Diplomarbeit

On the classification of total spaces of $S^7$-bundles over $S^8$

eingereicht von Matthias Grey
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Betreuer: Prof. Dr. Klaus Mohnke und Prof. Dr. Elmar Vogt
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1 Introduction

The bundle isomorphism classes of $S^7$-bundles over $S^8$ with structure groups $SO(8)$ are in correspondence with $\pi_7(SO(8)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus by picking an isomorphism and two integers we get a bundle isomorphism class. In each class we get a somehow canonical smooth structure on the total space. Our aim is to give results on the diffeomorphism and homeomorphism classification of these manifolds. The study of bundles played an important role in differential topology. Milnor for example discovered the exotic 7-spheres studying the total spaces of $S^3$-bundles over $S^4$ [Mil56]. The quaternions play an important role in his work because the quaternion multiplication gives nice generators of $\pi_3(SO(4))$ and also a fibration with total space $S^7$. In dimension 8 there are the octonions. Tamura used them and similar methods as Milnor to study the total spaces of $S^7$-bundles over $S^8$. He also gave more or less explicit homeomorphisms using foliations [Tam57, Tam58]. There are also results on the smooth structures on the total spaces of $S^7$-bundles over $S^8$ by Shimada [Shi57].

Lately Crowley and Escher [CE03] gave a full classification of the total spaces of $S^3$-bundles over $S^4$ up to homotopy equivalence, homeomorphism and diffeomorphism. We want to generalize the results on the diffeomorphism and homeomorphism classification to $S^7$-bundles over $S^8$. We will have to take a deep look into Crowley’s PhD thesis [Cro02]. He studied the almost diffeomorphism and diffeomorphism classification of highly connected 7- and 15-manifolds, where a highly connected manifold is a path connected manifold with all homotopy groups trivial up to half the dimension. This means for a $2k$ or $(2k + 1)$-manifold to be $(k - 1)$-connected. An almost diffeomorphism is a map between smooth manifolds that becomes a diffeomorphism after the addition of a homotopy sphere. Crowley used results by Wall and Wall’s student Wilkens. Wall gave results on the classification of highly connected $2n$-manifolds with non empty highly connected boundaries [Wal62] and used them in many cases for the almost diffeomorphism classification of highly connected $(2n + 1)$-manifolds [Wal67]. Wall’s methods break down in lower dimensions like 7 and 15. These cases were studied by Wall’s student Wilkens. Wilkens gave some incomplete results on their almost diffeomorphism classification [Wil71]. The unusual notion of almost diffeomorphism occurs in some sense naturally because in the dimensions of interest highly connected manifolds are boundaries of highly connected manifolds after the addition of a homotopy sphere.

This thesis consists of three parts. We start with the construction of the smooth structure on the total spaces and will see what we can get without applying Crowley’s thesis. Then we are going to give the background to understand Crowley’s classification and apply it to our case. In the end we will construct an invariant and apply it
to the homeomorphism classification.
We assume some knowledge on algebraic topology like covered in [Hat02] and some understanding of manifolds, vector bundles and characteristic classes like in [MS74]. Some familiarity with obstruction theory will also be useful ([DK01, Chapter 7] or [Ste51, Part III]).
In this thesis we are going to assume all manifolds to be compact and oriented unless noted otherwise. Manifolds will occur in the topological, piecewise linear (abbreviated PL) and smooth category and the morphisms are assumed to be orientation preserving. If the word manifold occurs alone, it will mean a smooth manifold.

Finally, I want to thank Klaus Mohnke because he agreed to supervise this thesis, Diarmuid Crowley for sharing his big knowledge with me and especially Elmar Vogt for giving me this interesting topic and his advice even regarding non thesis specific questions.
2 Preliminaries

In this Chapter we are going to review the classification up to fiber bundle isomorphism and calculate the cohomology and Pontryagin classes of the total spaces. The octonions play an important role. They are a non associative 8-dimensional divisor algebra. The two facts that we use most are that the multiplication by one element is a linear function and hence smooth and that any subalgebra with 1 spanned by two elements is associative. For more information check [Hat02, Example 4.47, p.378] or [Ste51, 20.5, p.108].

2.1 Classification as bundles

Consider fiber bundles with fiber $S^7$, base $S^8$ and structure group $SO(8)$. For the classification up to fiber bundle isomorphism we are going to use the so called clutching construction.

Let’s think of $S^8$ as two 8 disks $D^8$ identified at their boundary. Bundles over $D^8$ are trivial since $D^8$ is contractible. Thus, all bundles are isomorphic to $D^8 \times S^7$. When we want to obtain a bundle over $S^8$, we have to tell how to identify the fibers at the boundaries of the $D^8$s. We do this by giving a map

$$S^7 \xrightarrow{f} SO(8)$$

called clutching function. The total space of the bundle over $S^8$ is then given by

$$D^8_+ \times S^7 \sqcup D^8_- \times S^7 / \sim,$$

where

$$D^8_+ \times S^7 \supset S^7 \times S^7 \ni (x, f(x)(y)) \sim (x, y) \in S^7 \times S^7 \subset D^8_- \times S^7.$$

As projection we take the obvious map. The + and – shall indicate that we consider the disks with the standard and the opposite orientation. We orient the fibers of $D^8 \times S^7$ according to the standard orientation of $S^7$. This orientation carries over to the bundle over $S^8$ since we identify via orientation preserving maps.

It turns out that the fiber bundle isomorphism type only depends on the homotopy type of the clutching function and moreover, all bundles are obtained in this way (compare [Ste51, p.99]).

**Proposition 2.1.1.** The fiber bundle isomorphism classes of $S^7$-bundles over $S^8$ are in one to one correspondence with $\pi_7(SO(8))$. 

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Thus we have to investigate the structure of $\pi_7(SO(8))$.

**Proposition 2.1.2.** $\pi_7(SO(8)) \cong \pi_7(S^7) \oplus \pi_7(SO(7))$

**Proof.** Compare [Ste51, Example 8.6, p.37]. We are going to show that the principal bundle $SO(7) \hookrightarrow SO(8) \xrightarrow{\pi} S^7$, where $\pi(f) = f((1,0,...,0))$, is isomorphic to the product bundle. We can think of $S^7 \subset \mathbb{O}$ and hence get a map

$$s : S^7 \ni x \mapsto (y \mapsto xy) \in SO(8).$$

The linear maps defined by multiplications are in $SO(8)$ because $S^7$ is path-connected and $1_\mathbb{O}$ gets mapped to $id_{\mathbb{R}^8}$. $s$ is a section since $\pi(s(x)) = \pi((y \mapsto xy)) = x(1,0,...,0) = x1_\mathbb{O} = x$. Thus, the bundle is trivial. \qed

To use proposition 2.1.2 we define the following mappings:

$$\sigma : S^7 \rightarrow SO(8) \text{ and } \rho : S^7 \rightarrow SO(7) \subset SO(8),$$

where $\sigma(x)(y) := xy$ and $\rho(x)(y) := yx^{-1}$ for $x,y \in S^7 \subset \mathbb{O}$ with the octonion multiplication.

**Theorem 2.1.3** ([TSY57]). $\{[\sigma], [\rho]\}$ is a free generating set of $\pi_7(SO(8))$.

**Remark 2.1.4.** $[\sigma]$ is the image of a generator of $\pi_7(S^7)$ under the inclusion. The proof that $[\rho]$ is the image of a generator of $\pi_7(SO(7))$ is hard and involves knowledge about the double cover Spin(7) of SO(7).

We denote the isomorphism class of $S^7$-bundles over $S^8$ corresponding to an element $[m\rho + n\sigma] \in \pi_7(SO(8))$ by $B_{m,n}$. We will denote the total space of $B_{m,n}$ by $M_{m,n}$ and the projection by $\pi_{B_{m,n}}$. Similarly we can think of $SO(8)$ acting on $\mathbb{R}^8$ or the closed unit disk $D^8 \subset \mathbb{R}^8$ and obtain the associated bundles

$$C_{m,n} : \mathbb{R}^8 \hookrightarrow E_{m,n} \xrightarrow{\pi_{C_{m,n}}} S^8 \text{ and } D_{m,n} : D^8 \hookrightarrow L_{m,n} \xrightarrow{\pi_{D_{m,n}}} S^8.$$ 

We will leave out the decorations for the projection when it is clear which one we mean.

**Remark 2.1.5.** Note that $[m\rho + n\sigma] = [y \mapsto x^{m+n}yx^{-m}]$ since the point wise octonion multiplication and the homotopy group operation agree. By [Spa66, Theorem 8, p.43] it suffices to check for two composition laws $\ast$ and $\ast'$ to agree that they have a common two sided identity element and that they satisfy

$$(a \ast b) \ast' (c \ast d) = (a \ast' b) \ast (c \ast' d).$$

This is clearly the case.

In each bundle isomorphism class we now fix the one obtained via $y \mapsto x^{m+n}yx^{-m}$. The total spaces now have a canonical structure of a smooth oriented manifold since we can think of $M_{m,n}$ as two times the smooth manifold $D^8 \times S^7$ with the standard structure glued at its boundaries via the orientation reversing diffeomorphism
(x, y) \mapsto (x, x^{m+n}yx^{-m}) \) (because we considered one of the \( D^8 \times S^7 \) with the opposite orientation). When we mention the PL-manifold structure, we mean the unique one underlying the smooth structure by the Whitehead triangulation. We do the analogous things to give \( E_{m,n} \) and \( D_{m,n} \) a smooth manifold structure. Note that \( E_{m,n} \) is not compact and \( D_{m,n} \) has a boundary \( \partial D_{m,n} \cong M_{m,n} \). To orient \( D_{m,n} \) is a little more complicated. Denote by \( E^0_{m,n} \) the total space \( E_{m,n} \) with the zero section removed. The Thom class of \( C_{m,n} \) is the unique class \( u \in H^8(E_{m,n}, E^0_{m,n}) \) that gives the orientation of the fiber restricted to each fiber. The nice thing about the Thom class is that it defines an isomorphism \( H^i(E_{m,n}) \xrightarrow{\cup u} H^{i+8}(E_{m,n}, E^0_{m,n}) \). After choosing a Riemannian metric we can identify \( D_{m,n} \) with the unit disk bundle in \( E_{m,n} \). Since the inclusions \( D_{m,n} \hookrightarrow E_{m,n} \) and \( M_{m,n} \hookrightarrow E^0_{m,n} \) are homotopy equivalences, we get an isomorphism \( H^8(D_{m,n}, \partial D_{m,n}) \cong H^8(E_{m,n}, E^0_{m,n}) \) using the five lemma. Denote by \( \bar{u} \in H^8(D_{m,n}, \partial D_{m,n}) \) the element corresponding to \( u \). We orient \( D_{m,n} \) such that \( (\pi^*_{D_{m,n}} \zeta \cup \bar{u}, [D_{m,n}, \partial D_{m,n}]) = 1 \), where \( \zeta \in H^3(S^8) \) is the standard generator and \( (\ast, \ast) \) denotes the Kronecker pairing.

We denote by \(-M\) the manifold \( M \) with the opposite orientation.

**Lemma 2.1.6.** \( M_{m,n} \) is diffeomorphic to \(-M_{m, n} \) and \(-M_{m+n, n}\)

**Proof.** Consider \( M_{m,n} \) as

\[
D^8_+ \times S^7 \sqcup D^8_- \times S^7 / \phi, \quad \text{where} \quad \partial D^8_- \times S^7 \ni (x, y) \xmapsto{\phi} (x, x^{m+n}yx^{-m}) \in \partial D^8_+ \times S^7
\]

and \( M_{-m,-n} \) as

\[
D^8_+ \times S^7 \sqcup D^8_- \times S^7 / \psi, \quad \text{where} \quad \partial D^8_- \times S^7 \ni (x, y) \xmapsto{\psi} (x, x^{-m-n}yx^m) \in \partial D^8_+ \times S^7.
\]

We can define a diffeomorphism \( f : D^8_+ \times S^7 \sqcup D^8_- \times S^7 \to -(D^8_+ \times S^7 \sqcup D^8_- \times S^7) \) by changing the orientation in the \( D^8 \) by the octonion conjugation \( \mathbb{O} \ni x = (x_1, x_2, ..., x_8) \mapsto \tilde{x} = (x_1, -x_2, ..., -x_8) \in \mathbb{O} \). We are going to need that \( x^{-1} = \tilde{x} / ||x|| \).

This means for an \( x \in \partial D^8 \) that \( \tilde{x} = x^{-1} \). We need to see that \( f \) is compatible with the diffeomorphisms we use to glue the boundaries. This is the case since \( f^{-1} \circ \psi \circ f(x, y) = f^{-1} \circ \psi(x^{-1}, y) = f^{-1}(x^{-1}, x^{m+n}yx^{-m}) = (x, x^{-m-n}yx^m) = \phi(x, y) \).

Thus, we get a diffeomorphism \( F : M_{m,n} \to -M_{-m,n} \). To get the diffeomorphism from \( M_{m,n} \) to \(-M_{m+n,n}\), repeat the previous with an \( f' \), which is conjugation in the fibers. We get \( f'^{-1} \circ \psi' \circ f'(x, y) = f'^{-1} \circ \psi'(x, y^{-1}) = f'^{-1}(x, x^{m}yx^{-1}x^{-m-n}) = (x, x^{m+n}yx^{-m-n}) = \phi(x, y) \).

**Remark 2.1.7.** Lemma 2.1.6 is also true for the associated disk and vector bundles.

Applying both just constructed diffeomorphisms we get the following corollary:

**Corollary 2.1.8.** \( M_{m,n} \) is diffeomorphic to \( M_{-m,n} \).

For convenience we now assume \( n \geq 0 \) since we can obtain the classification for \( n < 0 \) by using Corollary 2.1.8.
2.2 The cohomology of the total spaces of $S^7$-bundles over $S^8$

Recall that a $2k$ or $(2k+1)$-manifold is highly connected if it is $(k-1)$-connected. Using the long exact sequence of homotopy groups of the fiber bundle over $S^8$ and hence fibration $S^7 \hookrightarrow M_{m,n} \rightarrow S^8$ we see that the total spaces are 6-connected and therefore highly connected. Using the Hurewicz theorem, the universal coefficient theorem and Poincare duality we get

$$H^0(M_{m,n}) \cong H^{15}(M_{m,n}) \cong \mathbb{Z} \text{ and } H^i(M_{m,n}) \cong 0 \text{ for } i \neq 0, 7, 8, 15.$$ 

To learn about the two missing groups we are going to use the Euler class of the $\mathbb{R}^8$-bundle $\mathcal{C}_{m,n}$. Let $\zeta \in H^8(S^8)$ denote the standard generator.

**Lemma 2.2.1.** The Euler class of $\mathcal{C}_{m,n}$ is given by $e(\mathcal{C}_{m,n}) = n\zeta \in H^8(S^8)$.

**Proof.** We want to use the fact that the Euler class can be identified with the primary obstruction to the existence of a non zero cross section ([MS74, Theorem 12.5, p.147]). We start with a short review of the construction of this obstruction. Let $\xi : F \rightarrow E \rightarrow B$ be a fiber bundle over a finite simply-connected CW-complex $B$. We assume that $\xi$ admits a section $s$ over the $q$-skeleton. The obstruction $Ob(\xi, s)$ to extend $s$ is constructed as follows: Restricted to each $(q+1)$-cell the bundle is trivial. Hence, we get a map

$$f_i : S^q \rightarrow \pi^{-1}(D^{q+1}) \cong D^{q+1} \times F \rightarrow F,$$

where $p_2$ is the projection on the second factor. This defines an element in $\pi_q(F)$ for each cell. The obstruction cocyle is now given by the element

$$\sum_i (e_i^{q+1} \mapsto [f_i]) \in \text{Hom}(H_{q+1}(B^{q+1}, B^q), \pi_q(F)),$$

mapping the generator belonging to a $(q+1)$-cell $e_i^{q+1}$ to the just constructed element in $\pi_q(F)$. (If we would not have assumed simply-connectedness, there were twisted coefficients.) $Ob(\xi, s) \in H^{q+1}(B, \pi_q(F))$ is now defined to be the element represented in cohomology by the obstruction cocycle. It vanishes if and only if one can extend $s|B^{(q-1)}$ over the $(q+1)$-skeleton.

