Classical and new log log-theorems

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April 16, 2010

Motivation

A unified approach to celebrated $\log\log$-theorems on majorants of analytic functions.
Actually, we obtain stronger results by replacing original pointwise bounds with integral ones.

Main tool: a description for radial projections of harmonic measures of bounded star-shaped domains in the plane (which, in particular, "explains" where the $\log\log$-conditions come from).

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Class $\mathcal{L}^{++}$

Definition

A nonnegative measurable function M on $[a,b] \subset \mathbb{R}$ belongs to the class $\mathcal{L}^{++}[a,b]$ if

$$\int_a^b \log^+ \log^+ M(t)\,dt < \infty.$$

(Here: $h^+ = \max\{h, 0\}$, $h^- = h^+ - h$.)

log log-theorems

Liouville setting:

Theorem

(T. Carleman 1926) *If $f \in \mathcal{O}(\mathbb{C})$, $|f(re^{i\theta})| \le M(\theta)$ $\forall\theta \in [0, 2\pi]$, and $\forall r > 0$, with $M \in \mathcal{L}^{++}[0, 2\pi]$, then $f \equiv const$.*

This is non-trivial if M is not bounded, because there exist nonconstant entire functions f such that $f(re^{i\theta})$ is bounded in r for every fixed θ. Moreover: $M^{1-\epsilon} \in \mathcal{L}^{++}$ does not imply $f \equiv const$.

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log log-theorems

Phragmén–Lindelöf setting:

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(F. Wolf 1939) *If* $f \in \mathcal{O}(\mathbb{C}_+)$ *in* $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, $\limsup_{z\to\mathbb{R}} |f(z)| \le 1$, *and*

$$|f(re^{i\theta})| \le [M(\theta)]^{\epsilon r} \quad \forall \epsilon > 0,\ \forall r > R(\epsilon),\ \forall \theta \in (0,\pi),$$

with $M \in \mathcal{L}^{++}[0,\pi]$, *then* $|f(z)| \le 1$ *on* $\mathbb{C}_+$.

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(A.W. McMillan 1944) *If* $f \in \mathcal{O}(\mathbb{C}_+)$, $\limsup_{z\to\mathbb{R}} |f(z)| \le 1$, *and*

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log log-theorems

Local setting:

Theorem

(N. Levinson 1939, N. Sjöberg 1939, F. Wolf 1942) *If $f \in \mathcal{O}(Q)$ in $Q = \{|x| < 1,\ |y| < 1\}$, has the bound $|f(x+iy)| \le M(y)\ \forall x+iy \in Q$, with $M \in \mathcal{L}^{++}[-1,1]$, then $\forall K \Subset Q$ there is a constant C_K, independent of the function f, such that $|f(z)| \le C_K$ in K.*

(Levinson and Sjöberg: M is even and non-increasing for $y > 0$, $M(0) = \infty$.)

Further developments of this theorem, including sharpness results and higher dimensional variants: Domar (1958, 1988), Gurarii (1960), Dyn'kin (1972), Beurling (1972), Rippon (1978).

Sharpness: $M \in \mathcal{L}^{++}$ is necessary, provided M is decreasing and continuous for $y > 0$ (Beurling); decreasing and satisfying $M(y) \ge [M(2y)]^C$ on $(0, 1/2)$ (Rippon).

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log log-theorems

A similar feature of majorants from the class $\mathcal{L}^{++}$ was discovered by Beurling (1971) in a *problem of extension of analytic functions*.

Let $Q_\pm = Q \cap \mathbb{C}_\pm$, and let $f \in \mathcal{O}(Q_\pm)$ have equal boundary values on $Q \cap \mathbb{R}$ in the sense of distributions from Q_+ and Q_-.
If and $|f(x+iy)| \le M(|y|)$ with $M \in \mathcal{L}^{++}[0,1]$, then $f \in \mathcal{O}(Q)$.

log log-theorems

It also appears in relation to holomorphic functions from the *MacLane class*: MacLane (1963, 1978), Hornblower (1971), Rippon (1978).

The class consists of functions in $\mathbb{D}$ with asymptotical boundary values on dense subsets of $\mathbb{T}$.
If $f \in \mathcal{O}(\mathbb{D})$ satisfies $f(re^{i\theta}) \le M(\theta)$ with $M \in \mathcal{L}^{++}[-\pi, \pi]$, then f belongs to the MacLane class.

log log-theorems

Next result does not look like a log log-theorem, however (as will be seen from what follows) it is also about the class $\mathcal{L}^{++}$.

