HOCHSCHILD HOMOLOGY OF STRUCTURED ALGEBRAS

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Abstract. We give a general method for constructing explicit and natural operations on the Hochschild complex of algebras over any PROP with $\mathcal{A}_\infty$–multiplication—we think of such algebras as $\mathcal{A}_\infty$–algebras "with extra structure". As applications, we obtain an integral version of the Costello-Kontsevich-Soibelman moduli space action on the Hochschild complex of open TCFTs, the Tradler-Zeinalian action of Sullivan diagrams on the Hochschild complex of strict Frobenius algebras, and give applications to string topology in characteristic zero. Our main tool is a generalization of the Hochschild complex.

The Hochschild complex of an associative algebra A admits a degree 1 self-map, Connes' boundary operator B. If A is Frobenius, the cyclic Deligne conjecture says that B is the Δ–operator of a BV-structure on the Hochschild complex of A, and in fact B is part of much richer structure, namely an action by the chain complex of Sullivan diagrams on the Hochschild complex [31]. A weaker version of Frobenius algebras, called $\mathcal{A}_\infty$–Frobenius algebras, yields instead an action by the chains on the moduli space of Riemann surfaces [5, 16]. In this paper we develop a general method for constructing explicit operations on the Hochschild complex of $\mathcal{A}_\infty$–algebras "with extra structure", which contains these theorems as special cases.

An $\mathcal{A}_\infty$–algebra can be described as an enriched symmetric monoidal functor from an certain dg-category $\mathcal{A}_\infty$ to Ch, the category of chain complexes over $\mathbb{Z}$. We consider here dg-functors $\Phi : \mathcal{E} \to \mathrm{Ch}$, where $\mathcal{E}$ is a monoidal dg-category equipped with a dg-functor $i : \mathcal{A}_\infty \to \mathcal{E}$. We are particularly interested in the case where $\mathcal{E}$ is a PROP[1] and think of Φ as an $\mathcal{A}_\infty$–algebra with an additional $\mathcal{E}$–structure. (Expanding on the terminology of McClure–Smith [20], one can call such an $\mathcal{E} = (\mathcal{E}, i)$ a *PROP with $\mathcal{A}_\infty$-multiplication.*) We introduce the Hochschild complex as a certain new functor $C(\Phi) : \mathcal{E} \to \mathrm{Ch}$. The assignment has the property that, for Φ symmetric monoidal, $C(\Phi)$ evaluated at 0 is the usual Hochschild complex of the underlying $\mathcal{A}_\infty$–algebra. Also if Φ is split monoidal[2], the iterated complex $C^n(\Phi)$ evaluated at 0 is the nth tensor power $(C(\Phi)(0))^{\otimes n}$. Our main theorem, Theorem 4.9, says that if the iterated Hochschild complexes

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[1] symmetric monoidal category with objects the natural numbers

[2] i.e. such that the maps $\Phi(n) \otimes \Phi(m) \to \Phi(n+m)$ are isomorphisms

of the functors $\Phi = \mathcal{E}(e,-)$ admit a natural action of a dg-PROP $\mathcal{D}$ of the form

$$C^n(\mathcal{E}(e,-)) \otimes \mathcal{D}(n,m) \to C^m(\mathcal{E}(e,-))$$

then the classical Hochschild complex of any split monoidal functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ is a $\mathcal{D}$–algebra, i.e. there are maps

$$\big(C(\Phi)(0)\big)^{\otimes n} \otimes \mathcal{D}(n,m) \to \big(C(\Phi)(0)\big)^{\otimes m}$$

associative with respect to composition in $\mathcal{D}$. This action is given explicitly and is natural in $\mathcal{E}$ and $\mathcal{D}$.

Before stating the theorem in more detail, we describe some consequences. Let $\mathcal{O}$ denote the *open cobordism category*, whose objects are the natural numbers and the morphisms from n to m are the chains on the moduli space of the Riemann surfaces that are cobordisms from n to m open strings. Taking $\mathcal{E} = \mathcal{O}$ and $\mathcal{D} = \mathcal{C}$, the closed co-positive[3] boundary cobordism category, Theorem 4.9 gives an integral version of Costello's main theorem in [5], i.e., an action of the chains of the moduli space of Riemann surfaces on the Hochschild chain of any $\mathcal{A}_\infty$–*Frobenius algebra*[4]. (See Theorem 5.3 and Corollary 5.5.) Reading off our action on the Hochschild chains, we recover the recipe for constructing such an action given by Kontsevich and Soibelman in [16]. We also get a version for non-compact $\mathcal{A}_\infty$–Frobenius algebras by replacing $\mathcal{O}$ by the positive boundary[5] open cobordism category and $\mathcal{C}$ with the positive and co-positive boundary category. (Corollary 5.7.)

Applying Theorem 4.9 to the category $\mathcal{E} = H_0(\mathcal{O})$, we obtain an action of the chain complex of Sullivan diagrams on the Hochschild complex of strict symmetric Frobenius algebras, recovering the main theorem of Tradler-Zeinalian in [31]. (See Theorem 5.11.) In particular, in genus 0, this gives the cyclic Deligne conjecture also proved in [15, 29]. (See Proposition 5.14.)

A consequence of our naturality statement, Theorem 4.11, is that the aforementioned HCFT structure constructed by Costello and Kontsevich-Soibelman factors through an action of Sullivan diagrams, when the $\mathcal{A}_\infty$–Frobenius algebra happens to be strict. This implies a collapse of a significant part of the structure in that case, in particular the action by stable classes. (See Proposition 5.12 and Corollary 5.13.)

We apply the above to the case of string topology for a simply-connected manifold M over a field of characteristic zero, using the strict Frobenius model of $C^*(M)$ given by Lambrechts-Stanley [17, 7], and obtain an HCFT structure on $H^*(LM,\mathbb{Q})$ factoring through an action of Sullivan diagrams. It is natural to conjecture that this structure is the same as the one defined by Godin in [8], and towards this we show in Proposition 5.15 that our structure also recovers the BV structure on $H_*(LM)$ originally introduced by Chas-Sullivan. The vanishing of the action of the stable classes in the

[3]where the components of morphism each have at least one *incoming* boundary

[4]called *extended Calabi-Yau $\mathcal{A}_\infty$ category* in [5]

[5]where the components of morphism each have at least one *outgoing* boundary

HCFT structure furthermore agrees with Tamanoi's vanishing result in [30]. Different approaches to Sullivan diagram action on $H^*(LM)$ can be found in [13, 14, 27].

We now describe our set-up and tools in a little more detail and give a more precise formulation of the main theorem.

Recall from above that $\mathcal{E}$ is a dg-category equipped with a functor $i : \mathcal{A}_\infty \to \mathcal{E}$, which, for ease of notation, will always be assumed to be the identity on the objects, the natural numbers. Recall also that the Hochschild complex of a functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ is defined here as a new functor $C(\Phi) : \mathcal{E} \to \mathrm{Ch}$.

Given a dg-category $\mathcal{E}$, we define its *Hochschild core category* $C\mathcal{E}$ which has objects $\left[\begin{smallmatrix} n \\ e \end{smallmatrix}\right] \in \mathrm{Obj}(\mathcal{E}) \times \mathbb{N} = \mathbb{N} \times \mathbb{N}$, has $\mathcal{E}$ as a full subcategory, and with the morphisms from $\left[\begin{smallmatrix} 0 \\ e_1 \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} n \\ e_2 \end{smallmatrix}\right]$ the iterated Hochschild complex $C^n(\mathcal{E}(e_1, -))$ evaluated at e_2. If $\mathcal{E}$ is the open cobordism category $\mathcal{O}$, then $C\mathcal{E}$ is the open-to-open and open-to-closed part of the open-closed cobordism category. Given a monoidal category $\widehat{\mathcal{E}}$ with the same objects as $C\mathcal{E}$, we call it an *extension of* $C\mathcal{E}$ if it agrees with $C\mathcal{E}$ on the morphisms with source $\left[\begin{smallmatrix} n \\ e \end{smallmatrix}\right]$ when $n = 0$. An extension of $\mathcal{E}$ can be thought of as the full open-closed cobordism category, also including the closed-to-closed morphisms.

Main Theorem (Theorem 4.9 for Φ split, C unreduced)**.** *Let $\widehat{\mathcal{E}}$ be an extension of $C\mathcal{E}$ for $\mathcal{E}$ as above, and let $\Phi : \mathcal{E} \to \mathrm{Ch}$ be a split monoidal functor, with $\Phi(1) = A$ its underlying $\mathcal{A}_\infty$–algebra and $C(A)$ its Hochschild complex. Then taking $F(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]) = A$ and $F(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]) = C(A)$ defines a split monoidal dg-functor $F : \widehat{\mathcal{E}} \to \mathrm{Ch}$, i.e. there are chain maps*

$$C(A)^{\otimes n_1} \otimes A^{\otimes m_1} \otimes \widehat{\mathcal{E}}(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]) \longrightarrow C(A)^{\otimes n_2} \otimes A^{\otimes m_2}$$

associative with respect to composition in $\widehat{\mathcal{E}}$. When $\mathcal{E}, \widehat{\mathcal{E}}, i$ and Φ are symmetric monoidal, F is also symmetric monoidal.

The same holds for a reduced version of the Hochschild complex.

This theorem applies to any category $\mathcal{E}$ and chosen extension $\widehat{\mathcal{E}}$. For each of the applications discussed above we construct explicit extension categories, and the PROP $\mathcal{D}$ mentioned above in each case is the "closed-to-closed" part of the extension category. A general, abstract construction of an extension also exists and is discussed in [32]. In the present paper, ad hoc methods coming from the geometry of the situation will however suffice.

The proof of the main theorem, inspired by [5], uses simple properties of the double bar construction (and a quotiented version of it to take care of the equivariant version of the theorem). Our action is explicit thanks to the construction of an explicit pointwise chain homotopy inverse to the quasi-isomorphism of functors $C(B(\Phi, \mathcal{E}, \mathcal{E})) \to C(\Phi)$. (See Proposition 4.7.) As an example of how our theory can be applied, we give in Section 5.5 explicit formulas for the product, coproduct, and Δ–operator on the Hochschild complex of strict Frobenius algebras.

The paper is organized as follows: Section 2 introduces the chain complexes of graphs used throughout the paper and the short Section 3 reviews a few properties of the double bar construction and its quotiented analog. Section 4 then defines the Hochschild operator, examines its properties, and proves the main theorem. The largest section, Section 5, gives applications: Section 5.1 gives the application to Costello's theorem, and Section 5.2 describes how to deduce the Kontsevich-Soibelman approach from it. Sections 5.3 and 5.4 take care of the twisting by the determinant bundle and the positive boundary variation. In Section 5.5, we treat the case of strict Frobenius algebras and Sullivan diagrams, with the application to string topology given in Section 5.6. Finally, Sections 5.7 and 5.8 consider $\mathcal{A}_\infty$ and $\mathcal{A}ss\times\mathcal{P}$–algebras for $\mathcal{P}$ an operad. Section 1 sets up some notation and the Appendix Section 6 explains how to compute signs given operations represented by graphs.

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1. Conventions and terminology

In the present paper, we work in the category Ch of chain complexes over $\mathbb{Z}$, unless otherwise specified. We use the usual sign conventions so that the differential $d_V + d_W$ on a tensor product $V \otimes W$ is $(d_V + d_W)(v \otimes w) = d_V(v) \otimes w + (-1)^{|v|} v \otimes d_W(w)$.

By a *dg-category*, we mean a category $\mathcal{E}$ whose morphism sets are chain complexes and whose composition maps $\mathcal{E}(m,n) \otimes \mathcal{E}(n,p) \to \mathcal{E}(m,p)$ are chain maps. A *dg-functor* $\Phi : \mathcal{E} \to \mathrm{Ch}$ is a functor such that the structure maps

$$c_\Phi : \Phi(m) \otimes \mathcal{E}(m,n) \to \Phi(n)$$

are chain maps.[6] For example, given any $r \in \mathrm{Obj}(\mathcal{E})$, the functor $\Phi(m) = \mathcal{E}(r,m)$ represented by r is a dg-functor.

If $\mathcal{E}$ is a symmetric monoidal category, we say that Φ is *symmetric monoidal* if there are maps $\Phi(n) \otimes \Phi(m) \to \Phi(n+m)$ natural in n and m and compatible with the symmetries of Ch and $\mathcal{E}$. We say that Φ is *split monoidal* if these maps are isomorphisms and *h-split* if they are quasi-isomorphisms.

[6]Equivalently, $\mathcal{E}$ is a category enriched in Ch, and Φ is an enriched functor.

2. Graphs and trees

In this section, we give the background definitions about graphs, chain complexes of graphs etc. necessary for the rest of the paper. In particular, we define $(\bar{p}, m)$–Graphs and the open cobordism category $\mathcal{O}$.

2.1. **Graph:** By a *graph* G we mean a tuple (V, H, s, i) where V is the set of *vertices*, H the set of *half-edges*, $s : H \to V$ is the *source map* and $i : H \to H$ is an involution. Fixed points of the involution are called *leaves*. A pair $\{h, i(h)\}$ with $i(h) \neq h$ is called an *edge*. We will consider graphs with vertices of any valence, also valence 1 and 2.

We allow the empty graph. We will also consider the following degenerate graphs which fail to fit the above description:

- The *leaf* consisting of a single leaf and no vertices.
- The *circle* with no vertices.

The leaf will appear in two flavors: as a *singly labeled leaf* and as a *doubly labeled leaf*. The circle will arise from gluing the doubly labeled leaf to itself.

2.2. **Fat graph:** A *fat graph* is a graph $G = (V, H, s, i)$ together with a cyclic ordering of each of the sets $s^{-1}(v)$ for $v \in V$. The cyclic orderings define *boundary cycles* on the graph, sequences of consecutive edges in the cyclic ordering, which correspond to the boundaries of the surface that can be obtained by thickening the graph. Figure 1(a) shows an example of a fat graph with two boundary cycles, where the cyclic ordering at vertices is that inherited from the plane.

2.3. (p, m)**–graph:** A (p, m)*–graph* is a fat graph with m of its leaves labeled $\{1, \ldots, m\}$ and with p of its vertices labeled $\{1, \ldots, p\}$. The labeled vertices are called *white vertices* and are allowed to have any valence (also 1 and 2). The other vertices of the graph are called *black vertices* and must be at least trivalent. We let V_w denote the set of white vertices and V_b denote the set of black vertices. Figure 1(b) shows an example of a $(2, 3)$–graph.

2.4. $(\bar{p}, m)$**–graph:** A $(\bar{p}, m)$*–graph* is a (p, m)–graph together with a chosen *start half-edge* for each of the sets $s^{-1}(v)$ with $v \in V_w$, so that we have an actual ordering rather than a cyclic ordering at the white vertices. Figure 1(c) shows an example of a $(\bar{2}, 3)$–graph, with the start half-edges marked by thick lines.

To define the Hochschild complex, we will use the $(\bar{1}, n)$–graph, denoted l_n, depicted in Figure 2, which has a single vertex which is white, and n leaves labeled cyclically, with the first leave as start-leaf.

Note that when $p = 0$, a (p, m)–graph or $(\bar{p}, m)$–graph is just an ordinary fat graph with m labeled leaves.

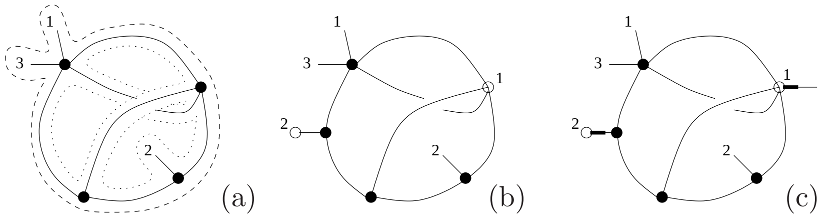

FIGURE 1. Fat graph with 3 labeled leaves, $(2,3)$–graph and $(\bar{2},3)$–graph.

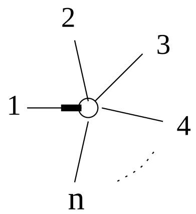

FIGURE 2. The $(\bar{1},n)$–graph l_n.

2.5. **Orientation:** An *orientation* of a graph G is a unit vector in $\det(\mathbb{R}(V \sqcup H))$. The degenerate graphs have a canonical positive orientation. Note moreover that any odd-valent (in particular trivalent) graph has a canonical orientation

$$v_1 \wedge h_1^1 \wedge \ldots \wedge h_{n_1}^1 \wedge \ldots \wedge v_k \wedge h_1^k \wedge \ldots \wedge h_{n_k}^k$$

where $v_1, \ldots, v_k$ is a chosen ordering of the vertices of the graph and $h_1^i, \ldots, h_{n_i}^i$ is the set of half-edges at v_i in their cyclic ordering.

2.6. **Edge collapse:** For a (p,m)–graph G and an edge e of G which is not a cycle and does not join two white vertices, we let G/e denote the (p,m)–graph obtained from G by collapsing the edge e, with the naturally induced cyclic orderings at its vertices and declaring the newly created vertex to be white if one of the end-vertices of e was white, with the same label.

For G oriented, G/e is oriented as follows. If $e = \{h_1, h_2\}$ with $s(h_1) = v_1$, $s(h_2) = v_2$, and writing the orientation of G in the form $v_1 \wedge v_2 \wedge h_1 \wedge h_2 \wedge x_1 \wedge \ldots \wedge x_k$, we define the orientation of G/e to be $v \wedge x_1 \wedge \ldots \wedge x_k$, where v is the vertex of G/e coming from identifying v_1 and v_2.

We note that edge collapses in a $(\bar{p},m)$–graph do not have a natural definition when they involve collapsing a start half-edge.

2.7. **Blow-up:** For an oriented (p,m)–graph G, we call a (p,m)–graph $\tilde{G}$ a *blow-up* of G if there exists an edge e of $\tilde{G}$ such that $G \cong \tilde{G}/e$ as oriented (p,m)–graphs. (Such an isomorphism is unique as soon as m is non-zero or p is non-zero and there is a chosen start half-edge like in a $(\bar{p},m)$–graph.)

For a $(\bar{p}, m)$–graph G, we call a $(\bar{p}, m)$-graph $\tilde{G}$ a *blow-up* of G if there exists an edge e of $\tilde{G}$ such that $G \cong \tilde{G}/e$ as oriented (p, m)–graphs respecting the start half-edges. Suppose $e = \{h_1, h_2\}$ with $s(h_1) = v_1 \in V_w$, $s(h_2) = v_2 \in V_b$, collapsing to v in G. If h_1 is the start-edge of v_1 in $\tilde{G}$, the requirement is that the start-edge of v in G lies in $s^{-1}(v_2)$ identified as a subset of $s^{-1}(v)$ under the (unique) isomorphism $G \cong \tilde{G}/e$. See Figure 3 for an example.

