THE GROMOV-WITTEN POTENTIAL ASSOC. TO A TCFT

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GOAL: CONSTRUCT SOLUTIONS TO THE BV MASTER EQUATION IN A FOCK SPACE ASSOCIATED TO A (CLOSED) TCFT.

I. MAURER - CARTAN EQUATIONS

- k = FIELD OF CHAR 0

DEF: A dg LIE ALGEBRA IS A CHAIN CX $(\mathfrak{g}_*, \partial)$ WITH A GRADED LIE BRACKET $\{,\}$ SATISFYING $\partial\{a,b\} = \{\partial a, b\} + (-1)^{|a|}\{a, \partial b\}$

THE MAURER-CARTAN EQUATION IN $\mathfrak{g}_*$ IS

(MC) $\partial S + \frac{1}{2}\{S, S\} = 0$

FOR SOME $S \in \mathfrak{g}_*$

- FOR TWO SOLUTIONS S_0, S_1 TO (MC) IN $\mathfrak{g}_*$, A HOMOTOPY (SOLUTION) BETWEEN THEM IS $S \in \mathfrak{g}_*[t, \epsilon]$, $|t|=0$, $|\epsilon| = -1$

MAKE $\mathfrak{g}_*[t,\epsilon] \cong \mathfrak{g}_* \otimes K[t,\epsilon]$ INTO A dgLIE ALG BY EXTENDING $\{,\}$ TO IT (WITH DIFF $= \epsilon \cdot \frac{d}{dt}$)

$\left(\{at^n\epsilon^m, bt^p\epsilon^q\} = \{a,b\}t^{n+p}\epsilon^{m+q}\right)$

REQUIRES
- $\partial S + \frac{1}{2}\{S, S\} = 0$ IN $\mathfrak{g}_*[t,\epsilon]$
- $S(0,0) = S_0$, $S(1,0) = S_1$

(IF S HOMOGENEOUS deg n, $|\partial S| = n-1$, $|\{S,S\}| = 2n$ $\rightarrow$ $n=-1$ FOR CANCELLATION)

DEFINE $\pi_0 MC(\mathfrak{g}) = \{$HOMOTOPY CLASSES OF SOLUTIONS TO MC IN $\mathfrak{g}\}$

GOVERNS DEFORMATIONS OF THE dgLIE ALG. STR. ON $\mathfrak{g}_*$

DEF: A dg BV ALG IS A TRIPLE $(\mathfrak{g}_*, d, \Delta)$ WHERE $(\mathfrak{g}_*, d)$ IS A DIFF GRADED COMM. ALG WITH 1

$\Delta : \mathfrak{g}_* \to \mathfrak{g}_{*+1}$ SATISFYING

1. $\Delta^2 = 0$
2. $d\Delta + \Delta d = 0$
3. $\Delta(abc) = \Delta(ab)c + (-1)^{|b|(|a|+1)} b\,\Delta(ac) + (-1)^{|a|} a\,\Delta(bc)$
   $- \Delta(a)bc - (-1)^{|a|} a \Delta b c - (-1)^{|a+b|} ab \Delta c$.

PROP: $(\mathfrak{g}, d, \Delta)$ IS A dg BV ALG, THEN $(\mathfrak{g}, \hat{d} = d + \Delta)$ IS A dgLIE ALG WITH

$\{a, b\} = \hat{d}(ab) - (-1)^{|a|} a \hat{d}(b) - \hat{d}(a) b$.

THE MAURER-CARTAN EQUATION IN $(\mathfrak{g}, \hat{d})$ IS CALLED THE BV (QUANTUM) MASTER EQUATION IN $(\mathfrak{g}, d, \Delta)$.

PROP: $S \in \mathfrak{g}$ SATISFIES THE BV MASTER EQUATION IFF

(BV) $\quad \hat{d}(\exp S) = 0$

ASSUMING THIS MAKES SENSE.

PROOF. $\hat{d}(\exp S) = d \exp S + \Delta \exp S$

$= (dS) \exp S + \left((\Delta S) + \tfrac{1}{2}\{S, S\}\right) \exp S$

$= \left(\hat{d} S + \tfrac{1}{2}\{S, S\}\right) \exp S$.

