \frac{\gamma_2}{1-\gamma_2}\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}}\frac{d}{dt}f_2^0(t) + \psi(t,x)\frac{d}{dt}a_2^0(t) \\
&+ \sup_{c^0,\pi^0,b^{01},b^{02}}\Bigg\{ \begin{array}{l} (rx + \pi^0 x(\alpha - r) + Y^0(t) - c^0 - \mu^{*01}(t)b^{01} - \mu^{*02}(t)b^{02})\psi(t,x) \\
-\frac{1}{2}(\pi^0)^2 x^2 \frac{\partial}{\partial x}\psi(t,x) + u^0(t,c^0) \\
+\mu^{01}(t)\Big(V_1^1(t,x_1+b^{01}) + V_2^1(t,x_2) + V_0^1(t,x_0) \\
-V_0^0(t,x_0) - V_1^0(t,x_1) - V_2^0(t,x_2)\Big) \\
+\mu^{02}(t)\Big(V_2^2(t,x_2+b^{02}) + V_1^2(t,x_1) + V_0^2(t,x_0) \\
-V_0^0(t,x_0) - V_1^0(t,x_1) - V_2^0(t,x_2)\Big)\Bigg\}. \end{array} \\
0 = &V^0(T,x).
\end{aligned}
$$

Using $x_i = x - \sum_{k:k\neq i} x_k$, for $i = 1,2,3$ we find the optimal controls, conditional on being in state 0, as

$$
\begin{aligned}
c^{*0}(t,x) &= g^0(t)\psi(t,x)^{-\frac{1}{\gamma_0}}, \qquad \pi^{*0}(t,x) = -\frac{(\alpha-r)}{\sigma^2}\frac{\psi(t,x)}{x\frac{\partial}{\partial x}\psi(t,x)}, \\
b^{*01}(t,x) &= \left(\frac{\mu^{*01}(t)}{\mu^{01}(t)}\right)^{-\frac{1}{\gamma_1}}\psi(t,x)^{-\frac{1}{\gamma_1}}f_1^1(t) - (a_1^1(t) + x - x_2 - x_0), \\
b^{*02}(t,x) &= \left(\frac{\mu^{*02}(t)}{\mu^{02}(t)}\right)^{-\frac{1}{\gamma_2}}\psi(t,x)^{-\frac{1}{\gamma_2}}f_2^2(t) - (a_2^2(t) + x - x_1 - x_0).
\end{aligned}
$$

By inserting the optimal controls into the Hamilton-Jacobi-Bellman equation we find,

$$
\begin{aligned}
0 = &\frac{\gamma_0}{1-\gamma_0}\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}}\frac{d}{dt}f_0^0(t) + \psi(t,x)\frac{d}{dt}a_0^0(t) + \frac{\gamma_1}{1-\gamma_1}\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}}\frac{d}{dt}f_1^0(t)\\
&+ \psi(t,x)\frac{d}{dt}a_1^0(t) + \frac{\gamma_2}{1-\gamma_2}\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}}\frac{d}{dt}f_2^0(t) + \psi(t,x)\frac{d}{dt}a_2^0(t)\\
&+ rx\psi(t,x) - \frac{(\alpha-r)^2}{2\sigma^2}\frac{\psi(t,x)^2}{\frac{\partial}{\partial x}\psi(t,x)} + Y^0(t)(\psi(t,x) + \frac{\gamma_0}{1-\gamma_0}g^0(t)\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}}\\
&- \mu^{*01}(t)^{\frac{\gamma_1-1}{\gamma_1}}\mu^{01}(t)^{\frac{1}{\gamma_1}}f_1^1(t)\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}} - \mu^{*02}(t)^{\frac{\gamma_2-1}{\gamma_2}}\mu^{02}(t)^{\frac{1}{\gamma_2}}f_2^2(t)\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}}\\
&+ \mu^{*01}(t)(x-x_0-x_2+a_1^1(t))\psi(t,x) + \mu^{*02}(t)(x-x_0-x_1+a_2^2(t))\psi(t,x)\\
&+ \frac{1}{1-\gamma_1}\mu^{*01}(t)^{\frac{\gamma_1-1}{\gamma_1}}\mu^{01}(t)^{\frac{1}{\gamma_1}}f_1^1(t)\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}}\\
&+ \frac{1}{1-\gamma_2}\mu^{*02}(t)^{\frac{\gamma_2-1}{\gamma_2}}\mu^{02}(t)^{\frac{1}{\gamma_2}}f_2^2(t)\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}}\\
&+ \mu^{01}(t)V_0^1(t,x_0) + \mu^{02}(t)V_0^2(t,x_0) + \mu^{01}(t)V_2^1(t,x_2) + \mu^{02}(t)V_1^2(t,x_1)\\
&- (\mu^{01}(t)+\mu^{02}(t))\Big(\frac{1}{1-\gamma_0}f_0^0(t)\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}}\\
&+ \frac{1}{1-\gamma_1}f_1^0(t)\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}} + \frac{1}{1-\gamma_2}f_2^0(t)\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}}\Big).\\
0 = &V^0(T,x).
\end{aligned}
$$

Note that we have changed the order of summation to illustrate clearly the similar change of order of summation in the general case. Further we use that $x_i =$

$f_i^j(t)\psi(t,x)^{-\frac{1}{\gamma_i}} - a_i^j$, for all $i \in \{0,1,2\}$.

$$\begin{aligned}
0 &= \frac{\gamma_0}{1-\gamma_0}\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}}\frac{d}{dt}f_0^0(t) + \psi(t,x)\frac{d}{dt}a_0^0(t) + \frac{\gamma_1}{1-\gamma_1}\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}}\frac{d}{dt}f_1^0(t)\\
&+ \psi(t,x)\frac{d}{dt}a_1^0(t) + \frac{\gamma_2}{1-\gamma_2}\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}}\frac{d}{dt}f_2^0(t) + \psi(t,x)\frac{d}{dt}a_2^0(t)\\
&+ (r + \mu^{*01}(t) + \mu^{*02}(t))\Big(f_0^0(t)\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}} - a_0^0(t) + f_1^0(t)\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}} - a_1^0(t)\\
&+ f_2^0(t)\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}} - a_2^0(t)\Big)\\
&+ \frac{(\alpha-r)^2}{2\sigma^2}\left(\frac{1}{\gamma_0}f_0^0(t)\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}} + \frac{1}{\gamma_1}f_1^0(t)\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}} + \frac{1}{\gamma_2}f_2^0(t)\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}}\right)\\
&+ Y^0(t)(\psi(t,x) + \frac{\gamma_0}{1-\gamma_0}g^0(t)\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}}\\
&+ \frac{\gamma_1}{1-\gamma_1}\mu^{*01}(t)^{\frac{\gamma_1-1}{\gamma_1}}\mu^{01}(t)^{\frac{1}{\gamma_1}}f_1^1(t)\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}}\\
&+ \frac{\gamma_2}{1-\gamma_2}\mu^{*02}(t)^{\frac{\gamma_2-1}{\gamma_2}}\mu^{02}(t)^{\frac{1}{\gamma_2}}f_2^2(t)\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}}\\
&- \mu^{*01}(t)(f_0^1(t)\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}} + f_2^1(t)\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}})\\
&+ \mu^{*01}(t)(a_1^1(t) + a_0^1(t) + a_2^1(t))\psi(t,x)\\
&- \mu^{*02}(t)(f_0^2(t)\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}} + f_1^2(t)\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}})\\
&+ \mu^{*02}(t)(a_2^2(t) + a_0^2(t) + a_1^2(t))\psi(t,x)\\
&+ \mu^{01}(t)\frac{1}{1-\gamma_0}f_0^1(t)\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}} + \mu^{02}(t)\frac{1}{1-\gamma_0}f_0^2(t)\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}}\\
&+ \mu^{01}(t))\frac{1}{1-\gamma_2}f_2^1(t)\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}} + \mu^{02}(t))\frac{1}{1-\gamma_1}f_1^2(t)\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}}\\
&- (\mu^{01}(t) + \mu^{02}(t))\Big(\frac{1}{1-\gamma_0}f_0^0(t)\psi(t,x)^{\frac{\gamma_0-1}{\gamma_0}} + \frac{1}{1-\gamma_1}f_1^0(t)\psi(t,x)^{\frac{\gamma_1-1}{\gamma_1}}\\
&\qquad + \frac{1}{1-\gamma_2}f_2^0(t)\psi(t,x)^{\frac{\gamma_2-1}{\gamma_2}}\Big).\\
0 &= V^0(T,x).
\end{aligned}$$

Using the notation of $\tilde{r}^{ji}$ and $\tilde{\mu}_i^{jl}$, the solution can be expressed as the solution to following differential equations for f and a, respectively,

$$\begin{aligned}
&\tfrac{d}{dt}f_0^0(t) = f_0^0(t)\tilde{r}^{00}(t) - g^0(t) - \textstyle\sum_{k:k\neq 0}\tilde{\mu}_0^{0k}(t)(f_0^k(t) - f_0^0(t)), && f_0^0(T) = 0,\\
&\tfrac{d}{dt}f_1^0(t) = f_1^0(t)\tilde{r}^{01}(t) - \textstyle\sum_{k:k\neq 0}\tilde{\mu}_1^{0k}(t)(f_1^k(t) - f_1^0(t)), && f_1^0(T) = 0,\\
&\tfrac{d}{dt}f_2^0(t) = f_2^0(t)\tilde{r}^{02}(t) - \textstyle\sum_{k:k\neq 0}\tilde{\mu}_2^{0k}(t)(f_2^k(t) - f_2^0(t)), && f_2^0(T) = 0,\\
&\tfrac{d}{dt}a_0^0(t) = ra_0^0(t) - \textstyle\sum_{k:k\neq 0}\mu^{*0k}(t)(a_0^k(t) - a_0^0(t)) - Y^0(t), && a_0^0(T) = 0,\\
&\tfrac{d}{dt}a_1^0(t) = ra_1^0(t) - \textstyle\sum_{k:k\neq 0}\mu^{*0k}(t)(a_1^k(t) - a_1^0(t)), && a_1^0(T) = 0,\\
&\tfrac{d}{dt}a_2^0(t) = ra_2^0(t) - \textstyle\sum_{k:k\neq 0}\mu^{*0k}(t)(a_2^k(t) - a_2^0(t)), && a_2^0(T) = 0.
\end{aligned}$$

The differential equations for f are three three-dimensional system of differential equations, namely one system consisting of (f_0^0, f_0^1, f_0^2), another of (f_1^1, f_1^0, f_1^2), and the last system of (f_2^2, f_2^1, f_2^0). The three systems are not intertwined, but each system must be solved numerically as a three-dimensional system of the ordinary differential equation.

If; however, we consider the special case where one the policyholder does not return to a left state the numerical calculation becomes easier as then the three-dimensional system reduces to three one-dimensional ordinary differential equations that can be solved one at a time. This is what occurs if *"recovery"* intensities are zero, $\mu^{20} = \mu^{10} = \mu^{21} = 0$. This is the model implemented in the numerical illustration below.

The intuitive interpretation of the f functions is that f_i^j measures how important it is to consume in state i in the future given being in state j today. The jump from state j to state i occur with preference-weighted-intensity $\tilde{\mu}_i^{jl}$ or passing through another state l.

Chapter 3

A State-Dependent Approach to Optimal Consumption, Investment, and Life Insurance by Risk-Adjusted Utilities

Abstract

By incorporating state dependence, we investigate optimal consumption, investment, and life insurance problems. State dependence typically expands state variables, leading to a more intricate problem. We work with an approach to state dependence that streamlines the problem rather than complicates the results and solutions. The idea is to quantify the financial worth of payments and compare these. Instead of evaluating the utility and, consequently, the moral value of the money, we take cues from derivative pricing techniques and compare payments by price or financial value. Within the financial state dependence, we further include the risk associated with life insurance and allow the agent to factor in the attitude to risk that the economic and insurance markets have established before making their decisions on life insurance, investments, and consumption. In the risk-adjusted framework, we solve the problem using dynamic programming and interpret solutions to the optimal consumption, investment, and life insurance problem for general utility functions and CRRA (constant relative risk aversion) power utility functions.

3.1 Introduction

We delve into the intricate problem of optimal consumption, investment, and life insurance in a particular case of state-dependent utility. Our approach is grounded in the notion that an agent's preferences are intricately linked to the financial and insurance markets, assuming that the agent's actions do not influence market prices. The concept of state-dependent utility is particularly significant as it vividly reflects the agent's preferences concerning future cash flows, evaluated in the context of the prevailing financial market. This implies that individuals assess money in alignment with their financial environment rather than merely focusing on numerical values. We propose adjusting for the state of the market, a concept aptly termed "risk-adjusted" utility, as first introduced by Steffensen, 2001.

The modern academic tradition of solving continuous-time optimal consumption and investment problems began with the seminal work Merton, 1969, 1971, where the agent seeks to optimize controls with a constant relative risk aversion in a Merton market. Since then, researchers have conducted several generalizations of the model in numerous directions. As we investigate the life insurance market, a noteworthy generalization for our work is the inclusion of lifetime uncertainty and access to life insurance by Richard, 1975 and even earlier by Yaari, 1965.

Merton, 1969, 1971 and Richard, 1975 tackled these problems using dynamic programming techniques, while the martingale method, commonly known from finance, has also been widely used. Both approaches have undergone significant refinements. One such generalization involves considering state-dependent utilities. When discussing state dependence, it is crucial to emphasize the precise nature of the referred state. For instance, Kraft and Steffensen, 2008a solved the optimal insurance and consumption problem in a Markovian multi-state framework, allowing income and consumption to depend on the insured's life state, such as disability or activity. Later, Steffensen and Søe, 2023 refined these findings to include state-dependent risk aversion, particularly in the health dimension.

Another form of potential state dependence pertains to the multiplicative state or the state being the financial state of the market. Within this context, Karni, 1983 established a partial ordering on state-dependent preferences concerning risk aversion for a binary state space. Karni subsequently extended this framework to the multivariate case in Karni, 1985, 1989. Numerous further generalizations have extended these insights, notably Nordquist, 1985, who formulated a comparative risk aversion measure for only two states. Expanding, Kelsey and Nordquist, 1991 and Kelsey, 1992 extended further for preferences with multiple states.

For more recent theoretical insight into the framework, Jarrow and Li, 2021. Wang et al., 2019 investigated the optimal consumption, investment, and insurance problem where a self-exciting threshold model governed the price process of the risky asset.

Doctor, 2021, further generalized the financial market and optimized the problem under inflation risk. Additionally, Wang et al., 2021 explored the problem in a household setting that allowed for random and non-hedgeable income increases, as well as market ambiguity.

The augmentation of state variables intensifies the complexity of the problem accordingly. We propose exploring an alternative perspective on state dependence that, rather than complicating the problem, streamlines it. An early exploration of this concept can be found in the work of Steffensen, 2001, which entails separating the investment and the consumption problems, enabling their independent, separate solutions. Based on the assumption that one agent's decision does not affect the market's attitude to risk. Fundamentally, the idea is that the agent's preferences factor in the market's pre-established risk attitudes and determinations. So, the agent considers the state of the market when making decisions. Later, Londono, 2009 formalized this using the martingale method and, in Gómez and Londoño, 2022; Londoño, 2023, explored the problem of including lifetime uncertainty and the possibility of buying life insurance.

We build on these findings and include the risk associated with life insurance, ensuring that the agent's preferences factor in both the financial market's and the insurance market's pre-established attitudes toward risk. In simpler terms, the idea is to encompass life insurance's risk elements, including the risks involved in buying life insurance policies, the possibility of death, and the concept of mortality credits that arise from these financial arrangements. We solve this problem by applying dynamic programming techniques as Steffensen, 2001, conversely Londono, 2009, and Gómez and Londoño, 2022; Londoño, 2023 use the martingale method.

The structure of this paper is as follows: We begin by introducing the general optimization problem and the setting in Section 2. Section 3 explains the concept of risk-adjusting the utility by building a deflator. In Section 4, we reformulate the optimization problem in the deflated setting, define our controls, and solve the problem for general, strictly concave utility functions. Section 5 presents solutions for CRRA power utility functions, investigates the optimal control rates and their dynamics, and compares them to previous findings. Finally, in Section 6, we explore specific corner cases.

3.2 A traditional problem

We consider an agent who makes decisions in a standard Black-Scholes market. We assume this epitome of simplicity for its computational advantages and to interpret results more transparently. Thereby, the market consists of a risk-free asset, the bond, and a risky asset, the stock, at a certain time t on $0 \leq t \leq T$, where T is the time horizon, which gives the price dynamics

$$dB(t) = rB(t)dt, \qquad B(0) = 1, \tag{3.2.1}$$

$$dS(t) = S(t)\alpha dt + S(t)\sigma dW(t), \qquad S(0) = s. \tag{3.2.2}$$

Here, r, α and σ are constants, W a Brownian Motion defined on an abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We adopt the classical survival model and consider a framework where the agent continuously decides about intermediate consumption, investment, and life insurance. Letting $N(t)$ and $I(t) = 1 - N(t)$ indicate the life situation of the agent, where $I(t) = 1$ indicates that the agent is alive at time t. We treat N and I as stochastic processes on the same abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where the event of death of the agent is modeled by the deterministic mortality intensity λ assumed bounded and away from zero and infinity as

$$\mathbb{P}(I(t) = 1) = e^{-\int_0^t \lambda(v)dv}, \qquad t \geq 0.$$

The abstract probability space is thought of and modeled as the combined financial and actuarial world, as in Dhaene et al., 2013. Taking a step back and explaining, we model the financial world from a filtered probability space with constant interest rates and stock prices as stochastic processes adapted to the filtration in a probability space. Furthermore, we describe the actuarial risk via adapted stochastic processes to a second filtered probability space. Then, the overall used abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the combined space of the financial and the actuarial risks, assuming that the dynamics of the financial risks and the dynamics of the actuarial risks are mutually independent. We equip the probability space with the natural filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ induced by both the risk of the financial world and of the actuarial world, meaning $\mathcal{F}_t, 0 \leq t \leq T$ is the σ-algebra of both processes completed with the zero-probability events of $\mathbb{P}$. More elaborate, the actuarial risk is considered the risk behind securities such as longevity bonds and the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ is home to a market with constant risk-free interest rate r in which a financial asset and an actuarial asset are traded. Assuming an arbitrage-free market, we know there exists an equivalent martingale measure, the pricing measure $\mathbb{P}^*$, for which Dhaene et al., 2013 shows that the assumed independence between financial risk and actuarial risk holds under both the objective probability measure $\mathbb{P}$ and the equivalent pricing probability measure $\mathbb{P}^*$ used for pricing both financial market risk and insurance market risk, with corresponding deterministic intensity λ^*, again assumed bounded and away from zero and infinity.

By this model, we thereby assume the existence of a pricing measure, ensuring linearity and uniqueness in prices. As academic tradition and formulated by Richard, 1975, we also acknowledge the potential difference between the pricing and objective measures, where both λ and λ^* are assumed continuous[1]. The relationship between the counting process $N(t)$ counting transitions and the transition intensities are formally defined as

$$E[N(t) - N(t-h)|N(t-h) = 0] = \lambda(t)h + o(h).$$

Where the expectation is under the objective probability measure $\mathbb{P}$ and $o(h)$ is a function that grows slower than h as h approaches zero. It follows that there existsλ^* such that

$$E^*[N(t) - N(t-h)|N(t-h) = 0] = \lambda^*(t)h + o(h),$$

where E^* denotes the expectation under then pricing measure $\mathbb{P}^*$. For positive intensities bounded away from zero and infinity, assuming both measures are equivalent implies that λ^* is zero if and only if λ is zero. The difference between the two transition intensities represents an insurance risk loading, which we describe later.

For our framework, we fix a time horizon of T, and initial wealth x'_0 and describe the financial wealth of the agent by the SDE as long as he is alive

$$\begin{aligned} dX'(t) &= \big(-c'(t) + a'(t) - \lambda^*(t)b'(t) + X'(t)(r + \theta(t)(\alpha - r))\big)dt \quad (3.2.3) \\ &+ X'(t)\theta(t)\sigma dW(t), \\ X'(0) &= x'_0. \quad (3.2.4) \end{aligned}$$

The agent consumes with rate $c'(t)$, receives income defined by the deterministic rate $a'(t)$, and pays premium-rate $\lambda^*(t)b'(t)$ for the life insurance coverage $b'(t)$, received upon death. Furthermore, θ is the proportion of wealth invested in the risky asset. We introduce the human capital $h(t)$, defined as the financial value of future labor income; by accessing the insurance market, the agent can hedge their future income. This means that the agent can offset the risk associated with future income through the market consisting of the financial asset, an insurance asset, and the risk-free asset, with no limitations on trading or access to the financial instruments. Therefore, the sum of financial wealth $X'(t)$ and human capital $h(t)$ at time t is the agent's total wealth, where human capital is the expected present value of the future income under the pricing measure.

In this setting, an interesting optimization problem is the future expected utility of intermediate consumption, investment, and wealth paid out upon death as life insurance, given by a time-additive utility function u for consumption and v for terminal wealth. Formulated as

$$\sup_{c',\theta,b'} E\Big[\int_0^T u(t, c'(t))I(t)dt + v(t, X'(t) + b'(t))dN(t)\Big]. \quad (3.2.5)$$

[1]We will not be overly concerned about regularity conditions on non-stochastic exogenous functions. In general, we will assume them to be continuous unless otherwise noted.

Where $u(t,x)$ and $v(t,x)$ are separable utility functions assumed to be concave on consumption and wealth, two times differentiable, and the inverse exists, i.e., the power function, the logarithmic, or the negative exponential function. This problem is previously solved by, among others, i.e., Kraft and Steffensen, 2008a and explored in many variations, Kraft and Steffensen, 2008a look at constant risk aversion but allow the utility and consumption to depend upon the life state of the insured, whereas, Steffensen and Søe, 2023 solves the problem for state-dependent risk aversion, again, where the state is in the health dimension. Naturally, the attitude to risk taken up by the market influences the preferences. The subject of state-dependent utility, where the state is the state of the financial market, is widely investigated, as explained in the introduction. Generally speaking, the fact that the problem includes state-dependency makes it much more complex, but we get inspiration from Londono, 2009, Steffensen, 2001, and Gómez and Londoño, 2022; Londoño, 2023 to simplify the problem instead.

3.3 Risk-adjusted utility

In this section, we explain the concept of risk-adjusted utility and introduce a new framework for solving the optimization problem.

The idea is to utilize the complete market and that it takes a unique position on the price of risk from the Brownian motion W, to quantify the financial worth of payments and compare them. Instead of evaluating the utility and, consequently, the moral value of the money, we take cues from derivative pricing techniques and compare the financial value instead. As mentioned, this has been done before by Steffensen, 2001, Londono, 2009, and Gómez and Londoño, 2022; Londoño, 2023, but their primary focus has been the financial market, which we expand. However, to do so, we first establish the idea in the financial market.

3.3.1 A deflator

All three previously mentioned papers that have examined this idea use that the time 0 the financial value or price of a payment $c'(t)$, that is $\mathcal{F}_t^W$-measurable, at time t is given by $E(\Lambda(t)c'(t))$, with the deflator

$$d\Lambda(t) = -\Lambda(t)\big(rdt + \frac{\alpha - r}{\sigma}dW(t)\big), \quad \Lambda(0) = 1,$$

where we assume $\frac{\alpha-r}{\sigma} \neq 0$ so we have a non-degenerate process. From here, instead of measuring the utility of the payment, we measure at time zero, the moral value of a payment $u'(t)$ at time t by $E(u(\Lambda(t)c'(t)))$ where u is a utility function. As it turns out, this has several advantages: First, the agent considers the market's attitudes to risk before making decisions based on his utility function. Second, the deflator $\Lambda(t)$ makes payments time-additive such that the payment at time s; $c'(s)$, added to the payment at time t; $c'(t)$, equals $E(\Lambda(s)c'(s) + \Lambda(t)c'(t))$. Further, it simplifies the problem considerably. We use the name 'deflator' for the process Λ to illustrate the dependence on the state of the market because one can think of the prices as being deflated by this process; this also relates to the assertion that the prices go up when the market goes up or vice versa. By this deflation, one way of thinking is to consider an agent with preferences over purchasing power, defined by the utility function u; then, this approach evaluates the risk-adjusted purchasing power instead of the nominal amount of money. The relation with price fluctuations seems natural, but it is noted that the choice of deflator is by mathematical convenience.

Throughout, we denote the nominal value by prime and the risk-adjusted values without. We incorporate the risk component associated with the insurance market into the framework, extending beyond the focus on insurance access, as explored by Gómez and Londoño, 2022; Londoño, 2023. The interest is in insurance loading, also known as the typical deviation between insurance premiums, which is calculated using a mortality rate that can be different from the objective mortality rate. This difference between pricing and objective mortality can lead to variations in how insurance is priced and perceived. It introduces the risk that the pricing measure, which reflects the insurer's assessment of policyholders' mortality risk, may differ from the objective measure, representing the actual mortality rate.

In an ideal, utterly fair market, the objective mortality rate should ideally align with the pricing mortality rate, resulting in a fair and balanced pricing of insurance policies. However, deviations between these two measures may occur due to various factors, including insurer assessments, legislation, individual perceptions of mortality risk, and market dynamics.

We enhance the deflator by including mortality credits. When deflated payments are assessed, the base payment is reflected, and mortality credits are factored in, which corresponds to the potential difference between the objective and pricing measures. This adjustment ensures that the deflator comprehensively accounts for the risk associated with insurance, considering disparities in the mortality rate used for pricing and the objective mortality rate. This also means the agent takes into consideration that there is risk associated with life insurance before making decisions. The preferences consider improvements in life expectancy or other risks associated with purchasing life insurance.

3.3.2 The new deflator

To establish the deflator in a market that includes insurance risk, we draw inspiration from Björk, 2021, where a derivative pricing framework is developed for jump-diffusion processes similar to those we encounter. While it is possible to adjust for each source of randomness sequentially, it is more efficient to do so simultaneously. Our model assumes an arbitrage-free market with a risky asset representing financial risk (the stock) and an asset representing actuarial risk (such as a longevity bond), thus incorporating two sources of randomness. Consequently, the agent operates in a complete market. Another perspective is that the additional risk introduced by mortality involves the potential loss of future income, which can be hedged by access to the insurance market. Following the derivative pricing approach, we propose a deflator as outlined in Chapter 19 of Björk, 2021, given by

$$\begin{aligned} d\Lambda(t) &= -\Lambda(t)(rdt + h(t)dW(t) - g(t)(dN(t) - \lambda(t)dt)), && (3.3.1) \\ \Lambda(0) &= 1. && (3.3.2) \end{aligned}$$

With appropriate conditions on h and g we can define

$$\Lambda(t) = e^{-rt}\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t\right].$$

This means that our deflator Λ is the product of a discount factor and the likelihood function of an equivalent pricing measure $\mathbb{Q}$ with respect to the objective measure $\mathbb{P}$. In an incomplete market, the risk-adjusting factor, the deflator Λ, would be ambiguous, with one risk-adjusting deflator for each martingale measure. However, in our complete market, the pricing measure is defined exogenously as $\mathbb{P}^*$ since the market takes a unique position on the pricing of the W risk and the pricing of the insurance risk. The attitude to risk, represented by the difference in the pricing and objective measures, is the difference between the pricing and objective mortality rates defined by $g > -1$ as $\lambda(t)(1 + g(t)) = \lambda^*(t)$. Thus, the functions are defined by

$$\begin{aligned} h(t) &= \frac{\alpha - r}{\sigma}, \\ g(t) &= \frac{\lambda^*(t)}{\lambda(t)} - 1. \end{aligned}$$

We conclude that the deflator Λ is a risk-adjusting process in the form

$$\begin{aligned} d\Lambda(t) &= -\Lambda(t)(rdt + \frac{\alpha - r}{\sigma}dW(t) - g(t)(dN(t) - \lambda(t)dt)), && (3.3.3) \\ \Lambda(0) &= 1. \end{aligned}$$

3.3.3 The deflated payments

Defining the deflated income, consumption, insurance sum, and wealth abbreviated as

$$\begin{aligned} a(t) &= \Lambda(t)a'(t), \\ c(t) &= \Lambda(t)c'(t), \\ b(t) &= \Lambda(t)b'(t), \\ X(t) &= \lambda(t)X'(t). \end{aligned}$$

We use Ito's lemma for a jump-diffusion process and have the dynamics

$$\begin{aligned} d(X'(t)\Lambda(t)) = dX(t) &= (a(t)-c(t))I(t)dt && (3.3.4)\\ &+ b(t)(1+g(t))(dN(t)-I(t)\lambda(t)dt) && (3.3.5)\\ &+ X(t)\big(\theta(t)\sigma - \frac{\alpha-r}{\sigma}\big)dW(t) \\ &+ X(t)g(t)(dN(t)-I(t)\lambda(t)dt), \\ X(0) &= x_0. && (3.3.6) \end{aligned}$$

All are still defined with respect to the objective measure $\mathbb{P}$, but now the wealth is the risk-adjusted wealth.

3.4 The optimization problem

With the deflated framework established, we fix a time horizon T and optimize intermediate deflated consumption $c(t)$, thereby the purchasing power, the part invested in the risky asset $\theta(t)$, and the bequest wealth, all illustrated by the policyholder's wealth (3.3.4). If there is a risk loading, thereby the pricing mortality is not the same as the objective mortality, then $g(t) \neq 0$, and the policyholder does only receive the financial value of the lump sum but additionally the $g(t)(X(t)+b(t))$ marking the extra mortality credits collected as the difference between the pricing and objective mortality arises. This additional part means that by looking at the risk-adjusted utility, we do not only account for the risk that the value of the money changes by looking at purchasing power but also account for the risk associated with having different mortality than the pricing mortality by adding the effect of the collected mortality credits. Concluding that the bequest wealth upon death is not as previously, $X'(t) + b'(t)$, but now $(X(t)+b(t))(1+g(t))$.

Since the agent does not know the future, using a simplified approach, we may assume the set of controls processes (c, θ, b) is adapted to the wealth process X. However, for computational convenience, we go one step further and require that the controls (c, θ, b) is of feedback form such that

$$(c(t), \theta(t), b(t)) = (\bar{c}(t, X(t)), \bar{\theta}(t, X(t)), \bar{b}(t, X(t))), \quad t \in [0, T].$$

Where $(\bar{c}, \bar{\theta}, \bar{b})$ are deterministic measurable functions $(\bar{c}, \bar{\theta}, \bar{b}) : [0, T] \times \mathbb{R} \to \mathbb{R}$, to not carry too much notation around, we redefine $(c, \theta, b) := (\bar{c}, \bar{\theta}, \bar{b})$, so now the SDE describing the wealth of the agent while he is alive (3.3.4) is written as

$$\begin{aligned} dX(t) &= \Big(a(t) - c(t, X(t)) + X(t)g(t)) && (3.4.1)\\ &- \lambda(t)(b(t, X(t))(1+g(t))\Big)dt \\ &+ X(t)\big(\theta(t, X(t))\sigma - \frac{\alpha-r}{\sigma}\big)dW(t), \\ X(0) &= x_0. \end{aligned}$$

Due to the dependency, the notation of $X(t)$ should be $X^{c,\theta,b}(t)$, but we suppress the notation such that $X^{c,\theta,b}(t) = X(t)$. Therefore, the deflated problem to solve is

$$\sup_{(c,\theta,b)} E\Big[\int_0^T u(t, c(t, X(t)))I(t)dt \\ + v(t, (1 + g(t))(X(t) + b(t, X(t))))dN(t)\Big].$$

We solve this by using dynamic programming techniques; let

$$\begin{aligned} J(t, x, c, \theta, b) &= E_{t,x}\Big[\int_t^T u(s, c(s, X(s)))I(s)ds \\ &+ v(s, (1 + g(s))(X(s) + b(s, X(s))))dN(s)\Big], \end{aligned} \tag{3.4.2}$$

where $E_{t,x}$ is the conditional expectation given $X(t) = x$ at time t, and u and v are defined as in section 2. Before solving the problem, we specify the set of admissible controls.

