A Landing Theorem of Periodic Rays for a Class of ETF

A.Deniz
C.L.Petersen

Roskilde University, University of Barcelona

Landing Theorem in Polynomial Dynamics:

Let $P(z)$ be a polynomial with degree $d \geq 2$. Then there exists a neighborhood U of ∞ and $R > 1$, and a unique conformal isomorphism Φ tangent to the identity at ∞ that conjugates P restricted to U to $z \rightarrow z^d$ restricted to $\mathbb{C} \setminus \overline{\mathbb{D}}_R$.

Landing Theorem in Polynomial Dynamics:

Let $P(z)$ be a polynomial with degree $d \geq 2$. Then there exists a neighborhood U of ∞ and $R > 1$, and a unique conformal isomorphism Φ tangent to the identity at ∞ that conjugates P restricted to U to $z \longrightarrow z^d$ restricted to $\mathbb{C} \backslash \overline{\mathbb{D}}_R$.

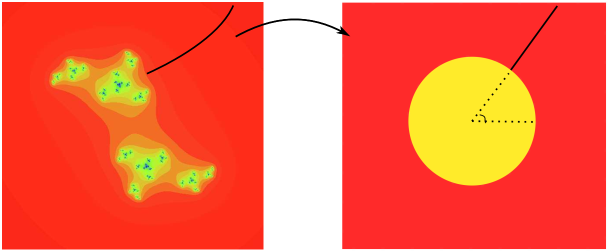
A ray of external argument θ is defined by

$$R_P(\theta) = \Phi^{-1}(\{re^{2i\pi\theta}, r > R > 1\})$$

Landing Theorem in Polynomial Dynamics:

Let $P(z)$ be a polynomial with degree $d \geq 2$. Then there exists a neighborhood U of ∞ and $R > 1$, and a unique conformal isomorphism Φ tangent to the identity at ∞ that conjugates P restricted to U to $z \to z^d$ restricted to $\mathbb{C} \backslash \overline{\mathbb{D}}_R$.
A ray of external argument θ is defined by

$$R_P(\theta) = \Phi^{-1}(\{re^{2i\pi\theta}, r > R > 1\})$$



Landing Theorem in Polynomial Dynamics:

If $K(P)$ is connected, then Φ extends to a conformal isomorphism:

$$\Phi : \mathbb{C} \backslash K(P) \to \mathbb{C} \backslash \overline{\mathbb{D}}.$$

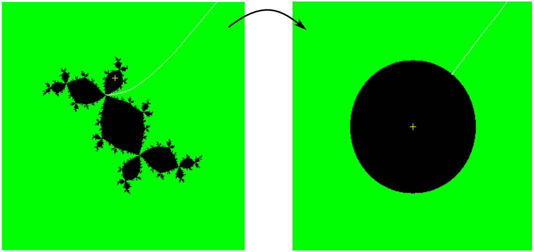
and $R_P(\theta) = \Phi^{-1}(\{re^{2i\pi\theta}, r > 1\})$.

Landing Theorem in Polynomial Dynamics:

If $K(P)$ is connected, then Φ extends to a conformal isomorphism:

$$\Phi : \mathbb{C} \backslash K(P) \to \mathbb{C} \backslash \overline{\mathbb{D}}.$$

and $R_P(\theta) = \Phi^{-1}(\{re^{2i\pi\theta}, r > 1\})$.

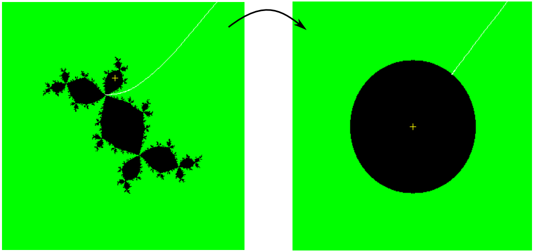


Landing Theorem in Polynomial Dynamics:

If $K(P)$ is connected, then Φ extends to a conformal isomorphism:

$$\Phi : \mathbb{C} \backslash K(P) \to \mathbb{C} \backslash \overline{\mathbb{D}}.$$

and $R_P(\theta) = \Phi^{-1}(\{re^{2i\pi\theta}, r > 1\})$.