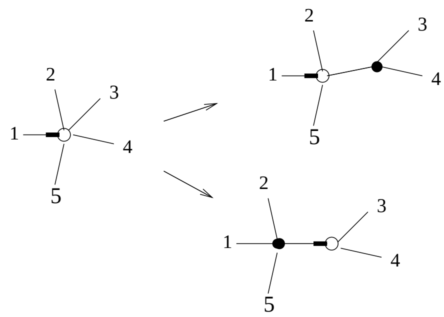

FIGURE 3. Two possible blow-ups of l_5

2.8. **The chain complex $(\bar{p}, m)$–Graphs:** Let $(\bar{p}, m)$–Graphs denote the chain complex generated as a $\mathbb{Z}$–module by isomorphism classes of (not necessarily connected, possibly degenerate) oriented $(\bar{p}, m)$–graphs with no unlabeled leaves which are not start-leaves of white vertices. We quotient by the relation that -1 acts by reversing the orientation. The degree of a graph is $\sum_{v \in V_b}(|v| - 3) + \sum_{v \in V_w}(|v| - 1)$ (the degenerate graphs having degree 0). The differential is defined on generators by

$$d \colon G \longrightarrow \sum_{\substack{(\tilde{G}, e) \\ \tilde{G}/e \cong G}} \lfloor \tilde{G} \rfloor$$

where $\tilde{G}$ runs over all possible blow-ups of G and $\lfloor \tilde{G} \rfloor = \tilde{G}$ unless $\tilde{G}$ has an unlabeled leaf l at a black vertex v. In this case, if v is trivalent, v and $s^{-1}(v)$ are collapsed in $\lfloor \tilde{G} \rfloor$, i.e. v and the leaf are removed, and if v is of higher valence, $\lfloor \tilde{G} \rfloor = 0$. (Unlabeled leaves at black vertices may occur if G had an unlabeled start-leaf.) The orientation of $\lfloor \tilde{G} \rfloor$ when $\lfloor \tilde{G} \rfloor \neq \tilde{G}$ (or 0) is obtained by first rewriting the orientation of G in the form $v \wedge l \wedge h_1 \wedge h_2 \wedge \ldots$ for $s^{-1}(v) = (l, h_1, h_2)$ in that cyclic ordering, and then removing the first 4 terms.

Figure 4 below shows two examples of differentials.

Lemma 2.1. *The map d is a differential.*

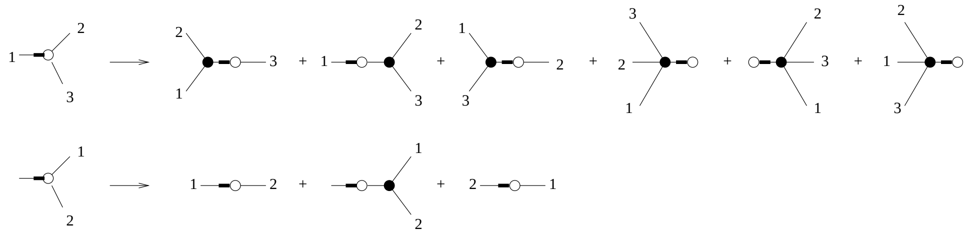

FIGURE 4. Differential applied to the $(\bar{1},3)$–graph l_3 and to a graph with an unlabeled start-leaf.

Proof. We have

$$d^2(G) = \sum_{\substack{(\tilde{G},e)\\ \tilde{G}/e\cong G}} \Big(\sum_{\substack{(\widehat{G},f)\\ \widehat{G}/f\cong\lfloor\tilde{G}\rfloor}} \lfloor\widehat{G}\rfloor \Big) = \sum_{\substack{(\tilde{G},e)\\ \tilde{G}/e\cong G\\ \lfloor\tilde{G}\rfloor\neq 0}} \Big(\sum_{\substack{(\widehat{G},f)\\ \widehat{G}/f\cong\tilde{G}}} \lfloor\widehat{G}\rfloor \Big) = \sum_{\substack{(\widehat{G},f,e)\\ \widehat{G}/f/e\cong G\\ \lfloor\widehat{G}/f\rfloor\neq 0}} \lfloor\widehat{G}\rfloor$$

where the second equality follows from the fact that if $G \neq \lfloor G\rfloor \neq 0$, then the difference between G and $\lfloor G\rfloor$ is a trivalent vertex with an unlabeled leaf which is forgotten in $\lfloor G\rfloor$. As the vertex is trivalent, it cannot be the collapse of f and there is therefore a unique way to lift that vertex to $\widehat{G}$—which now may have two unlabeled leaves, at different vertices.

One then checks that the orientations of $\widehat{G}/f/e$ and $\widehat{G}/e/f$ are opposite so that each term $(\widehat{G},f,e)$ cancels with the term $(\widehat{G},e,f)$, unless $\lfloor\widehat{G}/e\rfloor = 0$ while $\lfloor\widehat{G}/f\rfloor \neq 0$. This happens when $\widehat{G}/e$ has a valence 4 or higher vertex v with an unlabeled leaf attached to it, and $\widehat{G}$ is a blow-up of that vertex taking the leaf to a trivalent vertex. There are always exactly two ways of blowing-up such a vertex v to create a trivalent vertex with the leaf (coupled with its left or right neighbor), and the term $(\widehat{G},e,f)$ in this case cancels with the corresponding other term $(\widehat{G}',e,f')$. (Here $\widehat{G}\neq\widehat{G}'$ but $\lfloor\widehat{G}\rfloor = \lfloor\widehat{G}'\rfloor$.) □

2.9. **The open cobordism category $\mathcal{O}$:** For $p = 0$, $(\bar{0},m)$–Graphs is a chain complex of fat graphs with m labeled leaves. Fat graphs (without leaves) can be used to define a cell decomposition of Teichmuller space (see the work of Bowditch-Epstein [2], Harer [11], Penner [24, 25]), and the chain complex of isomorphism classes of graphs $(\bar{0},0)$–Graphs is the corresponding cellular complex of the quotient of Teichmuller space by the action of the mapping class group, namely the (coarse) moduli space of Riemann surfaces. Similarly, graphs with leaves define a chain complex for the moduli space of surfaces with fixed boundaries, or with fixed intervals in its boundaries, (see Penner [26, 23], Godin [9], Costello [5, Sect. 6] and [6]). Note that as soon as some part of a boundary in a surface is fixed (resp. a leaf), the Riemann surface (resp. the graph) has no symmetries, and the moduli space is a classifying space for the corresponding mapping class group, so that the chain complex computes the homology of the moduli space as well as the homology of the corresponding mapping class group.

Let S be a surface and I a collection of intervals in its boundary. If we denote by $\mathcal{M}(S, I)$ the moduli space of Riemann surfaces with a fixed structure on I (though with the convention that $\mathcal{M}(S^1 \times I, \varnothing) = *$), we have the following:

Theorem 2.2. *The homology of $(\bar{0}, m)$–Graphs is isomorphic to*

$$\bigoplus_{(S,I) \neq (D^2, \varnothing)} H_*(Mod(S, I))$$

where (S, I) ranges over all (possibly disconnected) oriented surfaces S with I a collection of m intervals on ∂S.

Hence $(\bar{0}, m)$–Graphs provide a model for the open cobordism category, as follows.

Let $\mathcal{O}$ be the symmetric monoidal dg-category with objects the natural numbers (including 0) and morphisms from m to n the chain complex $\mathcal{O}(m, n) = (\bar{0}, m + n)$–Graphs of fat graphs with $m + n$ labeled leaves. See Figure 5 for examples of morphisms in $\mathcal{O}$.

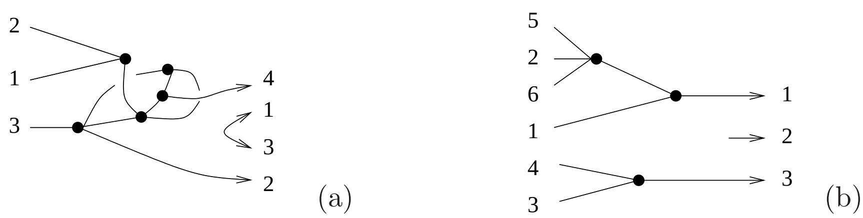

FIGURE 5. Morphisms of $\mathcal{O}(3, 4)$ and $\mathcal{A}_\infty(6, 3)$.

Labeling the leaves $in_1, \ldots, in_m$ and $out_1, \ldots, out_n$, composition is defined by gluing the leaves out_j and in_j together, so that the two leaves form an edge in the glued graph. Formally, we compose graphs by unioning vertices and half edges and altering the involution so that $i(in_j) = out_j$. The orientation is obtained by juxtaposition (wedge product). The rule for gluing the exceptional graphs is as follows:

- Gluing a leaf labeled on one side has the effect of removing the corresponding leaf of the other graph if this is a degree 0 operation (i.e. if the leaf was attached to a trivalent vertex)—otherwise the gluing just gives 0. If the trivalent vertex is v with half edges h_1, h_2, h_3 attached to it in that cyclic order, and the graph has orientation $v \wedge h_1 \wedge h_2 \wedge h_3 \wedge x_1 \wedge \ldots \wedge x_k$, then the glued graph has orientation $x_1 \wedge \ldots \wedge x_k$.
- Gluing a doubly labeled leaf has the effect of relabeling the leaf of the other graph if the labels of the leaf are incoming and outgoing. If both labels are incoming or outgoing, it attaches the corresponding leaves of the other graph

together so they form an edge. In particular (possibly empty) unions of doubly labeled leaves define the identity morphisms and the symmetries in the category.

2.10. **The category $\mathcal{A}_\infty$:** We let $\mathcal{A}_\infty$ denote the subcategory of directed forests in $\mathcal{O}$, i.e. $\mathcal{A}_\infty$ has the same objects as $\mathcal{O}$, the natural numbers, and the chain complex $\mathcal{A}_\infty(m,n)$ of morphisms from m to n is generated by graphs which are disjoint unions of n trees with a total of $m_1+\cdots+m_n=m$ incoming leaves in addition to the root of the tree which is labeled as an outgoing leaves. Here again we allow the degenerate graphs consisting of single leaves labeled on one side (as an output) or two sides (as an input and an output). See Figure 5 (b) for an example of a morphism in $\mathcal{A}_\infty$.

A symmetric monoidal functor $\Phi : \mathcal{A}_\infty \to \mathrm{Ch}$ corresponds precisely to giving a (unital) A_∞–structure on $\Phi(1)$. (This comes from the fact that planar, or equivalently "fat" trees define a cellular decomposition of Stasheff's polytopes. See for example [19, C.2, 9.2.8].) The kth multiplication of the A_∞–structure is given by the morphism $m_k : k \to 1$ of degree $k-2$ for $k \geq 2$ represented by a graph with a single vertex and $k+1$ leaves. Together with the map $u : 0 \to 1$ of degree 0 (singly labeled outgoing leaf), which is a unit for the multiplication m_2, these morphisms generate $\mathcal{A}_\infty$ as a symmetric monoidal category.

2.11. **$\mathcal{A}_\infty$–Frobenius algebras:** We will call a split symmetric monoidal functor $\Phi : \mathcal{O} \to \mathrm{Ch}$ (or its value at 1, $\Phi(1)$) an *$\mathcal{A}_\infty$–Frobenius algebra*. If Φ is h–split, Φ could be called an *extended $\mathcal{A}_\infty$–Frobenius algebra*, following [5, 7.3]. In either case, by restriction along $i : \mathcal{A}_\infty \to \mathcal{O}$, Φ equips $\Phi(1)$ with the structure of an $\mathcal{A}_\infty$–algebra.

In addition to the $\mathcal{A}_\infty$–structure, the morphism $tr : 1 \to 0$ in $\mathcal{O}$ given by a single incoming labeled leaf (u, backwards) gives a map $tr : \Phi(1) \to \Phi(0)$. When Φ is h–split, $\Phi(0)$ is quasi-isomorphic to $\mathbb{Z}$ (concentrated in degree 0). The map induced by the trace in homology

$$tr : H_*(\Phi(1)) \to H_*(\Phi(0)) = \mathbb{Z},$$

along with the associative multiplication coming from the $\mathcal{A}_\infty$ structure equips $H_*(\Phi(1))$ with the structure of a Frobenius algebra. When Φ is *split*, $\Phi(0) = \mathbb{Z}$, so one gets a trace defined on $\Phi(1)$, which is non-degenerate.

The structure of an $\mathcal{A}_\infty$–Frobenius algebra is generated by this $\mathcal{A}_\infty$–structure together with the trace; that is, all chain level operations from the moduli of surfaces in the open category can be derived from these operations, as is indicated in section 7.3 of [5]. Roughly speaking, having a non-degenerate trace allows one to construct the pairing and the copairing. Together with the $\mathcal{A}_\infty$–structure, one can recover any fat graph. We expand upon this in the following section.

2.12. **Positive boundary or "noncompact" $\mathcal{A}_\infty$–Frobenius algebras:** Define the *positive boundary open cobordism category* $\mathcal{O}^b$ to be the subcategory of $\mathcal{O}$ with the same objects and whose morphisms are given by the subcomplex of fat graphs whose associated topological type is a disjoint union of surfaces, all of which have at least one outgoing boundary.

There are certain morphisms in $\mathcal{O}^b$ whose role should be highlighted. Certainly, $\mathcal{O}^b$ contains all of the category $\mathcal{A}_\infty$, and in particular the corollas $m_k : k \to 1$. It also contains the *coproduct* ν – the morphism from 1 to 2 given by the corolla with one incoming and two outgoing leaves.

Proposition 2.3. *The category $\mathcal{O}^b$ is generated as a symmetric monoidal category by the operad $\mathcal{A}_\infty$ (along with the unit $u : 0 \to 1$), and the coproduct ν.*

Proof. First, define the *copairing* $C := \nu \circ u : 0 \to 2$; this is an exceptional graph with no vertices. Composing a disjoint union of $n-1$ copies of C with m_{k+n-1} gives the corolla[7] $c_{k,n} : k \to n$ for any $k \geq 0$ and $n \geq 1$. Note that we can write $m_k = c_{k,1}$, $u = c_{0,1}$, $\nu = c_{1,2}$, and $C = c_{0,2}$. Then the symmetric monoidal subcategory generated by $\mathcal{A}_\infty$, u, and ν is the same as the one generated by all of the $c_{k,n}$.

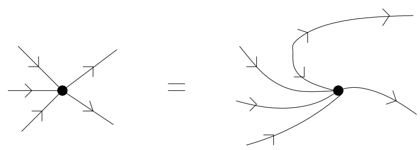

Figure 6. the corolla $c_{3,2}$ as a composition $m_4 \circ (C \sqcup id)$

Now let $\Gamma : m \to n$ be an arbitrary graph in $\mathcal{O}^b$; we may assume that Γ is connected and non-empty, and so $n \geq 1$. Pick a maximal tree T of edges of Γ and choose an outgoing leaf of Γ attached at a vertex v (which is included in T by maximality). There is a unique way to orient the edges of T to make it rooted at v. Extend that orientation (arbitrarily) to an orientation of Γ, though keeping the "in" and "out" orientations of the leaves. Since T includes all of the vertices of Γ, there is always at least one outgoing edge (or leaf) at each vertex. Thus the star of each vertex is $c_{k,n}$ for some value of k and n. Consequently Γ is obtained as an iterated composition of (disjoint unions of) the $c_{k,n}$, and so is in the symmetric monoidal subcategory generated by them. $\square$

The relations between these generators can be summarized (in a pithy if not particularly helpful way) by saying that two compositions of generators are equal if the fat graphs that they define are the same. For instance, the

[7]We should be careful to indicate the labeling of the leaves in $c_{k,n}$, but since we will consider the *symmetric* monoidal category generated by these, any choice will suffice.

Frobenius relation

$$(\text{coproduct} \sqcup id) \circ (id \sqcup \text{product}) = (id \sqcup \text{coproduct}) \circ (\text{product} \sqcup id)$$

expresses the fact that the fat graphs in Figure 7 are isomorphic.

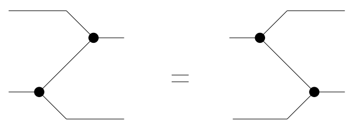

Figure 7. Frobenius relation

Noting that $\mathcal{O}^b$ contains a copy of $\mathcal{A}_\infty^{op}$, extending the coproduct (but not a counit!), Proposition 2.3 gives us:

Corollary 2.4. *A split symmetric monoidal functor* $\Phi : \mathcal{O}^b \to \mathrm{Ch}$ *makes* $A := \Phi(1)$ *into a unital* $\mathcal{A}_\infty$*–algebra and non-counital* $\mathcal{A}_\infty$*–coalgebra.*

2.13. **Decomposing** $(\bar{p}, m)$**–Graphs using** $\mathcal{O}$**:** We give here a construction which will play an essential role in our definition and use of the Hochschild complex.

For $I \subset \{1, \dots, n\}$, let $U_I \subset \mathcal{O}(n, m)$ denote the subcomplex of morphisms which have a component which is a singly-labeled leaf with label out_i for some $i \in I$. In other words, these are the graphs which are a disjoint union of a graph with the leaf $0 \to 1$ which defines the unit for the $\mathcal{A}_\infty$ multiplication.

Let L_n be the graded $\mathbb{Z}$–module generated by the $(\bar{1}, n)$–graph l_n of Figure 2 in degree $n-1$. Let $I = I(n_1, \dots, n_p)$ be the complement of $\{1, n_1+1, \dots, n_1+\dots+n_{p-1}+1\}$ in $\{1, \dots, n_1+\dots+n_p\}$. Then gluing the outgoing leaf out_j in $\mathcal{O}$ to the ith leaf of l_{n_k} when $j = n_1 + \dots + n_{k-1} + i$, defines a map

$$\bigoplus_{n_k \geq 1} \mathcal{O}(m, n_1 + \dots + n_p)/U_I \otimes L_{n_1} \otimes \ldots \otimes L_{n_p} \longrightarrow (\bar{p}, m)-\text{Graphs}.$$

This map is an isomorphism of graded vector spaces. The inverse is obtained by cutting the edges attached to the white vertices of a $(\bar{p}, m)$–graph, and adding labels cyclically starting by 1 at the start half-edge of the first white vertex.

The submodule

$$\bigoplus_{n_i \geq 1} \mathcal{A}_\infty(m, n_1 + \dots + n_p)/U_I \otimes L_{n_1} \otimes \ldots \otimes L_{n_p} \hookrightarrow (\bar{p}, m)-\text{Graphs}$$

is actually a sub-chain complex. (We will later think of the white vertices as modeling outgoing closed boundaries in an open-closed cobordism category. This sub-chain complex will then correspond to cobordisms which are unions of annuli, each with exactly one outgoing closed boundary—see Section 5.7.) In particular, the boundary map in $(\bar{1}, n)$–Graphs restricts to a map

$$d_L : L_n \to \bigoplus_{1 \leq k < n} \mathcal{A}_\infty(n, k) \otimes L_k,$$

Explicitly on the generator l_n of L_n,

$$d_L(l_n) = \sum_{\substack{2\leq r\leq n\\ 1\leq j\leq n}} \epsilon_r^j \langle m_r^j\rangle \otimes l_{n+1-r}$$

where m_r^j denotes multiplication m_r in $\mathcal{A}_\infty$ in position j, i.e. multiplying the entries $j,\dots,j+r-1 \subset_{mod\ n} \{1,\dots,n\}$, and ϵ_r^j is a sign which can be made explicit once a choice of orientation is made for the graph representing each m_k (see Section 6). Writing down actual signs in this differential depends on a choice of orientation for m_k. We will use the map d_L to define the Hochschild complex in Section 4.