Definition 3.4.1. *(Admissible controls) Considering controls (c, θ, b), we say they are admissible if they satisfy the following conditions*

1. *The insured cannot have a negative total capital in the sense of wealth, including human capital, that is, $X(t) + h(t) \geq 0$.*
2. *For each initial point $(t, x) \in [0, T] \times \mathbb{R}$ the SDE (3.4.1) has a unique strong solution and $\theta(t, X(t))$ is continuous and bounded.*
3. *For each initial point $(t, x) \in [0, T] \times \mathbb{R}$ the objective function (3.4.2), based on a specific strategy, is bounded.*
4. *Finally, we assume to have a martingale such that*

$$E\left[\int_0^T \sigma\theta(t, X(t))X(t)I(t)dW(t)\right] = 0,$$
$$E\left[\int_0^T b(t, X(t))(1 + g(t))\left(dN(t) - \lambda(t)I(t)dt\right)\right] = 0.$$

We denote the set of admissible controls by $\mathcal{A}$. Note that the compensator for the jump process $N(t)$ is $\lambda(t)I(t)$, unlike a standard Poisson process that has a deterministic compensator.

We define the value function as

$$V(t, x) = \sup_{(c,\theta,b)\in\mathcal{A}} J(t, x, c, \theta, b). \tag{3.4.3}$$

We only consider controls in the set of admissible controls defined in Definition 3.4.1. The corresponding Hamilton-Jacobi-Bellman equation for the optimization problem is

$$\begin{aligned}
-\frac{\partial}{\partial t}V(t,x) &= \sup_{(c,\theta,b)\in\mathcal{A}} \Big\{ (a(t) - c - b(1+g(t))\lambda(t) - \lambda(t)g(t)x)\frac{\partial}{\partial x}V(t,x) \\
&+ \frac{1}{2}x^2\left(\theta\sigma - \frac{\alpha - r}{\sigma}\right)\frac{\partial^2}{\partial x^2}V(t,x) + u(t,c) \\
&+ \lambda(t)\left(v(t,(1+g(t))(x+b)) - V(t,x)\right)\Big\}. \qquad (3.4.4)\\
V(T,x) &= 0. \qquad (3.4.5)
\end{aligned}$$

The following theorem gives the optimal controls.

Theorem 3.4.2. *Consider the wealth X defined by the equation* (3.4.1), *if u and v are strictly concave in the second argument, then V is concave and we find optimal controls $(c^*, \theta^*, b^*) \in \mathcal{A}$ such that for any $(c, \theta, b) \in \mathcal{A}$*

$$J(t, x, c^*, \theta^*, b^*) \geq J(t, x, c, \theta, b).$$

Furthermore, we find the optimal controls by the first-order conditions

$$\begin{aligned}
\theta^* &= \frac{\alpha - r}{\sigma^2}, \qquad (3.4.6)\\
0 &= -\frac{\partial}{\partial x}V(t,x) + \frac{\partial}{\partial c}u(t,c^*), \qquad (3.4.7)\\
0 &= -\frac{\partial}{\partial x}V(t,x) + \frac{\partial}{\partial b}v(t,(1+g(t))(x+b^*)). \qquad (3.4.8)
\end{aligned}$$

Proof. First, we prove the concavity of the value function V. Consider two points x_1 and x_2 and two strategies (c_1, θ_1, b_1) and (c_2, θ_2, b_2) and define $\eta \in (0, 1)$, then we define a strategy

$$(c, \theta, b) = (\eta c_1 + (1-\eta)c_2, \eta\theta_1 + (1-\eta)\theta_2, \eta b_1 + (1-\eta)b_2).$$

For a point $x = \eta x_1 + (1-\eta)x_2$, by linearity of X we have $X(t) = \eta X_1(t) + (1-\eta)X_2(t)$ and by linearity of the transformation $x \mapsto x(1+g(t))$, since $g(t) : [0,T] \to (-1,1]$, then

$$\begin{aligned}
(1+g(t))(X(t)+b) &= \eta(1+g(t))(X_1(t)+b_1) \\
&+ (1-\eta)(1+g(t))(X_2(t)+b_2).
\end{aligned}$$

By strict concavity of u and v it holds

$$\begin{aligned}
u(t,c) &> \eta u(t,c_1) + (1-\eta)u(t,c_2). \\
v(t,(1+g(t))(X(t)+b)) &> \eta v(t,(1+g(t))(X_1(t)+b_1)) \\
&+ (1-\eta)v(t,(1+g(t))(X_2(t)+b_2)).
\end{aligned}$$

Then it follows that

$$J(t,x,c,\theta,b) > \eta J(t,x,c_1,\theta_1,b_1) + (1-\eta)J(t,x,c_2,\theta_2,b_2).$$

Further, for any $\varepsilon > 0$ we can choose (c_1, θ_1, b_1) such that $J(t, x, c_1, \theta_1, b_1) \geq V(t, x_1) - \varepsilon$ and (c_2, θ_2, b_2) such that $J(t, x, c_2, \theta_2, b_2) \geq V(t, x_2) - \varepsilon$. But since (c, θ, b) is merely suboptimal in the sense $V(t, x) = \sup_{c,\theta,b} J(t, x, c, \theta, b)$ we have

$$\begin{aligned} V(t,x) &\geq J(t,x,c,\theta,b) > \eta J(t,x,c_1,\theta_1,b_1) + (1-\eta) J(t,x,c_2,\theta_2,b_2) \\ &\geq \eta V(t,x_1) + (1-\eta) V(t,x_2) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, we have concavity of V.

Moreover, the second-order conditions are

$$0 > x^2\sigma^2 \frac{\partial^2}{\partial x^2} V(t,x), \tag{3.4.9}$$

$$0 > \frac{\partial^2}{\partial c^2} u(t,c^*), \tag{3.4.10}$$

$$0 > \frac{\partial^2}{\partial b^2} v(t,(1+g(t))(x+b^*)). \tag{3.4.11}$$

Which, then, are satisfied if u and v are concave in the second argument and the optimal controls $(c^*, \theta^*, b^*) \in \mathcal{A}$ are found by the first order conditions (3.4.6), (3.4.7), and (3.4.8). □

It follows from Theorem 3.4.2, that the optimal part invested in the risk asset is generally $\theta^* = \frac{\alpha - r}{\sigma^2}$ (see (3.4.6)), as long as the two utility functions are strictly concave in the second argument. Simplifying the problem considerably and results in the consumption and legacy can be optimized separately from the portfolio selection.

Inserting this in (3.4.4), simplifies the Hamilton-Jacobi-Bellman equation simplifies to

$$\begin{aligned} -\frac{\partial}{\partial t} V(t,x) &= \sup_{(c,b)\in\mathcal{A}} \Big\{ (a(t) - c - b(1+g(t))\lambda(t) - \lambda(t)g(t)x) \frac{\partial}{\partial x} V(t,x) \\ &+ u(t,c) + \lambda(t)\left(v(t,(1+g(t))(x+b)) - V(t,x)\right) \Big\}. \\ V(T,x) &= 0. \end{aligned} \tag{3.4.12}$$

By the separability, we find the optimal controls by (3.4.7) and (3.4.8). Plugging these in, what is to be solved is

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} V(t,x) + \Big(a(t) - \frac{\partial}{\partial c} u^{-1}(t, \frac{\partial}{\partial x} V(t,x)) \\ &- \lambda(t)(\frac{\partial}{\partial b} v^{-1}(t, \frac{\partial}{\partial x} V(t,x)) - x)\Big) \frac{\partial}{\partial x} V(t,x) \\ &+ u(t, \frac{\partial}{\partial c} u^{-1}(t, \frac{\partial}{\partial x} V(t,x))) \\ &+ \lambda(t)\Big(v(t, \frac{\partial}{\partial b} v^{-1}(t, \frac{\partial}{\partial x} V(t,x))) - V(t,x)\Big). \end{aligned}$$

The latter is to be solved subject to the constraint $V(T,x) = 0$. By the pleasant simplicity of the problem, we recognize this as a simplified version of the problem in chapter 6 of Asmussen and Steffensen, 2020, and thereby, the verification theorem (Theorem 6.1) and proof follow directly. Hence, showing a candidate value function solves the Hamilton-Jacobi-Bellman equation, which is sufficient to ensure that this is the value function.

3.5 Solutions for CRRA

We solve the simplified problem for consumption and bequest for the case of constant relative risk aversion, where we specify the concave utility functions as

$$u(t,c) = \frac{1}{1-\gamma} c^{1-\gamma} \omega(t)^{\gamma}, \tag{3.5.1}$$

$$v(t,x) = \frac{1}{1-\gamma} x^{1-\gamma} w(t)^{\gamma}. \tag{3.5.2}$$

Here $\omega(t)$ is a time-dependent weight on the intermediate consumption, and $w(t)$ is a time-dependent weight on money at the time of death. We take both to the power γ for mathematical convenience without loss of generality.

We inherited the structure of the utility functions into our candidate value function as

$$V(t,x) = \frac{1}{1-\gamma} (x + h(t))^{1-\gamma} f(t)^{\gamma}. \tag{3.5.3}$$

Here $h(t)$ is the human capital, as previously defined, the expected actuarial-discounted future income, where the income is assumed deterministic, and together with $f(t)$ captures the time dependence of the candidate value function. The optimal part invested in the risky asset is unchanged and defined by (3.4.6) throughout the paper.

By the utility function (3.5.1) and the value function candidate (3.5.3), the optimal consumption rate from (3.4.7) is

$$c^*(t,x) = (x + h(t)) \frac{\omega(t)}{f(t)}, \tag{3.5.4}$$

which can be interpreted as a weighted proportion of the current wealth in terms of risk-adjusted purchasing power, the weight being $\frac{\omega(t)}{f(t)}$. To indicate the value of current consumption relative to the expected value of future consumption, it is essential to consider consumption in terms of risk-adjusted purchasing power, which contrasts with earlier papers that did not take this approach. Therefore, the optimal consumption rate depends on how much the agent presently desires to consume, evaluated as risk-adjusted purchasing power, compared to future consumption.

Correspondingly, inserting the utility function (3.5.2) and the value function candidate (3.5.3) in (3.4.8), the optimal insurance is

$$b^*(t,x) = (x+h(t))\frac{w(t)}{f(t)}\frac{1}{(1+g(t))} - x. \tag{3.5.5}$$

Here, $\frac{1}{1+g(t)}$ is the pure pricing impact which can be interpreted as a discounting, with $g(t)$ as the discount rate, resulting in the price-impact adjusted bequest wealth $X(t) + b^*(t, X(t))$. However, as the agent receives the bequest wealth in monetary values, $(1 + g(t))(b^*(t, x) + x))$, that by (3.5.5) is equal to

$$(x + b^*(t,x))(1+g(t)) = (x+h(t))\frac{w(t)}{f(t)}.$$

This results in the elimination of the pricing impact. By the risk adjusting, the bequest received only depends on the fraction $\frac{w(t)}{f(t)}$ multiplied by the total wealth and interpreted as the ratio between the importance of leaving behind and the value placed by the agent on future consumption. At first glance, one may think these solutions are equal to Kraft and Steffensen, 2008a, apart from the eliminated price impact, but we will investigate these similarities later.

3.5.1 The Solution

In order to verify a candidate value function and optimal controls as a solution, we solve the Hamilton-Jacobi-Bellman equation, which entails finding the solution to the Hamilton-Jacobi-Bellman equation, we first find the partial derivatives of the value function as

$$\begin{aligned}
\tfrac{\partial}{\partial t}V(t,x) = &\quad (x+h(t))^{-\gamma}f(t)^{\gamma}\frac{d}{dt}h(t) + \frac{\gamma}{1-\gamma}(x+h(t))^{1-\gamma}f(t)^{\gamma-1}\frac{d}{dt}f(t), \\
\tfrac{\partial}{\partial x}V(t,x) = &\quad (x+h(t))^{-\gamma}f(t)^{\gamma}.
\end{aligned}$$

Note that the twice derived in the second argument is irrelevant since the problem has simplified to (3.4.12).

We find the solution to the Hamilton-Jacobi-Bellman equation by inserting the optimal controls (3.5.4) and (3.5.5), our value function candidate (3.5.3) and the derivatives seen above and expressed as the solution to the following two differential equations

$$\frac{d}{dt}h(t) = \lambda(t)h(t) - a(t), \qquad h(T) = 0. \tag{3.5.6}$$

$$\frac{d}{dt}f(t) = \lambda(t)(f(t) - w(t)) - \omega(t), \qquad f(T) = 0. \tag{3.5.7}$$

When we look at these results, both the solution to the Hamilton-Jacobi-Bellman equation and the optimal controls, it is noteworthy that there is no γ present, meaning the preference for risk aversion is not here. The functions $\omega(t)$ and $w(t)$ represent an individual weight on the utility function. The simplicity we mentioned is the absence of the risk aversion parameter and the separation of solving the optimal portfolio and the optimal consumption and legacy. By including the market's attitude to risk and the insurance risk before making decisions, the relative risk aversion widely studied in many variations is no longer present in our solutions.

3.5.2 The optimal control rates

With the simplicity of our solutions to the Hamilton-Jacobi-Bellman equation, a natural question arises: How does this affect the optimal control rates? As optimal consumption is a function of time and wealth, so is the optimal bequest wealth. Therefore, we look at the optimal consumption rate and the optimal bequest wealth. We calculate the dynamic of the optimal consumption rate as

$$\begin{aligned} dc^*(t,X^*(t)) &= \frac{\partial}{\partial t}c^*(t,X^*(t))dt + \frac{\partial}{\partial x}c^*(t,X^*(t))d(X^*(t)), \\ &= \frac{\omega(t)}{f(t)}\Big(\frac{d}{dt}h(t)\Big)dt + (X^*(t)+h(t))\frac{\omega(t)}{f(t)}\frac{\frac{d}{dt}\omega(t)}{\omega(t)}dt \\ &- (X^*(t)+h(t))\frac{\omega(t)}{f^2(t)}\Big(\frac{d}{dt}f(t)\Big)dt \\ &+ \frac{\omega(t)}{f(t)}dX^*(t), \\ c^*(0,X^*(0)) &= c^*(0,x_0). \end{aligned}$$

We denote $X^{c^*,\theta^*,b^*}(t) = X^*(t)$ for notational ease. By inserting the partial derivatives, (3.5.6), the optimal controls (3.5.4), (3.5.5), and (3.4.6), and the dynamics of the wealth, (3.4.1), the dynamic of the optimal consumption rate as long as the agent is alive is

$$\begin{aligned} dc^*(t,X^*(t)) &= c^*(t,X^*(t))\frac{\frac{d}{dt}\omega(t)}{\omega(t)}dt, && (3.5.8) \\ c^*(0,X^*(0)) &= c^*(0,x_0). && (3.5.9) \end{aligned}$$

By looking at the purchasing power and deflating additionally for the insurance risk, we have found that only the agent's subjective value placed upon current utility matters concerning how the consumption rate evolves.

By defining the weight as a commonly studied exponential function $\omega(t) = e^{-\rho t}$, where we interpret ρ as the individual impatience parameter or a type of subjective discounting, the dynamics of the optimal consumption rate is

$$\begin{aligned} dc^*(t,X^*(t)) &= -\rho c^*(t,X^*(t))dt, \\ c^*(0,X^*(0)) &= c^*(0,x_0), \end{aligned}$$

where only the agent's impatience affects the consumption rate in terms of purchasing power. This means that if the agent is impatient and does not value future purchasing power as much as the present purchasing power, the consumption rate has a negative drift.

The dynamic of the optimal bequest wealth (3.5.5) is calculated in the fashion

$$d(X^*(t) + b^*(t, X^*(t))) = (X^*(t) + b^*(t, X^*(t)))\Big(\frac{\frac{d}{dt}w(t)}{w(t)} - \frac{\frac{d}{dt}g(t)}{1+g(t)}\Big)dt. \quad (3.5.10)$$

$$X^*(0) + b^*(0, X^*(0)) = x_0 + b^*(0, x_0). \quad (3.5.11)$$

The impact on the drift is similar to the optimal consumption rate by the agent's weight placed upon leaving a legacy, but additionally, the price impact is present. We can define the weight placed on having a legacy as $w(t) = \iota\omega(t)$, where ι is a scaling factor that adjusts the importance of leaving a legacy relative to consumption. Additionally, ι can represent the trade-off between consumption and legacy.

$$d(X^*(t) + b^*(t, X^*(t))) = (X^*(t) + b^*(t, X^*(t)))\left(-\rho - \frac{\frac{d}{dt}g(t)}{1+g(t)}\right)dt.$$

$$X^*(0) + b^*(0, X^*(0)) = x_0 + b^*(0, x_0).$$

The agent's impatience factor ρ serves as a reference level. The price impact $g(t)$ could be a constant, or another simple price impact could be $g(t) = \kappa t$, then we would have $\frac{\frac{d}{dt}g(t)}{1+g(t)} = \frac{\kappa}{1+\kappa t}$. If, however, we were to look at the bequest wealth $(1 + g(t))(b^*(t, X^*(t)) + X^*(t))$, the price impact would not be present, and the dynamics would only depend on the weight $\omega(t)$, comparable to the consumption.

3.5.3 Comparison

We finalize this section on the specific solutions for constant relative risk aversion by comparing the differential equation for $f(t)$ and $h(t)$, the optimal controls $c^*(t, x)$, and $b^*(t, x)$ and the dynamics of the optimal control rates with those obtained from Kraft and Steffensen, 2008a. They present different examples of the general solution; we look at Chapter 5 regarding the survival model, where the optimization problem is

$$\sup_{c^0, b^{01}} E\Big[\int_0^n \frac{1}{1-\gamma}(c^0(t))^{1-\gamma}w^0(t)^{\gamma}I(t)dt + \frac{1}{1-\gamma}(X(t) + b^{01}(t))^{1-\gamma}w^{01}(t)^{\gamma}dN^{01}(t)\Big].$$

Corresponding to the problem considered in this paper, where the bequest is $(x+b(t))$, the price impact is still present due to the lack of risk adjustment. Their solution to the optimization problem is the solution to the two equations

$$\frac{d}{dt}f^0(t) = -w^0(t) + f^0(t)\tilde{r}(t) - \tilde{\mu}(t)(w^{01}(t) - f^0(t)), \qquad f^0(T) = \Delta W^0(T).$$

$$\frac{d}{dt}h^0_t(t) = rh^0(t) - a^0(t) + \mu^*(t)h^0(t), \qquad h^0(T) = 0,$$

where $w^0(t)$ and $w^{01}(t)$ are the utility weights we denote by $\omega(t)$ and $w(t)$, respectively, and $\mu(t)$ and $\mu^*(t)$ are the transitions intensities corresponding to $\lambda(t)$ and $\lambda^*(t)$. Furthermore, $a^0(t)$ is the income we denote $a(t)$. The preference-weighted interest rate and mortality are

$$\begin{aligned}\tilde{r}(t) &= -\delta r - \delta(\mu^*(t) - \mu(t)) + \mu(t) - \tilde{\mu}(t),\\ \delta &= \frac{1-\gamma}{\gamma}, \qquad \tilde{\mu}(t) = \mu^*(t)\left(\frac{\mu(t)}{\mu^*(t)}\right)^{\frac{1}{\gamma}}.\end{aligned}$$

One of the noteworthy simplicities of our solutions is that the relative risk aversion parameter is not present for comparison, letting the preference parameter go to 1, essentially looking at the logarithmic utility, the results from Kraft and Steffensen, 2008a become

$$\begin{aligned}\frac{d}{dt}f^0(t) &= -w^0(t) - \mu(t)(w^{01}(t) - f^0(t)), \qquad f^0(T) = \Delta W^0(n).\\ \frac{d}{dt}h^0(t) &= rh^0(t) - a^0(t) + \mu^*(t)h^0(t), \qquad h^0(T) = 0.\end{aligned}$$

Here, the similarities of the differential equation for $f^0(t)$ and the differential equation for $f(t)$ from (3.5.6) are remarkable. However, the solution to $h^0(t)$ is still calculated differently than our differential equation for $h(t)$, since Kraft and Steffensen, 2008a does not work with a deflated wealth.

Next, we compare the optimal controls from Kraft and Steffensen, 2008a denoted $c(t,x)$ and bequest wealth $b^{01}(t,x)$ corresponding to $c(t,x)$ and $b(t,x)$. They are given as

$$\begin{aligned}c^0(t,x) &= \frac{w^0(t)}{f^0(t)}(x + h^0(t)),\\ b^{01}(t,x) &= \frac{w^{01}(t)}{f^0(t)}\frac{\mu(t)}{\mu^*(t)}(x + h^0(t)) - x.\end{aligned}$$

By the definition $\mu^*(t) = (1 + g(t))\mu(t)$, we obtain the same equations as in (3.5.4) and (3.5.5) at first look. However, the solution to the differential equation for the human capital $h(t)$ remains different since we are considering purchasing power, while Kraft and Steffensen, 2008a considers the monetary value. The differences are easier to see by comparing the dynamics of the controls. We could also compare with P. and Ye, 2007, which includes the stochastic market, and show the same solutions in Chapter 4 for a constant relative risk aversion. They do not investigate the dynamics of the optimal controls, which we want to investigate further.

From Kraft and Steffensen, 2008a, still looking at the case where the relative risk aversion, $\gamma \to 1$, they have the dynamics of the optimal consumption

$$dc^0(t, X(t)) = c^0(t, X(t))\left((r + \mu^*(t) - \mu(t)) + \frac{\frac{d}{dt}w^0(t)}{w^0(t)}\right), \tag{3.5.12}$$

$$c^0(0, X(0)) = c^0(0, x_0). \tag{3.5.13}$$

Further, the legacy is equal to $x + b^{01}$, with the dynamics as

$$\begin{aligned} d(X(t) + b^{01}(t, X(t))) &= (X(t) + b^{01}(t, X(t)))\Big((r + \mu^*(t) - \mu(t)) \\ &+ \frac{\frac{d}{dt}w^{01}(t)}{w^{01}(t)}\frac{\frac{d}{dt}h^{01}(t)}{h^{01}(t)}\Big), \\ X(0) + b^{01}(0, X(0)) &= x_0 + b^{01}(0, x_0). \end{aligned}$$

Here $h^{01}(t)$ is the function representing the price impact; in our case, $h^{01}(t) = \frac{1}{1+g(t)}$. Comparing these two dynamics to our (3.5.8) and (3.5.10), we note that the only difference is the presence of interest and mortality, simply due to the deflation. Therefore, we now look at the nominal dynamics of our solutions instead of the purchasing power. However, remember that in our problem, we also look at the risky investment market. Recalling $\Lambda(t)c'(t,x) = c(t,x)$ and $\Lambda(t)X'(t) = X(t)$ and by Ito, the dynamics of the nominal consumption rate is

$$\begin{aligned} d\left(\frac{c^*(t, X(t))}{\Lambda(t)}\right) &= d(c'^*(t, X'(t)) \\ &= c'^*(t, X'(t))\Big(\Big(r + g(t)\lambda(t) + \left(\frac{\alpha - r}{\sigma}\right)^2 + \frac{\frac{d}{dt}\omega(t)}{\omega(t)}\Big)dt \\ &\quad + \frac{\alpha - r}{\sigma}dW(t)\Big), \\ c'^*(0, X'(0)) &= c'^*(0, x'_0). \end{aligned}$$

Note here $\lambda^*(t) - \lambda(t) = (1 + g(t))\lambda(t) - \lambda(t) = g(t)\lambda(t)$ and disregarding the possibility for investing in the risky asset, we have the same dynamics as (3.5.12). The same goes for the dynamics of the optimal bequest wealth

$$\begin{aligned} d\left(\frac{X(t) + b^*(t, X(t))}{\Lambda(t)}\right) &= d(b'^*(t, X'(t)) + X'(t)) \\ &= (b'^*(t, X'(t)) + X'(t))\Big(\Big(r + g(t)\lambda(t) + \left(\frac{\alpha - r}{\sigma}\right)^2 \\ &+ \frac{\frac{d}{dt}w(t)}{w(t)} - \frac{\frac{d}{dt}g(t)}{1 + g(t)}\Big)dt + \frac{\alpha - r}{\sigma}dW(t)\Big), \\ X'(0) + b'^*(0, X'(0)) &= x'_0 + b'^*(0, x'_0). \end{aligned}$$

Again, we find the same dynamics as Kraft and Steffensen, 2008a with the additional part from the risky market. We are concluding that the nominal optimal consumption $c'(t, x')$ and bequest $x' + b'(t, x')$ have the same dynamics as the optimal consumption and bequest found in Kraft and Steffensen, 2008a. Therefore, by looking at a risk-adjusted consumption, taking the utility of the deflated consumption $E[u(t, \Lambda(t)c'(t))]$ and thereby investigating the risk-adjusted purchasing power, we find the exact solutions as if we risk-adjusted the optimal consumption and bequest from Kraft and Steffensen, 2008a.

As mentioned, we contribute with a new angle on the consumption-investment-insurance optimization problems by looking at a risk-adjusted utility, not only risk-adjusted on the financial risk but adding the risk of mortality, explicit that the pricing mortality might not be equal to the actual objective mortality. Further, we allow for a different weight on the utility while alive and upon death, inspired by Kraft and Steffensen, 2008a. By risk adjusting, we find more straightforward, more direct solutions without the relative risk aversion parameter present and eliminate the price impact from the solutions as we measure the monetary values.

3.6 Special cases

When investigating the possibility of a risk-adjusted optimal consumption-investment-insurance problem, we found inspiration from Steffensen, 2001, Gómez and Londoño, 2022, and Londoño, 2023, among others. From there, the results were developed to entail the risk associated with the difference in mortalities. This section examines some corner cases to illustrate the results, their contribution, and interpretation. For the rest of this paper, we continue to use the class of power (CRRA) utility functions, as in the previous section.

We look at three prominent cases; first, where there is no price loading, assuming the mortality intensities are the same $\lambda(t) = \lambda^*(t)$, corresponding to $g(t) = 0$. The second case is where the utility of consumption is measured by the same utility function as the utility of the legacy, letting $\omega(t) = w(t)$. Third, we combine the previous and look at the corner case where there is no price impact, and the utility is consistent regarding consumption and bequest.

3.6.1 No price loading

In the first case, we assume the pricing measure is the same as the objective measure, excluding the risk associated with the life insurance contract and simplify the deflator to

$$d\Lambda(t) = -\Lambda(t)(rdt + \frac{\alpha - r}{\sigma} dW(t)). \tag{3.6.1}$$

Correspondingly, we have the deflated wealth

$$dX(t) \quad = \quad (a(t) - c(t))dt + X(t)(\theta\sigma - \frac{\alpha - r}{\sigma})dW(t) + b(t)(dN(t) - \lambda(t)dt).$$

Here, the individual's objective is to maximize the expected utility from consumption and bequest until death. Thus, the value function is

$$V(t, x) = \sup_{c,\theta,b} E_{t,x} \left[\int_t^T u(s, c(s))I(s)ds + v(s, X(s) + b(s))dN(s) \right]. \tag{3.6.2}$$

Since the only change is to the pricing impact on life insurance, the optimal consumption rate (3.5.4) and optimal part invested in the risky asset (3.4.6) remain unchanged, but the optimal bequest wealth is now

$$b^*(t,x) \quad = \quad (x+h(t))\frac{w(t)}{f(t)} - x.$$

This optimal bequest wealth is the same as found in Gómez and Londoño, 2022, in the special case with no constraints on life insurance purchase. Thus, in the case of a completely fair contract, our results coincide with their previous findings.

3.6.2 One utility function

Secondly, we assume the utility function is the same for intermediate consumption while alive and bequest wealth upon death. Thus, the insured's preference for consumption is the same as the preference for how much wealth is left upon death, formalized by the utility functions as

$$v(t,x) = \frac{1}{1-\gamma}x^{1-\gamma}\omega(t)^{\gamma} = u(t,x).$$

In this case, the pricing impact is still present, and thus, we have the same deflator, (3.3.3), and correspondingly deflated wealth, (3.3.4), with the value function as previous (3.4.3). Consequently, the optimal part invested in the risky asset θ^* and optimal consumption c^* remain unchanged, but as the utility function for bequest is changed (simplified), the optimal bequest wealth is

$$x + b^*(t,x) \quad = \quad (x+h(t))\frac{\omega(t)}{f(t)}\frac{1}{(1+g(t))}.$$

Here, including the price impact as in (3.5.5), but the exciting difference is the weight-factor $\frac{\omega(t)}{f(t)}$ that is the same as in the solution for optimal consumption (3.5.4). In this case, the optimal bequest wealth is the optimal consumption times a price impact.

The change in utility function also shows in the solution to the Hamilton-Jacobi-Bellman that differs from (3.5.6) and is expressed as the solution to the following two differential equations

$$\frac{d}{dt}h(t) \quad = \quad \lambda(t)h(t) - a(t), \qquad h(T) = 0. \tag{3.6.3}$$

$$\frac{d}{dt}f(t) \quad = \quad \lambda(t)(f(t) - \omega(t)) - \omega(t), \qquad f(T) = 0. \tag{3.6.4}$$

As seen, the $w(t)$ is replaced by $\omega(t)$.

3.6.3 No price impact and continuation after death

The last and simplest case combines the previous. We let the mortality intensities coincide, eliminating the price impact, and further, the object is to maximize the expected utility of consumption until the end of the contract, regardless of death. Imagine that the insured decide that their future inheritors continue the consumption with the same utility preferences, which corresponds to the value function

$$V(t,x) = \sup_{c,\theta,b} E_{t,x}\left[\int_t^T u(s,c(s))ds\right]. \tag{3.6.5}$$

Upon death, the benefit is added to the wealth, and thereby it does not affect the deflator (3.3.3) nor the deflated wealth (3.3.4). However, the solutions would be affected as the difference in preference for the optimal bequest wealth is eliminated, corresponding to $\omega(t) = f(t)$ such that

$$b^*(t,x) \quad = \quad h(t).$$

Here the coverage is the potential loss in human capital in case of death, and it shows that if the contract is fair and the preferences are unchanged upon death, the optimal life insurance coverage is the potential loss in income.

Numerical studies

This section illustrates the expected optimal controls, as seen in Section 5. We illustrate the relative optimal controls as; optimal consumption $\frac{c'(t,X'(t))}{X'(t)+h'(t)}$ and optimal bequest $\frac{b'(t,X'(t))+X'(t)}{X'(t)+h'(t)}$, both values with the log-utility preferences as seen in Kraft and Steffensen, 2008a, but including the risky asset as in P. and Ye, 2007. The optimal deflated consumption $\frac{c^*(t,X(t))}{X(t)+h(t)}$ and optimal deflated bequest $\frac{b^*(t,X(t))+X(t)}{X(t)+h(t)}$, where the analytical difference between the two cases, are seen in the soultion to the function f, since for example the expected relative optimal consumption is defined by the fraction ω/f, interpreted as the weight of placed upon consumption diveded the conditional expected present value of future weights, both conusmption prior to death and consumption upon death. Comparing the two then equals comparing how much (risk-adjusted or not), consumption per unit of (risk-adjusted or not) expected wealth.

We choose some baseline parameters to isolate the desired effects; these are seen in Table 3.1. Here, we defined the weights in the utility functions as

$$\omega(t) = e^{-\rho t}, \quad w(t) = \iota \cdot \omega(t).$$

The interest rate is assumed to be constant.

In all illustrations, we look at the relative optimal controls as a function of age. Therefore, we can compare the nominal controls, marked with $'$, and the deflated controls, marked with $*$.