Theorem (Sullivan-Douady-Hubbard)

If $K(P)$ is connected, then every periodic external ray lands at a periodic point, which is either repelling or parabolic.

What is a dynamic ray tail?

Injective, continuous curve $g_{\underline{s}} : (t, \infty) \to \mathbb{C}$ *in escaping set.*

What is a dynamic ray tail?

Injective, continuous curve $g_{\underline{s}} : (t, \infty) \to \mathbb{C}$ *in escaping set.*

- **Devaney,Krych (1984):** escaping set $I(f)$ consists of curves to ∞ for $z \to \lambda e^z$, $\lambda \in (0, 1/e)$.
- **Devaney,Goldberg,Hubbard (1986):** existence of certain curves to ∞ in $I(f)$ for arbitrary exponential maps.
- **Devaney,Tangerman (1985):** generalize this result to a subclass of B.
- **Schleicher,Zimmer (2003):** escaping points can be connected to ∞ by a curve consisting of escaping points for any exponential map.
- **Baranski (2007):** for a subclass of ETF, every component of the Julia set is a curve tending to ∞.
- **Rottenfusser,Ruckert,Rempe,Schleicher (2009):** for $f \in B$ of finite order or a finite composition of such maps, every point in $I(f)$ can be connected to ∞ by a curve γ, such that $f^n|_\gamma \to \infty$ uniformly.

What is ray landing?

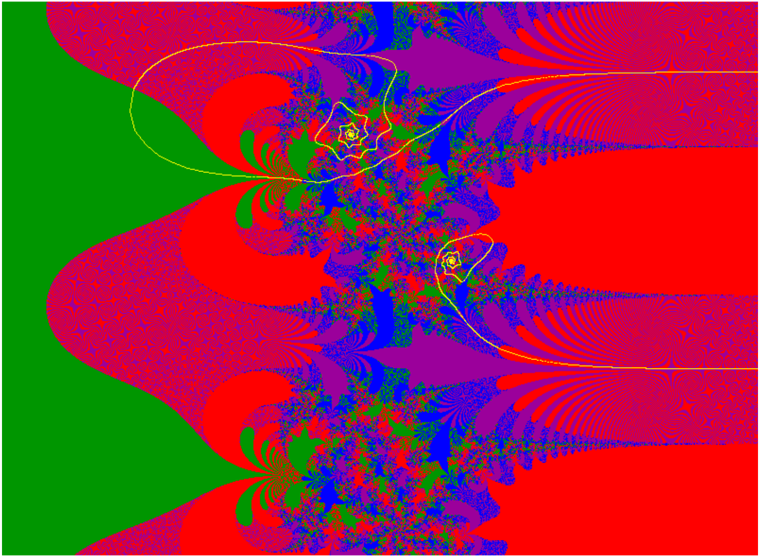
$g_{\underline{s}}$ lands if $\lim_{t \to 0} g_{\underline{s}}(t)$ exists.

What is ray landing?

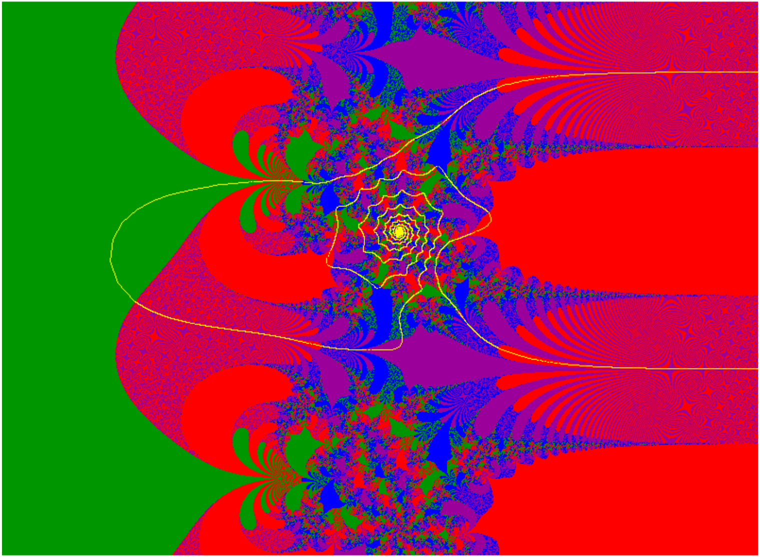
$g_{\underline{s}}$ *lands if* $\lim_{t\to 0} g_{\underline{s}}(t)$ *exists.*