Take the cell decomposition of $S^8$ with the 8-cells $D_+$ and $D_-$, one 1-cell and a 7-cell. Let $\pi_0 : E_{0,m,n}^0 \rightarrow S^8$ denote the restriction of the vector bundle $\mathcal{C}_{m,n}$ to the total space with the zero section removed. Note that there is always a section from the 7-skeleton since $\mathbb{R}^8 - \{0\}$ 6-connected. The obstruction to existence of a non zero cross section is the obstruction to extend the section over the 7-skeleton. We can take the constant 1-section. It is possible to extend it constant over $D_-$. Thus, the corresponding generator is mapped to 0. The generator corresponding to $D_+$ is given by the map $[x \mapsto x^{m+n}1x^{-m} = x^n]$. Using 2.1.5 we have $[x \mapsto x^n] = n[id_{\mathbb{R}^8 - \{0\}}] \in \pi_7(\mathbb{R}^8 - \{0\})$. This yields $e(\mathcal{C}_{m,n}) = n\zeta \in H^8(S^8, \pi_7(\mathbb{R}^8 - \{0\}) \cong \mathbb{Z}).$
Now the interesting part of the Gysin sequence is given by:

\[
\begin{align*}
H^0(S^8) \xrightarrow{\cup e(C_{m,n})} H^8(S^8) \xrightarrow{\pi_0^*} H^8(E_{m,n}^0) \longrightarrow \cdots \\
\downarrow \cong \quad \downarrow \cong \quad \downarrow \cong \\
\mathbb{Z} \xrightarrow{-n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow \cdots
\end{align*}
\]

Since \( E_{m,n}^0 \cong M_{m,n} \), using Poincare duality and universal coefficients for the case \( n = 0 \) this proofs:

**Proposition 2.2.2.** The non trivial cohomology groups of \( M_{m,n} \) are given by

\[
H^7(M_{m,0}) \cong H^8(M_{m,0}) \cong \mathbb{Z}
\]

and for \( n \neq 0 \) \( H^8(M_{m,n}) \cong \mathbb{Z}/n \).

Knowing the cohomology of the \( M_{m,n} \) we see that \( M_{m,n} \) can not even be homotopy equivalent to \( M_{m',n'} \) if \( n \neq n' \). Moreover, we get that all \( M_{m,1} \) are homeomorphic to the standard sphere since any simply connected homology sphere is a homotopy sphere.

### 2.3 The calculation of Pontryagin classes and a first partial classification

**Proposition 2.3.1.** The only non trivial Pontryagin class of \( C_{m,n} \) is given by

\[
p_2(C_{m,n}) = \pm 6(2m + n)\zeta \in H^8(S^8),
\]

where \( \zeta \) denotes the standard generator.

**Proof.** Compare [Shi57, Proof of Lemma 2, p.63] (please note that Shimada considers \([\rho + \sigma]\) and \([\rho]\) as generators of \( \pi_7(SO(8)) \)). The Pontryagin classes are linear in \( m \) and \( n \) and not affected by a change of orientation of the fiber. As we have seen in the proof of 2.1.6 this means that \( p_2(C_{m,n}) = p_2(C_{m+n,-n}) \). Therefore, \( p_2 \) is given by \( p_2(C_{m,n}) = c(2m + n)\zeta \in H^8(S^8) \), where \( c \in \mathbb{Z} \) is a constant. Thus, we need to know the Pontryagin class of one bundle \( C_{m,n} \). The bundle will be \( C_{0,1} \).

In [BH85] Borel and Hirzebruch develop a theory to calculate characteristic classes of coset spaces of compact connected Lie groups modulo a closed subgroup. They do it by interpreting the characteristic classes as elementary symmetric functions and identifying them with certain roots of the subgroup. Borel and Hirzebruch also apply their theory to \( F_4/\text{Spin}(9) \) (\( F_4 \), the exceptional Lee group), which is known to be diffeomorphic to the octonion projective plane \( \mathbb{O}P^2 \). They show that \( p_2(\mathbb{O}P^2) = 6u \in H^8(\mathbb{O}P^2) \cong \mathbb{Z} \), where \( u \) is a generator (see [BH85, 19.4. Theorem, p.537]). \( \mathbb{O}P^2 \) can also be described as \( S^8 \) with an 16-cell attached via the Hopf map \( p : S^{15} \to S^8 \). Consider \( S^{15} \subset \mathbb{O}^2 \) and \( S^8 = \mathbb{O} \cup \infty \) as the one point compactification, then \( p(x, y) := \ldots \)
We can define charts \( \phi_p \) be the complements of 0 respectively. Bundle of the boundary is trivial and hence \( p \) be the complements of 0 respectively \( \infty \) in \( S^8 \). Think about \( D^8 \) as \( S^7 \times I/S^7 \times \{0\} \).

We can define charts \( \phi_i : V_i \times D^8 \rightarrow \pi^{-1}(V_i) \), which are

\[
\phi_0(a, (b, t)) := \left( \frac{b, a-1b}{\| (b, a-1b) \|}, t \right) \text{ where we set } \infty^{-1} := 0
\]

and

\[
\phi_1(a, (b, t)) := \left( \frac{ab, b}{\| (ab, b) \|}, t \right).
\]

Of course it is not obvious that these are charts, but the calculations are not enlightening. The interesting part is now \( \phi_0^{-1} \circ \phi_1 | S^7(a, (b, t)) = (a, (ab, b)) \), where we think of \( S^7 \subset V_i \). This is the clutching function of \( D_{0,1} \).

Thus, we know that the second Pontryagin class of \( L_{0,1} \) is \( i^*p_2(\Omega P^2) \) because the inclusion \( i : L_{0,1} \rightarrow \Omega P^2 \) is covered by a bundle map. Since \( i^* : H^8(\Omega P^2) \rightarrow H^8(L_{0,1}) \) is an isomorphism, \( p_2(L_{0,1}) = \pm 6\pi_{D_{0,1}} \). This implies \( p_2(E_{0,1}) = \pm 6\pi_{C_{0,1}} \) since the inclusion is a homotopy equivalence and again covered by a bundle map. Denote by \( TM \) the tangent bundle of a manifold \( M \). The tangent bundle of \( E_{0,1} \) splits after choosing a Riemannian metric as \( TE_{0,1} \cong \pi_{C_{0,1}}^* (C_{0,1}) \oplus \nu \pi_{C_{0,1}}^* (C_{0,1}) \), where \( \nu \) denotes the normal bundle. \( \nu \pi_{C_{0,1}}^* (C_{0,1}) \cong \pi_{C_{0,1}}^* TS^8 \). Since the Pontryagin classes of spheres are trivial, we get

\[
p_2(\pi_{C_{0,1}}^* (C_{0,1})) = \pi_{C_{0,1}}^* \circ p_2 (C_{0,1}) \). \quad \pi_{C_{0,1}} : E_{0,1} \rightarrow S^8 \text{ is a homotopy equivalence and hence } p_2(\pi_{C_{0,1}}) = \pm 6\zeta.
\]

Thus, the constant is \( c = \pm 6 \).

Remark 2.3.2. Actually, Tamura calculates the Pontryagin classes in a different way and uses them to recalculate \( p_2(\Omega P^2) \) ([Tam58, Theorem 2.3, p.32]).

Corollary 2.3.3.

1. The only non trivial Pontryagin class of \( L_{m,n} \) is given by

\[
p_2(L_{m,n}) = \pm 6(2m+n)\pi_{D_{m,n}}^* \zeta \in H^8(L_{m,n}).
\]

2. The only non trivial Pontryagin class of \( M_{m,n} \) is given by

\[
p_2(M_{m,n}) = \pm 12m\pi_{E_{m,n}}^* \zeta \in H^8(M_{m,n}).
\]

Proof. We need to show that \( p_2(E_{m,n}) = \pm 6(2m+n) \zeta \in H^8(E_{m,n}) \). As in the proof before we use \( p_2(TE_{m,n}) = p_2(\pi_{C_{m,n}}^* (C_{m,n}) \oplus \pi_{C_{m,n}}^* TS^8) = \pi_{C_{m,n}}^* (p_2(C_{m,n})) \pm (2m+n)\pi_{C_{m,n}}^* \zeta \). 1. follows by pulling back via the inclusion. 2. follows since the normal bundle of the boundary is trivial and hence \( p_2(M_{m,n}) = p_2(TM_{m,n} \oplus \nu TM_{m,n}) = p_2(i^* TL_{m,n}) = i^* p_2(L_{m,n}) \).

Using the Pontryagin classes we can formulate a first classification result for \( n = 0 \).
**Theorem 2.3.4.** $M_{m,0}$ is diffeomorphic to $M_{m',0}$ if and only if $m = \pm m'$.

**Proof.** We know by 2.1.6 that $M_{m,0}$ is diffeomorphic to $M_{-m,0}$. If $m \neq \pm m'$, then $M_{m,0}$ can not be homeomorphic to $M_{m',0}$ by the topological invariance of rational Pontryagin classes.  

**Remark 2.3.5.** Theorem 2.3.4 is also true if we consider homeomorphism or PL-homeomorphism instead of diffeomorphism.

Since we just finished the classification for $n = 0$, we can assume for the rest of this thesis $n > 0$. Therefore, we have to deal with highly connected closed 15-manifolds with the rational homology groups of a sphere - or in other words highly connected rational homology 15-spheres.
3 The classification of highly connected rational homology 15-spheres

In this chapter we are going to outline the background to understand the statement of the diffeomorphism and almost diffeomorphism classification of highly connected rational homology 15-spheres by Crowley in [Cro02]. Many things are true in greater generality than stated here, but for the purpose of a cleaner exposition we will concentrate on our special case. The interested reader is referred to [Wal62, Cro02].

3.1 Wall’s classification of a special kind of handlebodies and their relation to highly connected manifolds

We assume \( n > 2 \). Denote by \( \mathcal{H}(2n, k, n) \) the set of diffeomorphism classes of manifolds \( L = L(D^{2n}, f_1, \ldots, f_k, n) \), where \( f_i : \partial D^n \times D^n \to \partial D^{2n} \subset D^{2n} \) are disjoint embeddings. \( L \) is obtained from the closed disk \( D^{2n} \) by attaching \( k \) \( n \)-handles via the \( f_i \) and smoothing corners. We can see the attaching maps \( f_i \) as a map

\[
\bigcup_{i=1}^{k} f_i : \bigcup_{i=1}^{k} \partial D^n \times D^n \to \partial D^{2n} \subset D^{2n}.
\]

We call \( f \) a presentation of \( L \). We denote \( \mathcal{H}(n) := \bigcup_k \mathcal{H}(2n, k, n) \). This is the special kind of handlebodies mentioned in the headline. A characterization of these manifolds is given by Smale.

**Theorem 3.1.1** ([Sma62, Theorem 1.2]). Let \( L \) be a highly connected \( 2n \)-manifold with non empty highly connected boundary and \( n > 2 \), then \( L \in \mathcal{H}(n) \). Every element of \( \mathcal{H}(n) \) is a highly connected manifold with non empty highly connected boundary.

**Remark 3.1.2.** Note the abuse of notation. When we write \( L \in \mathcal{H}(n) \), we mean a highly connected \( 2n \)-manifold with non empty highly connected boundary. When we mean the equivalence class, we will denote it by \([L]\). When we write handlebody, we will always mean a highly connected \( 2n \)-manifold with non empty highly connected boundary.

To define the required invariants we will need the following theorem:

**Theorem 3.1.3** ([Hae61, Theorem 1]). Let \( N^n \) and \( M^m \) be two smooth manifolds that are respectively \((k - 1)\) and \(k\)-connected. Then
• Any continuous map from \( N \) to \( M \) is homotopic to a smooth embedding if \( m \geq 2n - k + 1 \) and \( 2k < n \).

• Two smooth embeddings of \( N \) in \( M \), which are homotopic, are smoothly isotopic if \( m \geq 2n - k + 2 \) and \( 2k < n + 1 \).

Moreover, we observe that an \( L \in H(2n, k, n) \) has as a deformation retract a wedge of \( k n \)-spheres [Sma61, p.400]. Thus, we know the homology groups:

\[
H_i(L) \cong Z \quad \text{for} \quad i \neq 1, n \quad \text{and} \quad H_n(L) \cong Z^k
\]

By the Hurewicz theorem we get \( H_n(L) \cong \pi_n(L) \) and by 3.1.3 each element in \( \pi_n(L) \) is uniquely embedded up to smooth isotopy. Therefore, the following definition makes sense.

**Definition 3.1.4** ([Wal62, p.168]). Let \( L \in H(n) \). We define its tangential invariant \( \alpha'(L) \) to be the function \( \alpha'(L) : H_n(L) \to \pi_{n-1}(SO(n)) \), which maps an element \( x \in H_n(L) \) to the clutching function of the normal bundle of an embedded \( n \)-sphere representing it.

The second invariant we need is the (homology) intersection form:

\[
\lambda(L) : H_n(L) \times H_n(L) \ni (x, y) \mapsto \langle PD(x) \cup PD(y), [L, \partial L] \rangle,
\]

where \( PD : H_n(L) \to H^n(L, \partial L) \) denotes the Poincaré-Lefschetz duality isomorphism, \([L, \partial L]\) the fundamental class and \( \langle *, * \rangle \) the Kronecker pairing.

These two invariants are already sufficient, but before we state the classification we are going to give some of their properties.

Consider the fibration \( S^n \hookrightarrow SO(n+1) \to SO(n) \). We are going to need the following maps from the long exact sequence of homotopy groups:

\[
\pi_n(S^n) \xrightarrow{\partial} \pi_{n-1}(SO(n)) \xrightarrow{\gamma} \pi_{n-1}(SO(n+1)) \cong \pi_{n-1}(SO)
\]

Denote by \( \iota_n \) the homotopy class of the identity \( id_{S^n} : S^n \to S^n \) and by \( \pi : \pi_{n-1}(SO(n)) \to \pi_{n-1}(S^{n-1}) \cong Z \) the map induced by the projection \( SO(n) \to SO(n)/SO(n-1) \cong S^{n-1} \).

**Lemma 3.1.5** ([Wal62, Lemma 2, p.167]). \( \alpha' \) satisfies:

1. \( \lambda(x, x) = \pi \alpha'(x) \)
2. \( \alpha'(x + y) = \alpha'(x) + \alpha'(y) + \lambda(x, y) \partial \iota_n \)

**Remark 3.1.6.** Actually, in 1. Wall considers \( H \circ J \), where \( H \) denotes the Hopf invariant and \( J \) the \( J \)-homomorphism, but it can be identified with \( \pi \).
We now consider triples of a free abelian group $H$, an $n$-symmetric product $\lambda : H \times H \to \mathbb{Z}$ and a map $\alpha' : H \to \pi_{n-1}(SO(n))$ satisfying the conditions of the Lemma 3.1.5. We will call two such triples $(H_0, \lambda_0, \alpha'_0)$ and $(H_1, \lambda_1, \alpha'_1)$ isometric if there is a group homomorphism $\Theta : H_0 \to H_1$ such that $\lambda_1 \circ (\Theta \times \Theta) = \lambda_0$ and $\alpha'_1 \circ \Theta = \alpha'_0$. We denote the isometry classes by $[s]$ and the set of isometry classes by $\mathcal{P}'(n)$. Their sum is given by

$$(H_0, \lambda_0, \alpha'_0) \oplus (H_1, \lambda_1, \alpha'_1) := (H_0 \oplus H_1, \lambda_0 \oplus \lambda_1, \alpha'_0 \oplus \alpha'_1),$$

where $\lambda_0 \oplus \lambda_1 : (H_0 \oplus H_1) \times (H_0 \oplus H_1) \in ((x_0, x_1), (y_0, y_1)) \mapsto \lambda_0(x_0, y_0) + \lambda_1(x_1, y_1) \in \mathbb{Z}$ and $\alpha'_0 \oplus \alpha'_1 : H_0 \oplus H_1 \ni (x, y) \mapsto \alpha'_0(x) + \alpha'_1(y) \in \pi_{n-1}(SO(n))$. The sum induces the structure of a monoid on $\mathcal{P}'(n)$, where the trivial triple $(\emptyset, 0, 0, 0)$ gives the neutral element.

