The homology of the Higman–Thompson groups

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November 2016

We prove that Thompson's group V is acyclic, answering a 1992 question of Brown in the positive. More generally, we identify the homology of the Higman–Thompson groups $V_{n,r}$ with the homology of the zeroth component of the infinite loop space of the mod $n-1$ Moore spectrum. As $V = V_{2,1}$, we can deduce that this group is acyclic. Our proof involves establishing homological stability with respect to r, as well as a computation of the algebraic K-theory of the category of finitely generated free Cantor algebras of any type n.

MSC: 19D23, 20J05.

Keywords: Higman–Thompson groups, Cantor algebras, homological stability, stable homology.

Introduction

About half a century ago, Thompson introduced a group V together with subgroups $F \leqslant T \leqslant V$ in order to construct examples of finitely presented groups with unsolvable word problem. Thompson's groups have since developed a life of their own, relating to many branches of mathematics. The homology of the group F was computed by Brown and Geoghegan [BG84]; it is free abelian of rank 2 in all positive degrees. The homology of the group T was computed by Ghys and Sergiescu [GS87]; it is isomorphic to the homology of the free loop space on the 3-sphere. As for Thompson's group V itself, Brown [Bro92] proved that it is rationally acyclic and suggested that it might even be integrally so. In the present paper, we prove that V is indeed integrally acyclic.

Thompson's group V fits into the more general family of the Higman–Thompson groups $V_{n,r}$ for $n \geqslant 2$ and $r \geqslant 1$, with $V = V_{2,1}$ as the first case: A Cantor algebra of type n is a set X equipped with an isomorphism $X^n \cong X$, and the Higman–Thompson group $V_{n,r}$ is the automorphism group of the free Cantor algebra $C_{n,r}$ of type n on r generators. The main result of this text is an identification of the homology of all of the groups $V_{n,r}$ in terms of a well-known object of algebraic topology: the mod $n-1$ Moore spectrum $\mathbb{M}_{n-1}$.

Theorem A. For any $n \geqslant 2$ and $r \geqslant 1$ there is a map $\mathrm{BV}_{n,r} \to \Omega^\infty_0 \mathbb{M}_{n-1}$ inducing an isomorphism

$$\mathrm{H}_*(\mathrm{V}_{n,r}; M) \xrightarrow{\cong} \mathrm{H}_*(\Omega^\infty_0 \mathbb{M}_{n-1}; M).$$

in homology for any trivial or abelian coefficient system M.

Here the space $\Omega^\infty_0 \mathbb{M}_{n-1}$ is the zeroth component of the infinite loop space $\Omega^\infty \mathbb{M}_{n-1}$ that underlies the mod $n-1$ Moore spectrum $\mathbb{M}_{n-1}$. Note that the target of the isomorphism does not depend on r.

In the case $n = 2$, the spectrum $\mathbb{M}_{n-1}$ is contractible and the above result answers Brown's question:

Corollary B (Theorem 6.4). Thompson's group $\mathrm{V} = \mathrm{V}_{2,1}$ is acyclic.

In [Bro92], Brown indicates that his argument for the rational acyclicity of V extends to prove rational acyclicity for all groups $\mathrm{V}_{n,r}$. When n is odd, the group $\mathrm{V}_{n,r}$ was known not to be integrally acyclic just from the computation of its first homology group, which is $\mathbb{Z}/2$ in that case. Our main theorem applied to the case $n \geqslant 3$ completes the picture, giving a proof of rational acyclicity for all $\mathrm{V}_{n,r}$, and at the same time showing that integral acyclicity only holds in the special case $n = 2$:

Corollary C (Theorem 6.5). For all $n \geqslant 3$, the group $\mathrm{V}_{n,r}$ is rationally but not integrally acyclic.

The proof of our main theorem rests on two pillars. The first is homological stability: For any fixed $n \geqslant 2$, the Higman–Thompson groups $\mathrm{V}_{n,r}$ fit into a canonical diagram

$$\mathrm{V}_{n,1} \longrightarrow \mathrm{V}_{n,2} \longrightarrow \mathrm{V}_{n,3} \longrightarrow \cdots \qquad (\star)$$

of groups and we show that the maps $\mathrm{V}_{n,r} \to \mathrm{V}_{n,r+1}$ induce isomorphisms in homology for large r in any fixed homological degree. The definition of Cantor algebras leads to isomorphisms $\mathrm{C}_{n,r} \cong \mathrm{C}_{n,r+(n-1)}$ for all $n \geqslant 2$ and $r \geqslant 1$, so that there are isomorphisms $\mathrm{V}_{n,r} \cong \mathrm{V}_{n,r+(n-1)}$ for all $n \geqslant 2$ and $r \geqslant 1$. Using these isomorphisms, we obtain that the stabilization maps are actually isomorphisms in homology in *all* degrees, see Theorem 3.6.

To prove homological stability, we use the framework of [R-WW]. The main ingredient for stability is the proof of high connectivity of a certain simplicial complex of independent sets in the free Cantor algebra $\mathrm{C}_{n,r}$. It follows from [R-WW] that homological stability also holds with appropriate abelian and polynomial twisted coefficients.

Our stability theorem can be reformulated as saying that the map

$$\mathrm{V}_{n,r} \longrightarrow \bigcup_{r \geqslant 1} \mathrm{V}_{n,r} = \mathrm{V}_{n,\infty}$$

is a homology isomorphism, where the union is defined using the maps in the diagram $(\star)$, and our second pillar is the identification of the homology of $\mathrm{V}_{n,\infty}$. This is achieved by the identification of the classifying spaces of the groupoid of free Cantor algebras of type n, as we describe now.

Let $\mathbf{Cantor}_n^\times$ denote the category of free Cantor algebras of type n with morphisms their isomorphisms. The category $\mathbf{Cantor}_n^\times$ is symmetric monoidal, and hence has an associated spectrum $\mathbb{K}(\mathbf{Cantor}_n^\times)$, its algebraic K-theory. We denote by $\Omega_0^\infty\mathbb{K}(\mathbf{Cantor}_n^\times)$ the zeroth component of its associated infinite loop space. Applying the group completion theorem, we get a map

$$\mathrm{BV}_{n,\infty} \longrightarrow \Omega_0^\infty\mathbb{K}(\mathbf{Cantor}_n^\times),$$

defined up to homotopy, that induces an isomorphism in homology with all local coefficient systems on the target (see Theorem 5.4). Now using a model of Thomason, we identify the classifying space $|\mathbf{Cantor}_n^\times|$ with that of a homotopy colimit $\mathbf{Tho}_n$ in symmetric monoidal categories build out of the category of finite sets and the functor that takes the product with a set of cardinality n:

$$|\mathbf{Cantor}_n^\times| \simeq |\mathbf{Tho}_n|,$$

where the equivalence respects the monoidal structure, see Theorem 4.1. In particular, the two categories have equivalent algebraic K-theory spectra: $\mathbb{K}(\mathbf{Cantor}_n^\times) \simeq \mathbb{K}(\mathbf{Tho}_n)$. The main theorem then follows from an identification

$$\mathbb{K}(\mathbf{Tho}_n) \simeq \mathbb{M}_{n-1},$$

see Theorem 5.1. The idea behind the last two equivalences is as follows. The category $\mathbf{Tho}_n$ is a homotopy mapping torus of the functor, defined on the category of finite sets and bijections, that takes the product with a set of size n. Thinking of the finite sets as the generating sets of free Cantor algebras of a given type n, this functor implements, for any r, the identification of a Cantor algebra $\mathrm{C}_{n,r}$ with the Cantor algebra $\mathrm{C}_{n,rn} = \mathrm{C}_{n,r+r(n-1)}$, which reflects the defining property of Cantor algebras. In spectra, this mapping torus equalizes multiplication by n with the identity on the sphere spectrum, which leads to the Moore spectrum $\mathbb{M}_{n-1}$.

We give in the last section of the paper some explicit consequences of our main theorem. In particular, we confirm and complete the known information about the abelianizations and Schur multipliers of the groups $\mathrm{V}_{n,r}$ (Propositions 6.1 and 6.3), and compute the first non-trivial homology group of $\mathrm{V}_{n,r}$ (Proposition 6.2). When n is odd, the commutator subgroups $\mathrm{V}_{n,r}^+$ is an index two subgroup, and our methods can also be applied to study this group (Corollary 6.7).

The paper is organized as follows. In Section 1, we introduce the Cantor algebras and the Higman–Thompson groups, and we give some of their basic properties that we will need later in the paper. In Section 2, we show how the groups $\mathrm{V}_{n,r}$ fit into the set-up for homological stability of [R-WW] and construct the spaces relevant to the proof of homological stability, which is given in Section 3. The following Section 4 is devoted to the homotopy equivalence $|\mathbf{Cantor}_n^\times| \simeq |\mathbf{Tho}_n|$, which is given as a composition of three homotopy equivalences. Section 5 then relates the first of these two spaces to $\mathrm{V}_{n,\infty}$ and the second to the Moore spectrum; the section ends with the proof of the main theorem. Finally, Section 6 draws the computational consequences of our main theorem.