- **Rempe (2002):** If a periodic dynamic ray does not intersect the post singular set of an exponential map, it lands.
- **Schleicher,Zimmer (2003):** If the singular orbit bounded for an exponential map, every periodic dynamic rays land at a periodic point, if it escapes, then periodic ray lands unless $g_{\sigma^n(\underline{s})}(t) = 0$.
- **Rempe (2005):** If singular value does not escape to ∞, then all periodic dynamic rays land for exponentials.
- **Rempe (2008):** Let $f \in B$, and $\gamma : (-\infty, 1] \to I(f)$ with $f(\gamma(t)) = \gamma(t+1)$. Then γ lands at a repelling or parabolic fixed point of f if and only if there exists some domain U such that $U \subset f(U)$, $f : U \to f(U)$ is a covering map and $\gamma(-\infty, T] \subset U$ for some $T < 0$.

Landing of rays



Landing together



Theorem

Let f be a transcendental entire map of finite order or finite composition of finite order maps, with bounded post-singular set. Then all periodic dynamic rays land and the landing points are either repelling or parabolic periodic points.

Theorem

Let f be a transcendental entire map of finite order or finite composition of finite order maps, with bounded post-singular set. Then all periodic dynamic rays land and the landing points are either repelling or parabolic periodic points.

- U_0 : unbounded connected component of $\mathbb{C} \backslash P$,
- U_1 : connected component of $f^{-k}(U_0)$, which contains periodic ray $g_{\underline{s}}$ with period k,
- $F : [0, \infty) \to [0, \infty)$: model dynamics satisfying:

$$f(g_{\underline{s}}(t)) = g_{\sigma(\underline{s})}(F(t))$$

Theorem

Let f be a transcendental entire map of finite order or finite composition of finite order maps, with bounded post-singular set. Then all periodic dynamic rays land and the landing points are either repelling or parabolic periodic points.

- U_0 : unbounded connected component of $\mathbb{C} \backslash P$,
- U_1 : connected component of $f^{-k}(U_0)$, which contains periodic ray $g_{\underline{s}}$ with period k,
- $F : [0, \infty) \to [0, \infty)$: model dynamics satisfying:

$$f(g_{\underline{s}}(t)) = g_{\sigma(\underline{s})}(F(t))$$

Lemma

Suppose there exists a sequence of points $\{t_n\}$, $t_n \in \mathbb{R}^+$ converging to 0, such that $\lim_{n \to \infty} g_{\underline{s}}(t_n) = w \in \overline{U_1}$. Then $\lim_{t \to 0} g_{\underline{s}}(t) = w$.

Is ∞ an accumulation point?

Is ∞ an accumulation point?

NO!! There exists strong contraction near ∞.

Is ∞ an accumulation point?

NO!! There exists strong contraction near ∞.

Proposition

Let U be a hyperbolic domain with an isolated boundary point z_0 and let $\{w_i\}_{i\in\mathbb{N}}$ be a sequence in U such that $w_i \to z_0$ as $i \to \infty$ and $d_U(w_i, w_{i+1}) \leq \delta$. Let $V = U \backslash \{w_i\}_{i\in\mathbb{N}}$. Given a neighborhood ω of z_0, which is simply connected and relatively compact in $U \cup \{z_0\}$, there exists $\kappa > 0$, such that

$$0 < \frac{\lambda_U(z)}{\lambda_V(z)} \leq \kappa < 1.$$

Is ∞ an accumulation point?

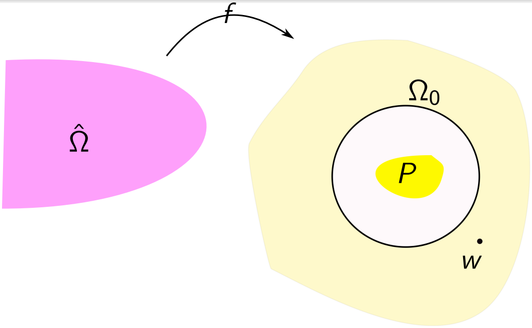
NO!! There exists strong contraction near ∞.