Let $L \in \mathcal{H}(n)$. We can associate a triple consisting of the homology group $H_n(L)$ the intersection form $\lambda(L)$ and the tangential invariant $\alpha'(L)$. On $\mathcal{H}(n)$ we also have a monoidal structure. The neutral element is $[D^{2n}]$. The sum is the boundary connected sum $\natural$. Let $M_0, M_1$ be two compact $m$-manifolds with non empty boundaries. Their sum is defined to be the manifold obtained by attaching a handle $D^{m-1} \times D^1$ via an embedding $f : D^{m-1} \times \{0, 1\}$, which maps $D^{m-1} \times i$ into $M_i$, and smoothing corners. This gives an up to diffeomorphism well defined manifold $[\partial M_0 \# \partial M_1]$. Another fact that we are going to need later is that $\partial(M_0 \natural M_1) \cong \partial M_0 \# \partial M_1$.

**Theorem 3.1.7** (compare [Wal62, p.168]). For $n > 2$

$$\mathcal{H}(n) \ni [L] \mapsto [(H_n(L), \lambda(L), \alpha'(L))] \in \mathcal{P}(n)$$

is an injective homomorphism of monoids.

**Sketch of Proof.** It is clear that the map is well defined since a diffeomorphism induces an isometry. The fact that it is injective is exactly the statement of [Wal62, p.168]. We are going to make it plausible. Smale observes that by the isotopy extension theorem the diffeomorphism types correspond to isotopy types of the presentations. He also shows that the linking numbers of the embedded spheres $f_i | \partial D^n \times \{0\}$ are a complete isotopy invariant for the restriction of a presentation $\tilde{f} := f | \bigsqcup_{i=1}^k D^n \times \{0\}$. Wall also identifies invariants of presentations restricting to one given $\tilde{f}$. Consider two presentations $f_1, f_2$ restricting to $\tilde{f}$. The normal bundle of an embedding of $S^{n-1}$ into $S^{2n-1}$ is stably trivial since the tangent bundle of $S^{2n-1}$ and hence the normal bundle of $S^{n-1} \to S^{2n-1}$ is stably trivial. The normal bundle is now also trivial since stably trivial bundles of rank greater than the dimension of the base are trivial (see [KM63, Lemma 3.5, p.509]). Thus, by the tubular neighborhood theorem $f_1$ and $f_2$ are isotopic by an isotopy fixing $\tilde{f}$. Hence, we get after isotopy:

$$f_2(x, y) = f_1(x, s(x)y)$$

for an $s : S^{n-1} \to SO(n)$.

Of course this map depends on the isotopy used, but its homotopy type does not. Thus, presentations corresponding to a fixed $\tilde{f}$ correspond to the choice of $k$ element
in \( \pi_{n-1}(SO(n)) \). By composing an \( f_i \) with an element of \( \pi_{n-1}(SO(n)) \) one sees that every set of elements can be realized. Wall shows that we can recover the invariants of a presentation from \( \lambda \) and \( \alpha' \). Consider \( L = L(D^{2n}, f_1, ..., f_k, n) \). \( f_i(\partial D^n_i \times \{0\}) \) bounds a disk in \( \text{int}(D^{2n}) \) and it also bounds \( f_i(D^n_i \times \{0\}) \). Thus, we get a sphere, which we may consider as embedded by 3.1.3, but this sphere is a basis element in \( H_n(L) \) because the \( f_i(D^n_i \times \{0\}) \) form a basis of \( H_n(L, D^{2n}) \). These spheres only intersect in \( \text{int}(D^{2n}) \) and their intersection numbers coincide with the linking numbers of the \( f_i(\partial D^n_i \times \{0\}) \). The element in \( \pi_{n-1}(SO(n)) \) corresponding to an \( f_i(\partial D^n_i \times \{0\}) \) is shown to be \( \alpha \) of the just constructed generating sphere in \( H_n(L) \).

It remains to check if it is a homomorphism of monoids. Let \( L_0, L_1 \in \mathcal{H}(n) \). Using the Mayer-Vietoris sequence we see that \( H^n(L_0 \# L_1) \cong H^n(L_0) \oplus H^n(L_1) \). We can assume that the boundary connected sum is taken at the complement of the embedded generators. Hence, we see that the normal bundles of the embedded generators do not change and thus \( \alpha'(L_0 \# L_1) = \alpha'(L_0) \oplus \alpha'(L_1) \). We can calculate \( \lambda(L_0 \# L_1) \) using the geometric intersection numbers of the embedded generators. If we take two in the same handlebody, the number does not change. If we take one in each handlebody, they do not intersect. Thus, \( \lambda(L_0 \# L_1) = \lambda(L_0) \oplus (L_1) \). Since \( H_n(D^{2n}) \cong 0 \), \( D^{2n} \) gets mapped to the neutral element.

In special cases Wall actually does some more calculations that will help us to formulate his classification in a final version.

**Proposition 3.1.8 ([Wal65, Proposition 4 (iv)])**. Let \( S \) and \( \pi \) be as before:

\[
\pi : \pi_7(SO(8)) \to \pi_7(SO(7)) \cong \mathbb{Z}
\]

\[
S : \pi_7(SO(8)) \to \pi_7(SO(9)) \cong \pi_7(SO) \cong \mathbb{Z}
\]

They induce an injective homomorphism \((\pi, S) : \pi_7(SO(8)) \to \mathbb{Z} \oplus \mathbb{Z} \) with image of index 2 and \( \pi(x) + S(x) \in \mathbb{Z} \) is always even.

Since \((\pi, S)\) is injective, \((\pi, S) \circ \alpha'\) contains the same information as \( \alpha' \). Moreover, it suffices to consider \( S \circ \alpha' \) since by Lemma 3.1.5 we can recover \( \pi \circ \alpha'(x) = \lambda(x, x) \).

This leads to the following definition:

**Definition 3.1.9. [Cro02]** We define the stable tangential invariant of an \( L \in \mathcal{H}(8) \) to be \( \alpha(L) := S \circ \alpha'(L) \).

Note that \( \alpha \) is a homomorphism by Lemma 3.1.5 since \( S \circ \partial = 0 \). Thus, we consider triples of a finitely generated free abelian group \( H \), a symmetric bilinear form \( \lambda : H \times H \to \mathbb{Z} \) and a homomorphism \( \alpha : H \to \mathbb{Z} \), which we will also call linear form. We can again define a sum and a notion of isometry as before. Again we get the structure of a monoid on the set of isometry classes, which we denote by \( \mathcal{P} \). We denote by \( \mathcal{P}^c \) the submonoid consisting of triples \((H, \lambda, \alpha)\), where \( \lambda(x, x) + \alpha(x) \) is always even.

Now we are able to state the classification of 16-handlebodies in a final version. It is formulated using the language of symmetric monoidal categories by Crowley (compare [Cro02, Proposition 2.13, p.20]), but it was most certainly already known to Wall.
Corollary 3.1.10.

\[ \mathcal{H}(8) \ni [L] \mapsto [(H_8(L), \lambda(L), \alpha(L))] \in \mathcal{P}^c \]

is an isomorphism of monoids.

**Sketch of Proof.** Compare [Cro02, Proposition 2.13, p.20]. The homomorphism is well defined since by 3.1.5, \( \pi \circ \alpha'(x) = \lambda(x, x) \) and by 3.1.8 \( \pi(x) + S(x) \) is always even. We need to see that it is surjective. In the sketch of proof of 3.1.7 we described that a presentation \( L = L(D^{2n}, f_1, ..., f_k, n) \) is determined by the linking numbers of the \( f_i(\partial D^n_i \times \{0\}) \) and a choice of \( k \) elements in \( \pi_{n-1}(SO(n)) \). We are going to use that every set of linking numbers can be realized and one can choose every element in \( \pi_{n-1}(SO(n)) \). Note that the linking form is determined by its values on basis elements. We have seen that there is a basis \( \{a_i\} \) of \( H_8(L) \) such that for \( i \neq j \) \( \lambda(a_i, a_j) \) is given by the linking number of \( f_i(\partial D^n_i \times \{0\}) \). \( \lambda(a_i, a_i) \) is the Euler number of the normal bundle of the embedded image of \( a_i \) in \( \pi_8(L) \) via the Hurewicz isomorphism. Since the normal bundle has an Euler number any element of \( \pi_{n-1}(SO(n)) \), we can produce any possible bundle and thus any Euler number. This implies that every possible symmetric \( \lambda \) and any \( \alpha \) can be realized.

\[ \square \]

What is the relation with highly connected closed 15-manifolds? We have already seen that the boundaries of elements in \( \mathcal{H}(n) \) are highly connected closed odd dimensional manifolds (see 3.1.1). In order to see in which sense highly connected closed 15-dimensional manifolds are boundaries we need to go back a little. Denote by \( \theta_n \) the group of exotic \( n \)-spheres with the connected sum and by \( bP_{n+1} \) the subgroup of exotic \( n \)-spheres bounding a parallelizable manifold. Kervaire and Milnor study these groups in their paper [KM63]. Brumfiel shows in [Bru68, Theorem 1.3] that \( bP_{4n} \) is actually a direct summand of \( \theta_{4n-1} \) by defining a section \( s : \theta_{4n-1} \to \mathbb{Z}/|bP_{4n}| \cong bP_{4n} \), where the isomorphism is given by mapping \([1] \in \mathbb{Z}/|bP_{4n}| \) to the Milnor sphere. Note that if we have a manifold \( W^{4n} \) bounded by a homotopy sphere, then \( H^i(W, \partial W) \to H^i(W) \to H^i(\partial W) \cong 0 \) is an isomorphism for \( 0 < i \leq n \) and we can define \( \hat{p}_i(W) \) to be the preimage of the Pontryagin class \( p_i(W) \).

**Theorem 3.1.11** ([Bru68, Theorem 1.5 and 1.6]). Every \( \Sigma \in \theta_{4n-1} \) bounds a spin manifold \( W \) with all Pontryagin numbers \( \langle \hat{p}_i(W) \cup \hat{p}_j(W) \cup \cdots \cup \hat{p}_k(W), [W, \partial W] \rangle \) with \( \sum_{j=1}^k i_j = n \), except possibly \( \langle \hat{p}_n(W), [W, \partial W] \rangle \), zero. Moreover, \( 8|\sigma(M) \), where \( \sigma \) denotes the signature.

\( s(\Sigma) \) is now defined to be \( s(\Sigma) := \frac{\sigma(W)}{8} \in \mathbb{Z}/|bP_{4n}| \) and shown to be independent of the choice of \( W \). Now we can formulate the following theorem:

**Theorem 3.1.12** ([Cro02, Corollary 2.17, p.23]). For every highly connected 15-manifold \( P \) there is a unique homotopy sphere \( \Sigma_P \) with \( s(\Sigma_P) = 0 \) such that \( P \# \Sigma_P = \partial L \) for a \( L \in \mathcal{H}(8) \). Moreover, for two highly connected 15-manifolds \( P_0, P_1 \):

\[ \Sigma_{P_0} \# \Sigma_{P_1} \cong \Sigma_{P_0 \# P_1} \]

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Sketch of Proof. The proof can be found in [Cro02, p.22 ff.]. We are going to sketch the main steps. Crowley starts with considering the set of highly connected cobordism classes \( \Omega_n^{HC} \), where a highly connected cobordism between two highly connected \( n \)-manifolds \( M_1, M_2 \) is a highly connected \((n + 1)\)-manifold \( W \) with \( \partial W \cong M_1 \sqcup -M_2 \). He shows that \( \Omega_n^{HC} \) is an abelian group for some values of \( n \) including \( n = 8j - 1 \).

The group operation is the connected sum, reversing the orientation gives the inverse. Consider \((M - D^{8j}) \times I\). It is a \((4j - 2)\)-connected manifold with boundary \( M \# - M \).

In a CW-complex we can always kill elements in homotopy groups by attaching a cell via a map representing the element. In manifolds we can in general not expect that the sum is well defined and that reversing the orientation gives an inverse.

Consider \((M - D^{8j}) \times I\). It is a \((4j - 2)\)-connected manifold with boundary \( M \# - M \).

Consider \( \pi_j \) can kill all elements of \( g \)-bundles [KM63, Lemma 3.3, p.509]. Therefore, we have framed embeddings and we need to show that there is a highly connected cobordism between \( M \# - M \).

Thus, we have a highly connected null-cobordism of \( M \# - M \).

Let \( M_0, M_1 \) be highly connected manifolds. To see that the sum is well defined, we need to show that there is a highly connected cobordism between \( M_0 \# M_1 \) and \( M_0' \# M_1 \), where \( M_0 \) is highly connected cobordant to \( M_0' \) via \( W_0 \). As before we can make \( M_1 \times I \) highly connected with surgeries in the interior. Thus, we have a highly connected cobordism between \( M_1 \) and \(-M_1 \) and we will call it \( W_1 \). Now we can take the band connected sum of \( W_0 \) and \( W_1 \) as in the proof of Lemma 2.2 in [KM63] and obtain a highly connected cobordism between \( M_0 \# M_1 \) and \( M_0' \# M_1 \). Associativity and commutativity follow by taking h-cobordisms and make them highly connected.

The next step is to calculate \( \Omega_S^{HC} \). To do this Crowley considers the homomorphism \( \Omega : \theta_{8j-1} \rightarrow \Omega_S^{HC} \) sending an exotic sphere to its cobordism class. Using surgery we can make a parallelizable manifold highly connected and hence \( bP_{8j} \) is in the kernel. It turns out that in the dimensions 7 and 15 it is actually the kernel. \( \Omega \) is surjective since again by surgery for every highly connected \((8j - 1)\)-manifold there exists a highly connected cobordism to an exotic sphere. Thus, we know that

\[
\Omega_{15}^{HC} \cong \theta_{15}/bP_{16}.
\]

If we now consider any highly connected compact 15-manifold \( P \), we know that there is an up to addition with elements of \( bP_{16} \) unique element \( \Sigma_P \in \theta_{15} \) such that \( P \) is
highly connected cobordant to the empty manifold. This means that there is a unique \( \Sigma_P \) with \( s(\Sigma_P) = 0 \) such that \( P \# \Sigma_P \) is the boundary of a handlebody. The second statement follows since \( \Omega_{15}^{HC} \) is a group and \( s(\Sigma_P \# \Sigma_R) = s(\Sigma_P) + s(\Sigma_R) \). 

In dimension 15 we have \( \theta_{15} \cong bP_{16} \oplus \mathbb{Z}_2 \) and \( |bP_{16}| = 8128 \). Thus, there are only two different homotopy spheres that occur, the standard sphere and the one not bounding a parallelizable manifold with \( s = 0 \) mod \( |bP_{16}| \).

Since we want to use the diffeomorphism classification of the handlebodies cobounding after the addition of a homotopy sphere, we should expect a classification up to the following concept:

**Definition 3.1.13.** We call a homeomorphism \( f : M_1 \to M_2 \) between smooth manifolds almost diffeomorphism if there is an exotic \( \dim(M_1) \)-sphere \( \Sigma \), such that \( f \) extends to a diffeomorphism \( f' : M_1 \# \Sigma \to M_2 \).

**Remark 3.1.14.** It is a nice exercise to check that the definition of almost diffeomorphism is equivalent to requiring the homeomorphism to be a diffeomorphism except in a finite number of points. The proof can also be found in \([\text{Cro02, p.12}]\).

**3.2 On the stable tangential invariant \( \alpha \)**

In this section we are going to identify \( \alpha \) with an obstruction to the extension of a section. Moreover, we show that this obstruction is a characteristic class.