1 Cantor algebra

In this section, we recall some facts we need about Cantor algebras, and their groups of automorphisms, the Higman–Thompson groups. We follow Higman's own account [Hig74]. See also [Bro87, Sec. 4] for a shorter survey.

Let A be a finite set of cardinality at least 2. Let

$$A^* = \bigsqcup_{n\geqslant 0} A^n$$

denote the word monoid on the set A. This is the free monoid generated by A, with unit $A^0 = \emptyset$ and multiplication by juxtaposition. By freeness, an action of the monoid A^* on a set S is determined by a map of sets $S \times A \to S$. Such a map has an adjoint

$$S \longrightarrow S^A = \mathrm{Map}(A, S). \tag{1.1}$$

Definition 1.1. A *Cantor algebra of type A* is an A^*–set S such that the adjoint structure map (1.1) is a bijection.

Such objects go also under the name *Jónsson–Tarski algebras*.

For any finite set X, there exists a free Cantor algebra $\mathrm{C}_A(X)$ of type A generated by X. It can be constructed from the free A^*–set with basis X, namely the set

$$\mathrm{C}_A^+(X) := X \times A^* = \bigsqcup_{n\geqslant 0} X \times A^n$$

by formally adding elements so as to make the adjoint of the map defining the action bijective. (See [Hig74, Sec. 2]; the set $\mathrm{C}_A^+(X)$ is denoted $X\langle A\rangle$ in [Hig74].) Throughout the paper we will work with free Cantor algebras, but only the elements of the canonical free A^*–set $\mathrm{C}_A^+(X) \subset \mathrm{C}_A(X)$ will play a direct role.

When $A = \{1, \ldots, n\}$ and $X = \{1, \ldots, r\}$, we will sometimes shorten the notation and just write $\mathrm{C}_{n,r}$ for $\mathrm{C}_A(X)$ and $\mathrm{C}_{n,r}^+$ for $\mathrm{C}_A^+(X)$.

Our main object of study, the Higman–Thompson groups $\mathrm{V}_{n,r}$, are the automorphism groups of the free Cantor algebras $\mathrm{C}_{n,r}$:

Definition 1.2. Given $n \geqslant 2$ and $r \geqslant 1$, the *Higman–Thompson group*

$$\mathrm{V}_{n,r} = \mathrm{Aut}(\mathrm{C}_{n,r})$$

is the automorphism group of the free Cantor algebra of type n on r generators.

1.1 Isomorphisms via bases and expansions

We need to understand isomorphisms of Cantor algebras. By freeness, a map of Cantor algebras from $\mathrm{C}_A(X)$ to a Cantor algebra S is determined by its value on the generating set X. For instance, the canonical map $X \times A \hookrightarrow \mathrm{C}_A^+(X) \hookrightarrow \mathrm{C}_A(X)$ induces a map $\mathrm{C}_A(X \times A) \to \mathrm{C}_A(X)$, which one can show is an isomorphism, using the Cantor algebra structure map of $\mathrm{C}_A(X \times A)$. We use Higman's description of isomorphisms in terms of expansions of the generating sets, which we recall now. We start by defining bases and expansions.

Definition 1.3. A set $S \subset \mathrm{C}_A(X)$ is called a *basis* for $\mathrm{C}_A(X)$ if the induced map $\mathrm{C}_A(S) \to \mathrm{C}_A(X)$ is an isomorphism.

Definition 1.4. Given a subset Y of a free Cantor algebra $\mathrm{C}_A(X)$, an *expansion* of Y is a subset of $\mathrm{C}_A(X)$ obtained from Y by applying a sequence of *simple expansions*, where a simple expansion replaces one element $y \in Y$ by the elements $\{y\} \times A \subset \mathrm{C}_A^+(Y) \subset \mathrm{C}_A(X)$, its "descendants" in $\mathrm{C}_A(X)$.

If Y was a basis, then so is any of its expansions [Hig74, Lem. 2.3]. In particular, all the expansions of X represent bases for $\mathrm{C}_A(X)$, and these are the bases we will work with. Note that any such basis is a finite subset of $\mathrm{C}_A^+(X)$. One can think of $\mathrm{C}_A^+(X)$ as the vertices of an infinite $|A|$–ary forest with roots the elements of X, and expansions of X as sets of leaves of finite $|A|$–ary forests, see for instance [Bro87, Sec. 4] for more details on this point of view.

Higman proved the following important fact about bases: Any two finite bases of a free Cantor algebra have a common expansion, see [Hig74, Cor. 1, p. 12]. It follows in particular that the cardinality of a finite basis for $\mathrm{C}_A(X)$ is congruent to $|X|$ modulo $|A|-1$. In fact, we have that $\mathrm{C}_A(X) \cong \mathrm{C}_A(Y)$ if and only if $X = \emptyset = Y$ or if X and Y are both non-empty of cardinality congruent modulo $|A|-1$; this is the condition that guarantees that the two Cantor algebras admit finite bases of the same cardinality.

Note that if Y is an expansion of X, then expansions of Y are also expansions of X, as $\mathrm{C}_A^+(Y)$ is naturally a subset of $\mathrm{C}_A^+(X)$.

Given a basis E of $\mathrm{C}_A(X)$, a basis F of $\mathrm{C}_A(Y)$ and a bijection $\lambda : E \to F$, there is a unique isomorphism of Cantor algebras $f : \mathrm{C}_A(X) \to \mathrm{C}_A(Y)$ that satisfies that $f|_E = \lambda$. But such a representing triple (E, F, λ) is far from unique. Indeed, any expansion E' of E defines a new such triple (E', F', μ) representing the same isomorphism f simply by taking $F' = f(E')$ and $\mu = f|_{E'}$. The following result, which we will make heavy use of, says that every isomorphism admits a canonical minimal such representative if we restrict ourselves to considering bases that lie inside $\mathrm{C}_A^+(X)$ and $\mathrm{C}_A^+(Y)$.

Proposition 1.5. *Any isomorphism of Cantor algebras $f : \mathrm{C}_A(X) \to \mathrm{C}_A(Y)$ has a canonical minimal presentation as triple (E, F, λ) with E an expansion of X, F an expansion of Y, and with $\lambda : E \to F$ a bijection.*

This is a slight generalization of [Hig74, Lem. 4.1], and it follows from the same proof. We give it for completeness.

Proof. Consider the subset U of $\mathrm{C}_A^+(X)$ defined by

$$U \;=\; \mathrm{C}_A^+(X) \,\cap\, f^{-1}\mathrm{C}_A^+(Y) \;=\; \mathrm{C}_A^+(X) \,\cap\, \mathrm{C}_A^+(f^{-1}(Y)).$$

By [Hig74, Lem. 2.4], there exists an expansion E of X such that $U = \mathrm{C}_A^+(E)$. As E lies inside U, it satisfies that $F = f(E)$ is an expansion of Y. Taking (E,F,λ) with $\lambda = f|_E$ yields a presentation of f with E and F expansions of X and Y.

Now assume that E' is another expansion of X satisfying that $f(E')$ is an expansion of X. Then E' must lie in U and hence be an expansion of E. Hence (E,F,λ) was minimal and unique with that property. □

1.2 Categories of Cantor algebras

We define the category $\mathbf{Cantor}_A$ to be the category of all finitely generated free Cantor algebras of type A, with morphisms the maps of A^*-sets, and $\mathbf{Cantor}_A^\times$ its subcategory of isomorphisms. Both categories are symmetric monoidal, with the symmetric monoidal structure, denoted $\oplus$, induced by the categorical sum in $\mathbf{Cantor}_A$. On objects we have

$$\mathrm{C}_A(X) \oplus \mathrm{C}_A(Y) = \mathrm{C}_A(X \sqcup Y)$$

and the symmetry

$$\mathrm{C}_A(X \sqcup Y) \to \mathrm{C}_A(Y \sqcup X)$$

is induced by the symmetry $X \sqcup Y \to Y \sqcup X$ in the category of sets.

Proposition 1.6. *The sum of Cantor algebras* $\mathrm{C}_A(X) \oplus \mathrm{C}_A(Y) = \mathrm{C}_A(X \sqcup Y)$ *induces a symmetric monoidal category* $\mathbf{Cantor}_A^\times$ *with the property that the map*

$$\mathrm{Aut}(\mathrm{C}_A(X)) \times \mathrm{Aut}(\mathrm{C}_A(Y)) \xrightarrow{\oplus} \mathrm{Aut}(\mathrm{C}_A(X \sqcup Y))$$

is injective.