3. Bar constructions

Given a dg-category $\mathcal{C}$ and dg-functors $\Phi : \mathcal{C} \to \mathrm{Ch}$ (which we can think of as a $\mathcal{C}$*–module*) and $\Psi : \mathcal{C}^{op} \to \mathrm{Ch}$ (a $\mathcal{C}^{op}$*–module*), define the pth simplicial level of the double bar construction

$$B_p(\Phi,\mathcal{C},\Psi) = \bigoplus_{\substack{m_0,\dots,m_p\\ \in \mathrm{Obj}(\mathcal{C})}} \Phi(m_0)\otimes\mathcal{C}(m_0,m_1)\otimes\ldots\otimes\mathcal{C}(m_{p-1},m_p)\otimes\Psi(m_p).$$

If $\mathcal{C}$ is symmetric monoidal with objects the natural numbers under addition, let $\Sigma \cong \coprod \Sigma_n$ denote the subcategory of $\mathcal{C}$ with the same objects and with morphisms the symmetries in $\mathcal{C}$. Then we can define similarly

$$B_p^\Sigma(\Phi,\mathcal{C},\Psi) = \bigoplus_{\substack{m_0,\dots,m_p\\ \in \mathrm{Obj}(\mathcal{C})}} \Phi(m_0)\otimes_\Sigma\mathcal{C}(m_0,m_1)\otimes_\Sigma\ldots\otimes_\Sigma\mathcal{C}(m_{p-1},m_p)\otimes_\Sigma\Psi(m_p)$$

where $X\otimes_\Sigma Y$ denotes the quotient of $X\otimes Y$ by $x.f\otimes y \sim x\otimes f.y$ for any $f\in\Sigma$ with f acting by pre- or post-composition on the middle factors and via $\Phi(f)$ and $\Psi(f)$ on the first and last factors.

Denoting elements of $B_p(\Phi,\mathcal{C},\Psi)$ by $a\otimes b_1\otimes\cdots\otimes b_p\otimes c$, let $d_i : B_p \to B_{p-1}$, the ith face map, be defined by

$d_0(a\otimes b_1\otimes\cdots\otimes b_p\otimes c) = \Phi(b_1)(a)\otimes b_2\cdots\otimes b_p\otimes c$

$d_i(a\otimes b_1\otimes\cdots\otimes b_p\otimes c) = a\otimes b_0\otimes\cdots\otimes b_{i+1}\circ b_i\otimes\cdots\otimes b_p\otimes c$ for $0<i<p$

$d_p(a\otimes b_1\otimes\cdots\otimes b_p\otimes c) = a\otimes b_0\cdots\otimes b_{p-1}\otimes\Psi(b_p)(c)$.

This makes $B(\Phi,\mathcal{C},\Psi) = \oplus_{p\geq 0}B_p(\Phi,\mathcal{C},\Psi)$, the *double bar construction*, into a semi-simplicial chain complex, and a chain complex with differential $D_p = (-1)^p\delta + d$ where δ denotes the differential of $B_p(\Phi,\mathcal{C},\Psi)$ as a tensor product of chain complexes, and $d = \sum_{i=0}^p(-1)^i d_i$ denotes the simplicial differential.

As all the face maps are well-defined over Σ, we have that $B^\Sigma(\Phi,\mathcal{C},\Psi) = \oplus_{p\geq 0}B_p^\Sigma(\Phi,\mathcal{C},\Psi)$ is also a semi-simplicial chain complex. (In fact, $B(\Phi,\mathcal{C},\Psi)$ is a simplicial chain complex, in that it admits well-defined degeneracies, but this is not true for $B^\Sigma(\Phi,\mathcal{C},\Psi)$.)

Taking $\Psi = \mathcal{C}(-,m)$ to be the $\mathcal{C}^{op}$–module represented by an object m of $\mathcal{C}$, we note moreover that the bar construction $B(\Phi,\mathcal{C},\mathcal{C}(-,m))$ is natural in m, i.e. we get a functor $B(\Phi,\mathcal{C},\mathcal{C}) : \mathcal{C} \to \mathrm{Ch}$ with value $B(\Phi,\mathcal{C},\mathcal{C}(-,m))$ at $m \in \mathrm{Obj}(\mathcal{C})$.

Proposition 3.1. *For any functor $\Phi : \mathcal{C} \to \mathrm{Ch}$ there are quasi-isomorphisms of functors*

$$\alpha\colon B(\Phi,\mathcal{C},\mathcal{C}) \xrightarrow{\simeq} \Phi \quad \textit{and} \quad \alpha^{\Sigma}\colon\ B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C}) \xrightarrow{\simeq} \Phi$$

In particular, $B(\Phi,\mathcal{C},\mathcal{C}(-,m)) \simeq B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C}(-,m))$ for each m.

The result is well-known for the usual bar construction B. We recall the proof here and show that it also applies to B^{Σ}.

Proof. Let $\alpha = \oplus_p\alpha_p : B(\Phi,\mathcal{C},\mathcal{C}(-,m)) = \oplus_p B_p(\Phi,\mathcal{C},\mathcal{C}(-,m)) \longrightarrow \Phi(m)$ be defined by $\alpha_0(a \otimes c) = \Phi(c)(a)$ and $\alpha_p = 0$ for $p > 0$. This is natural in m. Let $\beta : \Phi(m) \to B(\Phi,\mathcal{C},\mathcal{C}(-,m))$ be defined by $\beta(a) = a \otimes 1_m \in \Phi(m)\otimes\mathcal{C}(m,m)$, where 1_m here denotes the identity on m. We have $\alpha\circ\beta = id$ and $\beta\circ\alpha \simeq id$; an explicit chain homotopy is given by $h_i = s_p \circ\cdots\circ s_{i+1} \circ \eta \circ d_{i+1} \circ\cdots\circ d_p$, where s_i is the ith degeneracy, introducing an identity at the ith position, and η is the "extra degeneracy" which introduces an identity at the right-most spot. Explicitly, h_i takes $a \otimes b_1 \otimes\cdots\otimes b_p \otimes c$ to $a\otimes b_1\otimes\cdots\otimes b_i\otimes(c\circ b_p\circ\cdots\circ b_{i+1})\otimes 1_m\otimes\ldots\otimes 1_m$. Hence α gives a natural transformation by quasi-isomorphisms between the functors $B(\Phi,\mathcal{C},\mathcal{C})$ and Φ.

For B^{Σ}, we now just note that the maps α,β and h_i are well-defined over Σ. (For h_i, the degeneracies s_j are not well-defined but the above composition with η is.) $\square$

Remark 3.2. More generally, one can show that $B(M,\mathcal{C},N) \simeq B^{\Sigma}(M,\mathcal{C},N)$ if M or N is quasi-free (i.e., free as a $\mathcal{C}$–module, if one ignores the differential). Without any assumption on M and N, one can ask if $B^{\Sigma}_p(M,\mathcal{C},N)$ is quasi-isomorphic to $B_p(M,\mathcal{C},N)$ in characteristic 0.

Proposition 3.3. *If $\mathcal{C}$ is (symmetric) monoidal and $\Phi : \mathcal{C} \to \mathrm{Ch}$ is monoidal, then $B(\Phi,\mathcal{C},\mathcal{C})$ and $B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C})$ are monoidal. If Φ is symmetric monoidal, then so is $B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C})$. Moreover, if Φ is h-split, $B(\Phi,\mathcal{C},\mathcal{C})$ and $B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C})$ are both h-split.*

Proof. The monoidal structure of $B^{(\Sigma)}(\Phi,\mathcal{C},\mathcal{C})$ comes directly from that of Φ and $\mathcal{C}$, taking $(a \otimes f_1 \otimes\ldots\otimes f_{p+1}) \otimes (a' \otimes f'_1 \otimes\ldots\otimes f'_{p+1})$ to $(a \boxplus a') \otimes (f_1 \boxtimes f'_1) \otimes\ldots\otimes (f_{p+1} \boxtimes f'_{p+1})$, where $\boxplus$ denotes the monoidal structure of Φ and $\boxtimes$ that of $\mathcal{C}$.

We want to check that $B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C})$ is in fact symmetric monoidal, i.e. that the diagram

$$\begin{tikzcd}
B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C}(-,n)) \otimes B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C}(-,m)) \arrow[r] \arrow[d, "\tau_{\otimes}"'] & B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C}(-,n+m)) \arrow[d, "\tau_{\mathcal{C}}"] \\
B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C}(-,m)) \otimes B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C}(-,n)) \arrow[r] & B^{\Sigma}(\Phi,\mathcal{C},\mathcal{C}(-,m+n))
\end{tikzcd}$$

commutes, where $\tau_\otimes$ denotes the symmetry in the category of chain complexes and $\tau_{\mathcal{C}}$ the symmetry of $\mathcal{C}$. This means that we need
$(a' \boxplus a) \otimes_\Sigma (f_1' \boxtimes f_1) \otimes_\Sigma \ldots \otimes_\Sigma (f_{p+1}' \boxtimes f_{p+1})$ equal to
$(a \boxplus a') \otimes_\Sigma (f_1 \boxtimes f_1') \otimes_\Sigma \ldots \otimes_\Sigma (f_p \boxtimes f_p') \otimes_\Sigma ((f_{p+1} \boxtimes f_{p+1}') \circ \tau_{\mathcal{C}})$.
This holds because $(f_i \boxtimes f_i') \circ \tau_{\mathcal{C}} = \tau_{\mathcal{C}} \circ (f_i' \boxtimes f_i)$ in $\mathcal{C}$ and $\Phi(\tau_{\mathcal{C}})(a \boxplus a') = a' \boxplus a$ as Φ is symmetric monoidal.

The fact that Φ is h-split implies $B(\Phi, \mathcal{C}, \mathcal{C})$ and $B^\Sigma(\Phi, \mathcal{C}, \mathcal{C})$ are h-split; this follows from the commutativity of the following diagram:

$$\begin{tikzcd}
B(\Phi,\mathcal{C},\mathcal{C})(n) \otimes B(\Phi,\mathcal{C},\mathcal{C})(m) \arrow[r] \arrow[d, "\simeq"', "\alpha"] & B(\Phi,\mathcal{C},\mathcal{C})(n+m) \arrow[d, "\simeq"', "\alpha"] \\
\Phi(n) \otimes \Phi(m) \arrow[r, "\simeq"] & \Phi(n+m).
\end{tikzcd}$$

□

Note that in the above proposition, strengthening the assumption on Φ to be split still only yields $B^{(\Sigma)}(\Phi, \mathcal{C}, \mathcal{C})$ h-split.

4. Hochschild complex operator

Let $\mathcal{E}$ be a monoidal dg-category which admits a monoidal functor $i : \mathcal{A}_\infty \to \mathcal{E}$. For simplicity, we assume that $\mathrm{Obj}(\mathcal{E}) = \mathbb{N}$ and i is the identity on objects. Though this is not necessary, all of our examples will be of this sort. We define in this section a Hochschild complex operator C on dg-functors $\Phi : \mathcal{E} \to \mathrm{Ch}$ in such a way that the value of $C(\Phi)$ at 0 is a generalization of the usual Hochschild complex. In 4.1 we study its basic properties and in 4.2 we prove our main theorem, Theorem 4.9, which gives a way of constructing actions on Hochschild complexes.

Recall from 2.13 the complex L_n and the the differential d_L. Let $c_\Phi : \Phi \otimes \mathcal{E} \to \Phi$ denote the structure map of Φ. Lastly, write $i_m = i + id_m$ for the shift of the map $i : \mathcal{A}_\infty(n,k) \to \mathcal{E}(n,k)$ by the identity on the object m. We assemble these together in the following definition:

Definition 4.1 (Hochschild complex)**.** *Given $i : \mathcal{A}_\infty \to \mathcal{E}$ monoidal and $\Phi : \mathcal{E} \to \mathrm{Ch}$ a dg-functor, define $C(\Phi) : \mathcal{E} \to \mathrm{Ch}$ by*

$$C(\Phi)(m) \;=\; \bigoplus_{n \geq 1} \Phi(n+m) \otimes L_n$$

with differential $d_\Phi + d_L^\theta$, where d_Φ is the differential of Φ and $d_L^\theta = c_\Phi i_m d_L$:

$$\begin{array}{rcl}
\Phi(n+m) \otimes L_n & \xrightarrow{d_L} & \bigoplus_{1 \leq k < n} \Phi(n+m) \otimes \mathcal{A}_\infty(n,k) \otimes L_k \\
& \xrightarrow{i_m} & \bigoplus_{1 \leq k < n} \Phi(n+m) \otimes \mathcal{E}(n+m,k+m) \otimes L_k \\
& \xrightarrow{c_\phi} & \bigoplus_{1 \leq k < n} \Phi(k+m) \otimes L_k.
\end{array}$$

Define the reduced complex by

$$\overline{C}(\Phi)(m) \;=\; \bigoplus_{n\geq 1}\Phi(n+m)/_{U_n}\otimes L_n$$

with the same differential, where $U_n=\sum_{i=2}^n \mathrm{Im}(\Phi(i(u_i)+id_m))\subset\Phi(n+m)$ *with* $u_i=1\otimes\ldots\otimes u\otimes\ldots\otimes 1$ *in* $\mathcal{A}_\infty(n-1,n)$ *the morphism that inserts a unit at the* i*th position.*

In Lemma 4.2 below, we check that $d_\Phi+d_L^\theta$ is indeed a differential, and in Lemma 4.3 that it is well-defined on the reduced complex. The functoriality of $C(\Phi)$ is easy to check and is generalized in Proposition 4.4 to a left and right functoriality.

The construction is natural in Φ and $\mathcal{E}$ in the following sense: Given a factorization of i as $\mathcal{A}_\infty\xrightarrow{i'}\mathcal{E}'\xrightarrow{j}\mathcal{E}$ and a functor $\Phi:\mathcal{E}\to\mathrm{Ch}$, we have $C(j^*\Phi)\cong C(\Phi)$, and given two functors $\Phi,\Psi:\mathcal{E}\to\mathrm{Ch}$ and a natural transformation $\eta:\Phi\to\Psi$, we get a natural transformation $C(\Phi)\to C(\Psi)$ (and similarly for $\overline{C}$).

The operator C generalizes the usual Hochschild complex of A_∞–algebras in the sense that for $\Phi:\mathcal{A}_\infty\to\mathrm{Ch}$ split symmetric monoidal, $C_*(\Phi)(0)$ and $\overline{C}(\Phi)(0)$ are the usual Hochschild and reduced Hochschild complexes of the A_∞–algebra $\Phi(1)$. In the case of a strict graded algebra, taking as generator of L_n the graph of Figure 2 with orientation $v\wedge h_1\wedge\ldots\wedge h_n$ and using the convention for the product given in Figure 14, our differential is explicitely given by the following formula: for a n–chain $a_0\otimes\ldots\otimes a_n$ of the Hochschild complex of an algebra A, we have

$$\begin{aligned} d(a_0\otimes\ldots\otimes a_n) = \;& \textstyle\sum_{i=0}^n(-1)^{a_0+\cdots+a_{i-1}}a_0\otimes\ldots\otimes da_i\otimes\ldots\otimes a_n\\ &+\;(-1)^{a_0+\cdots+a_n}\textstyle\sum_{i=0}^{n-1}(-1)^{i+1}a_0\otimes\ldots\otimes a_ia_{i+1}\otimes\ldots\otimes a_n\\ &+\;(-1)^{n+1+(a_n+1)(a_0+\cdots+a_{n-1})+a_n}a_na_0\otimes a_1\otimes\ldots\otimes a_{n-1},\end{aligned}$$

where a_i in a superscript denotes the degree of a_i.

Note though that the Hochschild complex is defined for much more general functors, also non-monoidal functors. In particular, we will apply the Hochschild constructions to the (in general non-monoidal) representable functors $\Phi(m)=\mathcal{E}(-,m)$ and iteratively to $C(\Phi)$, given a functor Φ.

Lemma 4.2. $d_\Phi+d_L^\theta$ *is a differential on* $C(\Phi)$*.*

Proof. Recall from 2.13 that $\oplus_k\mathcal{A}_\infty(n,k)/U\otimes L_k$ (with U as in 2.13) can be identified with a subcomplex of $(\bar{1},n)$–Graphs. Rewrite the unreduced Hochschild complex of Φ as

$$C(\Phi)(m)\;\cong\;\bigoplus_{n\geq 1}\Phi(n+m)\otimes_{\mathcal{A}_\infty/U}(\oplus_{k\geq 1}\mathcal{A}_\infty(n,k)/U\otimes L_k)$$

with the $\mathcal{A}_\infty/U$–module structure of $\Phi(n+m)$ induced by identifying $\mathcal{A}_\infty/U$ as a subcomplex of $\mathcal{A}_\infty$, mapping it to $\mathcal{E}$ and using the structure map of

Φ. Defining the differential on the right hand side to be the quotient of the tensor product differential, the above is an isomorphism of chain complexes. As $d^2 = 0$ on the right hand side, we get $(d_\Phi + d_L^\theta)^2 = 0$ on $C(\Phi)(m)$. $\square$

For the reduced complex, Φ/U is not in general an $\mathcal{A}_\infty/U$–module so the situation is a little more subtle in that case. The following lemma justifies that the differential is indeed well-defined in that case.