Let \( L \) be a 16-handlebody. Recall that \( L \) contains a wedge of 8-spheres as a deformation retract. The inclusions of the 8-spheres, which we can assume to be embedded, form a set of generators of \( H_8(L) \). Thus, \( \alpha' \) is given by mapping these generators to the clutching function of their normal bundles. Since \( \text{Hom}_\mathbb{Z}(H_8(L), \pi_7(SO(8))) \cong H^8(L; \pi_7(SO(8))) \), we can think of \( \alpha' \) as an element of the cohomology group. Now consider the associated principal bundle of the normal bundles. By obstruction theory we have a section over any 7-skeleton. So consider a wedge of 7-spheres as 7-skeleton, where on each \( S^7 \) we attach two disks to get the wedge of 8-spheres. One way to think of the associated principal bundle is by constructing it using the clutching function for each pair of disks. The obstruction cocycle to extend a section to the 8-skeleton now is given the map sending the generator associated to one cell attached to an \( S^7 \) to zero and the other one to the homotopy class of the clutching function (compare to the calculation of the Euler class). Thus, in cohomology the generator coming from an 8-sphere gets mapped to the homotopy class of the clutching function of its normal bundle. Hence, we have identified \( \alpha' \) with the obstruction to extend a section of the 7-skeleton of the associated principal bundle. We will call this object primary obstruction to triviality. "Primary" means that it is first possibly non zero obstruction. An important property of the primary obstruction is that it does not depend on the section one wants to extend. The "to triviality" refers to the fact that if we can extend a section in a principal bundle it is trivial over the skeleton and thus this is also true for the bundle we started with. When we stabilize, we get that \( \alpha \) equals the primary obstruction to stable triviality of the tangent bundle. This is because the tangent
bundle of $L$ restricted to the wedge of 8-spheres is stably equivalent to the normal bundle. Actually, $\alpha$ and $\alpha'$ are the only obstructions to (stable) triviality because all higher cohomology groups of $L$ vanish.

To learn something about $\alpha$ and for calculations we need the following theorem:

**Proposition 3.2.1** ([Ker59, Lemma 1.1]). Let $X$ be a finite CW-complex and let $\xi$ be a principal $SO(n)$-bundle with $n > \dim(X)$ admitting a section $f$ over the $4k - 1 < \dim(X)$ skeleton. Then:

$$p_k(\xi) = (2k - 1)!a_k\text{Ob}(\xi, f) \in H^{4k}(X, \pi_{4k-1}(SO(n)) \cong \mathbb{Z}),$$

where $a_k$ is one for $k$ even and two for $k$ odd.

**Remark 3.2.2.** Actually, Kervaire talks about stable bundles, where he calls a bundle stable if $\pi_{q-1}(SO(n))$ is stable for $q \leq \dim(X)$, but $n > \dim(X)$ ensures that the bundle is stable in the sense of Kervaire. This implies that we can take $\pi_{4k-1}(SO)$ instead of $\pi_{4k-1}(SO(n))$ and the obstruction equals the obstruction to stable triviality.

Applying 3.2.1 to our situation yields that $6\alpha = p_2(L)$.

Now denote by $O(n)(7) \to SO(n)$ the 6-connected cover. It is also known as $String(n)$. Let $BO(n)(8) := B(SO(n)(7))$ be the classifying space for principal $O(n)(7)$-bundles and let $BO(n)(8) \xrightarrow{\pi_n} BSO(n)$ be the induced map between the classifying spaces. Denote by $\gamma_n : EO(n) \to BO(n)$ the universal bundle. Observe that an $O(n)(7)$-bundle over a finite CW-complex admits section over the 7-skeleton and thus the underlying $SO(n)$-bundle also does.

**Corollary 3.2.3.** Let $X$ be a finite CW-complex with $\dim(K) > 7$ and let $g : X \to BO(n)(8)$, where $n > \dim(K)$. There is an element $\alpha_n \in H^8(BO(n)(8))$ such that $\text{Ob}(\gamma_n) = g^*\alpha_n$, where $f$ is a section over the 7-skeleton. Moreover, $\pi_n^* : H^8(BSO(n)) \to H^8(BO(n)(8))$ maps $p_2$ to $6\alpha_n$, where $p_2$ denotes the universal second Pontryagin class.

**Proof.** Consider any CW-structure on $BO(n)(8)$ such that $BO(n)(8)$ has finitely many cells. We have a principal $SO(n)$-bundle obtained via $\pi_n^*\gamma_n$. It admits a section $s : BO(n)(8)^{(7)} \to \pi_n^*ESO(n)$. By 3.2.1 we get $6\text{Ob}(\pi_n^*\gamma_n, s) = \pi_n^*p_2$. Using $H^8(BO(n)(8)) \cong H^8(BO(n)(8))$ we consider $\alpha_n$ to be the image of $\text{Ob}(\pi_n^*\gamma_n, s)$ and thus $\pi_n^*p_2 = 6\alpha_n$. Note that $\text{Ob}(\pi_n^*\gamma_n|BO(n)(k)(8), s) = i^*\alpha(n)$ for all $k > 8$, where $i$ denotes the inclusion. By [Whi78, (6.2) (2), p.298] and considering $g$ to be a cellular map with image contained in the $k$-skeleton for $k$ large, we get $\text{Ob}((\pi_n \circ g)^*\gamma_n, f) = \text{Ob}((\pi_n \circ i \circ g)^*\gamma_n, f) = \text{Ob}((\pi_n \circ i)^*\gamma_n, f \circ g) = g^*\alpha_n$. \hfill $\square$

**Remark 3.2.4.** Denote by $j_n^{(8)} : BO(n)(8) \to BO(n + 1)(8)$ the inclusion. Using 3.2.3 we can show that $j_n^{(8)*}\alpha_{n+1} = \alpha_n$. Thus, it is a characteristic class of a stable $BO(8)$-bundle.

Let $L$ be a 16-handlebody and denote by $\tau(L) : L \to BSO(16)$ the classifying map of the tangent bundle. By obstruction theory we always have a lift $\bar{\tau}(L) : L \to BO(16)(8)$. Thus, we know that $\alpha$ is a stable characteristic class.
Now consider a highly connected 15-manifold $P$ and the associated principal bundle to the tangent bundle. Since $\pi_6(SO(15)) \cong \pi_6(SO) \cong 0$, we always have a section over the 7-skeleton. Note that $\pi_7(SO(15)) \cong \pi_7(SO)$ is in the stable range. Thus, we can define $\beta(P) \in H^8(P,\pi_7(SO) \cong \mathbb{Z})$ to be the primary obstruction to (stable) triviality of the tangent bundle. We will refer to $\beta(P)$ as the stable tangential invariant of $P$.

Again by obstruction theory we can lift classifying map $\tau(P)$ to $BO(15) \langle 8 \rangle$ and see that $\beta$ is also a stable characteristic class. A first consequence of this is that if $P \cong \partial L$, we get $i^*\alpha(L) = \beta(P)$, where $i : P \to L$ denotes the inclusion, since the normal bundle of a boundary is trivial.

### 3.3 Quadratic functions

The first step towards the classification by Crowley is to consider quadratic functions instead of triples. This is basically the same as we will see in Proposition 3.3.3, but it is a more convenient language. As before let $H$ be a finitely generated free abelian group, $\alpha : H \to \mathbb{Z}$ a linear form and $\lambda : H \times H \to \mathbb{Z}$ symmetric bilinear form.

**Definition 3.3.1** [(Cro02, p.16)]. We will call a function $\kappa : H \to \mathbb{Z}$ quadratic function if

$$\kappa(x) = \lambda(x, x) + \alpha(x)$$

for all $x \in H$, which we will denote by $\kappa = \kappa(H, \lambda, \alpha)$. For two groups $H_1$ and $H_2$ with quadratic functions $\kappa_1$ and $\kappa_2$ we define the sum $\kappa_1 + \kappa_2$ to be $H_1 \times H_2 \ni (x, y) \mapsto \kappa_1(x) + \kappa_2(y) \in \mathbb{Z}$. We call two quadratic functions isometric if there is an isomorphism $\theta : H_1 \to H_2$ such that $\kappa_2 \circ \theta = \kappa_1$ and denote by $\mathcal{F}$ the monoid of isometry classes of quadratic functions with the addition $[\kappa_1] \oplus [\kappa_2] := [\kappa_1 \oplus \kappa_2]$. It is easy to check that $\oplus$ is well defined and that the neutral element is the trivial quadratic function.

In analogy of 3.1.8 we need the following definition:

**Definition 3.3.2** [(Cro02, p.18)]. We will call a quadratic function $\kappa(H, \lambda, \alpha)$ characteristic quadratic function if $\kappa$ only takes even values in $\mathbb{Z}$. We will denote the monoid of isometry classes of characteristic quadratic functions by $\mathcal{F}^c$.

The next proposition shows that quadratic functions are basically the same as the triples Wall considered.

**Proposition 3.3.3.**

$$\mathcal{P} \ni [(H, \lambda, \alpha)] \mapsto [\kappa(H, \lambda, \alpha)] \in \mathcal{F}$$

and

$$\mathcal{P}^c \ni [(H, \lambda, \alpha)] \mapsto [\kappa(H, \lambda, \alpha)] \in \mathcal{F}^c$$

are isomorphisms of monoids.
Proof. The proof is simple algebra. To see that it is surjective use the relations
\[
\kappa(x, y) = \kappa(x) + \kappa(y) + 2\lambda(x, y)
\]
\[
\alpha(x) = \kappa(x) - \lambda(x, x)
\]
for all \(x, y \in H\) to first obtain \(\lambda\) and then \(\alpha\) (compare [Cro02, p.17]).

We want to restate Wall’s classification of 16-handlebodies in terms of quadratic functions. Thus, we need the following definition:

**Definition 3.3.4** ([Cro02, Definition 2.11, p.18]). Let \(L \in \mathcal{H}(8)\) with intersection form \(\lambda(L)\) and stable tangential invariant \(\alpha(L)\). We define the quadratic function of \(L\) to be
\[
\kappa(L) = \kappa(H_8(L), \lambda(L), \alpha(L)).
\]

Now we can restate 3.1.10 in terms of stable quadratic functions.

**Corollary 3.3.5.**
\[
\mathcal{H}(8) \ni [L] \mapsto [\kappa(L)] \in \mathcal{F}^c
\]
is an isomorphism of monoids.

**Remark 3.3.6.** An analogous theorem is also true for \(\mathcal{H}(4)\) (compare [Cro02, Proposition 2.13, p.20]).

Using 3.3.5 we define:

**Definition 3.3.7** ([Cro02, Definition 2.11, p.18]). For a characteristic quadratic function \(\kappa = \kappa(H, \lambda, \alpha)\) we define \(L(\kappa)\) to be the up to diffeomorphism unique handlebody in \(\mathcal{H}(8)\) with \(\kappa(L(\kappa))\) isometric to \(\kappa(H, \lambda, \alpha)\).

We are now going to introduce the so called fundamental sequence of a quadratic function \(\kappa = \kappa(H, \lambda, \alpha)\). Recall that the adjoint of a bilinear form \(\lambda\) is defined to be the linear transformation \(\lambda^* : H \ni x \mapsto \lambda(x, \star) \in H^*\), where \(H^*\) denotes the dual. We will call \(\text{Ker}(\lambda^*) = \{v \in H | \lambda^*(x)(y) = 0 \ \forall y \in H\}\) the radical of \(\kappa\) and \(\text{Cok}(\lambda^*) = H^*/\text{Im}(\lambda^*)\) the quotient of \(\kappa\). Then we have the following exact sequence, which we will call the fundamental sequence of \(\kappa = \kappa(H, \lambda, \alpha)\).

\[
0 \rightarrow \text{Ker}(\lambda^*) \ni H \xrightarrow{\lambda^*} H^* \xrightarrow{\pi} \text{Cok}(\lambda^*) \rightarrow 0,
\]

where \(\pi\) denotes the projection from \(H^*\) to \(H^*/\text{Im}(\lambda^*)\). We will denote \(\pi(x)\) also by \([x]\). A quadratic function \(\kappa\) is called nondegenerate if its radical is trivial. If the radical and the quotient are trivial, we call a quadratic function nonsingular.

Before we proceed we are going to take a look at the topological analogue of the fundamental sequence of a quadratic function. Let \(L \in \mathcal{H}(8)\). Denote by \(PD : H_8(L, \partial L) \rightarrow H^8(L)\) the Poincaré-Lefschetz duality isomorphism. Let \(j : (L, \emptyset) \rightarrow (L, \partial L)\) and \(i : \partial L \rightarrow L\) be the inclusions. Then we have a pair of isomorphic exact sequences [Cro02, p.18].
A direct consequence is that nonsingular characteristic quadratic functions correspond to 16-dimensional handlebodies bounded by homotopy spheres. A less obvious fact is that nondegenerate quadratic functions correspond to 16-dimensional handlebodies bounded by rational homology spheres. This follows since $\lambda^*$ is then an injective map between free abelian groups of the same rank and hence the cokernel $\text{Cok}(\lambda^*)$ is a torsion group.

In preparation of the main theorem of this section we need two more definitions.

**Definition 3.3.8** ([Cro02, Definition 2.14, p.21]). We will call two characteristic quadratic functions $\kappa_1, \kappa_2$ stably equivalent if there exist nonsingular characteristic quadratic functions $\mu_1, \mu_2$ and an isometry $\theta : \kappa_1 \oplus \mu_1 \cong \kappa_2 \oplus \mu_2$. We will call the equivalence classes $[\kappa_1]_{F^c}$ stable quadratic functions.

Again we can define a sum by $[\kappa_1]_{F^c} \oplus [\kappa_2]_{F^c} := [\kappa_1 \oplus \kappa_2]_{F^c}$, which is well defined since the sum of nonsingular quadratic functions is nonsingular.

**Definition 3.3.9** ([Cro02, Definition 2.25, p.32]). For a highly connected closed 15-manifold $P$ with $L \in H(8)$ such that $P \# \Sigma P \cong \partial L$, we define: $[\kappa(P)]_{F^c} := [\kappa(L)]_{F^c}$. We will call $[\kappa(P)]_{F^c}$ the stable quadratic function of $P$. Before we come to the main result of this section and the proof of that the stable quadratic function of a manifold $P$ is well defined, we need the following theorem by Wilkens:

**Theorem 3.3.10** ([Cro02, Theorem 2.24, p.31]). Let $W_0, W_1 \in \mathcal{H}(2n)$ with $n > 2$ and let $f : \partial W_0 \to \partial W_1$ be a diffeomorphism, then there exist $V_0, V_1 \in \mathcal{H}(2n)$ with boundaries diffeomorphic to the standard sphere and a diffeomorphism $g$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\partial W_0 & \xrightarrow{f} & \partial W_1 \\
\downarrow & & \downarrow \\
W_0 \oplus V_0 & \xrightarrow{g} & W_1 \oplus V_1
\end{array}
$$

Now we can formulate and proof the main result of this section.

**Theorem 3.3.11** ([Cro02, Corollary 2.26, p.32]). The stable quadratic function of a highly connected 15-manifold is a well defined and complete almost diffeomorphism invariant.
Proof. Consider two almost diffeomorphic manifolds $P_0, P_1$. We want to show that the stable quadratic function is an invariant. Thus, we need to show that the quadratic functions of handlebodies $L_0, L_1$ such that $P_0 \# \Sigma P_0 \cong \partial L_0$ and $P_1 \# \Sigma P_1 \cong \partial L_1$ are stably isometric. By definition there is a diffeomorphism 

$$ f : P_0 \to P_1 \# \Sigma $$

with $\Sigma \in \Theta_{15}$. This induces a diffeomorphism 

$$ f' : P_0 \# \Sigma P_0 \to P_1 \# \Sigma \# \Sigma P_0. $$

Since $\Sigma P_0 \cong \Sigma P_1 \# \Sigma$ by 3.1.12, we have a diffeomorphism 

$$ f'' : P_0 \# \Sigma P_0 \to P_1 \# \Sigma P_1 \# \Sigma \# \Sigma. $$

We denote by $L_\Sigma$ a handlebody bounded by $\Sigma \# \Sigma$. We note that $\partial (L_1 \natural L_\Sigma) \cong P_1 \# \Sigma P_1 \# \Sigma \# \Sigma$. Thus, we can apply Theorem 3.3.10 and obtain a commutative diagram:

$$
\begin{array}{ccc}
P_0 \# \Sigma P_0 & \xrightarrow{f''} & P_1 \# \Sigma P_1 \# \Sigma \# \Sigma \\
\downarrow & & \downarrow \\
L_0 \natural V_0 & \xrightarrow{g} & L_1 \natural L_\Sigma \natural V_1,
\end{array}
$$

where $V_0, V_1$ are handlebodies bounded by the standard sphere and $g$ is a diffeomorphism. By 3.3.5 we get $\kappa(L_0 \natural V_0) \cong \kappa(L_1 \natural L_\Sigma \natural V_1)$ and thus $\kappa(L_0) \oplus \kappa(V_0) \cong \kappa(L_1) \oplus \kappa(L_\Sigma \natural V_1)$. Since $V_0$ and $L_\Sigma \natural V_1$ are bounded by homotopy spheres, their linking functions are nonsingular and we are done.

