HOCHSCHILD HOMOLOGY OF STRUCTURED ALGEBRAS

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Abstract. We give a general method for constructing explicit and natural operations on the Hochschild complex of algebras over any PROP with \(A_\infty\)-multiplication—we think of such algebras as \(A_\infty\)-algebras “with extra structure”. As applications, we obtain an integral version of the Costello-Kontsevich-Soibelman moduli space action on the Hochschild complex of open TCFTs, the Tradler-Zeinalian action of Sullivan diagrams on the Hochschild complex of strict Frobenius algebras, and give applications to string topology in characteristic zero. Our main tool is a generalization of the Hochschild complex.

The Hochschild complex of an associative algebra \(A\) admits a degree 1 self-map, Connes-Rinehart’s boundary operator \(B\). If \(A\) is Frobenius, the (proven) cyclic Deligne conjecture says that \(B\) is the \(\Delta\)-operator of a BV-structure on the Hochschild complex of \(A\). In fact \(B\) is part of much richer structure, namely an action by the chain complex of Sullivan diagrams on the Hochschild complex [45]. A weaker version of Frobenius algebras, called here \(A_\infty\)-Frobenius algebras, yields instead an action by the chains on the moduli space of Riemann surfaces [8, 27]. In this paper we develop a general method for constructing explicit operations on the Hochschild complex of \(A_\infty\)-algebras “with extra structure”, which contains these theorems as special cases. Our method is global to local: we give conditions on a composable collection of operations that ensures that it acts on the Hochschild complex of algebras of a given type. We then show how to read-off the action explicitly, so that formulas for individual operations can be also obtained.

An \(A_\infty\)-algebra can be described as an enriched symmetric monoidal functor from a certain dg-category \(A_\infty\) to \(\text{Ch}\), the dg-category of chain complexes over \(\mathbb{Z}\). The category \(A_\infty\) is what is called a \(dg\)-PROP, a symmetric monoidal dg-category with objects the natural numbers. We consider here more generally dg-PROPs \(E\) equipped with a dg-functor \(i : A_\infty \rightarrow E\). Expanding on the terminology of McClure–Smith [32], we call such a pair \(E = (E, i)\) a PROP with \(A_\infty\)-multiplication. An \(E\)-algebra is a symmetric monoidal dg-functors \(\Phi : E \rightarrow \text{Ch}\). When \(E\) is a PROP with \(A_\infty\)-multiplication, any \(E\)-algebra comes with a specified \(A_\infty\)-structure by restriction along \(i\), and hence we can talk about the Hochschild complex of \(E\)-algebras.

We introduce in the present paper a generalization of the Hochschild complex which assigns to any dg-functor \(\Phi : E \rightarrow \text{Ch}\) a certain new functor \(C(\Phi) : E \rightarrow \text{Ch}\). The assignment has the property that, for \(\Phi\) symmetric monoidal, \(C(\Phi)\) evaluated at 0 is the usual Hochschild complex of the underlying \(A_\infty\)-algebra. (The evaluation of \(C(\Phi)(n)\) can more generally be interpreted in terms of higher Hochschild homology as in [39] associated to the union of a circle and \(n\) points.) This Hochschild complex construction can be iterated, and for \(\Phi\) split monoidal\(^1\), the iterated complex \(C^n(\Phi)\) evaluated at 0 is the \(n\)th tensor power \((C(\Phi)(0)) \otimes n\).

Our main theorem, Theorem 5.11, says that if the iterated Hochschild complexes of the functors \(\Phi = E(e, -)\) admit a natural action of a dg-PROP \(D\) of the form

\[
C^n(E(e, -)) \otimes D(n, m) \rightarrow C^m(E(e, -))
\]

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\(^1\)i.e. such that the maps \(\Phi(n) \otimes \Phi(m) \rightarrow \Phi(n + m)\) are isomorphisms (also known as strong monoidal)
then the classical Hochschild complex of any split monoidal functor \( \Phi : \mathcal{E} \to \text{Ch} \) is a \( \mathcal{D} \)-algebra, i.e. there are maps
\[
(C(\Phi)(0)) \otimes_n \mathcal{D}(n, m) \to (C(\Phi)(0)) \otimes^m
\]
associative with respect to composition in \( \mathcal{D} \). This action is given explicitly and is natural in \( \mathcal{E} \) and \( \mathcal{D} \).

Before stating the theorem in more detail, we describe some consequences. Let \( \mathcal{O} \) denote the open cobordism category, whose objects are the natural numbers and whose morphisms from \( n \) to \( m \) are chains on the moduli space of the Riemann surfaces that are cobordisms from \( n \) to \( m \) open strings. Taking \( \mathcal{E} = \mathcal{O} \) and \( \mathcal{D} = \mathcal{C} \), the closed co-positive\(^2\) boundary cobordism category, Theorem 5.11 gives an integral version of Costello’s main theorem in [8], i.e., an action of the chains of the moduli space of Riemann surfaces on the Hochschild chain complex of any \( \mathcal{A}_\infty \)-Frobenius algebra\(^3\). (See Theorem 6.2 and Corollary 6.3.) Reading off our action on the Hochschild chains, we recover the recipe for constructing such an action given by Kontsevich and Soibelman in [27], thus tying these two pieces of work together. We also get a version for non-compact \(^4\) \( \mathcal{A}_\infty \)-Frobenius algebras by replacing \( \mathcal{O} \) by the positive boundary\(^5\) open cobordism category and \( \mathcal{C} \) with the positive and co-positive boundary category. (See Corollary 6.5.)

Applying Theorem 5.11 to the category \( \mathcal{E} = H_0(\mathcal{O}) \), we obtain an action of the chain complex of Sullivan diagrams on the Hochschild complex of strict symmetric Frobenius algebras, recovering, with very different methods and after dualization, the main theorem of Tradler-Zeinalian in [45]. (See Theorem 6.7.) In particular, in genus 0, this gives the cyclic Deligne conjecture first proved in [21], see also [43]. (See Proposition 6.9.)

A consequence of our naturality statement, Theorem 5.13, is that the aforementioned HCFT structure constructed by Costello and Kontsevich-Soibelman factors through an action of Sullivan diagrams, when the \( \mathcal{A}_\infty \)-Frobenius algebra happens to be strict. Sullivan diagrams model the harmonic compactification of moduli space [11, Prop 5.1], so one can say that the action of moduli space compactifies in that case. New operations arise from the compactification, and we know that these act non-trivially already on very basic Frobenius algebras [49, Prop 4.1 and Cor 4.2]. On the other hand, a significant part of the homology of moduli space dies in the compactification, in particular the stable classes, which implies a significant collapse of the original structure when the algebra is strict. (See Proposition 2.11 and Corollary 6.8.)

We apply the above to the case of string topology for a simply-connected manifold \( M \) over a field of characteristic zero, using the strict Frobenius model of \( C^*(M) \) given by Lambrechts-Stanley [28, 12], and obtain an HCFT structure on \( H^*(LM, \mathbb{Q}) \) factoring through an action of Sullivan diagrams. It is natural to conjecture that this structure is the same as the one defined by Godin in [13], and towards this we show in Proposition 6.10 that our structure recovers the BV structure on \( H_s(LM) \) originally introduced by Chas-Sullivan. The vanishing of the action of the stable classes in the HCFT structure furthermore agrees with Tamanoi’s vanishing result in [44]. These vanishing results should though be contrasted with the non-vanishing results of [49] for classes coming from the compactification: these results give non-trivial higher operations on \( H^*(LS^n) \). A different approach to Sullivan diagram actions on \( H^*(LM) \) can be found in [40, 41] (see also [5] in the equivariant setting). The papers [6, 7] construct string topology actions using a more restricted definition of Sullivan diagrams. In [19, 20, 22], Kaufmann constructs an action of Sullivan diagrams on the \( E^1 \)-page of a spectral sequence converging to \( H_s(LM) \). (The PROP of open-closed Sullivan diagrams defined in [22]

\(^2\)where the components of morphism each have at least one incoming boundary
\(^3\)called an extended Calabi-Yau \( \mathcal{A}_\infty \) category in [8]
\(^4\)loosely, these are non-counital \( \mathcal{A}_\infty \)-Frobenius algebras
\(^5\)where the components of morphism each have at least one outgoing boundary
has its closed part isomorphic to the Sullivan diagrams considered here, but its open part is different from ours (see Remark 2.12).)

Further applications of our methods in the case of commutative and commutative Frobenius algebras where obtained by Klamt in [24, 25]. Other interesting examples of families of algebras to consider would be algebras over Kaufmann’s PROP of open Sullivan diagrams [22], Hopf algebras, Poisson algebras and $E_n$-algebras, to name a few.

We now describe our set-up and tools in a little more detail and give a more precise formulation of the main theorem.

Recall from above that $\mathcal{E}$ is a dg-category, in fact a dg-PROP, equipped with a functor $i : \mathcal{A}_\infty \to \tilde{\mathcal{E}}$, which will always be assumed to be the identity on the objects, the natural numbers. Recall also that the Hochschild complex of a functor $\Phi : \mathcal{E} \to \text{Ch}$ is defined here as a new functor $C(\Phi) : \mathcal{E} \to \text{Ch}$.

To any such dg-category $\mathcal{E}$, we associate in this paper a larger dg-category, its Hochschild core category $\mathcal{C}\mathcal{E}$. The category $\mathcal{C}\mathcal{E}$ has objects pairs of natural numbers $[n]$, has $\mathcal{E}$ as a full subcategory on the objects $[0]$, and with the morphisms from $[m]$ to $[n]$ the iterated Hochschild complex $C^n(\mathcal{E}(m_1, -))$ evaluated at $m_2$. If $\tilde{\mathcal{E}}$ is the open cobordism category $\mathcal{O}$, then $\mathcal{C}\mathcal{E}$ is the open-to-open and open-to-closed part of the open-closed cobordism category. Given a monoidal category $\tilde{\mathcal{E}}$ with the same objects as $\mathcal{C}\mathcal{E}$, we call it an extension of $\mathcal{C}\mathcal{E}$ if it agrees with $\mathcal{C}\mathcal{E}$ on the morphisms with source $[m]$ when $n = 0$. An extension of $\mathcal{C}\mathcal{E}$ can be thought of as the full open-closed cobordism category, also including the closed-to-closed and closed-to-open morphisms.

**Main Theorem** (Theorem 5.11 for $\Phi$ split symmetric monoidal, $C$ unreduced). Let $(\mathcal{E}, i)$ be a PROP with $\mathcal{A}_\infty$-multiplication and $\mathcal{C}\mathcal{E} \hookrightarrow \tilde{\mathcal{E}}$ an extension of $\mathcal{C}\mathcal{E}$ in the above sense. Then $\tilde{\mathcal{E}}$ acts naturally on the Hochschild complex of $\mathcal{E}$–algebras: For any $\mathcal{E}$–algebra $A$ with $C(A, A)$ its Hochschild complex, there are chain maps

$$C(A, A)^{\otimes m_1} \otimes A^{\otimes m_1} \otimes \tilde{\mathcal{E}}([m_1], [m_2]) \to C(A, A)^{\otimes m_2} \otimes A^{\otimes m_2}$$

which are natural in $A$ and associative with respect to composition in $\tilde{\mathcal{E}}$.

The same holds for a reduced version of the Hochschild complex.

This theorem applies to any PROP with $\mathcal{A}_\infty$–multiplication $\mathcal{E}$ and chosen extension $\tilde{\mathcal{E}}$. For each of the applications discussed above we have explicit extension categories, and the PROP $\mathcal{D}$ mentioned above in each case is the “closed-to-closed” part of the extension category. To prove that a candidate category acts on the Hochschild complex of a certain type of algebras, using the above theorem, all we need to do is check that they indeed are extension categories.

In the present paper, small extension categories are constructed using ad hoc methods coming from the geometry of the situation. Given any PROP with $\mathcal{A}_\infty$–multiplication $(\mathcal{E}, i)$, there exists a universal extension which is much larger than the extensions considered here (see [49]). In the particular cases where $\mathcal{E} = \mathcal{O}$ or $H_0(\mathcal{O})$, it is however shown in [49, Rem 2.4 and Thm B,C] that the universal extension is quasi-isomorphic to the PROPs constructed here, so on the level of homology we do actually construct all the operations.

The proof of the main theorem, inspired by, though independent of, [8], uses simple properties of the double bar construction, and a quotiented version of it to take care of the equivariant version of the theorem under the action of the symmetric groups. Our action is explicit thanks to the construction of an explicit pointwise chain homotopy inverse to the quasi-isomorphism of functors $C(B(\Phi, \mathcal{E})) \to C(\Phi)$. (See Proposition 5.9.) As an example of how our theory can be applied, we give in Section 6.5 explicit formulas for the product, coproduct, and $\Delta$–operator on the Hochschild complex of strict Frobenius algebras.
The paper is organized as follows: Section 2 introduces the chain complexes of graphs used throughout the paper. In particular, our graph model for the open-closed cobordism category and a category of Sullivan diagrams are constructed. Section 3 gives some background on types of algebras occurring in the paper. The short Section 4 reviews a few properties of the double bar construction and its quotiented analog. Section 5 then defines the Hochschild complex operator, examines its properties, and proves the main theorem. Section 6 gives applications: Section 6.1 gives the application to Costello’s theorem, and Section 6.2 describes how to deduce the Kontsevich-Soibelman approach from it. Sections 6.3 and 6.4 take care of the twisting by the determinant bundle and the positive boundary variation. In Section 6.5, we treat the case of strict Frobenius algebras and Sullivan diagrams, with the application to string topology given in Section 6.6. Finally, Sections 6.7 and 6.8 consider $A_\infty$ and $Ass \times P$–algebras for $P$ an operad.

Section 1 sets up some notation and the Appendix Section 7 explains how to compute signs given operations represented by graphs.

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category $OC$ and the category of Sullivan diagrams $SD$, which we identify as a quotient of $OC$.

2.1. Fat graphs. By a graph $G$ we mean a tuple $(V, H, s, i)$ where $V$ is the set of vertices, $H$ the set of half-edges, $s : H \rightarrow V$ is the source map and $i : H \rightarrow H$ is an involution. Fixed points of the involution are called leaves. A pair $\{h, i(h)\}$ with $i(h) \neq h$ is called an edge. We will consider graphs with vertices of any valence, also valence 1 and 2.

We allow the empty graph. We will also consider the following degenerate graphs which fail to fit the above description:

- The leaf consisting of a single leaf and no vertices.
- The circle with no vertices.

The leaf will appear in two flavors: as a singly labeled leaf and as a doubly labeled leaf. The circle will arise from gluing the doubly labeled leaf to itself.

A fat graph is a graph $G = (V, H, s, i)$ together with a cyclic ordering of each of the sets $s^{-1}(v)$ for $v \in V$. The cyclic orderings define boundary cycles on the graph, which are sequences of consecutive half-edges corresponding to the boundary components of the surface that can be obtained by thickening the graph. Figure 1(a) shows an example of a fat graph with two boundary cycles (the dotted and dashed lines), where the cyclic ordering at vertices is that inherited from the plane. (Formally, if $\sigma$ is the permutation of $H$ whose cycles are the cyclic orders at each vertex of the graph, then the boundary cycles of $G$ are the cycles of the permutation $\sigma \cdot i$ [14, Prop 1].)

![Figure 1](image_url)

**Figure 1.** Fat graph and black and white graphs.

2.2. Orientation. An orientation of a graph $G$ is a unit vector in $\det(\mathbb{R}(V \cup H))$. The degenerate graphs have a canonical formal positive orientation. Note moreover that any odd-valent (in particular trivalent) graph has a canonical orientation

$$v_1 \wedge h^1_1 \wedge \ldots \wedge h^1_{n_1} \wedge \ldots \wedge v_k \wedge h^k_1 \wedge \ldots \wedge h^k_{n_k}$$

where $v_1, \ldots, v_k$ is a chosen ordering of the vertices of the graph and $h^i_1, \ldots, h^i_{n_i}$ is the set of half-edges at $v_i$ in their cyclic ordering.

2.3. Black and white graphs. A black and white graph is a fat graph whose set of vertices is given as $V = V_b \bigsqcup V_w$, with $V_b$ the set of black vertices and $V_w$ the set of white vertices. The white vertices are labeled $1, 2, \ldots, |V_w|$ and are allowed to be of any valence (also 1 and 2). The black vertices are unlabeled and must be at least trivalent. Moreover, each white vertex is equipped with a choice of start half-edge, i.e. a choice of an element in $s^{-1}(v)$ for each $v \in V_w$. Equivalently, the set of half-edges $s^{-1}(v)$ at each white vertex $v$ has an actual ordering, not just a cyclic ordering.

We define a $[\binom{p}{m}]$–graph to be a black and white graph with $p$ white vertices and $m$ leaves labeled $\{1, \ldots, m\}$. A $[\binom{p}{m}]$–graph may have additional unlabeled leaves if they are the start half-edge of a white vertex. Figure 1(b) shows an example of a $[\binom{3}{3}]$–graph, with the start half-edges marked by thick lines.

To define the Hochschild complex, we will use the $[\binom{1}{n}]$–graph, denoted $l_n$, depicted in Figure 1(c) which has a single vertex which is white, and $n$ leaves labeled cyclically,
with the first leave as start half-edge. (As \( I_n \) has only one white vertex, we drop its label which is automatically 1 = |V_w|.)

A \([0|m]\)-graph is just an ordinary fat graph whose vertices are at least trivalent and which has \( m \) labeled leaves. Figure 1(a) gives an example of a \([0|3]\)-graph.

We will consider isomorphism classes of black and white graphs. If the graphs are oriented or have labeled leaves, we always assume this is preserved under the isomorphism. Note that when two black and white graphs are isomorphic, the isomorphism is unique whenever each component of the graph has at least one labeled leaf or at least one white vertex: starting with the leaf or the start half-edge of the white vertex, and using that the cyclic orderings at vertices are preserved, one can check by going around the corresponding component of the graph that the isomorphism is completely determined.

2.4. Edge collapses and blow-ups. For a black and white graph \( G \) and an edge \( e \) of \( G \) which is not a cycle and does not join two white vertices, we denote \( G/e \) the set of isomorphism classes of black and white graphs that can be obtained from \( G \) by collapsing the edge \( e \), identifying its two end-vertices, declaring the new vertex to be white with the same label if one of the collapsed vertices was white—in particular, the number of white vertices is constant under edge collapse. Graphs in \( G/e \) have naturally induced cyclic orderings at their vertices. If the new vertex is white, it has a well-defined start half-edge unless the start half-edge of the original white vertex is collapsed with \( e \), in which case there is a collection of possible collapses of \( G \) along \( e \), one for each choice of placement of the start half-edge at the new white vertex among the leaves originating from the collapsed black vertex of the original graph \( G \). (See Figure 2 for an example.)

![Figure 2](image)

**Figure 2.** The two possible collapses of \( e \) in \( G \)

If \( G \) is oriented, the graphs in \( G/e \) inherit an orientation as follows: If \( e = \{h_1, h_2\} \) with \( s(h_1) = v_1, s(h_2) = v_2 \), and writing the orientation of \( G \) in the form \( v_1 \wedge v_2 \wedge h_1 \wedge h_2 \wedge x_1 \wedge \ldots \wedge x_k \), we define the orientation of the collapsed graph to be \( v \wedge x_1 \wedge \ldots \wedge x_k \), where \( v \) is the vertex of the collapsed graph coming from identifying \( v_1 \) and \( v_2 \).

For an (oriented) black and white graph \( G \), we call an (oriented) black and white graph \( \tilde{G} \) a blow-up of \( G \) if there exists an edge \( e \) of \( \tilde{G} \) such that \( G \in \tilde{G}/e \). The first line in Figure 3 shows all the possible blow-ups of the graph \( l_3 \).