We observe that the relative optimal bequest gradually increases with age until around 80 years, where it starts to rise significantly as the risk of death similarly

Table 3.1: *Default parameters for numerical analysis*

r	ρ	ι	α	σ	Age at initialization
0.02	0.02	2	0.05	0.2	30

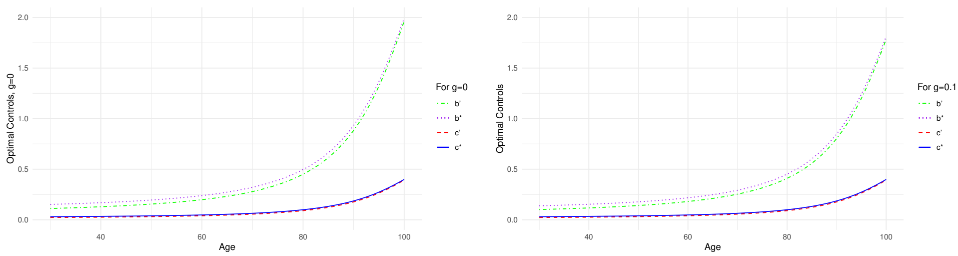

Figure 3.1: *The relative optimal controls as a function of age, left for* $g = 0$ *and right for* $g = 0.1$

increases. It also shows a difference between the relative nominal and deflated bequest, but the difference between the relative nominal and deflated optimal consumption is difficult to determine in Figure **??**.

Thus, we have investigated and shown the difference between the two relative optimal bequest and consumption in Figure **??**. Here, the relative deflated optimal bequest is approximately $3 - 5\%$ larger than the relative nominal optimal bequest, and the relative deflated optimal consumption is about 1% larger than the relative nominal optimal consumption.

This can be attributed to the fact that we have accounted for risk in the model, which manifests in a higher relative value in optimal controls, even when considering the expected controls. To illustrate the market's attitude towards actuarial risk, we present two scenarios: one with no pricing effect, so $g = 0$, and another where the pricing intensity exceeds the objective mortality, with $g = 0.1$. In Figure **??** on the right, the effect of a more expensive market is seen as a lower relative optimal bequest, whereas the relative optimal consumption curves remain the same. This is because the market's attitude towards actuarial risk is reflected in the bequest, as this is the part affected by actuarial pricing. In Figure **??** on the right, it is also observed that the relative difference is smaller for the lower relative optimal bequest, concluding that pricing has a greater effect on the deflated controls than on the nominal controls, as the difference between the two is smaller.

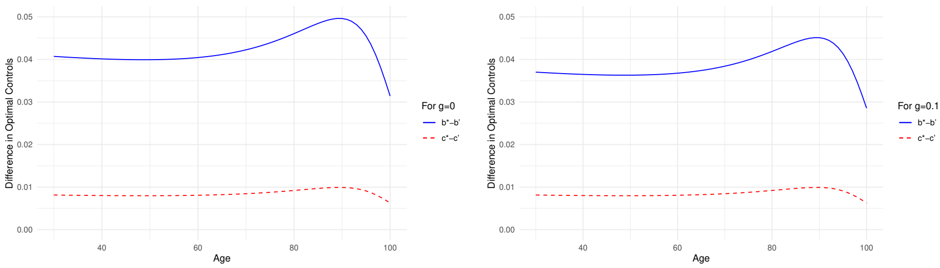

Figure 3.2: *The difference between the relative optimal deflated controls and the relative nominal values. Left for* $g = 0$ *and right for* $g = 0.1$

Chapter 4

Implicit Prioritization of Life Insurance Coverage: A Study of Customer Preferences in PFA Pension

Abstract

This study evaluates the utility derived by policyholders in a Danish pension company, from their life insurance coverages. We quantify the relative importance policyholders assign to their existing coverages versus a hypothetical complete coverage scenario, thereby measuring the implicit priority of their current coverage. By analyzing these implicit priorities based on individual attributes such as age, financial situation, and company agreement limitations, we gain a comprehensive understanding of policyholders' evaluations of their current life insurance coverage. Utilizing a continuous-time life cycle model, we optimize consumption and life insurance decisions during the accumulation phase, applying well-established theoretical findings to actual data. Our analysis identifies trends, outliers, and insights that can inform potential improvements in life insurance coverage. This tool aims to assist policyholders in prioritizing their coverage according to their life situations and provides a foundation for advisory dialogues and product development.

4.1 Introduction

We formalize and compare the policyholders' preferences for life insurance coverage within the context of a Danish pension company. This is formalized by quantifying the relative importance (weights) poliyholders assign to their existing coverages and juxtaposing these with the weights they would allocate to a hypothetical scenario of complete coverage. Thus, we measure the implicit priority they place on their current coverage. A novel aspect of this study is the examination of implicit preferences derived from real-life life insurance products offered by a pension company.

Subsequently, we analyze the implicit priority for each policyholder based on their individual attributes and parameters. This analysis involves evaluating various factors such as age, financial situation, the framework and limitations of the company agreement, and personal choices by the policyholder. By aggregating and examining these priorities across the entire portfolio, we gain a comprehensive understanding of policyholders' evaluations of their current life insurance coverage. This analysis enables us to identify trends, detect outliers, and uncover insights that can inform potential improvements or further investigations into policyholders' current coverage. The main contributions are establishing a framework handling implicit prioritization mechanisms in a life cycle model to evaluate preferences and explore from real data the practical consequences of decisions on life insurance products on an individual personal level.

When considering optimal consumption and insurance problems, practitioners commonly ask how to incorporate these theoretical results into business. *How can we gain practical tools and insights into our company through this largely investigated area?*

We address this question using a continuous-time life cycle model, which is a framework to describe how policyholders make financial decisions over time, taking into account the uncertainty of the future, such as mortality and income. Wherein policyholders optimize their consumption and life insurance decisions during the accumulation phase. Our analysis is confined to the accumulation phase, as policyholders primarily acquire life insurance coverage during this period.

Policyholders' preferences regarding risk, insurance, and consumption are fundamental to decision-making, and the measuring and quantifying of these pose significant challenges, as a standardized method is lacking. Preferences are a part of the utility functions, serving as a tool to rank different choices and clarify decisions in uncertain scenarios. Quantifying these preferences has traditionally been studied by the so-called risk aversion parameter, which is one or more parameters in the chosen utility functions. Quantification of risk aversion in decision-making was some of the first studies by Pratt, 1976 who, together with the work of Arrow, 1973, studied the concavity of utility functions and related it to the degree of risk aversion by the agent.

Our optimization uses a power utility function to model the policyholders' preferences and risk aversion. The power utility function is also known as the Constant Relative Risk Aversion utility function and is a frequently studied choice in modeling decisions, see for instance Wakker, 2008.

The academic tradition of considering consumption-investment-insurance problems dates back to the consumption-investment framework of Merton, 1969, 1971, which together with the discrete-time insurance one of Yaari, 1965 was combined by Richard, 1975 in his foundational work. Since then, the consumption-investment-insurance problems have been expanded in various directions, bridging the gap between financial and actuarial literature. To point out some P. and Ye, 2007 did notable work in the financial domain and Kraft and Steffensen, 2008a in the insurance. Further generalizations related to the market can be found in the works of Duarte et al., 2014 and Shen and Wei, 2016. For insights into the generalization of preferences, see Steffensen and Søe, 2023, Tang, Purcal, and Zhang, 2018, and Zhang, Purcal, and Wei, 2021. Additionally, significant contributions regarding health risk have been made by Koijen, Van Nieuwerburgh, and Yogo, 2016, Hambel et al., 2017, and Steffensen and Søe, 2023. These generalizations are crucial for bridging the gap between stylized theories and the complexities and heterogeneities of the real world. We propose leveraging the extensive knowledge around choices and decision-making to analyze real-world data, focusing on policyholders' actual preferences and implicit priorities.

The theory builds upon the Bellman principle of optimality, illustrating a relation between risk aversion and optimal investment. This concept is reversed in the work of Falden and Steffensen, 2024, who calibrate risk aversion, using the investor's real financial allocation to measure their implicit risk aversion under the assumption that the allocation is optimal. Each position of the agent leads to an equation for risk aversion, resulting in an overdetermined system. Allowing them to define a mutual fund that aligns with the agent's preferences, distinguishing it from previous solutions to estimate the risk aversion, such as Holt and Laury, 2002 Conine, McDonald, and Tamarkin, 2017 and Burgaard and Steffensen, 2020.

The majority of studies designed to estimate risk aversion employ revealed preference methodologies, particularly concerning decisions involving lotteries. These studies offer a framework for empirically assessing individuals' degree of risk aversion. Noteworthy contributions to this field include the works of Holt and Laury, 2002 Azar, 2006, and Wakker, 2008. In the realm of insurance Cohen and Einav, 2007, used data on policyholders' choices on deductible contracts to estimate risk aversion and later Barseghyan et al., 2013 elaborated on this. Eliciting risk aversion parameters based on questionnaires have been done by, among others, Barsky et al., 1997, and Burgaard and Steffensen, 2020 combined elements from the revealed preferences techniques with the propensity measure by asking agents to choose among certainty equivalents. Our approach is closer to that of Falden and Steffensen, 2024 in intuition and motivation but not in methodology; the novelty of our methodology is to consider policyholders' choices as optimal strategies and establish the concept of the implicit priority of life insurance coverage by the policyholder based on their current and possible maximum coverages, as a measure.

We obtained portfolio data from PFA Pension to analyze and compare. Due to confidentiality constraints, we do not present the raw data, but we present visual illustrations of the implicit priorities derived from the data and interpret their effects. We conducted this analysis with PFA Pension, incorporating mutual reflections on the results. Our study focuses exclusively on consumption and insurance, deliberately excluding the investment component to highlight and isolate the desired effects.

This paper is structured as follows: Section 2 presents the theoretical foundation and explains all components and variables used in the analysis. Section 3 introduces the concept of Implicit Priority, the conceptual idea behind the analysis and our main contribution to the evaluation of policyholders' choices. Section 4 presents the numerical analysis, discusses the assumptions and descriptions and presents and discuss the results. Finally, Section 5 concludes and adress the limitations on methodology and data, and suggest further improvement and research possibilities.

4.2 Set-up and formulation of the problems

In this section, we present the central optimization problem, with a corresponding solution we compare with a special case and utilize to define the Implicit Priority.

We consider a life insurance policyholder in a classical two-state life insurance model, where the policyholder can either be alive or deceased. We assume that the individual has an uncertain lifetime and denote by μ the individual's mortality rate. The insurance company uses a deterministic interest rate r, thereby excluding the portfolio optimization, and a deterministic mortality rate μ^* for valuation purposes. The mortality rate μ is assumed to be deterministic and increases with age, thus excluding the modeling of stochastic longevity risk. However, not modeling longevity risk does not preclude us from modeling longevity itself, where mortality for a given age decreases over calendar time. If the mortality rate is deterministic, we can implement this effect by allowing the age-dependent mortality rate to vary with birth year. It is important to note that the objective mortality intensity μ may differ from the pricing mortality intensity μ^*.

We specify this by the underlying stochastic processes. We let N denote the counting process, counting the number of deaths, either being 0 or 1. Thus, at time t, the process $N(t)$ equals the number of deaths. Letting $I(t)$ indicate whether the policyholder is alive at time t. The expected number of deaths in the interval $[t, s]$, given the policyholder is alive at time t, is thus

$$\mathbb{E}[N(s) - N(t)|I(t) = 1] = \int_t^s e^{-\int_t^\tau \mu(u)du} \mu(\tau)d\tau,$$

and the probability that the policyholder is alive at time s, given they are alive at time t is

$$\mathbb{E}[I(s)|I(t) = 1] = e^{-\int_t^s \mu(u)du}.$$

Corresponding using the pricing intensity, we denote with *

$$\mathbb{E}^*[N(s) - N(t)|I(t) = 1] = \int_t^s e^{-\int_t^\tau \mu^*(u)du} \mu^*(\tau)d\tau,$$
$$\mathbb{E}^*[I(s)|I(t) = 1] = e^{-\int_t^s \mu^*(u)du}.$$

In this setting, the wealth of the policyholder develops as

$$dX(t) = (rX(t) + I(t)(a(t) - \mu^*(t)b(t) - c(t)))dt + b(t)dN(t), \tag{4.2.1}$$
$$X(0) = x_0.$$

Where x_0 denotes the initial wealth at time 0. The policyholder earns interest at a constant rate of r. While alive, the policyholder's financial activities include earning income a, paying life insurance premiums $\mu^*(t)b(t)$, and consuming wealth at rate c. Upon the policyholder's death, the beneficiaries receive a lump sum payment of b and the remaining wealth x. From this point, we introduce the concept of human capital, denoted as $g(t)$. Human capital represents the financial value of expected future income. Consequently, an individual's total wealth is given by $X(t) + g(t)$. The individual can hedge their future income by accessing the insurance market, thereby facing a complete market. Human capital is the unique value of the future income hedging portfolio.

4.2.1 The central problem

Based on the established setup, we examine the objective of maximizing the expected utility of consumption while the insured is alive and the life insurance coverage upon death until termination n, i.e.

$$\sup_{c,b\in\mathcal{A}} \mathbb{E}\left[\int_0^n u(t, c(t))I(t)dt + v(t, b(t) + x(t))dN(t)\right]. \tag{4.2.2}$$

The supreme is taken over consumption and insurance processes in the set of $\mathcal{A}$ of admissible controls, and the utility functions are specified as

$$u(t, c) = \frac{1}{1-\gamma} c^{1-\gamma}(t)\omega^0(t)^\gamma,$$
$$v(t, x) = \frac{1}{1-\gamma} x^{1-\gamma}(t)\omega^1(t)^\gamma. \tag{4.2.3}$$

Here, ω represents the temporal aspect of the utility function, specifically the weight the insured places on money at time t, taken to the power of γ for mathematical convenience. This weight, ω, is particularly interesting for our analysis. The value function is correspondingly formulated as

$$V(t,x) = \sup_{c,b\in\mathcal{A}} \mathbb{E}_{t,x}\left[\int_t^n u(s,c(s))I(s)ds + v(s,b(s)+x(s))dN(s)\right], \tag{4.2.4}$$

where $\mathbb{E}_{t,x}$ denotes the conditional expectation, given that $X(t) = x$ and $I(t) = 1$.

We say the controls (c, b) are admissible if first, the insured does not have a negative total capital in the sense of wealth, including human capital, such that $X(t) + g(t) \geq 0$ for all $t \in [0, n]$. Second, (4.2.1) needs a unique solution, and third the expectation in (4.2.4) is well defined, and finally

$$\mathbb{E}[\int_0^n b(t)(dN(t) - \mu(t)I(t)dt)] = 0.$$

We denote by $\mathcal{A}$ the set of admissible controls. With the value function candidate

$$V(t,x) = \frac{1}{1-\gamma}(x+g(t))^{1-\gamma}f(t)^{\gamma}. \tag{4.2.5}$$

This is a specific instance of the problems addressed in, among others, Kraft and Steffensen, 2008b and Asmussen and Steffensen, 2020, where a verification theorem (Theorem 6.1 in Asmussen and Steffensen, 2020) along with a corresponding proof for the multi-state problem with constant risk aversion is provided. The problem is addressed using dynamic programming by solving the associated Hamilton-Jacobi-Bellman equation

$$\begin{aligned} V_t(t,x) &= \inf_{c,b}\Big\{ -(rx + a(t) - c - \mu^*(t)b)V_x(t,x) - u(t,c) \\ &\qquad - \mu(t)[v(t,x+b) - V(t,x)]\Big\}, \\ V(n,x) &= 0. \end{aligned}$$

Where the solution is presented as the solution to the differential equations for the g and f functions, which have the Feynman-Kac representation

$$g(t) = \mathbb{E}_t^*[\int_t^n e^{-\int_t^s r(u)du}a(s)I(s)ds], \tag{4.2.6}$$

$$f(t) = \tilde{\mathbb{E}}_t[\int_t^n e^{-\int_t^s \tilde{r}(u)du}(\omega^0(t)I(s)ds + w^1(s)dN(s))]. \tag{4.2.7}$$

where

$$\begin{aligned} \tilde{r} &= -\frac{\gamma}{1-\gamma}r - \frac{\gamma}{1-\gamma}(\mu^*(t) - \mu(t)) + \mu(t) - \tilde{\mu}(t), \\ \tilde{\mu}(t) &= \mu^*(t)\left(\frac{\mu(t)}{\mu^*(t)}\right)^{\frac{1}{\gamma}}. \end{aligned}$$

Such that g is the conditional expected present financial value of future income where the expectation is taken under the $\mathbb{P}^*$. And f is the conditional expected present value of future weights where the expectation is taken under an artificial measure where N admits the intensity process $\tilde{\mu}$. In other words an artificial financial value of future weights applying an artificial stochastic interest rate and valuation measure. Further, the corresponding optimal controls solving the problem are

$$c(t,x) = \frac{\omega^0(t)}{f(t)}(x + g(t)), \tag{4.2.8}$$

$$b(t,x) + x = \frac{\omega^1(t)}{f(t)}(x + g(t)) \left(\frac{\mu(t)}{\mu^*(t)}\right)^{\frac{1}{\gamma}}. \tag{4.2.9}$$

Here, the optimal consumption rate is a fraction of the total wealth, where the fraction measures the utility of present consumption against the utility of consumption in the future. Similarly, the optimal insurance and reserve payout upon death is a fraction of wealth, with an additional pricing factor. The fraction $\omega^1(t)/f(t)$ measures the utility of the lump sum upon dying against the utility of future consumption.

4.2.2 The special case

The idea of the special case is to examine the optimal bequest formula (4.2.9) which is the wealth just before death multiplied by two factors; $\frac{\omega^1(t)}{f(t)}$ and $\left(\frac{\mu(t)}{\mu^*(t)}\right)^{\frac{1}{\gamma}}$ and imagine the situation where both factors are one. Thus, having a completely fair insurance contract with no risk/premium loading, $\mu^* = \mu$, and the weight upon bequest ω^1 equal to the conditional expected present value of future weights, f; in other words, the significance of bequest would be the same as the significance of future consumption prior death. We speak of this as full or complete coverage inspired by Asmussen and Steffensen, 2020, with $b = g$. This is a full compensation of the potential financial loss the heirs of the policyholder would experience upon the death of the policyholder.

This is a corner result and allows us, in general, to evaluate and interpret the optimal bequest (4.2.9) dependent on the preferences and price of insurance since the two factors represent deviations from the complete coverage. If the insurance is expensive $\mu^* > \mu$, the factor regarding price, $\left(\frac{\mu(t)}{\mu^*(t)}\right)^{\frac{1}{\gamma}}$ defines depending on the preference γ how much to under-insure compared to the complete coverage, and corresponding how much to over-insure if the insurance is cheap. The other factor is of great interest to this investigation since it represents the appreciation of consumption of bequest upon death through the parameter ω^1, and the appreciation of accumulated consumption just before death is expressed by f. Thus, if the policyholder values the consumption of bequest upon death higher than consumption before death, one should over-insure compared to the complete coverage, and vice versa if consumption prior to death is valued higher than consumption upon death.

With this foundation established, we aim to isolate and emphasize the effects of interest. Therefore, instead of having two distinct weights, ω^0 and ω^1, in the utility functions (4.2.3), we set $\omega^0 = 1$ and redefine ω^1 as ω. This allows us to focus solely on the weight the insured places on their life insurance coverage. This weight, ω, is precisely the pivotal element of our analysis and this defined

Proposition 4.2.1. *For $t \in [0, n]$, assume $X(t) = x$, $\mu^* = \mu$, and $x + g(t) \geq 0$. Let everything known from data be denoted with a bar. Assume the functions $\bar{\bar{r}}(u)$, $\bar{\mu}(u)$, and $\bar{a}(s)$ are known and continuous over the interval $[0, n]$.*
The optimal bequest strategy is defined as

$$\bar{b} + x = \frac{\omega(t)}{f(t)}(\bar{x} + g(t)), \tag{4.2.10}$$

with the corresponding functions

$$f(t) = \int_t^n e^{-\int_t^s (\bar{\bar{r}}(u) + \bar{\mu}(u))\, du}(1 + \omega(s)\bar{\mu}(s))\, ds, \tag{4.2.11}$$

$$g(t) = \int_t^n e^{-\int_t^s (\bar{r}(u) + \bar{\mu}(u))\, du}\bar{a}(s)\, ds. \tag{4.2.12}$$

Here, $g(t)$ consists of known parameters and can be calculated by numerical integration. The function $f(t)$ is dependent on ω, and thus ω can be solved numerically.

4.3 The Implicit Priority

This section presents the general concept and establish the framework for constructing the Implicit Priority of life insurance coverage.

If we assume values for the relative risk-aversion parameter based on previous studies of the Danish population by Burgaard and Steffensen, 2020 and with the data we have avalible, the only unknowns in (4.2.10) are the weight, ω, quantifying the significance the policyholder places upon the consumption of bequest upon death, compared to the consumption prior to death. In two distinct scenarios, we examine the policyholder's choices as the optimal strategies, and with the data supplied, we can isolate the weight ω and asses the significance in the context of the insured's overall financial strategy and thus their implicit priorities.

1. In the first scenario, we consider the currently chosen life insurance payout to be the optimal life insurance coverage. We calculate backward to determine: *'If this life insurance coverage were optimal for the insured, what weight would correspond to the insured's valuation of it?'* In other words, we aim to understand how the insured values this life insurance payout relative to their intermediate money consumption. Thereby understanding the factor $\frac{\omega(t)}{f(t)}$ and the policyholder's appreciation for under og over insurance while noting that

ω itself plays a role in f as defined in (4.2.11). We denote the weight from this scenario $\omega^{current}$.

2. The second scenario extends the special case explained in the previous section 4.2.2 if, instead of the current life insurance coverage, the insured had the complete coverage equivalent to their human capital, g, we determine the weight, ω, that would correspond to the insured's valuation of this coverage. In other words, we aim to understand how the insured would value complete coverage relative to their intermediate consumption in this scenario. Serving as a reference to the examination of appreciation the policyholder has for over or under insurance, as this is exactly the value placed on the consumption of bequest if the policyholder had complete coverage. We denote the weight from this scenario $\omega^{complete}$.

From these two scenarios, we define the implicit priority of the current life insurance coverage as

$$\mathscr{I}(current) = \frac{\omega^{current}}{\omega^{complete}}. \tag{4.3.1}$$

Where the weights in each situaiton are defined as in Proposition 4.2.1. This implicit priority, $\mathscr{I}$, can be systematically compared across policyholders to evaluate how the insured prioritizes their current coverage relative to having all future income covered. The weight assigned to the expected future income as a life insurance payout is fixed, while the individual current payout can be adjusted. A high implicit prioritization of the life insurance payout indicates that the current payout is deemed highly valuable. Conversely, a low implicit prioritization of the life insurance payout indicates that the current coverage is considered lower value. The essential aspect is that the implicit priority is based on the policyholder's current situation, but it can be compared with the implicit priority of other policyholders.

This idea is based on several crucial assumptions. Firstly, it is assumed that the insured optimally consumes the funds until death, thereby reducing the discussion of consumption and with the weight placed on consumption being one still including it as a reference. Secondly, investments and an uncertain market are not considered. Lastly, the wealth used in the calculations is the accumulated value of the wealth in the pension. Our analysis focuses exclusively on policyholders who are not yet retired and are currently paying premiums.

4.4 Numerical analysis

We analyze 372667 policies with life insurance coverage from the Danish pension company PFA, with the relevant parameters chosen through mutual discussion. Our presentation focuses solely on the results derived from this data rather than the data itself. We first outline our assumptions and describe the selected parameters, and then present and discuss the results.

4.4.1 Assumptions and descriptions

We assume the mortality rates follow the Danish Financial Supervisory Authority's (Finanstilsynet) life expectancy model[1], using intensities provided by PFA Pension incorporating improvements in life expectancy. This means that the mortality intensity for women and men with age x at time t can be described as

$$\mu_{women}(x,t) = \mu_{women}(x,2023)(1 - R_{women}(x))^{t-2023},$$
$$\mu_{men}(x,t) = \mu_{men}(x,2023)(1 - R_{men}(x))^{t-2023}.$$

Where $\mu_{women}(x, 2023)$ and $\mu_{men}(x, 2023)$ for $x = 0, ..., 110$ are the current mortality for women and men, respectively, and $R_{women}(x)$ and $R_{men}(x)$ for $x = 0, ..., 110$ are the future expected life improvements for women and men respectively. As previously explained, it is possible to have a risk loading if $\mu^* \neq \mu$, but for simplicity, we exclude this in the calculations and assume entirely fair insurance $\mu^* = \mu$, simplifying the problem significantly. Based on Burgaard and Steffensen, 2020 we let the risk aversion for female be $\gamma_{women} := 2.25$ and for men $\gamma_{men} := 1.86$.

Moreover, we assume the wealth of the insured consists exclusively of the amount accumulated within the pension company. The payout upon death is equal to this accumulated wealth, the bequest wealth from the pension company, along with one other potential extra lump sum payment from outside the company made at the time of the policyholder's death.

In our analysis, we aim to isolate the effects of various attributes on policyholder behavior by segmenting the data into distinct groups. This approach allows us to identify patterns and correlations within the portfolio. The key covariates used for this segmentation are defined as follows: First, we consider each policyholder's coverage level. This is categorized into three distinct groups: those whose coverage is at the maximum allowable level under their agreement, those whose coverage falls between the minimum and maximum levels, and those whose coverage is at the minimum permissible level. Secondly, we classify policyholders based on their annual salary. The salary categories are defined as follows: less than 80,000 DKK, between 80,000 and 400,000 DKK, between 400,000 and 1,000,000 DKK, and greater than 1,000,000 DKK. These categories help us understand the financial background of the policyholders and its potential impact on their decisions.

Another significant covariate is the recommendation status. Policyholders who have completed an online questionnaire provided by the pension company and received

[1] `https://www.finanstilsynet.dk/finansielle-temaer/forsikring-og-pension/levetidsmodel`.

some coverage recommendations are classified as having ”Recommendation Viewed.” Conversely, those who have not completed the questionnaire are classified as having ”Recommendation Not Viewed.” We also differentiate policyholders based on the type of life insurance benefit they have. Those who retain the default coverage provided by the agreement are classified under ”Standard Life Insurance Benefit,” while those who have modified their coverage from the default are classified under ”Custom Life Insurance Benefit.” Lastly, we consider whether a policy is broker-assisted. Policies managed with a broker's assistance are classified as ”Broker-Assisted,” whereas those managed without broker assistance are classified as ”Non-Broker-Assisted.”

These covariates are crucial for our analysis as they allow us to group policyholders into homogeneous segments, facilitating a more granular examination of behavior and outcomes. By analyzing these segments, we can better understand the impact of different attributes on policyholder decisions and overall portfolio performance.

If a policyholder holds multiple policies, these are aggregated and evaluated cumulatively, as it is assumed that the policyholders would have rights to all policies upon death. When aggregating the policies, we have assumed some way of aggregating the variables we use to examine the results, and for there, we have assumed the following

- For the coverage level, it is assumed that all coverages must be at the minimum or maximum limit for the policyholder to be noted as being at the respective limit.

- The policyholder is cumulatively noted as ”Recommendation Viewed” if at least one of the policies has received a recommendation.

- The policyholder is noted as ”Custom Life Insurance Benefit” if this is the case for at least one of the policies.

- The policyholder is noted as ”Broker-Assisted” if this is the case for at least one of the policies.

Another critical assumption is that the provided salary is constant; extending it to an adjustable wage is not tricky. The discount rate is set at 3.9%, given that the analysis's purpose is to evaluate the value of future money placed in the PFA Pension internally, with the knowledge that it can carry risk. With the implicit priority as $\mathscr{I}(current) = w^{current}/w^{complete}$ and 110 used as the maximum possible living age. Composite Simpson has been used to calculate the numerical integrals; further explanation can be found in Appendix A.

4.4.2 Benchmark

To assess whether the insureds are generally over- or under-insured, a reference point is usually used to evaluate whether their coverage exceeds or falls short. This reference could be three times their annual income, with adjustments made in special cases where more detailed policyholder information is available. This evaluation method serves as a reference point, thus we introduce a benchmark coverage, and thereby a simplicity priority of this benchmark $\mathscr{I}(Benchmark)$. Representing the implicit prioritization a policyholder would have if their life insurance coverage were set at three times their annual income. In this calculation, we assume their life insurance coverage is precisely this benchmark and compare it in the same manner as their current coverage. Specifically, we analyze how the individual prioritizes this benchmark coverage relative to consumption, compared to having all future income covered as a life insurance sum.

4.4.3 Results and discussion

First, we examine the entire population and then break it down into individual components that we consider to be of greater significance. In all illustrations, the implicit priority $\mathscr{I}$ of the life insurance payout is depicted as a function of age. Each point represents a policyholder, illustrating how they individually prioritize their current death benefit relative to consuming funds compared to having full coverage of all future income.

When discussing implicit priority, $\mathscr{I}$, it is essential to note that we do not expect it to equal one or any other predetermined number. As previously described, having all expected future income covered is an extreme case. However, we do expect some form of consistency among groups of individuals with similar characteristics. Therefore, further investigation could be warranted if the data shows tendencies towards lower or higher prioritization for several groups or subdivisions. To illustrate the distribution of

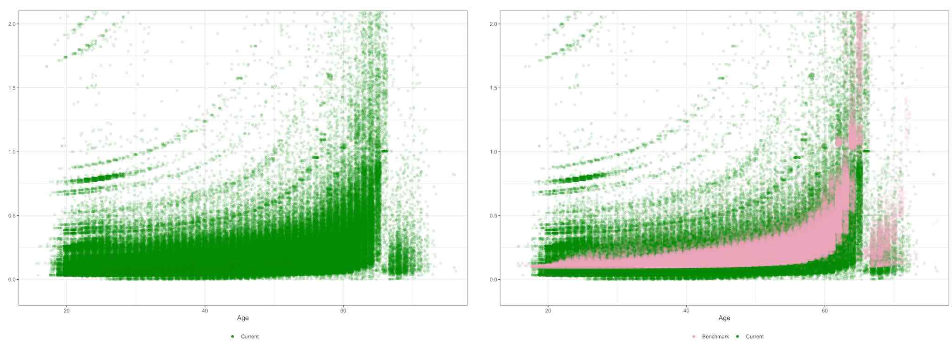

Figure 4.1: *$\mathscr{I}$ for the entire population. Left: $\mathscr{I}(current)$. Right: Compared with the corresponding $\mathscr{I}(Benchmark)$ in pink.*

the entire population, Figure 4.1 displays all $\mathscr{I}(current)$ aggregated for the currently selected coverage, compared with the benchmark implicit priority $\mathscr{I}(Benchmark)$ for the entire population, marked in pink. This results in a large green cloud, where we can observe several bands that lie higher than the majority. However, due to the high number of points, it is challenging to form a clear overview.

To establish an interpretative baseline, an implicit prioritization of exactly one can be understood as the policyholder implicitly prioritizing their current coverage as highly as having all their future income covered. Those with values above one implicitly prioritize their life insurance coverage more than all the income they can expect to earn until retirement.

We aim to demonstrate how policyholders individually prioritize their life insurance coverage within the population. Assuming that somewhat similar policyholders should exhibit similar $\mathscr{I}$, we group them to highlight outliers with two main points:

- Extra information can shed light on some outliers, and their implicit prioritization might indicate a deliberate and sensible choice.
- For some, it might be essential to ensure the policyholder is aware of the prioritization of their life insurance coverage and how they compare to similar policyholders.

To identify trends, we specify the analysis and attempt to find correlations and outliers that we can discuss. This will help us identify smaller groups with more similarities, allowing us to make more specific statements about how individuals prioritize their life insurance coverage compared to similar policyholders.