Proof. Note first that the symmetric monoidal structure of $\mathbf{Cantor}_A$ defined above restricts to a symmetric monoidal structure on $\mathbf{Cantor}_A^\times$. For the injectivity statement, we use a commutative diagram.

$$\begin{array}{ccc}
\mathrm{Aut}(\mathrm{C}_A(X)) \times \mathrm{Aut}(\mathrm{C}_A(Y)) & \longrightarrow & \mathbf{Cantor}_A(X, X \sqcup Y) \times \mathbf{Cantor}_A(Y, X \sqcup Y) \\
\downarrow & & \downarrow \\
\mathrm{Aut}(\mathrm{C}_A(X \sqcup Y)) & \longrightarrow & \mathbf{Cantor}_A(\mathrm{C}_A(X \sqcup Y), \mathrm{C}_A(X \sqcup Y))
\end{array}$$

The sum is the map on the left. It is sufficient to prove that the composition is injective, and we will do this for the composition through to top right corner. The first map is trivially injective if $Y = \emptyset$ and injectivity follows from the fact that it has a left inverse on each factor when $Y \neq \emptyset$. The second map is injective by the property of free algebras. □

2 Spaces associated to Higman–Thompson groups

This section and the next one are concerned with the proof of homological stability for the Higman–Thompson groups $\mathrm{V}_{n,r}$, the automorphism group of the Cantor algebra $\mathrm{C}_{n,r}$, with respect to the number r of generators. Given a family of groups satisfying a few properties, the paper [R-WW] yields a sequence of spaces whose high connectivity implies homological stability for the family of groups. In this section, we will show how the groups $\mathrm{V}_{n,r}$ fit in the framework of [R-WW] and construct spaces relevant to the proof of homological stability, which will be given in the following section.

For a fixed type n we collect the Higman–Thompson groups into a groupoid $\mathbf{V}_n$: The objects are the natural numbers r as placeholders for the free Cantor algebras $\mathrm{C}_r = \mathrm{C}_{n,r}$, and the morphism set $\mathbf{V}_n(\mathrm{C}_r, \mathrm{C}_s)$ is empty unless $r = s$ in which case $\mathbf{V}_n(\mathrm{C}_r, \mathrm{C}_r) = \mathrm{V}_{n,r}$. Recall that there are isomorphisms $\mathrm{C}_{n,r} \cong \mathrm{C}_{n,r+(n-1)}$ for any $r \geqslant 1$, but we do *not* include these isomorphisms into the groupoid $\mathbf{V}_n$. The groupoid $\mathbf{V}_n$ admits a symmetric monoidal structure $\oplus$ that is defined to be addition of natural numbers on objects: $\mathrm{C}_r \oplus \mathrm{C}_s := \mathrm{C}_{r+s}$. On morphisms, the monoidal structure is given by the homomorphisms $\mathrm{V}_{n,r} \times \mathrm{V}_{n,s} \to \mathrm{V}_{n,r+s}$ of groups as in $\mathbf{Cantor}_A$ for the set $A = \{1, \ldots, n\}$.

2.1 Quillen's bracket construction

We can apply Quillen's bracket construction [Gra76, p. 219] to the groupoid $\mathbf{V}_n$ to obtain a new category $\mathbf{Q}_n = \langle \mathbf{V}_n, \mathbf{V}_n \rangle$: The category $\mathbf{Q}_n$ has the same objects as $\mathbf{V}_n$ and there are no morphisms from C_r to C_s unless there exists a k such that $\mathrm{C}_k \oplus \mathrm{C}_r \cong \mathrm{C}_s$ in $\mathbf{V}_n$, i.e. $r \leqslant s$ in our case, with $k = s - r$. If this is the case, morphisms are equivalence classes $[f]$ of elements f in $\mathrm{V}_{n,s}$ with $f \sim f'$ if there exists an element g in $\mathrm{V}_{n,k}$ such that

$$f' = f \circ (g \oplus \mathrm{C}_r)\colon\ \mathrm{C}_s = \mathrm{C}_k \oplus \mathrm{C}_r \ \longrightarrow\ \mathrm{C}_s.$$

Here and in the following we employ the Milnor–Moore notation and denote the identity of an object by that object. Note that $\mathrm{C}_0 = \varnothing$ is now an initial object in the category $\mathbf{Q}_n$. We will write $\iota_r : \mathrm{C}_0 \to \mathrm{C}_r$ for the unique morphism, which we can represent as the equivalence class $[\mathrm{C}_r]$ of the identity in $\mathrm{V}_{n,r}$.

2.2 Homogeneous categories

Recall from [R-WW, Def. 1.3] that a monoidal category $(\mathbf{H}, \oplus, 0)$ is called *homogeneous* if the monoidal unit 0 is initial and the following two conditions are satisfied for every pair of objects A, B in $\mathbf{H}$: The set $\mathbf{H}(A,B)$ is a transitive $\mathrm{Aut}_{\mathbf{H}}(B)$–set under post-composition, and the homomorphism

$$\mathrm{Aut}_{\mathbf{H}}(A) \to \mathrm{Aut}_{\mathbf{H}}(A \oplus B)$$

that takes f to $f \oplus B$ is injective with image $\{\varphi \in \mathrm{Aut}_{\mathbf{H}}(A \oplus B) \mid \varphi \circ (\iota_A \oplus B) = \iota_A \oplus B\}$.

Proposition 2.1. *The category $\mathbf{Q}_n$ is symmetric monoidal and homogeneous with maximal subgroupoid $\mathbf{V}_n$.*

Proof. This is a direct application of three results in [R-WW]: Because $\mathbf{V}_n$ is a symmetric monoidal groupoid, [R-WW, Prop. 1.7] gives that $\mathbf{Q}_n$ (denoted $U\mathbf{V}_n$ in that paper) is a symmetric monoidal category, with its unit initial. We have that $\mathrm{Aut}(\mathrm{C}_0) = \{\mathrm{id}\}$ and that there are no zero divisors in $\mathbf{Q}_n$: If there is an isomorphism $\mathrm{C}_r \oplus \mathrm{C}_s \cong \mathrm{C}_0$ in $\mathbf{V}_n$, then we must have that $\mathrm{C}_r = \mathrm{C}_s = \mathrm{C}_0$. Then [R-WW, Prop. 1.6] gives that $\mathbf{V}_n$ is the maximal subgroupoid of $\mathbf{Q}_n$. Also, the groupoid $\mathbf{V}_n$ satisfies cancellation (by construction): If there exists an isomorphism $\mathrm{C}_r \oplus \mathrm{C}_s \cong \mathrm{C}_r \oplus \mathrm{C}_{s'}$ in $\mathbf{V}_n$, then $\mathrm{C}_s \cong \mathrm{C}_{s'}$ in $\mathbf{V}_n$, because we have $\mathrm{C}_{r+s} \cong \mathrm{C}_{r+s'}$ in the groupoid $\mathbf{V}_n$ if and only if $r+s = r+s'$. Finally, the groupoid $\mathbf{V}_n$ satisfies that the map $\mathrm{Aut}_{\mathbf{V}_n}(\mathrm{C}_r) \to \mathrm{Aut}_{\mathbf{V}_n}(\mathrm{C}_r \oplus \mathrm{C}_s)$ adding the identity on C_s is injective by Proposition 1.6. From [R-WW, Thm. 1.9] we see that $\mathbf{Q}_n$ is homogenous, which completes the proof. □

Recall from Proposition 1.5 that the elements f of $\mathrm{V}_{n,s}$ admit a unique minimal presentation (E,F,λ) where $E,F \subset \mathrm{C}_{n,s}^+$ are expansions of the standard generating set

$$[s] := \{1,\dots,s\}$$

of $\mathrm{C}_{n,s}$, and $\lambda\colon E \to F$ is an isomorphism. The morphism set $\mathbf{Q}_n(\mathrm{C}_s,\mathrm{C}_s)$ identifies by definition with $\mathrm{V}_{n,s}$. We will soon need the following description of the morphism sets $\mathbf{Q}_n(\mathrm{C}_r,\mathrm{C}_s)$ for $r < s$, analogous to the minimal presentations of the isomorphisms.

Definition 2.2. A subset $P \subset \mathrm{C}_{n,r}^+$ is called *independent* if there exists an expansion E of $[r]$ such that $P \subset E$.

It follows from [Hig74, Lem. 2.7 (i), (iii)] that there exists a unique minimal such expansion E containing P.

Consider the set $\mathrm{P}_n(r,s)$ of triples (E,P,λ), where $E \subset \mathrm{C}_{n,r}^+$ is an expansion of the canonical basis $[r]$, the set $P \subset \mathrm{C}_{n,s}^+$ is an independent set which is not a basis, and $\lambda : E \to P$ is bijection. We partially order $\mathrm{P}_n(r,s)$ by setting $(E,P,\lambda) \leqslant (F,Q,\mu)$ if F is an expansion of E, then Q is an expansion of P, and μ is the restriction to F of the map $\mathrm{C}_n^+(E) \to \mathrm{C}_n^+(P)$ induced by λ.

Lemma 2.3. *For all $0 \leqslant r < s$, the set of morphisms $\mathbf{Q}_n(\mathrm{C}_r,\mathrm{C}_s)$ is isomorphic to the set of minimal elements in $\mathrm{P}_n(r,s)$.*

Note that the statement is trivially true when $r = 0$ as both sets contain a single element.