Lemma 4.3. *$d_\Phi + d_L^\theta$ is well-defined on the reduced complex.*

Proof. Let $U_n \leq \Phi(n)$ be as in Definition 4.1. We first note that U_n is mapped to itself by d_Φ because the structure map c_Φ of Φ is by chain maps and $d(u_i) = 0$. We need to see that the same holds for d_L^θ. This follows from the commutativity of the following diagram (written in the case $m = 0$ for readability)

$$\begin{tikzcd}
\Phi(n-1) \otimes \langle u_i \rangle \otimes L_n \arrow[r, "c_\Phi"] \arrow[d, "d_L"] & \Phi(n) \otimes L_n \arrow[d, "d_L"] \\
\bigoplus_{\substack{2\leq r\leq n \\ 1\leq j\leq n}} \Phi(n-1) \otimes \langle u_i \rangle \otimes \langle m_r^j \rangle \otimes L_{n+1-r} \arrow[r, "c_\Phi"] \arrow[d, "c_{\mathcal{A}_\infty}"] & \bigoplus_{\substack{2\leq r\leq n \\ 1\leq j\leq n}} \Phi(n) \otimes \langle m_r^j \rangle \otimes L_{n+1-r} \arrow[d, "c_\Phi"] \\
\bigoplus_{r,j} \Phi(n-1) \otimes \langle u_i \oplus m_r^j \rangle \otimes L_{n+1-r} \arrow[r, "c_\Phi"] & \bigoplus_{k\geq 1} \Phi(k) \otimes L_k
\end{tikzcd}$$

where $m_r^j = 1 \oplus \cdots \oplus m_r \oplus \cdots \oplus 1$ denotes the multiplication m_r of the entries $j, \ldots, j+r-1 \pmod n$. The target of the map $c_{\mathcal{A}_\infty}$ is justified as follows. There are two cases when composing u_i and m_r^j: either $i \notin \{j, \ldots, j+r-1\}$ so that the composition $m_r^j \circ u_i$ is of the form $u_i \oplus m_r^j$. Otherwise, the composition $m_r^j \circ u_i$ is the identity map when $r = 2$ and 0 when $r > 2$. In the case $r = 2$, the term $m_2^{i-1} \circ u_i$ cancels with $m_2^i \circ u_i$. (The sign comes from the differential in L). $\square$

Let $\mathcal{E}, \mathcal{F}$ be dg-categories and suppose that $\Phi : \mathcal{E} \to \mathrm{Ch}$ is in fact a bifunctor $\Phi : \mathcal{F}^{op} \times \mathcal{E} \to Comp$. In this case, we also call Φ an $(\mathcal{F}^{op}, \mathcal{E})$–*bimodule.*

Proposition 4.4. *Suppose Φ is an $(\mathcal{F}^{op}, \mathcal{E})$–bimodule. Then its Hochschild complexes $C(\Phi)$ and $\overline{C}(\Phi)$ build using the $\mathcal{E}$–structure of Φ are again $(\mathcal{F}^{op}, \mathcal{E})$–bimodules.*

Proof. Given $f : m_1 \to m_2$ in $\mathcal{E}$ and $g : a_2 \to a_1$ in $\mathcal{F}^{op}$, $C(\Phi)(g, f)$ on the summand $\Phi(a_2, n + m_1) \otimes L_n$ is the map $(g, id_n + f)$. This is well-defined as the twisted part of the differential d_L^θ commutes with such maps. $\square$

The example we are interested in is the $(\mathcal{E}^{op},\mathcal{E})$–bimodule $\mathcal{E}$. By the proposition, its Hochschild and iterated Hochschild complexes $C(\mathcal{E}), C^n(\mathcal{E})$, and reduced versions, are again $(\mathcal{E}^{op},\mathcal{E})$–bimodules. This will allow us in particular to rewrite $C^n(B(\Phi,\mathcal{E},\mathcal{E}))$ as $B(\Phi,\mathcal{E},C^n\mathcal{E})$ and $\overline{C}^n(B(\Phi,\mathcal{E},\mathcal{E}))$ as $B(\Phi,\mathcal{E},\overline{C}^n\mathcal{E})$.

4.1. **Properties of the Hochschild operator.** We prove in this section that the Hochschild complex operator is homotopy invariant and we describe its behavior under iteration.

Recall that by a quasi-isomorphism of functors $\Phi \xrightarrow{\simeq} \Phi' : \mathcal{E} \to \mathrm{Ch}$, we mean a natural transformation by quasi-isomorphisms $\Phi(m) \xrightarrow{\simeq} \Phi'(m)$.

Proposition 4.5. *Let $\Phi,\Phi' : \mathcal{E} \to \mathrm{Ch}$. A quasi-isomorphism of functors $\Phi \xrightarrow{\simeq} \Phi'$ induces quasi-isomorphisms of functors $C_*(\Phi) \xrightarrow{\simeq} C_*(\Phi')$ and $\overline{C}_*(\Phi) \xrightarrow{\simeq} \overline{C}_*(\Phi')$.*

For the reduced part of the proposition, we need the following lemma.

Lemma 4.6. *Suppose $\Phi \xrightarrow{\simeq} \Phi' : \mathcal{E} \to \mathrm{Ch}$ are quasi-isomorphic functors. For any $J \subset \{1,\ldots,n\}$, let $U_J = \sum_{j\in J} \mathrm{Im}\big(\Phi(i(u_j))\big) \subset \Phi(n)$, and similarly for Φ'. Then*

$$\Phi(n)/U_J \xrightarrow{\simeq} \Phi'(n)/U_J.$$

If $\Phi \cong \Phi'$, these maps are also isomorphisms.

Proof. We prove the lemma by induction on the cardinality of J, for any n, starting with the case $J = \varnothing$ which is trivial.

Fix $J = \{j_1 \le \cdots \le j_s\} \subset \{1,\ldots,n\}$ and denote by U_i, U_i' the image of $i(u_i)$ in $\Phi(n)$ and $\Phi'(n)$ respectively. We want to show that $\Phi(n)/(U_{j_1} + \cdots + U_{j_s}) \xrightarrow{\simeq} \Phi'(n)/(U'_{j_1} + \cdots + U'_{j_s})$.

There is a short exact sequence

$$\begin{array}{ccc} \Phi(n-1)/(U_{j_1}+\cdots+U_{j_{s-1}}) & \xrightarrow{i(u_{j_s})} & \Phi(n)/(U_{j_1}+\cdots+U_{j_{s-1}}) \\ & & \downarrow \\ & & \Phi(n)/(U_{j_1}+\cdots+U_{j_s}). \end{array}$$

Indeed u_{j_s} is injective on $\Phi(n-1)/(U_{j_1}+\cdots+U_{j_{s-1}})$ with left inverse $i(m_2^{j_s})$ (where $m_2^{j_s}$ multiplies j_s and j_s+1 modulo n). The result then follows by induction by considering the map of short exact sequences induced by $\Phi \to \Phi'$. □

Proof of the Proposition. We filter the complexes $C_*(\Phi)(m) = \oplus\Phi(k+m)\otimes L_k$ and $\overline{C}_*(\Phi)(m) = \oplus\Phi(k+m)/U_k \otimes L_k$ by k and consider the resulting spectral sequence. In both cases the differential is $d_\Phi + d_L^\theta$ where d_L^θ decreases the filtration grading and d_Φ does not. Hence the E^1–terms of the spectral sequences are $E^1_{p,q} = H_p(\Phi(q+1+m))\otimes L_{q+1}$ and $E^1_{p,q} = H_p(\Phi(q+1+$

$m)/U_{q+1}) \otimes L_{q+1}$ in the reduced case. A quasi-isomorphism of functors induces a map of spectral sequences which is an isomorphism on the E^1–term by the assumption in the unreduced case and by Lemma 4.6 in the reduced case. □

Applying Proposition 4.5 to the map $\alpha : B(\Phi, \mathcal{E}, \mathcal{E}) \xrightarrow{\simeq} \Phi$ of Proposition 3.1, we get a quasi-isomorphism

$$C(\alpha) : C(B(\Phi, \mathcal{E}, \mathcal{E})) \xrightarrow{\simeq} C(\Phi).$$

The proof of Proposition 3.1 gives a pointwise homotopy inverse β to α which is not a natural transformation, so we cannot apply Proposition 4.5 to it. (In fact $C(\beta)$ does not define a chain map.) Instead, we construct now an explicit pointwise homotopy inverse $\tilde{\beta}$ to $C^n(\alpha)$, for any n, as this will be useful later.

Proposition 4.7. *For any n and m, there is a quasi-isomorphism of chain complexes*

$$\tilde{\beta} : C^n(\Phi)(m) \xrightarrow{\simeq} C^n(B(\Phi, \mathcal{E}, \mathcal{E}))(m)$$

natural both with respect to natural transformations $\Phi \to \Phi'$ and with respect to functors $j : \mathcal{E} \to \mathcal{E}'$ with $i' = j \circ i : \mathcal{A}_\infty \to \mathcal{E}'$.

Proof. We define below

$$\tilde{\beta} : C^n(\Phi)(m) \xrightarrow{\simeq} B(\Phi, \mathcal{A}_\infty, C^n(\mathcal{A}_\infty)(m)) \cong C^n(B(\Phi, \mathcal{A}_\infty, \mathcal{A}_\infty))(m)$$

for any $\Phi : \mathcal{A}_\infty \to \mathrm{Ch}$. The desired map $\tilde{\beta}$ is then obtained more generally for any $\mathcal{E}$ and $\Phi : \mathcal{E} \to \mathrm{Ch}$ by post-composition with the quasi-isomorphism $C^n(B(i^*\Phi, \mathcal{A}_\infty, \mathcal{A}_\infty))(m) \to C^n(B(\Phi, \mathcal{E}, \mathcal{E}))(m)$ induced by $i : \mathcal{A}_\infty \to \mathcal{E}$. The naturality of $\tilde{\beta}$ in $\mathcal{E}$ follows from the naturality of that second map.

Recall from 2.13 the map

$$d_L : L_k \to \bigoplus_{1 \le j < k} \mathcal{A}_\infty(k, j) \otimes L_j.$$

We consider here more generally the map

$$d_L : L_{k_1} \otimes \ldots \otimes L_{k_n} \to \bigoplus_{1 \le k < n} \mathcal{A}_\infty(k_1 + \cdots + k_n, k'_1 + \cdots + k'_n) \otimes L_{k'_1} \otimes \ldots \otimes L_{k'_n}$$

induced by the differential of the $(\bar{n}, k)$–graph which is the union $l_{k_1} \sqcup \cdots \sqcup l_{k_n}$, where $k = k_1 + \cdots + k_n$. We let $\tilde{\beta} := \sum_{p \ge 0} (d_L)^p$, where we interpret $(d_L)^p$ as the composition

$$\Phi(k+m) \otimes L_{\underline{k}} \xrightarrow{(d_L)^p} \bigoplus_{j_i} \Phi(k+m) \otimes \mathcal{A}_\infty(k, j_1) \otimes \ldots \otimes \mathcal{A}_\infty(j_{p-1}, j_p) \otimes L_{\underline{j}_p}$$

$$\xrightarrow{+id_m} \bigoplus_{j_i} \Phi(k+m) \otimes \mathcal{A}_\infty(k+m, j_1+m) \otimes \ldots \otimes \mathcal{A}_\infty(j_{p-1}+m, j_p+m) \otimes L_{\underline{j}_p}$$

with image in the pth simplicial level of $B(\Phi, \mathcal{A}_\infty, C^n(\mathcal{A}_\infty)(m))$, where $L_{\underline{k}} = L_{k_1} \otimes \ldots \otimes L_{k_n}$ and $L_{\underline{j}_p} = L_{j_1^p} \otimes \ldots \otimes L_{j_n^p}$ is identified with $\langle id_{j_p+m} \rangle \otimes \bar{L}_{\underline{j}_p}$

in $\mathcal{A}_\infty(j_p+m, j_p+m)\otimes L_{\underline{j}_p}$ in $C^n(\mathcal{A}_\infty)(m)$. Note that the sum is always finite as $(d_L)^p$ applied to L_k is 0 for all $p\geq k$.

We will show that the relation $d\tilde{\beta}=\tilde{\beta}d$ holds on each component as maps

$$\bigoplus_{(k)=(k_1,\dots,k_n)}\Phi(k+m)\otimes L_{k_1}\otimes\ldots\otimes L_{k_n}\quad\longrightarrow\quad\bigoplus_p B_p\big(\Phi,\mathcal{A}_\infty,C^n(\mathcal{A}_\infty)(m)\big)$$

i.e. that for each fixed (k), the images of $d\tilde{\beta}$ and $\tilde{\beta}d$ agree on the component of simplicial degree p. We first consider $\tilde{\beta}d$.

As $d=d_\Phi+c_\Phi d_L$, we have on the (k)th component

$${}_{(k)}(\tilde{\beta}d)=\sum_{i=0}^{K-1}(d_L)^i d_\Phi+\sum_{i=0}^{K-2}(d_L)^i c_\Phi d_L$$

with $K=\max(k_1,\dots,k_n)$, which can be rewritten as

$${}_{(k)}(\tilde{\beta}d)=d_\Phi\sum_{i=0}^{K-1}(-1)^i(d_L)^i+d_0\sum_{i=0}^{K-2}(d_L)^{i+1}$$

as $d_Ld_\Phi=-d_\Phi d_L$ and $\bar{d}_L^i c_\Phi\bar{d}_L=d_0\bar{d}_L^{i+1}$ with d_0 the 0–th face map in $B_i(\Phi,\mathcal{A}_\infty,C^n(\mathcal{A}_\infty)(m))$. Hence the component of ${}_{(k)}(\tilde{\beta}d)$ of simplicial degree p is

$${}_{(k)}(\tilde{\beta}d)_p=(-1)^p d_\Phi(d_L)^p+d_0(d_L)^{p+1}.$$

On the other hand, we have ${}_{(k)}(d\tilde{\beta})=d({}_{(k)}\tilde{\beta})$ where the differential on the pth component of ${}_{(k)}\tilde{\beta}$ is $(-1)^p(d_\Phi+(d_\mathcal{A})_1+\cdots+(d_\mathcal{A})_p+\widetilde{d_L})+\sum_{i=0}^p(-1)^i d_i$, where $(d_\mathcal{A})_i$ denotes the differential of the ith factor $\mathcal{A}_\infty(-,-)$ and $\widetilde{d_L}$ the map $d_{p+1}d_L$ which applies the differential to the factors L without increasing the simplicial degree. As the face maps d_i reduce the simplicial degree, we have

$${}_{(k)}(d\tilde{\beta})_p=(-1)^p\Big(d_\Phi+(d_\mathcal{A})_1+\cdots+(d_\mathcal{A})_p+d_{p+1}d_L\Big)(d_L)^p+\Big(\sum_{i=0}^{p+1}(-1)^i d_i\Big)(d_L)^{p+1}.$$

This is a sum of two compositions whose respective first terms are exactly ${}_{(k)}(\tilde{\beta}d)_p$, and whose last terms cancel. Hence

$${}_{(k)}(d\tilde{\beta})_p-{}_{(k)}(\tilde{\beta}d)_p=\sum_{i=1}^{p}\Big((-1)^p(d_\mathcal{A})_i(d_L)^p+(-1)^i d_i(d_L)^{p+1}\Big).$$

The ith term in the sum can be rewritten as

$$(-1)^i(d_L)^{p-i}\Big((d_\mathcal{A})_i+d_id_L\Big)(d_L)^i$$

which is 0 as the middle part $((d_\mathcal{A})_i+d_id_L)d_L$ is the square of a differential in the graph complex, which gives the desired equality.

As $C^n(\alpha)(m) \circ \tilde{\beta}$ is the identity and $C^n(\alpha)(m)$ is a quasi-isomorphism by Propositions 3.1 and 4.5, $\tilde{\beta}$ is also a quasi-isomorphism. The map $\tilde{\beta}$ is natural in Φ as d_L is natural in Φ. $\square$

Next we describe how the Hochschild operator behaves under iteration.

Recall from Section 1 that a monoidal functor $\Phi : \mathcal{E} \to \mathrm{Ch}$ is *h-split* if the maps $\Phi(n) \otimes \Phi(m) \to \Phi(n+m)$ are quasi-isomorphisms, and *split* if the maps are isomorphisms.

For $\Phi : \mathcal{E} \to \mathrm{Ch}$, we can consider the iterated Hochschild functor $C^n(\Phi) = C(C(\dots C(\Phi)\dots))$. When Φ is h-split monoidal, it computes the tensor powers of the Hochschild complex:

Proposition 4.8. *If $\Phi : \mathcal{E} \to \mathrm{Ch}$ is monoidal, then there are natural maps*

$$\lambda : C(\Phi)(0)^{\otimes n} \otimes \Phi(1)^{\otimes m} \longrightarrow C^n(\Phi)(m)$$

and

$$\bar{\lambda} : \overline{C}(\Phi)(0)^{\otimes n} \otimes \Phi(1)^{\otimes m} \longrightarrow \overline{C}^n(\Phi)(m).$$

These maps are quasi-isomorphisms if Φ is h-split, and isomorphisms if Φ is split.

Moreover, there exists an action of Σ_n on $C^n(\Phi)$ such that if $\mathcal{E}, \Phi$ and i are symmetric monoidal, these maps are $\Sigma_n \times \Sigma_m$–equivariant (where Σ_m acts on $C^n(\Phi)(m)$ via the symmetries of $\mathcal{E}$).

Proof. $C_*(\Phi)(0)^{\otimes n} = (\oplus_{k_1}\Phi(k_1) \otimes L_{k_1}) \otimes \dots \otimes (\oplus_{k_n}\Phi(k_n) \otimes L_{k_n})$ and $C^n_*(\Phi)(m) = \oplus_{k_n}(\dots(\oplus_{k_1}\Phi(k_1+\dots+k_n+m)\otimes L_{k_1})\otimes\dots\otimes L_{k_n})$. The maps λ and $\bar{\lambda}$ are then defined by appropriately permuting the factors and then using the monoidal structure of Φ. These maps are isomorphisms/quasi-isomorphisms in the unreduced case if the structure maps of Φ have that property. For the reduced complexes, we need $\Phi(k_1)/U_{k_1} \otimes \dots \otimes \Phi(k_n)/U_{k_n} \otimes \Phi(m) \to \Phi(k_1 + \dots + k_n + m)/U_{k_1}/\dots/U_{k_n}$ to be an isomorphism when Φ is split and a quasi-isomorphism when Φ is h-split. This follows from an iteration of Lemma 4.6: Consider the restriction of the natural transformation $\Phi\otimes\dots\otimes\Phi \to \Phi(\ +\dots+\ )$ to the first variable and apply the lemma with $J_1 = \{2, \dots, k_1\}$. This gives a quasi-isomorphism $\Phi(k_1)/U_{k_1}\otimes\Phi(k_2)\otimes\dots\Phi(m) \to \Phi(k_1+\dots+m)/U_{k_1}$. This quasi-isomorphism is functorial in the variables $k_2, \dots, m$ and we can repeat the process until we obtain the desired result. $\square$

4.2. **Action on Hochschild complexes.** Given a monoidal dg-category $\mathcal{D}$ with objects $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right] \in \mathbb{N} \times \mathbb{N}$, we say that a pair of chain complexes (V, W) is a *$\mathcal{D}$–module* if there is a split monoidal dg-functor $\Psi : \widehat{\mathcal{E}} \to \mathrm{Ch}$ with $\Psi(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]) = V$ and $\Phi(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]) = W$, i.e. if there are associative chain maps

$$\big(V^{\otimes n_1} \otimes W^{\otimes m_1}\big) \otimes \mathcal{D}(\left[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}\right]) \longrightarrow V^{\otimes n_2} \otimes W^{\otimes m_2}.$$

We say that (V, W) is a *homotopy $\mathcal{D}$–module* if the associativity is only satisfied up to homotopy. In particular, taking homology with field coefficients (or general coefficients but restricting to the "operadic part"), we get in both cases an honest action of $H_*(\mathcal{D})$ on $(H_*(V), H_*(W))$.

If $\mathcal{D}$ is symmetric monoidal, we say that the module structure is Σ–*equivariant* if the functor Ψ is symmetric monoidal.

Proposition 4.4 in the case where Φ is the $(\mathcal{E}, \mathcal{E}^{op})$–bimodule $\mathcal{E}$ can be reinterpreted as follows: Given $\mathcal{E}$, we can define its *Hochschild core category* $C\mathcal{E}$ with objects

$$[\begin{smallmatrix} n \\ m \end{smallmatrix}] = (m, n) \in \mathrm{Obj}(\mathcal{E}) \times \mathbb{N} \ (= \mathbb{N} \times \mathbb{N})$$

and morphisms

$$C\mathcal{E}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}]) = \begin{cases} C^{n_2}(\mathcal{E}(m_1, -))(m_2) & n_1 = 0 \\ 0 & n_1 \neq 0 \end{cases}$$

where C^0 means the identity operator, so that $C\mathcal{E}([\begin{smallmatrix} 0 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} 0 \\ m_2 \end{smallmatrix}]) = \mathcal{E}(m_1, m_2)$. The only possible non-trivial compositions in $C\mathcal{E}$ are given by the bimodule structure of $C^n(\mathcal{E}(m, -))$ described in Proposition 4.4. Moreover, $C\mathcal{E}$ is monoidal via the maps

$$C^n(\mathcal{E}(m_1, -))(m_2) \otimes C^{n'}(\mathcal{E}(m_1', -))(m_2') \to C^{n+n'}(\mathcal{E}(m_1 + m_1', -))(m_2 + m_2')$$

as in Proposition 4.8, and $C\mathcal{E}$ is symmetric monoidal when the same is true for $\mathcal{E}$.