II CATEGORIES OF SURFACES

DEF: S = 1+1 COBORDISM CATEGORY

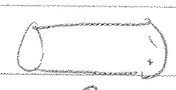
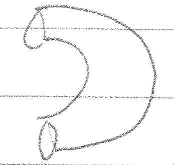
Obj = $\mathbb{Z}_{\geq 0}$

$S(n,m)$ = MODULI SPACES OF RIEMM. SURFACES WITH n INCOMING, m OUTGOING PARAMETRIZED BDRIES

REQUIRE: EACH COMPONENT HAS AT LEAST ONE INCOMING ∂

THE CATEGORY OF ANNULI $A \subsetneq S$ HAS THE SAME OBJECTS, BUT MORPHISMS ARE GIVEN BY SURFACES, EACH OF WHOSE COMPONENTS ARE ANNULI.

NOTE: THERE ARE EXACTLY 2 CONNECTED TYPES OF MORPHISMS:

 AND  $\in A(2,0)$

$\in A(1,1)$

($B(\ \ ) \simeq \mathbb{R}P^\infty_+ \wedge \mathbb{C}P^\infty$ (MAPPING THE CORE TO $\mathbb{C}P^\infty$))

REDEFINE TO ALLOW ONLY INFINITESIMAL ANNULI SO THAT

$A(1,1) = S^1 \leq \mathrm{Diff}(S^1)$

EX: $A(n,n) = \Sigma_n \wr S^1$

$\rightsquigarrow$ dg CATEGORIES C_*A, H_*A

LEMMA: THERE IS A q-ISO OF CATEGORIES $H_*A \to C_*A$

IDEA: S^1 IS FORMAL OVER ANY RING.

Prop. $H_*(A)$ is generated as a unital symm-monoidal cat on obj $= \mathbb{Z}_{\geq 0}$ by

- $D := [S^1] \in H_1(A(1,1))$
- $G := [*] \in H_0(A(2,0))$

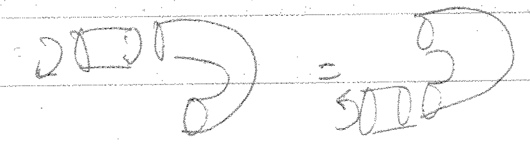
subject to

- $D^2 = 0$
- $G \circ (D \otimes 1 \pm 1 \otimes D) = 0$

Proof: A is generated by $A(1,1)$ and $A(2,0)$ as symm. monoidal cat $\rightarrow$ $H_*(A)$ gen. by $H_*(A(1,1))$, $H_*(A(2,0))$

$H_*(A(1,1)) = S^1$, $H_*(A(2,0)) \simeq S^1$

$H_1(A(2,0)) = \langle G \circ (D \otimes 1) \rangle$



Corollary: A strict monoidal functor (split)

$F : H_* A \longrightarrow \mathrm{Comp}$ is the same as

- A complex $(V_*, d) = F(1)$
- $D : V_* \to V_{*+1}$, $D^2 = 0$, $Dd - dD = 0$
- An even symmetric pairing $\langle , \rangle$ on V:
  - $\langle dv, w \rangle + (-1)^{|v|} \langle v, dw \rangle = 0$
  - $\langle Dv, w \rangle \pm (-1)^{|v|} \langle v, Dw \rangle = 0$

III Fock Spaces

Claim: One can construct a Fock space $\mathcal{F}(F)$ for any functor $F: C_*(A) \longrightarrow \mathrm{Comp}$

A BV-algebra

Let $F: H_*(A) \to \mathrm{Comp}$ be split, given by $V = F(1)$.

V is a $\Lambda[D]$-module. Define $V_{Tate} := V((t))$ = power series bounded below

$V^{hS^1} := V[[t]]$ power series starting at 0, $V_{hS^1} = V_{Tate} / V^{hS^1} = t^{-1}V[t^{-1}]$

with diff. given by

$$d(v \otimes f(t)) = dv \otimes f(t) + Dv \otimes t f(t)$$

Here $|t| = -2$

Def: The Fock space $\mathcal{F}(V_{Tate})$ is $\mathcal{F}(V_{Tate}) = \mathrm{Sym}^*(V_{hS^1})$

- Give this a diff d extending d on V_{hS^1} as a derivation.
- Define $\Delta: \mathcal{F}(V_{Tate}) \circlearrowleft$ by
  - $\Delta = 0$ on $\mathrm{Sym}^{\leq 1}(V_{hS^1})$
  - On $\mathrm{Sym}^2(V_{hS^1})$: $\Delta((v_1 f_1(t))(v_2 f_2(t)))$ $= \langle Dv_1, v_2 \rangle \underbrace{\mathrm{Res}(f_1(t_1) f_2(t_2)\, dt_1 dt_2)}_{\text{coef of } f_1 \text{ at } t_1^{-1} \text{ and } f_2 \text{ at } t_2^{-1}}$
  - Extend to higher products by BV formula.

