TCFT AND CALABI-YAU CATEGORY

Fix $\Lambda = \text{set of D-branes}$

$M^\Lambda = \text{topological category for "open-closed TCFT"}$

$\text{Obj}(M^\Lambda) = \{ (c, \theta, s, t) \mid c, \theta \in \mathbb{Z}_{\geq 0}, s, t, \theta \rightarrow \Lambda \}$

Closed/Open Boundary/Intervals/Strings

Morphism Spaces = Moduli Spaces of Riemann Surfaces with Open, Closed and/or Free Boundaries

Labelled by Elements of $\Lambda$

Incoming and Outgoing

EX:

$\begin{array}{c}
\text{1} \quad \text{2} \\
\text{3} \quad \text{4} \\
\text{5} \quad \text{6} \\
\text{7} \quad \text{8} \\
\text{9} \quad \text{10}
\end{array}$

$(1, [a, m]) \rightarrow ([m, a], [a, v], [v, a])$

Special Conditions:

1. $\Sigma$ has at least one incoming closed boundary or one free boundary.

2. $\odot$ and $\circ^m$ are allowed but with moduli replaced by $\star$ (because of instability).

3. Cylinders are moduli replaced by $\text{Diff}^+$ to have identity "infinite" moduli.

$M^\Lambda$ is symmetric monoidal under $\otimes$. 
\( \mathcal{O}_d^\Lambda = C_{\ast}(M^\Lambda, \text{det} \otimes d) \) \hspace{1cm} \text{sym. monoidal cat.}

\text{det local system on } M^\Lambda(q, 6)

\text{det } (\mathcal{E}) = \text{det } (H^0 \mathcal{E}) \otimes \text{det } (H^1 \mathcal{E}) \otimes \text{det } (H^0 \mathcal{E})

\text{satisfied } \det (\mathcal{E}_2 \otimes \mathcal{E}_1) \simeq \det (\mathcal{E}_2) \otimes \det (\mathcal{E}_1)

\det (\mathcal{E}_2 \otimes \mathcal{E}_1) \simeq

\mathcal{O}_d^\Lambda = \text{full subcategory with no closed boundary}

\mathcal{O}_d^\Lambda = \text{no open or free boundary}

\text{DEF: An open TCFT of dimension } d \text{ is a pair } (\Lambda, \Phi) \text{ where } \Lambda = \text{set of D-branes} \text{ and } \Phi: \mathcal{O}_d^\Lambda \rightarrow \text{Comp k} \text{ is } h\text{-split is. } \exists \text{ q-iso } \Phi(a) \otimes \Phi(b) \sim \Phi(a \otimes b) \quad a, b

\text{HTM (1) The category of open TCFT's of dimension } d \text{ with fixed set of D-branes } \Lambda \text{ is } h\text{-equiv. to the category of (unital) extended Calabi-Yau A}\infty\text{-categories of dimension } d \text{ with set of objects } \Lambda \text{.}

\text{DEF: A unital extended CY A}\infty\text{-category with obj. } \Lambda \text{ is an } h\text{-split functor } \Phi: D^d_{\text{open}, \Lambda} \rightarrow \text{Comp k}

\text{HTM: } D^d_{\text{open}, \Lambda} \sim \mathcal{O}_d^\Lambda

\text{Is. a zig-zag of functors which are q-iso's on the morphism complexes (have same objects)}
\( \forall \alpha, \beta \in \text{Ob}(\mathcal{M}^\Sigma) \) s.t. \( \alpha \) has no closed part, construct \( \tilde{G}(\alpha, \beta) \sim M_\alpha(\alpha, \beta) \) cellular cpx \( A_{\text{open}} \).\[ D_{\text{open}}^1 = C_\ast \left( \tilde{G}(\alpha, \beta)_{\text{open}}, \det \otimes d \right) \]

\[ \tilde{N}(\alpha, \beta) = \text{Moduli space of Riemann surfaces with} \]
- closed smooth boundary
- free possibly singular boundary with \( \delta(\alpha) + \delta(\beta) \)
- marked points representing inc. and cut. open bord.
- stability restrictions as before

\[ \tilde{G}(\alpha, \beta) \subset \tilde{N}(\alpha, \beta) \] subspace of surfaces whose irr. components are discs or annulus of modulus 1 with one outgoing closed bdy.