2.5. Chain complex of black and white graphs. Let \( BW-\text{Graphs} \) denote the chain complex generated as a \( \mathbb{Z} \)-module by isomorphism classes of (not necessarily connected, possibly degenerate) oriented black and white graphs, modulo the relation that \(-1 \) acts by reversing the orientation. The degree of a black and white graph is

\[
\text{deg}(G) = \sum_{v \in V_b} (|v| - 3) + \sum_{v \in V_w} (|v| - 1),
\]

where \(|v|\) denotes the valence of \( v \). The degenerate graphs have degree 0. The map

\[
d : G \mapsto \sum_{(G, e) \in \tilde{G}/e} \tilde{G}
\]
summing over all blow-ups of $G$ defines a differential on $BW$–Graphs. Indeed, we have

$$(d)^2(G) = \sum_{(\tilde{G}, e)} \left( \sum_{\tilde{G} / f} \tilde{G} \right) = \sum_{(\tilde{G}, f, e)} \tilde{G}$$

and one can check that the orientations of $\tilde{G} / f / e$ and $\tilde{G} / e / f$ are opposite so that each term $(\tilde{G}, f, e)$ cancels with the term $(\tilde{G}, e, f)$.

Let $[p]_m$–Graphs now denote the chain complex generated as a $\mathbb{Z}$–module by isomorphism classes of (not necessarily connected, possibly degenerate) oriented $[p]_m$–graphs, modulo the relation that $-1$ acts by reversing the orientation. Recall that $[p]_m$–graphs are black and white graphs with $p$ white vertices and $m$ labeled leaves, and that the only unlabeled leaves allowed in $[p]_m$–graphs are those which are start half-edge of a white vertex.

A black and white graph $G$ with $p$ white vertices and $m$ labeled leaves has an underlying $[p]_m$–graph $|G|$ defined by $|\tilde{G}| = \tilde{G}$ unless $\tilde{G}$ has unlabeled leaves which are not the start half-edge of a white vertex. In such a leaf $l$ is attached at a trivalent black vertex $v$, the vertex $v$ and the leaf are forgotten in $|\tilde{G}|$, and if such a leaf is attached at a white vertex (which will automatically be at least bivalent) or at black vertex of valence at least 4, we set $|\tilde{G}| = 0$. The orientation of $|\tilde{G}|$ when $|\tilde{G}| \neq \tilde{G}$ (or 0) is obtained by first rewriting the orientation of $G$ in the form $v \wedge l \wedge h_1 \wedge h_2 \ldots$ for $s^{-1}(v) = (l, h_1, h_2)$ in that cyclic ordering, and then removing the first 4 terms.

We now define the differential on $[p]_m$–Graphs as $dG = [d\tilde{G}]$. Figure 3 shows three examples of differentials.

![Figure 3. Differential applied to the graph $l_3$ and to two graphs with an unlabeled start-leaf.](image)

**Lemma 2.1.** $d|G| = |d\tilde{G}|$.

**Proof.** If $G$ has unlabeled leaves at trivalent black vertices which are forgotten in $|G|$, since trivalent black vertices cannot be expanded, $d\tilde{G}$ will also have such unlabeled leaves and it does not alter the differential if they are forgotten before or after we sum over all blow-ups. The case $|G| \neq 0$ follows.

Suppose now that $|G| = 0$. If $G$ has more than one unlabeled leaf at a high valent black vertex or non-start half-edge at a white vertex, it is immediate that $|d\tilde{G}|$ is also 0 as all terms in $d\tilde{G}$ will also have unlabeled leaves of that type. If $G$ has a single unlabeled leaf at a high valent black or white vertex, then $d\tilde{G}$ will have exactly two terms $G_l, G_r$ such that the unlabeled leaf is at a black trivalent vertex, namely the blow-ups of that the vertex blowing out the leaf together with its left and its right neighbor respectively. One has that $|G_l| = |G_r|$ with opposite orientations. □

**Proposition 2.2.** The map $d$ is a differential.

**Proof.** This follows directly from the lemma using the fact that $d$ is a differential: $d^2G = d[dG] = [(d)^2G] = 0$. □
2.6. The open cobordism category $O$. The open cobordism category is a dg-category with objects the natural numbers, thought of as representing disjoint unions of intervals, and morphism given by chain complexes with homology that of the moduli spaces of Riemann cobordisms between the intervals.

Fat graphs, without leaves, were invented to define a cell decomposition of Teichmüller space (see the work of Bowditch-Epstein [2], Harer [16], Penner [36, 37]), and the chain complex $\bigl[0^m\bigr] - Graphs$ defined above is the corresponding cellular complex of the quotient of Teichmüller space by the action of the mapping class group, namely the coarse moduli space of Riemann surfaces. Similarly, fat graphs with leaves define a chain complex for the moduli space of surfaces with fixed boundaries, or with fixed intervals in their boundaries (see Penner [38, 35], Godin [14], Costello [8, Sect. 6] and [9]). As already remarked in 2.3, graphs with leaves have no symmetries. The same holds for Riemann surfaces as soon as part of the boundary of the surface is assumed to have a fixed Riemann structure. It follows that the moduli space, being the quotient of Teichmüller space by a free action of the mapping class group of the surface, is a classifying space for that mapping class group, and the chain complex of $\bigl[0^m\bigr] - Graphs$ of that surface type when $m > 0$ computes the homology of the (now fine) moduli space as well as the homology of the corresponding mapping class group.

Let $S$ be a surface and $I$ a collection of intervals in its boundary. If we denote by $\mathcal{M}(S,I)$ the moduli space of Riemann surfaces with a fixed structure on an $\epsilon$–neighborhood of $I$ (with the convention that $\mathcal{M}(S^1 \times I, \emptyset) = * = \mathcal{M}(\mathbb{D}^2, \emptyset)$, and the moduli space is the coarse moduli space for other surfaces with no intervals in their boundary), we have the following:

**Theorem 2.3.** There is an isomorphism

$$H_\ast\bigl([0^m] - Graphs\bigr) \cong \bigoplus_{(S,I)} H_\ast(\mathcal{M}(S,I))$$

where $(S,I)$ ranges over all (possibly disconnected) oriented surfaces $S$ with $I$ a collection of $m$ labeled intervals in $\partial S$. Here, each component of $S$ must have nonempty boundary.

While the many references indicated above give similar such combinatorial models for moduli space, one may explicitly extract this result from [9], via the enumeration of the cells in Costello’s cellular model for moduli space after Proposition 2.2.3 in [9] with $s = 0$. An alternative reference is the restriction to the open part of [10, Thm A]. We can thus use $\bigl[0^m\bigr] - Graphs$ to provide a model for the open cobordism category, which we do now.

Let $\mathcal{O}$ be the symmetric monoidal dg-category with objects the natural numbers (including 0) and morphisms from $m$ to $n$ the chain complex

$$\mathcal{O}(m,n) := [0^m] - Graphs$$

of fat graphs with $m+n$ labeled leaves. See Figure 4 for examples of morphisms in $\mathcal{O}$.

![Figure 4. Morphisms of $\mathcal{O}(3,5)$ and $A_\infty(6,2)$.](image_url)
vertices and half edges and altering the involution so that the glued leaves are mapped
to each other under the involution.) The orientation is obtained by juxtaposition (wedge
product). The rule for gluing the exceptional graphs is as follows:
- Gluing a leaf labeled on one side has the effect of removing the corresponding leaf
of the other graph if this is a degree 0 operation (i.e. if the leaf was attached to a
trivalent vertex)—otherwise the gluing just gives 0. If the trivalent vertex is \( v \)
with half edges \( h_1, h_2, h_3 \) attached to it in that cyclic order, and the graph has orientation
\( v \land h_1 \land h_2 \land h_3 \land x_1 \land \ldots \land x_k \), then the glued graph has orientation \( x_1 \land \ldots \land x_k \).
- Gluing a doubly labeled leaf has the effect of relabeling the leaf of the other graph if
the labels of the leaf are incoming and outgoing. If both labels are incoming or outgoing,
it attaches the corresponding leaves of the other graph together so they form an edge.

The fact that this gluing is compatible with the gluing of Riemann surfaces along
intervals is [8, Prop 6.1.5]. This can be understood as follows: Fat graphs come from
a cell decomposition of Teichmüller space, a fat graph in a surface defining a dual
decomposition of the surface into polygones. Fat graphs with leaves can be interpreted
as corresponding to polygonal decompositions with an interval around each leave in their
boundary being always part of the dual decomposition. The gluing along leaves then
corresponds in Teichmüller space to gluing such polygonal decompositions along such
specified intervals, remembering the interval in the decomposition of the glued surface.

We note that this gluing along open boundaries does not agree with the one defined in
[22, 23].

The symmetric monoidal structure of \( \mathcal{O} \) is defined by taking disjoint union of graphs.
The identity morphisms and the symmetries in the category are given by (possibly empty)
unions of doubly labeled leaves.

2.7. The categories \( \mathcal{A}_\infty \) and \( \mathcal{A}_{\infty}^+ \). We let \( \mathcal{A}_\infty \) denote the subcategory of directed
forests in \( \mathcal{O} \), i.e. \( \mathcal{A}_\infty \) has the same objects as \( \mathcal{O} \), the natural numbers, and the chain
complex \( \mathcal{A}_{\infty}(m,n) \) of morphisms from \( m \) to \( n \) is generated by graphs which are disjoint
unions of \( n \) trees with a total of \( m_1 + \cdots + m_n = m \) incoming leaves, with each \( m_i > 0 \),
in addition to the root of the tree which is labeled as an outgoing leaves. Here we allow
the degenerate graphs consisting of single leaves labeled both sides (as one input and
one output), as well as the empty graph defining the identity morphism on 0. We let
\( \mathcal{A}_{\infty}^+ \) denote the slightly larger category where also the leaf labeled on one side as an
output is allowed. See Figure 4 (b) for an example of a morphism in \( \mathcal{A}_\infty \).

In 3.1, we will relate these categories to \( \mathcal{A}_\infty \)– and unital \( \mathcal{A}_\infty \)–algebras.

2.8. The open-closed cobordism category \( \mathcal{OC} \). The open-closed cobordism cate-
yogy is a dg-category with objects pairs of natural numbers, thought of as representing
disjoint unions of intervals (open strings) and circles (closed strings), and with morphism
defined as chain complexes on the moduli spaces of Riemann cobordisms between the
collections of intervals and circles. We give here a model of this category, the one used
by Costello and Kontsevich-Soibelman in [8, 27], in terms of black and white graphs.

Let \( \mathcal{OC} \) denote the dg-category with objects pairs of natural numbers \( [m]_n \), for \( m, n \geq 0 \)
representing \( m \) intervals and \( n \) circles, and with morphisms

\[
\mathcal{OC}([m]_1, [n]_2) \subset [m_1 + m_2 + m_3] - \text{Graphs}
\]

the subcomplex of \( [m_1 + m_2 + m_3] \)–graphs with the first \( n_1 \) leaves sole labeled leaves in their
boundary cycle, representing cobordisms from \( m_1 \) intervals and \( n_1 \) circles to \( m_2 \) intervals
and \( n_2 \) circles. Theorem 2.5 below says that the chain complex \( \mathcal{OC}([m]_1, [n]_2) \) does indeed
compute the homology of the moduli space of Riemann structures on such cobordisms.

Given graphs \( G_1 \in \mathcal{OC}([m]_1, [n]_2) \) and \( G_2 \in \mathcal{OC}([m]_2, [n]_1) \), their composition is defined
as the sum \( G_2 \circ G_1 = \sum [G] \) over all possible black and white graphs \( G \) that can be
obtained from \( G_1 \) and \( G_2 \) by:
Given 

Moreover it is associative.

Lemma 2.4. The composition of graphs defined above is a chain map
\[ \mathcal{OC}([m_1], [n_2]) \otimes \mathcal{OC}([n_2], [n_3]) \to \mathcal{OC}([m_1], [n_3]). \]
Moreover it is associative.

Proof. Given \( G_1, G_2 \) as above, we need to check that
\[ d(G_2 \circ G_1) = G_2 \circ dG_1 + (-1)^{|G_1|}dG_2 \circ G_1. \]
Recall from 2.5 that \( dG = [dG] \), for \( d \) the differential in black and white graphs. We have similarly \( G_2 \circ G_1 = [G_2 \circ G_1] \) where \( G_2 \circ G_1 \) denotes the composition of graphs as black and white graphs, without taking the underlying \([p_m]\)-graphs.

We first check that \( d(\hat{G}_2 \circ \hat{G}_1) = \hat{G}_2 \circ d\hat{G}_1 + (-1)^{|G_1|}d\hat{G}_2 \circ \hat{G}_1 \). Call a vertex of \( G_2 \circ G_1 \) special if it comes from a vertex of one of the first \( n_2 \) boundary cycles of \( G_2 \). The left-hand side has terms coming from

1. blowing up at a non-special vertex,
2. blowing up at a special vertex in such a way that the newly created vertices are either white,
   - black with no half-edges of \( G_2 \), or
   - black with at least two half-edges of \( G_2 \),
3. blowing up at a special vertex in such a way that one of the newly created vertices is black with exactly one half-edge of \( G_2 \) attached to it.
The terms of type (1) and (2) are exactly the terms occurring in $G_2 \hat{\circ} dG_1 + (-1)^{|G_1|} \hat{d}G_2 \hat{\circ} G_1$ as black and white graphs, i.e. before taking the underlying $[p_m]$-graphs $[G]$. Indeed, type (1) terms correspond to blowing up at vertices of $G_1$ or $G_2$ which are not affected by the gluing, and type (2) terms correspond either to blowing up a vertex of $G_2$ on an incoming cycle and then attach edges of $G_1$, or, in the case where one of the vertices is black with no half-edges of $G_2$ attached to it, this corresponds to blowing-up at a white vertex of $G_1$ and then glue the resulting graph to $G_2$. This covers all the possibilities.

The fact that the signs agree follows from the fact that the parity of the degree of a graph is the same as the parity of the number of vertices and half-edges in the graph, i.e. that $(-1)^{|G_1|} = (-1)^{|V_1|+|H_1|} = (-1)^{|V_1|+|H_1|-2|(V_1)_w|}$ for $(V_1)_w$, the set of white vertices of $G_1$. Indeed, a vertex contributes with an odd degree precisely when it has even valence, that is when the vertex plus its half-edges give an odd number.

We are left to check that the terms of type (3) cancel in pairs. A "bad" newly created vertex has exactly two half-edges attached to it which are not from $G_1$: one from $G_2$ and one newly created half-edge. Any such graph occurs a second time as a term of type (3) with the role of these two edges exchanged and one checks that the signs cancel.

Now $d(G_2 \circ G_1) = d(G_2 \hat{\circ} G_1) = [d(G_2 \hat{\circ} G_1)]$ by Lemma 2.1. Using the above calculation, we thus get $d(G_2 \circ G_1) = [\hat{d}G_2 \hat{\circ} G_1] + (-1)^{|G_1|}[\hat{d}G_2 \hat{\circ} G_1]$. Now $[\hat{d}G_2 \hat{\circ} G_1] = G_2 \circ dG_1$ as unlabeled leaves attached to trivalent black vertices of $G_1$ will still be attached at trivalent black vertices in the composition, and those attached to higher valent black vertices or to white vertices will still be attached to such. We are left to check that $[\hat{d}G_2 \hat{\circ} G_1] = G_2 \circ G_1$. In this case if $G_2$ has an unlabeled leaf at a trivalent black vertex of an incoming cycle, there will be terms in $\hat{d}G_2 \hat{\circ} G_1$ with this leaf attached to a higher valent black vertex, namely the terms where leaves of $G_1$ are attached at that vertex. These terms vanish in $[\hat{d}G_2 \hat{\circ} G_1]$ and are not present in $dG_2 \circ G_1$ as the vertex is forgotten in $dG_2$.

We check associativity. Suppose $G_1, G_2, G_3$ are three composable graphs and consider the compositions $G_3 \circ (G_2 \circ G_1)$ and $(G_3 \circ G_2) \circ G_1$. The identifications of leaves representing open boundaries will be the same in both cases. For closed boundaries, one checks that each term in the first composition corresponds exactly to a term in the second composition and vice versa: The identification of start-leaves is fixed, and the same in both cases. If we first remove the white vertices of $G_1$, some leaves of $G_1$ might be attached to white vertices of $G_2$. When those white vertices are removed in the further composition with $G_3$, the leaves of $G_1$ that were attached to a white vertex of $G_2$ will be attached in all possible ways, respecting their position in between leaves of $G_2$, to the corresponding boundary circle in $G_3$. If we start by composing $G_2$ and $G_3$, the incoming boundary cycles of $G_2$ with white vertices will become incoming boundary cycles of $G_3 \circ G_2$ partially in the old $G_3$. Attaching now $G_1$ along such a boundary cycle, we see exactly all the terms that occurred before. Indeed, the leaves of $G_1$ will either be attached only to black vertices of the old $G_2$, or some of them might be attached to vertices of $G_3$, in all possible ways, in between old edges of $G_2$ that where previously attached to a white vertex. Left if to check that taking the underlying black-and-white graph gives the same result in both cases: If $G$ in the composition $G_2 \circ G_1$ satisfies $|G| = 0$, then a start-leaf of $G_1$ was attached to a higher valence black vertex of $G_2$ or a white vertex of $G_2$. Any graph $G'$ obtained from $G$ by further attaching $G_3$ will then have that start-leaf attached to a higher valence black vertex of the composed graph, and hence also give 0. On the other hand, if $G'$ in the composition $G_3 \circ G_2$ satisfies $|G'| = 0$, then a start-leaf of $G_2$ was attached to a higher valence black vertex of $G_3$ or a white vertex of $G_3$, and it will remain attached to such a vertex after any further gluing of $G_1$. □
Finally, we verify that the morphism complexes in $\mathcal{OC}$ do indeed compute the homology of the moduli space of open-closed Riemann cobordisms.

**Theorem 2.5.** [8, Prop. 6.1.3] $\mathcal{OC}(\begin{bmatrix} n_1 \\ m_1 \end{bmatrix}, \begin{bmatrix} n_2 \\ m_2 \end{bmatrix})$ is the cellular complex of a space weakly homotopy equivalent to the disjoint union of coarse moduli spaces of Riemann surfaces of every genus, with

- $m_1 + m_2$ labeled open boundary components,
- $n_1 + n_2$ labeled closed boundary components,
- any number of free boundary components (at least one per component with no open or incoming closed boundary)

In particular, this theorem identifies the components of $\mathcal{OC}$ with the topological types of open-closed cobordisms.

Costello denotes this chain complex $\mathcal{G}$ in [8] and describes it in terms of discs (corresponding here to black vertices) and annuli with marked points (corresponding here to white vertices with start half-edges). The description in terms of graphs can be found in [9] as for the category $\mathcal{O}$ after Proposition 2.2.3, though in [9] the white vertices are used to model punctures and do not have start half-edges. In [8], Costello only states the above result in the case $n_1 = 0$, but the case $n_1 > 0$ follows from the fact that the moduli space of Riemann surfaces with $n_1$ open boundaries which are alone in their boundary component is homotopy equivalent to the moduli space with $n_1$ closed boundaries. This model of moduli space is very closely related to Penner’s original fat graph model, though it has the particularity of isolating $n_2$ of the closed boundary cycles in a way which is very similar to Godin’s admissible fat graphs [13, Sec 2.3]. The relationship to admissible fat graphs is made precise in [10, Thm B], which also gives an alternative proof of the above theorem.

Theorem B of [10] can be used to show that the gluing defined above is compatible with that of moduli space. Note also that Theorem 3.1 of [49] shows that $\mathcal{OC}$ identifies as a quasi-isomorphic subcategory of the category of natural operations on the Hochschild complex of $\mathcal{O}$–algebras, where a natural operation is defined as a natural transformation of the iterated generalized Hochschild complex functor, and composition is composition of natural transformations. This shows that the gluing defined here models the composition of the universal natural (formal) operations on the Hochschild complex of $\mathcal{O}$–algebras.