Proposition

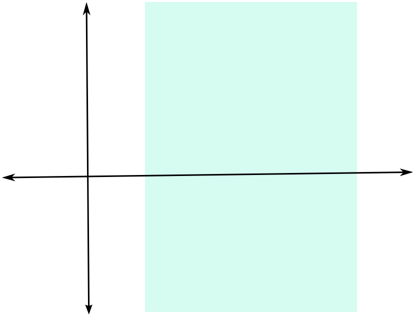
Let U be a hyperbolic domain with an isolated boundary point z_0 and let $\{w_i\}_{i\in\mathbb{N}}$ be a sequence in U such that $w_i \to z_0$ as $i \to \infty$ and $d_U(w_i, w_{i+1}) \leq \delta$. Let $V = U \backslash \{w_i\}_{i\in\mathbb{N}}$. Given a neighborhood ω of z_0, which is simply connected and relatively compact in $U \cup \{z_0\}$, there exists $\kappa > 0$, such that

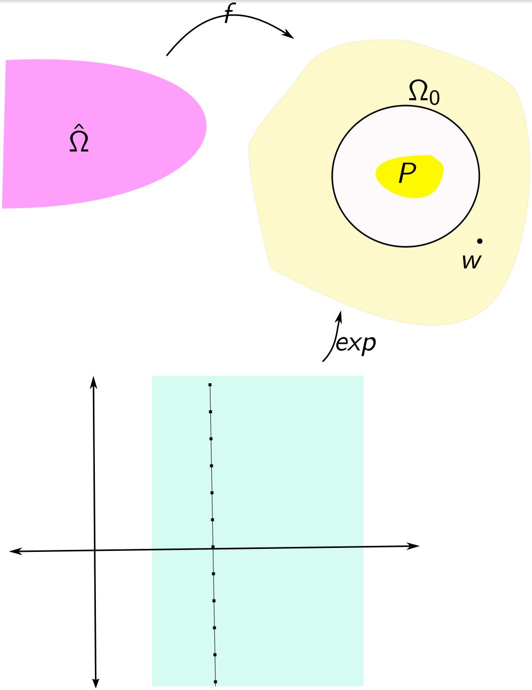
$$0 < \frac{\lambda_U(z)}{\lambda_V(z)} \leq \kappa < 1.$$

- $U_0 \backslash U_1$ contains a sequence of points $\{w_i\}$ with $d_{U_0}(w_i, w_{i+1}) < \delta$ for some $\delta > 0$ and $w_i \to \infty$ as $i \to \pm\infty$.