Proof. Given a morphism $[f] : \mathrm{C}_r \to \mathrm{C}_s$, with representative $f : \mathrm{C}_s = \mathrm{C}_k \oplus \mathrm{C}_r \to \mathrm{C}_s$, we want to associate a minimal element of $\mathrm{P}_n(r,s)$. We choose a minimal presentation (E,F,λ) of f. So the sets $E,F \subset \mathrm{C}_s^+$ are expansions of $[s]$ and $\lambda : E \to F$ is a bijection. Consider the inclusion $\mathrm{C}_r^+ \subset \mathrm{C}_s^+$ induced by $[s] \cong [k] \sqcup [r]$, and let $E_0 = E \cap \mathrm{C}_r^+$. We have that E_0 is an expansion of $[r]$, and hence a basis of C_r. Also $P = \lambda(E_0) \subset F$ is an independent set, and does not generate given that $k > 0$ under our assumption. Hence the triple $(E_0, P, \lambda|_{E_0})$ is an element of $\mathrm{P}_n(r,s)$. It is moreover minimal because (E,F,λ) is minimal.

We need to check that $(E_0, P, \lambda|_{E_0})$ is independent of the choice of representative f of $[f]$. So let $g \in \mathrm{V}_{n,k}$ with minimal presentation (A, B, μ). We have that $g \oplus \mathrm{C}_r$ has minimal presentation $(A \sqcup [r], B \sqcup [r], \mu \sqcup [r])$. Now if $f' = f \circ (g \oplus \mathrm{C}_r)$, we must have that f' has minimal presentation of the form $(A' \sqcup E_0, B' \sqcup P, \mu' \sqcup \lambda|_{E_0})$. In particular, the above construction will yield the same triple $(E_0, P, \lambda|_{E_0})$, showing that the association is independent of the choice of representative for $[f]$.

We check now that the association is injective. So suppose that we have two elements f and f' in $\mathrm{V}_{n,s}$, given minimally by $f = (E, F, \lambda)$ and $f' = (E', F', \lambda')$, and satisfying that $E_0 = E \cap \mathrm{C}_r^+ = E' \cap \mathrm{C}_r^+ = E_0'$ and $P = \lambda|_{E_0} = \lambda'|_{E_0'} = P'$. We would like to show that $[f] = [f']$ as elements of $\mathbf{Q}_n(\mathrm{C}_r, \mathrm{C}_s)$. Let H be the smallest basis of C_s which is a common expansion of F and F'. Note that H includes $P = P'$ as a subset. Then f and f' admit (non-minimal) representatives $(G, H, \mathrm{C}_s(\lambda)|_Z)$ and $(G', H, \mathrm{C}_s(\lambda')|_Z)$, for $G = \mathrm{C}_s(\lambda)^{-1}(H)$ and $G' = \mathrm{C}_s(\lambda')^{-1}(H)$, and we have that $E_0 \subset G$ and $E_0' \subset G'$. Note now that the complements G_0 and G_0' of X_0 in G and X_0' in G' are bases for $\mathrm{C}_k \subset \mathrm{C}_r$ and $\mu = \mathrm{C}_s(\lambda') \circ \mathrm{C}_s(\lambda)^{-1}$ restricts to a bijection between these two bases. The isomorphism $g : \mathrm{C}_k \to \mathrm{C}_k$ presented by (G_0, G_0', μ) has the property that $f' = f \circ g$, showing that $[f] = [f']$. This shows that the association defined above is injective.

Finally we show that the assignment is also surjective by constructing an inverse. So consider a minimal element (E, P, λ) of $\mathrm{P}_n(r, s)$. Let F be a basis of $\mathrm{C}_{n,s}$ containing P. We have that $|E| = r + a(n-1)$ for some $a \geqslant 0$ and $|F| = s + b(n-1)$ for some $b \geqslant 0$, with $s - r + (b-a)(n-1) > 0$ under our assumption. If $b - a < 0$, replace F by an expansion of F still containing P by expanding an element of $F \setminus P$ (which is non-empty by assumption) at least $(b-a)$ times. After doing this, we can assume moreover that $b - a \geqslant 0$. Then let G be a basis of C_{s-r} of cardinality $s - r + (b-a)(n-1)$ and pick a bijection $\mu : G \to F \setminus P$. Then $G \sqcup E$ is a basis of $\mathrm{C}_s = \mathrm{C}_{s-r} \oplus \mathrm{C}_s$ and $(G \sqcup E, \mu(G) \sqcup P, \mu \sqcup \lambda)$ represents an element $f \in V_{n,s}$ with the property that the triple associated to $[f] \in \mathbf{Q}_n(\mathrm{C}_r, \mathrm{C}_s)$, is precisely the pair (E, P, λ), finishing the proof of surjectivity. □

2.3 Associated spaces

In the general context of the paper [R-WW], given a pair (A, X) of objects in a homogeneous category, a sequence of semi-simplicial sets $\mathrm{W}_r(A, X)$ is defined, and the main theorem in that paper says that homological stability holds for the automorphism groups of the objects $A \oplus X^{\oplus r}$ as long as the associated semi-simplicial sets are highly connected. In good cases, the connectivity of the semi-simplicial sets $\mathrm{W}_r(A, X)$ can be computed from the connectivity of closely related simplicial complexes $\mathrm{S}_r(A, X)$.

We are here interested in the pair of objects $(A, X) = (\mathrm{C}_0, \mathrm{C}_1)$ in the homogeneous category $\mathbf{Q}_n$. Indeed, the automorphism group of $\mathrm{C}_0 \oplus \mathrm{C}_1^{\oplus r} = \mathrm{C}_r$ in the category $\mathbf{Q}_n$ is the Higman–Thompson group $\mathrm{V}_{n,r}$. We will therefore begin by describing the semi-simplicial sets $\mathrm{W}_r = \mathrm{W}_r(\mathrm{C}_0, \mathrm{C}_1)$ and the simplicial complexes $\mathrm{S}_r = \mathrm{S}_r(\mathrm{C}_0, \mathrm{C}_1)$ from Definitions 2.1 and 2.8 in [R-WW], and show that we are in a situation where we can use the connectivity of the latter to compute the connectivity of the former. In the following Section 3 we will estimate that connectivity.

Definition 2.4. Given $r \geqslant 1$, let $\mathrm{W}_r = \mathrm{W}_r(\mathrm{C}_0, \mathrm{C}_1)$ be the semi-simplicial set where the set of p–simplices is the set of maps $\mathbf{Q}_n(\mathrm{C}_{p+1}, \mathrm{C}_r)$ and the i–th boundary map $\mathbf{Q}_n(\mathrm{C}_{p+1}, \mathrm{C}_r) \to \mathbf{Q}_n(\mathrm{C}_p, \mathrm{C}_r)$ is defined by precomposing with $\mathrm{C}_i \oplus \iota_1 \oplus \mathrm{C}_{p-i}$.

Note that W_r has dimension $r-1$, and a top dimensional simplex in W_r is an automorphism of C_r, which is an element of $\mathrm{V}_{n,r}$. For $p < r-1$, Lemma 2.3 shows that p–simplices of W_r have a unique minimal presentation by a triple (E, P, λ) where E is an expansion of $\{p+1\}$, and P is an independent set of C_r^+ that is not a basis, and $\lambda : E \to P$ is a bijection.

To W_r we associate the following simplicial complex, of the same dimension $r-1$.

Definition 2.5. Given $r \geqslant 1$, let $\mathrm{S}_r = \mathrm{S}_r(\mathrm{C}_0, \mathrm{C}_1)$ be the simplicial complex with the same vertices as W_r, the set of maps $\mathbf{Q}_n(\mathrm{C}_1, \mathrm{C}_r)$, and where $p+1$ distinct vertices $[f_0], \dots, [f_p]$ form a p–simplex if there exists a p–simplex of W_r having them as vertices.

Given a simplicial complex X, one can build a semi-simplicial set X^{ord} that has a p–simplex for each ordering of the vertices of each p–simplex of X.

Proposition 2.6. *There is an isomorphism of semi-simplicial sets* $\mathrm{W}_r \cong \mathrm{S}_r^{\mathrm{ord}}$. *Moreover, if* S_r *is* $(r-3)$*–connected, then so is* W_r.

Proof. This follows from [R-WW, Prop. 2.9, Thm. 2.10], given that $\mathbf{Q}_n$ is symmetric monoidal, once we have checked that it is *locally standard* at $(\mathrm{C}_0, \mathrm{C}_1)$ in the sense of [R-WW, Def. 2.5]. This means two things: Firstly, the morphisms $\mathrm{C}_1 \oplus \iota_1$ and $\iota_1 \oplus \mathrm{C}_1$ are distinct in $\mathbf{Q}_n(\mathrm{C}_1, \mathrm{C}_2)$, and secondly, for all $r \geqslant 1$, the map $\mathbf{Q}_n(\mathrm{C}_1, \mathrm{C}_{r-1}) \to \mathbf{Q}_n(\mathrm{C}_1, \mathrm{C}_r)$ that takes a morphism $[f]$ to $[f] \oplus \iota_1$ is injective.