2.9. Annuli. To define the Hochschild complex, we will use a chain complex of annuli. Let $\mathcal{OC}_A(\begin{bmatrix} 0 \\ m \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \subset \mathcal{OC}(\begin{bmatrix} 0 \\ m \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix})$ denote the component of the annuli with $m$ open incoming boundaries on one side, and one closed outgoing boundary on the other side. Each generating graph in this chain complex is build from a white vertex (the outgoing circle) by attaching trees, with possibly one unlabeled leaf as start half-edge for the white vertex. Inside this chain complex, we can consider the sub-chain complex $\mathcal{L}(m)$ of graphs with no unlabeled leaf. Figure 6 shows an example of an graph in $\mathcal{L}(12)$. Let

$L_n = \langle l_n \rangle$ denote the free graded $\mathbb{Z}$-module on a single generator in degree $n - 1$, the

\footnote{Here we employ again the convention that the moduli space of a disk with a single free boundary is a point, as is the moduli of an annulus with two free boundaries.}
graph \( l_n \) of Figure 1(c). By cutting the graphs around their white vertex, the complex \( \mathcal{L}(m) \) can be described as

\[
\mathcal{L}(m) = \bigoplus_{n \geq 1} A_\infty(m, n) \otimes L_n
\]

with differential \( d = d_{A_\infty} + d_L \), where \( d_{A_\infty} \) is the differential of \( A_\infty \) and

\[
d_L : A_\infty(m, n) \otimes L_n \to A_\infty(m, n) \otimes \bigoplus_{1 \leq k < n} A_\infty(n, k) \otimes L_k \to \bigoplus_{1 \leq k < n} A_\infty(m, k) \otimes L_k,
\]

where the first map takes the differential of \( l_n \) in \( OC([0]^m, [1]^n] \) and reads off the blown-up graphs as elements of \( A_\infty(n, k) \otimes L_k \) for various \( k < n \), and the second map is composition in \( A_\infty \).

We will use the notation

\[
d_L(l_n) = \sum_{k < n} f_{n,k} \otimes l_k \in \bigoplus_{1 \leq k < n} A_\infty(n, k) \otimes L_k
\]

for this decomposition of the differential of \( l_n \) in \( OC \).

2.10. The category of Sullivan diagrams \( SD \). Sullivan chord diagrams are usually defined as fat graphs built from a disjoint union of circles by attaching chords, or trees, which should be thought of as “length 0” edges, in such a way that the original circles are still cycles in the resulting graph. One has to be aware that authors sometimes restrict to non-degenerate diagrams, those such that collapsing the chords does not change the homotopy type of the graph (as in for example [6, 7, 19]). There are also marked and unmarked versions, and there can be variations in the way the markings are handled (as in e.g. [40]). We consider here a chain complex of general Sullivan diagrams, also degenerate ones. Our definition is in the spirit of [5] and agrees with that of [45].

We start the section by giving a formal definition, relate it to the informal definition above, and build a chain complex of such diagrams. Then we will show that Sullivan diagram can be identified with a quotient complex of black and white graphs, which is the way they occur in the present paper. This will enable us to define an “open-closed” category \( SD \) of Sullivan diagrams directly as a quotient of the category \( OC \). We will then explain how our category of Sullivan diagrams relates to the one defined in [22], and prove a few facts about the map \( OC \to SD \).

We call a fat graph \( p \)-admissible (in the spirit of [13]) if \( p \) of its boundary cycles are disjoint embedded circles in the graph. We call these \( p \) special cycles admissible cycles and represent them as round circles when drawing such a graph.

**Definition 2.6.** An (oriented) \( [p]^m \)-Sullivan diagram is an equivalence class of (oriented) \( p \)-admissible fat graphs with \( p+m \) leaves, where the first \( p \) leaves are distributed as 1 per admissible boundary cycle and the remaining \( m \) leaves lie in the other cycles. Two such graphs \( G_1, G_2 \) are equivalent if they are connected by a zig-zag of edge collapses between \( p \)-admissible fat graphs, collapsing edges which are not in the \( p \) admissible cycles, for edge collapses as defined in 2.4. (Figure 7 shows four equivalent \( [3]^1 \)-Sullivan diagrams.)

![Figure 7. Equivalent Sullivan diagrams, with one admissible cycle which in each case is the outside of the round circle](image-url)
To a fat graph, one can associate a surface by thickening the graph. As edge collapses respect the topological type of the associated surface, a Sullivan diagram still has an associated topological type.

For a \([p]_{\mathbb{m}}\)-Sullivan diagram \(G\), we let \(E_a\) denote the set of edges lying on the admissible cycles of \(G\). The degree of \(G\) is then defined as

\[
\deg(G) = |E_a| - p.
\]

For example, the Sullivan diagrams of Figure 7 are of degree \(4 - 1 = 3\), while the left picture in Figure 9 is a Sullivan diagram of degree \(6 - 2 = 4\).

Let \([p]_{\mathbb{m}}-SD\) denote the chain complex generated as a graded \(\mathbb{Z}\)-module by all oriented \([p]_{\mathbb{m}}\)-Sullivan diagrams, modulo the relation that \(-1\) acts by reversing the orientation. The boundary map in \([p]_{\mathbb{m}}-SD\) is defined on generators by

\[
dG = \sum_{e \in E_a} G/e,
\]

the sum of all collapses of \(G\) along edges in the admissible boundary cycles\(^8\) of \(G\), with \(G/e\) defined in 2.4.

This chain complex is isomorphic to the complex Cyclic Sullivan Chord Diagrams considered by Tradler-Zeinalian in [45, Def 2.1]. Their diagrams are build from disjoint circles (our admissible cycles) by attaching trees (the chords or non-admissible edges), whereas we allow the chords to be unions of graphs. However, collapsing non-admissible edges, one can alway push the vertices of the representing graph to only lie on the admissible cycles. Hence a Sullivan diagram in our sense is always equivalent to one as in [45] which is a union of admissible cycles together with chords which are edges attached directly to the cycles. The equivalence relation in [45] corresponds this way to the one defined here.

Remark 2.7. The chain complex of \([p]_{\mathbb{m}}\)-Sullivan diagrams of topological type a surface of genus 0 with \(p + 1\) boundary components is a cellular complex for the \(p\)th space of the cactus operad [48, 18]. Indeed, such a Sullivan diagram is made out of \(p\) circles attached to each other in a tree-like fashion, exactly representing a cell in the cactus operad. (See Figure 8.)

\[\text{Figure 8. A Sullivan diagram with 6 admissible cycles modelling a 6-lobed cactus.}\]

The following theorem relates Sullivan diagrams to black and white graphs. In the proof, we will use the equivalence relation in Sullivan diagrams in the opposite way from what we used to relate our definition to that of [45]: we will represent Sullivan diagrams by the graphs in their equivalence class with the maximum number of vertices not on the admissible cycles.

\[\text{It is worth noting that we have already effectively collapsed the remaining edges by the equivalence relation.}\]
Theorem 2.8. The chain complex $[p_m] - SD$ of $[p_m] -$Sullivan diagrams is the quotient of $[p_m] -$Graphs by the graphs with black vertices of valence at least 4 and by the boundaries of such graphs.

Proof. We first note that every Sullivan diagram can be represented by a graph with only trivalent vertices, except for the vertices where the leaf of an admissible cycle is attached, which may be 4-valent. Indeed, if the graph has a higher valence vertex away from the admissible cycles, one can blow it up in any manner one likes and obtain an equivalent graph with trivalent vertices replacing the higher valence vertex. If there is a higher valence vertex on an admissible cycle, it has exactly two contiguous half-edges of that admissible cycle attached to it, unless the leaf of the admissible cycle is at that position, in which case it has three such. Any blow-up of that vertex which keeps the half-edges of the admissible cycle together produces an equivalent graph with the property we want. For the purpose of the proof, we call such graphs essentially trivalent.

Two essentially trivalent Sullivan diagrams are equivalent if and only if they are equivalent through such Sullivan diagrams and diagrams with exactly one 4-valent vertex which is away from the cycles: a single valence 4 vertex at a time suffices since we can do collapses and blow-ups one at a time, and no additional valence 4 (or 5) vertices on the admissible cycles are necessary because there is only one way of blowing-up such a vertex if the two (or three) half-edges of the admissible cycles have to stay together, up to collapses and blow-ups away from the admissible cycle.

Given an essentially trivalent $[p_m] -$Sullivan diagram, we get a $[p_m] -$graph by collapsing the admissible cycles to white vertices. If the leaf of the $i$th admissible cycle is at a 3-valent vertex, we place an unlabeled start-leaf at that position on the $i$th white vertex, and if it is at a 4-valent vertex, we remove it and define the remaining half-edge after the collapse to be start half-edge. (See Figure 9 for an example.)

![Figure 9](image)

Figure 9. Essentially trivalent Sullivan diagram (with admissible cycles the inside of the round circles) and the corresponding black and white graph

Given a $[p_m] -$graph, one can similarly obtain an essentially trivalent $[p_m] -$Sullivan diagram by expanding the white vertices to circles and placing leaves at the spots corresponding to start-edges. These two maps are inverses of one another, and the equivalence relations agree under the maps by the above remarks.

Note moreover that the degrees agree: the degree of a $[p_m] -$graph $G$ is $\sum_{v \in V_b} |v| - 3 + \sum_{v \in V_w} |v| - 1$. As all black vertices of the graphs occurring here are trivalent, the first sum gives 0. On the other hand, the valence of a white vertex in $G$ is the number of admissible edges on the corresponding admissible cycle of the associated Sullivan diagram.

We are left to check that the boundary maps also agree. Given an essentially trivalent Sullivan diagram, the boundary map in $[p_m] - SD$ is a sum of Sullivan diagrams, each with a higher valence vertex on an admissible cycle. Blowing up that vertex in the only possible manner to obtain an essentially trivalent graph corresponds exactly under the equivalence above to a term in the differential of the associated $[p_m] -$graph, and all the terms of the differential of this $[p_m] -$graph that do not have valence 4 or more black vertices will occur this way. □
Recall from 2.8 that the open-closed category \( OC \) has objects pairs of natural numbers \([n_1]\) and morphisms complexes \( OC([n_1],[n_2]) \subset [n_1+n_2,m_1+m_2] \) — Graphs, the subcomplex of graphs with the first \( n_1 \) leaves alone in their boundary cycles. As composition in \( OC \) can only increase the valence of black vertices, it still gives a well-defined composition when quotienting out by the graphs with black vertices of valence 4 or more. Hence, using the above theorem, we can simply define the category of Sullivan diagrams as a quotient category of \( OC \):  

**Definition 2.9.** Let \( SD \) be the category with objects pairs of natural numbers \([n_1]\), with \( m,n \geq 0 \), and morphisms from \([m_1]\) to \([m_2]\) the quotient of \( OC([m_1],[m_2]) \) by the graphs having black vertices of valence higher than 3 and by the boundary of such graphs.

Note that in terms of “classical” Sullivan diagrams, as in Definition 2.6, admissible cycles are considered here as outgoing boundary circles, while incoming boundary circles are ordinary cycles in the graph. The composition of Sullivan diagrams \( G_1, G_2 \) is defined in classical terms by gluing the \( i \)th admissible cycle \( G_1 \) to the \( i \)th incoming cycle of a graph \( G_2 \) by attaching the edges which had boundary points on this admissible cycle of \( G_1 \) to edges of admissible cycles of \( G_2 \). It is defined by attaching the edges of the first graph at vertices of the second along the corresponding cycle in all possible ways, but attaching edges at black vertices creates vertices of valence 4 or higher, and hence is trivial in Sullivan diagrams. On the other hand, attaching edges at white vertices corresponds to attaching at admissible edges in the classical picture.

By definition, we have a quotient functor  
\[
\pi : OC \to SD
\]

A direct consequence of Theorem 2.8 is the following:

**Proposition 2.10.** The quotient functor \( \pi : OC \to SD \) induces an isomorphism  
\[
H_0(OC([n_1],[n_2])) \cong H_0(SD([m_1],[m_2]))
\]

for each \( n_1, m_1, n_2, m_2 \).

The chain complex of Sullivan diagrams is a lot smaller than that of all fat graphs, or all \([n]\)-graphs, and hence computations of its homology are more approachable. It is for example not hard to compute that the component of the pair of pants in \( SD([3],[1]) \) is a complex that computes the homology of \( S^3 \times S^1 \). The corresponding component of \( OC \) computes the homology of the framed disk operad \( fD(2) \simeq S^1 \times S^1 \times S^1 \). The map \( OC \to SD \) in this case is induced by the canonical embedding of the first two \( S^1 \)–factors as a standard torus in the 3-sphere. Note that Sullivan diagrams are asymmetric in their inputs/outputs, and in fact it follows from Remark 2.7 that \( SD([3],[1]) \) is quasi-isomorphic to \( OC([3],[1]) \).

The fact that \( SD \) is a (quite drastic) quotient of \( OC \) makes one expect that, in homology, the projection \( \pi \) kills many things. We make this precise by analysing the map componentwise: denote by  
\[
\pi_S : OC_S([n_1],[n_2])) \to SD_S([m_1],[m_2]))
\]

the functor \( \pi : OC \to SD \) restricted to the component of morphisms of type \( S \), where \( S \) is a generator in \( H_0OC_S([n_1],[n_2])) \cong H_0SD_S([m_1],[m_2])) \), i.e. a topological type of cobordism.

We have the following general vanishing result:

**Proposition 2.11.** Suppose \( m_1 + m_2 + n_1 > 0 \) and \( S \) is a generator of \( H_0OC([n_1],[n_2])) \) which is a connected surface of genus \( g \). Then there exists \( S' \in H_0OC([n_1],[m_2])) \) and
a map \( f : OC_S([n_1], [m_2+1]) \to OC_S([n_1], [m_2]) \) which is an isomorphism in homology in degrees \( * \leq \frac{2g}{3} \) and such that the image of the composition

\[
OC_S([n_1], [m_2+1]) \xrightarrow{f} OC_S([n_1], [m_2]) \xrightarrow{\pi} SD_S([n_1], [m_2])
\]

is concentrated in degree 0. In particular, the stable classes of positive degree map to 0 under the map \( H_*(\pi) : H_*(OC) \to H_*(SD) \).

Here by a stable class, we mean a class in that lives in the stable range, i.e. a class in \( H_*(OC_S([n_1], [m_2])) \) of degree \( * \leq \frac{2g}{3} \) for \( g \) the genus of the component of lowest genus in \( S \).

For \( m_1 = m_2 = n_1 = 0 \), the situation is a little more subtle. An analogous statement can be made though using in place of \( f \) a map that replaces a fixed boundary by a free boundary, and hence is not an isomorphism in homology stably.

**Proof.** Suppose first that \( m_1 + m_2 > 0 \). Then \( S' \) can be obtained from \( S \) by gluing discs on the \( n_2 \) closed outgoing boundaries of \( S \) and adding a open outgoing boundary on a boundary component containing some other open boundary. We can reconstruct the topological type of \( S \) from \( S' \) by gluing a \( n_2 \)-legged pair of pants \( P \) along an open boundary as shown in Figure 10. Choosing a degree 0 representative of \( P \) in \( OC([0], [0]) \), the map \( f \) above is just induced by composition with \( P \) in \( OC \). The fact that it is an isomorphism in homology in the given range is part of Harer’s homological stability theorem ([15], with the improved range of [1, 42], see also [50] for a version with punctures). The fact that the composition \( \pi_S \circ f \) lands in degree 0 follows from the commutativity of the diagram

\[
\begin{array}{ccc}
OC([n_1], [m_2+1]) & \xrightarrow{f} & OC([n_1], [m_2]) \\
\downarrow \pi & & \downarrow \pi \\
SD([n_1], [m_2+1]) & \xrightarrow{\pi(f)} & SD([n_1], [m_2])
\end{array}
\]

and the fact that the complex \( SD([n_1], [m_2+1]) \) is concentrated in degree 0.

For \( n_1 > 0 \), we have an isomorphism \( OC_S([n_1], [m_2]) \cong OC_S([n_1-1], [m_2]) \), and similarly for \( SD \), for \( S \) the surface obtained from \( S \) by replacing an incoming closed boundary by an incoming open boundary, alone on that component. This reduces the case \( m_1 + m_2 = 0 \) with \( n_1 > 0 \) to the previous one.

We end by mentioning an alternative definition of Sullivan diagrams.

**Remark 2.12** (Sullivan diagrams as arcs in a surface). The cellular chains of the category of open-closed Sullivan diagrams \( Sull^{oc}_1 \) defined in [22, Sec 2.3,6.5] is not the same as the category \( SD \) we just define, though it is isomorphic to it when restricted to the closed part and switching the role of incoming and outgoing boundaries. Indeed, a Sullivan diagram in [22] with \( n_2 \) incoming closed boundaries and \( n_1 \) outgoing closed boundaries is defined as a weighted collection of arcs in a surface \( F \), where the arcs start at the incoming closed boundaries and end elsewhere, and such that the sum of the weights at each incoming closed boundary is equal to 1. One can reconstruct a classical
metric Sullivan diagram from such a collection of arcs by having a circle for each incoming closed boundary with an edge of length the associated weight for each arc starting at that boundary. The chords are then obtained by choosing a fat graph representative of each component of the surface $F$ cut along the arcs. (See Figure 11 for an example.) A collection of $k_1 + \cdots + k_{n_2}$ arcs corresponds to a cell $\Delta^{k_1-1} \times \cdots \times \Delta^{k_{n_2}-1}$ represented by the a diagram of degree $k_1 + \cdots + k_{n_2} - n_2$ in our definition above. The boundary in the arc description is defined by forgetting the arcs. This corresponds to gluing surfaces in the complement of the arcs, which corresponds to collapsing an edge in the classical description.

![Figure 11. A Sullivan diagram described by a system of arcs in a surface, the dual Sullivan diagram in the surface, and the same Sullivan diagram without the surface.](image)

3. Algebras

In this section, we describe the main types of algebras we will consider in the present paper. We use the formalism of PROPs of MacLane [31, §24], and describe algebraic structures via symmetric monoidal functors from given symmetric monoidal categories: recall that a PROP, product and permutation category, in the category $\text{Ch}$ is a symmetric monoidal dg-category with objects the natural numbers, and an algebra over that PROP is a symmetric monoidal functor from that category to $\text{Ch}$. We describe in this section the main PROPs we will use, and give descriptions of their algebras. (A good introductory reference for PROPs and operads is [47]).

If $E$ is a symmetric monoidal category and $\Phi : E \to \text{Ch}$ is a functor, we say that $\Phi$ is symmetric monoidal if there are maps $\Phi(n) \otimes \Phi(m) \to \Phi(n + m)$ natural in $n$ and $m$ and compatible with the symmetries of $\text{Ch}$ and $E$. We say that $\Phi$ is split monoidal if these maps are isomorphisms and $h$-split if they are quasi-isomorphisms.

3.1. $A_\infty$–algebras. Recall from 2.7 the symmetric monoidal dg-category $A_\infty$. This category is freely generated as a symmetric monoidal category by the morphisms from $k$ to 1, for each $k \geq 2$, represented by a tree (or rather a corolla) $m_k$ of degree $k - 2$ with a single vertex with $k$ incoming and 1 outgoing leaves. A symmetric monoidal functor

$$\Phi : A_\infty \to \text{Ch}$$

corresponds precisely to giving an $A_\infty$–structure on $\Phi(1)$ with multiplication and higher multiplications

$$\mu_k : \Phi(1)^{\otimes k} \to \Phi(k) \xrightarrow{\Phi(m_k)} \Phi(1)$$

for each $k \geq 2$, where the first map uses the monoidal structure of $\Phi$. The fact that this defines an $A_\infty$–structure comes from the fact that planar, or equivalently “fat” trees define a cellular decomposition of Stasheff’s polytopes. See for example [30, C.2, 9.2.7].

There is an additional generating map $u : 0 \to 1$ of degree 0 in the category $A_\infty^+$, a singly labeled outgoing leaf, which behaves as a unit for the multiplication $\mu_2$. So if $\Phi$ extends to a symmetric monoidal functor with source $A_\infty^+$, the $A_\infty$–algebra $\Phi(1)$ is
equipped with a unit for the multiplication $\mu_2$. This is what we will mean by a unital $A_\infty$–algebra.

More generally, we will consider in this paper symmetric monoidal dg-categories $\mathcal{E}$ equipped with a symmetric monoidal functor $i : A_\infty \to \mathcal{E}$, so that $\mathcal{E}$–algebras, i.e. symmetric monoidal functors $\mathcal{E} \to \text{Ch}$ have an underlying $A_\infty$–algebras by precomposition with $i$. We will call such a pair $(\mathcal{E}, i)$ a PROP with $A_\infty$–multiplication. If $\mathcal{E}$ admits a functor $i : A_\infty^+ \to \text{Ch}$, we call the pair $(\mathcal{E}, i)$ a PROP with unital $A_\infty$–multiplication.

3.2. Frobenius and $A_\infty$–Frobenius algebras. By a symmetric Frobenius algebra, or just Frobenius algebra for short, we mean a dg-algebra with a non-degenerate symmetric pairing. A Frobenius algebra can alternatively be defined as a chain complex with is just a Frobenius algebra.