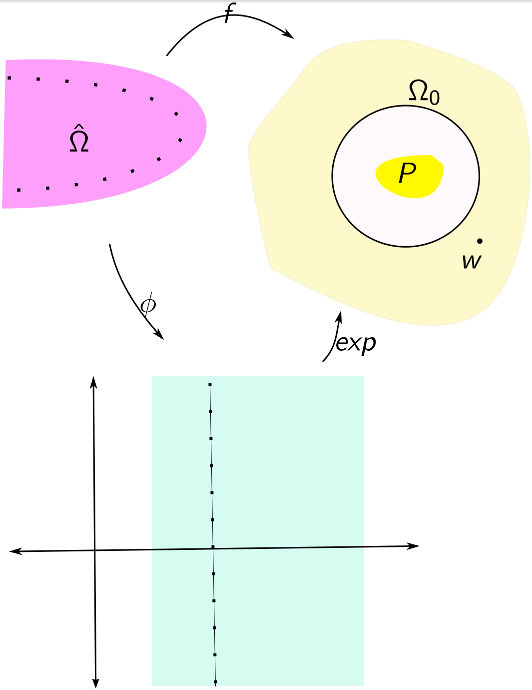


f
$\hat{\Omega}$
Ω_0
P
w

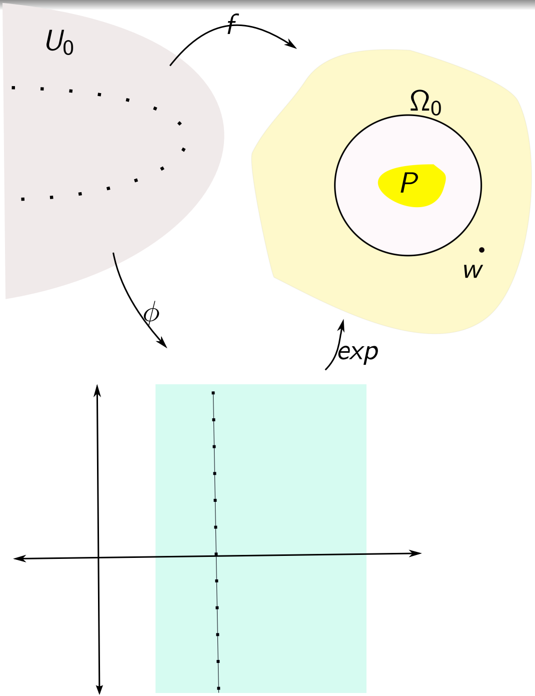




f
$\hat{\Omega}$
Ω_0
P
w
exp



f
$\hat{\Omega}$
Ω_0
P
w
ϕ
exp


f
U_0
Ω_0
P
w
ϕ
exp

Estimate an upper bound

Lemma

If $V \subset U \subsetneq \mathbb{C}$ are hyperbolic subsets, there exists continuous and increasing function $\kappa : [0, \infty[\to [0, 1[$ with $\kappa(0) = 0$, such that $\forall z \in V$,

$$0 < \frac{\lambda_U(z)}{\lambda_V(z)} \leq \kappa(d_U(z, \partial V)) < 1.$$

Estimate an upper bound

Lemma

If $V \subset U \subsetneq \mathbb{C}$ are hyperbolic subsets, there exists continuous and increasing function $\kappa : [0, \infty[\to [0, 1[$ with $\kappa(0) = 0$, such that $\forall z \in V$,

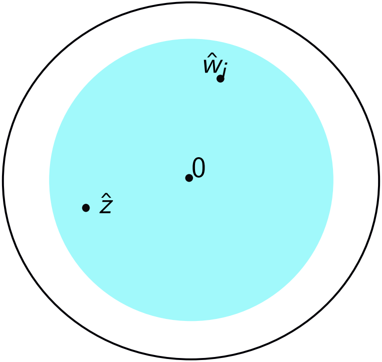
$$0 < \frac{\lambda_U(z)}{\lambda_V(z)} \leq \kappa(d_U(z, \partial V)) < 1.$$

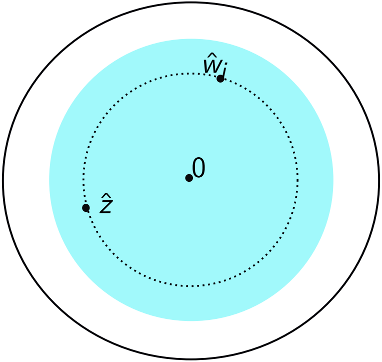
$$\kappa(d) = -\frac{e^{2d} - 1}{2e^d} \log(\frac{e^d - 1}{e^d + 1}), \quad d = d_U(z, \partial V).$$

Estimate an upper bound

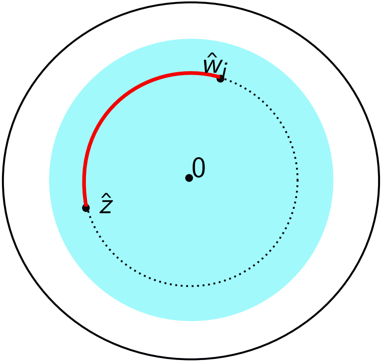
Lemma

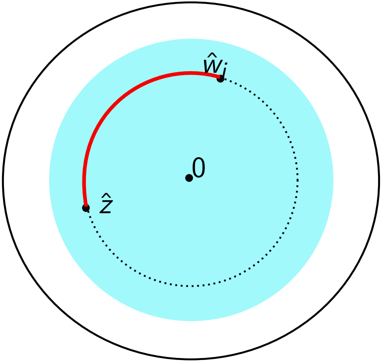
Let $U \subseteq \hat{\mathbb{C}}$ be hyperbolic, with an isolated boundary point z_0 and $\hat{U} = U \cup \{z_0\}$, be an open and connected domain. Then, there exists $\pi_* : \mathbb{D} \to \hat{U}$ holomorphic map with $\pi_*(0) = z_0$, $\pi'_*(0) \neq 0$, and $\pi_*| : \mathbb{D}^* \to U$ is a covering map with degree 1 at $z = 0$.