We call a monoidal category $\widehat{\mathcal{E}}$ with objects $\mathbb{N} \times \mathbb{N}$ an *extension* of $C\mathcal{E}$ if there is a monoidal inclusion $C\mathcal{E} \hookrightarrow \widehat{\mathcal{E}}$ with $\widehat{\mathcal{E}}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}]) = C\mathcal{E}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$ when $n_1 = 0$. We define the *reduced Hochschild core category* $\overline{C}\mathcal{E}$ and its extensions in the same way, replacing C by $\overline{C}$.

Our main result says that if $\widehat{\mathcal{E}}$ is an extension of $C\mathcal{E}$ (or $\overline{C}\mathcal{E}$), then $\widehat{\mathcal{E}}$ acts on the (reduced) Hochschild complex of split monoidal functors $\Phi : \mathcal{E} \to \mathrm{Ch}$ in the following sense:

Theorem 4.9. *Let $\widehat{\mathcal{E}}$ be an extension of $C\mathcal{E}$. Then for any monoidal functor $\Phi : \mathcal{E} \to \mathrm{Ch}$, there is a map*

$$\Big(C(\Phi)(0)^{\otimes n_1} \otimes \Phi(1)^{\otimes m_1}\Big) \bigotimes \widehat{\mathcal{E}}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}]) \xrightarrow{\ \gamma\ } C^{n_2}(\Phi)(m_2)$$

such that if Φ is split, post-composition with λ^{-1} of Proposition 4.8 makes the pair $(C(\Phi)(0), \Phi(1))$ into a $\widehat{\mathcal{E}}$–module, and a homotopy $\widehat{\mathcal{E}}$–module if Φ is h-split. Moreover, if $\mathcal{E}, \Phi, i$ and λ^{-1} are symmetric monoidal, the module structure is Σ–equivariant.

The same holds for the reduced case, replacing C by $\bar{C}$.

This theorem is only an improvement upon Proposition 4.8 when the complex $\widehat{\mathcal{E}}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$ is not identically 0 for $n_1 \neq 0$. Of particular interest is the action on the Hochschild complex, when $m_i = 0$.

The map γ is explicit, given by the big diagram below, which allows to write down actual operations given representatives of homology classes (see Section 5.2 and the end of Section 5.5).

Note that restricting to $n_2 = 1$ (i.e. the operad part of the action on the Hochschild complex) avoids having to invert λ, and restricting to $n_i = 1$ and $m_i = 0$ avoids needing λ at all. In particular, $C(\Phi)(0)$ is a $\widehat{\mathcal{E}}([\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}1\\0\end{smallmatrix}])$–module without any monoidal assumption on Φ. Alternatively, one can use $C^n(\Phi)(m)$ as a model of $C(\Phi)(0)^{\otimes n} \otimes \Phi(1)^{\otimes m}$ which admits an action of $\widehat{\mathcal{E}}$ without reference to λ, as in the following:

Corollary 4.10. *For any $\Phi : \mathcal{E} \to \mathrm{Ch}$ and any extension $\widehat{\mathcal{E}}$ of $C\mathcal{E}$, taking $C_\Phi([\begin{smallmatrix}n\\m\end{smallmatrix}]) = C^n(\Phi)(m)$ defines a dg-functor $C_\Phi : \widehat{\mathcal{E}} \to \mathrm{Ch}$ (and the same in the reduced case).*

This corollary is a direct corollary of the proof of Theorem 4.9.

Proof of Theorem 4.9. The action is defined by the following diagram:

$$\begin{tikzcd}
C(\Phi)(0)^{\otimes n_1} \otimes \Phi(1)^{\otimes m_1} \otimes \widehat{\mathcal{E}}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \arrow[d, "\lambda\otimes id"] \arrow[dr, dashed, "\gamma"] & C(\Phi)(0)^{\otimes n_2} \otimes \Phi(1)^{\otimes m_2} \arrow[d, "\lambda"'] \\
C^{n_1}(\Phi)(m_1) \otimes \widehat{\mathcal{E}}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \arrow[d, "\tilde{\beta}\otimes id", "\simeq"'] & C^{n_2}(\Phi)(m_2) \\
C^{n_1}\big(B(\Phi, \mathcal{E}, \mathcal{E})\big)(m_1) \otimes \widehat{\mathcal{E}}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \arrow[d, "\cong"'] & C^{n_2}\big(B(\Phi, \mathcal{E}, \mathcal{E})\big)(m_2) \arrow[u, "C(\alpha)"', "\simeq"] \\
B\big(\Phi, \mathcal{E}, C^{n_1}(\mathcal{E})(m_1)\big) \otimes \widehat{\mathcal{E}}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \arrow[d, "="'] & B\big(\Phi, \mathcal{E}, C^{n_2}(\mathcal{E})(m_2)\big) \arrow[u, "\cong"] \\
B\big(\Phi, \mathcal{E}, \widehat{\mathcal{E}}([\begin{smallmatrix}0\\-\end{smallmatrix}], [\begin{smallmatrix}n_1\\m_1\end{smallmatrix}])\big) \otimes \widehat{\mathcal{E}}([\begin{smallmatrix}n_1\\m_1\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}]) \arrow[r] & B\big(\Phi, \mathcal{E}, \widehat{\mathcal{E}}([\begin{smallmatrix}0\\-\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])\big) \arrow[u, "="]
\end{tikzcd}$$

The map $\tilde{\beta}$ is that of Proposition 4.7 and the map α is that of Proposition 3.1. They are quasi-isomorphisms for any Φ. The map λ is that of Proposition 4.8. It is an isomorphism whenever Φ is split and a quasi-isomorphism whenever Φ is h-split. The bottom horizontal arrow is induced by composition in $\widehat{\mathcal{E}}$.

Consider the composition with a further morphism in $\widehat{\mathcal{E}}([\begin{smallmatrix}n_2\\m_2\end{smallmatrix}], [\begin{smallmatrix}n_3\\m_3\end{smallmatrix}])$. Note now that the failure of $\tilde{\beta} \circ C^{n_2}(\alpha)$ to be the identity lies in the non-zero simplicial degrees of $B\big(\Phi, \mathcal{E}, \widehat{\mathcal{E}}([\begin{smallmatrix}0\\-\end{smallmatrix}], [\begin{smallmatrix}n_2\\m_2\end{smallmatrix}])\big)$. As the simplicial degree is constant when applying the composition with $\widehat{\mathcal{E}}([\begin{smallmatrix}n_2\\m_2\end{smallmatrix}], [\begin{smallmatrix}n_3\\m_3\end{smallmatrix}])$, this difference is killed when we apply $C^{n_3}(\alpha)$ at the end of the action. Hence, when Φ is split monoidal, the action is strictly associative.

Let B^Σ denote the quotiented bar construction defined in Section 3. If $\mathcal{E}, i$ and Φ are symmetric monoidal, then using B^Σ instead of B, replacing $\tilde{\beta}$ with its composition with the quotient map $B \to B^\Sigma$, makes the diagram above equivariant under the action of $\Sigma_{m_1} \times \Sigma_{n_1}$, by Proposition 3.3 and 4.8, and the fact that this action is given by morphisms of $\widehat{\mathcal{E}}$.

For the reduced version, we need to check that this composition of maps is well-defined. (The map $\tilde\beta$ is in fact not well-defined in that case.)

Consider the action of some $f \in \widehat{\mathcal{E}}([{}^{n_1}_{m_1}],[{}^{n_2}_{m_2}])$ on some $x\otimes l_{\underline{k}} \in C^{n_1}(\Phi)(m_1)$ with $x\otimes l_{\underline{k}}$ identified with 0 in $\overline{C}^{n_1}(\Phi)(m_1)$, i.e.

$$x\otimes l_{\underline{k}} = c_\Phi(y\otimes u_j)\otimes l_{k_1}\otimes\ldots\otimes l_{k_{n_1}}$$

for $y\in\Phi(k-1+m_1)$, with $k=k_1+\cdots+k_{n_1}$ and $u_j = i(u_j)\in\mathcal{E}(k-1+m_1,k+m_1)$ introducing a unit in the jth position for $j\in\{2,\ldots,k_1,k_1+2,\ldots,k_{n_1}\}$.

Following the diagram defining the action, we have

$$\begin{aligned}(x\otimes l_{k_1}\otimes\ldots\otimes l_{k_{n_1}})\otimes f &\overset{\tilde\beta}{\mapsto} x\otimes(id_{k+m_1}\otimes l_{k_1}\otimes\ldots\otimes l_{k_{n_1}})\otimes f + \text{higher order}\\ &\overset{c_{\widehat{\mathcal{E}}}}{\mapsto} x\otimes(\sum g\otimes l_{k'_1}\otimes\ldots\otimes l_{k'_{n_2}}) + \text{higher order}\\ &\overset{\alpha}{\mapsto} \sum c_\Phi(x\otimes g)\otimes l_{k'_1}\otimes\ldots\otimes l_{k'_{n_2}}\end{aligned}$$

for some maps $g\in\mathcal{E}(k+m_1,k'+m_2)$. Now

$$c_\Phi(x\otimes g) = c_\Phi(c_\Phi(y\otimes u_j)\otimes g) = c_\Phi(y\otimes c_{\mathcal{E}}(u_j\otimes g))$$

so it is enough to know that $\sum c_{\mathcal{E}}(u_j\otimes g)$ is of the form $\sum c_{\mathcal{E}}(g'\otimes u_{j'})$ for some g', j' whenever g comes from a composition as above. We have (in abreviated notation)

$$\sum c_{\mathcal{E}}(u_j\otimes g)\otimes l_{\underline{k}'} = c_{\widehat{\mathcal{E}}}(u_j\otimes c_{\widehat{\mathcal{E}}}((id_{k+m_1}\otimes l_{\underline{k}})\otimes f)) = c_{\widehat{\mathcal{E}}}((u_j\otimes l_{\underline{k}})\otimes f)$$

by definition and associativity of composition in $\widehat{\mathcal{E}}$. As $u_j\otimes l_{\underline{k}}$ is identified with 0 in $\overline{C}^{n_1}(\mathcal{E}(k-1+m_1,-))(m_1) = \widehat{\mathcal{E}}([{}^{0}_{k-1+m_1}],[{}^{n_1}_{m_1}])$, we must have that $c_{\widehat{\mathcal{E}}}((u_j\otimes l_{\underline{k}})\otimes f)$ is identified with 0 in the reduced Hochschild complex $\widehat{\mathcal{E}}([{}^{0}_{k-1+m_1}],[{}^{n_2}_{m_2}])$, which means precisely that $\sum c_{\mathcal{E}}(u_j\otimes g)$ is of the form $\sum c_{\mathcal{E}}(g'\otimes u_{j'})$ as required. □

The next result says that the action of Theorem 4.9 is natural.

Theorem 4.11. *Suppose that we have a factorization $\mathcal{A}_\infty \xrightarrow{i} \mathcal{E} \xrightarrow{j} \mathcal{E}'$ with j monoidal, which extends to a morphism of extensions $\tilde{j}:\widehat{\mathcal{E}}\to\widehat{\mathcal{E}}'$. Then for any (h-)split monoidal functor $\Phi:\mathcal{E}'\to \mathrm{Ch}$, the (homotopy) $\widehat{\mathcal{E}}$–action of Theorem 4.9 on the pair $\big(j^*\Phi, C(j^*\Phi)\big)\cong\big(\Phi, C(\Phi)\big)$ factors through the $\widehat{\mathcal{E}}'$–action.*

The same holds in the reduced case, replacing C by $\overline{C}$.

Proof. This follows directly from the naturality of the maps defining the action. □

5. Examples and applications

In this section, we apply Theorem 4.9 to specific categories $\mathcal{E}$.

In 5.1, we consider the case $\mathcal{E}=\mathcal{O}$, the open cobordism category built using fat graphs. We construct an explicit extension $\widehat{\mathcal{E}}=\mathcal{OC}$ of $\mathcal{O}$, which is a fat graph model of the open-closed cobordism category (with co-positive boundary restrictions on the closed part). The main theorem of that section, Theorem 5.3, can be interpreted as a reformulation of Costello's Theorem A (2-3) in [5].

In 5.2, we explain how reading off the action of $\mathcal{OC}$ obtained in the previous section on an $\mathcal{A}_\infty$–Frobenius algebra (i.e. a split monoidal functor $\Phi : \mathcal{O} \to \mathrm{Ch}$) recovers the recipe given by Kontsevich-Soibelman in [16]. Then sections 5.3 and 5.4 give the determinant-twisted and positive boundary versions of Theorem 5.3.

Section 5.5 considers the case of strict Frobenius algebras, with $\mathcal{E} = H_0(\mathcal{O})$. We construct an extension $\widehat{\mathcal{E}} = \mathcal{SD}$ of $H_0(\mathcal{O})$ which is an "open-closed cobordism category of Sullivan diagrams", given as a quotient of $\mathcal{OC}$. Theorem 5.11 then recovers Theorem 3.3 of [31], giving an action of Sullivan diagrams on the Hochschild complex of strict Frobenius algebras. Proposition 5.12 and Corollary 5.13 describe how this collapses a significant part of the associated homological conformal field theory on the Hochschild homology of such algebras. Finally, at the end of the section, we give explicit formulas for the product, coproduct, and Δ– (or B–)operator on the Hochschild complex.

Finally, sections 5.7 and 5.8 consider the cases of $\mathcal{E} = \mathcal{A}_\infty$ and $\mathcal{E} = \mathcal{A}ss \times \mathcal{P}$ for $\mathcal{P}$ an operad.

5.1. **Open topological conformal field theories.** Consider the dg-category of graphs $\mathcal{O}$ defined in 2.9. As $\mathcal{O}$ models the moduli space of surfaces with fixed intervals (*open strings*) in their boundary, a split monoidal functor $\Phi : \mathcal{O} \to \mathrm{Ch}$ is an object that can be called an *open topological conformal field theory.* We consider here an application of Theorem 4.9 to the case $\mathcal{E} = \mathcal{O}$.

There is a functor $i : \mathcal{A}_\infty \to \mathcal{O}$ including forests into graphs, as $\mathcal{A}_\infty$ is a subcategory of $\mathcal{O}$. Now note that the decomposition of $(\bar{p}, m)$–graphs in terms of $\mathcal{O}$ and L_n given in 2.13 can be reinterpreted as follows:

Proposition 5.1. *The map of section 2.13 is an isomorphism of chain complexes*

$$\overline{C}^p(\mathcal{O}(m,-))(n) \xrightarrow{\cong} (\bar{p}, m+n) - \textit{Graphs}.$$

To use Theorem 4.9, we need a category $\widehat{\mathcal{E}}$ with objects $\mathbb{N} \times \mathbb{N}$ and such that $\widehat{\mathcal{E}}([{}^{0}_{m_1}], [{}^{n_2}_{m_2}]) = \overline{C}^{n_2}(\mathcal{O}(m_1,-))(m_2)$. We will take $\widehat{\mathcal{E}}$ to be an *open-closed cobordism category of graphs* $\mathcal{OC}$ which we define now.

Let $\mathcal{OC}$ denote the dg-category with objects $\mathbb{N} \times \mathbb{N}$ and with morphisms $\mathcal{OC}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$ the subcomplex of $(\bar{n}_2, n_1 + m_1 + m_2)$–Graphs of graphs with the first n_1 leaves $\lambda_1, \ldots, \lambda_{n_1}$ sole labeled leaves in their boundary cycle[8]. Composition is defined on generators $G_1 \in \mathcal{OC}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$, $G_2 \in$

[8]Although this construction appears asymmetrical in its treatment of incoming and outgoing closed boundaries, it is justified by Theorem 5.4 below.

$\mathcal{OC}([{n_2 \atop m_2}],[{n_3 \atop m_3}])$ as the sum $G_2 \circ G_1 = \sum G$ over all possible graphs G that can be obtained from G_1 and G_2 by:

(1) removing the n_2 white vertices of G_1, -
(2) identifying the start half-edge of the ith white vertex v_i of G_1 with the leaf λ_i of G_2,
(3) attaching the remaining leafs in $s^{-1}(v_i)$ to vertices of the boundary cycle of G_2 containing λ_i, respecting the cyclic ordering.
(4) attaching the last m_2 labeled leaves of G_1 to the leaves of G_2 labeled $n_2+1,\dots,n_2+m_2$.

The orientation is obtained by juxtaposition after removing the white vertices w_i and their start half-edges h_i from the orientation of G_1 ordered as pairs $w_i \wedge h_i$.

Note that composition is determined by composition in $\mathcal{O}$ and by composition into a single white vertex (i.e., the case G_1 is the graph l_i of Figure 2). The number of graphs in the sum is a sum of ordered partitions of each $s^{-1}(v_i)\backslash\{*\}$ into however many slots there are in the ith boundary cycle of G_2. Figure 8 give two examples of compositions in $\mathcal{OC}$.

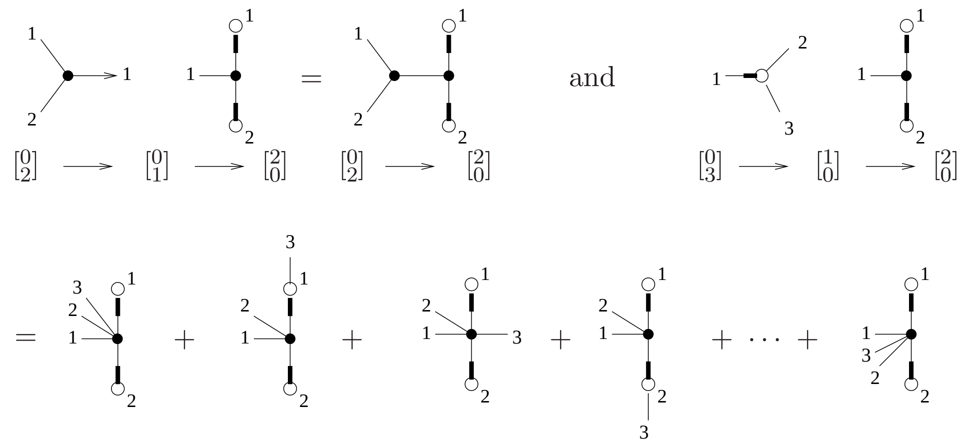

FIGURE 8. Compositions in $\mathcal{OC}$. (Note that we interpret the same graph in two different ways in the first and second composition.)