For the first statement, we need to decribe the morphisms $\mathrm{C}_1 \oplus \iota_1$ and $\iota_1 \oplus \mathrm{C}_1$. Both C_1 and ι_1 are represented by the identity on C_1, but the first up to an automorphism of the trivial complement C_0 in C_1 and the second up to an automorphism of the complement C_1 of C_0 in C_1. By definition of the monoidal structure of $\mathbf{Q}_n$, we have

$$\mathrm{C}_1 \oplus \iota_1 = [\tau_{1,1}] \colon \mathrm{C}_0 \oplus \mathrm{C}_1 \oplus \mathrm{C}_1 \oplus \mathrm{C}_0 \xrightarrow{\mathrm{C}_0 \oplus \tau_{1,1} \oplus \mathrm{C}_0} \mathrm{C}_0 \oplus \mathrm{C}_1 \oplus \mathrm{C}_1 \oplus \mathrm{C}_0 \xrightarrow{\mathrm{id} \oplus \mathrm{id}} \mathrm{C}_1 \oplus \mathrm{C}_1,$$

where the symmetry $\tau_{1,1}$ is induced by the transposition (12), while

$$\iota_1 \oplus \mathrm{C}_1 = [\mathrm{C}_2] \colon \mathrm{C}_1 \oplus \mathrm{C}_0 \oplus \mathrm{C}_0 \oplus \mathrm{C}_1 \xrightarrow{\mathrm{C}_1 \oplus \tau_{0,0} \oplus \mathrm{C}_1} \mathrm{C}_1 \oplus \mathrm{C}_0 \oplus \mathrm{C}_0 \oplus \mathrm{C}_1 \xrightarrow{\mathrm{id} \oplus \mathrm{id}} \mathrm{C}_1 \oplus \mathrm{C}_1 \ ,$$

with $\tau_{0,0} = \mathrm{id}$, is the isomorphism class of the identity on C_2. Hence these two elements are indeed distinct in $\mathbf{Q}_n(\mathrm{C}_1, \mathrm{C}_2)$, the first one having minimal presentation $(\{1\}, \{2\}, \lambda)$ while the second has presentation $(\{1\}, \{1\}, \mu)$, for λ, μ the unique maps.

For the second statement, we have

$$[f] \oplus \iota_1 = [(f \oplus \mathrm{C}_1) \circ (\mathrm{C}_{r-2} \oplus \tau_{1,1})].$$

If $[f]$ is minimally presented by (E,P,λ), then one can check that $[f]\oplus\iota_1$ is minimally presented by $(E,i(P),i\circ\lambda)$ for $i:\mathrm{C}_{r-1}\to\mathrm{C}_r$ induced by the isomorphism $[r-1]\sqcup[1]\cong[r]$. As i is injective, the result follows.

By [R-WW, Prop. 2.9] we now know that W_r "satisfies condition (A)", which means that it is isomorphic to $\mathrm{S}_r^{\mathrm{ord}}$, and [R-WW, Thm. 2.10] of the same paper gives the second part of the statement. □

2.4 Variations

We will have use for variants U_r, U_r^∞ and T_r^∞ of the simplicial complex S_r that we will introduce now.

Definition 2.7. Let U_1 to be the simplicial complex of dimension 0 consisting of all the expansions of $\{1\}$ inside $\mathrm{C}_{n,1}^+$. For $r\geqslant 2$, let U_r be the simplicial complex of dimension $r-1$ with vertices the independent subsets P of $\mathrm{C}_{n,r}^+$ of cardinality congruent to 1 modulo $n-1$ which are *not* bases. A set of $p+1$ vertices $P_0,\dots,P_p$ forms a p–simplex of U_r if the sets P_i are pairwise disjoint and

- $p<r-1$ and $P_0\sqcup\cdots\sqcup P_p$ is an independent set that is *not* a basis, or
- $p=r-1$ and $P_0\sqcup\cdots\sqcup P_p$ is a basis of $\mathrm{C}_{n,r}$.

Using Lemma 2.3, we see that there is a forgetful map $\mathrm{S}_r\to\mathrm{U}_r$ that takes a vertex (E,P,λ) of S_r to the independent set P. We in fact have the following:

Proposition 2.8. *The simplicial complex* S_r *is a* complete join complex *over* U_r *in the sense of* [HW10, Def. 3.2].

Proof. Given the projection $\mathrm{S}_r\to\mathrm{U}_r$ just described, we need to verify that a set of $p+1$ vertices forms a p–simplex of the simplicial complex S_r if and only if their images in U_r form a simplex there. So suppose $(E_0,P_0,\lambda_0),\dots,(E_p,P_p,\lambda_p)$ are vertices of S_r. We first consider the case $p=r-1$. We have that $P_0,\dots,P_{r-1}$ forms an $(r-1)$–simplex of U_r if and only if the sets are disjoint and their disjoint union $P_0\sqcup\cdots\sqcup P_{r-1}$ is a basis, which is the case if and only if $(E_0\sqcup\cdots\sqcup E_{r-1},P_0\sqcup\cdots\sqcup P_{r-1},\lambda_0\sqcup\cdots\sqcup\lambda_{r-1})$ defines an element of the set $\mathrm{V}_{n,r}=\mathbf{Q}_n(\mathrm{C}_r,\mathrm{C}_r)$, which is the case if and only if the vertices $(E_0,P_0,\lambda_0),\dots,(E_{r-1},P_{r-1},\lambda_{r-1})$ form an $(r-1)$–simplex in the simplicial complex S_r. Similarly, for $p<r-1$, using Lemma 2.3, we see that the vertices $P_0,\dots,P_p$ form a p–simplex of the simplicial complex U_r if and only if $(E_0\sqcup\cdots\sqcup E_p,P_0\sqcup\cdots\sqcup P_p,\lambda_0\sqcup\cdots\sqcup\lambda_p)$ defines an element of the set $\mathbf{Q}_n(\mathrm{C}_{p+1},\mathrm{C}_r)$, which is the case if and only if $(E_0,P_0,\lambda_0),\dots,(E_p,P_p,\lambda_p)$ forms an p–simplex in the simplicial complex S_r. □

In contrast to the simplicial complexes S_r and U_r, the variants U_r^∞ and T_r^∞ that we will now introduce are both infinite-dimensional (as the notation suggests).

Definition 2.9. For $r\geqslant 1$, let U_r^∞ be the simplicial complex whose vertices are the independent subsets of $\mathrm{C}_{n,r}^+$ of cardinality congruent to 1 modulo $n-1$ that are *not* bases. Distinct vertices $P_0,\dots,P_p$ form a p-simplex U_r^∞ if the subsets are pairwise disjoint, their union $P_0 \sqcup \cdots \sqcup P_p$ is still independent, and they do *not* form a basis.

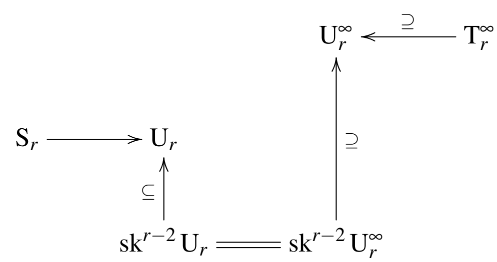
Note that U_r^∞ and U_r have the same set of vertices when $r \geqslant 2$ but not when $r = 1$. In fact we have the following:

Lemma 2.10. *The simplicial complexes* U_r *and* U_r^∞ *share the same* $(r-2)$*–skeleton.*

Proof. This follows immediately from the definitions. □

Definition 2.11. For $r \geqslant 1$, let T_r^∞ denote the full subcomplex of U_r^∞ on the vertices that have cardinality 1.

The diagram



indicates the relations between the spaces that we have introduced so far, with the height reflecting their dimension. In the next section, we will prove high connectivity of S_r using this sequence of maps.

3 Homological stability

In this section, we estimate the connectivity of the simplicial complexes defined in the previous section, and use these to deduce homological stability for the canonical diagrams ($\star$) of the Higman–Thompson groups. All of this will be based on the following two results.

Proposition 3.1. *For each integer* $r \geqslant 1$ *the simplicial complex* T_r^∞ *is contractible.*

Proposition 3.2. *For each integer* $r \geqslant 1$ *the simplicial complex* U_r^∞ *is contractible.*

Before we give proofs of these two propositions in Section 3.3, let us state and prove their consequences that lead to homological stability.

3.1 Consequences of Propositions 3.1 and 3.2

Recall from [HW10, Def. 3.4] that a simplicial complex is called *weakly Cohen–Macaulay of dimension n* if it is $(n-1)$–connected and the link of every p–simplex of it is $(n-p-2)$–connected.