Given that PFA Pension is a private pension company with a substantial portion of its population drawn from various companies, it is common for coverages to be determined by agreements specific to each company. Consequently, we aim to examine the parameters of these agreements and analyze how policyholders have made choices and prioritized within these constraints. In Figure 4.2, we examine the minimum and maximum boundaries of the company agreements that policyholders are mandated to follow. For policyholders whose coverage lies on the minimum limit or between the minimum and maximum boundaries, we further illustrate where the middle 90% of all data points for each age group are located, marked by black vertical lines. The prominent point here is the median for each age group. There is a wide dispersion in the cloud of points, but the band is considerably narrower when focusing on the middle 90%. Additionally, a relatively stable median gradually increasing until a few years before retirement age is visible. A possible interpretation could be policyholders not regularly updating their life insurance coverage, which may become less accurate over time.

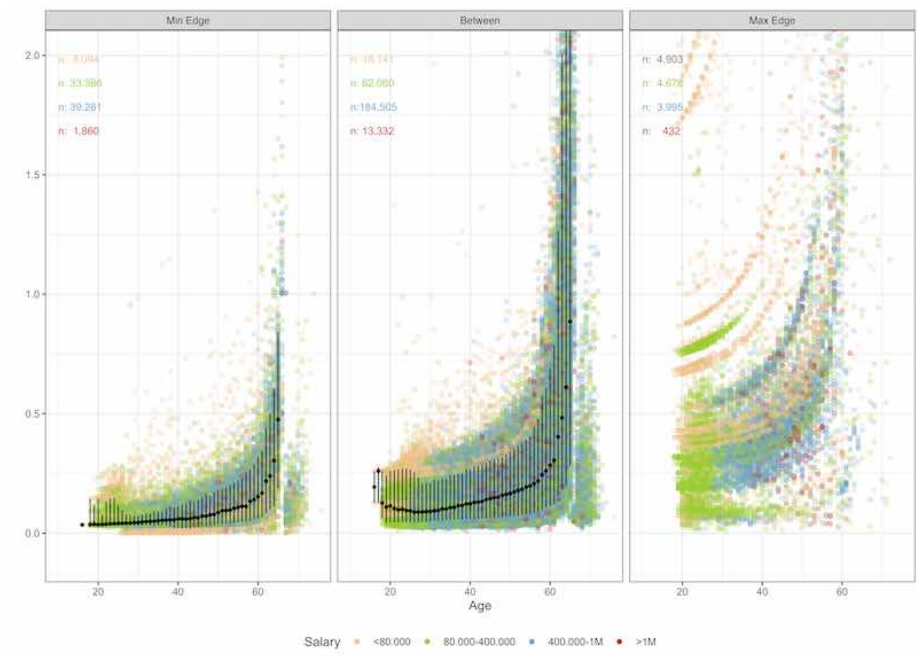

Figure 4.2: *$\mathscr{I}(current)$, categorized by the minimum and maximum coverage limits, along with yearly salary. Middle 90% and median are highlighted where relevant.*

It is evident that almost all notable outliers of $\mathscr{I}(current)$ in the previous plots are at the maximum boundary of their agreement, with all income levels are represented. The same division is shown in Figure 4.3, additionally split by income instead of overlaying the colors. Here, the middle 90% of the points are between the solid lines, with the median marked by the dashed line. Allowing for a clearer view of how the median and the concentration of points change across different income groups. Assuming the boundaries are a potential limitation on the policyholder's choices, it further depends on whether the policyholder has consciously chosen to have maximum or minimum coverage or simply stayed with a default coverage. Therefore, we examine whether the policyholder has the standard life insurance sum as coverage specified in the agreement or has chosen something different. When a policyholder has received a recommendation, it indicates that they have completed a questionnaire regarding coverage available on PFA's website. We have included markers for the middle 90% points and the median to better illustrate the development and the concentration of policyholders. In most cases, policyholders who have chosen something other than the standard coverage have opted for higher coverage. This may be due to the greater distance from the standard to the maximum compared to the standard to the minimum. Since receiving a recommendation only indicates whether the policyholder has completed PFA's questionnaire, we have in Figure 4.5 included whether the policyholder is broker-administered, as shown in the graph on the left.

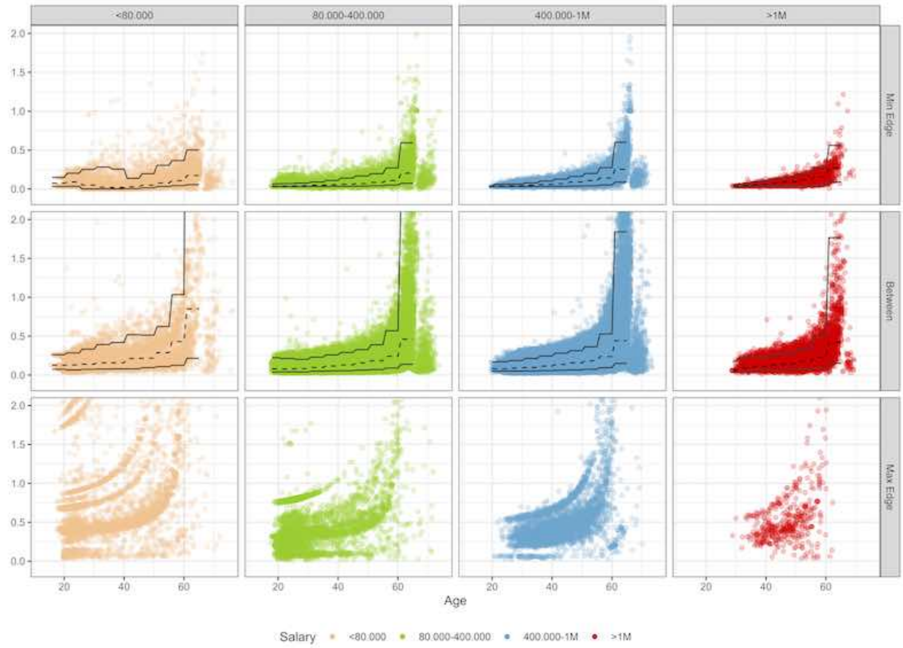

Figure 4.3: *$\mathscr{I}(current)$, categorized by the minimum and maximum coverage limits, along with yearly salary. Middle 90% and median are highlighted where relevant.*

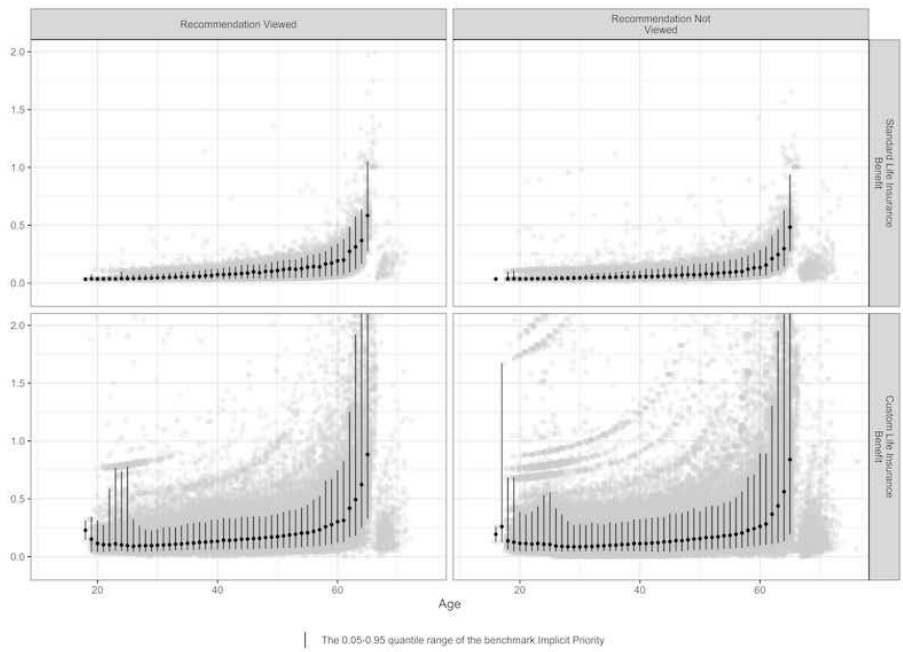

Figure 4.4: *$\mathscr{I}(current)$, categorized on recommendation and standard coverage or not, with the central 90% range and median highlighted.*

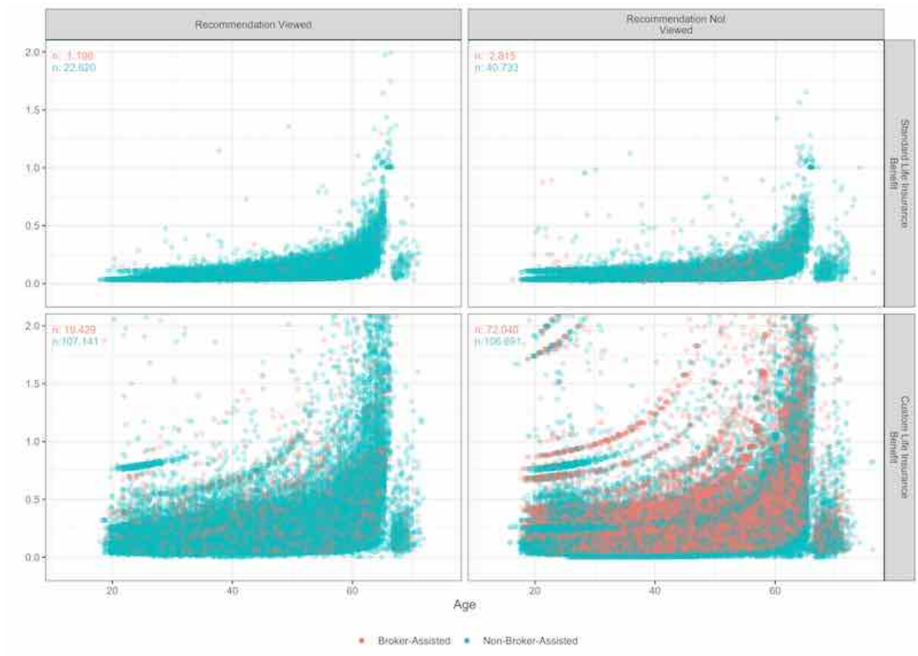

Figure 4.5: $\mathscr{I}(current)$, *categorized on recommendation and standard coverage or not, and indicated if broker-assisted or not.*

We assume that if they are broker-administered, the brokers have had a dialogue with the policyholder and have, therefore, gone through a form of recommendation marked in pink. The most interesting points from these graphs are possibly the policyholders who have chosen something other than the standard coverage, indicating a conscious decision, which results in a very high or very low implicit priority of life insurance coverage. Furthermore, those who are neither broker-administered nor have received a recommendation but have chosen to prioritize their coverage very highly can be seen in the lower right corner with the prominent blue high points.

Therefore, we have chosen to delve deeper and examine only those who have chosen something other than the standard coverage and how their implicit prioritization of life insurance coverage relates to the boundaries of the agreement. This is shown in Figure 4.6. Here, all those who have chosen something other than the standard coverage are shown, representing a form of choice by the policyholder. Even with a conscious choice of the highest possible coverage within the agreement, many have a low implicit prioritization of their life insurance coverage compared to all their future income. This indicates that they may be under-insured in their economic situation and desire higher coverage but cannot choose it due to the agreement's constraints; conversely, for the few who implicitly prioritize their life insurance coverage highly but have chosen the agreement's minimum boundary.

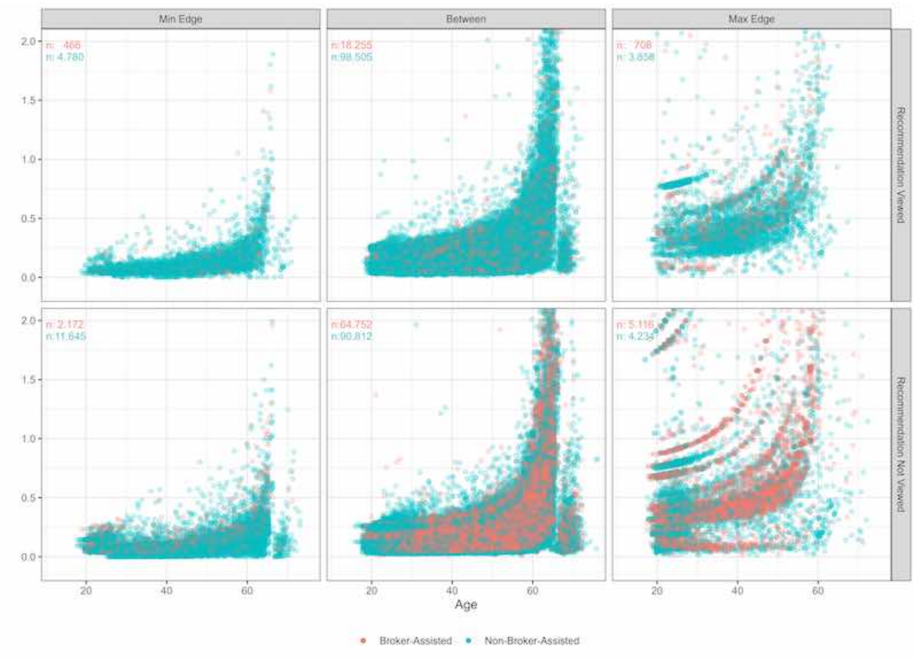

Figure 4.6: *$\mathscr{I}$(current) for those with Custom Life Insurance Benefit, categorized on coverage, recommendation and broker-assistance.*

Additionally, it remains interesting to investigate the very high implicit priorities, where policyholders prioritize their life insurance coverage much higher than all their future income. To provide an overview and a comparison with the benchmark of three times annual income, Figure 4.7 shows that all the gray dots represent the policyholders' implicit priority of the benchmark coverage, $\mathscr{I}(benchmark)$ relative to all their future income. The black vertical markers represent the middle 50% of the $\mathscr{I}(Benchmark)$ values, and the purple marker illustrates where, for each group, the current middle 50% of implicit prioritizations of current coverage $\mathscr{I}(current)$ are located. This cle arly shows that the benchmark encapsulates policyholders who fall between the agreement's minimum and maximum boundaries with all different income levels. However, for policyholders at the agreement's minimum boundary, the benchmark is much too high. For the maximum boundary, the current middle 50% is almost entirely outside the interval where the benchmark values lie.

This demonstrates how the benchmark can serve as a good starting point for a general guideline for most policyholders. However, to guide and examine the entire population, it may be beneficial to go down to a more individual level and find a guideline for how more policyholder can be advised based on their individual situations.

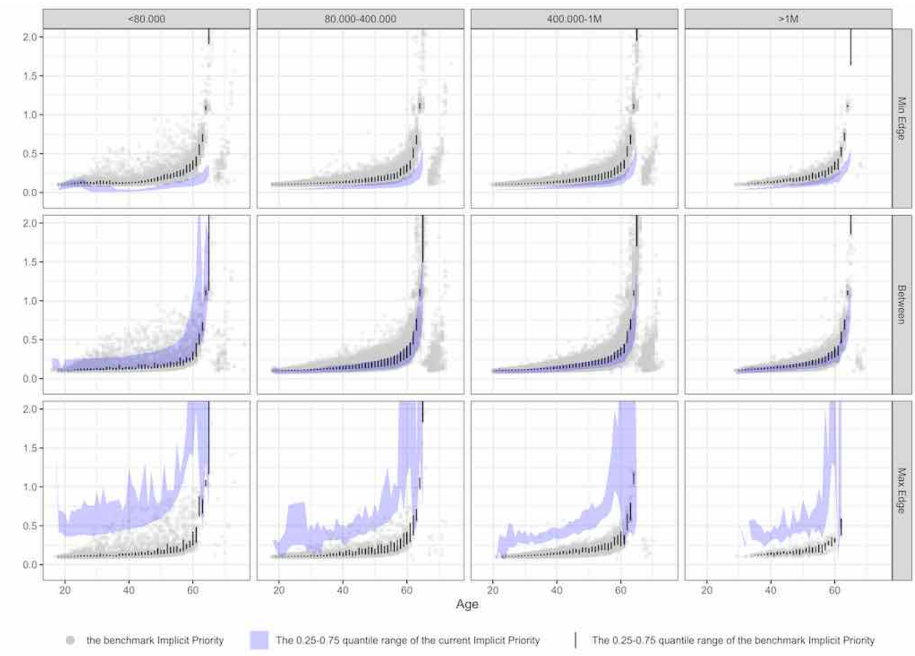

Figure 4.7: *$\mathscr{I}(current)$ compared with $\mathscr{I}(Benchmark)$ categorized by yearly salary and coverage.*

4.5 Conclusion and future work

We formalize the implicit priority based on the policyholder's choices within a continuous-time life-cycle model. Initially, we have established this framework specifically for decisions related to life insurance coverage. However, extending it to include other insurance products, such as disability coverage, is of significant interest, as this would offer a more nuanced understanding of the policyholder's actual choices. After all, the policyholder does not merely choose between life insurance coverage and immediate consumption.

Assuming the policyholder's preferences follow a power utility function, the optimal consumption and bequest strategies are derived using the Hamilton-Jacobi-Bellman equation and used to measure the appreciation policyholders place upon their life insurance coverage compared to intermediate consumption.

We suggest a method to evaluate the implicit priority of a policyholder's current coverage by measuring the appreciation of the life insurance coverage in two different scenarios, first assuming the current coverage is optimal and second concerning the complete coverage. Real data on each policyholder allows us to evaluate and compare the implicit priority across different sub-portfolios.

The study relies on the assumption of power utility functions and constant relative risk aversion. Together with the inclusion of optimal investment, it would be an intuitive step to include more refined optimal consumption and insurance solutions with more real-world aspects to reflect reality better. The exclusion of investment optimization and limiting the insurance framework to that of life insurance are deliberate choices to develop the concept of the implicit priority in the simplest possible setting, making it a starting point and it would be a natural extension to investigate.

The data for our study is extensive but limited to one pension company. As a result, policyholders may have savings and other financial dependencies outside of this closed environment. Additionally, the specific living situations of the policyholders can significantly influence their preferences and would add depth to the analysis of implicit priorities.

4.6 Acknowledgements

I would like to express my sincere gratitude to PFA Pension for granting me access to their data and for their excellent cooperation throughout this project. Their support has been invaluable in conducting the analysis.

I am also deeply thankful to my colleagues at Mancofi for their dedication and generosity in sharing their time and expertise. Their contributions have been instrumental in making this project possible.

This work would not have been achievable without the support and collaboration of both PFA Pension and Mancofi, and I am truly grateful for their assistance.

Appendix A

Composite 1/3 Simpson

We have used the composite Simpson's rule to compute numerical integrals in implementing various formulas.

The interval $[a, b]$ is divided into n sub-intervals of length $h = \frac{b-a}{n}$. Note that n must be even. Since $x_i = a + ih$ for $0 \leq i \leq n$ with $x_0 = a, x_n = b$.

$$\int_a^b f(x)\,dx \approx \frac{1}{3}h\left[f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + f(x_n)\right] \tag{4.A.1}$$

The error in this method is given by

$$-\frac{1}{180}h^4(b-a)f^{(4)}(\xi) \tag{4.A.2}$$

where ξ is a number between a and b. If we choose $n = (b-a)^2$, we will obtain the same error every time. This error will be bounded by

$$\frac{1}{180}\frac{1}{(b-a)^3}\max_{\xi\in[a,b]}\left[f^{(4)}(\xi)\right] \tag{4.A.3}$$

Since we have chosen n in this way, it may turn out that n is odd. In that case, we make n even by adding 1.

Chapter 5

Introduction to time-inconsistent optimization problems

We explored time-consistent optimization problems through dynamic programming in the first three chapters. In this chapter we introduce the concept of time-inconsistent optimization problems and usage in two different questions regarding life insurance optimization problems and the contributions to these concepts from chapter 6 and chapter 7. As in chapter 1, this chapter serves to build an intuition and background to comprehend the forthcoming chapters on the topic, both being independent manuscripts thus containing an elaborate introduction and relevant references.

5.1 Time-inconsistent preferences

The time-inconsistent optimization problems arise when preferences or constraints change over time, so that the optimal strategy at one point may not remain optimal later. To address these challenges, we turn to equilibrium theory, which provides a framework for finding consistent solutions over time despite the inherent inconsistencies in preferences or constraints.

Imagine an individual who decides to save money for a dream vacation. On Sunday night, she plans to save a certain amount of money each week. She sets up a budget and decides to cut back on non-essential spending. On Monday, her motivation is high, and she sticks to the budget and avoids unnecessary purchases. However, by Wednesday, she sees a sale on something she has desired for a while and decides to indulge and spend some of the money she had planned to save.

Illustrating, comprehensively, the concept of time inconsistency. The optimal strategy for spending Sunday night is not consistent with that of Wednesday. Equilibrium theory deals with these problems and helps us find solutions that remain consistent over time despite changing preferences. Continuing with the example, this could be setting aside a smaller portion so she could indulge in some spending occasionally without disrupting the complete plan and providing a balance between long-term and short-term goals and desires, making an individual consistent with their current and future strategies.

Regarding time-inconsistent control theory, handled by equilibrium theory, we address two approaches: first, separating preferences by aggregating certainty equivalents, and second, the problem of optimizing random preferences. Both methodologies leverage the concept of certainty equivalents yet diverge conceptually in their application and underlying rationale.

The first approach, the separation of preferences by aggregation of certainty-equivalents, addresses the question of disentangling the risk aversion from the aversion to variation over time by the notion of elasticity of inter-temporal substitution. This method, formalized first in discrete time and later in continuous time, introduces an aggregator function that can capture the effect of the underlying preferences toward risk and the time variation. By introducing the elasticity of inter-temporal substitution as a notion of preference for aversion towards variation over time, one can construct a value function that optimizes decisions over a policyholder's uncertain lifespan. The certainty equivalents are here used to transform uncertain future consumption rates into certain equivalents, to say what uncertain future consumption amount they would be willing to exchange for a specific current rate, thus simplifying the decision-making process in the face of temporal inconsistencies.

The second approach handles the optimization of random preferences, allowing preferences to vary in various unknown ways and acknowledging how policyholders are often unaware of their future risk aversion preferences. Imagine having to choose between different investment strategies but, at the same time, being uncertain about the preferences for these investments, which are expected to change over time. A way to handle this is by certainty equivalents so that instead of averaging preferences that lack real-world applicability and interpretation, one can use certainty equivalents. These have the same interpretation as in the first approach, namely the amount of money a policyholder would accept today instead of a future uncertain, risky amount. Thus providing a monetary, clear economic, and interpretable value. Once again, the introduction of certainty equivalents introduces mathematical complexity and time-inconsistent optimization problems.

Both approaches offer unique insights and solutions to the complexities of time-inconsistent controls, each with its distinct perspective on the role of certainty equivalents in managing temporal decision-making challenges.

5.1.1 Separation of Preferences by Aggregation of Certainty-Equivalents

When studying optimal consumption and investment strategies, the separation of preferences by aggregation of certainty-equivalents offers a powerful framework for addressing the complexities of policyholders' preferences in life and pension by disentangles, as mentioned, risk aversion and aversion towards variation over time.

A fundamental method is the construction of the aggregator function of the two underlying preferences. This was first formalized in discrete time with an objective function that separates these preferences; when translated to continuous time, a corresponding value function could have implicit representation as

$$V(t,x) = \sup_{c,\pi} E_{t,x}\left[\int_t^\infty f(s, V(s))ds\right].$$

Here, the aggregator function $f(c, V)$ depends on the structure of preferences towards risk and variation over time. The parameter ϕ represents the elasticity of intertemporal substitution, which is the reciprocal of variation aversion. For constant relative risk aversion and variation aversion, the aggregator can be formulated as

$$f(c,V) = \frac{1-\gamma}{1-\phi}\delta V\left(\left(\frac{\frac{1}{1-\gamma}c^{1-\gamma}}{V}\right)^{\frac{1-\phi}{1-\gamma}} - 1\right).$$

This framework allows for generalizing the disentanglement of risk and time preferences. The generalization to lifetime uncertainty is a bit more complicated, since the value function itself appears as an argument in the value function, so if one updates its with mortality uncertainty there is no clear argument for why the updated argument considers the aversion to lifetime uncertainty.

A generalization, that forms a global objective with global risk and time variation disentanglement without lifetime uncertainty can therefore instead read

$$V(t,x) \quad = \quad \int_t^\infty v(u^{-1}(E_{t,x}\left[u(t,s,c(s))\right]))ds,$$

where u is the utility function containing risk aversion and v is a time preference function containing variation aversion. The consumption-investment strategy this problem coincides with that from the previous suggested form of separating preferences. This value function is constructed in two steps: The argument of v, namely $u^{-1}(E_{t,x}\left[u(t,s,c(s))\right])$, is the certainty equivalent, turning uncertain future consumption rates into certain rates. This 'deletes' uncertainty from the objective. The function v expresses preferences concerning time variation of certain consumption rates. Unlike recursive utility, this certainty equivalent is based on actual consumption utility, not indirect utility. Mathematically, this changes the optimization problem. If v and u are the same, $v(u^{-1}(\cdot))$ vanishes, and the expectation goes outside the integral, reverting to a standard objective. With different v and u, the integral forms a sum of non-linear functions of conditional expectations, breaking time consistency and standard dynamic programming. Then addressed by equilibrium theory, akin to how a sophisticated individual handles time-inconsistent problems.

The contributions from chapter 6

In Chapter 6, we compare the indirect utility of individuals with and without access to annuities in two frameworks: time-additive utility and separated time and risk preferences. Time-additive utility combines aversion to risk and variation over time in consumption over time into a single parameter, as explained in the first chapters. In comparison, the second framework has a separation of these preferences and provides a more nuanced analysis, distinguishing between the elasticity of inter-temporal substitution and risk aversion.

We focus solely on the decumulation phase, since it is here the annuity market becomes valuable to the individual, simply because mortality rates in the accumulation phase are low. Further, we can avoid the mathematical complication it would take to study the accumulation phase since we need to be able to calculate lifetime consumption in a market without access to annuitization to compare the situations with and without annuitization.

This chapter contributes to the understanding of annuitization decisions by explicit characterization of the relative loss of wealth for individuals who lose access to annuities. We specify the optimal payout profiles of retirement products with and without mortality credits. The preference parameters, as well as the insurance and financial market parameters, determine the optimal drift and volatility of the consumption/benefit profiles in the two frameworks. It turns out that in the optimal payout profiles are determined by the proportion invested in the risky asset, the interest, and the mortality basis used in the annuity when spreading out current wealth during the 'rest of the life'.

We compare the cases with and without access to the annuity numerically and find for realistic parameters a considerable loss of wealth if an individual without utility from bequest does not annuitize. This we study and discuss its dependence on preference and the parameters of the market.

Furthermore, we compare these findings with a particular suboptimal consumption plan that mimics an annuity-certain based on the conditional residual expected lifetime. Although this plan is not optimal, it offers valuable insights into alternative product designs that may appeal to individuals without access to annuities.

The results generally contribute to the optimal design of annuity contracts, both life annuities offered by pension funds and annuity contracts offered by banks. The results can help both financial regulators in reconsidering their framework for annuity designs and financial institutions in redesigning their product range and arguing for (or against) life annuitization.

5.1.2 Optimization under Random Preferences

Regarding optimal investment and life insurance strategies, the uncertainty of preferences is a vital question, *how do we know our current or future preferences for risk?*. Preferences vary significantly across different states, as discussed and motivated in the previous chapter on state-dependent utilities, but further than including the possibility of variation, there is also an unknown of the values. This heterogeneity necessitates a framework that can handle state-dependent risk aversion, where a stochastic process represents the state, and the values are unknown.

By, considering a finite-state continuous-time Markov Chain to model the biometric states, we account for the randomness in preferences. Thus, adjusting the utility function to reflect the state-dependent risk aversion allows a more realistic representation of the insured's preferences. The problem of optimizing terminal wealth, investment, and life insurance is formulated as

$$\sup_{\pi,\mathbf{b}} E\left[\frac{1}{1-\gamma(Z(T))} X(T)^{1-\gamma(Z(T))}\right],$$

where $\gamma(Z(T))$ represents the state-dependent relative risk aversion at time T. The wealth process $X(t)$ is influenced by both investment decisions $\pi(t)$ and insurance decisions $\mathbf{b}(t)$, with the dynamics described by

$$\begin{aligned} dX^{\pi,\mathbf{b}}(t) &= (r+\pi(t)(\alpha-r))X^{\pi,\mathbf{b}}(t)dt - \sum_{k:k\neq j, Z(t)=j} \mu^{*jk}(t) b^{jk}(t) I^j(t) dt \\ &\quad + \sum_{k:k\neq j, Z(t)=j} b^{jk}(t) dN^{jk}(t) + \pi(t)\sigma X^{\pi,\mathbf{b}}(t) dW(t), \\ X^{\pi,\mathbf{b}}(0) &= x_0. \end{aligned}$$

The objective here is a simplification of the problems from previous chapters, in the sense of merely maximizing the expected utility of terminal wealth. However, the previous approach of utility conditional on preferences and thereby ordering different strategies, has only relative meaning and the value functions based on its preferences are not comparable with other value functions with differrent preferences. Therefore, the expectiation of such value functions conditional on preferences gives little economically meaning, since each element only have relative meaning. Instead we adopt the idea to take the expectation over risk aversions of the certainty equivalents conditional on risk aversion. The certainty equivalents are monetary and thus the expectation makes economic sense and can be formulated as

$$J^{\pi,\mathbf{b}}(t,x,z) = E_{t,x,z}\left[v((u^{Z_T})^{-1}(E_{t,x,Z}\left[u^{Z_T}(X_T^{\pi,\mathbf{b}})\right]))\right].$$

Classical dynamic programming cannot solve this problem since the transformation $v \circ u^{-1}$ of the expectation interrupts the nice property of linearity of the expectation operator. This is a known consequence of using certainty equivalents.

The contributions of chapter 7

In Chapter 7, we address a decision problem related to life insurance, including the uncertainty of preferences in an optimal investment and the life insurance problem with state-dependent risk aversion, where the state is the biometric state of the insured. In this setting, the problem of optimal terminal wealth is formulated and solved by controlling investment and insurance. This builds on the intuitive rationality that preferences can be heterogeneous across different states, whether for single-agent models with health states, multi-agent, or multi-generation models, and includes the uncertainty of preferences. Including the randomness of preferences and evaluating certainty equivalents makes the problem time-inconsistent, which we deal with using the equilibrium approach.

This chapter aims to contribute to understanding optimal investment and life insurance decisions by incorporating state-dependent risk aversion into the utility function. We build on a lineage of continuous-time decision problems, integrating the Markov Chain framework to account for state dependency. By letting the risk aversion parameter be state-dependent within a general J-state Markov chain model, we provide a more realistic representation of preferences that vary with the insured's biometric state.

Chapter 6

What is the value of the annuity market?

Abstract

In the decumulation phase of a pension plan, consumption depends on the level of annuitization. We measure the welfare loss of an individual with a demand for annuitization if he has no access to annuitization or, equivalently, does not use such access. Unlike earlier studies of the value of the annuity option, both individuals with and without access to annuitization, respectively, are offered complete flexibility in the consumption/payout profile. In that sense, we assume that the financial institutions (are allowed to) design the best possible products in the two regimes, with and without annuitization. We find for realistic parameters that a patient individual with time-additive preferences loses 22% of wealth upon retirement if not annuitizing. Sensitivity studies show that the relative loss decreases with a higher interest rate, a higher market price of financial risk, a higher market price of mortality risk, more certainty in the lifetime distribution, and a lower elasticity of intertemporal substitution. Further, we analyze a suboptimal bank product based on conditional expected residual lifetime.

6.1 The Introduction

We compare the indirect utility of individuals without and with access to annuities and measure the relative loss of wealth for an individual with access to annuities who loses that access. We characterize the loss explicitly and study its sensitivity towards the interest rate, the market price of financial risk, the market price of insurance risk, and the level of uncertainty in the survival model. We characterize the loss for individuals with time-additive utility and individuals with separated time and risk preferences. Finally, we include in the comparison a particular suboptimal consumption plan without access to annuities, which might seem to be an appealing product design. All value functions are characterized explicitly such that all comparisons can be calculated directly as a solution to a non-linear equation, and all numerical illustrations are presented accordingly.