Frobenius and

3.2.

where $a, b$ are elements of the algebra, $\nu$ is the coproduct, $\nu(a) = \sum_i a_i \otimes a_i''$, and $\nu(b) = \sum_j b_j \otimes b_j''$. (See [26, 2.2.2.9.2.3.24] for the various equivalent definitions of (symmetric) Frobenius algebras.)

The cohomology of a closed manifold is an example of a Frobenius algebra, though with a pairing of degree $-d$ for $d$ the dimension of a manifold. Because of this, Frobenius algebras are sometimes called a Poincaré duality algebra (see e.g. [28, Def 2.1] in the commutative setting).

Recall from 2.6 the open cobordism category $\mathcal{O}$ with objects the natural numbers and morphisms the chain complexes of moduli spaces of open cobordisms. We denote by $H_0(\mathcal{O})$ the dg-category with the same objects but with morphisms from $n$ to $m$ concentrated in degree 0, given by $H_0(\mathcal{O})$. In other words, the morphisms from $n$ to $m$ is the free module on the topological types of cobordisms from $n$ to $m$ intervals. Corollary 4.5 of [29] says that split symmetric monoidal functors $\Phi : H_0(\mathcal{O}) \to \text{Ch}$ are in 1-1 correspondence with symmetric Frobenius algebras.

Replacing $H_0(\mathcal{O})$ in the above by the original open cobordism category $\mathcal{O}$, we get an $A_\infty$–version of Frobenius algebras: We call a split symmetric monoidal functor

$$\Phi : \mathcal{O} \to \text{Ch}$$

(or by abuse of language its value at 1, $\Phi(1)$) an $A_\infty$–Frobenius algebra. If $\Phi$ is $h$–split, $\Phi$ could be called an extended $A_\infty$–Frobenius algebra, following [8, 7.3]. In either case, note that by restriction along $i : A_\infty \to \mathcal{O}$, $\Phi$ equips $\Phi(1)$ with the structure of an $A_\infty$–algebra (in fact an $A_\infty^+$–algebra).

In addition to the $A_\infty$–structure, the morphism $tr : 1 \to 0$ in $\mathcal{O}$ given by a single incoming labeled leaf (the mirror of the unit $u$) gives a map $tr : \Phi(1) \to \Phi(0)$. When $\Phi$ is $h$–split, $\Phi(0)$ is quasi-isomorphic to $\mathbb{Z}$ (concentrated in degree 0). The map induced by the trace in homology

$$tr : H_s(\Phi(1)) \to H_s(\Phi(0)) = \mathbb{Z},$$

which, along with the associative multiplication coming from the $A_\infty$–structure, equips $H_s(\Phi(1))$ with the structure of a Frobenius algebra. When $\Phi$ is split, $\Phi(0) = \mathbb{Z}$, so one gets a trace defined on $\Phi(1)$, which is non-degenerate.

The structure of an $A_\infty$–Frobenius algebra is generated by this $A_\infty$–structure together with the trace; that is, all chain level operations from the moduli of surfaces in the open category can be derived from these operations, as is indicated in section 7.3 of [8]. Roughly speaking, having a non-degenerate trace allows one to construct the pairing and the copairing. Together with the $A_\infty$–structure, one can recover any fat graph. We expand upon this in the following section.
3.3. Positive boundary or “noncompact” $A_\infty$–Frobenius algebras. Define the positive boundary open cobordism category $O^b$ to be the subcategory of $O$ with the same objects and whose morphisms are given by the subcomplex of fat graphs whose associated topological type is a disjoint union of surfaces, all of which have at least one outgoing boundary.

There are certain morphisms in $O^b$ whose role should be highlighted. Certainly, $O^b$ contains all of the category $A_\infty^+$, and in particular the corollas $m_k : k \to 1$. It also contains the coproduct $\nu$ — the morphism from 1 to 2 given by the corolla with one incoming and two outgoing leaves.

**Proposition 3.1.** The category $O^b$ is generated as a symmetric monoidal category by its subcategory $A_\infty^+$ and the coproduct $\nu$.

**Proof.** First, define the copairing $C := \nu \circ u : 0 \to 2$; this is an exceptional graph with no vertices. Composing a disjoint union of $n - 1$ copies of $C$ with $m_{k+n-1}$ gives the corolla $c_{k,n} : k \to n$ for any $k \geq 0$ and $n \geq 1$. Note that we can write $m_k = c_{k,1}$, $u = c_{0,1}$, $\nu = c_{1,2}$, and $C = c_{0,2}$. Then the symmetric monoidal subcategory generated by $A_\infty$, $u$, and $\nu$ is the same as the one generated by all of the $c_{k,n}$.

**Figure 12.** the corolla $c_{3,2}$ as a composition $m_4 \circ (C \sqcup id)$

Now let $\Gamma : m \to n$ be an arbitrary graph in $O^b$; we may assume that $\Gamma$ is connected and non-empty, and so $n \geq 1$. Pick a maximal tree $T$ of edges of $\Gamma$ and choose an outgoing leaf of $\Gamma$ attached at a vertex $v$ (which is included in $T$ by maximality). There is a unique way to orient the edges of $T$ to make it rooted at $v$. Extend that orientation (arbitrarily) to an orientation of $\Gamma$, though keeping the “in” and “out” orientations of the leaves. Since $T$ includes all of the vertices of $\Gamma$, there is always at least one outgoing edge (or leaf) at each vertex. Thus the star of each vertex is $c_{k,n}$ for some value of $k$ and $n$. Consequently $\Gamma$ is obtained as an iterated composition of (disjoint unions of) the $c_{k,n}$, and so is in the symmetric monoidal subcategory generated by them. $\square$

The relations between these generators can be summarized (in a pithy if not particularly helpful way) by saying that two compositions of generators are equal if the fat graphs that they define are the same. For instance, the Frobenius relation

$$(\text{coproduct} \sqcup id) \circ (id \sqcup \text{product}) = (id \sqcup \text{coproduct}) \circ (\text{product} \sqcup id)$$

expresses the fact that the fat graphs in Figure 13 are isomorphic.

**Figure 13.** Frobenius relation

Noting that $O^b$ contains a copy of $A_\infty^{op}$, extending the coproduct (though with no counit!), Proposition 3.1 gives us:

**Corollary 3.2.** A split symmetric monoidal functor $\Phi : O^b \to \text{Ch}$ makes $A := \Phi(1)$ into a unital $A_\infty$–algebra and non-counital $A_\infty$–coalgebra.

---

We should be careful to indicate the labeling of the leaves in $c_{k,n}$, but since we will consider the symmetric monoidal category generated by these, any choice will suffice.
4. Bar constructions

In this section, we define the classical double bar construction, as studied by many authors, and a quotient version of it by symmetries occurring in [8]. This less well-known bar construction has the advantage of providing resolutions of symmetric monoidal functors. (See Proposition 4.3.)

Given a dg-category $C$ and dg-functors $\Phi : C \to \text{Ch}$ (which we can think of as a $C$-module) and $\Psi : C^{op} \to \text{Ch}$ (a $C^{op}$-module), define the $p$th simplicial level of the double bar construction

$$B_p(\Phi, C, \Psi) = \bigoplus_{m_0, \ldots, m_p \in \text{Obj}(C)} \Phi(m_0) \otimes C(m_0, m_1) \otimes \cdots \otimes C(m_{p-1}, m_p) \otimes \Psi(m_p).$$

If $C$ is symmetric monoidal with objects the natural numbers under addition, let $\Sigma \equiv \prod \Sigma_n$ denote the subcategory of $C$ with the same objects and with morphisms the symmetries in $C$. Then we can define similarly

$$B_p^\Sigma(\Phi, C, \Psi) = \bigoplus_{m_0, \ldots, m_p \in \text{Obj}(C)} \Phi(m_0) \otimes_{\Sigma} C(m_0, m_1) \otimes_{\Sigma} \cdots \otimes_{\Sigma} C(m_{p-1}, m_p) \otimes_{\Sigma} \Psi(m_p)$$

where $X \otimes_{\Sigma} Y$ denotes the quotient of $X \otimes Y$ by $x \cdot f \otimes y \sim x \otimes f \cdot y$ for any $f \in \Sigma$ with $f$ acting by pre- or post-composition on the middle factors and via $\Phi(f)$ and $\Psi(f)$ on the first and last factors.

Denoting elements of $B_p(\Phi, C, \Psi)$ by $a \otimes b_1 \otimes \cdots \otimes b_p \otimes c$, let $d_i : B_p \to B_{p-1}$, the $i$th face map, be defined by

$$d_0(a \otimes b_1 \otimes \cdots \otimes b_p \otimes c) = \Phi(b_1)(a) \otimes b_2 \otimes \cdots \otimes b_p \otimes c,$$

$$d_i(a \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes b_i \otimes b_{i+1} \otimes \cdots \otimes b_p \otimes c) = a \otimes b_1 \otimes \cdots \otimes b_{i-1} \otimes b_{i+1} \otimes \cdots \otimes b_p \otimes c \text{ for } 0 < i < p,$$

$$d_p(a \otimes b_1 \otimes \cdots \otimes b_{p-1} \otimes c) = a \otimes b_1 \otimes \cdots \otimes b_{p-1} \otimes \Psi(b_p)(c).$$

This makes $B(\Phi, C, \Psi) = \bigoplus_{p \geq 0} B_p(\Phi, C, \Psi)$, the double bar construction, into a semi-simplicial chain complex, and a chain complex with differential $D_p = (-1)^i d_i$ where $\delta$ denotes the differential of $B_p(\Phi, C, \Psi)$ as a tensor product of chain complexes, and $d = \sum_{i=0}^p (-1)^i d_i$ denotes the simplicial differential.

As all the face maps are well-defined over $\Sigma$, we have that $B^\Sigma(\Phi, C, \Psi) = \bigoplus_{p \geq 0} B_p^\Sigma(\Phi, C, \Psi)$ is also a semi-simplicial chain complex. (In fact, $B(\Phi, C, \Psi)$ is a simplicial chain complex, in that it admits well-defined degeneracies, but this is not true for $B^\Sigma(\Phi, C, \Psi)$.)

Taking $\Psi = C(-, m)$ to be the $C^{op}$-module represented by an object $m$ of $C$, we note moreover that the bar construction $B(\Phi, C, C(-, m))$ is natural in $m$, i.e. we get a functor $B(\Phi, C, C) : C \to \text{Ch}$ with value $B(\Phi, C, C(-, m))$ at $m \in \text{Obj}(C)$.

**Proposition 4.1.** For any functor $\Phi : C \to \text{Ch}$ there are quasi-isomorphisms of functors

$$\alpha : B(\Phi, C, C) \xrightarrow{\simeq} \Phi \quad \text{and} \quad \alpha^\Sigma : B^\Sigma(\Phi, C, C) \xrightarrow{\simeq} \Phi$$

In particular, $B(\Phi, C, C(-, m)) \cong B^\Sigma(\Phi, C, C(-, m))$ for each $m$.

The result is well-known for the usual bar construction $B$. We recall the proof here and show that it also applies to $B^\Sigma$.

**Proof.** Let $\alpha = \oplus_{p \geq 0} : B(\Phi, C, C(-, m)) = \oplus_{p \geq 0} B_p(\Phi, C, C(-, m)) \to \Phi(m)$ be defined by $\alpha_0(a \otimes c) = \Phi(c)(a)$ and $\alpha_p = 0$ for $p > 0$. This is natural in $m$. Let $\beta : (\Phi(m) \to B(\Phi, C, C(-, m))$ be defined by $\beta(a) = a \otimes 1_m \in \Phi(m) \otimes C(m, m)$, where $1_m$ here denotes the identity on $m$. We have $\alpha \circ \beta = id$ and $\beta \circ \alpha \simeq id$; an explicit chain homotopy is given by $h_i = s_p \circ \cdots \circ s_{i+1} \circ \eta \circ d_{i+1} \circ \cdots \circ d_p$, where $s_i$ is the $i$th degeneracy, introducing an identity at the $i$th position, and $\eta$ is the “extra degeneracy” which introduces an identity at the right-most spot. Explicitly, $h_i$ takes $a \otimes b_1 \otimes \cdots \otimes b_p \otimes c$ to $a \otimes b_1 \otimes \cdots \otimes b_i \otimes \circ b_{i+1} \otimes \cdots \otimes b_p \otimes \cdots \otimes 1_m$. Hence $\alpha$ gives a natural transformation by quasi-isomorphisms between the functors $B(\Phi, C, C)$ and $\Phi$. 
For $B^Σ$, we now just note that the maps $α, β$ and $h_i$ are well-defined over $Σ$. (For $h_i$, the degeneracies $s_j$ are not well-defined but the above composition with $η$ is.)

**Remark 4.2.** More generally, one can show that $B(M, C, N) \simeq B^Σ(M, C, N)$ if $M$ or $N$ is quasi-free (i.e., free as a $C$-module, if one ignores the differential).

**Proposition 4.3.** If $C$ is (symmetric) monoidal and $Φ : C \to Ch$ is monoidal, then $B(Φ, C, C)$ and $B^Σ(Φ, C, C)$ are monoidal. If $Φ$ is symmetric monoidal, then so is $B^Σ(Φ, C, C)$. Moreover, if $Φ$ is $h$-split, $B(Φ, C, C)$ and $B^Σ(Φ, C, C)$ are both $h$-split.

**Proof.** The monoidal structure of $B^Σ(Φ, C, C)$ comes directly from that of $Φ$ and $C$, taking $(a ⊗ f_1 ⊗ \ldots ⊗ f_{p+1}) ⊗ (a' ⊗ f_1' ⊗ \ldots ⊗ f'_{p+1})$ to $(a ⊗ a') ⊗ (f_1 ⊗ f_1') ⊗ \ldots ⊗ (f_{p+1} ⊗ f'_{p+1})$, where $⊗$ denotes the monoidal structure of $Φ$ and $⊗$ that of $C$.

We want to check that $B^Σ(Φ, C, C)$ is in fact symmetric monoidal, i.e. that the diagram

$$B^Σ(Φ, C, C(−, n)) ⊗ B^Σ(Φ, C, C(−, m)) \twoheadrightarrow B^Σ(Φ, C, C(−, n + m))$$

$$\tau_⊗ \downarrow \quad \tau_⊗$$

$$B^Σ(Φ, C, C(−, m)) ⊗ B^Σ(Φ, C, C(−, n)) \twoheadrightarrow B^Σ(Φ, C, C(−, m + n))$$

commutes, where $τ_⊗$ denotes the symmetry in the category of chain complexes and $τ_⊗$ the symmetry of $C$. This means that we need

$$(a' ⊗ a) ⊗ (f_1' ⊗ f_1) ⊗ (f_2 ⊗ f_2') ⊗ \ldots ⊗ (f_{p+1} ⊗ f_{p+1})$$

$$(a ⊗ a') ⊗ (f_1 ⊗ f_1') ⊗ (f_2 ⊗ f_2') ⊗ \ldots ⊗ (f_{p+1} ⊗ f_{p+1})$$

This holds because $(f_1 ⊗ f_1') ⊗ τ_⊗ = τ_⊗ ⊗ (f_1' ⊗ f_1)$ in $C$ and $Φ(τ)(a ⊗ a') = a' ⊗ a$ as $Φ$ is symmetric monoidal.

The fact that $Φ$ is $h$-split implies $B(Φ, C, C)$ and $B^Σ(Φ, C, C)$ are $h$-split; this follows from the commutativity of the following diagram:

$$B(Φ, C, C)(m) \twoheadrightarrow B(Φ, C, C)(n + m)$$

$$\alpha \downarrow \quad \alpha$$

$$Φ(m) \twoheadrightarrow Φ(n + m).$$

Note that in the above proposition, strengthening the assumption on $Φ$ to be split still only yields $B^Σ(Φ, C, C)$ $h$-split.

5. Hochschild complex operator

Let $E$ be a symmetric monoidal dg-category which admits a symmetric monoidal functor $i : A_∞ \to E$, for $A_∞$ the category defined in 2.7. For simplicity, and because all our examples are of this sort, we assume that $i$ is the identity on objects, i.e. that $E$ is a PROP with $A_∞$-multiplication. Recall from 3.1 that $E$-algebras, i.e. symmetric monoidal functors $E \to Ch$, have an underlying $A_∞$-algebra structure by precomposition with $i$, and hence have a well-defined Hochschild complex. We define in this section a generalization of the Hochschild complex in the form of an operator $C$ on dg-functors $Φ : E \to Ch$ with the property that, if $Φ$ is symmetric monoidal, the value of $C(Φ)$ at $0$ is the usual Hochschild complex of the underlying $A_∞$-algebra. The value of $C(Φ)$ at $n$ can more generally be identified with the higher Hochschild homology à la Pirashvili [39] associated to the simplicial set which is a union of a circle and $n$ points.

In 5.1 we study the basic properties of our Hochschild complex operator and in 5.2 we prove our main theorem, Theorem 5.11, which gives a way of constructing actions on Hochschild complexes.
Recall from 2.9 the functor $L : \mathcal{A}_\infty^{op} \to \text{Ch}$ defined by

$$L(k) = \bigoplus_{n \geq 1} \mathcal{A}_\infty(k,n) \otimes L_n$$

for $L_n = (l_n)$.

Let $\mathcal{E}$ be a monoidal dg-category. Given a functor $\Phi : \mathcal{E} \to \text{Ch}$ and an object $m \in \mathcal{E}$, we can define a new functor

$$\Phi(- + m) : \mathcal{E} \to \text{Ch}$$

by setting $\Phi((- + m)(n)) = \Phi(n + m)$ and $\Phi((- + m)(f)) = \Phi(f + id_m)$. Note that for any morphism $g \in \mathcal{E}(m,m')$, $\Phi(id + g)$ induces a natural transformation $\Phi((- + m) \to \Phi((- + m'))$.

Given functors $F : \mathcal{C} \to \text{Ch}$ and $G : \mathcal{C}^{op} \to \text{Ch}$, we denote by

$$F \otimes_{\mathcal{C}} G = \bigoplus_{k \in \text{Obj}(\mathcal{C})} F(k) \otimes G(k)/\sim$$

the tensor product of $F$ and $G$, where the equivalence relation is given by $f(x) \otimes y \sim x \otimes f(y)$ for any $x \in F(k)$, $y \in G(l)$ and $f \in \mathcal{C}(k,l)$. This is a chain complex with differential $d = d_F + d_G$ (with the usual Koszul sign convention).

**Definition 5.1 (Hochschild complex).** Let $(\mathcal{E}, i)$ be a PROP with $\mathcal{A}_\infty$–multiplication. For a functor $\Phi : \mathcal{E} \to \text{Ch}$, define its Hochschild complex as a functor $C(\Phi) : \mathcal{E} \to \text{Ch}$ given on objects by

$$C(\Phi)(m) := i^*\Phi((- + m) \otimes_{\mathcal{A}_\infty} \mathcal{L}$$

and on morphisms by

$$C_s(\Phi)(f) := i^*\Phi(id + f) \otimes id.$$

Note that $\mathcal{L}$ is free as a functor to graded vector spaces, so as a graded vector space,

$$C(\Phi)(m) \cong \bigoplus_{n \geq 1} \Phi(n + m) \otimes L_n \cong \bigoplus_{n \geq 1} \Phi(n + m)[n - 1]$$

where the second isomorphism comes from the fact that each $L_n$ is generated by a single element in degree $n - 1$. The differential is given, for $x \in \Phi(n + m)$, by

$$d(x \otimes l_n) = d_{\Phi} x \otimes l_n + (-1)^{|z|} \sum_{k=1}^{n-1} \Phi(i(f_{n,k}) + id_m)(x) \otimes l_k$$

with $f_{n,k}$ the terms of the differential of $L_n$ as defined in 2.9.

The construction is natural in $\Phi$ and $\mathcal{E}$ in the following sense: Given a factorization of $i$ as $\mathcal{A}_\infty \xrightarrow{i'} \mathcal{E}' \xrightarrow{i} \mathcal{E}$ and a functor $\Phi : \mathcal{E} \to \text{Ch}$, we have $C(j^*\Phi) \cong j^*C(\Phi)$, and given two functors $\Phi, \Psi : \mathcal{E} \to \text{Ch}$ and a natural transformation $\eta : \Phi \to \Psi$, we get a natural transformation $C(\eta) : C(\Phi) \to C(\Psi)$.