Corollary 3.3. *For all $r \geqslant 2$ the simplicial complexes U_r are weakly Cohen–Macaulay of dimension $r-2$.*

Proof. A space is $(r-3)$–connected if and only if its $(r-2)$–skeleton is. Since the simplicial complexes U_r and U_r^∞ share the same $(r-2)$–skeleton by Lemma 2.10, we see that the simplicial complex U_r is $(r-3)$–connected if the simplicial complex U_r^∞ is, and the latter is even contractible by Proposition 3.2. Let now σ be a p–simplex of U_r with vertices $P_0,\dots,P_p$. For $p \geqslant r-2$, there is nothing to check as $(r-2)-p-2 \leqslant -2$ and (-2)–connected is a non-condition. So assume $p \leqslant r-3$. Let $E \subset \mathrm{C}_s^+$ be an expansion of $\{s\}$ that contains the independent subset $P_0 \sqcup \dots \sqcup P_p$, and that is minimal with respect to that property. Write $Q = E \setminus (P_0 \sqcup \dots \sqcup P_p)$. Note that Q is non-empty. The link of σ is the subcomplex of vertices P that lie in $\mathrm{C}_n^+(Q) \subset \mathrm{C}_{n,s}^+$, as P must be independent of $P_0,\dots,P_p$, and the union $P \sqcup P_0 \sqcup \dots \sqcup P_p$ must not form a basis as $p+1 < r-1$, given that E was minimal. A set of $k+1$ vertices of the link forms a k–simplex if and only if the corresponding subsets are disjoint, their union is independent, and they together with $P_0,\dots,P_p$ form a basis if and only if $k+p+1 = r-1$, or $k = r-p-2$. Hence the $(r-p-3)$–skeleton of the link is isomorphic to the $(r-p-3)$–skeleton of $\mathrm{U}_{|Q|}^\infty$. As the latter space is contractible by Proposition 3.2, it follows that the link is at least $(r-p-4)$–connected, which is what we needed for the Cohen–Macaulay condition as $r-p-4 = (r-2)-p-2$. □

Corollary 3.4. *For each $r \geqslant 2$ the spaces S_r and W_r are $(r-3)$–connected.*

Proof. By Proposition 2.8), the simplicial complex S_r is a complete join complex over U_r, and U_r is weakly Cohen–Macaulay of dimension $r-2$ by Corollary 3.3. It thus follows from [HW10, Prop. 3.5] that S_r is also weakly Cohen–Macaulay of that dimension, and so in particular $(r-3)$–connected. Then, by Proposition 2.6, it follows that the semi-simplicial sets W_r are also $(r-3)$–connected. □

Corollary 3.5. *The semi-simplicial set W_{r+1} is $(\frac{r-2}{2})$–connected for all $r \geqslant 0$.*

Proof. There is a morphism $\mathrm{C}_1 \to \mathrm{C}_{r+1}$ in the category $\mathbf{Q}_n$ as soon as $r+1 \geqslant 1$, or equivalently $r \geqslant 0$. This shows that the semi-simplicial set W_{r+1} is non-empty for all $r \geqslant 0$. This gives the starting cases $r = 0,1$. For $r \geqslant 2$, Corollary 3.4 gives that W_{r+1} is $(r-2)$–connected, which, under the assumption on r, implies that it is at least $(\frac{r-2}{2})$–connected. □

3.2 Stability theorem

The stabilization homomorphism $\sigma_r \colon \mathrm{V}_{n,r} \to \mathrm{V}_{n,r+1}$ takes an element f to $f \oplus \mathrm{C}_1$, leading to the canonical diagram $(\star)$ of groups.

Theorem 3.6. *The stabilization homomorphisms induce isomorphisms*

$$\sigma_r \colon \mathrm{H}_d(\mathrm{V}_{n,r};M) \longrightarrow \mathrm{H}_d(\mathrm{V}_{n,r+1};M)$$

in homology in all dimensions $d \geqslant 0$*, for all* $r \geqslant 1$*, and for all* $\mathrm{H}_1(\mathrm{V}_{n,\infty})$*–modules* M*.*

Proof. First, we can apply the stability result of [R-WW] to the category $\mathbf{Q}_n$ with the choice of objects $A = \mathrm{C}_1$ and $X = \mathrm{C}_1$: our Corollary 3.5 shows that the complexes $\mathrm{W}_r(\mathrm{C}_1,\mathrm{C}_1) \cong \mathrm{W}_{r+1}(\mathrm{C}_0,\mathrm{C}_1)$ are $(\frac{r-2}{2})$–connected, which implies, by [R-WW, Thm. 3.4], that σ_r is an isomorphism in the range of dimensions $d \leqslant \frac{r-4}{3}$, that is a range that increases with the rank r. Recall now that the group $\mathrm{V}_{n,r}$ is isomorphic to the group $\mathrm{V}_{n,r+(n-1)}$ as soon as $r \geqslant 1$. We choose an isomorphism between the two groups as follows. Let $h \colon \mathrm{C}_{n,1} \to \mathrm{C}_{n,n}$ be the isomorphism with minimal presentation $(\{1\} \times [n], [n], \mathrm{id})$. Then $h_r = h \oplus \mathrm{C}_{r-1} \colon \mathrm{C}_r \to \mathrm{C}_{r+n-1}$ is also an isomorphism. Let $\gamma(h_r)$ denote conjugation with h_r. Then we get a commutative diagram

$$\begin{array}{ccc} \mathrm{V}_{n,r} & \xrightarrow{\sigma_r} & \mathrm{V}_{n,r+1} \\ {\scriptstyle \gamma(h_r)}\downarrow & & \downarrow{\scriptstyle \gamma(h_{r+1})} \\ \mathrm{V}_{n,r+(n-1)} & \xrightarrow{\sigma_{r+1}} & \mathrm{V}_{n,r+(n-1)+1}, \end{array}$$

as $(h \oplus \mathrm{C}_r)^{-1}(f \oplus \mathrm{C}_1)(h \oplus \mathrm{C}_r) = \big((h \oplus \mathrm{C}_{r-1})^{-1} f (h \oplus \mathrm{C}_{r-1})\big) \oplus \mathrm{C}_1$ for all $r \geqslant 1$. Given that the vertical maps are isomorphisms and increase the rank of the group, while the horizontal maps induce an isomorphism in homology when the rank of the group is large enough, we get that the horizontal maps must always induce an isomorphism in homology. □

Remark 3.7. For the purposes of the present paper, we will only need homological stability with respect to trivial, or potentially abelian, coefficients, in the form stated. Applying the more general Theorem A of [R-WW] instead of Theorem 3.4 in that paper, one obtains that stability also holds with finite degree coefficient systems. We refer the interested reader to [R-WW] for the definitions.

3.3 Proofs of Propositions 3.1 and 3.2

In the rest of this section we give the proofs of Propositions 3.1 and 3.2.

Proof of Proposition 3.1. We will show that all maps $\mathrm{S}^k \to \mathrm{T}_r^\infty$ from spheres into the space T_r^∞ are null-homotopic. Recall that a vertex of T_r^∞ is a set of cardinality 1 in C_r^+, that is an element of C_r^+. The set

$$\mathrm{C}_r^+ = \coprod_{h \geqslant 0} [r] \times [n]^h$$

is canonically graded by the *height* h of its elements. The number of vertices in T_r^∞ of a given height h is strictly increasing with h. In fact, more is true: There is a strictly increasing function w such that, given any non-generating independent set P of at most k (the dimension of S^k) vertices in T_r^∞, there are at least $w(h)$ vertices in T_r^∞ of height h that are in the complement of $[n]^*$–set $\mathrm{C}_n^+(P)$ generated by P inside C_r^+. To see this, notice that a non-generating subset admits an independent element in the complement. If that element has height h_0, then for all $h \geqslant h_0$ there are at least n^{h-h_0} vertices of height h in the complement, namely the descendants of the chosen vertex of height h_0.

Let us now be given a map $f\colon \mathrm{S}^k \to \mathrm{T}_r^\infty$. We can assume that the sphere S^k comes with a triangulation such that the map f is simplicial. Let v be the (total) number of simplices of all dimensions of that triangulation. In particular, and this is obviously a very crude estimate, this triangulation has at most v vertices. We choose an integer h such that $w(h) \geqslant v+2$, and we call a simplex of the sphere S^k *bad* for f if all of its vertices are mapped to vertices in T_r^∞ that have height less than h.

If there is a bad simplex for f with respect to the given triangulation of the sphere, then we will see that we can find a homotopy from f to a map that is simplicial with respect to another triangulation of the sphere that still has at most v vertices and that has fewer bad simplices. To do so, we will choose a bad simplex σ of maximal dimension p among all bad simplices, and modify f and the triangulation of the sphere in the star of that simplex. In the process, we will increase the number of vertices by at most 1, and not at all if σ was a vertex. (This implies that we will always have at most v vertices in the triangulation of the sphere.) There are two cases:

The case $p = k$. If the bad simplex σ is of the dimension k of the sphere S^k, then its image $f(\sigma)$ is a non-generating independent subset of $\mathrm{C}_{n,r}^+$. Hence we can choose a vertex y that has height at least h and that is not a descendant of $f(\sigma)$ and still, together with $f(\sigma)$, gives a non-generating independent subset. As the union $f(\sigma) \cup \{y\}$ is again a simplex of T_r^∞, we can add a vertex x in the center of σ, replacing σ by $\partial\sigma * x$ and replace f by the map $f|_{\partial\sigma} * (x \mapsto y)$ on $\partial\sigma * x$. This map is homotopic to f through the simplex $f(\sigma) \cup \{y\}$. We have added a single vertex to the triangulation and removed one bad simplex without adding any new one.