Theorem

(V.I. Matsaev 1960) *If an entire function f satisfies the relation*

$$\log|f(re^{i\theta})| \geq -Cr^{\alpha}|\sin\theta|^{-k} \quad \forall\theta\in(0,\pi),\ \forall r>0,$$

with some $C>0$, $\alpha>1$, and $k\geq 0$, then it has at most normal type with respect to the order α, that is, $\log|f(re^{i\theta})| \leq Ar^{\alpha}+B$.

All these theorems can be formulated in terms of subharmonic functions (by taking $u(z) = \log|f(z)|$ as a pattern), however our main goal is to replace the *pointwise* bounds with some *integral* conditions.

A model situation is the following form of the Phragmén–Lindelöf theorem.

Theorem

(Ahlfors 1937) *If* $u \in \mathrm{SH}(\mathbb{C}_+)$ *with nonpositive boundary values on* $\mathbb{R}$ *satisfies*

$$\lim_{r\to\infty} r^{-1} \int_0^{\pi} u^+(re^{i\theta}) \sin\theta \, d\theta = 0,$$

then $u \le 0$ *in* $\mathbb{C}_+$.

Will show: all these theorems are particular cases of results on a class $\mathcal{A}$ defined below, and the log log-conditions appear as conditions for continuity of certain logarithmic potentials.

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Class $\mathcal{A}$

Definition

Let ν be a probability measure on $[a, b]$. Suppose $\nu(t) := \nu([a, t])$ is strictly increasing and continuous, and μ is its inverse (extended as $\mu(t) = a$ for $t < 0$ and $\mu(t) = b$ for $t > 1$).
We will say that $\nu \in \mathcal{A}[a, b]$ if

$$\lim_{\delta \to 0} \sup_{x} \int_0^{\delta} \frac{\mu(x+t) - \mu(x-t)}{t}\, dt = 0.$$

Relation of the class $\mathcal{A}$ to the $\log\log$-theorems

Definition

$\mathcal{L}^-[a,b]$ is the class of all nonnegative integrable functions g on $[a,b]$, such that

$$\int_a^b \log^- g(s)\,ds < \infty. \tag{1}$$

Proposition

If the density ν' of an absolutely continuous increasing function ν belongs to $\mathcal{L}^-[a,b]$, then $\nu \in \mathcal{A}[a,b]$.
Consequently, if a holomorphic function f has a majorant $M \in \mathcal{L}^{++}[a,b]$, then $\log^+|f|$ has the corresponding integral bound with

$$\nu(t) = \int_a^t \min\{1, 1/M(s)\}\,ds \in \mathcal{A}[a,b].$$

Statements

Entire functions (Carleman+):

Theorem

Let $u \in \mathrm{SH}(\mathbb{C})$ *satisfy*

$$\int_0^{2\pi} u^+(te^{i\theta})\, d\nu(\theta) \le V(t)$$

with $\nu \in \mathcal{A}[0, 2\pi]$ *and a nondecreasing function* V *on* $\mathbb{R}_+$*. Then there exist constants* $c > 0$ *and* $A \ge 1$*, independent of* u*, such that*

$$u(te^{i\theta}) \le c\, V(At).$$

Statements

Phragmén-Lindelöf (Wolf+):

Theorem

If $u \in \mathrm{SH}(\mathbb{C}_+)$ *satisfies* $\limsup_{z\to\mathbb{R}} u(z) \le 0$ *and*

$$\lim_{t\to\infty} t^{-1} \int_0^{\pi} u^+(te^{i\theta})\, d\nu(\theta) = 0$$

with $\nu \in \mathcal{A}[0,\pi]$, *then* $u(z) \le 0$ $\forall z \in \mathbb{C}_+$.