If we repeat the above with $P_0 = P_1$, we can show that the stable quadratic function does not depend on the handlebody chosen and it is thus well defined.

Now we are going to show that it is also a complete invariant. Consider $P_0, P_1$ with $[\kappa(P_0)]_{\mathcal{F}} = [\kappa(P_1)]_{\mathcal{F}}$. Take handlebodies $L_0, L_1$ such that $P_0 \# \Sigma P_0 \cong \partial L_0$ and $P_1 \# \Sigma P_1 \cong \partial L_1$. We then have

$$ [\kappa(L_0)]_{\mathcal{F}} = [\kappa(P_0)]_{\mathcal{F}} = [\kappa(P_1)]_{\mathcal{F}} = [\kappa(L_1)]_{\mathcal{F}}. $$

Thus, there are nonsingular characteristic quadratic functions $\nu_0, \nu_1$ such that $\kappa(L_0) \oplus \nu_0 \cong \kappa(L_1) \oplus \nu_1$. Hence, by 3.3.5 we get a diffeomorphism 

$$ f : L_0 \natural L(\nu_0) \to L_1 \natural L(\nu_1). $$

Restricted to the boundary we have a diffeomorphism 

$$ P_0 \# \Sigma P_0 \# \partial L(\nu_0) \cong P_1 \# \Sigma P_1 \# \partial L(\nu_1). $$

Since $\nu_0$ and $\nu_1$ are nonsingular, they are bounded by homotopy spheres and thus $P_0$ and $P_1$ are almost diffeomorphic. \qed

The problem is to decide when two quadratic functions are stably equivalent. When we restrict our attention to nondegenerate characteristic quadratic functions or on the topological side highly connected rational homology 15-spheres, there is a nice answer as we will see soon.
3.4 Invariants of stable quadratic functions

In this chapter we review (almost diffeomorphism) invariants of highly connected rational homology 15-spheres $P$ and translate them into the language of stable characteristic quadratic functions.

The first object we want to consider is $H_7(P) \cong H_8(P)$. It is clearly invariant under almost diffeomorphism. As we have already seen in the discussion of the fundamental sequence, the corresponding object for a quadratic function $\kappa = \kappa(H, \lambda, \alpha)$ is $\text{Cok}(\lambda^*)$. It does not change when we add a nonsingular quadratic function $\nu = (H_0, \lambda_0, \alpha_0)$ since $\text{Cok}((\lambda_0)^*) \cong 0$. Thus, it only depends on the stable quadratic function $\kappa_f$.

When we add two highly connected rational homology 15-spheres $P_0$ and $P_1$, the Maier-Vietoris sequence gives us $H_8(P_0 \# P_1) \cong H_8(P_0) \oplus H_8(P_1)$.

The second object is the primary obstruction to triviality of the tangent bundle $\beta(P)$, which we have defined in Section 3.2. When we add two highly connected rational homology 15-spheres $P_0$ and $P_1$, we get $\beta(P_0 \# P_1) = \beta(P_0) \oplus \beta(P_1)$. One can see this by choosing a CW-structure on $P_0 \# P_1$ having only 8-cells in $P_0$ or $P_1$ (compare [Wil71, Lemma 1.3, p.25]). This yields that $\beta$ is an almost diffeomorphism invariant since a homotopy sphere $\Sigma$ is stably parallelisable and hence $\beta(\Sigma)$ is zero.

This also implies that $i^*\alpha(L) = \beta(P)$ for any handlebody $L$ bounded by $P \# \Sigma_P$. Thus, its algebraic analogue is given by the image of $\alpha$ via $\pi : H^* \to \text{Cok}(\lambda^*)$ for any $\kappa(H, \lambda, \alpha) \in [\kappa(H, \lambda, \alpha)]_F$. Actually, Wilkens shows that $\beta$ is always even [Wil71, Lemma 1.2, p.23].

The third is the linking form $b(\lambda)$, which we are going to deal with for the rest of this chapter.

**Definition 3.4.1** ([Cro02, p.26]). Let $G$ be a finite abelian group. A function

$$b : G \times G \to \mathbb{Q}/\mathbb{Z}$$

is called linking form if the following holds:

1. Symmetry: $b(x, y) = b(y, x)$
2. Bilinearity: $b(x, y + z) = b(x, y) + b(x, z)$
3. Nonsingularity: $b(x, y) = 0 \forall x \in G, \text{ then } y = 0$

We define the sum of two linking forms $b_0 : G_0 \times G_0 \to \mathbb{Q}/\mathbb{Z}$ and $b_1 : G_1 \times G_1 \to \mathbb{Q}/\mathbb{Z}$ to be:

$$b_0 \oplus b_1 : (G_0 \oplus G_1) \times (G_0 \oplus G_1) \ni ((x_0, x_1), (y_0, y_1)) \mapsto b_0(x_0, y_0) + b_1(x_1, y_1) \in \mathbb{Q}/\mathbb{Z}.$$ 

$b_0 \oplus b_1$ is a linking form on $G_0 \oplus G_1$. We call two linking forms isometric if there is an isomorphism $\theta : G_0 \to G_1$ such that $b_1 \circ (\theta \times \theta) = b_0$. Then we denote $b_0 \cong b_1$. 

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We are now going to see how to obtain a linking form from a nondegenerate quadratic function \( \kappa = \kappa(H, \lambda, \alpha) \) on the torsion subgroup of its quotient \( TCok(\lambda^*) \). This can be found in [Cro02, p.27f]. We have already noted that in our case \( Cok(\lambda^*) \) is a torsion group. Since \( \lambda^* \) is injective, the fundamental sequence reduces to:

\[
0 \to H \xrightarrow{\lambda^*} H^* \xrightarrow{\pi} Cok(\lambda^*) \to 0
\]

Tensoring the sequence with \( \mathbb{Q} \) respectively \( \text{id}_\mathbb{Q} \) leads to:

\[
0 \to H \otimes \mathbb{Q} \xrightarrow{\lambda^* \otimes \text{id}_\mathbb{Q}} H^* \otimes \mathbb{Q} \xrightarrow{\pi \otimes \text{id}_\mathbb{Q}} Cok(\lambda^*) \otimes \mathbb{Q} \cong 0
\]

Thus we can invert \( \lambda^* \otimes \text{id}_\mathbb{Q} \). Now we identify \( H^* \) with the image of \( H^* \ni x \mapsto x \otimes 1 \in \mathbb{Q} \). We define:

\[
(\lambda^*)^{-1} := (\lambda^* \otimes \text{id}_\mathbb{Q})^{-1}|H^*: H^* \ni x \mapsto (\lambda^* \otimes \text{id}_\mathbb{Q})^{-1}(x \otimes 1) \in H \otimes \mathbb{Q}
\]

This defines a bilinear pairing:

\[
\lambda^{-1}: H^* \times H^* \ni (x, y) \mapsto y((\lambda^*)^{-1}(x)) \in \mathbb{Q}, \quad (3.1)
\]

where we set \( y(x \otimes \frac{p}{q}) := \frac{p}{q}y(x) \) for \( \frac{p}{q} \in \mathbb{Q}, \ y \in H^* \) and \( x \in H \).

There is a different way to look at \( \lambda^{-1} \). Consider \( x, y \in H^* \). Since \( Cok(\lambda^*) \) is a torsion group, we find \( v \in H \) such that \( rx = \lambda^*(v) \), where \( r \in \mathbb{Z} \). Then the following holds:

\[
\frac{y(v)}{r} = \frac{y}{r}(v) = (\lambda^*)^{-1}(x)(y) = \lambda^{-1}(x, y). \quad (3.2)
\]

We now define:

\[
b(\lambda) : Cok(\lambda^*) \times Cok(\lambda^*) \ni ([x], [y]) \mapsto \lambda^{-1}(x, y) \in \mathbb{Q}/\mathbb{Z}.
\]

**Proposition 3.4.2.** \( b(\lambda) \) is a well defined linking form.

**Proof.** Consider \( x, x + \lambda^*(z) \in [x] \) and \( y, y + \lambda^*(z') \in [y] \), then

\[
\lambda^{-1}(x + \lambda^*(z), y + \lambda^*(z')) = \lambda^{-1}(x, y) + \lambda^{-1}(x, \lambda^*(z')) + \lambda^{-1}(\lambda^*(z), y) + \lambda^{-1}(\lambda^*(z), \lambda^*(z')) \equiv \lambda^{-1}(x, y) + x(z') + y(z) + \lambda(z, z') \equiv \lambda^{-1}(x, y) \text{ mod } 1.
\]

Hence \( b(\lambda) \) is well defined. Symmetry and bilinearity follow from these properties of \( \lambda^{-1} \). Thus, it is left to show that it is nonsingular. Let \( rx = \lambda^*(v) \) with \( v \) without loss of generality divisible. If \( v \) were divisible, it could be written as \( sv' = v \) with \( s \in \mathbb{Z} \) and \( v' \) indivisible. In this case set \( r = rs \) and \( v = v' \). Assume that \( b([x], [y]) \equiv 0 \mod 1 \) for all \([y] \in Cok(\lambda^*) \). Thus, \( \frac{y(v)}{r} \in \mathbb{Z} \) for all \( y \in H^* \). Since we assumed \( v \) indivisible, there is \( y \in H^* \) such that \( y(v) = 1 \). This implies that \( r = \pm 1 \) and thus \( [x] = 0 \).

**Remark 3.4.3.** There is also an algebraic procedure to obtain a linking function from general quadratic functions [Cro02, p.28f].
It is not hard to see that \( b(\lambda_0) \oplus b(\lambda_1) = b(\lambda_0 \oplus \lambda_1) \) and that isometric bilinear forms give isometric linking forms.

**Definition 3.4.4** ([Cro02, Definition 2.22, p.31]). Let \( P \) be a highly connected rational homology 15-sphere and let \( L \) be a handlebody such that \( \partial L \cong P \# \Sigma_r \), then we define

\[
\lambda(L) := b(\kappa(L)).
\]

This is well defined up to isometry because if we have two handlebodies \( L_0 \) and \( L_1 \) cobounding \( P \# \Sigma_r \) then there are nonsingular quadratic functions \( \nu_i : H_i \times H_i \to \mathbb{Q}/\mathbb{Z} \) such that \( \kappa(L_0) \oplus \nu_0 \cong \kappa(L_1) \oplus \nu_1 \) and hence \( \lambda(L_0) \oplus \lambda_0 \cong \lambda(L_1) \oplus \lambda_1 \). Since the \( \lambda_i \) are nonsingular, \( \text{Cok}(\lambda^*_i) \cong 0 \) and hence \( b(\lambda(L_0)) \cong b(\lambda(L_0) \oplus \lambda_0) \cong b(\lambda(L_1) \oplus \lambda_0) \cong b(\lambda(L_1)) \).

**Remark 3.4.5.** There are several equivalent definitions of the linking form of an odd dimensional manifold. Compare for example [KM63, p.524], where the intersection pairing and the Bockstein sequence associated to \( 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \) are used.

We will call the triple \( W(\kappa) := (\text{Cok}(\lambda^*), b(\lambda), [\alpha]) \) the Wilkens invariant of \( \kappa(H, \lambda, [\alpha]) \). Two triples are isomorphic if there is an isomorphism respecting the linking forms and the \([\alpha]\). We denote the isometry classes by \([\ast]\). We define the Wilkens invariant of a highly connected rational homology 15-sphere to be the isometry class of the triple \([W(\kappa(P))]\).

Wilkens used the invariant \( W \) to obtain partial results on the classification of highly connected closed 15-manifolds \( P \), which we will state here for completeness. We call a linking form \( b : H \to \mathbb{Q}/\mathbb{Z} \) indecomposable if it is not isometric to the sum \( b_0 \oplus b_1 \) of two linking forms \( b_0 : H_0 \times H_0 \to \mathbb{Q}/\mathbb{Z} \) and \( b_1 : H_1 \times H_1 \to \mathbb{Q}/\mathbb{Z} \) with \( H_0 \oplus H_1 \cong H \).

A manifold \( P \) is called indecomposable if its linking form is indecomposable. Wilkens shows that a manifold \( P \) splits into the connected sum of indecomposable manifolds corresponding to the splitting of its linking form. He uses this to obtain the following theorem:

**Theorem 3.4.6** (Compare [Wil72, Theorem 2 and 3]). If the order of \( H^8(P) \) is odd, then the Wilkens invariant is a complete almost diffeomorphism invariant. If \( P \) is indecomposable and \( H^8(P) \) is even, then there are at most two almost diffeomorphic manifolds with the same Wilkens invariants.

**Remark 3.4.7.** Actually, a slightly larger class of manifolds \( P \) is classified by their Wilkens invariant, but we keep it this way for simplicity. Theorem 3.4.6 is also true for highly connected closed 7 manifolds after adapting the invariants.

### 3.5 Quadratic linking functions

In this section we want to give the relevant definitions and state some lemmas regarding quadratic linking functions. Quadratic linking functions will turn out to be the required algebraic objects to finish the classification of stable characteristic quadratic functions. Let \( G \) denote a finite abelian group.
**Definition 3.5.1** ([Cro02, Definition 2.29, p.34]). A quadratic linking function \( q : G \to \mathbb{Q}/\mathbb{Z} \) is a function such that

\[
b(q) : G \times G \ni (x, y) \mapsto q(x + y) - q(x) - q(y) \in \mathbb{Q}/\mathbb{Z}
\]

is a linking form. We say that \( q \) is a quadratic refinement of \( b(q) \).

Now let \( \kappa = \kappa(H, \lambda, \alpha) \) be a nondegenerate characteristic function. We can define a quadratic linking function on \( \text{Cok}(\lambda^*) \) using \( \kappa \).

**Definition 3.5.2** ([Cro02, Definition 2.29, p.34]). For a nondegenerate characteristic quadratic function \( \kappa = \kappa(H, \lambda, \alpha) \) we define:

\[
q^\kappa(x) : \text{Cok}(\lambda^*) \ni [x] \mapsto \frac{\lambda^{-1}(x, x) + \lambda^{-1}(x, \alpha)}{2} \in \mathbb{Q}/\mathbb{Z}
\]

**Lemma 3.5.3** ([Cro02, Lemma 2.34 2., p.37]). \( q^\kappa(x) \) is a well defined quadratic refinement of \( b(\lambda) \).