$\hat{w}_j$
0
$\hat{z}$

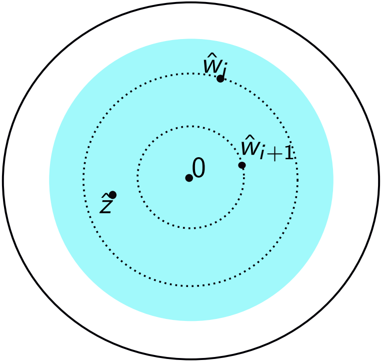


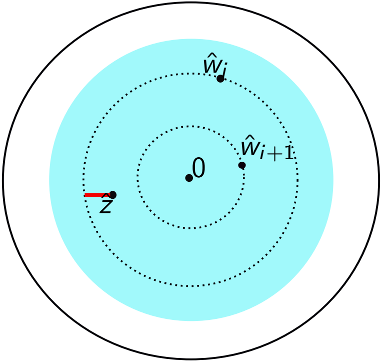
$\hat{w}_j$
0
$\hat{z}$

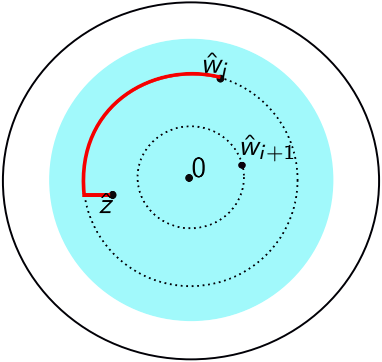

$\hat{w}_j$
0
$\hat{z}$

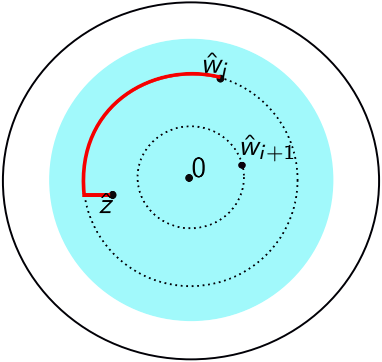


$$d_{\mathbb{D}^*}(\hat{z}, \hat{w}_i) < -\frac{\pi}{\log r}$$


$\hat{w}_i$
$\hat{w}_{i+1}$
0
$\hat{z}$


$\hat{w}_i$
$\hat{w}_{i+1}$
0
$\hat{z}$


$\hat{w}_i$
$\hat{w}_{i+1}$
0
$\hat{z}$



$$d_{\mathbb{D}^*}(\hat{z}, \hat{w}_i) < \delta - \frac{\pi}{\log r}$$

For all z in a simply connected neighborhood of ∞:

- $$\frac{\lambda_{U_0}(z)}{\lambda_{U_1}(z)} \leq \kappa(r,\delta) = -\frac{e^{-2\frac{\pi}{\log r}} - 1}{2e^{-\frac{\pi}{\log r}}} \log\left(\frac{e^{-\frac{\pi}{\log r}} - 1}{e^{-\frac{\pi}{\log r}} + 1}\right) < 1$$

For all z in a simply connected neighborhood of ∞:

- $$\frac{\lambda_{U_0}(z)}{\lambda_{U_1}(z)} \leq \kappa(r,\delta) = -\frac{e^{-2\frac{\pi}{\log r}} - 1}{2e^{-\frac{\pi}{\log r}}} \log\left(\frac{e^{-\frac{\pi}{\log r}} - 1}{e^{-\frac{\pi}{\log r}} + 1}\right) < 1$$

- $$\frac{\lambda_{U_0}(z)}{\lambda_{U_1}(z)} \leq \kappa(r,\delta) = -\frac{e^{2(\delta-\frac{\pi}{\log r})} - 1}{2e^{(\delta-\frac{\pi}{\log r})}} \log\left(\frac{e^{(\delta-\frac{\pi}{\log r})} - 1}{e^{(\delta-\frac{\pi}{\log r})} + 1}\right) < 1$$

THANK YOU FOR YOUR ATTENTION!