(McMillan+)

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Statements

Local setting (Levinson-Sjöberg+):

Theorem

Let $u \in \mathrm{SH}(Q)$ *in* $Q = \{x + iy : |x| < 1, \ |y| < 1\}$ *satisfy*

$$\int_{-1}^{1} u^+(x + iy)\, d\nu(y) \le 1 \quad \forall x \in (-1, 1)$$

with $\nu \in \mathcal{A}[-1, 1]$*. Then for each compact set* $K \subset Q$ *there is a constant* C_K*, independent of the function* u*, such that* $u(z) \le C_K$ *on* K*.*

Statements

Functions with a lower bound (Matsaev+)

Theorem

Let a function $u \in \mathrm{SH}(\mathbb{C})$*, harmonic in* $\mathbb{C} \setminus \mathbb{R}$*, satisfy*

$$\int_{-\pi}^{\pi} u^{-}(re^{i\theta})\Phi(|\sin\theta|)\, d\theta \le V(r),$$

where $\Phi \in \mathcal{L}^{-}[0,1]$ *is nonnegative and nondecreasing, and the function* V *is such that* $r^{-1-\delta}V(r)$ *is increasing for some* $\delta > 0$*. Then there are constants* $c > 0$ *and* $A \ge 1$*, independent of* u*, such that*

$$u(re^{i\theta}) \le cV(Ar).$$

Remark. We do not know if the condition on u^{-} can be replaced by a more general one in terms of the class $\mathcal{A}$.

Radial projections of harmonic measures

Proofs of the theorems rest on a presentation of measures of the class $\mathcal{A}[0, 2\pi]$ as radial projections of harmonic measures of star-shaped domains.

Let Ω be a bounded Jordan domain containing the origin.

$\omega(z, E, \Omega) \equiv$ harmonic measure of $E \subset \partial\Omega$ at $z \in \Omega$: the solution of the Dirichlet problem in Ω with the boundary data 1 on E and 0 on $\partial\Omega \setminus E$

$\omega(0, E, \Omega)$ generates a measure on the unit circle $\mathbb{T}$ by means of the radial projection $\zeta \mapsto \zeta/|\zeta|$, which we consider as a measure on $[0, 2\pi]$:

$$\widehat{\omega}_\Omega(F) = \omega(0, \{\zeta \in \partial\Omega : \arg\zeta \in F\}, \Omega), \quad F \subset [0, 2\pi].$$

The inverse problem: *Given a probability measure on the unit circle $\mathbb{T}$, is it the radial projection of the harmonic measure of any domain Ω?*

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Radial projections of harmonic measures (cont'd)

We specify Ω to be *strictly star-shaped*:

$$\Omega = \{re^{i\theta} : \ r < r_\Omega(\theta), \ 0 \le \theta \le 2\pi\}$$

with r_Ω a positive continuous function on $[0, 2\pi]$, $r_\Omega(0) = r_\Omega(2\pi)$.

Theorem

Radial projection theorem: *A continuous probability measure ν on $[0, 2\pi]$ is the radial projection of the harmonic measure of a strictly star-shaped domain if and only if $\nu \in \mathcal{A}[0, 2\pi]$.*

The theorem was proved (1990) by a method originated by B.Ya. Levin in theory of majorants in classes of subharmonic functions.

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Proof of Radial Projection Theorem

Step 1: *continuity of a potential*

Proposition

Let ν be a strictly increasing continuous function $[0, 2\pi] \to [0, 1]$, and $\mu : [0, 1] \to [0, 2\pi]$ be its inverse. Then the function

$$h(z) = \int_0^{2\pi} \log |e^{i\theta} - z| \, d\mu(\theta/2\pi)$$

is continuous if and only if $\nu \in \mathcal{A}[0, 2\pi]$.

Proof: Basically, integration by parts and Evans' theorem (continuity on the support of the measure implies continuity everywhere). □

Remark. Recall that $\nu' \in \mathcal{L}^-$ implies $\nu \in \mathcal{L}^{++}$. On the other hand, $\nu' \in \mathcal{L}^-$ iff μ' belongs to the Zygmund class $\mathbf{L} \log \mathbf{L}$ appearing in continuity problems for the Hilbert transform.

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Proof of Radial Projection Theorem (cont'd)

Step 2: *Sufficiency*

Proposition

Every $\nu \in \mathcal{A}[0, 2\pi]$ has the form $\nu = \widehat{\omega}_\Omega$ for some strictly star-shaped domain Ω.