A standard question in pensions is whether and when to annuitize pension savings. If annuities are flexible and market returns drive payments, the question is whether the individual should put his wealth at stake and pick up so-called mortality credits. This depends on what the individual wants to leave behind when he dies. If he does not annuitize, his wealth goes to his inheritors. If he annuitizes, he leaves nothing. What is optimal depends on the individual's so-called utility from a bequest.

When discussing annuitization, it is relevant to quantify the benefit of annuitization to the annuitant. For example, if he has no demand for annuitization because his utility from bequest urges him not to annuitize, the value of the annuity market is zero. But what about the other extreme, where an annuitant wishes to annuitize fully? What is the maximal value of the annuity market to the annuitant? And how does this value depend on various parameters of the market and the mortality? These are the questions we wish to answer in this paper.

We answer these questions in a continuous-time life-cycle model where the annuitant can choose optimal consumption and investment in the decumulation phase. First, we calculate the individual's lifetime utility in the decumulation phase in markets without and with access to annuitization. We then compare these lifetime utility measures by translating them into wealth proportions using certainty equivalents.

We focus on the decumulation phase exclusively. It is not until the decumulation phase that the annuity market becomes valuable to the individual, simply because mortality rates in the accumulation phase are so low that they can probably be partly neglected. Then we can appropriately avoid the mathematical complication it would take to study the accumulation phase. The point is that we need to be able to calculate lifetime consumption in a market without access to annuitization to compare the situations with and without annuitization. However, with uncertain lifetime and labor income in the saving phase, this problem has no explicit solution. This is disturbing both if the saving rate is residual to optimal consumption and if it is a fixed ratio of the labor income. In both cases, we need a unique financial value of a fixed payment stream, which does not exist in a market without access to annuitization. Therefore, we focus on the decumulation phase and avoid resorting to numerics for non-explicit solutions.

For optimizing consumption and investment, we work with the power utility function. We consider two different cases. One case is the so-called time-additive utility, where the risk aversion parameter covers both aversions toward risk and variation in consumption over time. Aversion towards time variation is the reciprocal of the so-called elasticity of intertemporal substitution. In another case, we separate risk and time preferences. There is no consensus about how this should be done under an uncertain lifetime. We briefly review the different proposals in the literature and continue to work with one of them.

The optimal consumption and investment without access to life insurance are purely flexible bank savings products. Therefore, one can speak of the optimal consumption and investment plan as an optimal banking product design. There are, of course, many different suboptimal product designs of both bank and annuity products, but we pay attention to a particular suboptimal bank product. That behaves as an annuity-certain based on the conditional residual expected lifetime. However, since the conditional residual expected lifetime does not decrease linearly with age, it works as an annuity-certain with a moving time horizon. We compare the individuals consuming optimally with and without access to annuitization, respectively, to the individual offered the ingenious suboptimal product.

The bibliographical starting point of our work is Richard, 1975, who merged the consumption-insurance results by Yaari, 1965 with the consumption-investment results by Merton, 1971. This consumption-investment-insurance problem has been generalized in various directions since its revival in both the financial literature, see P. and Ye, 2007, and in the insurance literature, see Kraft and Steffensen, 2008a, contributing to closing some gaps between the financial and the insurance literature. These directions of generalizations include the market, see Duarte et al., 2014 and Shen and Wei, 2016; the preferences, see Tang, Purcal, and Zhang, 2018, Zhang, Purcal, and Wei, 2021 and Steffensen and Søe, 2023; the inclusion of health risk, see Kraft and Steffensen, 2008a, Koijen, Van Nieuwerburgh, and Yogo, 2016, Hambel et al., 2017, and Steffensen and Søe, 2023; and constraints, see Nielsen and Steffensen, 2008, Hambel et al., 2017, and Di Liddo and Striani, 2022.

Duffie and Epstein, 1992 formalized the continuous-time version of the separation of time and risk preferences introduced as a recursive utility by Epstein and Zin, 1989 and Epstein and Zin, 1991. Local separation of time and risk preferences under lifetime uncertainty is studied in Aase, 2016 and Jensen, 2019. In contrast, the approach taken in Jensen and Steffensen, 2015 is based on the global separation and the equilibrium control, also analyzed by Fahrenwaldt, Jensen, and Steffensen,

2020. These are significant background results as we wish to analyze the impact of separation.

Mitchell et al., 1999 also quantified the welfare loss of not annuitizing. That work initiated a vast amount of economic literature with positivistic explanations of the lack of annuitization, including Einav, Finkelstein, and Schrimpf, 2010, Hosseini, 2015, and Brown et al., 2017. In contrast to Mitchell et al., 1999, we work in continuous time, study the sensitivity to market and mortality parameters, analyze the performance of a particular suboptimal bank product, and pay special attention to non-time additive utility.

Other studies related to our scope are Milevsky and Young, 2007 and Milevsky and Huang, 2018. Common for Mitchell et al., 1999, Milevsky and Young, 2007, Milevsky and Huang, 2018 is that the annuity market does not give full flexibility of the consumption-investment profile as we have in our setup. Annuities are there either fixed annuities or variable annuities with some degree of investment freedom but not the full freedom to choose both the investment portfolio and payout profile optimally. This means that annuitization in their works is always a tradeoff between losing flexibility and gaining access to mortality credits. Annuitization in our work, in contrast, means full flexibility. Annuitization is, therefore, in their works, in general, less attractive than in our work, where there is no tradeoff; annuitizing has no downside, only the upside of getting access to mortality credits.

The paper is structured as follows. In Section 2, we present the four optimization problems that later are the fundament of our comparison. In Section 3, we offer the solutions to their problems. Section 4 presents the suboptimal banking products we include in the comparison, which is then performed in Section 5. Section 6 concludes.

6.2 The Problems

In this section, we present the various optimization problems, the solutions of which we will later give and compare. The problems have two variations in the insurance market's and objective function's dimensions. Thus, we face four different problems.

For all four different problems, the underlying financial market is the same. Thus, only the insurance market depends on the market available to the investor. In all four problems, the underlying financial market is a classical Black-Scholes financial market with price processes,

$$\begin{aligned} dS^0(t) &= S^0(t) r dt, \\ dS(t) &= S(t) \left(\alpha dt + \sigma dW(t) \right). \end{aligned}$$

Here, W is a Brownian motion, and $r, \alpha, \sigma > 0$ are constants. We assume that $\alpha \geq r$ such that the market price of risk defined by $\theta := (\alpha - r)/\sigma$ is non-negative.

The individual invests a proportion $\pi(t)$ of his wealth in the stock at time t, and the process π is called the stock proportion. The individual consumes at rate $c(t)$ at time t, and the process c is called the consumption rate.

We assume that the individual has an uncertain lifetime and denote by $\mu(t)$ the individual's mortality rate. Furthermore, we assume that the mortality rate is deterministic and increases with age. Thus, we do not model the so-called longevity risk where the mortality rate is stochastic. However, not modeling longevity risk does not mean we cannot model longevity, i.e., that mortality for a given age decreases with calendar time. If it is deterministic, we can quickly implement such an effect by letting the age-dependent mortality rate vary with birth year.

We distinguish between two different situations in the insurance market. In one case, there exists no insurance market. Thus the market is fully described by the financial market above. On the other hand, with lifetime uncertainty present in the individual's objective, the market is incomplete, and we can formulate contingent claims that are not hedgeable in the market. An example is so-called pure endowment insurance that pays out one unit upon survival until time n. Letting I indicate survival such that $I(t) = 1$ if the individual is alive at time t, the non-hedgeable claim payable at time n is $I(n)$. This claim is not hedgeable in the Black-Scholes market, where one cannot trade the survival risk of the individual. However, our goal is not to price contingent claims. Instead, our goal is to make optimal decisions, and the investment-consumption problem below is well-posed in this incomplete market.

We consider an individual after retirement when labor income has fallen away. The retirement phase is crucial for accessing the explicit solution for the investment-consumption problem below. Otherwise, the non-hedgeability of the labor income, which is only earned before retirement as long as the individual is alive, prevents an explicit solution. Of course, it is possible to work with a problem with non-hedgeable income and no access to insurance, but then one has to resort to a numerical solution of the HJB equation. The alternative idea is to assume that the mortality rate is zero until retirement. In that case, the otherwise incomplete market is, in a sense, 'sufficiently complete' to make the labor income hedgeable; we still have access to solutions in closed form. However, since we are interested in understanding the value of access to insurance, which comes from lifetime uncertainty, modeling over ages with zero mortality does not add value to our study. Therefore, we entirely disregard labor income by moving our problem's starting point to retirement age.

In the problem described above, the individual consumes his wealth invested in the financial market. The dynamics of the wealth of that individual, as long as he is alive, becomes

$$dX(t) = X(t)(r + \pi(t)(\alpha - r))dt + X(t)\pi(t)\sigma dW(t) - c(t)dt, \quad (6.2.1)$$

where $X(0) = x_0 > 0$ is the given initial wealth.

The individual's objective is to maximize the expected utility from consumption until death. Thus, we have a value function in the following form

$$V(t,x) = \sup_{c,\pi} E_{t,x}\left[\int_t^\infty u(t,s,c(s))I(s)ds\right], \tag{6.2.2}$$

where we remind the reader that the process I indicates survival. The subscript (t,x) denotes that the expectation is taken conditional on $X(t) = x$ and $I(t) = 1$, i.e., conditional on the individual being alive at time t.

Note that there is no so-called utility from the bequest. Then the retiree does not achieve any utility from leaving money behind. In (6.2.2), this appears as the individual gets utility during survival only. The no-bequest case is a corner case that has several benefits. First, it prevents us from discussing what that utility from bequest different from zero should be. Second, it severely simplifies some elegant solutions to the consumption problem, as they appear in the coming sections. Finally, we can say that this is a clear case where we can measure the value of the annuity market in a situation where the individual has no economic dependants that he also has to take into account in his objective.

We are going to work with a constant relative risk aversion γ in combination with exponential discounting of the utility with the discount rate ρ such that

$$u(t,s,c) = e^{-\rho(s-t)}\frac{1}{1-\gamma}c^{1-\gamma}. \tag{6.2.3}$$

We speak of the individual with wealth dynamics given by (6.2.1), the value function (6.2.2), and the utility function (6.2.3) as the *uninsured individual with time-additive preferences.*

The next individual has the same objective as the first, namely the one presented through the value function (6.2.2). Thus, neither he has any utility from the bequest. He distinguishes himself from the first individual by having access to the insurance market. Instead of introducing the insurance sum as a decision process and optimizing it, we implement the optimal solution directly. The optimal solution for an individual with no utility from the bequest and access to life insurance is to sell an insurance contract that pays out current wealth at any point in time. This position is called annuitization. Access to an insurance market and an annuity market are two sides of the same story and are just a matter of the sign of the insurance sum paid out. If the individual is willing to give up his wealth upon death, he receives a premium based on the pricing mortality rate used by the life annuity provider. We denote that mortality by μ^* and the premium rate he receives at time t is $\mu^*(t)X(t)$ where $X(t)$ is current wealth. The dynamics of the wealth of that individual, as long as he is alive, then becomes

$$\begin{aligned} dX(t) &= X(t)(r+\mu^*(t)+\pi(t)(\alpha-r))dt \\ &+ X(t)\pi(t)\sigma dW(t) - c(t)dt, \end{aligned} \tag{6.2.4}$$

where $X(0) = x_0 > 0$ is the given initial wealth. The premium rate from annuitization appears in the return term of the dynamics because the premium is assumed to be proportional to wealth itself, corresponding to the linear pricing of the insurance contract.

We speak of the individual with wealth dynamics given by (6.2.4) in combination with the value function (6.2.2) as the *insured individual with time-additive preferences*. The appearance of μ^* in the dynamics (6.2.4) makes it seem as if the insured behaves as the uninsured with an addition of the mortality rate to the interest rate. But we have to be careful here. Since the interest rate also appears in the term stemming from stock investment, without the mortality rate, the correct statement is instead: The insured individual behaves as the non-insured individual with an addition of the mortality rate to both the interest rate and the stock return. Then, these additions offset correctly in the term $\alpha - r$.

The uninsured and the insured individual above share the objective formalized through the value function (6.2.2). It is, however, well-known that this objective misses an essential point about time and risk preferences. It assumes the parameter γ, spoken of as risk aversion, as a parameter that characterizes preferences towards both risks, i.e., variation of consumption over outcomes of stochastic variables, and time, i.e., variation of consumption over time. We see this quickly by considering the particular case of no mortality risk (μ=0) and no financial risk ($\alpha = r$ and $\sigma = 0$). Given the objective of the paper, this is an odd particular case. Still, it unveils the role of γ as a parameter that (in general but in this case of no risk only) characterizes preferences concerning time variation. That odd version of the problem has an internal solution that depends on γ. The parameter γ reflects aversion towards the variation of consumption over time. If γ is large, the investor is not as willing to postpone consumption to pick up (deterministic) capital gains from interest payments as if γ is smaller. That is true, even if that would allow him to consume more. Note that this pattern of thinking works well without risk. We speak of the parameter as covering both risk aversion and variation aversion.

Epstein and Zin, 1989, 1991 formalized the disentanglement of risk and variation aversion in the objective formalized by (6.2.2) in discrete time and Duffie and Epstein, 1992 translated the concepts to continuous time. In the case of no lifetime uncertainty, they derive a so-called aggregator $f(c, v)$ such that the value function has the implicit representation,

$$V(t,x) = \sup_{c,\pi} E_{t,x}\left[\int_t^\infty f(c(s), V(s, X(s)))ds\right]. \qquad (6.2.5)$$

The aggregator function f depends on the underlying structure of preferences towards risk and variation, respectively. They work with a parameter ϕ for the elasticity of intertemporal substitution, which is the reciprocal of variation aversion. If both the relative risk aversion and the relative variation aversion are constant, they derive the aggregator

$$f(c,v) = \frac{1-\gamma}{1-\phi}\delta v\left(\left(\frac{\frac{1}{1-\gamma}c^{1-\gamma}}{v}\right)^{\frac{1-\phi}{1-\gamma}}-1\right). \tag{6.2.6}$$

Now comes the question of how to generalize the disentanglement of risk and time preferences by Duffie and Epstein, 1992 to the case of an uncertain lifetime. They showed how to construct the aggregator for diffusive markets only. Others have worked on generalizing to other markets and more general risk and variation preferences. Yet, there is no consensus about how to implement lifetime uncertainty. The literature contains (at least) three proposals that we now explain.

The simplest generalization to an uncertain lifetime is the one obtained by simply replacing the value function (6.2.5) by

$$V(t,x) = \sup_{c,\pi} E_{t,x}\left[\int_t^\infty f(c(s),V(s,X(s)))I(s)ds\right]. \tag{6.2.7}$$

Again, the expectation is conditional on both current wealth and upon survival until time t, like how we read it in (6.2.2). Aase, 2016 proposed this and studied the impact of insurance markets. However, the value function appearing as an argument in the aggregator in (6.2.7) is certainly different from the value function in (6.2.5). So, is there an argument that the same aggregator f with an updated argument V properly considers possible aversion towards lifetime uncertainty?

Jensen and Steffensen, 2015 proposed a different generalization. They drop the idea of working with (local) aggregators as the fundamental ingredient in the (local) disentanglement. Instead, they form a global objective with a global risk and time variation disentanglement. Their version without mortality risk reads

$$V(t,x) = \int_t^\infty v(u^{-1}(E_{t,x}[u(t,s,c(s))]))ds, \tag{6.2.8}$$

where u is the utility function containing risk aversion and v is a time preference function containing variation aversion. It is probably an insinuation to call what we construct below a generalization of recursive utility to an uncertain lifetime. However, the consumption-investment strategy formed from the value function (6.2.8) does coincide with the consumption-investment strategy formed from (6.2.5). At least, this is the case for the Black-Scholes market. Both Fahrenwaldt, Jensen, and Steffensen, 2020, and Jensen and Steffensen, 2015 obtain this result. So, in that sense, we present a generalization of the consumption-investment strategy obtained in recursive utility.

In two steps, we construct the value function in (6.2.8). The argument of the function v, $u^{-1}(E_{t,x}[u(t,s,c(s))])$, is the so-called certainty equivalent. The first step is to form these certainty equivalents. They turn the utility of uncertain future consumption rates into certain consumption rates from which the individual obtains the same utility. Thus, in a sense, these certainty equivalence operations 'delete' uncertainty from the objective. The function v expresses preferences concerning time variation of certain (or rather certainty equivalent) consumption rates. There is a crucial difference between the recursive utility approach to the disentanglement of time and risk preferences and ours. The certainty equivalent is here based on the utility of actual consumption. In contrast, the certainty equivalent in recursive utility is based on indirect utility.

From a mathematical point of view, this construction radically changes the optimization problem. Suppose v and u are the same functions, such that the operation $v(u^{-1}(\cdot))$ vanishes. In that case, the expectation goes outside the integral, and we are back with a standard objective (corresponds to (6.2.2) without the survival indicator). But v and u being different functions, the integral forms a sum of non-linear functions of conditional expectations. Time consistency and standard dynamic programming break down. But other methods are ready to take over. Jensen and Steffensen, 2015 attack the problem with equilibrium theory, corresponding to how the so-called sophisticated individual thinks when facing a time-inconsistent problem. The technical details are beyond the level of ambition in that direction for this exposition. But this explains why we add *in the equilibrium sense* whenever we speak of an optimal solution below.

The following is the generalization of (6.2.8) to include lifetime uncertainty suggested by Jensen and Steffensen, 2015. We compose the optimal value function in the equilibrium sense as

$$V(t,x) \quad = \quad \int_t^\infty v(u^{-1}(E_{t,x}\left[u(t,s,c(s))I(s)\right]))ds, \tag{6.2.9}$$

and we see how the utility function operates on both financial risk and lifetime uncertainty.

We mention that Jensen and Steffensen, 2015 also works with utility from a bequest. Their approach to this and its consequences for consumption and insurance is a crucial idea of their work. They introduce an elasticity between consuming as dead (the bequest) or alive (like our consumption above). When working with time-additive utility, one usually works with additive utility across the states, dead and alive. But suppose we, in addition to disentangling time and risk preferences, also introduce an elasticity between consuming as dead or alive. Following Jensen and Steffensen, 2015, this has exciting consequences and interpretations. However, when there is no utility from the bequest, the elasticity between consuming as dead or alive vanishes from the problem and, therefore, without utility from the bequest, only the parameters of the functions u and v appear in the solution.

Jensen, 2019 proposes the third generalization of recursive utility to lifetime uncertainty and separation of preferences. Jensen, 2019 extends the original derivation of the aggregator by Duffie and Epstein, 1992. We shall not present the formalism behind it. But as mentioned above, a certainty equivalent based on indirect utility appears in classical recursive utility. Similarly to when Jensen and Steffensen, 2015 introduced elasticity between consumption as dead or alive, it is natural in recursive utility to introduce elasticity between bequest (consumption as dead) and indirect utility conditional on surviving the next small time interval. However, since indirect utility upon survival contains both future consumption and future bequest, that elasticity does not explicitly concern bequest and consumption. Therefore, it should also be clear, as is also discussed in Jensen, 2019, that Jensen and Steffensen, 2015 and Jensen, 2019 are fundamentally different approaches. In contrast to Jensen and Steffensen, 2015, the elasticity between a bequest and indirect utility given survival appears in the solution by Jensen, 2019, even in the case of no utility from the bequest we consider here.

When we work with separated preferences in the next section, our individual has an objective corresponding to (6.2.9). Suppose this individual does not have access to life insurance (unlike the situation in Jensen and Steffensen, 2015) and therefore cannot annuitize and must realize the wealth dynamics (6.2.1). In that case, we speak of the *uninsured individual with separated preferences.* Finally, suppose the individual with an objective corresponding to (6.2.9) has access to life insurance (like the situation in Jensen and Steffensen, 2015) and therefore fully annuitizes and realizes the wealth dynamics (6.2.4). In that case, we speak of the *insured individual with separated preferences.*

We have now presented four different individuals, namely *the uninsured individual with time-additive preferences*, *the insured individual with time-additive preference*, *the uninsured individual with separated preferences*, and *the insured individual with separated preferences.* In the next section, we present and discuss their optimal investment and consumption processes.

6.3 The Solutions

In this section, we present the solutions to the problems presented in the previous section. These problems can be seen as special cases of Jensen and Steffensen, 2015, here presented in our setting to enable an easier and more comprehendible comparison.

We start by considering the investment decision. The solutions for all four individuals are the same well-known Merton proportion given by

$$\pi = \frac{1}{\gamma}\frac{\alpha - r}{\sigma^2} = \frac{1}{\gamma}\frac{\theta}{\sigma}. \tag{6.3.1}$$

The expected return from the investment equals $r + \pi(\alpha - r) = r + \frac{1}{\gamma}\theta^2$. However, a return rate with half of the excess return obtained from stock investments added to the risk-free return shows up in the solution again and again, and we denote this by R such that $R = r + \frac{1}{2}\pi(\alpha - r) = r + \frac{1}{2\gamma}\theta^2$.

We now turn to the consumption rate. All individuals withdraw a fraction of their wealth for consumption. The fraction for all individuals is the reciprocal of an annuity, i.e., for all individuals, we can write $c(t) = X(t)/a(t)$, where $a(t)$ is a specific annuity that depends on which individual we consider. *The uninsured individual with time-additive preferences* withdraws optimally in accordance with the annuity

$$a^{ua}(t) = \int_t^\infty e^{-\int_t^s \delta^a + \mu^{ua}} ds, \tag{6.3.2}$$

where

$$\delta^a = \frac{1}{\gamma}\rho + (1 - \frac{1}{\gamma})R, \tag{6.3.3}$$

$$\mu^{ua} = \frac{1}{\gamma}\mu \tag{6.3.4}$$

The letters in the top script u and a abbreviate *uninsured* and *time-additive*. Decorating δ with just an a reflects that this is the same for uninsured and insured individuals. Only the μ in the annuity depends on whether the individual is insured. In the annuity, we use the slightly informal notation $e^{-\int_t^s \delta^a + \mu^{ua}}$ representing $e^{-\int_t^s (\delta^a + \mu^{ua}(\tau)})d\tau$, for notational ease, where the transition intensity is time-dependent, but δ is not, we continue to use the abbreviated notation. We recognize the annuity formula as the actuarial formula for a life annuity with the design interest rate δ^a and the design mortality rate μ^{ua}. We call these elements design elements as they form different annuity product designs. In that formula, the interest rate is a weighted average of the impatience rate ρ and the return rate R. The weights are $\frac{1}{\gamma}$ and $1 - \frac{1}{\gamma}$, respectively. We also note how mortality impacts ρ. Namely, we cover the case with an uncertain lifetime by the case without uncertainty lifetime by simply adding μ to ρ.

We also present the optimal consumption rate dynamics for each individual. For all four individuals, the optimal consumption rate follows a geometric Brownian motion. *The uninsured individual with time-additive preferences* consumes according to the dynamics

$$\frac{dc^{ua}(t, X(t))}{c^{ua}(t, X(t))} = \frac{r - \rho - \mu + \frac{1}{2\gamma}(1+\gamma)\theta^2}{\gamma} dt + \frac{\theta}{\gamma} dW(t). \tag{6.3.5}$$

We now consider *the uninsured individual with separated preferences.* He also consumes a proportion of his wealth corresponding to the annuity,

$$a^{us}(t) = \int_t^\infty e^{-\int_t^s \delta^s + \mu^{us}} ds, \tag{6.3.6}$$

where

$$\delta^s = \frac{1}{\phi}\rho + (1 - \frac{1}{\phi})R, \tag{6.3.7}$$

$$\mu^{us} = \frac{1}{\phi}\frac{1-\phi}{1-\gamma}\mu. \tag{6.3.8}$$

Again, we have the actuarial formula for the life annuity formed by a design interest rate and a design mortality rate. However, the weights on ρ and R in forming the design interest rate are now replaced by $\frac{1}{\phi}$ and $1 - \frac{1}{\phi}$. The design mortality rate is replaced by $\frac{1}{\phi}\frac{1-\phi}{1-\gamma}\mu$. As it should be, we find that the design rates of the uninsured individual with time-additive preferences equal those of the uninsured individual with separated preferences in the special case $\phi = \gamma$.

The dynamics of consumption for the uninsured individual with separated preferences are

$$\frac{dc^{us}(t, X(t))}{c^{us}(t, X(t))} = \frac{r - \rho - \frac{1-\phi}{1-\gamma}\mu + \frac{1}{2\gamma}(1+\phi)\theta^2}{\phi}dt + \frac{\theta}{\gamma}dW(t). \tag{6.3.9}$$

Again, we note how the dynamics of the uninsured individual with separated preferences collapse into those of the uninsured individual with time-additive preferences in the particular case $\phi = \gamma$.

We now turn to the individuals with access to the life annuity market. We first consider *the insured individual with time-additive preferences.* He consumes a fraction of his wealth based on the annuity

$$a^{ia}(t) = \int_t^\infty e^{-\int_t^s \delta^a + \mu^{ia}} ds, \tag{6.3.10}$$

where

$$\mu^{ia} = \frac{1}{\gamma}\mu + (1 - \frac{1}{\gamma})\mu^*. \tag{6.3.11}$$

Thus, we base the annuity of the insured individual with time-additive preferences on a design interest rate that is the same as that of the uninsured with the same preferences. However, we replace the design mortality rate of the uninsured individual μ/γ by a weighted average of the actual mortality intensity and the pricing mortality intensity with weights given by $\frac{1}{\gamma}$ and $1 - \frac{1}{\gamma}$. Note that the design mortality rate of the uninsured individual is obtained as the design mortality rate of the insured individual in the particular case $\mu^* = 0$.

The dynamics of the consumption rate for the insured individual with time-additive preferences are

$$\frac{dc^{ia}(t, X(t))}{c^{ia}(t, X(t))} = \frac{r - \rho + \mu^* - \mu + \frac{1}{2\gamma}(1+\gamma)\theta^2}{\gamma}dt + \frac{\theta}{\gamma}dW(t). \tag{6.3.12}$$

Note that the dynamics of consumption for the uninsured individual follow from the special case $\mu^* = 0$.

Finally, we consider *the insured individual with separated preferences.* He consumed a fraction of his wealth based on the annuity

$$a^{is}(t) \quad = \int_t^\infty e^{-\int_t^s \delta^s + \mu^{is}} ds, \tag{6.3.13}$$

with

$$\mu^{is} \quad = \quad \frac{1}{\phi}\frac{1-\phi}{1-\gamma}\mu + (1-\frac{1}{\phi})\mu^*. \tag{6.3.14}$$

Thus, as was the case for time-additive utility, we can reuse the design interest rate δ^s for this uninsured individual. And, as was the case for time-additive utility, we have to update the design mortality rate. Now, for the case of separated preferences, the introduction of the life annuity market allows us to replace the design mortality rate $\frac{1}{\phi}\frac{1-\phi}{1-\gamma}\mu$ by $\frac{1}{\phi}\frac{1-\phi}{1-\gamma}\mu + (1-\frac{1}{\phi})\mu^*$. Note how we obtain the design mortality rate of the insured individual with time-additive preferences as a particular case of the insured individual with separated preferences in the case of $\phi = \gamma$. Also, note how we obtain the design mortality rate of the uninsured individual with separated preferences as a particular case of the insured individual with separated preferences in the specific case of $\mu^* = 0$.

The dynamics of the consumption rate of the insured individual with separated preferences are

$$\frac{dc^{is}(t, X(t))}{c^{is}(t, X(t))} \quad = \quad \frac{r - \rho + \mu^* - \frac{1-\phi}{1-\gamma}\mu + \frac{1}{2\gamma}(1+\phi)\theta^2}{\phi}dt + \frac{\theta}{\gamma}dW(t). \tag{6.3.15}$$

Note that we obtain the consumption dynamics for the individual with time-additive preferences by the particular case $\phi = \gamma$. Note that the dynamics of consumption for the uninsured individual is the specific case $\mu^* = 0$.

All the above results follow Jensen and Steffensen, 2015 with properly specifying special cases. As discussed in Section 6.2, Jensen, 2019 provides a different disentanglement of time and risk preferences under lifetime uncertainty than Jensen and Steffensen, 2015. In the case of no consumption upon death, the additional parameter considered to deal with an uncertain lifetime does not appear in the optimal controls in Jensen and Steffensen, 2015, in contrast to what Jensen, 2019 obtains. It is difficult to unravel the optimal control of Jensen, 2019's approach in the case of no utility from a bequest since the utility from a bequest there cannot immediately be set to zero. However, it seems that we can base the optimal control on the annuity

$$a^{is}(t) \quad = \int_t^\infty e^{-\int_t^s \delta^s + \mu^{is}} ds, \tag{6.3.16}$$

with

$$\mu^{is} = \frac{1}{\phi}\frac{1-\phi}{1-\kappa}\mu + (1-\frac{1}{\phi})\mu^*, \qquad (6.3.17)$$

where κ is what Jensen, 2019 speaks of as the reciprocal elasticity of substitution between a bequest and future utility. Jensen, 2019 also provides a different interpretation of κ. We can think of the distinction between γ and κ as working with different risk aversion concerning market risk and mortality risk, respectively, where γ is the former, and κ is the latter. Based on the annuity above, the dynamics of consumption are then in Jensen, 2019 given by

$$\frac{dc^{is}(t,X(t))}{c^{is}(t,X(t))} = \frac{r-\rho+\mu^* - \frac{1-\phi}{1-\kappa}\mu + \frac{1}{2\gamma}(1+\phi)\theta^2}{\phi}dt + \frac{\theta}{\gamma}dW(t). \qquad (6.3.18)$$

In our numerical studies, we stick to the approach by Jensen and Steffensen, 2015, i.e., corresponding to (6.3.13), (6.3.14), and (6.3.15). However, as it can be seen through (6.3.16), (6.3.17), and (6.3.18), this can be thought of as a particular case of Jensen, 2019 where $\kappa = \gamma$, i.e., according to the interpretation by Jensen, 2019, as the particular case where the preferences for financial and insurance risk are identical.

In all the annuity formulas above, we recognize the actuarial life annuity formula with specific design interest and mortality rates that depend on the individual and whether he has access to an annuity market. With access to an annuity market, the design mortality rate is a weighted average of the actual mortality rate and the pricing mortality rate, depending on whether the individual has time-additive or separated preferences.

One may think that such a construction can be generalized to multi-state models. Indeed, it can. But from Kraft and Steffensen, 2008a and Steffensen and Søe, 2023, one can learn that the design mortality rates cannot be directly generalized based on the construction of a weighted average. They both work with time-additive utility and access to insurance, so we should compare with (6.3.10), (6.3.11), and (6.3.12). From there, one learns that the more general representation follows from adjusting the calculation interest rate by the difference between the arithmetic and the geometric weighted mean of mortalities and then using the geometric weighted mean as the mortality rate in the actuarial formula. I.e. we should redefine δ^a and μ^{ia} by

$$\delta^{ag} = \delta^a + \mu^{ia} - \mu^{iag}, \qquad (6.3.19)$$

$$\mu^{iag} = \mu^{\frac{1}{\gamma}}(\mu^*)^{1-\frac{1}{\gamma}}. \qquad (6.3.20)$$

These design interest and mortality rates can be directly generalized to multi-state models. Obviously, in our studies, they form the same control processes since $\delta^a + \mu^{ia} = \delta^{ag} + \mu^{iag}$.