**Remark 5.2.** The operator $C$ generalizes the usual Hochschild complex of $A_\infty$–algebras in the sense that for $\Phi : \mathcal{A}_{\infty} \to \text{Ch}$ symmetric monoidal, $C_s(\Phi)(0)$ is the usual Hochschild complex of the $A_\infty$–algebra $\Phi(1)$ as in e.g. [27, 7.2.4]. In the case of a strict graded algebra, taking as generator of $L_n$ the graph $l_n$ of Figure 1 with orientation $v \wedge h_1 \wedge \ldots \wedge h_n$ and using the sign convention for the product given in Figure 17, our differential is explicitly given by the following formula: for a $n$–chain $a_0 \otimes \ldots \otimes a_n$ of the Hochschild complex of an algebra $A$, we have

$$d(a_0 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} (-1)^{a_0 + \ldots + a_{i-1}} a_0 \otimes \ldots \otimes da_i \otimes \ldots \otimes a_n$$

$$+ (-1)^{a_0 + \ldots + a_n} \sum_{i=0}^{n-1} (-1)^{i+1} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n$$

$$+ (-1)^{n+1} (a_{n+1})(a_0 + \ldots + a_{n-1}) a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1},$$

where $a_i$ in a superscript denotes the degree of $a_i$. 

Note though that we have defined the Hochschild complex for any functor $\Phi : \mathcal{E} \to \text{Ch}$, not just for monoidal ones. In particular, we will apply the Hochschild constructions to the (in general non-monoidal) representable functors $\Phi(m) = \mathcal{E}(m, -)$, which can be thought of as “generalized free $\mathcal{E}$–algebras”. Also, even for $\Phi$ monoidal, $C(\Phi)$ will in general not be monoidal, but we can nevertheless iterate the construction and talk about $C(C(\Phi)) = C^2(\Phi), C^3(\Phi)$, etc.

**Definition 5.3** (Reduced Hochschild complex). Let $(\mathcal{E}, i)$ be a PROP with unital $A_{\infty}$–multiplication and $\Phi : \mathcal{E} \to \text{Ch}$ a functor. Define the reduced Hochschild complex of $\Phi$ as the quotient functor $\overline{C}(\Phi) = C(\Phi)/U : \mathcal{E} \to \text{Ch}$ given on object by

$$\overline{C}(\Phi)(m) = \bigoplus_{n \geq 1} \Phi(n + m)/U_n \otimes L_n$$

where $U_n = \sum_{i=2}^{n} \text{Im}(\Phi(i(u_i) + id_m)) \subset \Phi(n + m)$ with $u_i = 1 \otimes \ldots \otimes u \otimes \ldots \otimes 1$ in $A_{\infty}^+(n-1,n)$ the morphism that inserts a unit at the $i$th position.

As the quotient does not affect the variable part of $C(\Phi)$, it is clear that $\overline{C}(\Phi)$ is still defines a functor $\mathcal{E} \to \text{Ch}$. On the other hand, we need to check that the differential is well-defined on the quotient, which is done in the following lemma:

**Lemma 5.4.** The differential of $C(\Phi)(m)$ induces a well-defined differential on $\overline{C}(\Phi)(m)$ for each $m$.

**Proof.** Let $U_n \leq \Phi(n)$ be as in Definition 5.3. We first note that $U_n$ is mapped to itself by $d_{q}$ because the structure map $c_{q}$ of $\Phi$ is by chain maps and $d(u_i) = 0$. We need to see that the same holds for the Hochschild part of the differential. This follows from the commutativity of the following diagram (written in the case $m = 0$ for readability)

$$
\begin{array}{ccc}
\bigoplus_{2 \leq r \leq n} \Phi(n-1) \otimes \langle u_i \rangle \otimes m^r \otimes L_{n+1-r} & c_{\Phi} & \bigoplus_{2 \leq r \leq n} \Phi(n) \otimes \langle m^{2}_r \rangle \otimes L_{n+1-r} \\
\Phi(n-1) \otimes \langle u_i \rangle \otimes L_n & d_{L} & \Phi(n) \otimes L_n \\
\bigoplus_{\sum_{i,j} \leq n} \Phi(n-1) \otimes \langle u_i \rangle \otimes m^r \otimes L_{n+1-r} & c_{\Phi} & \bigoplus_{k \geq 1} \Phi(k) \otimes L_k
\end{array}
$$

where $m^r = 1 \oplus \cdots \oplus m_r \oplus \cdots \oplus 1$ denotes the multiplication $m_r$ of the entries $j, \ldots, j+r-1$ (mod $n$). The target of the map $c_{A_{\infty}}$ is justified as follows. There are two cases when composing $u_i$ and $m^r$: either $i \notin \{j, \ldots, j+r-1\}$ so that the composition $m^r \circ u_i$ is of the form $u_i \oplus m^r$. Otherwise, the composition $m^r \circ u_i$ is the identity map when $r = 2$ and 0 when $r > 2$. In the case $r = 2$, the term $m^{r-1}_2 \circ u_i$ cancels with $m^r_2 \circ u_i$. (The sign comes from the differential in $L$).

Let $\mathcal{E}, \mathcal{F}$ be dg-categories and suppose that $\Phi : \mathcal{E} \to \text{Ch}$ in fact extends to a bifunctor $\Phi : \mathcal{F}^{op} \times \mathcal{E} \to \text{Ch}$. In this case, we also call $\Phi$ an $(\mathcal{F}^{op}, \mathcal{E})$–bimodule.\(^{10}\)

**Proposition 5.5.** Let $(\mathcal{E}, i)$ be a PROP with (unital) $A_{\infty}$–multiplication and suppose $\Phi$ is an $(\mathcal{F}^{op}, \mathcal{E})$–bimodule. Then the Hochschild complexes $C(\Phi(a, -))$ and $\overline{C}(\Phi(a, -))$ built using the $\mathcal{E}$–structure of $\Phi$ pointwise on objects $a$ of $\mathcal{F}$ assemble again to $(\mathcal{F}^{op}, \mathcal{E})$–bimodules.

\(^{10}\)Here, to correctly work out the signs in the differential, we take the structure maps of the bimodule to be in the form $\mathcal{F}(m_1, m_2) \times \Phi(m_2, n_1) \times \mathcal{E}(n_1, n_2) \to \Phi(m_1, n_2)$ and apply the usual sign convention.
Proof. Given \( f : m_1 \to m_2 \) in \( \mathcal{E} \) and \( g : a_2 \to a_1 \) in \( \mathcal{F}^{op} \), \( C(\Phi)(g, f) \) on the summand \( \Phi(a_2, n + m_1) \otimes L_n \) is the map \((-1)^{(n-1)f}(g, id_n + f)\). This is well-defined as the Hochschild part of the differential commutes with such maps. \( \Box \)

Example 5.6. The example we are interested in is the \((\mathcal{E}^{op}, \mathcal{E})\)–bimodule \( \mathcal{E} \). By the proposition, its Hochschild and iterated Hochschild complexes \( C(\mathcal{E}), C^n(\mathcal{E}) \), and reduced versions when relevant, are again \((\mathcal{E}^{op}, \mathcal{E})\)–bimodules. Given any \( \Phi : \mathcal{E} \to \text{Ch} \), this allows to consider the double bar construction \( B(\Phi, \mathcal{E}, C^n\mathcal{E}) \) (as in Section 4), which in fact identifies with \( C^n(\mathcal{E}^{op})(\Phi, \mathcal{E}, \mathcal{E}) \) as both have value at \( m \) given by

\[
\bigoplus_{p \geq 0, n \geq 1, \sum m_i \geq 0} \Phi(m_0) \otimes \mathcal{E}(m_0, m_1) \otimes \cdots \otimes \mathcal{E}(m_p, n + m) \otimes L_n
\]

(and similarly for the reduced constructions).

5.1. Properties of the Hochschild Operator. We prove in this section that the Hochschild complex operator is homotopy invariant and we describe its behavior under iteration. Throughout the section, we assume that \((\mathcal{E}, i)\) is a PROP with \( A_{\infty} \)–multiplication when we consider the Hochschild complex \( C \), and that \((\mathcal{E}, i)\) is a PROP with unital \( A_{\infty} \)–multiplication when we consider its reduced version \( \overline{C} \).

Recall that by a quasi-isomorphism of functors \( \Phi \xrightarrow{\simeq} \Phi' : \mathcal{E} \to \text{Ch} \), we mean a natural transformation by quasi-isomorphisms \( \Phi(m) \xrightarrow{\simeq} \Phi'(m) \).

Proposition 5.7. Let \( \Phi, \Phi' : \mathcal{E} \to \text{Ch} \). A quasi-isomorphism of functors \( \Phi \xrightarrow{\simeq} \Phi' \) induces quasi-isomorphisms of functors \( C_\ast(\Phi) \xrightarrow{\simeq} C_\ast(\Phi') \) and \( \overline{C}_\ast(\Phi) \xrightarrow{\simeq} \overline{C}_\ast(\Phi') \).

For the reduced part of the proposition, we need the following lemma.

Lemma 5.8. Suppose \( \Phi \xrightarrow{\simeq} \Phi' : \mathcal{E} \to \text{Ch} \) are quasi-isomorphic functors. For any \( J \subset \{1, \ldots, n\} \), let \( U_J = \sum_{j \in J} \text{Im}(\Phi(i(u_j))) \subset \Phi(n) \), and similarly for \( \Phi' \). Then

\[
\Phi(n)/U_J \xrightarrow{\simeq} \Phi'(n)/U_J.
\]

If \( \Phi \cong \Phi' \), these maps are also isomorphisms.

Proof. We prove the lemma by induction on the cardinality of \( J \), for any \( n \), starting with the case \( J = \emptyset \) which is trivial.

Fix \( J = \{j_1 \leq \cdots \leq j_s\} \subset \{1, \ldots, n\} \) and denote by \( U_J, U'_J \) the image of \( i(u_j) \) in \( \Phi(n) \) and \( \Phi'(n) \) respectively. We want to show that \( \Phi(n)/(U_{j_1} + \cdots + U_{j_s}) \xrightarrow{\simeq} \Phi'(n)/(U'_{j_1} + \cdots + U'_{j_s}) \).

There is a short exact sequence

\[
\Phi(n-1)/(U_{j_1} + \cdots + U_{j_{s-1}}) \xrightarrow{i(u_{j_s})} \Phi(n)/(U_{j_1} + \cdots + U_{j_s-1}) \quad \Phi(n)/(U_{j_1} + \cdots + U_{j_s}).
\]

Indeed \( u_{j_s} \) is injective on \( \Phi(n-1)/(U_{j_1} + \cdots + U_{j_{s-1}}) \) with left inverse \( i(m_{j_s}) \) (where \( m_{j_s} \) multiplies \( j_s \) and \( j_s + 1 \) modulo \( n \)). The result then follows by induction by considering the map of short exact sequences induced by \( \Phi \to \Phi' \).

Proof of the Proposition. We filter the complexes \( C_\ast(\Phi)(m) = \oplus \Phi(k + m) \otimes L_k \) and \( \overline{C}_\ast(\Phi)(m) = \oplus \Phi(k + m)/U_k \otimes L_k \) by \( k \) and consider the resulting spectral sequence. In both cases the differential is \( d_\Phi + d_H \) where \( d_H \) decreases the filtration grading and \( d_\Phi \) does not. Hence the \( E^1 \)–terms of the spectral sequences are \( E^1_{p,q} = H_p(\Phi(q + 1 + m))/U_{q+1} \otimes L_{q+1} \) and \( E^1_{p,q} = H_p(\Phi(q + 1 + m)/U_{q+1}) \otimes L_{q+1} \) in the reduced case. A quasi-isomorphism of functors induces a map of spectral sequences which is an isomorphism.
on the $E^1$–term by the assumption in the unreduced case and by Lemma 5.8 in the reduced case.

Applying Proposition 5.7 to the map $\alpha : B(\Phi, E) \xrightarrow{\sim} \Phi$ of Proposition 4.1, we get a quasi-isomorphism

$$C(\alpha) : C(B(\Phi, E)) \xrightarrow{\sim} C(\Phi).$$

The proof of Proposition 4.1 gives a pointwise homotopy inverse $\beta$ to $\alpha$ which is not a natural transformation, so we cannot apply Proposition 5.7 to it. (In fact $C(\beta)$ does not define a chain map.) Instead, we construct now an explicit pointwise homotopy inverse $\tilde{\beta}$ to $C^n(\alpha)$, for any $n$, as this will be useful later to produce explicit actions on the Hochschild complex of $E$–algebras.

**Proposition 5.9.** For any $n$ and $m$, there is a quasi-isomorphism of chain complexes

$$\tilde{\beta} : C^n(\Phi)(m) \xrightarrow{\sim} C^n(B(\Phi, E))(m)$$

natural both with respect to natural transformations $\Phi \to \Phi'$ and with respect to functors $j : E \to E'$ with $i' = j \circ i : A_\infty \to E'$. Moreover, $\tilde{\beta}$ is a right inverse to $C(\alpha)$ for $\alpha$ as in Proposition 4.1.

**Proof.** We first define $\tilde{\beta}$ in the case $E = A_\infty$, and using the identification

$$C^n(B(\Phi, A_\infty))(m) \cong B(\Phi, A_\infty, C^n(A_\infty)(m))$$

of Example 5.6. The map $\tilde{\beta}$ for a general $E$ and $\Phi : E \to \text{Ch}$ is then obtained by post-composition with the quasi-isomorphism

$$C^n(B(i^*\Phi, A_\infty))(m) \to C^n(B(\Phi, E))(m)$$

induced by $i : A_\infty \to E$. The naturality of $\tilde{\beta}$ in $E$ follows from the naturality of that second map.

Recall from 2.9 the map

$$d_L : L_k \to \bigoplus_{1 \leq j < k} A_\infty(k, j) \otimes L_j.$$  

We consider here more generally the map

$$d_L : L_{k_1} \otimes \ldots \otimes L_{k_n} \to \bigoplus_{1 \leq k < n} A_\infty(k_1 + \cdots + k_n, k_1' + \cdots + k_n') \otimes L_{k_1'} \otimes \ldots \otimes L_{k_n'}$$

induced by the differential of the $[n]$–graph which is the union $l_{k_1} \sqcup \ldots \sqcup l_{k_n}$, where $k = k_1 + \cdots + k_n$. We let $\tilde{\beta} := \sum_{p \geq 0} (d_L)^p_n$, where we interpret $(d_L)^p_n$ as the composition

$$\Phi(k + m) \otimes L_k \xrightarrow{(d_L)^p_n} \bigoplus_{j_i} \Phi(k + m) \otimes A_\infty(k, j_1) \otimes \ldots \otimes A_\infty(j_{p-1}, j_p) \otimes L_{j_p}$$

$$+ \text{id}_{\Phi} \bigoplus_{j_i} \Phi(k + m) \otimes A_\infty(k + m, j_1 + m) \otimes \ldots \otimes A_\infty(j_{p-1} + m, j_p + m) \otimes L_{j_p}$$

with image in the $p$th simplicial level of $B(\Phi, A_\infty, C^n(A_\infty)(m))$, where $L_k = L_{k_1} \otimes \ldots \otimes L_{k_n}$ and $L_{j_p} = L_{j_{p+1}} \otimes \ldots \otimes L_{j_n}$ is identified with $(\text{id}_{j_p} + m) \otimes L_{j_p}$ in $A_\infty(j_p + m, j_p + m) \otimes L_{j_p}$ in $C^n(A_\infty)(m)$. Note that the sum is always finite as $(d_L)^p_n$ applied to $L_k$ is 0 for all $p \geq k$.

We will show that the relation $d\tilde{\beta} = \tilde{\beta}d$ holds on each component as maps

$$\bigoplus_{(k) = (k_1, \ldots, k_n)} \Phi(k + m) \otimes L_{k_1} \otimes \ldots \otimes L_{k_n} \longrightarrow \bigoplus_p B_p(\Phi, A_\infty, C^n(A_\infty)(m))$$

i.e. that for each fixed $(k)$, the images of $d\tilde{\beta}$ and $\tilde{\beta}d$ agree on the component of simplicial degree $p$. We first consider $\tilde{\beta}d$. 

As \( d = d_{\Phi} + c_{\Phi}d_L \), we have on the \((k)\)th component

\[
(k)(\tilde{\beta}d) = \sum_{i=0}^{K-1} (d_L)^i d_{\Phi} + \sum_{i=0}^{K-2} (d_L)^i c_{\Phi} d_L
\]

with \( K = \max(k_1, \ldots, k_n) \), which can be rewritten as

\[
(k)(\tilde{\beta}d) = d_{\Phi} \sum_{i=0}^{K-1} (-1)^i (d_L)^i + d_0 \sum_{i=0}^{K-2} (d_L)^i + 1
\]

as \( d_L d_{\Phi} = -d_{\Phi}d_L \) and \( \delta_L c_{\Phi} d_L = d_0 d_L^{p+1} \) with \( d_0 \) the 0-th face map in \( B_i(\Phi, A_{\infty}, C^n(\mathcal{A}_{\infty})(m)) \). Hence the component of \((k)(\tilde{\beta}d)\) of simplicial degree \( p \) is

\[
(k)(\tilde{\beta}d)_p = (-1)^p d_{\Phi}(d_L)^p + d_0(d_L)^{p+1}.
\]

On the other hand, we have \((k)(d\tilde{\beta}) = d((k)\tilde{\beta})\) where the differential on the \(p\)th component of \((k)\tilde{\beta}\) is \((-1)^p(d_{\Phi} + (d_A)_1 + \cdots + (d_A)_p + \tilde{d}_L) + \sum_{i=0}^{p} (-1)^i d_i\), where \((d_A)_i\) denotes the differential of the \(i\)th factor \(A_{\infty}(-, -)\) and \(d_L\) the map \(d_{p+1}d_L\), which applies the differential to the factors \(L\) without increasing the simplicial degree. As the face maps \(d_i\) reduce the simplicial degree, we have

\[
(k)(d\tilde{\beta})_p = (-1)^p(d_{\Phi} + (d_A)_1 + \cdots + (d_A)_p + d_{p+1}d_L)(d_L)^p + \sum_{i=0}^{p+1} (-1)^i d_i(d_L)^{p+1}.
\]

This is a sum of two compositions whose respective first terms are exactly \((k)(\tilde{\beta}d)_p\), and whose last terms cancel. Hence

\[
(k)(d\tilde{\beta})_p - (k)(\tilde{\beta}d)_p = \sum_{i=1}^{p} \left((-1)^p(d_A)_i(d_L)^p + (-1)^i d_i(d_L)^{p+1}\right).
\]

The \(i\)th term in the sum can be rewritten as

\[
(-1)^i(d_L)^{p-i}(d_A)_i + d_i d_L(d_L)^i
\]

which is 0 as the middle part \(((d_A)_i + d_id_L)d_L\) is the square of a differential in the graph complex, which gives the desired equality.

As \( C^n(\alpha)(m) \circ \tilde{\beta} \) is the identity and \( C^n(\alpha)(m) \) is a quasi-isomorphism by Propositions 4.1 and 5.7, \( \tilde{\beta} \) is also a quasi-isomorphism. The map \( \tilde{\beta} \) is natural in \( \Phi \) as \( d_L \) is natural in \( \Phi \).

\( \square \)

Next we describe how the Hochschild operator behaves under iteration. Recall from Section 3 that a monoidal functor \( \Phi : E \to \text{Ch} \) is \( h \)-split if the maps \( \Phi(n) \otimes \Phi(m) \to \Phi(n+m) \) are quasi-isomorphisms, and \( \text{split} \) if the maps are isomorphisms.

For \( \Phi : E \to \text{Ch} \), we can consider the iterated Hochschild functor \( C^n(\Phi) = C(C(...C(\Phi)...)) \).