Proof: By Step 1, the function

$$u(z) = \frac{1}{\pi}\int_0^{2\pi} \log|e^{i\theta} - z|\, d\mu(\theta/2\pi) \in SH(\mathbb{C}) \cap C(\mathbb{C}).$$

Let v be harmonic conjugate to u in $\mathbb{D}$, $v(1) = 0$.
By the C-R condition, $v \in C(\overline{\mathbb{D}})$ and $v(e^{i\theta}) = \theta - \mu(\theta/2\pi)$. Therefore, the function $w(z) := z\exp\{-u(z) - iv(z)\} \in C(\overline{\mathbb{D}})$, $\arg w(e^{i\theta}) = \mu(\theta/2\pi)$.
By the boundary correspondence principle, w maps $\mathbb{D}$ conformally to

$$\Omega = w(\mathbb{D}) = \{re^{i\theta} : \ r < \exp\{-u(\exp\{2\pi i\nu(\theta)\})\},\ 0 \le \theta \le 2\pi\},$$

so that $\omega(0, E, \Omega) = \nu(\arg E)$ for any Borel $E \subset \partial\Omega$.

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Every $\nu \in \mathcal{A}[0, 2\pi]$ has the form $\nu = \widehat{\omega}_\Omega$ for some strictly star-shaped domain Ω.

Proof: By Step 1, the function

$$u(z) = \frac{1}{\pi} \int_0^{2\pi} \log |e^{i\theta} - z| \, d\mu(\theta/2\pi) \in \mathrm{SH}(\mathbb{C}) \cap C(\mathbb{C}).$$

Let v be harmonic conjugate to u in $\mathbb{D}$, $v(1) = 0$.
By the C-R condition, $v \in C(\overline{\mathbb{D}})$ and $v(e^{i\theta}) = \theta - \mu(\theta/2\pi)$. Therefore, the function $w(z) := z \exp\{-u(z) - iv(z)\} \in C(\overline{\mathbb{D}})$, $\arg w(e^{i\theta}) = \mu(\theta/2\pi)$.
By the boundary correspondence principle, w maps $\mathbb{D}$ conformally to

$$\Omega = w(\mathbb{D}) = \{re^{i\theta} : \ r < \exp\{-u(\exp\{2\pi i \nu(\theta)\})\}, \ 0 \le \theta \le 2\pi\},$$

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Proof of Radial Projection Theorem (cont'd)

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Proof of Radial Projection Theorem (cont'd)

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Proposition

If $\Omega = \{re^{i\theta} : r < r_\Omega(\theta)\}$, then $\widehat{\omega}_\Omega \in \mathcal{A}[0, 2\pi]$.

Proof: Let w be the conformal map of $\mathbb{D}$ to Ω, $w(0) = 0$, $\arg w(1) = 0$.

Define $f(z) = u(z) + iv(z) = \log \frac{w(z)}{z}$ for $|z| \leq 1$ and $f(z) = f(|z|^{-2} z)$ for $|z| > 1$. It is analytic in $\mathbb{D}$ and continuous in $\mathbb{C}$.

The function $\lambda(z) = u(z) + \frac{1}{\pi} \int_0^{2\pi} \log |e^{i\psi} - z| \, dv(e^{i\psi})$ can be shown to be harmonic ($\Rightarrow$ continuous) in $\mathbb{C}$. Therefore, the potential is continuous.

By invariance of harmonic measure, $\widehat{\omega}_\Omega([0, \arg w(e^{i\psi})]) = \psi/2\pi$, so $v(e^{i\psi}) = \arg w(e^{i\psi}) - \psi = \mu(\psi/2\pi) - \psi$, where $\mu(\psi/2\pi)$ is the inverse to the function $\widehat{\omega}_\Omega([0, \arg w(e^{i\psi})])$.

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Proof of the Integral Variant for Carleman Theorem

Theorem

Let $u \in \mathrm{SH}(\mathbb{C})$ satisfy $\int_0^{2\pi} u^+(te^{i\theta})\,d\nu(\theta) \le V(t)$ with $\nu \in \mathcal{A}[0,2\pi]$ and a nondecreasing function V on $\mathbb{R}_+$. Then there exist constants $c > 0$ and $A \ge 1$, independent of u, such that $u(te^{i\theta}) \le c\,V(At)$.

Proof: By the Radial Projection Theorem, there exists a strictly star-shaped domain Ω such that $\omega(z,E,\Omega) \le c\,\nu(\arg E)$ for all $z \in K \Subset \Omega$, $E \subset \partial\Omega$.