6.4 The Sub-Optimal Product Design

The uninsured individuals above decide optimally in the market they face. Of course, there are many sub-optimal ways to determine, e.g., the consumption plan. We now pay special attention to one of them, which has some merits in its construction. Although the construction is sub-optimal, we derive the dynamics of the consumption plan such that we can compare its structure to the optimal one. The idea behind the design is to have a problem with deterministic finite time horizon n in mind. We use the letter n for the finite time point to reserve the otherwise frequently used T as the stochastic lifetime of the individual. For the problem with finite-time horizon n, the solution is to consume a fraction of your wealth according to the annuity

$$a(t) \quad = \int_t^n e^{-\int_t^s \tilde{r}} ds. \tag{6.4.1}$$

As interest rate in the annuity, we introduce a rate of return $\tilde{r}$, which we can adjust to accommodate the individual's preferences. If, e.g., the individual has time-additive preferences, and n is the actual time horizon, then $\tilde{r} = \delta^a$ is optimal.

Now, we acknowledge that the lifetime is uncertain, but what is our best estimate of that lifetime? The answer is the conditional expectation $E_t[T]$ where the subscript t denotes survival until time t. We have that

$$E_t[T] \quad = t + \int_t^\infty e^{-\int_t^s \mu} ds, \tag{6.4.2}$$

Now we replace n in our annuity construction by $E_t[T]$ as this is our best estimate of our time horizon. Thus, we suggest the consumption rate $X(t)/a(t)$ with

$$a(t) \quad = \int_t^{E_t[T]} e^{-\int_t^s \tilde{r}} ds. \tag{6.4.3}$$

However, this is not an optimal consumption under any problem with a stochastic lifetime. It just seems to be a good idea. Note carefully that the expected lifetime is continuously updated with the conditioning on survival. This means that there is no risk of outliving your wealth, and there is nothing particular about dying before or after the expected lifetime, conditional on survival to some earlier age.

To derive the dynamics of c, we have to decide which dynamics of X to use. Since the product is proposed here as an alternative to the optimal consumption for the uninsured individual, we go with the dynamics in (6.2.1). We can then derive the dynamics of the consumption rate to be

$$\frac{dc(t, X(t))}{c(t, X(t))} \quad = \quad (r - \tilde{r} - \tilde{\mu} + \frac{\theta^2}{\gamma})dt + \frac{\theta}{\gamma} dW(t), \tag{6.4.4}$$

where

$$\tilde{\mu} = \frac{e^{-\int_t^{E_t[T]} \tilde{r}} (E_t[T] - t)}{a(t)} \mu, \tag{6.4.5}$$

since

$$\frac{da(t)}{dt} = \mu(E_t[T]-t)e^{-\int_t^{E_t[T]}\tilde{r}} - 1 + \tilde{r}a(t).$$

We formulate the dynamics of the consumption rate by constructing an odd mortality rate $\tilde{\mu}$ to make it comparable with the optimal consumption patterns we have seen in the previous section.

We want to compare the performance of the proposed consumption strategy with those with and without insurance access. We must calculate the sub-optimal value function based on the suboptimal consumption strategy. We compare the suboptimal strategy to the optimal one under time-additive preferences. Thus, the objective is as defined in (6.2.2), such that

$$\begin{aligned} V(t,x) &= E_{t,x}\left[\int_t^\infty u(t,s,c(s))I(s)ds\right] \\ &= E_{t,x}\left[\int_t^\infty \frac{1}{1-\gamma}c(s)^{1-\gamma}e^{-\rho(s-t)}I(s)ds\right] \\ &= \int_t^\infty \frac{1}{1-\gamma}E_{t,x}\left[c(s)^{1-\gamma}\right]e^{-\rho(s-t)}e^{-\int_t^s\mu}ds. \end{aligned}$$

To calculate the expectation, we write down the solution to (6.4.4) as

$$c(s,X(s)) = c(t,X(t))e^{\int_t^s((r-\tilde{r}-\tilde{\mu}+\frac{\theta^2}{\gamma}-\frac{1}{2}\frac{\theta^2}{\gamma^2})du+\frac{\theta}{\gamma}dW(t))}.$$

We achieve

$$\begin{aligned} &E_{t,x}[c(s,X(s))^{1-\gamma}] \\ =&E_{t,x}[c(t,X(t))^{1-\gamma}e^{(1-\gamma)\int_t^s((r-\tilde{r}-\tilde{\mu}+\frac{\theta^2}{\gamma}-\frac{1}{2}\frac{\theta^2}{\gamma^2})du+\frac{\theta}{\gamma}dW(u))}] \\ =&c(t,x)^{1-\gamma}E_{t,x}[e^{(1-\gamma)\int_t^s((r-\tilde{r}-\tilde{\mu}+\frac{\theta^2}{\gamma}-\frac{1}{2}\frac{\theta^2}{\gamma^2})du+\frac{\theta}{\gamma}dW(u))}] \\ =&c(t,x)^{1-\gamma}e^{\int_t^s(1-\gamma)(r-\tilde{r}-\tilde{\mu}+\frac{1}{2}\frac{\theta^2}{\gamma})du}. \end{aligned}$$

By rewriting the power coefficient,

$$(1-\gamma)(r-\tilde{r}-\tilde{\mu}+\frac{\theta^2}{2\gamma}) = -\gamma\delta^a+\rho-(1-\gamma)\tilde{\mu}-(1-\gamma)\tilde{r},$$

we conclude that

$$\begin{aligned} &\int_t^\infty \frac{1}{1-\gamma}E_{t,x}[c(s,X(s))^{1-\gamma}]e^{-\rho(s-t)}e^{-\int_t^s\mu}ds \\ =&\frac{1}{1-\gamma}c(t,x)^{1-\gamma}\int_t^\infty e^{-\int_t^s\mu+\gamma\delta^a+(1-\gamma)(\tilde{r}+\tilde{\mu})}ds \\ =&\frac{1}{1-\gamma}x^{1-\gamma}a(t)^\gamma\frac{\int_t^\infty e^{-\int_t^s\mu+\gamma\delta^a+(1-\gamma)(\tilde{r}+\tilde{\mu})}ds}{a(t)}. \end{aligned}$$

Note we have used the consumption rate $c(t,x) = x/a(t)$. We define a function $f(t)$ to simplify notation,

$$f(t) = \frac{\int_t^\infty e^{-\int_t^s \mu + \gamma\delta^a + (1-\gamma)(\tilde{r}+\tilde{\mu})} ds}{a(t)},$$

such that we can write the sub-optimal value function as

$$V(t,x) = \frac{1}{1-\gamma} x^{1-\gamma} a(t)^\gamma f(t).$$

This expression deviates from the structure of the other presented value functions because of the extra function f.

6.5 The Comparison

In this section, we compare, formally as well as numerically, the individuals. In particular, we measure the welfare lost from losing access to an annuity market. We calculate the welfare loss for individuals with time-additive and separated preferences.

6.5.1 Comparison of the optimal solutions

We can compare the uninsured and insured individuals with either time-additive or separated preferences by comparing their optimal value functions, called indirect utility. One should be careful with comparing optimal value functions. Only if the same preferences underlie the optimal value functions are they comparable. This is the case for the uninsured and insured individuals with time-additive and separated preferences, respectively. Only the markets are different, namely, through access to life annuities. However, for the same reason, we cannot, e.g., compare the value functions from the time-additive and separated preferences since the preferences are not the same.

We have presented the uninsured individual as an individual who behaves optimally in a market without insurance. However, we can also think of that individual as an individual who behaves sub-optimally in a market with insurance. His sub-optimal decision is not to buy any insurance. In that sense, we calculate the welfare loss from deciding sub-optimally rather than optimally in the market with insurance. Also, in such cases, one can compare the value functions corresponding to optimal and sub-optimal decisions. For most sub-optimal decisions, this is an utterly complicated numerical task. Whereas some problem formulations allow for closed-form expressions for the optimal value function, value functions for most sub-optimal decisions cannot be calculated directly. In our case, we can because the sub-optimal decision is optimal in the restricted market, and we have access to an explicit value function there.

The value functions are given by

$$
\begin{aligned}
V^{ua}(t,x) =&(a^{ua}(t))^{\gamma}\frac{1}{1-\gamma}x^{1-\gamma},\\
V^{us}(t,x) =&(a^{us}(t))^{\frac{1-\gamma}{1-\phi}\phi}\frac{1}{1-\gamma}x^{1-\gamma},\\
V^{ia}(t,x) =&(a^{ia}(t))^{\gamma}\frac{1}{1-\gamma}x^{1-\gamma},\\
V^{is}(t,x) =&(a^{is}(t))^{\frac{1-\gamma}{1-\phi}\phi}\frac{1}{1-\gamma}x^{1-\gamma},
\end{aligned}
$$

respectively, for the four individuals we study. Thus, by specifying all the annuities in the previous section, we have all the ingredients we need to compare and calculate welfare gains from access to the insurance market.

For the individual with time-additive preferences, we form the equation

$$
V^{ia}(0,x(1-\epsilon)) =V^{ua}(0,x),
$$

which we then want to solve concerning ϵ. This is the relative loss of wealth that the insured individual would suffer from losing access to the insurance market. By plugging in the value functions above, it is easy to obtain

$$
\epsilon = 1-\left(\frac{a^{ua}(0)}{a^{ia}(0)}\right)^{\frac{\gamma}{1-\gamma}}.
$$

We speak of ϵ as the relative value of the annuity market. We calculate it here as the value of that market to someone who has access to it. We could have calculated the relative value in terms of the gain experienced by an individual without access if that individual would get this access. It is just a convention whether to use one or the other as long as we use the same one in all calculations.

Correspondingly, for the individual with separate preferences, we form the equation

$$
V^{is}(0,x(1-\epsilon)) =V^{us}(0,x),
$$

again solving for ϵ. The relative value in terms of the loss experienced by someone with separated preferences and access in case they lose this access is

$$
\epsilon = 1-\left(\frac{a^{us}(0)}{a^{is}(0)}\right)^{\frac{\phi}{1-\phi}}.
$$

In the forthcoming illustrations, we use parameters from Table 6.1, with true mortality intensity defined by the Gompertz law with $\mu(t) = A\cdot B^{(z+t)}$. These parameters and transition intensities are chosen as a baseline case, where we study the effect of their values in the numerical examples by varying them. By studying the variations of the parameters, we can isolate and evaluate their effect and impact on

Table 6.1: *The parameters used in the numerical examples. Note r, α, and σ are thought of as corrected for inflation.*

Parameters	Description	Value
z	Age at initialization/retirement	65
ρ	Impatience factor for all states	0.02
r_0	The constant drift of the risk-free asset	0.02
α	The constant drift of the risky asset	0.05
σ	The constant volatility of the risky asset	0.2
A	Parameter for pricing mortality intensity	0.0000005
B	Parameter for pricing mortality intensity	1.14

the relative loss. The illustrations below show the time-additive case with $\gamma = 2$. The separated case is calculated with $\gamma = 2$ and $\phi = 6$. These values are chosen based on previously performed studies, such as Burgaard and Steffensen, 2020 where the average risk aversion for males is 1.9 and for females 2.3. In Burgaard and Steffensen, 2020, they also discuss the values of ϕ and that it should be greater than the risk aversion. Further, we remind the reader that the time-additive preferences correspond to the case of the separated preferences where $\gamma = \phi$. Thus, in the illustration, the individual with separated preferences has a stronger aversion towards time variability than risk.

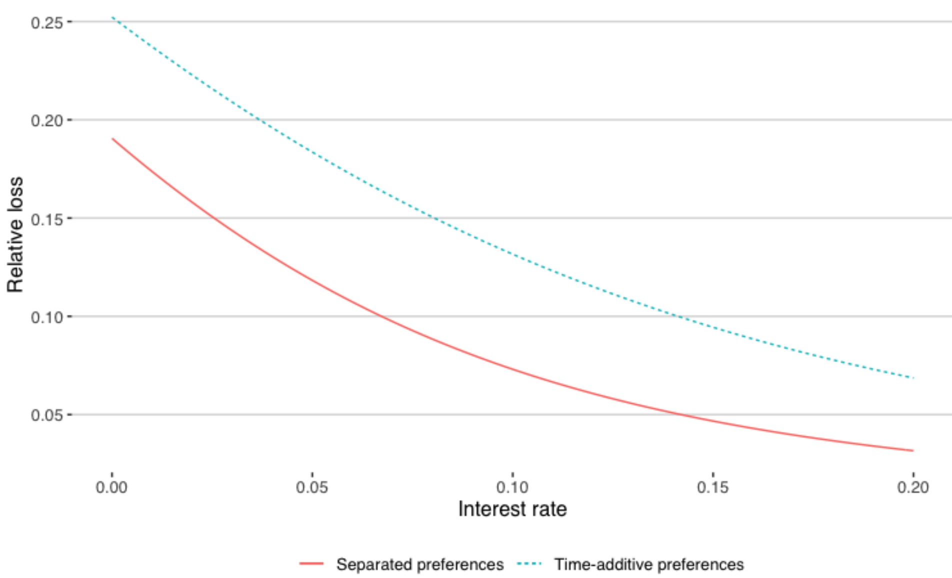

Figure 6.1: *The relative loss ϵ, as a function of the interest rate for a fixed market price of risk. For time-additive preferences ($\gamma = \phi = 2$) and for separated preferences ($\gamma = 2$, $\phi = 6$).*

In the baseline case, the relative loss is 22,2% for time-additive preferences and 15,8% for separated preferences. Thus, separated preferences lead here to a reduction of the relative loss. The reason is that the individual with separated preferences demands, for our choice of parameters, growth in the consumption rate, which is relatively smaller. Therefore, he consumes faster than the individual with time-additive preferences, and his capital is generally lower. But then his mortality credits are relatively lower, and his loss from giving up the annuity option is smaller.

We start by varying the interest rate, r, in Figure 6.1. The market price of risk, defined by $\theta_0 = \frac{\alpha - r_0}{\sigma}$, where r_0, is defined in Table 6.1. The relative loss decreases with the interest rate since a higher interest rate means that (risk-free) capital gains finance a higher proportion of total income. When capital gains finance a higher proportion of total income, the additional return from mortality credits plays a smaller role, and the relative loss from losing the annuity becomes smaller. The line for separated preferences is lower than the line for time-additive preferences with the same argument as the one for the baseline case above.

Now we vary the market price of risk and keep the interest rate constant as r_0 in Figure 6.2. We see how the relative loss decreases with the increasing market price of risk. Again the explanation is that a higher market price of risk leads to a higher part of the consumption being financed by (risky) capital gains.

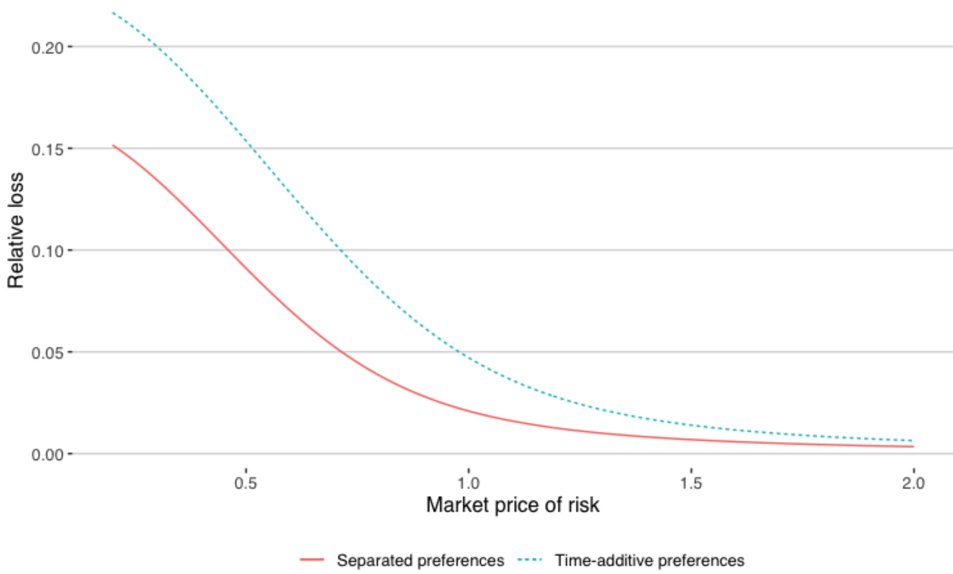

Figure 6.2: *The relative loss ϵ, as a function of the market price of risk for a fixed interest rate, for time-additive preference ($\gamma = \phi = 2$) and separated preferences ($\gamma = 2$, $\phi = 6$).*

In Figure 6.3, we study the impact of insurance pricing. So far, we have assumed that $\mu^* = \mu$. Now we define $\mu^* = (1-\xi)\mu$ with $\xi \in [0,1]$. When $\xi = 0$, there is no risk loading in the price, and we are back with $\mu^* = \mu$. When $\xi \geq 0$, the insurance company has a risk loading in the pricing. We can see that the larger the risk loading, the less attractive the mortality credits and, thus, the less is lost if we lose the annuity market. When $\xi = 1$, there are no mortality credits. In that case, there is no benefit from access to the annuity market.

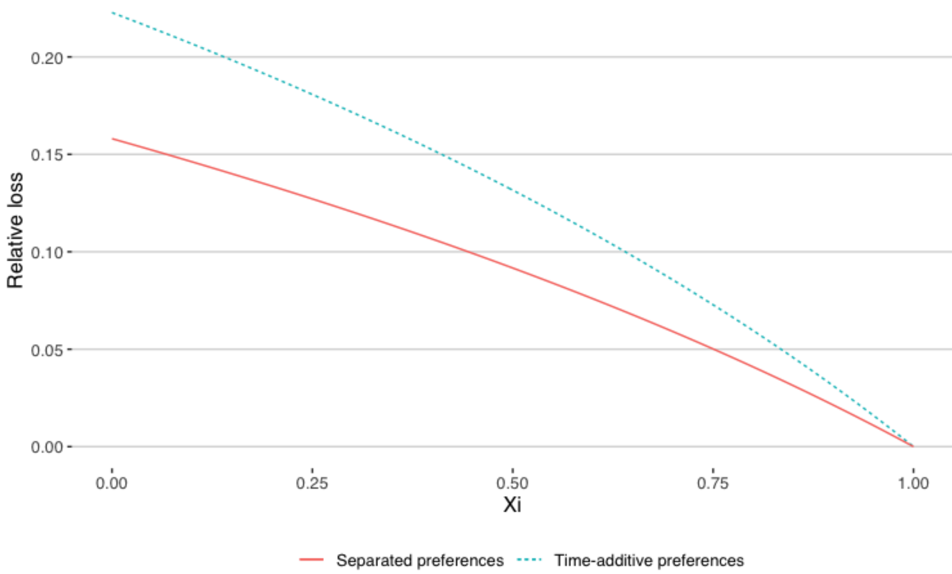

Figure 6.3: *The relative loss ϵ, for time-additive preferences ($\gamma = \phi = 2$) and separated preferences ($\gamma = 2$, $\phi = 6$) as a function of the reduction from μ to μ^*.*

It is also interesting to study the loss as a function of the lifetime's uncertainty. Intuitively, if the lifetime were certain, there should be no difference between insured and uninsured individuals as both would have to buy the same annuity-certain. In the Gompertz model we have used so far, it is difficult to control the uncertainty level by changing the parameters. We, therefore, consider a so-called hyperbolic mortality model defined by $\mu_K(t,n) = \frac{1}{K(n-t)}$, where n is the maximum age possible (we let $n = 120$) and where K is a measure of certainty. When K increases, the mortality for all ages earlier than n decreases. However, the maximum age is still n. Thus, we see how the uncertainty decreases in K, and for K increasing, we approach a model where the lifetime ends deterministically at age n.

Figure 6.4 shows how the relative loss decreases when K increases. The intuition is that the larger the K, the less lifetime uncertainty, and, naturally, the less is lost from losing access to the insurance market.

Finally, we study the sensitivity of the relative loss towards the assumptions

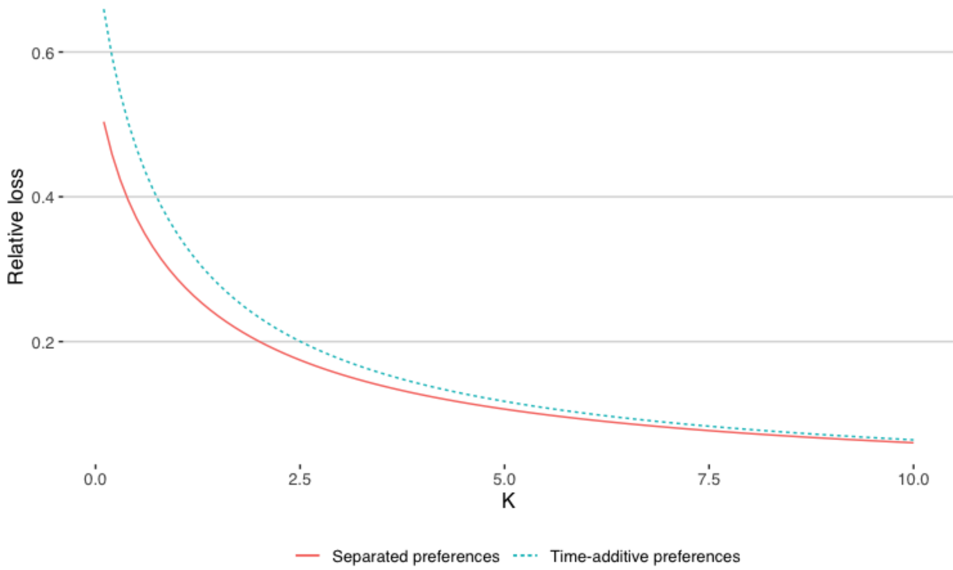

Figure 6.4: *The relative loss ϵ, for time-additive preferences ($\gamma = \phi = 2$) and separated preferences ($\gamma = 2$, $\phi = 6$) as a function of the constant K defining the certainty of survival until time $n = 120$.*

about risk aversion and EIS. We perform two different sensitivity analyses. For time-additive preferences, we vary γ. For separated preferences, we fix $\gamma = 2$ and vary ϕ.

Figure 6.5 the sensitivity towards these assumptions. Note that the $x-$axis means something different for the two curves. We see that the loss is relatively robust with respect to risk aversion in the case of time-additive preferences. A slight upward trend can be explained by the fact that the bank actually uses smaller risk aversions to slope the consumption profile as well as they can. If the risk aversion is high, this feature vanishes as μ^{ia} tends to zero for the banking case, $\mu^* = 0$. For separated preferences, we see that the loss is relatively robust but slightly decreasing in ϕ when this is larger than γ. However, it increases drastically for ϕ smaller than γ. As ϕ tends to 1, μ^{is} tends to zero for both the bank and insurance products. Then the loss is the pure impact of mortality credits since the bank does not deviate from the insurance company's assumption about μ^{is} to compensate for the loss of mortality credits. It should also be noted, as mentioned earlier, the value of ϕ is known to be bigger than γ.

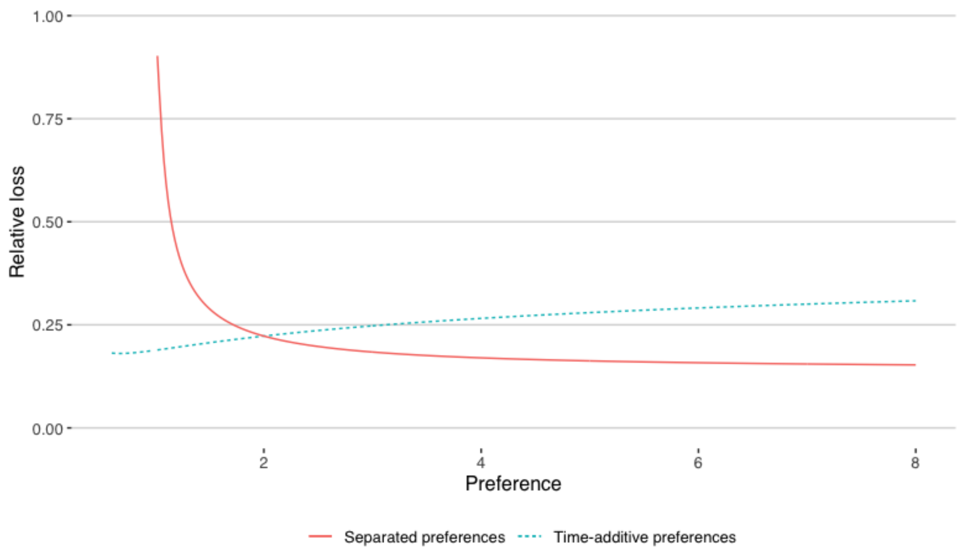

Figure 6.5: *The relative loss ϵ, for time-additive preferences and separated preferences as a function of either γ or fixing $\gamma = 2$ and as a function of ϕ.*

6.5.2 Comparison to the sub-optimal product design

We now wish to compare the relative loss of being equipped with the suboptimal product design. We want to calculate both the loss from optimality *with* access to life insurance to the suboptimal product and the loss from optimality *without* access to life insurance to the suboptimal product.

The value function for the suboptimal consumption is formulated as

$$V(t,x) = (a(t))^{\gamma} f(t) \frac{1}{1-\gamma} x^{1-\gamma}, \tag{6.5.1}$$

such that we can form the two loss quantification problems as

$$\begin{aligned} V^{ia}(0, x(1-\epsilon)) &= V(0,x), \\ V^{ua}(0, x(1-\epsilon)) &= V(0,x), \end{aligned}$$

respectively. Corresponding to when comparing the optimal controls, we get relative losses in the form,

$$\begin{aligned} \epsilon &= 1 - \left(\frac{a(0)}{a^{ia}(0)}\right)^{\frac{\gamma}{1-\gamma}} f(0)^{\frac{1}{1-\gamma}}, \\ \epsilon &= 1 - \left(\frac{a(0)}{a^{ua}(0)}\right)^{\frac{\gamma}{1-\gamma}} f(0)^{\frac{1}{1-\gamma}}, \end{aligned}$$

respectively.

When comparing the suboptimal product design, we must decide which interest rate $\tilde{r}$ to use. There are two natural alternatives. One is to determine with which interest rate the product performs the best. With the baseline parameters, we have calculated that by static optimization to be $\tilde{r} = -0.00835$. An alternative is, of course, to use δ^a. For the baseline case, this is $\delta^a = 0.02281$.

In Table 6.2, we present the results for the two interest rate choices in the two lines. The loss from insured to uninsured is unrelated to the suboptimal product and is in the baseline case 22%. This corresponds to, e.g., the point in Figure 6.3 for time-additive preferences and $\xi = 0$. If the insured is offered the suboptimal design with the best possible interest rate instead of optimality with insurance, he loses 29%. If the uninsured individual is provided a suboptimal design with the best possible interest rate instead of the best possible design without insurance, he loses 9%. These two losses, 22% and 9%, do not add up to the 29% since they are relative losses stemming from non-linear functions.

Table 6.2: *Table for values of relative loss in the three cases with $\gamma = 2$, age at 65 and in the case of using $\tilde{r}$ and δ^a*

Interest rate	Insured to Uninsured	Insured to Suboptimal	Uninsured to Suboptimal
optimal	22 %	29 %	9 %
δ^a	22 %	35 %	16 %

It is clear from the suboptimal control that this performs optimally if the mortality is deterministic. To compare the three consumption plans as a function of the level of lifetime uncertainty, we consider the hyperbolic mortality rate underlying Figure 6.4 again. In Figure 6.6, all three relative losses converge towards zero as K becomes larger and mortality becomes less uncertain. The interest rate level $\tilde{r}$ is chosen equal to δ^a for greater comparability to the optimal and suboptimal consumption. Like in Table 6.2, we do not have additivity in the sense that the relative loss from insured to suboptimal is not the sum of the relative loss from insured to uninsured and the relative loss from uninsured to suboptimal. The relative loss from uninsured to suboptimal decreases steeply towards zero. Similarly, the loss of the life annuity is not so different depending on whether the alternative is the optimal consumption plan without insurance or the suboptimal plan. For $K < 2.5$, the loss is more than 20% in both cases.

6.5.3 Comparison of consumption profiles

It is interesting to study the consumption profiles from each of the three cases, insured, uninsured, and sub-optimal. This adds time to the dimension, and we limit ourselves to the baseline assumptions of the time-additive individual. For the insured individual, the uninsured individual, and the sub-optimal product design, we get from (6.3.5), (6.3.12), and (6.4.4), where the sub-optimal consumption is not

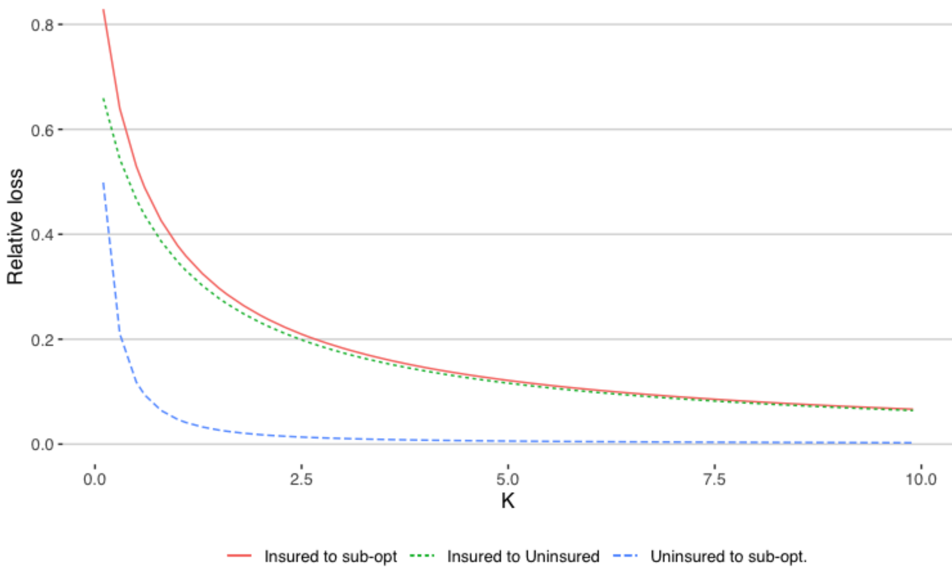

Figure 6.6: *The relative loss for the individual in the three situations, as a function of the constant K defining the certainty of surviving until time $T = 120$ for $\gamma = 2$ and $\tilde{r} = \delta^a$.*

decorated with topscripts,

$$\begin{aligned} E_{t,x}\left[c^{ua}(s, X(s))\right] &= c^{ua}(t,x)e^{\int_t^s \frac{1}{\gamma}(r-\rho-\mu+\frac{(1+\gamma)\theta^2}{\gamma}du)} \\ &= \frac{a^{ua}(t)}{x}e^{\int_t^s \frac{1}{\gamma}(r-\rho-\mu+\frac{(1+\gamma)\theta^2}{\gamma}du)}, \\ E_{t,x}\left[c^{ia}(s, X(s))\right] &= c^{ia}(t,x)e^{\int_t^s \frac{1}{\gamma}(r-\rho+\mu^*-\mu+\frac{(1+\gamma)\theta^2}{\gamma})du} \\ &= \frac{a^{ia}(t)}{x}e^{\int_t^s \frac{1}{\gamma}(r-\rho+\mu^*-\mu+\frac{(1+\gamma)\theta^2}{\gamma})du}, \\ E_{t,x}\left[c(s, X(s))\right] &= c(t,x)e^{\int_t^s (r-\tilde{r}-\tilde{\mu}+\frac{\theta^2}{\gamma})du} \\ &= \frac{a(t)}{x}e^{\int_t^s (r-\tilde{r}-\tilde{\mu}+\frac{\theta^2}{\gamma})du}. \end{aligned}$$

The expected consumption profiles are illustrated in Figure 6.7, assuming the parameters presented in Table 6.1, $\gamma = 2$, and the interest rate in the sub-optimal case chosen as δ^a. The wealth at time 0 is taken to be 1. The insured individual demands an exponentially increasing consumption rate. The uninsured individual starts out at a lower level because $\gamma > 1$ and demands a hump-shaped as a consequence of the drift of c^{ua} crossing zero from above when mortality increases. The sub-optimal design starts out at a higher level than the uninsured individual. However, after approximately ten years, the consumption rate of the uninsured individual becomes larger than that in the sub-optimal design. The value of the insured benefit rate equals the initial wealth of 1. The values of the two other benefit rates are smaller and mutually different since in both cases, some wealth is left behind upon death.