For the above definition to make sense, we need a lemma:

Lemma 5.2. *The composition of graphs defined above is a chain map.*

Proof. Given G_1, G_2 as above, we need to check that

$$d(G_2 \circ G_1) = G_2 \circ dG_1 + (-1)^{|G_1|} dG_2 \circ G_1.$$

Call a vertex of $G_2 \circ G_1$ *special* if it comes from a vertex of one of the first n_2 boundary cycles of G_2. The left-hand side has terms coming from

(1) blowing up at a non-special vertex,

(2) blowing up at a special vertex in such a way that the newly created vertices are either white,
black with no half-edges of G_2, or
black with at least two half-edges of G_2,

(3) blowing up at a special vertex in such a way that one of the newly created vertices is black with exactly one half-edge of G_2 attached to it.

The terms of type (1) and (2) are exactly the terms occurring in $G_2 \circ dG_1 + (-1)^{|G_1|} dG_2 \circ G_1$. Indeed, type (1) terms correspond to blowing up at vertices of G_1 or G_2 which are not affected by the gluing, and type (2) terms correspond either to blowing up a vertex of G_2 on a special cycle and then attach edges of G_1, or, in the case where one of the vertices is black with no half-edges of G_2 attached to it, this correspond to blowing-up at a white vertex of G_1 and then glue the resulting graph to G_2. This covers all the possibilities.

The fact that the signs agree follows from the fact that the parity of the degree of a graph is the same as the parity of the number of vertices and half-edges in the graph, i.e. that $(-1)^{|G_1|} = (-1)^{|V_1|+|H_1|} = (-1)^{|V_1|+|H_1|-2|(V_1)_w|}$ for $(V_1)_w$ the set of white vertices of G_1. Indeed, a vertex contributes with an odd degree precisely when it has even valence, that is when the vertex plus its half-edges give an odd number.

We are left to check that the terms of type (3) cancel in pairs. A "bad" newly created vertex has exactly two half-edges attached to it which are not from G_1: one from G_2 and one newly created half-edge. Any such graph occurs a second time as a term of type (3) with the role of these two edges exchanged and one checks that the signs cancel. $\square$

By Proposition 5.1, $\mathcal{OC}$ is an extension of $\overline{\mathcal{C}}\mathcal{O}$. Hence we can apply Theorem 4.9 to the categories $\mathcal{E} = \mathcal{O}$ and $\widehat{\mathcal{E}} = \mathcal{OC}$.

Theorem 5.3. *Let $\Phi : \mathcal{O} \to \mathrm{Ch}$ be an (h-)split symmetric monoidal functor. Then the pair $\big(\overline{C}(\Phi)(0), \Phi(1)\big)$ is a Σ–equivariant (homotopy) $\mathcal{OC}$–module.*

The above theorem is a reformulation of Costello's theorem [5, Thm. A (2-3)]. We give a more precise description of the action of the open-closed cobordism category on the pair $\big(\overline{C}(\Phi)(0), \Phi(1)\big)$, for Φ an *open TCFT* (i.e. (h-)split monoidal functor from $\mathcal{O}$). This allows us to recover the recipe given by Kontsevich-Soibelman in Section 11.6 of [16] (see Section 5.2).

To interpret the above in terms of moduli spaces of Riemann surfaces, we need the following: -

Theorem 5.4. [5, Prop. 6.1.3] *$\mathcal{OC}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$ is the cellular complex of a space weakly homotopy equivalent to the disjoint union of coarse moduli spaces[9] of Riemann surfaces of every genus, with*

[9]Here we employ the convention that the moduli space of a disk with a single free boundary is a point, as is the moduli of an annulus with two free boundaries.

- *$m_1 + m_2$ labeled open boundary components,*
- *$n_1 + n_2$ labeled closed boundary components,*
- *any number of free boundary components (at least one if $m_1 = m_2 = n_1 = 0$)*

Note that this category does *not* include morphisms associated to the disk with one outgoing closed boundary component. Consequently, algebras over the closed sector of this theory are not necessarily unital (the unit in the algebra would come from the generator of H_0 of the moduli of such disks). That is, algebras over $\mathcal{OC}$ are inherently "co-positive boundary" topological conformal field theories.

Costello only states this last result in the case $n_1 = 0$, but the case $n_1 > 0$ follows directly from the fact that the moduli space of Riemann surfaces with n_1 open boundaries which are alone on their boundary components is homotopy equivalent to the moduli space with n_1 closed boundaries. This model of moduli space is very closely related to Penner's original fat graph model, though it has the particularity of isolating n_2 of the closed boundary cycles in a way which is very similar to Godin's admissible fat graphs [8, Sect. 2.3].

We note Theorem 5.4 is a statement on the level of *categories*: not only does $\mathcal{OC}$ compute the homology of the moduli space of Riemann surfaces, it does so in a way that preserves the composition of morphisms. Specifically, the gluing of graphs corresponds to the geometric gluing of Riemann surfaces as varying the metrics of the graphs to be glued along a cycle corresponds in the glued graph to varying its metric and varying the relative position of the edges attached to this cycle.

5.2. **Making the action explicit: the Kontsevich-Soibelman recipe.** Let $\Phi : \mathcal{O} \to \mathrm{Ch}$ be a split monoidal functor with $\Phi(1) = A$. Given n_1 Hochschild chains in A, m_1 elements A and a graph Γ in $\mathcal{OC}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$, that is:

$$(a_0^1 \otimes \ldots \otimes a_{k_1}^1), \ldots, (a_0^{n_1} \otimes \ldots \otimes a_{k_{n_1}}^{n_1}),\ b_1, \ldots, b_{m_1} \text{ and } \Gamma$$

the diagram in the proof of Theorem 4.9 gives an explicit way to obtain a sum of n_2 Hochschild chains in A and m_2 elements of A. We apply here the sequence of maps to these elements and show how this recovers the recipe given by Kontsevich and Soibelman in [16, pp 58–62].

The first map in the diagram assembles all these terms as

$$a_0^1 \otimes \ldots \otimes a_{k_1}^1 \otimes \ldots \otimes a_0^{n_1} \otimes \ldots \otimes a_{k_{n_1}}^{n_1} \otimes b_1 \otimes \ldots \otimes b_{m_1} \otimes l_{k_1+1} \otimes \ldots \otimes l_{k_{n_1}+1}$$

The following map, $\tilde{\beta}$, embeds these into the Hochschild complex of the bar construction. It gives terms of simplicial degree 0 coming from the canonical inclusion (adding an identity map in $\mathcal{O}(k + m_1.k + m_1)$ to the above), plus additional terms of higher simplicial degrees. These elements of $C^{n_1}(B(\Phi, \mathcal{O}, \mathcal{O}))(m_1)$ are now reinterpreted as lying in $B(\Phi, \mathcal{O}, \mathcal{OC}(-, [{}^{n_1}_{m_1}]))$

just by considering $id_{k+m_1} \otimes l_{k_1+1} \otimes \ldots \otimes l_{k_{n_1}+1}$ as a graph with n_1 disjoint white vertices of valences $k_1+1, \ldots, k_{n_1}+1$ and m_1 additional disjoint leaves.

The bottom horizontal map in the diagram now glues this last graph to Γ. The result of gluing is a sum of graphs Γ' which are obtained from Γ by adding k_i labeled leaves cyclically in all possible manners on the ith closed incoming cycle of Γ for each i. After reinterpreting the new graphs as morphisms in $\mathcal{O}$ attached to n_2 white vertices (as in 2.13), the map α—in simplicial degree 0—applies these morphisms of $\mathcal{O}$ to the elements of A and kills terms of higher simplicial degree. Finally, the resulting chain of $\Phi((k_1'+1)+\cdots+(k_{n_2}'+1)+m_2)$ is reinterpreted as n_2 Hochschild chains in A and m_2 elements of A. The terms of higher simplicial degrees produces by $\tilde{\beta}$ are killed by α.

The appendix explains how to read signs for the operations. For concrete examples of these operations in the case of a strict Frobenius algebra, we refer the reader to the end of section 5.5.

5.3. **Twisting by the determinant bundle.** For a graph G defining a morphism in $\mathcal{OC}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])$, we define its outgoing boundary $\partial_{out}G$ to be the union of its n_2 white vertices and the endpoints of its m_2 outgoing leaves, regarded as a subspace of the corresponding topological graph, also denoted G. We write $\det(G, \partial_{out})$ for the Euler characteristic of the relative homology $H_*(G, \partial_{out})$, regarded as a graded abelian group:

$$\det(G, \partial_{out}) := \det(H_*(G, \partial_{out})) = \det(H_0(G, \partial_{out})) \otimes \det(H_1(G, \partial_{out}))^*$$

seen here as a graded $\mathbb{Z}$–module, in degree $-\chi(G, \partial_{out})$.

For $d \in \mathbb{Z}$, define a *d–orientation* for G to be an element of

$$\det(\mathbb{R}(V \sqcup H)) \otimes \det(G, \partial_{out})^{\otimes d}.$$

We define new categories $\mathcal{O}_d$ and $\mathcal{OC}_d$ just like $\mathcal{O}$ and $\mathcal{OC}$ but replacing the previously defined orientation of graphs by a d–orientation. So the objects of $\mathcal{O}_d$ and $\mathcal{OC}_d$ are the same as those of $\mathcal{O}$ and $\mathcal{OC}$, but the morphisms are now chain complexes generated by pairs $(G, o_d(G))$ for G a graph representing a morphism in $\mathcal{O}$ or $\mathcal{OC}$ and $o_d(G)$ a d–orientation of G. The boundary of a d–oriented graph $(G, o_d(G))$ is the boundary of the graph G as before together with the d–orientation induced as before its $\det(\mathbb{R}(V \sqcup H))$–part, and by choosing a topological map realizing the blow-up of a vertex with support in a small neighborhood of that vertex for its $\det(G, \partial_{out})$–part.

To define composition in $\mathcal{O}_d$ and $\mathcal{OC}_d$, we need the following. Let G_1, G_2 be two graphs representing composable morphisms in $\mathcal{OC}$, with $(G_2 \circ G_1) = \sum G'$ their composition. As G_2 is a subgraph of each G', we have a triple $(G', G_2, \partial_{out})$. Note also that $H_*(G', G_2) \cong H_*(G_1, \partial_{out})$ as collapsing the copy of G_2 in any term G' of $G_2 \circ G_1$ will exactly recreate G_1 with its outer boundary collapsed. Splitting the long exact sequence in homology for each triple $(G', G_2, \partial_{out})$ into short exact sequences and choosing splittings of those, one gets a natural isomorphism

$$\det(G_1,\partial_{out})\otimes\det(G_2,\partial_{out})\to\det(G',\partial_{out})$$

for each term in the composition. This isomorphism is associative (see [5, Sect. 3] and [22]). One then can define composition in $\mathcal{O}_d$ or $\mathcal{OC}_d$ as composition in $\mathcal{O}$ or $\mathcal{OC}$, tensored with the d^{th} power of this isomorphism. More precisely, the composition of d–oriented graphs $(G_1, o_d(G_1))$ and $(G_2, o_d(G_2))$ is by the same gluing as before on the graphs, and via the composition

$$\begin{array}{ll} & \det(\mathbb{R}(V_1\sqcup H_1))\otimes\det(G_1,\partial_{out})^{\otimes d}\otimes\det(\mathbb{R}(V_2\sqcup H_2))\otimes\det(G_2,\partial_{out})^{\otimes d}\\ \to & \det(\mathbb{R}(V_1\sqcup H_1))\otimes\det(\mathbb{R}(V_2\sqcup H_2))\otimes\det(G_1,\partial_{out})^{\otimes d}\otimes\det(G_2,\partial_{out})^{\otimes d}\\ \to & \det(\mathbb{R}(V_1\sqcup H_1\sqcup V_2\sqcup H_2))\otimes\det(G',\partial_{out})^{\otimes d}\end{array}$$

for each term G' in $G_2\circ G_1$, where the first arrow introduces a sign $(-1)^{d|G_2|\chi(G_1,\partial_{out})}$ and the second map is juxtaposition on the first factors as in $\mathcal{O}$ and $\mathcal{OC}$, and the d–power the above isomorphism on the last factors.

The resulting categories $\mathcal{O}_d$ and $\mathcal{OC}_d$ are symmetric monoidal.

Note that $\mathcal{O}_d$ admits a symmetric monoidal functor $i:\mathcal{A}_\infty\to\mathcal{O}_d$, since $\det(G,\partial_{out})$ is trivial, lying in degree 0, for any graph $G\in\mathcal{A}_\infty$ (unions of trees, each with exactly one outgoing boundary point), with canonical, and compatible, generators. Thus we are entitled to form the Hochschild complex of any functor $\Phi:\mathcal{O}_d\to\mathrm{Ch}$. As in Section 5.1, $\mathcal{OC}_d$ is an extension of $\overline{C}\mathcal{O}_d$, and so, by Theorem 4.9, we have

Corollary 5.5. *Let $\Phi:\mathcal{O}_d\to\mathrm{Ch}$ be an (h-)split symmetric monoidal functor. Then the pair $\big(\overline{C}(\Phi)(0),\Phi(1)\big)$ is a Σ–equivariant (homotopy) $\mathcal{OC}_d$–module.*

5.4. **Positive boundary variations.** We now restrict to the subcategory $\mathcal{E}=\mathcal{O}^b\subseteq\mathcal{O}$ with the same objects and whose morphisms are those satisfying that their underlying surface has at least one outgoing boundary in each component. We have the following variation on Proposition 5.1:

Proposition 5.6. *The map of section 2.13 identifies $\overline{C}^p(\mathcal{O}^b(m,-))(n)$ with the subcomplex of $(\bar{p},m+n)-$ Graphs consisting of graphs with at least one outgoing boundary (closed or open) in each component.*

If we define $\mathcal{OC}^b\subseteq\mathcal{OC}$ to be the subcategory consisting of graphs with at least one outgoing boundary in each component, then $\mathcal{OC}^b$ is an extension of $\mathcal{O}^b$. Recalling that the closed-to-close morphisms of $\mathcal{OC}$ satisfy a "co-positive" boundary condition, namely that every component of the underlying surface has at least one incoming boundary, we have that the closed-to-closed part of $\mathcal{OC}^b$ satisfies both the positive and co-positive boundary conditions.

Corollary 5.7. *If $\Phi:\mathcal{O}^b\to\mathrm{Ch}$ is an (h-)split symmetric monoidal functor, then the pair $\big(\overline{C}(\Phi)(0),\Phi(1)\big)$ is a Σ–equivariant (homotopy) $\mathcal{OC}^b$–module.*

5.5. **Strict Frobenius algebras and Sullivan diagrams.** We consider now the category $\mathcal{E} = H_0\mathcal{O}$, and d–shifted versions of it, whose split monoidal functors are (dimension d) strict symmetric Frobenius algebras. We will relate the Hochschild complex of $\mathcal{E}$ to Sullivan diagrams.

Let $\mathcal{E}_d$ be the category with objects the natural numbers and morphisms the chain complexes

$$\mathcal{E}_d(n,m) = \coprod_{\Sigma\in\pi_0(\mathcal{O}(n,m))} H_{-d\cdot\chi(\Sigma,\partial_{out})}(\mathcal{O}_{d,\Sigma}(n,m))$$

where Σ runs over the topological types of cobordisms, $\mathcal{O}_{d,\Sigma}(n,m)$ denotes the path component of Σ in $\mathcal{O}_d$, and $\chi(\Sigma,\partial_{out}) = \chi(G,\partial_{out})$ for any graph in that component is the degree shift induced by the determinant. Composition in $\mathcal{E}_d$ is induced from composition in $\mathcal{O}_d$ as described in Section 5.3.

When $d = 0$, morphisms are just the 0th homology of the morphisms in the category $\mathcal{O}$. As the components of $\mathcal{O}(n,m)$ are the topological types of cobordisms, this is the category modeling open TQFT's as studied for example in [18]. As shown in that paper, a split symmetric monoidal functor $\Phi : \mathcal{E}_0 \to \mathrm{Ch}$ is a (strict) symmetric Frobenius algebra.

We call a split monoidal functor $\Phi : \mathcal{E}_d \to \mathrm{Ch}$ a *symmetric Frobenius algebra of dimension d.* This structure is generated by a unital product of degree 0 and a trace of degree $-d$. The main example of such algebras is the cohomology $H^*(M)$ of a manifold of dimension d (see Section 5.6).

We call a fat graph *p–admissible* (in the spirit of [8]) if p of its boundary cycles are disjoint embedded circles in the graph—and we call these p special cycles *admissible cycles.*

Definition 5.8. *An* (oriented) (p,m)–Sullivan diagram *is an equivalence class of (oriented) p–admissible fat graphs with $p+m$ leaves, where the first p leaves are distributed as 1 per admissible boundary cycle and the remaining m leaves lie in the other cycles. Two such graphs G_1, G_2 are equivalent if they are connected by a zig-zag of edge collapses of p–admissible fat graphs, collapsing edges which are not in the p admissible cycles.*

See Figure 9 for an example of a $(2,2)$–Sullivan diagram.

We define the *degree* of a (p,m)–Sullivan diagram to be $E^a - p$, where E^a is the number of edges in its p admissible cycles. (The Sullivan diagram of Figure 9 is of degree 3.)

We denote by (p,m)–SD the chain complex of (p,m)–Sullivan diagrams, whose generators are oriented (p,m)–Sullivan diagrams, with -1 acting by reversing the orientation and with boundary map defined on generators by

$$dG = \sum_{e\in E^a} G/e,$$

the sum of all collapses of G along edges in the admissible boundary cycles[10] of G.

This chain complex is isomorphic to the complex *Cyclic Sullivan Chord Diagrams* considered by Tradler-Zeinalian in [31, Def 2.1]. Sullivan diagrams are usually defined as unions of circles (the admissible cycles above) together with "length zero" chords which are unions of trees. We allow the chords to be unions of graphs, but pushing the vertices to only be on the cycles, a Sullivan diagram in our sense is a union of cycles together with chords which are edges attached to the cycles—with possibly coinciding endpoints. In [31], the trees are not assumed to be attached to the cycles at distinct vertices, which makes their definition equivalent to ours.

Theorem 5.9. *The complex $\overline{C}^n(\mathcal{E}_0(m_1,-))(m_2)$ is isomorphic to the chain complex (n, m_1+m_2)–SD of (n, m_1+m_2)–Sullivan diagrams.*

Proof. We first note that every Sullivan diagram is equivalent to one with only trivalent vertices, except for the vertices where the leaf of an admissible cycle is attached, which may be 4-valent. Indeed, if the graph has a higher valence vertex away from the admissible cycles, one can blow it up in any manner one likes and obtain an equivalent graph with trivalent vertices replacing the higher valence vertex. If there is a higher valence vertex on an admissible cycle, it has exactly two (contiguous) half-edges of that admissible cycle attached to it, unless the leaf of the admissible cycle is at that position, in which case it has three such. Any blow-up of that vertex which keeps the half-edges of the admissible cycle together produces an equivalent graph with the property we want. For the purpose of the proof, we call such graphs *essentially trivalent*.

Now two essentially trivalent Sullivan diagrams are equivalent if and only if they are equivalent through such Sullivan diagrams and diagrams with exactly one 4-valent vertex which is away from the cycles: a single valence 4 vertex at a time suffices since we can do collapses and blow-ups one at a time, and no additional valence 4 (or 5) vertices on the admissible cycles is necessary because there is only one way of blowing-up such a vertex if the two (or three) half-edges of the admissible cycles have to stay together, up to collapses and blow-ups away from the admissible cycle.