**Proof.** We need to show that \( q^\kappa(x) \) is independent of the representative \( x \in [x] \) and that \( b(q^\kappa(x)) = b(\lambda) \). So let \( x + \lambda^*(z) \in [x] \) with \( z \in H \) be a different representative.

\[
\frac{\lambda^{-1}(x + \lambda^*(z), x + \lambda^*(z)) + \lambda^{-1}(x + \lambda^*(z), \alpha)}{2} = \frac{\lambda^{-1}(x, x) + \lambda^{-1}(x, \alpha)}{2} + \frac{2\lambda^{-1}(x, \lambda^*(z)) + \lambda^{-1}(\lambda^*(z), \lambda^*(z)) + \lambda^{-1}(\lambda^*(z), \alpha)}{2}
\]

\[
\equiv \frac{\lambda^{-1}(x, x) + \lambda^{-1}(x, \alpha)}{2} + \frac{\lambda(z) + \alpha(z)}{2} \equiv \frac{\lambda^{-1}(x, x) + \lambda^{-1}(x, \alpha)}{2} \mod 1,
\]

since \( \lambda(z) + \alpha(z) \) is always even.

\[
b(q^\kappa)([x], [y]) \equiv \frac{\lambda^{-1}(x + y, x + y) + \lambda^{-1}(x + y, \alpha)}{2} - \frac{\lambda^{-1}(x, x) + \lambda^{-1}(x, \alpha)}{2} - \frac{2\lambda^{-1}(x, y) + \lambda^{-1}(y, \alpha)}{2} \equiv -\frac{\lambda^{-1}(x, y)}{2} \equiv b(\lambda)([x], [y]) \mod 1
\]

\[\square\]

**Lemma 3.5.4** ([Cro02, Compare Lemma 2.30, p.34]). Let \( b : G \to \mathbb{Q}/\mathbb{Z} \) be a linking form. Then there is a quadratic linking form \( q_0 \) refining it such that \( q_0(x) = q_0(-x) \) and if the order of \( G \) is odd then \( q \) is unique.

For a linking function \( b \) on \( G \) we define \( Q(b) \) to be the set of quadratic linking functions refining \( b \).

**Lemma 3.5.5** (compare [Cro02, Lemma 2.34, p.37]). Consider that \( q \in Q(b) \). Then for any \( a \in G \)

\[
q_a(x) := q(x) + b(x, a)
\]

defines a quadratic linking form refining \( b \). If \( q = q_0^0 \), where \( q_0^0 \) is such that \( q_0^0(x) = q_0^0(-x) \ \forall x \in G \), then \( \beta(q) := 2a \) is independent of \( q_0^0 \) and thus an invariant of \( q \). Moreover, the action of \( G \) on \( Q(q) \) defined above is free and transitive.
We call two linking forms \( q_0 : G_0 \to \mathbb{Q}/\mathbb{Z} \) and \( q_1 : G_1 \to \mathbb{Q}/\mathbb{Z} \) isometric if there is an isomorphism \( \theta : G_0 \to G_1 \) such that \( q_1 \circ \theta = q_0 \). Denote the isometry classes by \([*]\).

Now we come to the theorem that motivates our interest in quadratic linking functions.

**Theorem 3.5.6** (compare [Cro02, Theorem A, p.5]). Let \( \kappa \in \mathcal{F}^c \). \([q^c(\kappa)]\) is a complete invariant of \([\kappa] \in \mathcal{F}^c \).

**Remark 3.5.7.** This theorem is not stated in this form by Crowley, but it follows from [Cro02, Theorem A, p.5] using Corollary 3.3.5. There is a generalization of quadratic linking functions: quadratic linking families. They are needed for the classification of general (not necessarily nondegenerate) stable quadratic functions and are also the language used in the proof.

### 3.6 The classification of quadratic linking functions

The last theorem again translates one algebraic problem to another, but this time Crowley is able to solve it. We are going to state the classification of quadratic linking functions and we will have a look at its topological interpretation.

The first isometry invariant of a quadratic linking function \( q : G \to \mathbb{Q}/\mathbb{Z} \) we need is the linking form \( b(q) \), which it refines. By 3.5.4 and 3.5.5 we always have a second invariant \( \beta(q) \). We will call \((G, b(q), \beta(q))\) the Wilkens invariant of a quadratic linking function. Two such triples are isometric if there is an isomorphism of the groups respecting the linking form and mapping the \( \beta \) to each other. We denote the isometry classes by \([*]\).

Note that for a characteristic \( \kappa = \kappa(H, \lambda, \alpha) \) we get \( \beta(q^c(\kappa)) = [\alpha] \) because

\[
q^c(\kappa)([x]) = \frac{\lambda^{-1}(x, x)}{2} + \frac{\lambda^{-1}(x, \alpha)}{2} = \frac{b(\lambda)([x], [x])}{2} + \frac{b(\lambda)([x], [\alpha]/2)}{2}
\]

and \( b(\lambda)([x], [x])/2 \) is a quadratic refinement of \( b(\lambda) \) such that

\[
b(\lambda)([x], [x])/2 = b(\lambda)(-[x], -[x])/2.
\]

We also used that \( \beta(\kappa) = [\alpha] \) is always even (see [Wil71, Lemma 1.2, p.23]). Thus, we have just identified the Wilkens invariants of a quadratic function and its quadratic refinement.

**Lemma 3.6.1** ([Cro02, Lemma 5.21, p.89]). Let \( q : G \to \mathbb{Q}/\mathbb{Z} \) be a quadratic linking function. If the order of \( G \) is odd, then \([G, b(q), \beta(q)]\) is a complete isometry invariant.

**Remark 3.6.2.** Lemma 3.6.1 proofs the first statement of Wilkens’ classification result 3.4.6.

For the classification of general quadratic linking forms we need another invariant. We define the Gauss-sum invariant of a quadratic linking function \( q \) as

\[
GS(q) := \sum_{x \in G} \exp(2\pi i q(x)) \in \mathbb{C}
\]
and the Kervaire-Arf invariant as

\[ K(q) := \frac{\text{Arg}(GS(Q))}{2\pi}. \]

**Theorem 3.6.3** (compare [Cro02, Theorem 5.22, p.90]). Let \( q_0, q_1 : G \to \mathbb{Q}/\mathbb{Z} \) be two quadratic linking functions. They are isometric if and only if

\[ (G, b(q_0), \beta(q_0)) \cong (G, b(q_1), \beta(q_1)) \text{ and } K(q_0) = K(q_1). \]

Crowley calculates the Kervaire-Arf invariant of a quadratic linking function \( q^c(\kappa) \) coming from a quadratic function \( \kappa = \kappa(H, \lambda, \alpha) \). It turns out that

\[ -K(q^c(\kappa)) \equiv \frac{\lambda - 1(\alpha, \alpha) - \sigma(\lambda)}{8} \mod 1, \tag{3.3} \]

where \( \sigma \) denotes the signature [Cro02, Proposition 5.19, p.88].

**Remark 3.6.4.** By [Cro02, Lemma 2.36, p.38] every quadratic linking function is isometric to a \( q^c(\kappa) \) for a characteristic \( \kappa = \kappa(H, \lambda, \alpha) \). Hence, we could use (3.3) for all quadratic linking functions.

One may ask if there is a topological interpretation of the Kervaire-Arf invariant. Indeed there is one. We start with the definition of a diffeomorphism invariant, called the Eells-Kuiper \( \mu \)-invariant [EK62]. It is a refinement of Milnor’s \( \lambda \)-invariant, which he used for the discovery of the exotic sphere [Mil56]. It is defined for closed \((4k-1)\)-manifolds \( M \) satisfying the following conditions:

- There is a compact spin manifold \( W \) with boundary \( \partial W \cong M \).
- Let \( j^* \) be the map in the long exact sequence of the pair \( (W, M) \) then:

  \[ j^* : H^{2k}(W, M; \mathbb{Q}) \to H^{2k}(W; \mathbb{Q}) \]

  \[ j^* : H^{2i}(W, M; \mathbb{Q}) \to H^{2i}(W; \mathbb{Q}) \]

  for \( 0 < i < k \) are isomorphisms.
- Let \( i : M \hookrightarrow W \) be the inclusion. Then

  \[ i^* : H^2(W; \mathbb{Z}/2) \to H^2(M; \mathbb{Z}/2) \]

  is surjective.

The second condition enables us to pull back the Pontryagin classes in a unique way. We denote by \( \tilde{p}_i(W) := (j^*)^{-1}(p_i(W) \otimes \mathbb{Q}) \).

**Definition 3.6.5** ([EK62, p.97]). Let \( M \) and \( W \) be as above then we define:

\[ \mu(W, M) := \frac{(N_k(\tilde{p}_1, ..., \tilde{p}_{k-1})(W), [W, \partial W]) + t_k\sigma(W)}{a_k} \in \mathbb{Q}/\mathbb{Z}, \]

where \( a_k \) is 1 for \( k \) even and 2 for \( k \) odd and \( \sigma \) is the signature. \( N_k \) is a certain polynomial with rational coefficients evaluated at the \( \tilde{p}_i(W) \) with the cup product derived from the so called \( A- \) and \( L \)-genus and \( t_k \) is a rational number.
Eells and Kuiper show that \( \mu \) does not depend on the chosen manifold \( W \). Thus, we denote it by \( \mu(M) := \mu(M, W) \). Moreover, they show that if \( M_1, M_2 \) are two manifolds satisfying the conditions, then \( M_1 \# M_2 \) also satisfy the conditions and \( \mu(M_1 \# M_2) = \mu(M_1) + \mu(M_2) \in \mathbb{Q}/\mathbb{Z} \) and \( \mu(-M_1) = -\mu(M_1) \). They also do calculations and show that in our case \( \mu \) is given by:

**Proposition 3.6.6 ([EK62, p.14]).** Let \( M \) be a 15-manifolds satisfying the conditions with coboundary \( W \), then

\[
\mu(M) = \frac{(12096p_3p_1 + 5040p_2^2 - 22680p_2p_1^2 + 9639p_1^4)(W)[W, \partial W] - 181440\sigma(W)}{2^{15}.3^4.5^2.7.127} \in \mathbb{Q}/\mathbb{Z}
\]

Note that the conditions are satisfied for highly connected rational homology 15-spheres bounding a handlebody. For a highly connected rational homology 15-spheres \( P \) with \( \Sigma_P \not\cong S^{15} \) we get by 3.1.11 a manifold \( W \) bounded by \( \Sigma_P \) satisfying the conditions. Thus, by additivity of the \( \mu \) invariant it is also defined for \( P \). We are now going to calculate it.

**Proposition 3.6.7** (compare [Cro02, Proposition 2.57, p.53]). Let \( P \) be a highly connected rational homology 15-sphere and let \( L \) be a handlebody such that \( \partial L \cong P \# \Sigma_P \). Then

\[
\mu(P) = \frac{1}{8128} \frac{\lambda^{-1}(\alpha(L), \alpha(L)) - \sigma(\lambda(L))}{8} \in \mathbb{Q}/\mathbb{Z}.
\]

**Proof.** If \( P \) bounds a handlebody \( L \), we can use \( L \) to calculate \( \mu(P) \). The only non-trivial Pontryagin class is given by \( p_2(L) = 6\alpha(L) \). Thus, we only have to calculate

\[
\langle (j^*)^{-1}(p_2(L)) \cup (j^*)^{-1}(p_2(L)), [L, \partial L] \rangle = \langle (j^*)^{-1}(p_2(L)) \cup p_2(L), [L, \partial L] \rangle
\]

\[
= \langle (j^*)^{-1}(p_2(L)), p_2(L) \cap [L, \partial L] \rangle = \langle p_2(L), (j_*)^{-1}PD^{-1}(p_2(L)) \rangle
\]

\[
= \lambda^{-1}(p_2(L), p_2(L)) = 36\lambda^{-1}(\alpha(L), \alpha(L)).
\]

Now using 3.6.5 we get:

\[
\mu(P) = \frac{36 \cdot 5040\lambda^{-1}(\alpha(L), \alpha(L)) - 181440\sigma(L)}{2^{15}.3^4.5^2.7.127} = \frac{1}{8128} \frac{\lambda^{-1}(\alpha(L), \alpha(L)) - \sigma(\lambda(L))}{8}.
\]

When \( \Sigma_P \not\cong S^{15} \), we use that there is by 3.1.11 a manifold \( W \) cobounding \( \Sigma_P \) satisfying the conditions and decomposable Pontryagin numbers zero. Thus, \( \mu(\Sigma_P) = \sigma(W)/(8 \cdot 8128) \). Since \( \sigma(W)/(8 \cdot 8128) \equiv 0 \mod \mathbb{Z} \) and only if \( \sigma(W)/8 \equiv 0 \mod 8128 \) and \( s(\Sigma_P) = \sigma(W)/8 \equiv 0 \mod 8128 \), we get \( \mu(\Sigma_P) = 0 \). Hence, \( \mu(P) = \mu(\partial L) \) for an \( L \) with \( \partial L \cong \Sigma_P \# P \).

**Remark 3.6.8.** \( \mu \) can actually distinguish all elements of \( bP_{16} \). We know that \( M_{1,1} \in bP_{16} \) and we will see that \( \mu(M_{1,1}) = \frac{1}{8128} \in \mathbb{Q}/\mathbb{Z} \). Thus, by successively adding \( M_{1,1} \) we get 8128 different elements of \( bP_{16} \).
How do we get an almost diffeomorphism invariant? Using 3.1.11 we see that for $\Sigma \in \theta_{15}$ there exists a manifold $W$ satisfying the conditions with vanishing decomposable Pontryagin numbers and $8|\sigma(W)$. Thus, we get $|bP_{16}|\mu(\Sigma) = \frac{8128\sigma(W)}{8128} \equiv 0 \mod 1$ and thus $\bar{\mu} := |bP_{16}|\mu$ is an almost diffeomorphism invariant because $\mu$ is additive.

Since $\bar{\mu}(P) = \lambda^{-1}(\alpha(L),\alpha(L)) - \sigma(\lambda(L))$, we see using 3.3 that the $\bar{\mu}$ and $KS$ give the same information. To conclude this section we gather everything together in the following corollary:

**Corollary 3.6.9** (compare [Cro02, Theorem A, p.5]). Let $P_0, P_1$ be highly connected rational homology 15-spheres. $P_0$ is almost diffeomorphic to $P_1$ if and only if

$$[(H^8(P_0),b(P_0),\beta(P_0))] = [(H^8(P_1),b(P_1),\beta(P_1))] \quad \text{and} \quad \bar{\mu}(P_0) = \bar{\mu}(P_1).$$

**Remark 3.6.10.** Actually, Crowley uses the Stolz invariant $s_1 ([Sto87])$. Stolz defines the invariant only for homotopy spheres, but Crowley generalizes it. The $\mu$-invariant and the Stolz invariant coincide in dimension 15 [Cro02, p.53].

### 3.7 The diffeomorphism classification

In some sense the stable quadratic function of a highly connected closed 15-manifold $[\kappa(P)]_{Fc}$ is an almost diffeomorphism invariant because we allow the addition of handlebodies bounded by homotopy spheres. When we restrict to the addition of handlebodies bounded by the standard sphere, we actually obtain a diffeomorphism invariant.

**Definition 3.7.1** ([Cro02, p.79]). We call a nonsingular characteristic quadratic function $\nu$ spherical if $L(\nu)$ is bounded by the standard sphere.

As we have seen in the proof of Theorem 3.1.12 all homotopy spheres bounding handlebodies lie in $bP_{16}$. Thus, the $\mu$-invariant suffices to tell us when the homotopy sphere is actually the standard sphere:

**Lemma 3.7.2** ([Cro02, Lemma 4.7, p.79]). A nonsingular characteristic quadratic function $\nu$ is spherical if and only if $\mu(L(\nu)) \equiv 0 \mod 1$.

This leads to the definition of smooth equivalence of quadratic functions.

**Definition 3.7.3** ([Cro02, p.79]). Two characteristic quadratic functions $\kappa_0, \kappa_1$ are called smoothly equivalent if there are spherical quadratic functions $\nu_0, \nu_1$ such that $\kappa_0 \oplus \nu_0 \cong \kappa_1 \oplus \nu_1$. This is an equivalence relation and we denote the equivalence classes by $[\nu]_{sFc}^\kappa$. For a highly connected rational homology 15-sphere $P$ we call $[\kappa(P)]_{sFc}$ the smooth quadratic function of $P$.

Now it is easy to see that two highly connected closed 15-manifolds $P_0, P_1$ are diffeomorphic if their smooth quadratic functions agree and $\Sigma_{P_0} \cong \Sigma_{P_1}$. The converse is also true:
3.8 Application to the classification of total spaces of $S^7$-bundles over $S^8$

We need to calculate the linking form, the stable tangential invariants and Stolz invariant $\mu$. Recall that we assumed $n > 0$.