The case $p < k$. If the bad simplex σ is a p–simplex for some $p < k$, by maximality of its dimension, the link of σ is mapped to vertices of height at least h in the complement of the $[n]^*$–set $\mathrm{C}_n^+(f(\sigma))$ generated by $f(\sigma)$. The simplex σ has $p+1$ vertices whose images form an independent subset of $\mathrm{C}_{n,r}^+$ of cardinality at most $p+1 \leqslant k$. Hence there are at least $w(h) \geqslant v+2$ vertices of height h in the complement of $\mathrm{C}_n^+(f(\sigma))$. As there are fewer vertices in the link than in the whole sphere, and the whole sphere has at most v vertices, there are at least two vertices y and y' of height h that are not descendent of $f(\sigma)$ and that are not already in the link. It follows that for any simplex τ of the link the union $f(\tau) \cup f(\sigma) \cup \{y\}$ is independent and non-generating, and hence forms a simplex of T_r^∞. We can then replace f inside the star

$$\mathrm{Star}(\sigma) = \mathrm{Link}(\sigma) * \sigma \simeq \mathrm{S}^{k-p-1} * \mathrm{D}^p$$

through the cone on a new vertex x

$$\mathrm{Link}(\sigma) * x * \sigma \simeq \mathrm{D}^{k-p} * \mathrm{D}^p$$

by the map $f|_{\mathrm{Link}(\sigma)} * (x \mapsto y) * f|_\sigma$ on

$$\mathrm{Link}(\sigma) * x * \partial\sigma \simeq \mathrm{D}^{k-p} * \mathrm{S}^{p-1}.$$

Now $\mathrm{Link}(\sigma) * x * \partial(\sigma)$ has exactly one extra vertex compared to the star of σ, unless σ was just a vertex, in which case its boundary is empty, and it has the same number of vertices. And again, we have reduced the number of bad simplices by one.

By induction, we can now assume that there are no bad simplices for f with respect to a triangulation with at most v vertices. With this assumption, we can cone off f as follows. We have at least $w(h) \geqslant v+2$ vertices of height h in T_r^∞, and at most v vertices in the sphere. These vertices are mapped to vertices of height at least h, that is to descendants of the vertices of height h. By the pigeonhole principle, we know that there are at least two vertices, say y and y', of height h such that no vertex of the sphere is mapped to any of their descendants. Hence we can cone off the sphere using $\{y\}$. Indeed, this $\{y\}$ is disjoint and independent from the set $f(\sigma)$ for every simplex σ of the sphere, ensuring that the union $f(\sigma) \cup \{y\}$ still forms a simplex of T_r^∞. □

Proof of Proposition 3.2. Consider a map $f : \mathrm{S}^k \to \mathrm{U}_r^\infty$. Again we can assume that it is simplicial for some triangulation of the sphere S^k. We will show that there is a homotopy from f to a map that lands inside T_r^∞. This will prove the result by Proposition 3.1.

We will, just like in the previous proof, modify f by a homotopy on the stars of the *bad* simplices in S^k, namely those whose vertices are all mapped to $\mathrm{U}_r^\infty \backslash \mathrm{T}_r^\infty$. We will show how to reduce their number one by one, so that the result follows by induction.

Let σ be a bad simplex of maximal dimension, say p, in the sphere S^k. By maximality, the link of σ is mapped to $\mathrm{T}_r^\infty \cap \mathrm{Star}(f(\sigma))$, which itself equals $\mathrm{T}_r^\infty \cap \mathrm{Link}(f(\sigma))$ as the vertices of $f(\sigma)$ are mapped to $\mathrm{U}_r^\infty \setminus \mathrm{T}_r^\infty$. We now argue that the intersection $\mathrm{T}_r^\infty \cap \mathrm{Link}(f(\sigma))$ is isomorphic to $\mathrm{T}_{r'}^\infty$ for some $r' \geqslant 1$.

The simplex $f(\sigma)$ is a set of disjoint subsets of $\mathrm{C}_{n,r}^+$ that together form a non-generating independent subset. Let E be the minimal basis of C_r containing these. Let $\mathrm{C}_n^+(P)$ be the $[n]^*$–invariant subset of $\mathrm{C}_{n,r}^+$ that is generated by $P = E \setminus f(\sigma)$. Note that P is non-empty as $f(\sigma)$ was non-generating. In particular, we have $r' = |P| \geqslant 1$. The link of σ is mapped by f to independent subsets of $\mathrm{C}_n^+(P)$, because E is the minimal basis containing $f(\sigma)$, and any simplex of the link is mapped to a set of subsets of $\mathrm{C}_{n,r}^+$ that, together with $f(\sigma)$, forms an independent subset, and hence must lie in an expansion of E. It follows that $\mathrm{Link}(f(\sigma)) \cong \mathrm{U}_{r'}^\infty$ and hence also that $\mathrm{T}_r^\infty \cap \mathrm{Link}(f(\sigma)) \cong \mathrm{T}_{r'}^\infty$.

Let us consider the restriction of the map f to the star of the simplex σ. Since we are working inside a k-sphere, we have

$$\mathrm{Star}(\sigma) \cong \mathrm{Link}(\sigma) \star \sigma \cong \mathrm{S}^{k-p-1} \star \sigma.$$

The simplicial complex $\mathrm{T}_{r'}^\infty$ is contractible by Proposition 3.1. Therefore, the restriction of the map f to the star of the simplex σ can be extended to get a map $\hat{f} : \mathrm{D}^{k-p} * \sigma \to \mathrm{U}_r^\infty$ on the ball $\mathrm{D}^{k-p} * \sigma$ with D^{k-p} mapped to $T_{r'}^\infty$. This extension defines of homotopy on the star of σ, relative to the boundary link, from f to a map with strictly fewer bad simplices: The simplex σ is gone, and we have not introduced any new bad simplex, because new simplices are joins of simplices of $\partial\sigma$ and simplices in the disc, but the latter are mapped to good vertices. The result follows by induction. □

4 The classifying space for the Cantor groupoid

Let $\mathbf{Set}^\times$ denote the groupoid of finite sets and their isomorphisms and recall from Section 1 the groupoid $\mathbf{Cantor}_A^\times$ with objects the free Cantor algebras of type A, where A is a fixed set of cardinality at least 2, and morphisms their isomorphisms. The groupoids $\mathbf{Set}^\times$ and $\mathbf{Cantor}_A^\times$ are both symmetric monoidal with respect to categorical sums formed in the larger categories $\mathbf{Set}$ and $\mathbf{Cantor}_A$, where all maps of sets and all morphisms of Cantor algebras are allowed. The monoidal unit in $\mathbf{Set}^\times$ is the empty set $\varnothing$, and in $\mathbf{Cantor}_A^\times$ it is the empty algebra $\mathrm{C}_A(\varnothing)$. The purpose of this section is to describe the homotopy type of the classifying space $|\mathbf{Cantor}_A^\times|$ of the symmetric monoidal groupoid $\mathbf{Cantor}_A^\times$ and its associated infinite loop space in terms of the of the classifying spaces $|\mathbf{Set}^\times|$.

There are algebraic K-theory "machines" that produce for every symmetric monoidal category $\mathbf{C}$ a spectrum $\mathbb{K}(\mathbf{C})$ whose underlying infinite loop space $\Omega^\infty\mathbb{K}(\mathbf{C})$ is a group completion of the classifying space $|\mathbf{C}|$ of $\mathbf{C}$. (See [Tho82] for the particular machine that we will be using here.) The Barratt–Priddy–Quillen theorem says that for $\mathbf{C} = \mathbf{Set}^\times$, this spectrum is the sphere spectrum $\mathbb{S} \simeq \mathbb{K}(\mathbf{Set}^\times)$, so that there is a group completion $|\mathbf{Set}^\times| \to \Omega^\infty\mathbb{S}$ (see [BP72]).

In this section we will use a construction of Thomason to produce from $\mathbf{Set}^\times$ and a set A a symmetric monoidal category $\mathbf{Tho}_A$ and prove the following result.