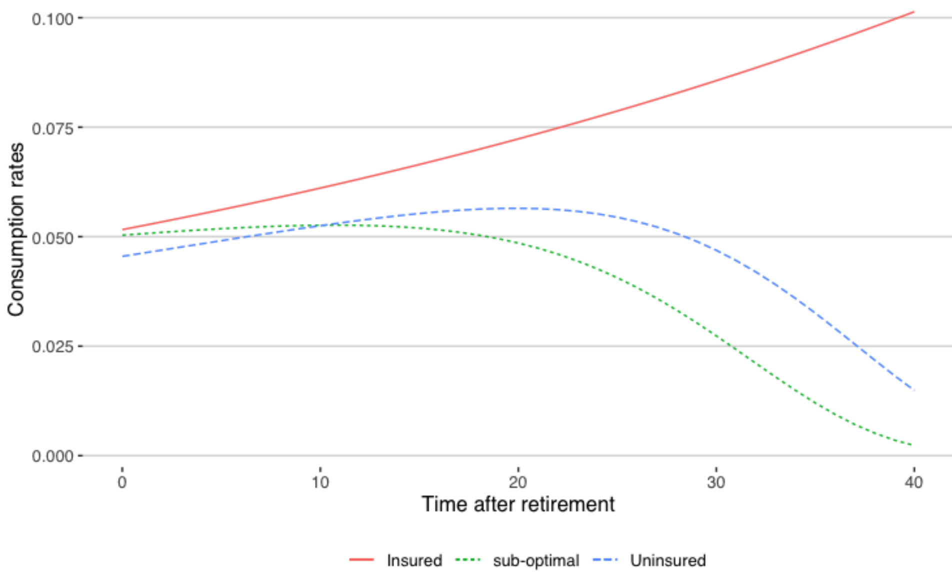

Figure 6.7: *The expected consumption rates as a function of time, with the preference* $\gamma = 2$ *and wealth equal to 1 at retirement.*

6.6 The Conclusion

We have specified the optimal payout profiles of retirement products with and without mortality credits. The preference parameters, as well as the insurance and financial market parameters, determine the optimal drift and volatility of the consumption/benefit profiles in the two cases. In the product design, these are determined by the proportion invested in risky assets and the interest and mortality basis used in the annuity when spreading out current wealth during the 'rest of the life'.

We have compared the cases with and without mortality credits numerically. We found, for realistic parameters, a considerable loss of wealth if an individual without utility from bequest does not annuitize, and we studied and discussed its dependence on preference and market parameters.

The results generally contribute to the optimal design of annuity contracts, both life annuities offered by pension funds and annuity contracts offered by banks. The results can help both financial regulators in reconsidering their framework for annuity designs and financial institutions in redesigning their product range and arguing for (or against) life annuitization. All of this contributes to the generation of welfare for retirees.

Future works along the lines include, in unprioritized order, a) evaluation of the sub-optimal product design under separated preferences; b) more fundamental understanding and comparison of the different approaches to separated preference under mortality risk; c) further numerical studies to illustrate the various consumption profiles arising from different market conditions; d) formalizing and discussing the impact of longevity risk in the sense of a stochastic process for μ, both with and without the individual's access to longevity derivatives; e) the impact of asymmetric information about health that could undermine the extreme payout flexibility assumed; f) more profound discussions about institutional and policy impact of the study.

Inserting (6.3.1), and by defining (6.4.5), obtaining the dynamics of the consumption rate in the sub-optimal situation (6.4.4).

Chapter 7

Optimal Equilibrium Investment and Insurance with State-Dependent Risk Aversion

Abstract

We formalize a consumption-investment-insurance problem with state-dependent relative risk aversion, where the state is modeled by a finite-state Markov chain representing insurable risks such as health and lifetime uncertainty. We use certainty equivalents to quantify the economic value of uncertain preferences. This introduces time inconsistency, which we address using the equilibrium approach. We formulate and prove the verification theorem, deriving solutions for two different scenarios: logarithmic preferences and indifference to state changes. In both cases, it demonstrates how optimal strategies involve state-dependent adjustments in investment and insurance decisions.

7.1 Introduction

We address a decision problem related to life insurance, including uncertainty of preferences in an optimal investment and life insurance problem with state-dependent risk aversion, where the state is the biometric state of the insured. In this setting, the problem of optimal terminal wealth is formulated and solved by controlling investment and insurance. This builds on the intuitive rationality that preferences can be heterogeneous across different states, whether for single-agent models with health states, multi-agent, or multi-generation models, and includes the uncertainty of preferences. Including the randomness of preferences and evaluating certainty equivalents makes the problem time-inconsistent, which we deal with using the equilibrium approach.

This research builds on a lineage of continuous-time decision problems that originated with Merton, 1971 and later expanded by Richard, 1975, introducing life insurance and uncertain lifetimes. Later, a significant milestone was reached with Kraft and Steffensen, 2008a, which integrated the Markov chain framework from life insurance setups, also investigated by Hoem, 1988 and Norberg, 1991. This incorporated state dependency into the utility function within the Markov chain's context by letting the utility function be state-dependent by the weights placed upon each state of the Markov chain. Steffensen and Søe, 2023 further expanded on this concept, not only incorporating state dependency into the weights as Kraft and Steffensen, 2008a but additionally letting the risk aversion parameter be state-dependent within a general J-state Markov chain model.

Incorporating the uncertainty of preferences and randomness is highly relevant, as policyholders are unaware of their preferences' current and future specific values. Various methods have been proposed to estimate these uncertainties, such as questionnaires (e.g., Burgaard and Steffensen, 2020). However, the equilibrium approach is novel and crucial for this research. Desmettre and Steffensen, 2023 addressed a portfolio optimization problem under random preferences by considering a distribution over certainty equivalents. They illustrated this approach by comparing it to choosing between different investment strategies while being uncertain about one's preferences, which are expected to change over time. Instead of averaging all preferences, which might result in a solution lacking real-world applicability, they proposed using certainty equivalents. This method involves determining the amount of money one would accept today instead of taking the risk, providing a clear monetary value and economic clarity. We adopt this approach to deal with uncertain preferences. However, this technique introduces mathematical complexity due to non-linear functions, which can result in time inconsistency. As a result, we deviate from conventional dynamic programming principles and instead adopt the equilibrium approach which was formalized by, among others, Björk, Murgoci, and Zhou, 2014, Björk, Khapko, Murgoci, et al., 2021, and Björk, Khapko, and

Murgoci, 2017, which aligns with research conducted by Jensen and Steffensen, 2015, Jensen, 2019, and Fahrenwaldt, Jensen, and Steffensen, 2020. This shift allows us to formulate equilibrium controls consistent with future decisions, creating a time-consistent strategy. However, it is essential to note that this approach does not necessarily result in optimality in the traditional sense.

This paper outlines the problem and solution process. We present and motivate the problem formulation in Section 7.2 and explain the equilibrium concept in Section 7.3. In Section 7.4, we present and prove the verification theorem, and in Section 7.5, we solve the optimization problem for specific choices of utility functions.

7.2 Set-up and formulation of the problem

This section provides an abstract and non-technical presentation of the optimization framework and problem, where we formulate and motivate the problem.

7.2.1 The financial and insurance market

We consider an insurance policy until time T in a standard Black-Scholes market, with the bond B as the risk-free asset and the stock S as the risky asset. The price dynamics are thus

$$\begin{aligned} dB(t) &= rB(t)dt, & B(0) &= 1, \\ dS(t) &= \alpha S(t)dt + \sigma S(t)dW(t), & S(0) &= s_0. \end{aligned}$$

Here, r, α and σ are constants, and W is a standard Brownian motion. For a classical survival model, where the insured can be alive or dead, we introduce the process $N(t)$ counting the number of deaths and $I(t)$ indicating whether the insured is alive, such that $N = 1 - I$. In addition, while the insured is alive, they can make investment decisions by managing the portion invested in the risky asset, $\pi(t)$ at time $t \in [0, T]$ and decisions about the life insurance payout by overseeing the amount paid out upon death, $b(t)$. This lump sum is paid out along with the remaining wealth. For this insurance, the insured is responsible for paying the corresponding premium $\mu^*(t)b(t)$ at time $t \in [0, T]$ while alive. Then the wealth process while the insured is alive under admissible controls π, b, is given by the solution of the stochastic differential equation

$$\begin{aligned} dX^{\pi,b}(t) &= \big((r + \pi(t)(\alpha - r))X^{\pi,b}(t) - \mu^*(t)b(t)\big)\, dt + \pi(t)\sigma X^{\pi,b}(t)dW(t), \\ X^{\pi,b}(0) &= x_0. \end{aligned}$$

Moreover, to allow for a state-dependent risk aversion, where the state refers to the life state of the insured as Steffensen and Søe, 2023, we let the problem be governed by the state of the finite-state continuous-time Markov chain Z in the state space $\mathcal{J}$. For the remainder of this paper, we require that for each initial point

$(t,z) \in [0,T] \times \mathcal{J}$, the stochastic differential equation (above) has a unique strong solution.

In a more general setting, beyond the survival model, the position of the insured and affiliated insurance policy is described by the finite-state continuous-time Markov chain, Z, on the state space $\mathcal{J}$. For a policy that ends at time T, we use $Z(t)$ to represent the state at time $t \in [0,T]$. The Brownian motion W and Markov chain Z are independent and exist within the measurable space $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the natural filtration of (Z,W), see Dhaene et al., 2013 for further explanation regarding the independence and measures.

We consider life insurance policies as tradable market contracts, similar to Richard, 1975 and Kraft and Steffensen, 2008a. Thus, we define two equivalent probability measures on $(\Omega, \mathcal{F})$, the objective measure $\mathbb{P}$ and the pricing measure $\mathbb{P}^*$, for both the market risk and the insurance risk. We assume that the pricing measure exists so that the pricing is linear and unique with respect to insurance risk. Now let N^{jk} denote the counting process counting the number of transitions $j \to k$, let μ^{jk} be the objective transition intensity under $\mathbb{P}$ and μ^{*jk} the pricing intensity under $\mathbb{P}^*$ of the state process for jumping from state j to state k, allowing for a risk loading such that, possibly, $\mu^{jk}(t) \neq \mu^{*jk}(t)$. Formally described as

$$\lim_{t\to 0} \frac{E[dN^{jk}(t)|Z(t-)]}{dt} = \mu^{jk}(t),$$
$$\lim_{t\to 0} \frac{E^*[dN^{jk}(t)|Z(t-)]}{dt} = \mu^{*jk}(t).$$

The expectation E^* is taken under the pricing measure. Furthermore, for those pairs (j,k) where the transition $j \to k$ is possible, we assume $\mu^{jk}, \mu^{*jk} \to \infty$ for $t \to \infty$ additional for those pairs (j',k') where the transition $j' \to k'$ is not possible we let the transition intensities be zero, $\mu^{j'k'}(t) = \mu^{*j'k'}(t) = 0$ for any $t \in [0,T]$. We assume the state process Z is Markovian under both measures.

In this generalized setting, the wealth under admissible controls π and $\mathbf{b}$, where $\mathbf{b} = (b^{jk})_{j,k\in\mathcal{J}, j\neq k}$ of the insured is described by the equation

$$\begin{aligned} dX^{\pi,\mathbf{b}}(t) &= (r + \pi(t)(\alpha - r))X^{\pi,\mathbf{b}}(t)dt - \sum_{k:k\neq j, Z(t)=j} \mu^{*jk}(t) b^{jk}(t) I^j(t)dt \\ &\quad + \sum_{k:k\neq j, Z(t)=j} b^{jk}(t)dN^{jk}(t) + \pi(t)\sigma X^{\pi,\mathbf{b}}(t)dW(t), \qquad (7.2.1) \\ X^{\pi,\mathbf{b}}(0) &= x_0. \end{aligned}$$

7.2.2 Introduction to the heterogeneity of preferences

Our goal is to include the heterogeneity of preferences by allowing them to be state-dependent, as in Steffensen and Søe, 2023, with a state-dependent relative risk aversion, but also letting the state be random such that the relative risk aversion depends on the random state of the Markov process Z at time T, that is, $\gamma(Z(T))$.

The problem of optimizing terminal wealth, investment, and life insurance while taking into account state-dependent relative risk aversion can be expressed as

$$\sup_{\pi,\mathbf{b}} E\left[\frac{1}{1-\gamma(Z(T))}X(T)^{1-\gamma(Z(T))}\right].$$

The problem is time-consistent; however, regular separation is not feasible; see Steffensen and Søe, 2023 for a further explanation. To simplify the following notation, we introduce $Z(t) = Z_t$ and $X(t) = X_t$ for $t \in [0,T]$. By Steffensen and Søe, 2023, we know the above problem can be dealt with by a value function expressed as

$$V(t,x,z) = \sup_{\pi,\mathbf{b}} E_{t,x,z}\left[\frac{1}{1-\gamma(Z_T)}X_T^{1-\gamma(Z_T)}\right],$$

where $E_{t,x,z}$ is the conditional expectation given $X(t) = x$ and $Z(t) = z$. The value function is characterized by a PDE containing local optimization problems at each time point, solved by dynamic programming techniques. Due to linearity, the solution for the controls π and $\mathbf{b}$ to the local optimization problem can also be proven to solve the global optimization problem. This linearity is essential for the equality between the local and global optimization problems; we stress this because it is exactly the linearity we disrupt later.

7.2.3 Certainty equivalents

We introduce certainty equivalents using the same method as Desmettre and Steffensen, 2023 and Jensen and Steffensen, 2015. The certainty equivalent is formulated as

$$u^{-1}(E\left[u(X_T)\right]).$$

Think of the certainty equivalent as the specific amount of terminal wealth an investor needs at time 0 to forego the uncertain terminal wealth at time T, X_T. The utility function expressing an investor's risk aversion preferences is denoted by the variable u, which is formulated using the power utility

$$u^{Z_T}(X_T) = \frac{1}{1-\gamma(Z_T)}X_T^{1-\gamma(Z_T)}. \tag{7.2.2}$$

As a consequence of introducing certainty equivalents, they are added instead of utility, which has its economic upsides in terms of adding a monetary value instead of a utility value, but still being dependent on the stochastic value of the state Z_T manifested in the preference $\gamma(Z_T)$.

7.2.4 The General problem

We now present the objective as the problem has been motivated and explained. The objective is to maximize the expected utility of terminal wealth, with random state-dependent relative risk aversion in an equilibrium sense

$$J^{\pi,\mathbf{b}}(t,x,z) = E_{t,x,z}\left[v((u^{Z_T})^{-1}(E_{t,x,Z}\left[u^{Z_T}(X_T^{\pi,\mathbf{b}})\right]))\right]. \tag{7.2.3}$$

Classical dynamic programming cannot solve this problem since the transformation $v \circ u^{-1}$ of the expectation interrupts the nice property of linearity of the expectation operator. This is a known consequence of using certainty equivalents, but before presenting an alternate method as in Desmettre and Steffensen, 2023, Jensen and Steffensen, 2015, Björk, Murgoci, and Zhou, 2014, and Fahrenwaldt, Jensen, and Steffensen, 2020, we want to give some intuition on the utility functions.

7.2.5 Utility functions

When working with two utility functions, u and v, it is essential to understand their interpretations. The function u represents the baseline utility and reflects the preferences and values for a specific outcome, assuming no uncertainty about the terminal state. Essentially, u measures how much value is placed on the terminal wealth and can also be considered a ranking of the terminal wealth. On the other hand, v is used to account for and evaluate the uncertainty of the states associated with the utility function u. By modifying the baseline preference function u, v quantifies how to assess the impact of uncertainty. It represents a transformation of u, reflecting how one's preferences change when considering the possibility of a changing state.

Let u be the power utility function defined in (7.2.2) and consider three different cases of the function v. First, we consider the indifferent case, where v is the identity function $v(x) = x$, and the insured is indifferent regarding the uncertainty related to the state change. This means that the insured is indifferent regarding the terminal state. The second case is where $v = u$, the state preferences are the same as the risk aversion preferences, and γ_{Z_T} describes these preferences. This is the same problem as in Steffensen and Søe, 2023, where equivalence is particularly interesting since it emphasizes the importance of the results in Steffensen and Søe, 2023. The last case investigated is $v(x) = \log(x)$, and is a special case of power utility for $\lim_{\gamma\to 1} \frac{x^{1-\gamma}}{1-\gamma} = \log(x)$. Representing states' preferences for the insured, indicating a logarithmic relationship in this final and third case, can describe the transformation that reflects how the preferences change when the insured considers the possibility of a change in states.

Thereby, there is a logarithmic aversion toward the uncertainty of the final state, so the insured changes their preferences in a logarithmic manner when considering the possibility of ending up in another state. More concretely, when insureds incorporate the possibility that they might terminate at time T in a disability state, an active state, a reactivated state, or even a free policy state, their utility function is transformed by a logarithmic function, altering the ranking.

Throughout this paper, we present results in the first and third cases and relate them to the results in the second. The first case is known as the *indifferent case* and the third as the *logarithmic case.*

7.3 Equilibrium

In this section, we define the equilibrium in our setting. For derivatives, we shall use the notation $V_t = \frac{\partial}{\partial t}V$, $V_x = \frac{\partial}{\partial x}V$ and $V_{xx} = \frac{\partial^2}{\partial x^2}V$, to ease the notation as much as possible.

Definition 7.3.1 (Admissible Control Law)**.** *An admissible control law* $\hat{\pi}, \hat{\mathbf{b}}$*, where* $\hat{\mathbf{b}} = (\hat{b}^{jk})_{k:k\neq j,j,k\in\mathcal{J}}$*, is each a map so that* $\hat{\pi} : [0,T]\times\mathbb{R}\times\mathcal{J}\to\mathbb{R}$ *and for each possible transition* $k\neq j$ *then the* $\hat{b}^{jk} : [0,T]\times\mathbb{R}\times\mathcal{J}\to\mathbb{R}$ *satisfying the following conditions*

- *For each initial point* (t,x)*, the SDE (eq.ref) has a unique strong solution denoted* $X^{\hat{\pi},\hat{\mathbf{b}}}$
- *For each initial point* $(t,x,z)\in[0,T]\times\mathbb{R}\times\mathcal{J}$ *we have*

$$E_{t,x,z}\Big[v((u^{Z_T})^{-1}(E_{t,x,Z}\Big[u^{Z_T}(X_T^{\hat{\pi},\hat{\mathbf{b}}})\Big]))\Big] < \infty.$$

- $\hat{\pi}, \hat{\mathbf{b}}$ *is continuous.*

The set of admissible control laws is denoted by $\mathcal{U}$*.*

Informally speaking, the argument can describe an equilibrium control: Given a control strategy $(\pi', \mathbf{b}')$ the value function $V(t,x,z)$ is defined by the function $J^{\pi',\mathbf{b}'}(t,x,z)$ dependent on the control strategy restricted to the time interval $[t,T]$ and the state $Z_t = z$. An equilibrium control $(\hat{\pi},\hat{\mathbf{b}})(t,x,z)$ must satisfy; at any arbitrary point $t\in[0,T]$, any future decisions made by the insured at time $s\in(t,T]$ must use the control strategy $(\hat{\pi},\hat{\mathbf{b}})(s,\cdot,\cdot)$, then the optimal strategy at time t, given the objective function $J^{\hat{\pi},\hat{\mathbf{b}}}(t,x,z)$ is to use the control $(\hat{\pi},\hat{\mathbf{b}})(t,\cdot,\cdot)$. A more formal definition is

Definition 7.3.2 (Equilibrium control law)**.** *Consider an admissible control law* $(\hat{\pi},\hat{\mathbf{b}})$ *(informally viewed as a candidate control law). Choose an arbitrary admissible control law* $(\pi,\mathbf{b})\in\mathcal{U}$ *and a fixed real number* $h>0$*, and fix an arbitrary chosen* *initial point* (t,x,z)*. Define the control law* $(\pi_h, \mathbf{b}_h)$ *by*

$$(\pi_h, \mathbf{b}_h)(s,y,\zeta) = \begin{cases} (\pi, \mathbf{b})(s,y,\zeta) & \text{for } t \leq s < t+h, \quad y \in \mathbb{R}, \quad \zeta \in \mathcal{J} \\ (\hat{\pi}, \hat{\mathbf{b}})(s,y,\zeta) & \text{for } t+h \leq s \leq T, \quad y \in \mathbb{R}, \quad \zeta \in \mathcal{J} \end{cases},$$

if

$$\liminf_{h \to 0} \frac{J^{\hat{\pi},\hat{\mathbf{b}}}(t,x,z) - J^{\pi_h,\mathbf{b_h}}(t,x,z)}{h} \geq 0,$$

for all $(\pi, \mathbf{b}) \in \mathcal{U}$*.*

We say that $(\hat{\pi}, \hat{\mathbf{b}})$ is an equilibrium control law. Corresponding to the equilibrium law $(\hat{\pi}, \hat{\mathbf{b}})$ we define the equilibrium value function by

$$V(t,x,z) := J^{\hat{\pi},\hat{\mathbf{b}}}(t,x,z). \tag{7.3.1}$$

Remark 7.3.3. Note that in a standard time-consistent case, the definition of an equilibrium coincides with the definition of an optimal strategy, and the equilibrium value function is the optimal value function. This means that if the insured has the same preferences for risk aversion and towards state $v(x) = u(x)$, we have a time-consistent problem. In this case, the equilibrium controls are the optimal controls, and the corresponding equilibrium value function is the optimal value function.

For notational ease in later calculations, we define the infinitesimal operator; for further information, look to (Björk: Jump-diffusion process)

Definition 7.3.4 (Infinitesimal Operator)**.** *The infinitesimal operator* $\mathcal{A}$ *for the process* $X^{\pi,\mathbf{b}}$ *is given by*

$$\begin{aligned} \mathcal{A}^{\pi,\mathbf{b}} F^z(t,x) &= F_t^z(t,x) + ((r + \pi(\alpha - r))x - \textstyle\sum_{k:k\neq z} \mu^{*zk}(t) b^{zk}) F_x^z(t,x) \\ &\quad + \tfrac{1}{2}\pi\sigma x F_{xx}^z(t,x) + \textstyle\sum_{k:k\neq z} \mu^{zk}(t)(F^k(t,x+b^{zk}) - F^z(t,x)). \end{aligned}$$

Furthermore, we need one more definition for the set to which our important functions belong.

Definition 7.3.5 (L^2-space, cf. Def. 15.5 in Björk, Khapko, Murgoci, et al., 2021)**.** *Consider an arbitrary admissible control* $\pi, \mathbf{b} \in \mathcal{U}$*. A function* $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ *is said to belong to the space* $L^2(X^{\pi,\mathbf{b}})$ *if it satisfies the condition*

$$E_{t,x}\left[\int_t^T ||h_x(s, X^{\pi,\mathbf{b}})\, \sigma\pi\,(s)\, X^{\pi,\mathbf{b}}\,(s)\,||^2\, ds\right] < \infty, \tag{7.3.2}$$

for every (t,x)*. In this expression,* h_x *denotes the gradient of* h *in the* $x-$*variable.*

7.4 Verification

Definition 7.4.1. *Given an object function*

$$\begin{aligned} J^{\pi,\mathbf{b}}(t,x,z) &= E_{t,x,z}[F^{\pi,\mathbf{b},Z_T}(t,x,X_T^{\pi,\mathbf{b}})], \\ &= E_{t,x,z}[v\left((u^{Z_T})^{-1}\left(E_{t,x,Z}[u^{Z_T}(X_T^{\pi,\mathbf{b}})]\right)\right)]. \end{aligned}$$

The extended Hamilton-Jacobi-Belman system for V is given by the following system of equations

1. *The function V is defined by*

$$\inf\Big\{-(\mathcal{A}^{\pi,\mathbf{b}}V)(t,x,z)+(\mathcal{A}^{\pi,\mathbf{b}}f)(t,t,x,x,z) - (\mathcal{A}^{\pi,\mathbf{b}}f^{t,x})(t,x,z)\Big\}=0, \tag{7.4.1}$$

with the boundary condition

$$V(T,x,z)=F(T,x,x). \tag{7.4.2}$$

2. *For each fixed s and y the function $f^{s,y}(t,x,z)$ is defined as*

$$\begin{aligned} (\mathcal{A}^{\hat\pi,\hat{\mathbf{b}}}f^{s,y})(t,x,z) &= 0, \qquad 0\le t<T, \\ f^{s,y}(T,x,z) &= F^{\hat\pi,\hat{\mathbf{b}},Z_T}(s,y,X_T^{\pi,\mathbf{b}}) = l^{Z_T}(a^{\hat\pi,\hat{\mathbf{b}},Z_T}(s,y)). \end{aligned}$$

The above the $\hat\pi,\hat{\mathbf{b}}$ always denotes the control law that realizes the supremum in 1). Namely,

$$\hat\pi,\hat{\mathbf{b}} = \arg\inf_{\pi,\mathbf{b}}\Big\{-(\mathcal{A}^{\pi,\mathbf{b}}V)(t,x,z)+(\mathcal{A}^{\pi,\mathbf{b}}f)(t,t,x,x,z) - (\mathcal{A}^{\pi,\mathbf{b}}f^{t,x})(t,x,z)\Big\} \tag{7.4.3}$$

with the boundary condition

$$V(T,x,z)=v(x).$$

In this, we have used the notation

$$f(t,s,x,y,z)=f^{s,y}(t,x,z).$$

Theorem 7.4.2 (Verification)**.** *Assume that for all (s,y) the functions $V(t,x,z)$, $f^{s,y}(t,x,z)$, $\hat{\mathbf{b}}$, and, $\hat\pi$ have the following properties*

1. *V and $f^{s,y}$ constitute a solution to the extended Hamilton-Jacobi-Bellman equation in* (7.4.1)*.*

2. V and $f^{s,y}$ are smooth in the sense that they are in $C^{1,2}$, for all $z \in \mathcal{J}$.

3. The function $\hat{\mathbf{b}}$, and $\hat{\pi}$ is defined by (7.4.3) and thereby fulfills the supremum in (7.4.1) and is an admissible control law.

4. V and $f^{s,y}$ and $(t,x,z) \mapsto f(t,t,x,x,z)$ all belong to $\mathcal{L}^2(X^{\hat{\mathbf{b}},\hat{\pi}})$ defined in Definition 7.3.5.

Then $\hat{\mathbf{b}}, \hat{\pi}$ is an admissible control law, and V is the corresponding equilibrium value function.

Further, f har probabilistic representation

$$
\begin{aligned}
f^{s,y}(t,x,z) &= E_{t,x,z}\big[l^{Z_T}(E_{s,y,Z}[u^{Z_T}(X_T^{\hat{\pi},\hat{\mathbf{b}}})])\big], \\
&= E_{t,x,z}\big[F^{Z_T,\hat{\pi},\hat{\mathbf{b}}}(s,y)\big].
\end{aligned}
$$

Proof:

The proof consists of 3 steps:

1. Show the probabilistic representation

$$
\begin{aligned}
f^{t,x}(t,x,z) &= E_{t,x,z}\big[l^{Z_T}(E_{t,x,Z}[u^{Z_T}(X_T^{\hat{\pi},\hat{\mathbf{b}}})])\big], \\
&= E_{t,x,z}\big[F^{Z_T,\hat{\pi},\hat{\mathbf{b}}}(t,x)\big].
\end{aligned}
$$

2. Show that $V(t,x,z) = J^{\hat{\pi},\hat{\mathbf{b}}}(t,x,z)$.
3. Show that $\hat{\pi}, \hat{\mathbf{b}}$ is indeed an equilibrium control law.