Cor. $\mathcal{F}(V_{Tate})$ is a BV algebra.

Fock spaces from TCFT's

If Φ is a split TCFT,

$$\begin{tikzcd} C_*S \arrow[r, "\Phi"] & \mathrm{Comp} \\ C_*A \arrow[u, phantom, "\cup"] & \\ H_*A \arrow[u, "\simeq"] \arrow[uur] & \end{tikzcd}$$

$\Phi|_{H^*(A)} \longleftrightarrow V_{Tate}$

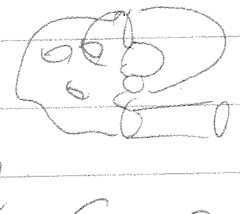
$\mathcal{F}(\Phi) = \mathcal{F}(V_{Tate})$ is a dg BV alg

Other Fock spaces

Def: $\mathcal{M}(m)$ = moduli space of Riem. surfaces with m borders, all outgoing.

Prop: There is a functor $\mathcal{A} \to \mathrm{Top}$ given by

- Obj: $m \longmapsto \mathcal{M}(m)$
- Morph: $\mathcal{A}(m,n) \times \mathcal{M}(m) \to \mathcal{M}(n)$



$\rightsquigarrow C_*\mathcal{A} \xrightarrow{C_*\mathcal{M}} \mathrm{Comp}$ (not h-split)

Fock space $\mathcal{F}(\mathcal{M}) := \bigoplus_{m \geq 0} C_*(\mathcal{M}(m)) / \Sigma_m \wr S^1$

$\left(\mathrm{Sym}(V_{hS^1}) = \bigoplus_{m \geq 0} \left((V^{\otimes m})_{hS^1} \right)_{\Sigma_m} \right.$

$\left. V = F(1) \quad F \text{ split } H_*(\mathcal{A}) \to \mathrm{Comp} \right)$

Prop: $\mathcal{F}(M)$ is a dg BV algebra

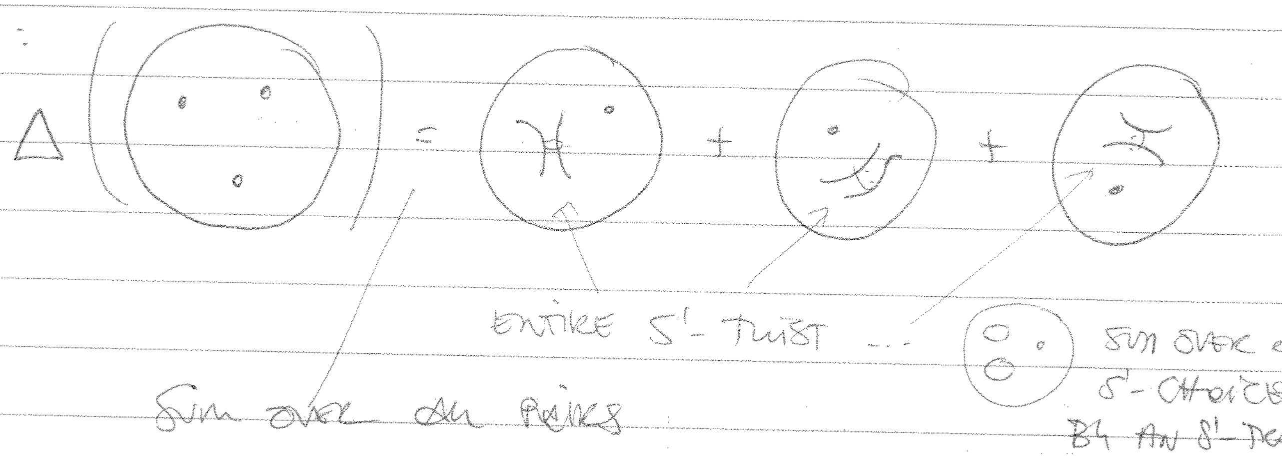
Proof: Mult = disjoint union of surfaces

Generally, Δ is given by $G(D\otimes 1) = \pm G(1\otimes D)$ $\in H_1(A(2,0))$

And an S^1-transfer.