Given an essentially trivalent (n, m_1+m_2)–Sullivan diagram, we get an element of $\overline{C}^n(H_0\mathcal{O}(m_1,-))(m_2)$ by collapsing the admissible cycles to white vertices (the remainder of the graph define elements of $H_0\mathcal{O}$ since $H_0\mathcal{O}$ is given by equivalence classes of trivalent graphs). If the leaf of the ith admissible cycle is at a 3-valent vertex, we place a start-leaf at that position on the ith white vertex, and if it is at a 4-valent vertex, we remove it and define the remaining half-edge after to be start half-edge. (See Figure 9 for an example.)

[10]It is worth noting that we have already effectively collapsed the remaining edges by the equivalence relation.

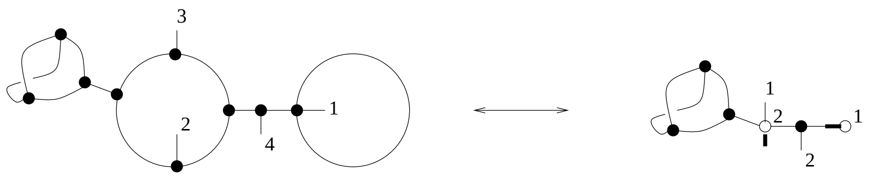

FIGURE 9. Essentially trivalent $(2,2)$–Sullivan diagram and the corresponding $(\bar{2},2)$–graph

Given a graph in $\overline{C}^n(H_0\mathcal{O}(m_1,-))(m_2)$, one can similarly obtain an essentially trivalent (n, m_1+m_2)–Sullivan diagram by blowing up the white vertices and placing leaves at the spots corresponding to start-edges. These two maps are inverse to each other, and the equivalence relations agree under the maps by the above remarks.

Note moreover that the degrees agree: the degree of a graph G in the Hochschild complex of $\mathcal{E}_0$ is the degree of the Hochschild chain (which is the number of edges attached to white vertices of G minus the number of white vertices, that is exactly the degree of the corresponding Sullivan diagram), plus the degree coming from the elements of $\mathcal{E}_0$ attached to the white vertices, but this is just always zero here.

We are left to check that the boundary maps also agree. Given an essentially trivalent Sullivan diagram, the boundary map in (n, m_1+m_2)–SD is a sum of Sullivan diagrams, each with a higher valence vertex on an admissible cycle. Blowing up that vertex in the only possible manner to obtain an essentially trivalent graph corresponds exactly under the equivalence above to a term in the differential of the associated element of $\overline{C}^n(\mathcal{E}_0(m_1,-))(m_2)$. $\square$

Note that $(n, m_1+m_2) - SD \cong \overline{C}^n(H_0\mathcal{O}(m_1,-))(m_2)$ is the quotient of the complex $\overline{C}^n(\mathcal{O}(m_1,-))(m_2)$ of $(\bar{n}, m_1+m_2)$–graphs by the graphs having black vertices of valence higher than 3, and by the boundaries of such graphs. Indeed, a non-trivial element in this quotient is a graph with all its black vertices trivalent, and the equivalence relation comes from the boundary of graphs having a single valence 4 black vertex. In the boundary of such a graph, only the two terms coming from the blow up of the valence 4 vertex are non-zero in the quotient complex.

Recall that the open-closed category $\mathcal{OC}$ is an extension of $\overline{C}\mathcal{O}$, so that $\mathcal{OC}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}]) = \overline{C}^{n_2}(\mathcal{O}(m_1,-))(m_2)$ when $n_1 = 0$. Theorem 5.9 and the previous discussion then justify defining the following quotient category:

Definition 5.10. *Let $\mathcal{SD}$ be the category with objects $\mathbb{N}\times\mathbb{N}$ and morphisms from $[\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}]$ to $[\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}]$ the quotient of $\mathcal{OC}([\begin{smallmatrix} n_1 \\ m_1 \end{smallmatrix}], [\begin{smallmatrix} n_2 \\ m_2 \end{smallmatrix}])$ by the graphs having black vertices of valence higher than 3 and by the boundary of such graphs.*

As composition in $\mathcal{OC}$ can only increase the valence of black vertices, it is still well-defined on the quotient and so $\mathcal{SD}$ is a category. It naturally forms an extension of $\overline{C}(H_0\mathcal{O})$ by Theorem 5.9. More generally, we may define a category $\mathcal{SD}_d$ by the same quotient construction, though twisted by the determinant bundle of section 5.3.

For $\mathcal{E} = \mathcal{E}_d = H_{-d\cdot\chi}(\mathcal{O}_d)$, taking $\widehat{\mathcal{E}} = \mathcal{SD}_d$, Theorem 4.9 thus gives

Theorem 5.11. *Let A be a symmetric Frobenius algebra of dimension d, then the pair $(\overline{C}(A), A)$ is a Σ–equivariant $\mathcal{SD}_d$–module, where $\overline{C}(A)$ denotes the reduced Hochschild complex of the algebra A.*

As a differential graded algebra with a non-degenerate inner product defines a symmetric Frobenius algebra, this recovers Theorem 3.3 of [31].

For a concrete interpretation of this result, one needs to know what the homology of SD is, or what the quotient map $\pi : \mathcal{OC} \to \mathcal{SD}$ is in homology. The chain complex of Sullivan diagrams is a lot smaller than that of all fat graphs, or all $(\bar{p}, m)$–graphs, and hence computations of its homology are more approachable. It is for example not hard to compute that the component of the pair of pants in $\mathcal{SD}([{}^2_0], [{}^1_0])$ is a complex that computes the homology of $S^3 \times S^1$. The corresponding component of $\mathcal{OC}$ computes the homology of the framed disk operad $fD(2) \simeq S^1 \times S^1 \times S^1$. The map $\mathcal{OC} \to \mathcal{SD}$ in this case is induced by the canonical embedding of the first two S^1–factors as a torus in the 3-sphere. Since that embedding is null-homotopic, this puts strong restrictions on what BV operations on $HH_*(A, A)$ can be nontrivial. See the computations following Figure 11 for concrete examples of this phenomenon.

More generally, we can consider the restriction of π to components:

$$\pi_\Sigma : \mathcal{OC}_\Sigma([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])) \longrightarrow \mathcal{SD}_\Sigma([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}]))$$

where $\Sigma \in \pi_0\mathcal{OC}_\Sigma([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}])) \cong \pi_0\mathcal{SD}_\Sigma([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}]))$ is a topological type of cobordism.

We have the following general result:

Proposition 5.12. *Suppose $m_1 + m_2 + n_1 > 0$ and $\Sigma \in \pi_0\mathcal{OC}([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}]))$ is a connected surface of genus g. Then there exists $\Sigma' \in \pi_0\mathcal{OC}([{}^{n_1}_{m_1}], [{}^{0}_{m_2+1}]))$ and a map $f : \mathcal{OC}_{\Sigma'}([{}^{n_1}_{m_1}], [{}^{0}_{m_2+1}])) \to \mathcal{OC}_\Sigma([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}]))$ which is an isomorphism in homology in degrees $* \le \frac{2g}{3}$ and such that the image of $\pi_\Sigma \circ f$ in $\mathcal{SD}_\Sigma([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}]))$ is concentrated in degree 0. In particular, the stable classes (of positive degree) map to 0 under the map $H_*(\pi) : H_*(\mathcal{OC}) \to H_*(\mathcal{SD})$.*

Here by a *stable class*, we mean a class in that lives in the stable range, i.e. a class $H_*(\mathcal{OC}_\Sigma([{}^{n_1}_{m_1}], [{}^{n_2}_{m_2}]))$ of degree $* \le \frac{2g}{3}$ for g the genus of the component of lowest genus in Σ.

For $m_1 = m_2 = n_1 = 0$, the situation is a little more subtle. An analogous statement can be made though using in place of f a map that replaces a fixed boundary by a free boundary, and hence is not an isomorphism in homology stably.

Proof. Suppose first that $m_1 + m_2 > 0$. Then Σ' can be obtained from Σ by gluing discs on the n_2 closed outgoing boundaries of Σ and adding a open outgoing boundary on a boundary component containing some other open boundary. We can reconstruct the topological type of Σ from Σ' by gluing a n_2–legged pair of pants P along an open boundary as shown in Figure 10. Choosing a degree 0 representative of P in $\mathcal{OC}([{0 \atop 1}],[{n_2 \atop 0}]))$, the map f above

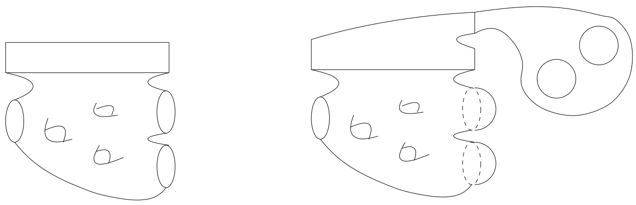

FIGURE 10. The surfaces Σ and $P \circ \Sigma' \cong \Sigma$

is just induced by composition in $\mathcal{OC}$. The fact that it is an isomorphism in homology in the given range is part of Harer's homological stability theorem ([10], with the improved range of [1, 28]). The fact the the composition $\pi_\Sigma \circ f$ lands in degree 0 follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathcal{OC}([{n_1 \atop m_1}],[{0 \atop m_2+1}])) \arrow[r, "f"] \arrow[d, "\pi"] & \mathcal{OC}([{n_1 \atop m_1}],[{n_2 \atop m_2}])) \arrow[d, "\pi"] \\
\mathcal{SD}([{n_1 \atop m_1}],[{0 \atop m_2+1}])) \arrow[r, "\pi(f)"] & \mathcal{SD}([{n_1 \atop m_1}],[{n_2 \atop m_2}]))
\end{tikzcd}$$

and the fact that the complex $\mathcal{SD}([{n_1 \atop m_1}],[{0 \atop m_2+1}]))$ is concentrated in degree 0.

For $n_1 > 0$, we have an isomorphism $\mathcal{OC}_\Sigma([{n_1 \atop m_1}],[{n_2 \atop m_2}])) \cong \mathcal{OC}_{\bar\Sigma}([{n_1-1 \atop m_1+1}],[{n_2 \atop m_2}]))$, and similarly for $\mathcal{SD}$, for $\bar\Sigma$ the surface obtained from Σ by replacing an incoming closed boundary by an incoming open boundary, alone on that component. The reduces the case $n_1 > 0$ to the previous one. □

Using Theorem 4.11, a consequence of the above result and of Theorem 5.11 is the following:

Corollary 5.13. *For strict symmetric Frobenius algebras A, the TCFT structure on $\overline{C}_*(A)$ defined by Costello and Kontsevich-Soibelman factors through an action of Sullivan diagrams. In particular, stable classes in the homology of the moduli space act trivially.*

This results puts together the work of Costello and Kontsevich-Soibelman with that of Tradler-Zeinalian: We show that Costello's construction (which translates to that of Kontsevich-Soibelman when made explicit) of an action of moduli space on the Hochschild homology of a strict Frobenius algebra factors through an action of the complex of Sullivan diagrams as constructed by Tradler-Zeinalian [31, Thm 3.3].

The action on the Hochschild complex given by Theorem 4.9 is easy to implement explicitly in the case of strict Frobenius algebras because operations involve fewer terms than in the general case. Figure 11 (a-c) gives examples of graphs representing the product (pair of pants with two inputs and one output), the coproduct (pair of pants with one input and two outputs) and the Δ–operator (degree 1 operator with one closed input and one closed output). We give now the explicit formulas for the action of these graphs on the Hochschild complex of a strict Frobenius algebra.

Let A be a strict symmetric Frobenius algebra. To obtain the action of a (sum of) graph(s) G representing a chain in $\mathcal{SD}_d$, on a chain in the Hochschild complex of A, we need to follow the prescription laid out in section 5.2 (together with the appendix section 6 for the signs). In Figure 11(a-c), we have made a choice of an ordering of the vertices, and of an orientation of the edges. The chosen orientation of each graph is then the orientation corresponding to considering the graph as a composition of the operations at each vertex in this ordering, with their canonical orientation (see Section 6). Figures (a'-c') show the non-trivial graphs created when applying the procedure described in section 5.2.

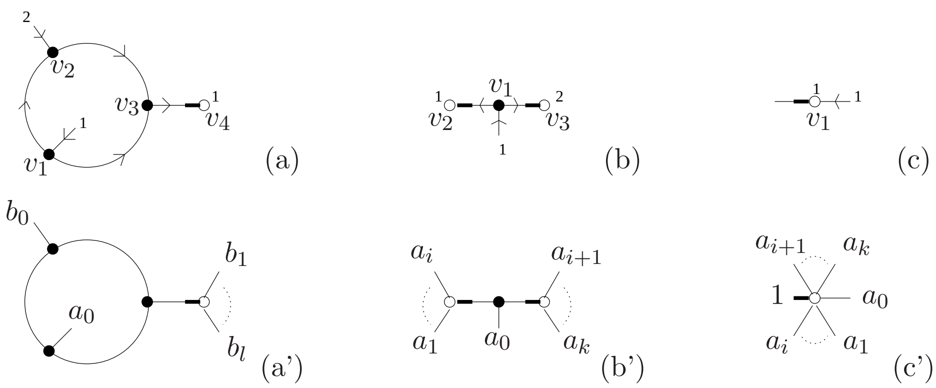

FIGURE 11. Representing graphs for the product, coproduct and Δ–operator

We denote as before a k–chain in the Hochschild complex of A by $a_0 \otimes \ldots \otimes a_k$. Using the convention for the product and coproduct given in Section 6, the graphs of Figure 11 induce the following operations on $C_*(A)$:

(a) Product:

$$(a_0 \otimes \cdots \otimes a_k) \otimes (b_0 \otimes \cdots \otimes b_l) \mapsto \begin{cases} 0 & k > 0 \\ \sum (-1)^\epsilon a_0'' a_0' b_0 \otimes b_1 \otimes \cdots \otimes b_l & k = 0 \end{cases}$$

where $\sum a_0' \otimes a_0''$ denotes the coproduct of a_0 as an element of the Frobenius algebra A and

$$\epsilon = |a_0'||a_0''| + d(|b_0| + \cdots + |b_l| + l).$$

(The main part of this computation is done in detail in the appendix.) Note in particular that, as the product is homotopy commutative, in homology it is 0 except on $HH_0(A, A) \otimes HH_0(A, A)$.

(b) Coproduct:

$$(a_0 \otimes \cdots \otimes a_k) \mapsto \sum (-1)^\epsilon (a_0'' \otimes a_1 \otimes \cdots \otimes a_i) \otimes (a_0' \otimes a_{i+1} \cdots \otimes a_k)$$

where $\epsilon = d(|a_1| + \cdots + |a_k| + k)$.

(c) Δ–operator:

$$(a_0 \otimes \cdots \otimes a_k) \mapsto \sum (-1)^\epsilon 1 \otimes a_{i+1} \otimes \cdots \otimes a_k \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_i$$

where $\epsilon = (|a_0| + \cdots + |a_i|)(|a_{i+1}| + \cdots + |a_k|) + ik$.

Proposition 5.14. *If A is a strict graded symmetric Frobenius algebra over a field k, the coproduct and Δ make $HH_*(A,A)$ into a Batalin-Vilkovisky coalgebra. Moreover, this structure is dual to the BV-algebra structure on $HH^*(A,A)$, where the product is the cup product of Hochschild cochains, and the BV operator is dual to Connes' B–operator.*

The first part of this proposition, before going to homology, recovers the cyclic Deligne conjecture as proved in [15, 31, 29].

The duality in this proposition is given on the chain level by a chain isomorphism $CH^*(A,A) \to \mathrm{Hom}(CH_*(A,A),k)$. Degree-wise this is given by the map

$$\mathrm{Hom}(A^{\otimes n}, A) \to \mathrm{Hom}(A^{\otimes n+1}, k), \quad f \mapsto \widetilde{f}$$

where $\widetilde{f}(a_0, \ldots, a_n) = \langle a_0, f(a_1, \ldots, a_n) \rangle$.

Proof. A BV coalgebra is an algebra over the cooperad whose k–ary operations are given by the homology of the moduli space of Riemann surfaces of genus 0 with one incoming and k outgoing closed boundary components, with composition induced by gluing. As the corresponding component of $\mathcal{SD}([{1 \atop 0}], [{k \atop 0}])$ is quasi-isomorphic to that of $\mathcal{OC}([{1 \atop 0}], [{k \atop 0}])$, the first part of the statement follows, independently of the second part, from Theorems 5.4 and 5.11.

Now the duality carries Δ to B, since the Δ–operator in $HH_*(A,A)$ given in (c) is precisely B, and the Δ–operator on $HH^*(A,A)$ is defined by transferring B^* via $f \mapsto \widetilde{f}$. (The signs in the formula for B given in [7, Sect. 2.4] differs from ours due to different conventions. They match if we introduce a factor $(-1)^{a_0 + \cdots + a_k + k}$ passing the generator of $H_1(S^1)$ on the other side of the Hochschild complex, and a factor $(-1)^{a_1 + 2a_2 + \cdots + ka_k}$ before and after the operation to compare the Hochschild complexes—this last factor sets the degree k shift of the Hochschild complex in between the a_i's instead of at the end as we have it).

So it suffices to check that the coproduct in (b) (which we will write as ν) is dual to the Hochschild cup product. Let f and g be two Hochschild cochains; then (up to sign issues as above)

$$\begin{aligned}\widetilde{f \cup g}\,(a_0, \ldots, a_{p+q}) &= \pm\langle a_0, f(a_1, \ldots, a_p) \cdot g(a_{p+1}, \ldots, a_{p+q})\rangle \\ &= \pm\sum \langle a_0'', f(a_1, \ldots, a_p)\rangle \cdot \langle a_0', g(a_{p+1}, \ldots, a_{p+q})\rangle \\ &= \pm\,\nu^*(\widetilde{f} \otimes \widetilde{g})(a_0, \ldots, a_{p+q})\end{aligned}$$

where the first equality is the definition and the third from the formula given in (b) above. The second follows from Figure 12, below, which relates the coproduct and product in the Frobenius algebra A via the pairing. $\square$

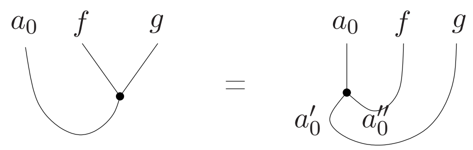

FIGURE 12. Duality of the cup product and coproduct

5.6. **String topology.** We apply in this section the results of the previous sections – particularly Theorem 4.9 and Corollary 5.13 – in order to control the (not entirely understood) operations in string topology in characteristic 0. Let $C^*(M)$ denote the *rational* singular cochain complex of a compact, oriented, simply connected manifold, and $H^*(M)$ its cohomology.

It is well known (see e.g. [12]) that there is an isomorphism

$$H^*(LM) \cong HH_*(C^*(M), C^*(M))$$

from the cohomology of the free loop space LM to the Hochschild homology of $C^*(M)$. $H_*(LM)$ is equipped with the structure of a (positive boundary) homological conformal field theory through papers beginning with Chas-Sullivan [4], and reaching its most complete form in the work of Godin [8].