When \( \Phi \) is \( h \)-split monoidal, it computes the tensor powers of the Hochschild complex:

**Proposition 5.10.** If \( \Phi : E \to \text{Ch} \) is monoidal, then there are natural maps

\[
\lambda : C(\Phi)(0)^{\otimes n} \otimes \Phi(1)^{\otimes m} \to C^n(\Phi)(m)
\]

and

\[
\bar{\lambda} : \overline{C}(\Phi)(0)^{\otimes n} \otimes \Phi(1)^{\otimes m} \to \overline{C^n}(\Phi)(m).
\]

These maps are quasi-isomorphisms if \( \Phi \) is \( h \)-split, and isomorphisms if \( \Phi \) is \( \text{split} \).

Moreover, there exists an action of \( \Sigma_n \times \Sigma_m \) on \( C^n(\Phi)(m) \) such that if \( E, \Phi \) and \( i \) are symmetric monoidal, these maps are \( \Sigma_n \times \Sigma_m \)-equivariant (where \( \Sigma_m \) acts on \( C^n(\Phi)(m) \) via the symmetries of \( E \)).
the reduced Hochschild core category
where monoidal inclusion
\( C \)
\( C \)
\( C \) described in Proposition 5.5. Moreover, \( \Phi \) is symmetric monoidal. This follows from an iteration of Lemma 5.8: Consider the restriction of the natural transformation \( \Phi \) to the first variable and apply the lemma with \( J_1 = \{ 2, \ldots, k_1 \} \).

This gives a quasi-isomorphism \( \Phi(k_1)/U_{k_1} \otimes \Phi(k_2) \otimes \cdots \Phi(m) \to \Phi(k_1 + \cdots + m)/U_{k_1}/\ldots/U_{k_m} \) to be an isomorphism when \( \Phi \) is split and a quasi-isomorphism when \( \Phi \) is h-split. This quasi-isomorphism is functorial in the variables \( k_2, \ldots, m \) and we can repeat the process until we obtain the desired result. \( \square \)

5.2. Action on Hochschild complexes. Given a monoidal dg-category \( D \) with objects pairs of natural numbers \( [m] \), we say that a pair of chain complexes \( (V, W) \) is a \( D \)-module if there is a split monoidal dg-functor \( \Psi : D \to \operatorname{Ch} \) with \( \Psi([1]) = V \) and \( \Phi([0]) = W \), i.e. if there are chain maps

\[
(V^\otimes n_1 \otimes W^\otimes m_1) \otimes D([m_1], [m_2]) \to V^\otimes n_2 \otimes W^\otimes m_2
\]

compatible with composition in \( D \). We say that \( (V, W) \) is a homotopy \( D \)-module if the compatibility condition is only satisfied up to homotopy, that is if \( \Psi \) is only a functor up to homotopy, satisfy the equation \( \Psi(f \circ g) \simeq \Psi(f) \circ \Psi(g) \) for any pair of composable morphisms \( f, g \) in \( D \). In particular, taking homology with field coefficients (or general coefficients but restricting to the “operadic part” with \( [m_2] = [1] \) or \( [1] \)), we get in both cases an honest action of \( H_*(D) \) on \( (H_*(V), H_*(W)) \).

If \( D \) is symmetric monoidal, we say that the module structure is \( \Sigma \)-equivariant if the functor \( \Psi \) is symmetric monoidal.

Proposition 5.5 in the case where \( \Phi \) is the \( (\mathcal{E}, \mathcal{E}^{op}) \)-bimodule \( \mathcal{E} \) can be reinterpreted as follows: Given \( \mathcal{E} \), we can define its Hochschild core category \( CE \) with objects

\[
[n] = (m, n) \in \operatorname{Obj}(\mathcal{E}) \times \mathbb{N} (= \mathbb{N} \times \mathbb{N}),
\]

for \( \mathbb{N} \) the natural numbers including 0, and morphisms

\[
CE([m_1], [m_2]) = \begin{cases} 
C^m(\mathcal{E}(m_1, -))(m_2) & n_1 = 0 \\
0 & n_1 \neq 0
\end{cases}
\]

where \( C^0 \) means the identity operator, so that \( CE([0], [0]) = \mathcal{E}(m_1, m_2) \). The only possible non-trivial compositions in \( CE \) are given by the bimodule structure of \( C^n(\mathcal{E}(m, -)) \) described in Proposition 5.5. Moreover, \( CE \) is monoidal via the maps

\[
C^n(\mathcal{E}(m_1, -))(m_2) \otimes C^{n'}(\mathcal{E}(m_1', -))(m_2') \to C^{n+n'}(\mathcal{E}(m_1 + m_1', -))(m_2 + m_2')
\]

as in Proposition 5.10, and \( CE \) is symmetric monoidal when the same is true for \( \mathcal{E} \).

We call a monoidal bimodule \( \tilde{\mathcal{E}} \) with objects \( \mathbb{N} \times \mathbb{N} \) an extension of \( CE \) if there is a monoidal inclusion \( CE \to \tilde{\mathcal{E}} \) with \( \tilde{\mathcal{E}}([m_1], [m_2]) = CE([m_1], [m_2]) \) when \( n_1 = 0 \). We define the reduced Hochschild core category \( \overline{CE} \) and its extensions in the same way, replacing \( C \) by \( \overline{C} \).

Our main result says that if \( \tilde{\mathcal{E}} \) is an extension of \( CE \) (or \( \overline{CE} \)), then \( \tilde{\mathcal{E}} \) acts on the (reduced) Hochschild complex of split monoidal functors \( \Phi : \mathcal{E} \to \operatorname{Ch} \) in the following sense:
Theorem 5.11. Let $(\mathcal{E}, i)$ be a PROP with $A_\infty$–multiplication and $\tilde{\mathcal{E}}$ an extension of $C\mathcal{E}$. Then for any monoidal functor $\Phi : \mathcal{E} \to \text{Ch}$, there is a diagram

$$
(\Phi(0) \otimes^{m_1} \Phi(1) \otimes^{m_1}) \otimes \tilde{\mathcal{E}}((m_1], [m_2]) \xrightarrow{\gamma} C^{m_2}(\Phi)(m_2)
$$

natural in $\Phi$, with $\lambda$ as in Proposition 5.10. If $\Phi$ is split, the composition $\lambda^{-1} \circ \gamma$ makes the pair $(\Phi(0), \Phi(1))$ into a $\tilde{\mathcal{E}}$–module, and a homotopy $\tilde{\mathcal{E}}$–module for any choice of $\lambda^{-1}$ if $\Phi$ is $h$-split. Moreover, if $\mathcal{E}, \Phi, i$ and $\lambda^{-1}$ are symmetric monoidal, the module structure is $\Sigma$–equivariant.

If $(\mathcal{E}, i)$ is a PROP with unital $A_\infty$–multiplication, the same holds for the reduced case, replacing $C$ by $\overline{C}$.

An extension $\tilde{\mathcal{E}}$ of $C\mathcal{E}$ can be thought of as a way to encode a natural action on the Hochschild complex of the representable functors $\mathcal{E}(n, -)$, and the above theorem is only non-trivial when the complex $\tilde{\mathcal{E}}((m_1], [m_2])$ are not identically 0 for $n_1 \neq 0$. Thinking of the representable functors as generalized free algebras, the theorem can be interpreted as saying that an natural/compatible action on the Hochschild complex of free algebras induces an action on the Hochschild complex of all algebras.

The map $\gamma$ in the statement is explicit, given by the big diagram in the proof of the theorem below. This allows to write down formulas for operations given cycles in the extension category (see Section 6.2 and the end of Section 6.5).

Note that restricting to $n_2 = 1$ and $m_2 = 0$ avoids having to invert $\lambda$, and restricting further to $n_1 = 1$ and $m_1 = 0$ avoids needing $\lambda$ at all. In particular, $C(\Phi)(0)$ is a $\tilde{\mathcal{E}}((0], [0])$–module without any monoidal assumption on $\Phi$. Alternatively, one can use $C^n(\Phi)(m)$ as a model of $C(\Phi)(0)^\otimes n \otimes (\Phi(1)^\otimes m)$ which admits an action of $\tilde{\mathcal{E}}$ without reference to $\lambda$, as in the following:

Corollary 5.12. Let $(\mathcal{E}, i)$ be a PROP with (unital) $A_\infty$–multiplication. For any $\Phi : \mathcal{E} \to \text{Ch}$ and any extension $\tilde{\mathcal{E}}$ of $C\mathcal{E}$, taking $C_\Phi((n]) = C^n(\Phi)(m)$ defines a dg-functor $C_\Phi : \tilde{\mathcal{E}} \to \text{Ch}$ extending $\Phi$ on $\mathcal{E}$ (and the same in the reduced case). Moreover, the association $\Phi \mapsto C_\Phi$ defines a functor $\text{Fun}(\mathcal{E}, \text{Ch}) \to \text{Fun}(\tilde{\mathcal{E}}, \text{Ch})$. The this corollary is a direct corollary of the proof of Theorem 5.11.

Proof of Theorem 5.11. The action is defined by the following diagram:

$$
\begin{array}{ccc}
C(\Phi)(0)^\otimes m_1 \otimes (\Phi(1)^\otimes m_1) & \otimes & \tilde{\mathcal{E}}((m_1], [m_2]) \\
\xrightarrow{\lambda \otimes \text{id}} & & \xrightarrow{\lambda}
\end{array}
$$
The map $\tilde{\beta}$ is that of Proposition 5.9 and the map $\alpha$ is that of Proposition 4.1. They are quasi-isomorphisms for any $\Phi$. The map $\lambda$ is that of Proposition 5.10. It is an isomorphism whenever $\Phi$ is split and a quasi-isomorphism whenever $\Phi$ is h-split. The bottom horizontal arrow is induced by composition in $\tilde{E}$.

Consider the composition with a further morphism in $\tilde{E}([n_2^1],[n_3^1])$. Note now that the failure of $\tilde{\beta} \circ C^m(i)(\alpha)$ to be the identity lies in the non-zero simplicial degrees of $B(\Phi, E, \tilde{E}, [0],[n_2^1])$. As the simplicial degree is constant when applying the composition with $\tilde{E}([n_2^1],[n_3^1])$, this difference is killed when we apply $C^m(i)(\alpha)$ at the end of the action. Hence, when $\Phi$ is split monoidal, the action is strictly associative.

Let $B^\Sigma$ denote the quotiented bar construction defined in Section 4. If $E, i$ and $\Phi$ are symmetric monoidal, then using $B^\Sigma$ instead of $B$, replacing $\tilde{\beta}$ with its composition with the quotient map $B \to B^\Sigma$, makes the diagram above equivariant under the action of $\Sigma_{m_1} \times \Sigma_{m_2}$, by Proposition 4.3 and 5.10, and the fact that this action is given by morphisms of $\tilde{E}$.

For the reduced version, we need to check that this composition of maps is well-defined. (The map $\tilde{\beta}$ is in fact not well-defined in that case.)

Consider the action of some $f \in \tilde{E}([n_1^1],[n_2^1])$ on some $x \otimes l_k \in C^{m_1}(\Phi)(m_1)$ with $x \otimes l_k$ identified with $0$ in $C^{m_1}(\Phi)(m_1)$, i.e.

$$x \otimes l_k = c_\Phi(y \otimes u_j) \otimes l_{k_1} \ldots \otimes l_{k_{n_1}}$$

for $y \in \Phi(k-1+m_1)$, with $k = k_1 + \ldots + k_{n_1}$ and $u_j = i(u_j) \in E(k-1+m_1,k+m_1)$ introducing a unit in the $j$th position for $j \in \{2, \ldots, k_1, k_1 + 2, \ldots, k_{n_1}\}$.

Following the diagram defining the action, we have

$$(x \otimes l_{k_1} \otimes \ldots \otimes l_{k_{n_1}}) \otimes f \xrightarrow{c_{\beta}} x \otimes (id_{k+m_1} \otimes l_{k_1} \otimes \ldots \otimes l_{k_{n_1}}) \otimes f + \text{higher order}$$

$$\xrightarrow{c_\Phi} x \otimes \left(\sum g \otimes l_{k_1'} \otimes \ldots \otimes l_{k_{n_2}'}\right) + \text{higher order}$$

$$\xrightarrow{\alpha} \sum c_\Phi(x \otimes g) \otimes l_{k_1'} \otimes \ldots \otimes l_{k_{n_2}'}$$

for some maps $g \in E(k+m_1,k'+m_2)$. Now

$$c_\Phi(x \otimes g) = c_\Phi(c_\Phi(y \otimes u_j) \otimes g) = c_\Phi(y \otimes c_\Phi(u_j \otimes g))$$

so it is enough to know that $\sum c_\Phi(u_j \otimes g)$ is of the form $\sum c_\Phi(g' \otimes u_j')$ for some $g', j'$ whenever $g$ comes from a composition as above. We have (in abbreviated notation)

$$\sum c_\Phi(u_j \otimes g) \otimes l_k' = c_\Phi(u_j \otimes c_\Phi(id_{k+m_1} \otimes l_k) \otimes f) = c_\Phi((u_j \otimes l_k) \otimes f)$$

by definition and associativity of composition in $\tilde{E}$. As $u_j \otimes l_k$ is identified with $0$ in $C^{m_1}(E(k-1+m_1,1),l_k)$, we must have that $c_\Phi((u_j \otimes l_k) \otimes f)$ is identified with $0$ in the reduced Hochschild complex $\tilde{E}([k-1+m_1],[n_2^1])$, which means precisely that $\sum c_\Phi(u_j \otimes g)$ is of the form $\sum c_\Phi(g' \otimes u_j')$ as required. 

The next result says that the action of Theorem 5.11 is also natural in $(E, \tilde{E})$ in the following sense:

**Theorem 5.13.** Let $(E,i),(\tilde{E},i')$ be PROPs with (unital) $A_\infty$-multiplication and $\tilde{E}, \tilde{E}'$ be extensions of $E,E'$. Suppose that there is a symmetric monoidal functor $\tilde{j} : \tilde{E} \to \tilde{E}'$ such that $i' = j \circ i : A_\infty \to E \to \tilde{E}'$ for $j$ the restriction of $\tilde{j}$ to $E$. Then for any (h-)split monoidal functor $\Phi : E' \to \text{Ch}$, the (homotopy) $\tilde{E}$-action of Theorem 5.11 on the pair $(j^*\Phi, C(j^*\Phi)) \cong (\Phi, C(\Phi))$ factors through the $\tilde{E}'$-action.

The same holds in the reduced case, replacing $C$ by $C$. 

**Proof.** This follows directly from the naturality of the maps defining the action. 

\[\square\]
6. Examples and applications

In this section, we apply Theorem 5.11 to specific categories \( \mathcal{E} \). In 6.1, we consider the case \( \mathcal{E} = \mathcal{O} \), the open cobordism category of 2.6. We show that the open-closed category \( \mathcal{OC} \) of section 2.8 is an extension of \( \mathcal{CO} \) in the sense of Section 5.2. The application of Theorem 5.11 to this extension, stated as Theorem 6.2, can be interpreted as a reformulation of Costello’s Theorem A (2-3) in [8]. In 6.2, we explain how reading off the action of \( \mathcal{OC} \) obtained in the previous section on open field theories \( \Phi : \mathcal{O} \to \text{Ch} \) recovers the recipe given by Kontsevich-Soibelman in [27]. Sections 6.3 and 6.4 give determinant-twisted and positive boundary versions of Theorem 6.2.

In 6.5, we consider the case of strict Frobenius algebras, with \( \mathcal{E} = \mathcal{H}_0(\mathcal{O}) \). We show that the category \( \mathcal{SD} \) of Sullivan diagrams defined in 2.10 is an extension of \( \mathcal{CH}_0(\mathcal{O}) \). The application of Theorem 5.11 in this case yields Theorem 6.7, which recovers Theorem 3.3 of [45], giving an action of Sullivan diagrams on the Hochschild complex of strict Frobenius algebras. Using the projection \( \mathcal{OC} \to \mathcal{SD} \), this produces an open-closed field theory though with much of the structure collapsed. At the end of the section, we give explicit formulas for the product, coproduct, and \( \Delta - \) (or \( B - \)) operator on the Hochschild complex in this case. In Section 6.6 then gives an application to string topology in characteristic 0 using the models of Lambrechts-Stanley [28].

Finally, sections 6.7 and 6.8 consider the cases of \( \mathcal{E} = \mathcal{A}_\infty^+ \) and \( \mathcal{E} = \text{Ass}^+ \times \mathcal{P} \) for \( \mathcal{P} \) an operad.

6.1. Open topological conformal field theories. Let \( \mathcal{O} \) be the open cobordism category defined in 2.6, with \( i : \mathcal{A}_\infty^+ \to \mathcal{O} \) the inclusion of trees into all graphs, and \( \mathcal{OC} \) the open-closed cobordism category of 2.8. We have that \( \mathcal{O} \) is a subcategory of \( \mathcal{OC} \). The following lemma shows that \( \mathcal{OC} \) is in fact an extension of the Hochschild core category of \( \mathcal{O} \):

\[
\begin{align*}
\text{Figure 14. Black and white graphs as elements in the iterated Hochschild complex of } \mathcal{O}
\end{align*}
\]

**Lemma 6.1.** The category \( \mathcal{OC} \) is an extension of \( \mathcal{CO} \).

**Proof.** We need to check that \( \mathcal{OC}([m_1],[n_2]) \cong \mathcal{CO}(m_1,-)(m_2) \). Now

\[
\mathcal{CO}(m_1,-)(m_2) = \bigoplus_{k_1,\ldots,k_n \geq 1} \mathcal{O}(m_1, k_1 + \cdots + k_n + m_2)/U \otimes L_{k_1} \otimes \cdots \otimes L_{k_n}.
\]

We describe a bijection between the generators of this complex and the generators of \( \mathcal{OC} \): a generator of the complex above is identified with a black and white graph with \( n \) white vertices and \( m_1 + m_2 \) leaves by attaching the first \( k_1 + \cdots + k_n \) outgoing leaves of generating graphs in \( \mathcal{O} \) to the leaves of the generating graphs \( L_{k_1}, \ldots, L_{k_n} \), respecting the ordering. (An example of this procedure is shown in Figure 14.) The fact that the only units allowed in \( \mathcal{O} \) are at the positions corresponding to the first leaf of an \( L_{k_i} \) corresponds to the fact that the only unlabeled leaves allowed in \( \mathcal{OC} \) are those that are start-edges of white vertices. As the graphs \( L_{k_i} \) have a start-leaf, this is a reversible process whose target is exactly the generator of \( \mathcal{OC}([m_1],[n_2]) \). \( \square \)

Applying Theorem 5.11 to \( \mathcal{E} = \mathcal{O} \) with \( \tilde{\mathcal{E}} = \mathcal{OC} \) then yields:
Theorem 6.2. Let $\Phi : \mathcal{O} \to \text{Ch}$ be an (h-)split symmetric monoidal functor. Then the pair $(\overline{\mathcal{C}}(\Phi)(0), \Phi(1))$ is a $\Sigma$-equivariant (homotopy) $\mathcal{O}C$–module.

As morphisms in $\mathcal{O}$ models the moduli space of cobordisms between (open strings) (see Theorem 2.3), a split monoidal functor $\Phi : \mathcal{O} \to \text{Ch}$ is a model of an open topological conformal field theory. Algebraically, such an object is an $A_{\infty}$–version of a Frobenius algebra (see Section 3.2). Similarly, Theorem 2.5 shows that an equivariant $\mathcal{O}C$–module can be thought of as a model for an open-closed topological conformal field theory. In particular, it includes an action of a chain model of the moduli space of Riemann surfaces with fixed boundary parametrization on the value of the module at the circle.

We note that the category $\mathcal{O}C$ does not include morphisms associated to the disk with one outgoing closed boundary component. Consequently, algebras over the closed sector of this theory are not necessarily unital (the unit in the algebra would come from the generator of $H_0$ of the moduli of such disks). That is, algebras over $\mathcal{O}C$ are inherently “co-positive boundary” topological conformal field theories.

The above theorem is essentially a reformulation of Costello’s theorem [8, Thm. A (2-3)], though we obtain a more precise description of the action of the open-closed cobordism category. This allows us to recover the recipe given by Kontsevich-Soibelman for such an action in Section 11.6 of [27], which we expand on in the next section. Restricting to genus 0 surfaces, the statement includes the “$Ai$–cyclic Deligne conjecture”, which was also proved in [51].