The Poisson-Jensen formula for the function $u^+(t\,z)$ in the domain $s\,\Omega$, $t \ge 1$, $s \ge 1$, implies

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For a transition from $\partial(st\Omega)$ to $At\mathbb{T}$, integrate this w.r.t. s. □

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Proof of the Integral Variant for Wolf Theorem

Theorem

If $u \in \mathrm{SH}(\mathbb{C}_+)$ *satisfies the conditions* $\limsup_{z\to x_0} u(z) \le 0 \quad \forall x_0 \in \mathbb{R}$ *and*

$$\lim_{t\to\infty} t^{-1} \int_0^{\pi} u^+(te^{i\theta})\, d\nu(\theta) = 0$$

with $\nu \in \mathcal{A}[0,\pi]$, *then* $u(z) \le 0\ \forall z \in \mathbb{C}_+$.

Proof: this follows from the previous Theorem applied to the function u_+ extended to $\mathbb{C}$ by 0, and standard Phragmén–Lindelöf theorem. □

McMillan+: similar proof

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Proof of Integral Variant for Levinson-Sjöberg Theorem

Theorem

Let $u \in \mathrm{SH}(Q)$ in $Q = \{|x| < 1,\ |y| < 1\}$ satisfy $\int_{-1}^{1} u^+(x+iy)\, d\nu(y) \le 1$ $\forall x \in (-1,1)$ with $\nu \in \mathcal{A}[-1,1]$. Then for each compact set $K \subset Q$ there is a constant C_K, independent of u, such that $u(z) \le C_K$ on K.

Proof: Same idea as for the theorem on entire functions (Carleman+), refined adaptation.

By using the Radial Projection Theorem, construct a domain $\Omega_0 = \{x + iy : t_1(y) < x < t_2(y),\ -1 < x < 1\} \subset Q$ such that the harmonic measure of any subset E of the curvilinear part of $\partial\Omega_0$ at a given point $z_0 \in \Omega_0$ equals $\nu(\mathrm{Im}\, E)$.

In order to replace the integration over $\partial\Omega_0$ by the integration over vertical intervals, a partition needed. □

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Proof of Integral Variant for Matsaev Theorem

Theorem

Let a function $u \in \mathrm{SH}(\mathbb{C})$*, harmonic in* $\mathbb{C} \setminus \mathbb{R}$*, satisfy* $\int_{-\pi}^{\pi} u^-(re^{i\theta})\Phi(|\sin\theta|)\,d\theta \le V(r)$*, where* $\Phi \in \mathcal{L}^-[0,1]$ *is nonnegative and nondecreasing, and the function* V *is such that* $r^{-1-\delta}V(r)$ *is increasing for some* $\delta > 0$*. Then there are constants* $c > 0$ *and* $A \ge 1$*, independent of* u*, such that* $u(re^{i\theta}) \le cV(Ar)$*.*

Proof: Using Carleman's formula for the function u in the domains $\{r < |z| < R,\ |\pm\arg z - \frac{\pi}{2}| < \frac{\pi}{2} - a\}$, multiplied by $\Phi(|\sin\theta|)$ and integrated in $a \in (0,\tau)$ for a sufficiently small $\tau > 0$, one can show there is a constant $C > 0$ such that

$$\int_{-\pi}^{\pi} u^+(re^{i\theta})\sin^2\theta\,\Phi(|\sin\theta|)\,d\theta \le o(V(r)) + C\int_{-\pi}^{\pi} u^-(re^{i\theta})\Phi(|\sin\theta|)\,d\theta,$$

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Proof: Using Carleman's formula for the function u in the domains $\{r < |z| < R,\ |\pm\arg z - \frac{\pi}{2}| < \frac{\pi}{2} - a\}$, multiplied by $\Phi(|\sin\theta|)$ and integrated in $a \in (0,\tau)$ for a sufficiently small $\tau > 0$, one can show there is a constant $C > 0$ such that

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Questions

1. Does an integral variant of Matsaev's theorem hold for measures $\nu \in \mathcal{A}$?

2. Is $\mathcal{A}$ the largest class of measures for the results to hold?

3. A description of the radial projections for harmonic measures of general star-shaped domains (including the case of non-bounded domains)?

4. Radial projections for arbitrary domains?

5. Higher dimensional analogues for the Radial Projection Theorem?

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