Step 1: Show the probabilistic representation

We want to show that $f^{t,x}(t,x,z) = E_{t,x,z}\big[l^{Z_T}(E_{t,x,Z}[u^{Z_T}(X_T^{\hat{\pi},\hat{\mathbf{b}}})])\big]$, we know that $f, V \in \mathcal{L}^2(X^{\hat{\pi},\hat{\mathbf{b}}})$ and that $(\mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}} f^{s,y})(t,x,z) = 0$ for $0 \le t \le T$ and $f^{s,y}(T,x,z) = F^{Z_T,\hat{\pi},\hat{\mathbf{b}}}(s,y,X^{\hat{\pi},\hat{\mathbf{b}}}(T))$ meaning that

$$
\begin{aligned}
0 = \mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}} F(t,x,z) = F_t(t,x,z) &+ \big((r + \hat{\pi}(\alpha - r))x \\
&- \sum_{k:k \neq z} \mu^{*zk}(t)\hat{b}^{zk}\big) F_x(t,x,z) + \frac{1}{2}\hat{\pi}\sigma x F_{xx}(t,x,z) \\
&+ \sum_{k:k \neq z} \mu^{zk}(t)(F(t,x+\hat{b}^{zk},k) - F(t,x,z)).
\end{aligned}
$$

With Z being a Markov chain and

$$
\begin{aligned}
&E[\int_0^T \hat{\pi}\sigma X^{\hat{\pi},\hat{\mathbf{b}}}(s) f^{s,y}(s, X^{\hat{\pi},\hat{\mathbf{b}}}(s)) dW(s)] = 0, \\
&E[\int_0^T \hat{b}^{jk}(s) f^{s,y}(s, X^{\hat{\pi},\hat{\mathbf{b}}}(s))(dN^{jk}(s) - \mu^{jk}(s)\mathbb{1}_{\{Z_s=j\}}ds)] = 0.
\end{aligned}
$$

Then by Feynman-Kac theorem (see e.g. Björk, 2021)

$$f^{s,y}(t,x,z) = E_{t,x,z}[F^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,y,X^{\hat{\pi},\hat{\mathbf{b}}}(T))] = E_{t,x,z}[l^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,y))]. \tag{7.4.4}$$

Here, $a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x) = E_{t,x,Z}[u^{Z_T}(X^{\hat{\pi},\hat{\mathbf{b}}}(T)]$, such that

$$f^{t,x}(t,x,z) = E_{t,x,z}[l^{Z_T}(E_{t,x,Z}[u^{Z_T}(X^{\hat{\pi},\hat{\mathbf{b}}}(T)]]. \tag{7.4.5}$$

Step 2: Show that $V(t,x,z) = J^{\hat{\pi},\hat{\mathbf{b}}}(t,x,z)$

To do so, we use the equation defining the function V (7.4.1) and get

$$-(\mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}}V)(t,x,z) + (\mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}}f)(t,t,x,x,z) - (\mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}}f^{t,x})(t,x,z) = 0,$$

and by the definition of $f^{t,x}(t,x,z)$, we know that $(\mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}}f^{t,x})(t,x,z) = 0$ such that for all t and x we have

$$(\mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}}V)(t,x,z) = (\mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}}f)(t,t,x,x,z).$$

Further, by using Ito's lemma on the pro"cess $V(s,X^{\hat{\pi},\hat{\mathbf{b}}}(s),z)$, remembering $V \in \mathcal{L}^2(X^{\hat{\pi},\hat{\mathbf{b}}})$, we have that

$$E_{t,x,z}[V(T,X^{\hat{\pi},\hat{\mathbf{b}}}(T),Z_T)] = V(t,x,z) + E_{t,x,z}[\int_t^T (\mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}}V)(s,X^{\hat{\pi},\hat{\mathbf{b}}}(s),Z(s))ds].$$

Inserting from the previous equation gives us

$$E_{t,x,z}[V(T,X^{\hat{\pi},\hat{\mathbf{b}}}(T),Z_T)] = V(t,x,z) + E_{t,x,z}[\int_t^T (\mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}}f)(s,s,X^{\hat{\pi},\hat{\mathbf{b}}}(s),X^{\hat{\pi},\hat{\mathbf{b}}}(s),Z(s))ds].$$

But we can also apply Ito's lemma to $f(s,s,X^{\hat{\pi},\hat{\mathbf{b}}}(s),X^{\hat{\pi},\hat{\mathbf{b}}}(s),Z(s))$ as

$$E_{t,x,z}[f(T,T,X^{\hat{\pi},\hat{\mathbf{b}}}(T),X^{\hat{\pi},\hat{\mathbf{b}}}(T),Z_T)] = f(t,t,x,x,z) + E_{t,x,z}[\int_t^T (\mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}}f)(s,s,X^{\hat{\pi},\hat{\mathbf{b}}}(s),X^{\hat{\pi},\hat{\mathbf{b}}}(s),Z(s))ds].$$

Now, inserting this into the last term of the previous equation yields

$$E_{t,x,z}[V(T,X^{\hat{\pi},\hat{\mathbf{b}}}(T),Z_T)] = V(t,x,z) + E_{t,x,z}[f(T,T,X^{\hat{\pi},\hat{\mathbf{b}}}(T),X^{\hat{\pi},\hat{\mathbf{b}}}(T),Z_T)] - f(t,t,x,x,z).$$

Remembering that by the boundary condition, we have, in general, that

$$E_{t,x,z}[V(T,X^{\hat{\pi},\hat{\mathbf{b}}}(T),Z_T)] = E_{t,x,z}[f(T,T,X^{\hat{\pi},\hat{\mathbf{b}}}(T),X^{\hat{\pi},\hat{\mathbf{b}}}(T),Z_T)].$$

Concluding that

$$V(t,x,z) = f(t,t,x,x,z) = f^{t,x}(t,x,z) = E_{t,x,z}[l^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x))] = J^{\hat{\pi},\hat{\mathbf{b}}}(t,x,z).$$

Step 3: Show that $(\hat{\pi}, \hat{\mathbf{b}})$ is indeed an equilibrium control law

For an admissible control law $(\hat{\pi}, \hat{\mathbf{b}})$ and for any $h > 0$, we apply Lemma 3.8 og Björk and Murgoci, 2014 to the points h and $t+h$ with

$$f^{\hat{\pi},\hat{\mathbf{b}},s,y}(t,x,z) = E_{t,x,z}[l^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}}}(s,y))] = E_{t,x,z}[F^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,y,X_T^{\hat{\pi},\hat{\mathbf{b}}})].$$

Thus we obtain

$$\begin{aligned} J^{\pi_h,\mathbf{b}_h}(t,x,z) = &E_{t,x,z}[J^{\pi_h,\mathbf{b}_h}(t+h, X^{\pi_h,\mathbf{b}_h}(t+h), Z(t+h)], \\ &- E_{t,x,z}[f^{\pi_h,\mathbf{b}_h}(t+h,t+h,X^{\pi_h,\mathbf{b}_h}(t+h), X^{\pi_h,\mathbf{b}_h}(t+h), Z(t+h)], \\ &+ E_{t,x,z}[f^{\pi_h,\mathbf{b}_h,t,x}(t+h, X^{\pi_h,\mathbf{b}_h}(t+h), Z(t+h)]. \end{aligned}$$

Since on $[t, t+h)$ we know $\pi_h = \pi$ and $\mathbf{b}_h = \mathbf{b}$, then by continuity of $X^{\pi_h,\mathbf{b}_h}(s)$, for $s \le t$ we have that $X^{\pi_h,\mathbf{b}_h}(t+h) = X^{\pi,\mathbf{b}}(t+h)$ and since $\pi_h = \hat{\pi}$ and $\mathbf{b}_h = \hat{\mathbf{b}}$ on $[t+h, T]$ then by Step 2

$$J^{\pi_h,\mathbf{b}_h}(t+h, X^{\pi_h,\mathbf{b}_h}(t+h), Z(t+h)) = V(t+h, X^{\pi,\mathbf{b}}(t+h), Z(t+h)),$$

and furthermore

$$\begin{aligned} &f^{\pi_h,\mathbf{b}_h}(t+h,t+h,X^{\pi_h,\mathbf{b}_h}(t+h), X^{\pi_h,\mathbf{b}_h}(t+h), Z(t+h)) \\ &\quad = f^{\pi,\mathbf{b}}(t+h,t+h,X^{\pi,\mathbf{b}}(t+h), X^{\pi,\mathbf{b}}(t+h), Z(t+h)). \end{aligned}$$

Putting everything together yields

$$\begin{aligned} J^{\pi_h,\mathbf{b}_h}(t,x,z) = &E_{t,x,z}[V(t+h, X^{\pi,\mathbf{b}}(t+h), Z(t+h)] \\ &- E_{t,x,z}[f^{\pi,\mathbf{b}}(t+h,t+h,X^{\pi,\mathbf{b}}(t+h), X^{\pi,\mathbf{b}}(t+h), Z(t+h)] \\ &+ E_{t,x,z}[f^{\pi,\mathbf{b},t,x}(t+h, X^{\pi,\mathbf{b}}(t+h), Z(t+h)]. \end{aligned} \tag{7.4.6}$$

Now from (7.4.1) with $\pi = \pi(t,x,z)$ and $\mathbf{b} = \mathbf{b}(t,x,z)$ we have

$$\begin{aligned} &(\mathcal{A}^{\pi(t,x,z),\mathbf{b}(t,x,z)}V)(t,x,z) - (\mathcal{A}^{\pi(t,x,z),\mathbf{b}(t,x,z)}f)(t,t,x,x,z) \\ &\qquad + (\mathcal{A}^{\pi(t,x,z),\mathbf{b}(t,x,z)}f^{t,x})(t,x,z) \le 0. \end{aligned} \tag{7.4.7}$$

Now by applying Ito's lemma for $h > 0$, taking the limit $h \to 0$, and remembering the admissible controls on the $\pi(t,x,z)$ and $\mathbf{b}(t,x,z)$ are continuous we get the following three equalities

$$\begin{aligned} E_{t,x,z}[V(t+h, X^{\pi,\mathbf{b}}(t+h), Z(t+h))] = &V(t,x,z) \\ &+ h(\mathcal{A}^{\pi(t,x,z),\mathbf{b}(t,x,z)}V)(t,x,z) + o(h), \end{aligned} \tag{7.4.8}$$

$$\begin{aligned} E_{t,x,z}[f(t+h,t+h,X^{\pi,\mathbf{b}}(t+h), X^{\pi,\mathbf{b}}(t+h), Z(t+h))] = &f(t,t,x,x,z) \\ &+ h(\mathcal{A}^{\pi(t,x,z),\mathbf{b}(t,x,z)}f)(t,t,x,x,z) + o(h), \end{aligned} \tag{7.4.9}$$

$$\begin{aligned} E_{t,x,z}[f^{t,x}(t+h, X^{\pi,\mathbf{b}}(t+h), Z(t+h))] = &f^{t,x}(t,x,z) \\ &+ h(\mathcal{A}^{\pi(t,x,z),\mathbf{b}(t,x,z)}f^{t,x})(t,x,z) + o(h). \end{aligned} \tag{7.4.10}$$

Combining (7.4.8), (7.4.9), (7.4.10) with (7.4.7) gives

$$\begin{aligned}
o(h) \geq\ & E_{t,x,z}[V(t+h, X^{\pi,\mathbf{b}}(t+h), Z(t+h))] - V(t,x,z) \\
& - \Big(E_{t,x,z}[f(t+h, t+h, X^{\pi,\mathbf{b}}(t+h), X^{\pi,\mathbf{b}}(t+h), Z(t+h))] \\
& \qquad - f(t,t,x,x,z) \Big) \\
& + E_{t,x,z}[f^{t,x}(t+h, X^{\pi,\mathbf{b}}(t+h), Z(t+h))] - f^{t,x}(t,x,z).
\end{aligned}$$

Now, since $f(t,t,x,x,z) = f^{t,x}(t,x,z)$, these equal out, and further, we remind ourselves of the calculation (7.4.6) to obtain

$$\begin{aligned}
V(t,x,z) \quad \geq \quad & E_{t,x,z}[V(t+h, X^{\pi,\mathbf{b}}(t+h), Z(t+h))] \\
- \quad & E_{t,x,z}[f(t+h, t+h, X^{\pi,\mathbf{b}}(t+h), X^{\pi,\mathbf{b}}(t+h), Z(t+h))] \\
+ \quad & E_{t,x,z}[f^{t,x}(t+h, X^{\pi,\mathbf{b}}(t+h), Z(t+h))] = J^{\pi_h,\mathbf{b}_h}(t,x,z) + o(h).
\end{aligned}$$

Now from the second step we know that $V(t,x,z) = J^{\hat{\pi},\hat{\mathbf{b}}}(t,x,z)$ and from the last equation we have that

$$V(t,x,z) \geq J^{\pi_h,\mathbf{b}_h}(t,x,z) + o(h).$$

Concluding that

$$J^{\hat{\pi},\hat{\mathbf{b}}}(t,x,z) - J^{\pi_h,\mathbf{b}_h}(t,x,z) \geq o(h).$$

Thus

$$\liminf_{h\to 0} \frac{J^{\hat{\pi},\hat{\mathbf{b}}}(t,x,z) - J^{\pi_h,\mathbf{b}_h}(t,x,z)}{h} \geq o(h). \tag{7.4.11}$$

With equilibrium controls $\hat{\pi}, \hat{\mathbf{b}}$, we conclude that V is the optimal value function.

7.5 Solving the optimization problem

To solve the optimization problem, we aim to address the extended Hamilton-Jacobi-Bellman equation in (7.4.1). This requires reformulating the problem into a more tractable form. To achieve this, we introduce the following auxiliary functions for general utility functions v and u

$$\begin{aligned}
a^{\pi,\mathbf{b},Z_T}(t,x) \quad &:= \quad E_{t,x,Z}\left[u^{Z_T}(X_T^{\pi,\mathbf{b}})\right], && (7.5.1) \\
F^{\pi,\mathbf{b},Z_T}(t,x) \quad &:= \quad v\left((u^{Z_T})^{-1}\left(E_{t,x,Z}[u^{Z_T}(X_T^{\pi,\mathbf{b}})]\right)\right), && (7.5.2) \\
l^{Z_T}(y) \quad &:= \quad v\left((u^{Z_T})^{-1}(y)\right). && (7.5.3)
\end{aligned}$$

Using these definitions, the reward function can be expressed as

$$J^{\pi,\mathbf{b}}(t,x,z) = E_{t,x,z}[l^{Z_T}(a^{\pi,\mathbf{b},Z_T}(t,x)] = E_{t,x,z}[F^{\pi,\mathbf{b},Z_T}(t,x)].$$

According to the verification theorem, we have

$$(\mathcal{A}^{\hat{\pi},\hat{\mathbf{b}}} f^{s,y})(t,x,z) = 0,$$

where $f(t,s,x,y,z) = f^{s,y}(t,x,z)$, under boundary condition

$$f^{s,y}(T,x,z) = l^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,y)).$$

Following Step 1 of the proof, we obtain

$$\begin{aligned} f^{s,y}(t,x,z) &= E_{t,x,z}[f^{s,y}(T, X^{\hat{\pi},\hat{\mathbf{b}}}(T), Z_T)], \\ &= E_{t,x,z}[l^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}}}(s,y))]. \end{aligned}$$

Here,

$$a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x) = E_{t,x,Z}[u^{Z_T}(X^{\hat{\pi},\hat{\mathbf{b}}}(T))].$$

Thus,

$$f^{t,x}(t,x,z) = E_{t,x,z}\Big[l^{Z_T}(E_{t,x,Z}[u^{Z_T}(X^{\hat{\pi},\hat{\mathbf{b}}}(T))])\Big].$$

Consequently, we derive the boundary condition

$$f^{s,y}(T,x,z) = l^{Z_T}(E_{s,y,Z}[u^{Z_T}(X^{\hat{\pi},\hat{\mathbf{b}}}(T))]).$$

This leads us to the definition

$$G^{Z_T}(t,x) := l^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)).$$

Between the event times of N^{jk} for all $k \neq j \in \mathcal{J}$ we have the standard expression

$$\begin{aligned} da^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x) = {}& ((r + \hat{\pi}(\alpha - r))x - \sum_{k:k\neq j, Z_t = j} \mu^{*jk}(t)\hat{b}^{jk}) a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)dt \\ &+ \frac{1}{2}\sigma^2\hat{\pi}^2 x^2 a_{xx}^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)dt \\ &+ a_t^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)dt + \sigma\hat{\pi} x a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)dW(t) \\ a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(T,x) = {}& u^{Z_T}(x). \end{aligned}$$

If N^{jk} has an event at time t, the process X will experience a jump of size b^{jk}. Consequently the jump of $a^{\hat{\pi},\hat{\mathbf{b}},Z_T}$ is defined by

$$a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t, X(t-) + b^{jk}(t)) - a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t-, X(t-)) = 0.$$

This equals zero for all $k \neq j$ in $\mathcal{J}$ because we have a martingale when conditioning on the entire trajectory of Z. Thus,

$$\begin{aligned} a_t^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x) = {}& -((r + \hat{\pi}(\alpha - r))x - \sum_{k:k\neq j, Z_t = j} \mu^{*jk}(t)\hat{b}^{jk}) a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x) \\ &- \frac{1}{2}\sigma^2\hat{\pi}^2 x^2 a_{xx}^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x). \end{aligned}$$

Using $G^{Z_T}(t,x) = l^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x))$, the partial derivatives of G are

$$
\begin{aligned}
G_t^{Z_T}(t,x) &= l_t^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}) = (l^{Z_T})'(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x))a_t^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x),\\
G_x^{Z_T}(t,x) &= l_x^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)) = (l^{Z_T})'(a^{\hat{\pi},\hat{\mathbf{b}},Z_T})a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x),\\
G_{xx}^{Z_T}(t,x) &= l_{xx}^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)) = (l^{Z_T})'(a^{\hat{\pi},\hat{\mathbf{b}},Z_T})a_{xx}^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)\\
&\qquad + (l^{Z_T})''(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x))(a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x))^2.
\end{aligned}
$$

This implies that between the N^{jk} events, we have for $G^{Z_T}(t,x)$

$$
\begin{aligned}
dG^{Z_T}(t,x) &= G_t^{Z_T}(t,x)dt + \Big((r+\hat{\pi}(\alpha-r))x\\
&\qquad - \sum_{k:k\neq j, Z_t=j} \mu^{*jk}(t)\hat{b}^{jk}\Big)G_x^{Z_T}(t,x)dt\\
&\qquad + \frac{1}{2}\sigma^2\hat{\pi}^2x^2G_{xx}^{Z_T}(t,x)dt + \sigma\hat{\pi}xG_x^{Z_T}(t,x)dW(t),\\
G^{Z_T}(T,x) &= v(x).
\end{aligned}
$$

Thus, by the partial derivatives, we can write

$$
\begin{aligned}
dG^{Z_T}(t,x) &= (l^{Z_T})'(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x))a_t^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)dt\\
&\quad + \Big((r+\hat{\pi}(\alpha-r))x\\
&\quad - \sum_{k:k\neq j, Z_t=j} \mu^{*jk}(t)\hat{b}^{jk}\Big)(l^{Z_T})'(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x))a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)dt\\
&\quad + \frac{1}{2}\sigma^2\hat{\pi}^2x^2\Big((l^{Z_T})'(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x))a_{xx}^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)\\
&\quad + (l^{Z_T})''(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x))(a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x))^2\Big)dt\\
&\quad + \sigma\hat{\pi}x(l^{Z_T})'(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x))a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)dW(t),\\
G^{Z_T}(T,x) &= v(x).
\end{aligned}
$$

Inserting our knowledge of $a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)$ between the jumps, and considering that at event times of N^{jk} the process X experiences a jump of size b^{jk}, the jump is defined by

$$
l^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t, X(t-)+b^{jk}(t))) - l^{Z_T}(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t-, X(t-))) = 0.
$$

Following the same argument, and applying the Feynman-Kac theorem, we have

$$
\begin{aligned}
G^{Z_T}(t,x) = E_{t,x,Z}\Big[\int_t^T &-\frac{1}{2}\sigma^2\hat{\pi}^2(s)X^2(s)(l^{Z_T})''(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s)))\\
&\times (a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s)))^2ds + v(X(T))\Big].
\end{aligned}
$$

Here,

$$
\begin{aligned}
f(s,t,x,y,z) &= E_{t,x,z}[G^{Z_T}(s,y)],\\
f(t,t,x,x,z) &= E_{t,x,z}[G^{Z_T}(t,x)].
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
f(t,t,x,x,z) &= E_{t,x,z}[E_{t,x,Z}[\int_t^T -\frac{1}{2}\sigma^2\hat{\pi}^2(s)X^2(s)(l^{Z_T})''(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s))) \\
&\qquad \times (a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s)))^2 ds + v(X(T))]], \\
&= E_{t,x,z}[\int_t^T -\frac{1}{2}\sigma^2\hat{\pi}^2(s)X^2(s)(l^{Z_T})''(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s))) \\
&\qquad \times (a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s)))^2 ds + v(X(T))].
\end{aligned}
$$

Depending on the choice of v we obtain two equations, one with running costs that follow from the Feynman-Kac result above as an expression for $(\mathcal{A}^{\pi,\mathbf{b}}f)(t,t,x,x,z)$ and one without the running costs where

$$
(\mathcal{A}^{\pi,\mathbf{b}}f^{s,y})(t,x,z) = 0.
$$

Hence, we can write the Hamilton-Jacobi-Bellman equation as

$$
0 = \inf_{\pi,b}\Big\{ -(\mathcal{A}^{\pi,\mathbf{b}}V)(t,x,z) - (\mathcal{A}^{\pi,\mathbf{b}}f)(t,t,x,x,z)\Big\}.
$$

Here we encounter a special case of a fixed point problem because, regardless of the chosen v function, the $(\mathcal{A}^{\pi,\mathbf{b}}f)(t,t,x,x,z)$ part will include $\hat{\pi}$.

7.5.1 fixed point explanation

To explain why the special case of the fixed point does not affect our problem, we will explore two seemingly different systems. First, consider a system representing the special fixed-point case

$$
\begin{aligned}
0 &= \inf_{\pi} f(\pi,\hat{\pi}), \\
\hat{\pi} &= \arg\inf_{\pi} f(\pi,\hat{\pi}),
\end{aligned}
$$

where the right-hand side of the first equation presents a unique situation: we aim to minimize a function that is itself a function of the argument where it attains its minimum. The key question is how to minimize such a function. By substituting the $\arg\inf$ from the second line back into the Hamilton-Jacobi-Bellman (HJB) equation in the first line, we obtain

$$
0 = f(\arg\inf_{\pi} f(\pi,\hat{\pi}),\hat{\pi}) = f(\hat{\pi},\hat{\pi}).
$$

We introduce $g(\hat{\pi}) = f(\hat{\pi},\hat{\pi})$ such that $0 = g(\hat{\pi})$. Using the notations $\arg_1$ and $\arg_2$ for the arguments in a two-dimensional function's first and second dimensions, we substitute the $\arg\inf$ into the equation for $\hat{\pi}$ and get

$$
\hat{\pi} = \arg_1 f(\arg\inf_{\pi} f(\pi,\hat{\pi}),\hat{\pi}) = \arg_1 f(\hat{\pi},\hat{\pi}).
$$

Since $\hat{\pi} = \arg_2 f(\hat{\pi}, \hat{\pi})$, we can write $\hat{\pi} = \arg g(\hat{\pi})$, leading to the conclusion

$$\hat{\pi} = \arg(0),$$

for the function g.

Next, we consider the second system, where we replace the appearance of $\hat{\pi}$ in the HJB-system with π over which we optimize, resulting in the standard system

$$0 = \inf_{\pi} f(\pi, \pi) = g(\pi),$$
$$\pi^* = \arg\inf_{\pi} f(\pi, \pi) = \arg\inf_{\pi} g(\pi),$$

We now explore this system. By substituting the arg inf back into the HJB, we use the relation for π^*

$$0 = g(\arg\inf g(\pi)) = g(\pi^*).$$

Then, substituting the arg inf into the equation for π^*, we get

$$\pi^* = \arg g(\arg\inf_{\pi} g(\pi)) = \arg g(\pi^*).$$

From $0 = g(\pi^*)$, we conclude

$$\pi^* = \arg(0),$$

for the function g.

Thus, we conclude from the two systems that since $\hat{\pi} = \arg(0)$ for the function g and $\pi^* = \arg(0)$ for the function g, it follows that $\hat{\pi} = \pi^*$. Therefore, if we find a solution to the standard version

$$0 = \inf_{\pi} f(\pi, \pi),$$
$$\pi^* = \arg\inf_{\pi} f(\pi, \pi),$$

This solution coincides with the solution to the alternative version

$$0 = \inf_{\pi} f(\pi, \hat{\pi}),$$
$$\hat{\pi} = \arg\inf_{\pi} f(\pi, \hat{\pi}).$$

7.5.2 Logarithmic Preferences

In the case of logarithmic preferences, as previously discussed, we have $v(x) = \log(x)$. Thus

$$l^{Z_T}(y) = v((u^{-1})(y)) = \log(((1-\gamma(Z_T))y)^{\frac{1}{1-\gamma(Z_T)}}),$$
$$\left(l^{Z_T}\right)'(y) = \frac{1}{(1-\gamma(Z_T))y},$$
$$\left(l^{Z_T}\right)''(y) = \frac{-1}{(1-\gamma(Z_T))y^2}.$$

To find a candidate value function and solve the HJB in this particular case, we use the auxiliary function (7.5.1) described by

$$a^{\pi,\mathbf{b},Z_T}(t,x) = \frac{1}{1-\gamma(Z_T)} x^{1-\gamma(Z_T)} k^{\pi,\mathbf{b},Z_T}(t)^{\gamma(Z_T)}.$$

Here, the controls' dependency shifts to the time-dependent function k, an exponential function that depends on the jump-diffusion process and can be found using Black-Scholes results. This function is an intermediate result, leaving out the specific form but emphasizing the dependence on the controls and the stochastic preference at termination.

Using these specifications, we get

$$\begin{aligned}
&-\int_t^T \frac{1}{2}\sigma^2 X(s)^2 \hat{\pi}^2(s) \left(l^{Z_T}\right)'' (a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s)))(a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s)))^2 ds \\
&= -\int_t^T \frac{1}{2}\sigma^2 \hat{\pi}^2(s)(\gamma(Z_T)-1)ds.
\end{aligned}$$

Then, by the tower property

$$\begin{aligned}
f(t,t,x,x,z) &= E_{t,x,z}[\int_t^T -\frac{1}{2}\sigma^2\hat{\pi}^2(s)X^2(s)(l^{Z_T})''(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s))) \\
&\quad \times (a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s)))^2 ds + v(X(T))], \\
&= E_{t,x,z}[\int_t^T -\frac{1}{2}\sigma^2\hat{\pi}^2(s)(\gamma(Z_T)-1)ds + v(X(T))], \\
&= E_{t,x,z}[\int_t^T -\frac{1}{2}\sigma^2\hat{\pi}^2(s)(E_{s,Z(s)}[\gamma(Z_T)]-1)ds + v(X(T))].
\end{aligned}$$

Thus, the equilibrium Hamilton-Jacobi-Bellman equation is

$$0 = \inf_{\pi,b}\Big\{ -(\mathcal{A}^{\pi,\mathbf{b}}V)(t,x,z) + \frac{1}{2}\hat{\pi}^2\sigma^2(E_{t,z}[\gamma(Z_T)]-1)\Big\}.$$

A qualified guess for the value function candidate is

$$V(t,x,z) = \log x + h(t,z),$$

with $h(T,z)=0$. This structure originates from the transformation by the certainty equivalents and the logarithmic function as

$$E_{t,x,z}[v((u^{-1})(a^{\pi,\mathbf{b},Z_T}(t,x))] = E_{t,x,z}[\log x + \log(k^{\pi,\mathbf{b},Z_T}(t)^{\frac{\gamma(Z_T)}{1-\gamma(Z_T)}})].$$

The partial derivatives are

$$V_t(t,x,z) = h_t(t,z), \quad V_x(t,x,z) = \frac{1}{x}, \quad V_{xx}(t,x,z) = \frac{-1}{x^2}.$$

Importantly h is independent of x, yielding the following HJB equation

$$\begin{aligned} h_t(t,z) = &\inf_{\pi,b}\Big\{ -((r+\pi(\alpha-r))x - \sum_{k:k\neq z}\mu^{zk*}(t)b^{zk})\frac{1}{x} - \frac{1}{2}\sigma^2\pi^2x^2\frac{-1}{x^2} \\ &- \sum_{k:k\neq z}\mu^{zk}(t)\Big(\log(x+b^{zk}) + h(t,k) - \log(x) - h(t,z)\Big) \\ &+ \frac{1}{2}\sigma^2\hat{\pi}^2(E_{t,z}[\gamma(Z_T)]-1)\Big\}. \end{aligned}$$

We then obtain candidate equilibrium strategies as

$$\begin{aligned} \pi^*(t,z) &= \frac{(\alpha-r)}{\sigma^2 E_{t,z}[\gamma(Z_T)]}, \\ b^{*zk}(t,x) &= \left(\frac{\mu^{zk}(t)}{\mu^{*zk}(t)} - 1\right)x, \qquad \text{for all } (z,k)\in\mathcal{J}. \end{aligned}$$

These are explicit, agreeable, and valuable optimal insurance and portfolio optimization control strategies comparable to the results from Desmettre and Steffensen, 2023 in their exponential case. The insurance lump sum is proportional to the wealth, and if the insurance is completely fair, the optimal choice is to annuitize and give up the rights to your wealth upon transition.

Inserting the optimal controls, we obtain the solution to the HJB as the solution to the differential equation

$$\begin{aligned} h_t(t,z) &= -r - \frac{1}{2}\left(\frac{\alpha-r}{\sigma}\right)^2\frac{1}{\mathbb{E}t,z[\gamma(Z_T)]} - \sum_{k:k\neq z}\mu^{zk*}(t) \\ &\quad + \sum_{k:k\neq z}\mu^{zk}(t)\left(\left(1-\log\frac{\mu^{zk}(t)}{\mu^{zk*}(t)}\right) + h(t,z) - h(t,k)\right), \\ h(T,z) &= 0. \end{aligned}$$

This equation is structured similarly to a Thiele equation, incorporating key components. It includes the effect of the risk-free rate, the adjusted impact of the market price of risk modulated by the stochastic relative risk aversion, and the pricing intensity adjustment. Additionally, it accounts for the effects of transition between states, notably where only the logarithmic preferences are present. These elements collectively ensure that the value function reflects the balancing of the risk and return in a stochastic environment.

7.5.3 The indifferent case

Now, if we instead use $v(x) = x$, with the same $a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,x)$, the insured is thereby indifferent regarding the uncertainty related to change of state. We have

$$\begin{aligned} \left(l^{Z_T}\right)'(y) &= \big((1-\gamma(Z_T))y\big)^{\frac{\gamma(Z_T)}{1-\gamma(Z_T)}}, \\ \left(l^{Z_T}\right)''(y) &= \gamma(Z_T)\big((1-\gamma(Z_T))y\big)^{\frac{2\gamma(Z_T)-1}{1-\gamma(Z_T)}}. \end{aligned}$$

Then,

$$
\begin{aligned}
&-\int_t^T \frac{1}{2}\sigma^2\hat{\pi}^2(s)\left(l^{Z_T}\right)''\left(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s))\right)\left(a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s))\right)^2 ds \\
&= -\int_t^T \frac{1}{2}\sigma^2\hat{\pi}^2(s)X(s)\gamma(Z_T)k^{Z_T}(s)^{\frac{\gamma(Z_T)}{1-\gamma(Z_T)}}ds.
\end{aligned}
$$

Then, by the tower property

$$
\begin{aligned}
f(t,t,x,x,z) &= E_{t,x,z}\Big[\int_t^T -\frac{1}{2}\sigma^2\hat{\pi}^2(s)X^2(s)(l^{Z_T})''(a^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s))) \\
&\qquad \times (a_x^{\hat{\pi},\hat{\mathbf{b}},Z_T}(s,X(s)))^2 ds + v(X(T))\Big], \\
&= E_{t,x,z}\Big[\int_t^T -\frac{1}{2}\sigma^2\hat{\pi}^2(s)X(s)\gamma(Z_T)k^{Z_T}(s)^{\frac{\gamma(Z_T)}{1-\gamma(Z_T)}}ds + v(X(T))\Big], \\
&= E_{t,x,z}\Big[\int_t^T -\frac{1}{2}\sigma^2\hat{\pi}^2(s)X(s)E_{s,Z(s)}\big[\gamma(Z_T)k^{Z_T}(s)^{\frac{\gamma(Z_T)}{1-\gamma(Z_T)}}\big]ds \\
&\qquad + v(X(T))\Big],
\end{aligned}
$$

Thus, the equilibrium Hamilton-Jacobi-Bellman equation can be written as

$$
0 = \inf_{\pi,b}\Big\{-(\mathcal{A}^{\pi,\mathbf{b}}V)(t,x,z) + \frac{1}{2}\sigma^2\hat{\pi}^2 x E_{t,z}[\gamma(Z_T)k^{\hat{\pi},\hat{\mathbf{b}},Z_T}(t,z)]\Big\}.
$$

Following the structure, we have the value function candidate

$$
V(t,x,z) = xd(t,z),
$$

with $d(T,z) = 1$ and partial derivatives

$$
V_t(t,x,z) = xd_t(t,z), \quad V_x(t,x,z) = d(t,z), \quad V_{xx}(t,x,z) = 0.
$$

Importantly d is independent of x, yielding the following HJB equation

$$
\begin{aligned}
d_t(t,z)x = \inf_{\pi,b}\Big\{ &-((r+\pi(\alpha-r))x - \sum_{k:k\neq z}\mu^{zk*}(t)b^{zk})d(t,z) \\
&- \sum_{k:k\neq z}\mu^{zk}(t)\Big((x+b^{zk})d(t,k) - xd(t,z)\Big) \\
&+ \frac{1}{2}\sigma^2\hat{\pi}^2 x E_{t,z}[\gamma(Z_T)k^{Z_T}(t)^{\frac{\gamma(Z_T)}{1-\gamma(Z_T)}}]\Big\}.
\end{aligned}
$$

We then obtain candidate equilibrium strategies as

$$
\begin{aligned}
\pi^*(t,z) &= \frac{(\alpha-r)}{\sigma^2}\frac{d(t,z)}{E_{t,z}[\gamma(Z_T)k^{Z_T}(t)^{\frac{\gamma(Z_T)}{1-\gamma(Z_T)}}]}, \\
b^{*zk}(t,z) &= \arg\inf_{b^{zk}}\Big\{b^{zk}\Big(\mu^{*zk}(t)d(t,z) - \mu^{zk}(t)d(t,k)\Big)\Big\}, \qquad \text{for all } (z,k)\in\mathcal{J}.
\end{aligned}
$$

There is no explicit solution to the optimal insurance coverage. Still, the optimal control b can be interpreted as how much the insured values the future in state k compared to the current state z while accounting for the coverage price by the premium loading. If the transition is to a state where all terminates, e.g., death, wealth is lost. It would be optimal for the insured to have no wealth left when death, and the optimal control would be thereafter. Thus we have

$$\begin{aligned} d_t(t,z) = &- \left(r + \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 \frac{d(t,z)}{E_{t,z}[\gamma(Z_T)k^{Z_T}(t)^{\frac{\gamma(Z_T)}{1-\gamma(Z_T)}}]} \right) d(t,z) \\ &- \sum_{k:k\neq z} \mu^{zk}(t)\Big(d(t,k) - d(t,z)\Big) \\ &+ \sum_{k:k\neq z} \frac{b^{zk*}(t,x)}{x} \left(\mu^{zk*}(t)d(t,z) - \mu^{zk}(t)d(t,k)\right). \end{aligned}$$

Here, the lump sum of b can be seen as compensation for the risk of transitioning between states. It ensures that the insured is adequately compensated for the change in their status, maintaining their overall utility. This leads to a strategy considering transition rates and stochastic preferences while ensuring fair and efficient insurance payouts.

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