6.2. Making the action explicit: the Kontsevich-Soibelman recipe. Let $\Phi : \mathcal{O} \to \text{Ch}$ be a split monoidal functor with $\Phi(1) = A$. Given $n_1$ Hochschild chains in $A$, $m_1$ elements $A$ and a graph $\Gamma$ in $\mathcal{O}C([m_1],[m_2])$, that is:

$$(a_0^1 \otimes \ldots \otimes a_k^1), \ldots, (a_0^{n_1} \otimes \ldots \otimes a_k^{n_1}), b_1, \ldots, b_{m_1} \text{ and } \Gamma$$

the diagram in the proof of Theorem 5.11 gives an explicit way to obtain a sum of $n_2$ Hochschild chains in $A$ and $m_2$ elements of $A$. We apply here the sequence of maps to these elements and show how this recovers the recipe given by Kontsevich and Soibelman in [27, pp 58–62].

The first map in the diagram assembles all these terms as

$$a_0^1 \otimes \ldots \otimes a_k^1 \otimes \ldots \otimes a_0^{n_1} \otimes \ldots \otimes a_k^{n_1} \otimes b_1 \otimes \ldots \otimes b_{m_1} \otimes l_{k_1+1} \otimes \ldots \otimes l_{k_{n_1}+1}$$

The following map, $\tilde{\beta}$, embeds these into the Hochschild complex of the bar construction. It gives terms of simplicial degree 0 coming from the canonical inclusion (adding an identity map in $\mathcal{O}(k + m_1, k + m_1)$ to the above), plus additional terms of higher simplicial degrees. These elements of $C^{m_1}(B(\Phi, \mathcal{O}, \mathcal{O}))(m_1)$ are now reinterpreted as lying in $B(\Phi, \mathcal{O}, \mathcal{O}C(-, [m_1]))$ just by considering $id_{k+m_1} \otimes l_{k_1+1} \otimes \ldots \otimes l_{k_{n_1}+1}$ as a graph with $n_1$ disjoint white vertices of valences $k_1 + 1, \ldots, k_{n_1} + 1$ and $m_1$ additional disjoint leaves.

The bottom horizontal map in the diagram now glues this last graph to $\Gamma$. The result of gluing is a sum of graphs $\Gamma'$ which are obtained from $\Gamma$ by adding $k_i$ labeled leaves cyclically in all possible manners on the $i$th closed incoming cycle of $\Gamma$ for each $i$. After reinterpreting the new graphs as morphisms in $\mathcal{O}$ attached to $n_2$ white vertices (as in Lemma 6.1), the map $\alpha$—in simplicial degree 0—applies these morphisms of $\mathcal{O}$ to the elements of $A$ and kills terms of higher simplicial degree. Finally, the resulting chain of $\Phi((k_1' + 1) + \ldots + (k_{n_2} + 1) + m_2)$ is reinterpreted as $n_2$ Hochschild chains in $A$ and $m_2$ elements of $A$. The terms of higher simplicial degrees produces by $\tilde{\beta}$ are killed by $\alpha$.

The appendix explains how to read signs for the operations. For concrete examples of these operations in the case of a strict Frobenius algebra, we refer the reader to the end of section 6.5.
6.3. Twisting by the determinant bundle. For a black and white graph $G$ defining a morphism in $\mathcal{OC}([m_1], [m_2])$, we define its outgoing boundary $\partial_{\text{out}} G$ to be the union of its $n_2$ white vertices and the endpoints of its $m_2$ outgoing leaves, regarded as a subspace of the corresponding topological graph, also denoted $G$. We write $\det(G, \partial_{\text{out}})$ for the Euler characteristic of the relative homology $H_*(G, \partial_{\text{out}})$, regarded as a graded abelian group:

$$\det(G, \partial_{\text{out}}) := \det(H_*(G, \partial_{\text{out}})) = \det(H_0(G, \partial_{\text{out}})) \otimes \det(H_1(G, \partial_{\text{out}}))^*$$

considered as a graded $\mathbb{Z}$-module, in degree $-\chi(G, \partial_{\text{out}})$.

For $d \in \mathbb{Z}$, define a $d$–orientation for $G$ to be a choice of generator of

$$\det(\mathbb{R}(V \sqcup H)) \otimes \det(G, \partial_{\text{out}})^{\otimes d}.$$ 

We define new categories $\mathcal{O}_d$ and $\mathcal{OC}_d$ just like $\mathcal{O}$ and $\mathcal{OC}$ but replacing the previously defined orientation of graphs by a $d$–orientation. So the objects of $\mathcal{O}_d$ and $\mathcal{OC}_d$ are the same as those of $\mathcal{O}$ and $\mathcal{OC}$, but the morphisms are now chain complexes generated by pairs $(G, o_d(G))$ for $G$ a graph representing a morphism in $\mathcal{O}$ or $\mathcal{OC}$ and $o_d(G)$ a $d$–orientation of $G$. The boundary of a $d$–oriented graph $(G, o_d(G))$ is the boundary of the graph $G$ as before together with the $d$–orientation induced as before for its $\det(\mathbb{R}(V \sqcup H))$–part, and by choosing a topological map realizing the blow-up of a vertex with support in a small neighborhood of that vertex for its $\det(G, \partial_{\text{out}})$–part.

To define composition in $\mathcal{O}_d$ and $\mathcal{OC}_d$, we need the following. Let $G_1, G_2$ be two graphs representing composable morphisms in $\mathcal{OC}$, with $(G_2 \circ G_1) = \sum G$ their composition in $\mathcal{OC}$. As $G_2$ is a subgraph of each $G$, we have a triple $(G, G_2, \partial_{\text{out}})$. Note also that $H_*(G, G_2) \cong H_*(G_1, \partial_{\text{out}})$ as collapsing the copy of $G_2$ in any term $G$ of $G_2 \circ G_1$ will exactly recreate $G_1$ with its outer boundary collapsed. Splitting the long exact sequence in homology for each triple $(G, G_2, \partial_{\text{out}})$ into short exact sequences and choosing splittings of those, one gets a natural isomorphism

$$\det(G_1, \partial_{\text{out}}) \otimes \det(G_2, \partial_{\text{out}}) \to \det(G, \partial_{\text{out}})$$

for each term in the composition. This isomorphism is associative (see [8, Sect. 3] and [34]). One then can define composition in $\mathcal{O}_d$ or $\mathcal{OC}_d$ as composition in $\mathcal{O}$ or $\mathcal{OC}$, tensored with the $d$th power of this isomorphism. More precisely, the composition of $d$–oriented graphs $(G_1, o_d(G_1))$ and $(G_2, o_d(G_2))$ is by the same gluing as before on the graphs, and via the composition

$$\det(\mathbb{R}(V_1 \sqcup H_1)) \otimes \det(G_1, \partial_{\text{out}})^{\otimes d} \otimes \det(\mathbb{R}(V_2 \sqcup H_2)) \otimes \det(G_2, \partial_{\text{out}})^{\otimes d}$$

$$\to \det(\mathbb{R}(V_1 \sqcup H_1)) \otimes \det(\mathbb{R}(V_2 \sqcup H_2)) \otimes \det(G_1, \partial_{\text{out}})^{\otimes d} \otimes \det(G_2, \partial_{\text{out}})^{\otimes d}$$

$$\to \det(\mathbb{R}(V_1 \sqcup H_1 \sqcup V_2 \sqcup H_2)) \otimes \det(G, \partial_{\text{out}})^{\otimes d}$$

for each term $G$ in $G_2 \circ G_1$, where the first arrow introduces a sign $(-1)^{d[G_2]}\chi(G_1, \partial_{\text{out}})$ and the second map is juxtaposition on the first factors as in $\mathcal{O}$ and $\mathcal{OC}$, and the $d$–power the above isomorphism on the last factors.

The resulting categories $\mathcal{O}_d$ and $\mathcal{OC}_d$ are symmetric monoidal.

Note that $\mathcal{O}_d$ admits a symmetric monoidal functor $i : \mathcal{A}_\infty \to \mathcal{O}_d$, since any graph $G \in \mathcal{A}_\infty$ is a union of trees, each with exactly one outgoing boundary point, so $\det(G, \partial_{\text{out}})$ is of degree 0, with a canonical generator, compatible under composition in $\mathcal{A}_\infty$. Thus we are entitled to form the Hochschild complex of any functor $\Phi : \mathcal{O}_d \to \text{Ch}$. Lemma 6.1 extends to show that $\mathcal{OC}_d$ is an extension of $\overline{\mathcal{C}}\mathcal{O}_d$, and so, by Theorem 5.11, we have

**Corollary 6.3.** Let $\Phi : \mathcal{O}_d \to \text{Ch}$ be an $(h)$–split symmetric monoidal functor. Then the pair $(\overline{\mathcal{C}}(\Phi)(0), \Phi(1))$ is a $\Sigma$–equivariant (homotopy) $\mathcal{OC}_d$–module.

6.4. Positive boundary variations. Recall from 3.3 the positive boundary subcategory $\mathcal{O}^b \subseteq \mathcal{O}$ whose morphisms are those satisfying that their underlying surface has at least one outgoing boundary in each component. Define now $\mathcal{OC}^b \subseteq \mathcal{OC}$ to be the subcategory consisting of graphs with at least one outgoing boundary in each component.
Recalling that the closed-to-closed morphisms of $\mathcal{OC}$ satisfy a “co-positive” boundary condition, namely that every component of the underlying surface has at least one incoming or free boundary, we have that the closed-to-closed part of $\mathcal{OC}^b$ satisfies both the positive and free/co-positive boundary conditions.

**Lemma 6.6.** The category $\mathcal{OC}^b$ is an extension of $\overline{\mathcal{C}}(\mathcal{O}^b)$.

**Proof.** Using the bijection in Lemma 6.1, we see that

$$\overline{\mathcal{C}}^b(\mathcal{O}^b(m_1, -))(m_2) = \oplus_{k_1, \ldots, k_n \geq 1} \mathcal{O}^b(m_1, k_1 + \cdots + k_n + m_2)/U \otimes L_{k_1} \otimes \cdots \otimes L_{k_n}$$

identifies with the subcomplex $\mathcal{O}^b([m_1], [n_2])$ of $\mathcal{OC}([m_1], [n_2])$ as outgoing boundary components in $\mathcal{OC}$ correspond to non-empty outgoing boundary in each component of $\mathcal{O}$ attached to it in the above decomposition. \hfill $\Box$

Applying Theorem 5.11 to $\mathcal{E} = \mathcal{O}^b$ and $\widehat{\mathcal{E}} = \mathcal{O}^b$ immediately gives

**Corollary 6.5.** If $\Phi : \mathcal{O}^b \to \operatorname{Ch}$ is an (h-)split symmetric monoidal functor, then the pair $(\overline{\mathcal{C}}(\Phi)(0), \Phi(1))$ is a $\Sigma$–equivariant (homotopy) $\mathcal{OC}^b$–module.

6.5. **Strict Frobenius algebras and Sullivan diagrams.** Recall from 3.2 the category $H_0(\mathcal{O})$, whose morphisms are the 0-th homology groups of those of $\mathcal{O}$, and which has the property that $H_0(\mathcal{O})$–algebras are exactly (strict) symmetric Frobenius algebras. We consider also the shifted version $H_{bot}(\mathcal{O}_d)$ whose morphisms are the bottom homology groups in each component of the morphisms of the category $\mathcal{O}_d$ of Section 6.3, i.e.

$$H_{bot}(\mathcal{O}_d) = \bigoplus_{S \in \pi_0(\mathcal{O}(n,m))} H_{-d, X(S, \partial_{out})}(\mathcal{O}_d, S(n, m)).$$

We call $H_{bot}(\mathcal{O}_d)$–algebras dimension $d$ Frobenius algebras.

We show in this section that the category of Sullivan diagrams $\mathcal{SD}$ of Section 2.10 is an extension of $\mathcal{C}(H_0(\mathcal{O}))$, and a shifted version $\mathcal{SD}_d$ of $\mathcal{SD}$ is an extension of $\mathcal{C}(H_{bot}(\mathcal{O}_d))$, which gives the action of Sullivan diagrams on the Hochschild complex of Frobenius algebras stated in Theorem 6.7. We then give explicit formulas for the product, coproduct and $\Delta$–operator on the Hochschild complex of Frobenius algebras coming out of our method, and check in Proposition 6.9 that, over a field, the Batalin-Vilkovisky coalgebra structure given by the coproduct and $\Delta$–operator on Hochschild homology is dual to the Batalin-Vilkovisky structure on the Hochschild cohomology of the algebra defined using the cup product and the dual to Connes’ operator $B$.

As already remarked in 2.10, the components of the category $\mathcal{SD}$ of Sullivan diagrams are in 1-1 correspondence with those of $\mathcal{OC}$, namely the topological types of open-closed cobordisms. For $S$ such a topological type, we denote by $\mathcal{SD}_S([m_1], [n_2])$ the corresponding component. We define $\mathcal{SD}_d$ to be the dg-category obtained from $\mathcal{SD}$ by shifting the degree of the component $\mathcal{SD}_S([m_1], [n_2])$ by $d_X(S, \partial_{out})$; note that the shifts in degree are consistent with composition. (The category $\mathcal{SD}_d$ is a quotient of the category $\mathcal{OC}_d$ of 6.3.)

**Lemma 6.6.** The category $\mathcal{SD}$ is an extension of $\overline{\mathcal{C}}(H_0(\mathcal{O}))$ and more generally, $\mathcal{SD}_d$ is an extension of $\overline{\mathcal{C}}(H_{bot}(\mathcal{O}_d))$.

**Proof.** We have

$$\overline{\mathcal{C}}^b(H_0(\mathcal{O})(m_1, -))(m_2) = \oplus_{k_1, \ldots, k_n \geq 1} H_0(\mathcal{O}(m_1, k_1 + \cdots + k_n + m_2)/U \otimes L_{k_1} \otimes \cdots \otimes L_{k_n}$$

whose generators, by gluing the graphs in $\mathcal{O}$ to the white vertices in the $L_{k_i}$’s, are black and white graphs with trivalent black vertices modulo the equivalence relation coming 1-cells in $\mathcal{O}(m_1, k_1 + \cdots + k_n + m_2)$, i.e. from blowing up 4-valent black vertices. But this corresponds exactly to the description of $\mathcal{SD}([m_1], [n_2])$ in terms of quotient of black and white graphs given by Theorem 2.8.
Replacing $H_0(\mathcal{O})$ by $H_{bot}(\mathcal{O}_d)$ in the above, we get $\mathcal{S}D_d([\alpha _{m_1}^n],[\beta _{m_2}^n])$ instead as the shifts in degree are the same. □

For $\mathcal{E} = H_{bot}(\mathcal{O}_d)$, taking $\tilde{\mathcal{E}} = \mathcal{S}D_d$, Theorem 5.11 thus gives

**Theorem 6.7.** Let $A$ be a symmetric Frobenius algebra of dimension $d$, then the pair $(\overline{C}(A), A)$ is a $\Sigma$–equivariant $\mathcal{S}D_d$–module, where $\overline{C}(A)$ denotes the reduced Hochschild complex of the algebra $A$.

As a differential graded algebra with a non-degenerate inner product defines a symmetric Frobenius algebra, this recovers Theorem 3.3 of [45] after dualization. (See also [46] which considers the open part as well as the closed part).

Using Theorem 5.13, a consequence of Proposition 2.11 and the above theorem is the following:

**Corollary 6.8.** For strict symmetric Frobenius algebras $A$, the TCFT structure on $\overline{C}_*(A)$ defined by Costello and Kontsevich-Soibelman factors through an action of Sullivan diagrams. In particular, stable classes in the homology of the moduli space act trivially.

This results puts together the work of Costello and Kontsevich-Soibelman with that of Tradler-Zeinalian: we have shown that Costello’s construction (which translates to that of Kontsevich-Soibelman when made explicit) of an action of moduli space on the Hochschild homology of a strict Frobenius algebra factors through an action of the complex of Sullivan diagrams as constructed by Tradler-Zeinalian [45, Thm 3.3].

The action on the Hochschild complex given by Theorem 5.11 is easy to implement explicitly in the case of strict Frobenius algebras because operations involve fewer terms than in the general case. Figure 15 (a–c) gives examples of graphs representing the product (pair of pants with two inputs and one output), the coproduct (pair of pants with one input and two outputs) and the $\Delta$–operator (degree 1 operator with one closed input and one closed output). We give now the explicit formulas for the action of these graphs on the Hochschild complex of a strict Frobenius algebra.

Let $A$ be a strict symmetric Frobenius algebra. To obtain the action of a (sum of) graph(s) $G$ representing a chain in $\mathcal{S}D_d$, on a chain in the Hochschild complex of $A$, we need to follow the prescription laid out in Section 6.2 (together with the appendix Section 7 for the signs). In Figure 15(a–c), we have made a choice of an ordering of the vertices, and of an orientation of the edges. The chosen orientation of each graph is then the orientation corresponding to considering the graph as a composition of the operations at each vertex in this ordering, with their canonical orientation (see Section 7). Figure 15(a’–c’) shows the non-trivial graphs created when applying the procedure described in Section 6.2.

![Graphs](image-url)
A BV coalgebra is an algebra over the cooperad whose conventions. They match if we introduce a factor \((-1)^{a_0 \cdots a_k} k\) outgoing closed boundary components, with composition induced by gluing. As the corresponding component of particular that, as the product is homotopy commutative, in homology it is 0 except on \(HH_0(A, A) \otimes HH_0(A, A)\).

(b) Coproduct:

\[
(a_0 \otimes \cdots \otimes a_k) \mapsto \sum (-1)^l (a''_0 \otimes a_1 \otimes \cdots \otimes a_i) \otimes (a'_0 \otimes a_{i+1} \cdots \otimes a_k)
\]

where \(\epsilon = d(|a_1| + \cdots + |a_k| + k)\).

(c) \(\Delta\)–operator:

\[
(a_0 \otimes \cdots \otimes a_k) \mapsto \sum (-1)^l 1 \otimes a_{i+1} \otimes \cdots \otimes a_k \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_i
\]

where \(\epsilon = (|a_0| + \cdots + |a_i|)(|a_{i+1}| + \cdots + |a_k|) + ik\).

**Proposition 6.9.** If \(A\) is a strict graded symmetric Frobenius algebra over a field \(k\), the coproduct and \(\Delta\) make \(HH_* (A, A)\) into a Batalin-Vilkovisky coalgebra. Moreover, this structure is dual to the BV-algebra structure on \(HH^* (A, A)\), where the product is the cup product of Hochschild cochains, and the BV operator is dual to Connes’ \(B\)–operator.

The first part of this proposition, before going to homology, recovers the cyclic Deligne conjecture as proved in [21, 45, 43].

The duality in this proposition is given on the chain level by a chain isomorphism \(CH^*(A, A) \to \text{Hom}(CH_*(A, A), k)\). Degree-wise this is given by the map

\[
\text{Hom}(A^{\otimes n}, A) \to \text{Hom}(A^{\otimes n+1}, k), \quad f \mapsto f
\]

where \(\tilde{f}(a_0, \ldots, a_n) = \langle a_0, f(a_1, \ldots, a_n) \rangle\).

**Proof.** A BV coalgebra is an algebra over the cooperad whose \(k\)–ary operations are given by the homology of the moduli space of Riemann surfaces of genus 0 with one incoming and \(k\) outgoing closed boundary components, with composition induced by gluing. As the corresponding component of \(\mathcal{SD}(\{\underline{1}\}, \{\underline{k}\})\) is quasi-isomorphic to that of \(\mathcal{OC}(\{\underline{1}\}, \{\underline{k}\})\), the first part of the statement follows, independently of the second part, from Theorems 2.5 and 6.7.

Now the duality carries \(\Delta\) to \(B\), since the \(\Delta\)–operator in \(HH_*(A, A)\) given in (c) is precisely \(B\), and the \(\Delta\)–operator on \(HH^*(A, A)\) is defined by transferring \(B^*\) via \(f \mapsto \tilde{f}\).

(The signs in the formula for \(B\) given in [12, Sect. 2.4] differs from ours due to different conventions. They match if we introduce a factor \((-1)^{a_0 \cdots a_k + k}\) passing the generator of \(H_1(S^1)\) on the other side of the Hochschild complex, and a factor \((-1)^{a_1+2a_2+\cdots+ka_k}\) before and after the operation to compare the Hochschild complexes—this last factor sets the degree \(k\) shift of the Hochschild complex in between the \(a_i\)’s instead of at the end as we have it).