**Lemma 3.8.1.** The linking form of $M_{m,n}$ is given by:

$$b(M_{m,n}) : H^8(M_{m,n}) \times H^8(M_{m,n}) \cong \mathbb{Z}/n \times \mathbb{Z}/n \ni ([x], [y]) \mapsto \frac{xy}{n} \in \mathbb{Q}/\mathbb{Z}$$

**Proof.** We will show that the fundamental sequence of $\kappa(L_{m,n})$ is given by

$$
\begin{array}{cccccc}
0 & \longrightarrow & H_8(L_{m,n}) & \xrightarrow{\lambda(L_{m,n})^*} & H_8(L_{m,n})^* & \longrightarrow & \mathrm{Coker}(\lambda(L_{m,n})^*) & \longrightarrow & 0 \\
& & \cong & & \cong & & & & \\
0 & \longrightarrow & H_8(L_{m,n}) & \xrightarrow{PD_{m,n}} & H^8(L_{m,n}) & \xrightarrow{\iota^*} & H^8(M_{m,n}) & \longrightarrow & 0 \\
& & \downarrow_{v \mapsto 1} & & \downarrow_{\pi_{PD_{m,n}}^*} & & \downarrow_{\pi_{PD_{m,n}}^* \circ v \mapsto 1} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{n} & \mathbb{Z} & \xrightarrow{1 \mapsto [1]} & \mathbb{Z}/n & \longrightarrow & 0,
\end{array}
$$

\[\text{Remark 3.7.5.}\] Of course $[\ast]^{\varphi_{\ast}}$ is well defined. As in the proof of 3.3.11 set $P_0 = P_1$ and repeat the last part of the proof.

**Theorem 3.7.6** (compare [Cro02, Theorem 4.9, p.79 and Theorem A, p.5]). Two highly connected rational homology 15-spheres $P_0, P_1$ are diffeomorphic if and only if

$$[H^8(P_0), b(P_0), \beta(P_0)] = ([H^8(P_1), b(P_1), \beta(P_1)]),\quad \mu(P_0) = \mu(P_1) \quad \text{and} \quad \Sigma_{P_0} \cong \Sigma_{P_1}.$$

**Proof.** Let $P_0, P_1$ be diffeomorphic. Then $P_0 \# \Sigma_{P_0}$ is diffeomorphic to $P_1 \# \Sigma_{P_0}$. Thus, $P_1 \# \Sigma_{P_0}$ bound a handlebody and hence $\Sigma_{P_0} \cong \Sigma_{P_1}$. Denote by $L_0$ and $L_1$ handlebodies bounded by $P_0 \# \Sigma_{P_0}$ respectively by $P_1 \# \Sigma_{P_0}$. Using the extension of diffeomorphism Theorem 3.3.10 we get handlebodies $V_0, V_1$ bounded by the standard sphere and a diffeomorphism $g : L_0 \# V_0 \to L_1 \# V_1$. Thus, $\kappa(L_0) \oplus \kappa(V_0) \cong \kappa(L_1) \oplus \kappa(V_1)$, where $\kappa(V_i)$ are spherical quadratic functions. Thus, $[\kappa(P_0)]^{\varphi_{\ast}} = [\kappa(P_1)]^{\varphi_{\ast}}$. \(\square\)
where $\iota$ denotes the generator getting mapped to the standard generator $[S^8]$ of $H_8(S^8)$ via $\pi_{D_{m,n}}^\ast$. We need to calculate $\lambda^\ast$. It suffices to calculate $\lambda^\ast(\iota)(\iota) = \langle PD(\iota) \cup PD(\iota), [L_{m,n}, \partial L_{m,n}] \rangle$ because the rest follows by bilinearity. $PD(\iota) = \bar{u}$, where $\bar{u}$ was the image of the Thom class $u \in H^8(E_{m,n}, E^0_{m,n})$. Thus, we know that $H^8(L_{m,n}) \xrightarrow{\cup \bar{u}} H^8(L_{m,n}, \partial L_{m,n})$ is an isomorphism and $j^\ast \bar{u} = n\pi_{D_{m,n}}^\ast \zeta$ (since $j^\ast u = \pi^\ast e(C_{m,n})$, where $j^\prime : (E_{m,n}, \emptyset) \hookrightarrow (E_{m,n}, E^0_{m,n})$ is the inclusion). So we can calculate

$$
\langle \bar{u} \cup \bar{u}, [L_{m,n}, \partial L_{m,n}] \rangle = n\langle \pi_{D_{m,n}}^\ast \zeta \cup \bar{u}, [L_{m,n}, \partial L_{m,n}] \rangle = n \langle \pi_{D_{m,n}}^\ast \zeta \cup \bar{u}, [L_{m,n}, \partial L_{m,n}] \rangle = n,
$$

by the choice of orientation on $L_{m,n}$ (see p.7).

Now we can calculate $b(M_{m,n})(\pi_{B_{m,n}}^\ast \zeta, \pi_{B_{m,n}}^\ast \zeta) = \langle \pi_{D_{m,n}}^\ast \zeta, \iota \rangle / n = 1 / n$ since $\langle \pi_{D_{m,n}}^\ast \zeta, \iota \rangle = \langle \zeta, [S^8] \rangle = 1$.

**Lemma 3.8.2.** The stable tangential invariants of $L_{m,n}$ and $M_{m,n}$ are given by:

$$
\alpha(L_{m,n}) = \pm (2m + n)\pi_{D_{m,n}}^\ast \zeta \in H^8(L_{m,n}) \quad \text{and} \quad \beta(M_{m,n}) = \pm 2m\pi_{B_{m,n}}^\ast \zeta \in H^8(M_{m,n})
$$

**Proof.** Using obstruction theory we see that $C_{m,n}$ has a $O(T)$-structure. Then use the in 3.2.3 established fact that $\alpha$ is a characteristic class of a stable $O(T)$-bundle. Since we may devide by 6 in $H^8(S^8)$, using 3.2.1 we see that $\alpha(C_{1,0}) = \pm \zeta$. Now doing the same as in the calculation of the Pontryagin classes, where we only used the fact that Pontryagin classes are stable characteristic classes, we get the desired results.

To calculate the $\mu$-invariant we use the coboundary $L_{m,n}$. The only thing left to calculate is the signature of $L_{m,n}$. The signature is defined to be the signature of the quadratic form $H_8(L_{m,n}, \mathbb{Q}) \ni x \mapsto \langle x \cup x, [L_{m,n}, \partial L_{m,n}] \rangle \in \mathbb{Q}$.

Since we know $\lambda(L_{m,n})(\iota, \iota) = n$, the signature is given by $+1$. Thus $\mu$ is:

$$
\mu(M_{m,n}) = \frac{1}{8128} \frac{1}{8n}((2m + n)^2 - n) \in \mathbb{Q}/\mathbb{Z}
$$

Now we have everything we need to proof the following theorem:

**Theorem 3.8.3** (Main Theorem I).

1. $M_{m,n}$ and $M_{m',n}$ are almost diffeomorphic if and only if $2m \equiv \epsilon 2m' \mod n$, where $\epsilon$ is such that $\epsilon^2 \equiv 1 \mod n$ and $\bar{\mu}(M_{m,n}) = \bar{\mu}(M_{m',n})$.

2. $M_{m,n}$ and $M_{m',n}$ are diffeomorphic if and only if they are almost diffeomorphic and $\mu(M_{m,n}) = \mu(M_{m',n})$. 

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Proof. Recall that Wilkins invariants of $M_{m,n}$ and $M_{m',n}$ agree if there is an isomorphism $\theta : H^7(M_{m,n}) \to H^7(M_{m',n})$ respecting the linking form and mapping $\beta(M_{m,n})$ to $\beta(M_{m',n})$. The isomorphisms respecting the standard linking form on $\mathbb{Z}/n$ are given by multiplication by $\epsilon \in \mathbb{Z}/n$, where $\epsilon^2 \equiv 1 \mod n$. This means that $\theta$ has to be $H^7(M_{m,n}) \cong \mathbb{Z}/n \xrightarrow{\sim} \mathbb{Z}/n \cong H^7(M_{m',n})$, where we identify via the generator $\pi^* \zeta \in H^7$. For a multiplication by $\epsilon$ to respect the tangential invariant we need $2m \equiv \epsilon 2m' \mod n$. The uncertainty about the sign of $\beta$ does not matter since $((-1)\epsilon)^2 \equiv 1 \mod n$. The rest follows by 3.6.9 and 3.7.6.

Actually, there is a little improvement of the presentation of the theorem. In [CE03, p.371] it is shown that ”$2m \equiv \epsilon 2m' \mod n$” can be replaced by ”$m \equiv \epsilon m' \mod n$.” This is purely algebraic and since we are not going to need this we will skip it.

Remark 3.8.4. Exactly the same theorem, modulo the improvement of presentation, is true for $S^3$-bundles over $S^4$ (see [CE03, Theorem 1.2 and 1.5]). Actually, Crowley and Escher consider homeomorphisms instead of almost diffeomorphisms, but they used the fact that for highly connected 7-manifolds the notions agree. Instead of Crowley’s classification they use Wilkens’ result 3.3.10 and show that $\bar{\mu}$ can solve the ambiguities.

Corollary 3.8.5. 4096 exotic 15-spheres are realized as total spaces of $S^7$-bundles over $S^8$.

Proof. We already know that the manifolds $M_{m,1}$ are homeomorphic to the sphere and that for $\Sigma \in bP_{16}$ we have $\mu(\Sigma) = \frac{k}{8128}$, where $k \in \mathbb{Z}$. Thus, we only need to count the number of $k \in \mathbb{Z}/|bP_{16}|$ such that there is an $m$ with $\mu(M_{m,0}) = \frac{m(m+1)-1}{8128} \equiv \frac{k}{8128} \mod 1$. □

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4 On the homeomorphism classification

In this chapter we will obtain partial results on the homeomorphism classification of highly connected rational homology spheres and thus also on the homeomorphism classification of total spaces of $S^7$-bundles over $S^8$. We use a PL-invariant and the fact that the topological and PL classification coincide.

4.1 Construction and calculation of a PL-invariant

In this section $M$ will denote a highly connected closed 15-manifold. Recall that a smooth manifold has a unique PL-structure via the Whitehead triangulation. We are going to construct and calculate an invariant of the PL-structure. When we want to have something similar to the tangential invariant $\beta$, we need a "PL-tangent bundle."

To define this we need the notion of PL-microbundles. A good introduction to the theory of PL-microbundles is [Mil61]. Sources for the information on the classifying spaces are [KS77, §9. p.187ff] and [LR65].

Definition 4.1.1. Let $B$ and $E$ be locally finite simplicial complexes and $i$ and $j$ PL-maps. We call a diagram

$$\xi : B \overset{i}{\to} E \overset{j}{\to} B$$

PL-microbundle of dimension $n$ if the following holds: For all $b \in B$ there are neighborhoods $B_0$ of $b$ and $E_0$ of $i(b)$ and a PL-homeomorphism $h : E_0 \to B_0 \times \mathbb{R}^n$ such that the following diagram commutes:

$$
\begin{array}{ccc}
B_0 & \overset{i|B_0}{\longrightarrow} & E_0 \\
\downarrow & & \downarrow h \\
B_0 \times \mathbb{R}^n & \overset{p_1}{\longrightarrow} & B_0 \\
\end{array}
$$

where $\times 0$ is the inclusion at 0 and $p_1$ the projection onto the first factor.

We call $B$ the base of $\xi$, $E$ the total space, $i$ the injection map or zero section and $j$ the projection map. A first example for a PL-microbundle is a smooth $n$-vector bundle $E \overset{j}{\to} B$ with the zero section. As neighborhoods $B_0$ we just take the coordinate neighborhoods and $E_0 := j^{-1}(B_0)$. The fact that in general we do not consider the whole preimage under $j$, but a possibly smaller neighborhood, is a first motivation for
the name microbundle. Another example is the so called trivial PL-microbundle given by:

\[ \ell^n : B \xrightarrow{x_0} B \times \mathbb{R}^n \xrightarrow{p_1} B \]

We now need to tell when we consider PL-microbundles to be the same.

**Definition 4.1.2.** We call two PL-microbundles \( \xi : B \xrightarrow{i} E \xrightarrow{j} B \) and \( \xi' : B \xrightarrow{i'} E' \xrightarrow{j'} B \) isomorphic if there exist neighborhoods \( E_1 \) of \( i(B) \) and \( E_1' \) of \( i'(B) \) and a PL-homeomorphism \( h \), such that the following diagram commutes:

\[ \begin{array}{ccc}
E_1 & \xrightarrow{j} & E_1' \\
\downarrow{h} & & \downarrow{j'} \\
B & \xrightarrow{i} & B \\
\end{array} \]

Another motivation for the name microbundle is the fact that not the total spaces but just neighborhoods of the zero section need to be PL-homeomorphic for the PL-microbundles to be isomorphic.

For two PL-microbundles \( \xi : B \xrightarrow{i} E \xrightarrow{j} B \) and \( \xi' : B \xrightarrow{i'} E' \xrightarrow{j'} B \) there is also a Whitney sum as for vector bundles.

\[ \xi \oplus \xi' : B \xrightarrow{(i,i')} \bar{E} \xrightarrow{j} B, \]

where \( \bar{E} := \{(x,y) \in E \times E' : j(x) = j'(y)\} \) and \( j(x,y) = j(x) \). Another construction we need is the pullback bundle. Let \( B' \xrightarrow{f} B \) be a PL-map then we call

\[ f^* \xi : B' \xrightarrow{i'} E' \xrightarrow{p_1} B' \]

the pullback bundle, where \( E' := \{(x,y) \in B \times E : j(x) = j'(y)\} \) and \( i'(b) = (b, i \circ f(b)) \).

For a PL-manifold \( M \) we now can define the tangent PL-microbundle.

\[ \tau_{PL}(M) : M \xrightarrow{\Delta} M \times M \xrightarrow{p_1} M, \]

where \( \Delta : M \ni x \mapsto (x,x) \in M \times M \) denotes the diagonal map and \( p_1 \) the projection on the first factor. This is actually a PL-microbundle (it suffices to show it for \( M = \mathbb{R}^n \)) and the PL-microbundle isomorphism type only depends on the PL-homeomorphism type of the PL-manifold.

Now let \( M \) be a smooth manifold. Identify \( M \) with the diagonal \( \Delta(M) \subset M \times M \) via \( p_1 \). The normal bundle of \( \Delta(M) \) is isomorphic to the tangent bundle of \( M \). Since the normal bundle of \( \Delta(M) \) is diffeomorphic to a tubular neighborhood by the tubular neighborhood theorem, we see that the tangent bundle becomes the PL-tangent microbundle when we consider it as a PL-microbundle.

For PL-microbundles there is also a classifying space and an universal bundle:
Theorem 4.1.3 (Universal bundle Theorem [Mil61]). There is a PL-microbundle

\[ \gamma_{PL}^n : BPL(n) \to EPL(n) \to BPL(n) \]

such that every PL-microbundle of dimension \( n \) is the pullback via an up to homotopy unique map.

Sadly the proof is not as nice as in the case of vector bundles, where one has the Grassmannian. It uses semi simplicial complexes and can be found in [Mil61].

Also using semi simplicial methods one finds a complex model for \( BO(n) \). Now considering the universal bundle as PL-microbundle we get an up to homotopy unique map \( \pi_n : BO(n) \to BPL(n) \). We also get maps \( i_n : BPL(n) \to BPL(n+1) \). We take \( i_n \) to be the classifying map of \( \gamma \oplus e^1 \). Similar maps we also have for \( BO(n) \) and we denote them by \( j_n \). The following diagram is commutative up to homotopy [KS77, p.190]:

\[ \begin{array}{cccccc}
\cdots & \longrightarrow & BO(n-1) & \overset{j_{n-1}}{\longrightarrow} & BO(n) & \overset{j_n}{\longrightarrow} & BO(n+1) & \longrightarrow & \cdots \\
\downarrow \pi_{n-1} & & \downarrow \pi_n & & \downarrow \pi_{n+1} & & & & \\
\cdots & \longrightarrow & BPL(n-1) & \overset{i_{n-1}}{\longrightarrow} & BPL(n) & \overset{i_n}{\longrightarrow} & BPL(n+1) & \longrightarrow & \cdots 
\end{array} \]  

(4.1)

Definition 4.1.4. We call two PL-microbundles \( \xi, \xi' \) over the same base \( B \) stably equivalent if there are \( m, n \in \mathbb{N} \) such that \( \xi \oplus e^m \) and \( \xi' \oplus e^n \) are isomorphic. We will call the equivalence classes stable PL-microbundles.