\( (\text{no annuli} \rightarrow \text{dual graph = fat graph}) \)

(partial) composition by identifying incoming and outgoing points, except for \( \rightarrow \circ \circ = \text{id} \)

\( G_{\text{open}}^{\text{KMS}} \) = category with \( n \) closed boundaries

\( D_{\text{open}}^1 = C_\ast \left( G_{\text{open}}, \det \right) \)

\( D^+_{\text{open}} \subset D_{\text{open}}^1 \) subcategory generated by \( \delta \text{-in} \text{d} \) under \( 0 \text{ and } \Pi \)

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\[ D^+(\alpha_0, \ldots, \alpha_n) \]
PROPOSITION
(i) \( \Phi : D^+ \to \text{Comp Split} \) \( \iff \) \( \text{Unital A\omega CAT.} \) with \( \text{Obj} = \Lambda \)

(ii) \( \Phi : D^+ \to \text{Comp Split} \) \( \iff \) \( \text{Unital CY A\omega CAT.} \) with \( \text{Obj} = \Lambda \)

\( \Phi \) splits if \( \Phi(a) \circ \Phi(b) \Rightarrow \Phi(a \circ b) \)

PROOF (i) GIVEN \( \Phi \) WANT TO CONSTRUCT \( B \)

\[ B(\lambda_0, \lambda_1) := \Phi([\lambda_0, \lambda_1]) \]

\[ B(\lambda_0, \lambda_1) \circ B(\lambda_1, \lambda_2) \cdots \circ B(\lambda_{n-1}, \lambda_n) \xrightarrow{m_n} B(\lambda_0, \lambda_n) \]

\[ \Phi([\lambda_0, \lambda_1]) \cdots \Phi([\lambda_{n-1}, \lambda_n]) \]

(ii) FOR EACH \( \lambda_0, \lambda_1 \in \text{Obj}(B) \) 3 NON-DEG. PAIRING

\[ \langle \lambda_0, \lambda_1 : \text{Hom}(\lambda_0, \lambda_1) \otimes \text{Hom}(\lambda_1, \lambda_0) \to WK[d] \]

\[ \text{which is symmetric } \langle \lambda_0, \lambda_1 \rangle = \langle \lambda_1, \lambda_0 \rangle \text{ AND st.} \]

\[ \langle m_{n-1}(f_1 \circ \cdots \circ f_{n-1}), f_n \rangle = (-1)^{n+1} f_1^{\circ n} \frac{\sum_{i=2}^{n} f_i^{\circ n-1}}{f_1^{\circ n-1}} \left\langle m_{n-1}(f_2 \circ \cdots \circ f_n), f_1 \right\rangle \]

GIVEN \( \Phi([\lambda_0, \lambda_1]) \otimes \Phi([\lambda_1, \lambda_0]) \to WK[d] \)

\[ \Phi(\lambda_0) \xrightarrow{\lambda_1} \Phi(\lambda_1) \]

WITH INVERSE \( \Phi(\lambda_1) \xleftarrow{\lambda_0} \Phi(\lambda_0) \)

WHICH ARE THE TWO EXTRA GENERATORS IN \( D^+ \)
THM (2) \( \Phi : \mathcal{O}^d \to \text{Comp} \) \( h \)-split (open TCFT)

\[ \text{Li}_X \Phi : \mathcal{O}^d \to \text{Comp} \] \( h \)-split (open-closed TCFT)

\[ j^* \text{Li}_X \Phi : \mathcal{C}^d \to \text{Comp} \] \( h \)-split (closed TCFT)

(3) Let \( \text{HH}^+_X(\Phi) \) denote the Hochschild homology of the \( A_\infty \)-category associated to \( \Phi \). Then

\[ H_x(Li_X \Phi)(0, c) = H_x(\Phi \circ 0) \otimes \text{HH}^+_X(\Phi) \]