We follow the prescription laid out by Lambrechts-Stanley [17] and Felix-Thomas [7] to construct this structure in Hochschild homology. We note that $C^*(M)$ is quasi-isomorphic to a (simply connected) commutative differential graded algebra A (e.g., the algebra of differential forms on M with rational coefficients), and that $H^*(A) \cong H^*(M)$ is a strict Frobenius algebra. Lambrechts-Stanley give a recipe for constructing, for any such A, a weakly equivalent algebra B which is itself a commutative differential graded Frobenius algebra; that is, B itself satisfies Poincaré duality prior to application of cohomology. Such an algebra B is a strict symmetric Frobenius algebra in our sense, that is it defines a functor $\Phi : H_0(O_d) \to \mathrm{Ch}$, for d the dimension of the manifold. In particular, we can apply Theorem 5.11 to B and get an action of Sullivan diagrams on its Hochschild homology.

Using the chain of isomorphisms

$$(*) \qquad H^*(LM) \cong HH_*(C^*(M), C^*(M)) \cong HH_*(A, A) \cong HH_*(B, B)$$

we get an action of Sullivan diagrams on $H^*(LM)$, and hence a homological conformal field theory (i.e. action of the moduli space of Riemann surfaces) by restriction. We do not know for sure that this (somewhat collapsed) action is the one constructed by Godin, but the next proposition says that it is an extension of Chas-Sullivan's string topology:

Proposition 5.15. *The co-BV operations on $H^*(LM,\mathbb{Q})$ dual to the string topology BV operations on $H_*(LM,\mathbb{Q})$ of* [4] *extend to an action of the closed part of $H_*(\mathcal{SD}_d,\mathbb{Q})$, for d the dimension of M.*

Proof. The co-BV structure (and Sullivan diagram structure) we define on $H^*(LM,\mathbb{Q})$ is defined via an action on $HH_*(B,B)$, hence it is equivalent to check that the action on $HH_*(B,B)$ is dual to the string topology action. By Proposition 5.14, our co-BV structure is dual to the BV structure on $HH^*(B,B)$ coming from the Hochschild cup product and the dual of Connes' operator B. Hence, by [7, Prop. 1], our structure is carried to the dual of the Chas-Sullivan structure by the isomorphism $(*)$. □

Our construction thus produces an HCFT structure on $H^*(LM,\mathbb{Q})$ for which a substantial part of the action is trivial (see Proposition 5.12). In particular, we know that all stable classes in the homology of the moduli space act trivially, a fact known in the string topology setting by work of Tamanoi [30].

Remark 5.16. It is worth issuing a caveat here: the main result of [21] implies that the BV structures on

$$H_*(LS^2;\mathbb{F}_2) \quad \textit{and} \quad HH^*(H^*(S^2;\mathbb{F}_2),H^*(S^2;\mathbb{F}_2))$$

cannot be isomorphic (even though the underlying Gerstenhaber structures are), if we equip $H^*(S^2;\mathbb{F}_2)$ with the Frobenius algebra structure coming from Poincaré duality. Consequently, we cannot expect the construction given above to yield the HCFT structure on string topology if done integrally.

5.7. **Hochschild homology of $\mathcal{A}_\infty$ algebras.** In this section, we briefly consider what our construction gives when applied to the category $\mathcal{E}=\mathcal{A}_\infty$, equipped with the identity functor $id:\mathcal{A}_\infty\to\mathcal{E}$.

Proposition 5.17. *The Hochschild complex $\overline{C}^p(\mathcal{A}_\infty(m,-))(n)$ is isomorphic to the (split) subcomplex of $(\bar{p},m+n)-$ Graphs consisting of fat graphs whose associated surface is a disjoint union of*

- *n disks, each with precisely one outgoing open boundary, and*
- *p annuli, each with precisely one closed outgoing boundary,*

and with m incoming open boundaries distributed on the free boundaries of these.

Proof. This is a restriction of Proposition 5.1 to $\mathcal{A}_\infty\subseteq\mathcal{O}$. The gluing map

$$\bigoplus_{n_i\geq 1}\mathcal{A}_\infty(m,n_1+\cdots+n_p+n)/U_I\otimes L_{n_1}\otimes\ldots\otimes L_{n_p}\to(\bar{p},m+n)-\text{Graphs}$$

produces graphs which are a disjoint union of trees and trees attached to white vertices (see Figure 13); the associated surfaces are as described. □

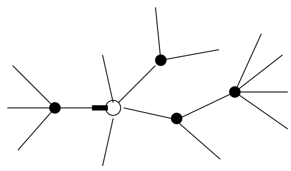

FIGURE 13. Annulus representing a morphism in $\mathcal{A}nn$ from open to closed

We therefore define an extension $\mathcal{A}nn$ of $\overline{C}\mathcal{A}_\infty$ to be the subcategory of $\mathcal{OC}$ consisting of graphs whose associated surface is a disjoint union of surfaces as in 5.17, or a closed-to-closed annulus. Note that we cannot introduce any closed-to-open annuli in $\mathcal{A}nn$, for composites would produce open-to-open morphisms that are not already present[11] in $\overline{C}\mathcal{A}_\infty$. As $\mathcal{A}nn$ is an extension of $\overline{C}\mathcal{A}_\infty$, by Theorem 4.9, we conclude:

Theorem 5.18. *For any $\mathcal{A}_\infty$–algebra A, the pair $(\overline{C}(A), A)$ is an $\mathcal{A}nn$–module.*

For simplicity, we will focus on $\mathcal{A}_\infty$–algebras, and examine the resulting $H_*(\mathcal{A}nn)$–structure on the pair $(HH_*(A,A), H_*(A))$.

$\mathcal{A}nn$ evidently contains $\mathcal{A}_\infty = \mathcal{A}nn \cap \mathcal{O}$, and so the open sector of an $\mathcal{A}nn$–module remains (unsurprisingly) an $\mathcal{A}_\infty$–algebra. This equips $H_*(A)$ with the structure of a unital associative ring. Write $m \in H_0(\mathcal{A}nn([\begin{smallmatrix}0\\2\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}]))$ for the class corresponding to the product, and $u \in H_0(\mathcal{A}nn([\begin{smallmatrix}0\\0\end{smallmatrix}], [\begin{smallmatrix}0\\1\end{smallmatrix}]))$ for the class corresponding to the unit.

Furthermore, since the mapping class group of an annulus with fixed boundaries is isomorphic to $\mathbb{Z}$, generated by the Dehn twist, the morphism complex $\mathcal{A}nn([\begin{smallmatrix}1\\0\end{smallmatrix}], [\begin{smallmatrix}1\\0\end{smallmatrix}])$ is quasi-isomorphic to $C_*(B\mathbb{Z}) = C_*(S^1)$. Up to homotopy, the only nontrivial operation $[\begin{smallmatrix}1\\0\end{smallmatrix}] \to [\begin{smallmatrix}1\\0\end{smallmatrix}]$ is thus a class Δ of degree 1, corresponding to the fundamental class of the circle. This is Connes' operator B explicitly given at the end of Section 5.5 (see Proposition 5.14)[12].

One should also consider the interaction of the open and closed sectors. There are no closed-to-open morphisms in $\mathcal{A}nn$, but there is a class $i \in H_0(\mathcal{A}nn([\begin{smallmatrix}0\\1\end{smallmatrix}], [\begin{smallmatrix}1\\0\end{smallmatrix}]))$ coming from the annulus with one open incoming and one closed outgoing boundary. This map $i : H_*(A) \to HH_*(A,A)$ is induced by the quotient map $A \to HH_0(A,A)$.

Proposition 5.19. *The category $H_*(\mathcal{A}nn)$ is generated as a symmetric monoidal category by the operations m, u, Δ, and i.*

[11] Similarly there are no disks with a closed incoming boundary, since compositions would produce an open-to-open morphism with codomain 0.

[12] Note here that the formula is the same for $\mathcal{A}_\infty$–algebras as for strictly associative algebras as there are no black vertices in the graph generating the operation Δ.

Remark 5.20. The Hochschild complex of a category $\mathcal{E}$ is functorial in $\mathcal{E}$; furthermore, it is not hard to see that a monoidal quasi-isomorphism $\mathcal{E} \to \mathcal{E}'$ induces a quasi-isomorphism of Hochschild complexes (using, e.g. the spectral sequence of a bicomplex). Consequently the results above apply equally to the category associated to the operad Ass of associative algebras, since it is quasi-isomorphic to $\mathcal{A}_\infty$.

5.8. **Algebras over $\mathcal{E} = Ass \otimes \mathcal{P}$ for an operad $\mathcal{P}$.** Let $\mathcal{P}$ be a chain operad, and consider the operad $Ass \otimes \mathcal{P}$ whose algebras are associative (unital) algebras together with a commuting $\mathcal{P}$–algebra structure. By the work of Brun, Fiedorowicz, and Vogt [3], if $\mathcal{P}$ is the chain complex of the little disks operad $\mathcal{C}_n$, the resulting tensor product is an E_{n+1}–operad. Furthermore, they show that the Hochschild complex of an $Ass \otimes \mathcal{P}$–algebra admits the structure of a $\mathcal{P}$–algebra.

Explicitly, the action of $\mathcal{P}$ on $C_*(A)$ is as follows: As A is a unital associative algebra, we can consider $C_*(A)$ as the chain complex associated to a simplicial chain complex $A_\bullet$ with $A_p = A^{\otimes p+1}$ and degeneracy s_i inserting a unit in position $i+1$. The $Ass \otimes \mathcal{P}$–structure of A defines a simplicial $\mathcal{P}$–structure on $A_\bullet$ by acting diagonally on $A^{\otimes p+1}$, and this in turn induces a $\mathcal{P}$–structure on the associated total chain complex $C_*(A)$. This last structure can be made explicit via the Eilenberg-Zilber maps. The action of a chain $p \in \mathcal{P}(k)$ on $(a_0^1 \otimes \ldots \otimes a_{p_1}^1) \otimes \ldots \otimes (a_0^k \otimes \ldots \otimes a_{p_k}^k)$ is of the form

$$\sum \pm\, p(a_0^1, \ldots, a_0^k) \otimes p(1, \ldots, a_1, \ldots, 1) \otimes \ldots \otimes p(1, \ldots, a_{p_1+\cdots+p_k}, \ldots, 1),$$

where the sum is over all possible shuffles of $(a_1^1, \ldots, a_{p_1}^1), \ldots, (a_1^k, \ldots, a_{p_k}^k)$, with the resulting sequence denoted $a_1, \ldots, a_{p_1+\cdots+p_k}$, and $p(1, \ldots, a_i, \ldots, 1)$ means take $a_i = a_k^j$ at the jth position and 1's everywhere else.

By the results of the previous section, $HH_*(A, A)$ is a $H_*(\mathcal{A}nn)$–module. It is natural, then, to ask how this interacts with the Brun-Fiedorowicz-Vogt $\mathcal{P}$–algebra structure. Comparing the above formula with the formula for Connes' B operator (given at the end of Section 5.5) shows though that these two structures do not interact very well, in particular because of the special role of the a_0^j's in the $\mathcal{P}$–action. One can though define an extension of the category $Ass \otimes \mathcal{P}$ with the free operad generated by $\mathcal{P}$ and B as "closed-to-closed" morphisms, subject to the relations in $\mathcal{P}$ and $B^2 = 0$.

6. Appendix: How to compute signs

Let $\Phi : \mathcal{E} \to \mathrm{Ch}$ be a split monoidal functor for $\mathcal{E} = \mathcal{O}, \mathcal{O}_d, \mathcal{OC}$ or $\mathcal{OC}_d$, with $\Phi(1) = A$ an $\mathcal{A}_\infty$–Frobenius algebra. Given an *oriented* graph Γ which is a morphism in $\mathcal{E}$, we want to read off an explicit formula of the associated operation on A or $C_*(A, a)$ *with signs.* The explicit formula will be given in terms of a chosen set of generating operations for $\mathcal{O}$, for example in terms of the (co)product and higher (co)products, the unit and the trace in $\mathcal{O}$ (or $\mathcal{O}_d$), and additionally the generator l_n of Figure 2 for $\mathcal{OC}$ (or $\mathcal{OC}_d$).

To be precise, one first needs to make a choice of which orientation should be thought of as the "positive" orientation for the graphs representing the chosen basic operations. For the products and coproducts, we choose here the orientation $v \wedge h_1 \wedge \ldots \wedge h_k$ for v the vertex and $h_1, \ldots, h_k$ the half edges in their cyclic order starting at the first incoming half-edge. The unit and the trace are exceptional graphs with a canonical positive orientation. For l_k, we take the orientation $w \wedge h_1 \wedge \ldots \wedge h_k$ for w the vertex, $h_1, \ldots, h_k$ the half edges in their cyclic order starting at the start half-edge.

Figure 14 gives as an example the convention we will use for the product in an algebra.

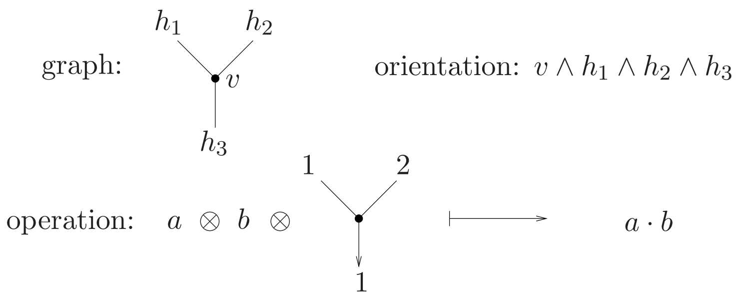

FIGURE 14. Sign convention for the product

Given a graph Γ, we first need to write it as a composition of the chosen generating operations. This means choosing an orientation of the internal edges and an ordering of the vertices, possibly introducing new vertices together with unit or trace operations, and possibly using the symmetries of the category. (See Figure 15 below for an example, and the proof of Proposition 2.3 for the case of $\mathcal{O}^b$.) Suppose Γ has vertices $v_1, \ldots, v_k$ with half-edges $h^i_1, \ldots, h^i_{n_i}$ at v_i and $v_i \wedge h^i_1 \wedge \ldots \wedge h^i_{n_i}$ the chosen orientation of the (chosen) operation μ_i associated to v_i. To interpret Γ as a composition of the operation at v_1, then at v_2 etc. requires writing the orientation of Γ as $\pm(v_1 \wedge h^1_1 \wedge \ldots \wedge h^1_{n_1}) \wedge \ldots \wedge (v_k \wedge h^k_1 \wedge \ldots \wedge h^k_{n_k})$.

Suppose we start from

$$a_1 \otimes \ldots \otimes a_n \otimes (\Gamma, o_d(\Gamma))$$

in $A^{\otimes n} \otimes \mathcal{O}_d(n, m)$, with Γ as above and

$$o_d(\Gamma) = (v_1 \wedge h^1_1 \wedge \ldots \wedge h^1_{n_1}) \wedge \ldots \wedge (v_k \wedge h^k_1 \wedge \ldots \wedge h^k_{n_k}) \otimes \det(\Gamma, \partial_{out})^{\otimes d}.$$

We rewrite this (with a Koszul sign!) as

$$a_1 \otimes \ldots \otimes a_n \otimes \big((v_1 \wedge h^1_1 \wedge \ldots \wedge h^1_{n_1}) \otimes \det(\mu_1)^{\otimes d}\big) \otimes \ldots \otimes \big((v_k \wedge h^k_1 \wedge \ldots \wedge h^k_{n_k}) \otimes \det(\mu_k)^{\otimes d}\big)$$

in $A^{\otimes n} \otimes \mathcal{O}_d(n, p_1) \otimes \ldots \otimes \mathcal{O}_d(p_r, m)$, from which we can apply the first operation and then the next etc. The final sign for the operation will come, in addition, from the signs occurring when using the symmetries in the category.

If the graph was an operation in $\mathcal{OC}_d$ instead, that is if we start with

$$(a_0^1 \otimes \ldots \otimes a_{k_1}^1 \otimes l_{k_1}) \otimes \ldots \otimes (a_0^{n_1} \otimes \ldots \otimes a_{k_n}^n \otimes l_{k_n}) \otimes \; b_1 \otimes \ldots \otimes b_m \otimes (\Gamma, o_d(\Gamma))$$

in $C(A,A)^{\otimes n}) \otimes A^{\otimes m} \otimes \mathcal{OC}_d([{n \atop m}],[{n' \atop m'}])$, the principle is the same, but we have in addition to apply the procedure described in Section 5.2.

We now give an explicit example with a graph of $\mathcal{O}_d(2,1)$ which is used in the computations at the end of section 5.5. In Figure 15, we give a graph with a choice of ordering of its vertices v_1, v_2, v_3, and a choice of orientation of its internal edges e_1, e_2, e_3. We choose the orientation of the graph that corresponds to writing it as a composition of the operation attached to v_1 (a coproduct), followed by the operation attached to v_2 and then v_3 (both products). Explicitly, it is given as

$$(v_1 \wedge h_1 \wedge e_1 \wedge e_2) \wedge (v_2 \wedge \bar{e}_2 \wedge h_2 \wedge e_3) \wedge (v_3 \wedge \bar{e}_1 \wedge \bar{e}_3 \wedge \bar{h}_1)$$

where e_i and $\bar{e}_i$ are the start and end half-edges of e_i, h_i is the ith incoming leaf, and $\bar{h}_1$ is the outgoing leaf.

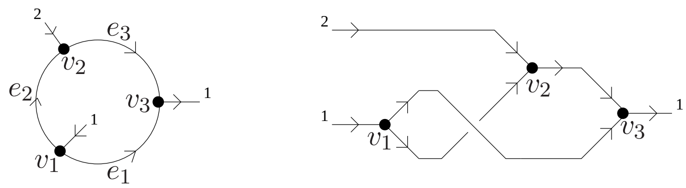

FIGURE 15. Writing a graph as a composition

The graph has relative Euler characteristic $\chi(\Gamma, \partial_{out}) = -1$ which is also the relative Euler characteristic $\det(c)$ of the coproduct, while the products have trivial relative Euler characteristic. As the products have degree 0, moving the determinant past the products does not produce a sign and the operation associated to Γ with the above orientation is that of the composition

$$\big(((v_1 \wedge h_1 \wedge e_1 \wedge e_2) \otimes (\det c)^{\otimes d}) \oplus id\big) \otimes (\tau \oplus id) \otimes (v_2 \wedge \bar{e}_2 \wedge h_2 \wedge e_3) \otimes (v_3 \wedge \bar{e}_1 \wedge \bar{e}_3 \wedge \bar{h}_1)$$

in $(\mathcal{O}_d(1,2) \oplus \mathcal{O}_d(1,1)) \otimes \mathcal{O}_d(3,3) \otimes \mathcal{O}_d(3,2) \otimes \mathcal{O}_d(2,1)$, where τ denotes the twist map.

The succession of operations (a comultiplication, a twist and two multiplications) applied to an pair $a \otimes b$ is

$$\begin{array}{rcl} a \otimes b & \mapsto & (-1)^{|b|d} \sum a' \otimes a'' \otimes b \\ & \mapsto & (-1)^{|b|d+|a'||a''|} \sum a'' \otimes a' \otimes b \\ & \mapsto & (-1)^{|b|d+|a'||a''|} \sum a'' \otimes a'b \\ & \mapsto & (-1)^{|b|d+|a'||a''|} \sum a''a'b \end{array}$$

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