So it suffices to check that the coproduct in (b) (which we will write as \(\nu\)) is dual to the Hochschild cup product. Let \(f\) and \(g\) be two Hochschild cochains; then (up to sign...
issues as above)
\[
\widetilde{f} \cup g(a_0,\ldots,a_{p+q}) = \pm \langle a_0, f(a_1,\ldots,a_p) \cdot g(a_{p+1},\ldots,a_{p+q}) \rangle \\
= \pm \sum (a''_0, f(a_1,\ldots,a_p)) \cdot \langle a'_0, g(a_{p+1},\ldots,a_{p+q}) \rangle \\
= \pm \nu^*(\tilde{f} \otimes \tilde{g})(a_0,\ldots,a_{p+q})
\]
where the first equality is the definition and the third from the formula given in (b) above. The second follows from Figure 16, below, which relates the coproduct and product in the Frobenius algebra \(A\) via the pairing.

\[\begin{array}{c}
a_0 \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
f \\
g \\
a_0' \end{array} \quad = \quad \begin{array}{c}
a_0 \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
f \\
g \\
a_0'
\end{array}\]

**Figure 16. Duality of the cup product and coproduct**

### 6.6. String topology

We apply in this section the results of the previous sections – particularly Theorem 5.11 and Corollary 6.8 – in order to control the (not entirely understood) operations in string topology in characteristic 0. Let \(C^*(M)\) denote the\( \text{ rational singular cochain complex of a compact, oriented, simply connected manifold, and } H^*(M)\) its cohomology.

It is well known (see e.g. [17]) that there is an isomorphism
\[
H^{-*}(LM) \cong HH_*(C^{-*}(M), C^{-*}(M))
\]
from the cohomology of the free loop space \(LM\) to the Hochschild homology of \(C^{-*}(M)\), the cochains of \(M\) seen as a chain complex in negative degree. \(H_*(LM)\) is equipped with the structure of a (positive boundary) homological conformal field theory (HCFT), i.e. action of the moduli space of Riemann surfaces, through papers beginning with Chas-Sullivan [4], and reaching its most complete form in the work of Godin [13].

We follow the prescription laid out by Lambrechts-Stanley [28] and Felix-Thomas [12] to construct this structure in Hochschild homology. We note that \(C^*(M)\) is quasi-isomorphic to a (simply connected) commutative differential graded algebra \(A\) (e.g., the algebra of differential forms on \(M\) with rational coefficients), and that \(H^*(A) \cong H^*(M)\) is a strict Frobenius algebra. Lambrechts-Stanley give a recipe for constructing, for any such \(A\), a weakly equivalent algebra \(B\) which is itself a commutative differential graded Frobenius algebra; that is, \(B\) itself satisfies Poincaré duality prior to application of cohomology. If \(M\) has dimension \(d\), such an algebra \(B\) is a dimension \(d\) symmetric Frobenius algebra in our sense, that is it defines a functor \(\Phi : H_{bot}(\mathcal{O}_d) \to Ch\). In particular, we can apply Theorem 6.7 to \(B\) and get an action of Sullivan diagrams on its Hochschild homology.

Using the chain of isomorphisms
\[
(*) \quad H^{-*}(LM) \cong HH_*(C^{-*}(M), C^{-*}(M)) \cong HH_*(A^{-*}, A^{-*}) \cong HH_*(B^{-*}, B^{-*})
\]
we get an action of Sullivan diagrams on \(H^*(LM)\), and hence an HCFT by precomposition with the map \(\mathcal{O}C \to SD\). We do not know for sure that this (somewhat collapsed) action is the one constructed by Godin, but the next proposition says that it is an extension of Chas-Sullivan’s string topology:

**Proposition 6.10.** The co-BV operations on \(H^*(LM, \mathbb{Q})\) dual to the Chas-Sullivan string topology BV operations on \(H_*(LM, \mathbb{Q})\) of [4] extend to an action of the closed part of \(H_{-*}(SD_{-d}, \mathbb{Q})\) (for \(d = \dim M\)) using Theorem 6.7.
Proof. The co-BV structure (and $H_*(SD_d)$-structure) we define on $H^*(LM, \mathbb{Q})$ is defined via an action on $HH_*(B, B)$, hence it is equivalent to check that the action on $HH_*(B, B)$ is dual to the string topology action. By Proposition 6.9, our co-BV structure is dual to the BV structure on $HH^*(B, B)$ coming from the Hochschild cup product and the dual of Connes’ operator $B$. Hence, by [12, Prop. 1], our structure is carried to the dual of the Chas-Sullivan structure by the isomorphism $(*)$. □

The HCFT structure we produce on $H^*(LM, \mathbb{Q})$ is an action of moduli spaces of Riemann surfaces factoring through an action of Sullivan diagrams, which immediately implies that a substantial part of the action is trivial (see Proposition 2.11). In particular, we know that all stable classes in the homology of the moduli space act trivially, a fact known in the string topology setting by work of Tamanoi [44].

Remark 6.11. It is worth issuing a caveat here: the main result of [33] implies that

$$
\text{Remark 6.11.} \quad \text{It is worth issuing a caveat here: the main result of [33] implies that}
$$

the BV structures on

$$
H_*(LS^2; \mathbb{F}_2) \quad \text{and} \quad HH^*(H^*(S^2; \mathbb{F}_2), H^*(S^2; \mathbb{F}_2))
$$

cannot be isomorphic (even though the underlying Gerstenhaber structures are), if we equip $H^*(S^2; \mathbb{F}_2)$ with the Frobenius algebra structure coming from Poincaré duality. Consequently, we cannot expect the construction given above to yield the HCFT structure on string topology if done integrally.

6.7. Hochschild homology of unital $A_\infty$ algebras. In this section, we briefly consider what our construction gives when applied to the category $\mathcal{E} = A_\infty^+, \text{ equipped with the identity functor } id : A_\infty^+ \to A_\infty^+.$

Proposition 6.12. The Hochschild complex $C^o(A_\infty^+(m, -))(n)$ is isomorphic to the (split) subcomplex of $(\bar{p}, m + n)$—Graphs consisting of fat graphs whose associated surface is a disjoint union of

- $n$ disks, each with precisely one outgoing open boundary, and
- $p$ annuli, each with precisely one closed outgoing boundary,

and with $m$ incoming open boundaries distributed on the free boundaries of these.

Proof. The gluing map

$$
\bigoplus_{n_i \geq 1} A_\infty^+(m, n_1 + \cdots + n_p + n)/U_1 \otimes L_{n_1} \otimes \ldots \otimes L_{n_p} \to (\bar{p}, m + n) - \text{Graphs}
$$

produces graphs which are a disjoint union of trees and trees attached to white vertices (see Figure 6); the associated surfaces are as described. □

We therefore define an extension $\text{Ann}$ of $\mathcal{CA}_\infty^+$ to be the subcategory of $\mathcal{OC}$ consisting of graphs whose associated surface is a disjoint union of surfaces as in 6.12, or a closed-to-closed annulus. Note that we cannot introduce any closed-to-open annuli in $\text{Ann}$, for composites would produce open-to-open morphisms that are not already present in $\mathcal{CA}_\infty^+$. As $\text{Ann}$ is an extension of $\mathcal{CA}_\infty^+$, by Theorem 5.11, we conclude:

Theorem 6.13. For any $A_\infty^+$-algebra $A$, the pair $(\mathcal{C}(A), A)$ is an $\text{Ann}$-module.

We examine the resulting $H_*(\text{Ann})$-structure on the pair $(HH_*(A, A), H_*(A))$, for $A$ a unital $A_\infty^+$-algebra.

$\text{Ann}$ evidently contains $\mathcal{A}_\infty^+ = \text{Ann} \cap \mathcal{O}$, and so the open sector of an $\text{Ann}$-module remains (unsurprisingly) a unital $A_\infty^+$-algebra. This equips $H_*(A)$ with the structure of a unital associative ring. Write $m \in H_0(\text{Ann}(\begin{smallmatrix} 0 \\ 12 \end{smallmatrix}), \begin{smallmatrix} 0 \\ 11 \end{smallmatrix}))$ for the class corresponding to the product, and $u \in H_0(\text{Ann}(\begin{smallmatrix} 0 \\ 12 \end{smallmatrix}), \begin{smallmatrix} 0 \\ 11 \end{smallmatrix}))$ for the class corresponding to the unit.

\footnote{Similarly there are no disks with a closed incoming boundary, since compositions would produce an open-to-open morphism with codomain 0.}
Furthermore, since the mapping class group of an annulus with fixed boundaries is isomorphic to \( \mathbb{Z} \), generated by the Dehn twist, the morphism complex \( \mathcal{A}nn([\Pi_1], [\Pi_2]) \) is quasi-isomorphic to \( C_*\mathcal{B}\mathbb{Z} = C_*S^1 \). Up to homotopy, the only nontrivial operation \( [\Pi] \to [\Pi] \) is thus a class \( \Delta \) of degree 1, corresponding to the fundamental class of the circle. This is Connes’ operator \( B \) explicitly given at the end of Section 6.5 (see Proposition 6.9)\(^{12}\).

One should also consider the interaction of the open and closed sectors. There are no closed-to-open morphisms in \( \mathcal{A}nn \), but there is a class \( i \in H_0(\mathcal{A}nn([\Pi_1], [\Pi_2])) \) coming from the annulus with one open incoming and one closed outgoing boundary. This map \( i : H_*(A) \to HH_*(A, A) \) is induced by the quotient map \( A \to HH_0(A, A) \).

**Proposition 6.14.** The category \( H_*(\mathcal{A}nn) \) is generated as a symmetric monoidal category by the operations \( m, u, \Delta, \) and \( i \).

**Remark 6.15.** The Hochschild complex of a category \( \mathcal{E} \) is functorial in \( \mathcal{E} \); furthermore, it is not hard to see that a monoidal quasi-isomorphism \( \mathcal{E} \to \mathcal{E}' \) induces a quasi-isomorphism of Hochschild complexes (using, e.g. the spectral sequence of a bicomplex). Consequently the results above apply equally to the category associated to the operad \( \mathcal{A}ss^+ \) of unital associative algebras, since it is quasi-isomorphic to \( \mathcal{A}ss^+_{\infty} \).

### 6.8. Algebras over \( \mathcal{E} = \mathcal{A}ss^+ \otimes \mathcal{P} \) for an operad \( \mathcal{P} \)

Let \( \mathcal{P} \) be a chain operad, and consider the operad \( \mathcal{A}ss^+ \otimes \mathcal{P} \) whose algebras are unital associative algebras together with a commuting \( \mathcal{P} \)-algebra structure. By the work of Brun, Fiedorowicz, and Vogt [3], if \( \mathcal{P} \) is the chain complex of the little disks operad \( \mathcal{C}n \), the resulting tensor product is an \( En_{n+1} \)-operad. Furthermore, they show that the Hochschild complex of an \( \mathcal{A}ss^+ \otimes \mathcal{P} \)-algebra admits the structure of a \( \mathcal{P} \)-algebra.

Explicitly, the action of \( \mathcal{P} \) on \( C_*(\mathcal{A}ss^+) \) is as follows: As \( A \) is a unital associative algebra, we can consider \( C_*(A) \) as the chain complex associated to a simplicial chain complex \( \mathcal{A}ss^+ \) with \( \mathcal{A}ss^+ = \mathcal{A}ss^+ \otimes \mathcal{B}n \) and degeneracy \( s_i \) inserting a unit in position \( i + 1 \). The \( \mathcal{A}ss^+ \otimes \mathcal{P} \)-structure of \( A \) defines a simplicial \( \mathcal{P} \)-structure on \( \mathcal{A}ss^+ \) by acting diagonally on \( \mathcal{A}ss^+ \otimes \mathcal{B}n \), and this in turn induces a \( \mathcal{P} \)-structure on the associated total chain complex \( C_*(A) \).

This last structure can be made explicit via the Eilenberg-Zilber maps. The action of a chain \( p \in \mathcal{P}(k) \) on \( (a_1^0 \otimes \cdots \otimes a_n^0) \otimes \cdots \otimes (a_1^k \otimes \cdots \otimes a_n^k) \) is of the form

\[
\sum \pm (p(a_1^0, \ldots, a_n^0) \otimes p(1, \ldots, a_1, \ldots, 1) \otimes \cdots \otimes p(1, \ldots, a_p, \ldots, 1),
\]

where the sum is over all possible shuffles of \( (a_1^0, \ldots, a_n^0), \ldots, (a_1^k, \ldots, a_n^k) \), with the resulting sequence denoted \( a_1, \ldots, a_p, \ldots, 1 \), and \( p(1, \ldots, a_1, \ldots, 1) \) means \( a_i = a_k^i \) at the \( j \)th position and \( 1's \) everywhere else.

By the results of the previous section, \( HH_*(A, A) \) is a \( H_*(\mathcal{A}nn) \)-module. It is natural, then, to ask how this interacts with the Brun-Fiedorowicz-Vogt \( \mathcal{P} \)-algebra structure. Comparing the above formula with the formula for Connes’ \( B \) operator (given at the end of Section 6.5) shows though that these two structures do not interact very well, in particular because of the special role of the \( a_0^0 \)'s in the \( \mathcal{P} \)-action. One can though define an extension of the category \( \mathcal{A}ss \otimes \mathcal{P} \) with the free operad generated by \( \mathcal{P} \) and \( B \) as “closed-to-closed” morphisms, subject to the relations in \( \mathcal{P} \) and \( B^2 = 0 \).

### 7. Appendix: How to compute signs

Let \( \Phi : \mathcal{E} \to \text{Ch} \) be a split monoidal functor for \( \mathcal{E} = \mathcal{O}, \mathcal{O}_d, \mathcal{O}_C \) or \( \mathcal{O}_C \), with \( \Phi(1) = A \) an \( \mathcal{A}ss^+ \)-Frobenius algebra. Given an oriented graph \( \Gamma \) which is a morphism in \( \mathcal{E} \), we want to read off an explicit formula of the associated operation on \( A \) or \( C_*(A, a) \) with signs. The explicit formula will be given in terms of a chosen set of generating operations

\(^{12}\text{Note here that the formula is the same for } \mathcal{A}ss^+ \text{–algebras as for strictly associative algebras as there are no black vertices in the graph generating the operation } \Delta.\)
for $\mathcal{O}$, for example in terms of the (co)product and higher (co)products, the unit and the trace in $\mathcal{O}$ (or $\mathcal{O}_d$), and additionally the generator $l_n$ of Figure 1 for $\mathcal{O}C$ (or $\mathcal{O}_dC$).

To be precise, one first needs to make a choice of which orientation should be thought of as the “positive” orientation for the graphs representing the chosen basic operations. For the products and coproducts, we choose here the orientation $v \wedge h_1 \wedge \ldots \wedge h_k$ for $v$ the vertex and $h_1, \ldots, h_k$ the half edges in their cyclic order starting at the first incoming half-edge. The unit and the trace are exceptional graphs with a canonical positive orientation. For $l_k$, we take the orientation $w \wedge h_1 \wedge \ldots \wedge h_k$ for $w$ the vertex, $h_1, \ldots, h_k$ the half edges in their cyclic order starting at the start half-edge.

Figure 17 gives as an example the convention we will use for the product in an algebra.

Given a graph $\Gamma$, we first need to write it as a composition of the chosen generating operations. This means choosing an orientation of the internal edges and an ordering of the vertices, possibly introducing new vertices together with unit or trace operations, and possibly using the symmetries of the category. (See Figure 18 below for an example, and the proof of Proposition 3.1 for the case of $\mathcal{O}^k$.) Suppose $\Gamma$ has vertices $v_1, \ldots, v_k$ with half-edges $h_1^i, \ldots, h_n^i$ at $v_i$ and $v_i \wedge h_1^i \wedge \ldots \wedge h_n^i$ the chosen orientation of the (chosen) operation $\mu_i$ associated to $v_i$. To interpret $\Gamma$ as a composition of the operation at $v_1$, then at $v_2$ etc. requires writing the orientation of $\Gamma$ as $\pm(v_1 \wedge h_1^1 \wedge \ldots \wedge h_n^1) \wedge \ldots \wedge (v_k \wedge h_1^k \wedge \ldots \wedge h_n^k)$.

Suppose we start from

$$a_1 \otimes \ldots \otimes a_n \otimes (\Gamma, o_d(\Gamma))$$

in $A^\otimes n \otimes \mathcal{O}_d(n, m)$, with $\Gamma$ as above and

$$o_d(\Gamma) = (v_1 \wedge h_1^1 \wedge \ldots \wedge h_n^1) \wedge \ldots \wedge (v_k \wedge h_1^k \wedge \ldots \wedge h_n^k) \otimes \det(\Gamma, \partial_{\text{out}})^{\otimes d}.$$ 

We rewrite this (with a Koszul sign!) as

$$a_1 \otimes \ldots \otimes a_n \otimes ((v_1 \wedge h_1^1 \wedge \ldots \wedge h_n^1) \otimes \det(\mu_1)^{\otimes d}) \otimes \ldots \otimes ((v_k \wedge h_1^k \wedge \ldots \wedge h_n^k) \otimes \det(\mu_k)^{\otimes d})$$

in $A^\otimes n \otimes \mathcal{O}_d(n, p_1) \otimes \ldots \otimes \mathcal{O}_d(p_r, m)$, from which we can apply the first operation and then the next etc. The final sign for the operation will come, in addition, from the signs occurring when using the symmetries in the category.

If the graph was an operation in $\mathcal{O}_dC$ instead, that is if we start with

$$(a^n_0 \otimes \ldots \otimes a^n_k \otimes l_k) \otimes \ldots \otimes (a^n_0 \otimes \ldots \otimes a^n_k \otimes l_k) \otimes b_1 \otimes \ldots \otimes b_m \otimes (\Gamma, o_d(\Gamma))$$

in $C(A, A)^{\otimes n} \otimes A^{\otimes m} \otimes \mathcal{O}_d([n], [i_1, i_2])$, the principle is the same, but we have in addition to apply the procedure described in Section 6.2.

We now give an explicit example with a graph of $\mathcal{O}_d(2, 1)$ which is used in the computations at the end of section 6.5. In Figure 18, we give a graph with a choice of ordering of its vertices $v_1, v_2, v_3$, and a choice of orientation of its internal edges $e_1, e_2, e_3$. We choose the orientation of the graph that corresponds to writing it as a composition of
the operation attached to $v_1$ (a coproduct), followed by the operation attached to $v_2$ and then $v_3$ (both products). Explicitly, it is given as

$$(v_1 \wedge h_1 \wedge e_1 \wedge e_2) \wedge (v_2 \wedge e_2 \wedge h_2 \wedge e_3) \wedge (v_3 \wedge e_1 \wedge e_3 \wedge \bar{h}_1)$$

where $e_i$ and $\bar{e}_i$ are the start and end half-edges of $e_i$, $h_i$ is the $i$th incoming leaf, and $\bar{h}_1$ is the outgoing leaf.

Figure 18. Writing a graph as a composition

The graph has relative Euler characteristic $\chi(\Gamma, \partial_{\text{out}}) = -1$ which is also the relative Euler characteristic $\det(c)$ of the coproduct, while the products have trivial relative Euler characteristic. As the products have degree 0, moving the determinant past the operations does not produce a sign and the operation associated to $\Gamma$ with the above orientation is that of the composition

$$((v_1 \wedge h_1 \wedge e_1 \wedge e_2) \otimes (\det c)^{\otimes d} \otimes id) \otimes (\tau \otimes id) \otimes (v_2 \wedge e_2 \wedge h_2 \wedge e_3) \otimes (v_3 \wedge \bar{e}_1 \wedge \bar{e}_3 \wedge \bar{h}_1)$$

in $(O_d(1,2) \oplus O_d(1,1)) \otimes O_d(3,3) \otimes O_d(3,2) \otimes O_d(2,1)$, where $\tau$ denotes the twist map.

The succession of operations (a comultiplication, a twist and two multiplications) applied to an pair $a \otimes b$ is

$$a \otimes b \mapsto (-1)^{|b+d|} \sum a' \otimes a'' \otimes b$$
$$(-1)^{|b+d+|a'||a''|} \sum a'' \otimes a' \otimes b$$
$$(-1)^{|b+d+|a'||a''|} \sum a'' \otimes a'b$$

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