**Cor.** \( H_x(\text{smooth spaces}) \) act on \( \text{HH}^+_X(\text{CY } A_\infty \text{-category}) \)

**Ex:** A Frobenius algebra \( A \) \( A_\infty \)-calabiyan cat with i object

\[ \text{HH}^+_X(A) \text{ is } H_x \text{(closed TCFT)} \]

\[ \begin{array}{c}
\Phi : \mathcal{O}^d \\
\downarrow F \Phi \\
\text{Comp}
\end{array} \]

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\downarrow F \Phi = L F \Phi \\
\text{Comp}
\end{array} \]

\[ (F \Phi(d)) = \bigoplus_{c_1, \ldots, c_p} \Phi(c_1) \otimes \Phi(c_1, c_1) \otimes \cdots \otimes \Phi(c_p, c_p) \otimes D(F(c), d) \]

\[ \Phi \text{ is a complex with } d = d_+ + d_{\text{ext}} \]

**B dg \( A_\infty \)-category. Define**

\[ C_*(B) = \bigoplus_{a_0, \ldots, a_n} (B(a_0, a_1) \otimes \cdots \otimes B(a_n, a_0)) [-n] \]

with

\[ d(F_0 \otimes \cdots \otimes F_n) = \sum_{i=0}^n \pm f_0 \otimes \cdots \otimes df_i \otimes \cdots \otimes f_n \]

\[ + \sum_{i=0}^{n-1} \pm f_0 \otimes \cdots \otimes (f_i \otimes f_{i+1}) \otimes \cdots \otimes f_n \pm (f_0 \otimes f_n) \otimes \cdots \otimes f_{n-2} \]

\[ \to \text{HH}^+_X(B) = H_*(C_*(B)) \]
Proof of Thin (3):

\[ \Phi : D_{\text{open}} \to \text{Comp} \]

Let \( \Phi \) be the closed curve.

\[ \{ \text{Li} \Phi (1) \} = \bigoplus_{p=0}^{a_0 \ldots a_p} \Phi (a_o) \circ \Theta (a_o, a_1) \circ \ldots \circ \Theta (a_{p-1}, a_p) \circ \Theta (a_p, 1) \]

Claim: This is an equivalence because \( D^d = D^d \) \( [ \Phi, \bar{\Phi} ] \)

\[ D^+ = D^+ \left[ \bigoplus_{p=0}^{a_0 \ldots a_p} \Phi (a_o) \circ D^d_{\text{open}} (a_o, a_1) \circ \ldots \circ D^d_{\text{open}} (a_{p-1}, a_p) \circ D^d (a_p, 1) \right] \]

\( D^d (a_p, 1) \) is cyclic if \( a_p \) is not cyclic!

\( D^d (a_p, 1) = \emptyset \) if \( a_p \) is cyclic.

\( D^d_{\text{open}} (a_o, a_1) \circ \ldots \circ D^d_{\text{open}} (a_{p-1}, a_p) \circ D^d (a_p, 1) \) non-empty if \( a_0 > a_1 > \ldots > a_p \) is cyclic.

Example: \( b = \{ \ldots, a_1, a_2, a_3, \ldots, b \} \) is cyclic iff \( a > b \) is also cyclic.

\( D^d_{\text{open}} (a_1, b) \neq \emptyset \) when \( b \) is cyclic.
\[ C_*(\mathcal{B}(\mathcal{F})) \cong \bigoplus_{p=0}^n (\mathcal{B}(\Lambda_0, \Lambda) \otimes \cdots \otimes \mathcal{B}(\Lambda_p, \Lambda_0)) \] for \[ p \geq 0 \quad \Lambda_0, \ldots, \Lambda_p \]

\[ \cong \bigoplus_{p=0}^n (\mathcal{F}(\Lambda_0, \Lambda_p) \otimes \cdots \otimes \mathcal{F}(\Lambda_p, \Lambda_0)) \] for \[ p \geq 0 \quad \Lambda_0, \ldots, \Lambda_p \]

Claim: Can assume split \[ \mathcal{F}(\Lambda_0, \ldots, \Lambda_p^\infty) \]

My essentially the same vector space showing up with the same differential ...