Concrete for $\mathcal{F}(M)$: Δ = sum of all possible ways to glue pairs of boundary components together with a full S^1-twist.

Eg:



IV Results

Thm 1: $\exists$ elements $S_{g,n} \in \mathcal{F}^{g,n}(M)$, $|S_{g,n}| = 6g-6+2n$

with 1) $S_{0,3}$ is a 0-chain of degree (= coeff. of generator) $\frac{1}{3!}$ in $\mathcal{M}_{0,3}/\Sigma_3$

2) Let $S := \sum_{\substack{g,n \\ 2g-2+n>0}} S_{g,n}\lambda^{2g-2+n} \in \lambda\mathcal{F}(M)[[\lambda]]$

Then S satisfies the BV master equation, and is unique up to homotopy through such series.

UNTHM 3 (IS MOTIVATION BUT NOT TRUE AS STATED)

FOR ANY TCFT $\Phi : C_*S \to \mathrm{Comp}$,

$\Phi|_{C_*(A)} : C_*(A) \to \mathrm{Comp}$, THERE IS A NATURAL

TRANSF. $C_*(\mathcal{M}) \to \Phi|_{C_*(A)}$ INDUCING A MAP

OF dg BV-ALG $\mathcal{F}(\mathcal{M}) \to \mathcal{F}(\Phi)$.

$\to$ CAN PUSH SOLUTION OF BV MASTER EQUATION:

$\underbrace{\mathrm{Im}(S)}_{\text{GROMOV-WITTEN POTENTIAL IN } \Phi} \in \mathcal{F}(\Phi)$ SATISFIES THE MASTER EQUATION.

BUT THIS ISN'T TRUE!

RIGHT VERSION: SOME SORT OF PROJECTIVE RESULT IN HOMOLOGY.

(THM 2: VERSION FOR COMPACTIFIED MODULI SPACE)

BEST EXAMPLE: A = FROBENIUS ALG

$\rightsquigarrow \underbrace{CH_*(A, A)}_{\text{HOCHSCHILD CPX}} = \Phi(1)$ IS A TCFT

$V_{hS^1} = CC_*^{\#}(A)$

$\mathcal{F}(V_{Tate}) = \mathrm{Sym}^*(CC_*(A))$ ADMITS A SOLUTION TO THE MASTER EQUATION.

EX. $A = C^*_{DR}(M)$, THEN $V_{hS^1} = CC_*(A) = C_*(LM_{hS^1})$

(OR: $C_*(LM^{hS^1})$)

$\mathcal{F}(V_{Tate}) \text{ "}=\text{" } C_* \Omega^\infty \Sigma^\infty LM_{hS^1}$

[IN COMPACTIFIED EXAMPLE: $S = \sum \underbrace{S_{g,n}}_{[\overline{\mathcal{M}}_{g,n}]} \lambda \in \lambda\,\mathcal{F}(\mathcal{M})[[\lambda]]$

BUT: $\Delta = 0 \Rightarrow \{,\} = 0 \Rightarrow$ SOLUTION TO MC $\Leftrightarrow \partial S = 0$ –]

$$
\begin{tikzcd}
 & & S \arrow[dll] \arrow[drr] & & \\
{[\bar{M}]} & & \mathcal{F}(M) \arrow[u, phantom, sloped, "\subset"] \arrow[dll, squiggly] \arrow[drr, squiggly] & & \mathrm{im}(S) \\
\mathcal{F}(\bar{M}) \arrow[u, phantom, sloped, "\subset"] \arrow[rrrr, "\text{WANT}"'] & & & & \mathcal{F}(\Phi) \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}
$$

WHICH WOULD ENCODE GROMOV-WITTEN THEORY ON A CY-MFD

$\leftrightarrow$ TCFT ASSOC. TO A DER. CAT OF COH. SHEAVES ON CY MFD.