We now restrict for simplicity to compact bases \( B \). The stable PL-microbundles form a group, which we will denote by \( k_{PL}(B) \), with the Whitney sum. Set \( BPL := \bigcup_n BPL(n) \) with the topology induced by the subspaces \( BPL(n) \). If we have maps \( f, g : B \to BPL \), we can assume them to be contained in a \( BPL(n_i) \) \( i = 0, 1 \). The Whitney sum \( f^*\gamma_{n_0} \oplus g^*\gamma_{n_1} \) now gives us a map into \( BPL(n_0 + n_1) \subset BPL \). This defines a sum on \( [B, BPL] \). \( BPL \) is the classifying space of stable PL-microbundles in the sense that there is an isomorphism of groups \( [B, BPL] \cong k_{PL}(B) \).

From now on we stop caring about the universal bundles and are only interested in the homotopy type of the classifying spaces. After homotopies we can assume (4.1) to be commutative and use it to define a map \( j = \bigcup_n j_n : BO \to BPL \). We replace \( j : BO \to BPL \) by a fibration and call the fiber \( PL/O \).

Remark 4.1.5. These are the beginnings of smoothing theory. One can show that there is a smooth structure on a PL-manifold \( N \) if and only if there is a lift \( \tau \) of the classifying map of the stable tangent bundle \( \tau_{PL} \).

If there is a smoothing, the smooth structures on \( N \) are in correspondence with \( [N, PL/O] \).
The long exact sequence of homotopy groups of the fibration $PL/O \to BO \overset{j}{\to} BPL$ splits and in some cases pieces are calculated in [Bru68, Fra73]. We are going to need the following piece that can be found in [CZ08, p.9].

\[
\begin{array}{cccc}
0 & \longrightarrow & \pi_8(BO) & \overset{j_*}{\longrightarrow} \pi_8(BPL) & \longrightarrow & \pi_7(PL/O) & \longrightarrow & 0 \\
\Big\downarrow{\cong} & & \Big\downarrow{\cong} & & \Big\downarrow{\cong} & & \\
0 & \longrightarrow & \mathbb{Z} & \overset{(7,1)}{\longrightarrow} & \mathbb{Z} \oplus \mathbb{Z}/4 & \overset{(-1,7)^T}{\longrightarrow} & \mathbb{Z}/28 & \longrightarrow & 0
\end{array}
\]

Now let $M$ denote a highly connected closed 15-manifold. Denote by $Ob_O : [M, BO] \to H^8(M, \pi_8(BO))$ and $Ob_{PL} : [M, BO] \to H^8(M, \pi_8(BPL))$ the primary obstructions to null-homotopy. Note that $\pi_i(BO) \cong \pi_i(BPL)$ for $i < 8$. And since $M$ is highly connected, the first non zero obstruction could live in $H^7(M, \pi_7(BO))$, but since $\pi_7(BO) \cong 0$ it is in $H^8(M, \pi_8(BO))$. The same is true for $Ob_{PL}$. We can summarize the information we need in the following commutative diagram:

\[
\begin{array}{ccc}
[M, BO] & \overset{j_*}{\longrightarrow} & [M, BPL] \\
\downarrow{Ob_O} & & \downarrow{Ob_{PL}} \\
H^8(M, \pi_8(BO)) & \overset{j_{**}}{\longrightarrow} & H^8(M, \pi_8(BPL))
\end{array}
\]

Denote by $\tau(M)$ the classifying map of the stable tangent bundle and by $\tau_{PL}(M)$ the classifying map of the stable tangent PL-microbundle. It is now clear that $j_{**} Ob_O(\tau(M)) = Ob_{PL}(\tau_{PL}(M))$. Before we calculate $Ob_O(\tau(M))$ and hence $Ob_{PL}(\tau_{PL}(M))$ we give the definition of the invariant we want to use.

**Definition 4.1.6.** We define the stable tangential PL-invariant of a highly connected closed 15 PL-manifold $M$ to be

$$\gamma(M) := p_* Ob_{PL}(\tau_{PL}(M)) \in H^8(M),$$

where $p_*$ is the change of coefficient map induced by $p : \mathbb{Z} \oplus \mathbb{Z}/4 \in (x, y) \mapsto x \in \mathbb{Z}$.

**Remark 4.1.7.** We waste some information contained in $Ob_{PL}(\tau_{PL}(M))$, but for our application and also for possible further ones it suffices.

So far we talked about the primary obstruction to null-homotopy. The next lemma shows that it is in principle the same as the primary obstruction to triviality. Denote for a topological group $G$ by $\gamma_G : G \hookrightarrow EG \overset{\pi_G}{\longrightarrow} BG$ the universal bundle. Let $X$ be a finite CW-complex. If $G$ is $(n - 1)$-connected we have for any $f : X \to BG$ a section $s : X^{(n)} \to f^* \gamma_G|X^{(n)}$.

**Lemma 4.1.8.** Let $G$ be an $(n - 1)$-connected topological group and $X$ a finite simply-connected CW-complex. Then $Ob(f^* \gamma_G, s)$ and the obstruction to null-homotopy $Ob_G(f)$ agree after identifying $\pi_{n+1}(BG) \cong \pi_n(G)$ via the boundary map $\partial$ of the fibration $G \hookrightarrow EG \overset{\pi_G}{\longrightarrow} BG$. 

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Proof. Since the primary obstruction to triviality does not depend on the section, it suffices to show that there is a section such that the obstruction cocycles agree. We can assume \( f | X^{(n)} \equiv * \) since \( f^* \gamma | X^{(n)} \) is trivial, where \( * \in BG \) is the base point. Thus, we can think of \( f \) restricted to each \((n + 1)\)-cell \( e_i^{(n+1)} \) as a map of pairs \( f_i : (D_i^{n+1}, \partial D_i^{n+1}) \to (BG, *) \) and hence as an element of \( \pi_{n+1}(BG) \). The obstruction cocycle of the obstruction to null-homotopy is given by the map sending the generator corresponding to the cell \( e_i^{(n+1)} \) to \([f_i] \in \pi_{n+1}(BG)\). Using the homotopy extension property we find a lift \( F \) making the following diagram commutative:

\[
\begin{array}{ccc}
X^{(n+1)} \times \{0\} & \xrightarrow{\text{const}} & EG \\
\downarrow \text{incl} & & \downarrow \\
X^{(n+1)} \times I & \xrightarrow{F} & BG
\end{array}
\]

We can use \( F \) to define a section \( s : X^{(n)} \to f^*EG | X^{(n)} \) by \( s(x) = (x, F(x, 1)) \in \{(x, y) \in S^n \times EG | f(x) = \pi_G(y)\} = f^*EG | X^{(n)} \). Because we assumed \( f | X^{(n)} \) to be constant, \( f^*EG | \partial D_i^{n+1} = \partial D_i^{n+1} \times G \) for every cell. Thus, the obstruction cocycle of \( Ob(f^*\gamma_G, s) \) now is given by the map sending the generator corresponding to the cell \( e_i^{n+1} \) to \([F(x, 1) | \partial D_i^{n+1}] \in \pi_n(G)\), but by definition this is also \( \partial[f_i] \). Thus, we have shown that there is a section such that the obstruction cocycles agree. \( \square \)

This Lemma implies that the primary obstruction to null-homotopy of the classifying map of the tangent bundle and the primary obstruction to stable triviality of the tangent bundle agree after identifying \( \pi_3(BO) \) and \( \pi_7(O) \). Thus, we know that \( Ob_O(\tau(M)) = \beta(M) \) and hence \( \gamma(M) = p_*j_* \beta(M) = 7\beta(M) \).

4.2 Identification of the topological and PL classification

There is also a notion of topological microbundles and tangent microbundles. Basically one has to replace all PL-objects and morphisms by topological versions. One can also obtain a classifying space for stable topological microbundles \( BTOP \) and a map \( \pi : BPL \to BTOP \) with homotopy fiber \( TOP / PL \). An analogous statement as mentioned in Remark 4.1.5 is true for PL-structures on a topological manifold \( M \). But since \( TOP / PL \) is a connected space with \( \pi_3(TOP / PL) \cong \mathbb{Z}/2 \) as the only nontrivial homotopy group, one can use obstruction theory to get a nice result: The obstructions to find a PL-structure lies in \( H^4(M; \pi_3(TOP / PL)) \cong H^4(M; \mathbb{Z}/2\mathbb{Z}) \). If there is a PL-structure, the PL-structures are in correspondence with \( H^3(M; \pi_3(TOP / PL)) \cong H^3(M; \mathbb{Z}/2\mathbb{Z}) \) (compare [KS77, p.300]). Using the Hurewicz theorem and the universal coefficient theorem we obtain that these groups are trivial for highly connected manifolds of dimension greater than 8. So we can formulate the following proposition:

**Proposition 4.2.1.** There is only one PL-structure on a highly connected topological manifold of dimension greater than 8. Two such manifolds are homeomorphic if and only if they are PL-homeomorphic.
4.3 Application to the classification of total spaces of $S^7$-bundles over $S^8$

Using the PL-invariant $\gamma$ we can now proof a partial result on the homeomorphism classification.

**Theorem 4.3.1.** Let $P_0, P_1$ be highly connected rational homology 15-spheres and let $H^8(P_0)$ be without 7-torsion and of odd order. Then $P_0, P_1$ are (PL) homeomorphic if and only if they are almost diffeomorphic.

**Proof.** We need to see that PL-homeomorphic implies almost diffeomorphic. But the induced map in cohomology respects the linking form and maps $\gamma(P_1) = 7\beta(P_1)$ to $\gamma(P_0) = 7\beta(P_0)$. Since we assumed that $H^8(P_0) \cong H^8(P_1)$ has no 7-torsion this implies that it maps $\beta(P_1)$ to $\beta(P_0)$. Thus, it respects the Wilkens invariant and by 3.6.1 they are almost diffeomorphic.

Applying 4.3.1 we get:

**Corollary 4.3.2** (Main Theorem II). Let $n$ be odd and $7 \nmid n$, then $M_{m,n}$ and $M_{m',n}$ are (PL) homeomorphic if and only if they are almost diffeomorphic.

I think that Corollary 4.3.2 is also true for $n$ even and $7 \nmid n$. But I haven’t found a nice proof yet. We will discuss some ideas how to proof it in the next chapter.

The condition $7 \nmid n$ cannot be removed because in [CZ08] the existence of non almost diffeomorphic PL-homeomorphic $M_{m,14}$ is proofed.

We discuss now a few thoughts on exotic structures on manifolds homeomorphic to the total spaces of $S^7$-bundles over $S^8$. We have already seen the most interesting result: the exotic structures on $S^{15}$ realized by bundles. Another somehow familiar manifold is the product $S^7 \times S^8 = M_{0,0}$, as for every manifold $M_{m,0}$ there is only one smooth structure realized as a total space by Theorem 2.3.4. But using the $\mu$-invariant it is not hard to see that there are multiple smooth structures on them. If we now consider the other cases with $7 \nmid n$ and $2 \nmid n$, there is possibly only one differentiable structure realized if $\tilde{\mu}(M_{m,n}) \equiv \tilde{\mu}(M_{m',n})$ implies $\mu(M_{m,n}) \equiv \mu(M_{m',n})$. But this is only the case if $\frac{m^2-m'^2}{n} + m - m'$ is a multiple of 16156. Thus, in general multiple structures are realized.
We were able to generalize the results of [CE03] on the almost diffeomorphism and diffeomorphism classification. As already mentioned in dimension 7 the notions of almost diffeomorphic and (PL) homeomorphic agree. Since this is not the case in dimension 15, we could not use the argumentation of Escher and Crowley. If the order of $H^8$ is odd, the linking form and the stable tangential invariant suffice to classify. Thus, we were able to obtain results using $\gamma = 7\beta$. If we consider $H^8$ of even order, we need something like a PL version of $\bar{\mu}$. Crowley suggested to take

$$\mu_{PL}(P) := \frac{\lambda(L)^{-1}(p_4\text{Ob}_{PL}(\tau_{PL}(L)), p_5\text{Ob}_{PL}(\tau_{PL}(L))) - \sigma(\lambda(L))}{8},$$

where $L$ is a handlebody bounded by $P\#\Sigma P$. We have to show that this is independent of the coboundary chosen. One idea to proof this is to repeat Chapter 3 in terms of PL-manifolds. The first thing we need is a classification of highly connected PL $2n$-manifolds $L$ with non empty highly connected boundaries. Currently Diarmuid Crowley and Peter Teichner are working on a paper on generalizations of Wall’s work [Wal62] and they show that the triples $(H_n(L), \lambda(L), \text{Ob}_{PL}(\tau_{PL}(L)))$ classify the PL-manifolds $L$. Two problems appear already at this point: The $\mathbb{Z}/4$ summand of $\pi_7(PL)$ and the fact that I have no idea which triples are realized. The $\mathbb{Z}/4$ summand forbids us to use the nice language of quadratic functions, but this does not stop us from trying to define a PL version of $[\ast]_{F^c}$. To do this, we need a PL version of Wilkens’ extension of diffeomorphism Theorem 3.3.10. Since the proof uses handle decomposition and the h-cobordism theorem, we should be able to proof a PL version. We define $\mathcal{P}_{PL}(n)$ to be the monoid of isometry classes of tripels $(H_n(L), \lambda(L), \text{Ob}_{PL}(\tau_{PL}(L)))$ and $\mathcal{P}_{PL}^b(n)$ to be the submonoid corresponding to PL-handlebodies bounded by the sphere. Thus, we can define two triples in $\mathcal{P}_{PL}(n)$ to be equivalent if they are isometric after the addition of triples in $\mathcal{P}_{PL}^b(n)$ and we denote the classes by $[\ast]_{\mathcal{P}_{PL}^b}$. If we assume that all highly connected odd dimensional PL-manifolds are boundaries of highly connected PL-manifolds, we would be able to proof that $[\ast]_{\mathcal{P}_{PL}}$ is a complete PL-invariant. This does not help us since I have no clue how to solve the algebraic problem now occurring. Another strategy could be to restrict the attention to smoothable PL-manifolds and forget the information in the $\mathbb{Z}/4$ summand. Many things would become easier, but in this case we would need a version of 3.3.10 stating that we can extend an PL-homeomorphism on the boundary to the entire manifold after the addition of smoothable handlebodies bounded by the sphere. Unfortunately, I have no idea if this is even true. If it was true, we could show that $[\kappa(H_8(L), \lambda(L), 7\alpha(L))]_{F^c}$ is a PL-invariant of a (smooth) $P$ bounding $L$ after the addition of $\Sigma P$ (note that
\( \kappa(H_8(L), \lambda(L), 7\alpha(L)) \) is characteristic. Now using the classification of characteristic quadratic linking functions we could define \( \mu_{PL} \) for smoothable PL-manifolds. We would get:

\[
\mu_{PL}(P) := \frac{49\lambda(L)^{-1}(\alpha(L), \alpha(L)) - \sigma(\lambda(L))}{8}
\]

This would enable us to prove that the odd order is not necessary for Corollary 4.3.2. On the other hand this invariant is probably not complete, because of the not considered information in the \( \mathbb{Z}/4 \) coefficient.

Crowley and Escher also give a classification up to homotopy equivalence. Given their results and results by Tamura [Tam57, Theorem 2.3] it seems reasonable to formulate the following conjecture:

**Conjecture 5.0.3.** \( M_{0,1} \) and \( M_{m',n} \) are orientation preserving homotopy equivalent if and only if \( m \equiv \epsilon n' \mod \gcd(n, 120) \), where \( \epsilon \) is such that \( \epsilon^2 \equiv 1 \mod \gcd(n, 120) \) and \( \gcd \) means greatest common divisor.
Bibliography


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Selbstständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe.

Berlin, den