Theorem 4.1. *There is a zigzag $\mathbf{Cantor}_A^\times \longleftrightarrow \mathbf{Tho}_A$ of symmetric monoidal functors that induce monoidal equivalences $|\mathbf{Cantor}_A^\times| \simeq |\mathbf{Tho}_A|$ between the classifying spaces.*

Corollary 4.2. *There is an equivalence $\mathbb{K}(\mathbf{Cantor}_A^\times) \simeq \mathbb{K}(\mathbf{Tho}_A)$ of spectra.*

The idea behind the symmetric monoidal category $\mathbf{Tho}_A$ is as follows. First of all, there is a symmetric monoidal functor from the symmetric monoidal category $\mathbf{Set}$ to $\mathbf{Cantor}_A$ that sends a set X to the free Cantor algebra $\mathrm{C}_A(X)$ of type A with basis X. It restricts to a symmetric monoidal functor $\mathrm{C}_A\colon \mathbf{Set}^\times \to \mathbf{Cantor}_A^\times$. There is also a symmetric monoidal endofunctor

$$\Sigma_A\colon \mathbf{Set}^\times \longrightarrow \mathbf{Set}^\times$$

that takes a set X to the set $X \times A$ and similarly for maps. The functor C_A has the property that it is insensitive to pre-composition with Σ_A: as we have already seen, there are isomorphisms $\mathrm{C}_A(X\times A) \cong \mathrm{C}_A(X)$, and these isomorphisms are essentially the defining property of Cantor algebras of type A. There is in fact a natural isomorphism $\mathrm{C}_A \circ \Sigma_A \cong \mathrm{C}_A$ of functors. This suggests that C_A extends over a "mapping torus of Σ_A" and the category $\mathbf{Tho}_A$, which we define below, will be precisely such a device.

Given a diagram $\mathbf{C}\colon \lambda \mapsto \mathbf{C}_\lambda$ of symmetric monoidal categories, indexed on a small category Λ, Thomason [Tho82, Sec. 3] defines a new symmetric monoidal category, denoted here by $\operatorname{hocolim}_\Lambda \mathbf{C}_\lambda$, with the property that

$$\mathbb{K}(\operatorname*{hocolim}_\Lambda \mathbf{C}_\lambda) \simeq \operatorname*{hocolim}_\Lambda \mathbb{K}(\mathbf{C}_\lambda). \tag{4.1}$$

In order to construct $\mathbf{Tho}_A$, we take the diagram that is indexed by the monoid $\mathbb{N}$ of natural numbers, thought of as a category with one object. Given a pair $(\mathbf{C},F)$ consisting of a symmetric monoidal category **C** together with a symmetric monoidal endo-functor F, we get a diagram on $\mathbb{N}$ that associates to the unique object the category **C** and to the morphism $k \in \mathbb{N}$ the functor F^k. Accordingly, we can define

$$\mathbf{Tho}_A = \operatorname*{hocolim}_{\mathbb{N}}(\mathbf{Set}^{\times}, \Sigma_A). \tag{4.2}$$

By the universal property of Thomason's construction [Tho82, p. 337], symmetric monoidal functors from $\mathbf{Tho}_A$ to $\mathbf{Cantor}_A^{\times}$ can be defined by the following data: A symmetric monoidal functor $\mathbf{Set}^{\times} \to \mathbf{Cantor}_A^{\times}$ and a symmetric monoidal natural transformation $\mathrm{C}_A \circ \Sigma_A \cong \mathrm{C}_A$. Since we already have these in our hands, this gives rise to a symmetric monoidal functor $\mathbf{Tho}_A \to \mathbf{Cantor}_A^{\times}$. It turns out that this functor induces an equivalence on the level of classifying spaces. To prove the equivalence between the two classifying spaces, we will proceed in three steps, as we explain now.

The idea behind the proof is as follows. Considering the functor $\mathbf{Tho}_A \to \mathbf{Cantor}_A^{\times}$, one sees that morphisms of $\mathbf{Cantor}_A^{\times}$ differ from morphisms of $\mathbf{Tho}_A$ in two ways. Firstly, the morphisms of $\mathbf{Tho}_A$ that are mapped to expansions are not invertible in $\mathbf{Tho}_A$. Secondly, only certain simple types of expansions occur directly as morphisms of $\mathbf{Tho}_A$. We will take care of these two issues one at a time, writing a homotopy equivalence in three steps: We will define a category $\mathbf{Exp}_A$ of finite sets, expansions and isomorphisms, where the expansions are not invertible anymore, and a category $\mathbf{Lev}_A$ of "level expansions" and isomorphisms, where level expansions are simpler types of expansions. These categories come with monoidal functors

$$\mathbf{Cantor}_A^{\times} \longleftarrow \mathbf{Exp}_A \longleftarrow \mathbf{Lev}_A \longrightarrow \mathbf{Tho}_A$$

and we will show that each functor in the sequence induces a homotopy equivalence by studying the fibers each time.

4.1 Expansions

We will now define the category $\mathbf{Exp}_A$ of expansions and we will then explain that passing from the groupoid $\mathbf{Cantor}_A^{\times}$ to the category $\mathbf{Exp}_A$ preserves the homotopy type of the classifying space.

Recall from Section 1 that a morphism from $\mathrm{C}_A(X)$ to $\mathrm{C}_A(Y)$ in $\mathbf{Cantor}_A^{\times}$ can be described by a triple (E,F,λ), with the set E an expansion of X, the set F an expansion of Y, and $\lambda\colon E \to F$ a bijection. The expansions of a given set X form a poset $\mathcal{E}(X)$: We write $E \leqslant F$ for two expansions E and F of X if F is also an expansion of E.

Proposition 4.3. *For all X the classifying space $|\mathcal{E}(X)|$ is contractible.*

Proof. The poset $\mathcal{E}(X)$ has a minimal element X. □

The poset $\mathcal{E}(X)$ of expansions depends functorially on the finite set $X \in \mathbf{Set}^{\times}$: To see that note that expansions of X are subsets of the free A^*–set $\mathrm{C}_A^+(X)$ and any bijection $\lambda\colon X \to Y$ induces an isomorphism $\mathrm{C}_A^+(\lambda)\colon \mathrm{C}_A^+(X) \to \mathrm{C}_A^+(Y)$ which takes expansions to expansions and induces in turn an isomorphism $\mathcal{E}(\lambda)\colon \mathcal{E}(X) \to \mathcal{E}(Y)$ of posets. More than that, if $E \in \mathcal{E}(X)$ and $F \in \mathcal{E}(Y)$ correspond under this isomorphism, then there is also an induced bijection $E \to F$ that we will denote by λ for short.

Definition 4.4. The *expansion category* $\mathbf{Exp}_A$ has objects the finite sets and morphism $X \to Y$ given by pairs (E,λ) of an expansion E of X and a bijection $\lambda\colon E \to Y$. This expansion category can be identified with the subcategory of $\mathbf{Cantor}_A^\times$, with the same objects, and with morphisms that can be presented as (E,Y,λ) with $F = Y$. This description also makes the composition obvious.

Proposition 4.5. *The inclusion $I\colon \mathbf{Exp}_A \to \mathbf{Cantor}_A^\times$ of categories induces an equivalence*

$$|\mathbf{Exp}_A| \simeq |\mathbf{Cantor}_A^\times|$$

on the level of classifying spaces.

Proof. By Proposition 4.3 and Quillen's Theorem A, the result will follow if we show that the fibers of I over $\mathrm{C}_A(X)$ are equivalent to $\mathcal{E}(X)$.

Consider the fiber $I/\mathrm{C}_A(X)$ over an object $\mathrm{C}_A(X)$ of $\mathbf{Cantor}_A^\times$. This category has objects the pairs (Y,y) with $y\colon \mathrm{C}_A(Y) \to \mathrm{C}_A(X)$ an isomorphism of Cantor algebras, and morphisms $(Y,y) \to (Z,z)$ given by maps $(E,\lambda)\colon Y \to Z$ in $\mathbf{Exp}_A$ such that $z \circ I(E,\lambda) = y$.

There is a functor $\Lambda\colon \mathcal{E}(X) \to I/\mathrm{C}_A(X)$ that is defined on objects by $\Lambda(E) = (E,e)$, where e is the isomorphism $\mathrm{C}_A(E) \to \mathrm{C}_A(X)$ described by the triple (E,E,id), and on morphisms by taking an expansion to the corresponding morphism of $I/\mathrm{C}_A(X)$.

We can also define a functor $\Pi\colon I/\mathrm{C}_A(X) \to \mathcal{E}(X)$ in the other direction: Given an object (Y,y) with $y\colon \mathrm{C}_A(Y) \to \mathrm{C}_A(X)$ an isomorphism of Cantor algebras, let (F,G,μ) be the minimal presentation of y. We define Π on objects by $\Pi(Y,y) = G$. On morphisms, we have no choice as the target category is a poset, but we have to check that, given a morphism $(E,\lambda)\colon (Y,y) \to (Z,z)$ in the fiber, the expansion $\Pi(Z,z)$ of X is an expansion of $\Pi(Y,y)$. This follows from the fact that $y = z \circ I(E,\lambda)$ and the fact that $I(E,\lambda)$ is an expansion: If y and z have minimal presentations (F,G,μ) and (H,L,ν), then the composition $z \circ I(E,\lambda) = (H,L,\nu) \circ (E,Z,\lambda)$ can be represented as $(E',L,\nu\circ\lambda)$ for E' an expansion of E. Given that this composition is equal to y, and that y has minimal presentation (F,G,μ), we necessarily have that L is an expansion of G.

The composition $\Pi\Lambda$ is clearly the identity. On the other hand, the composition $\Lambda\Pi$ takes an object $(Y,(F,E,\lambda))$ to $(E,(E,E,\mathrm{id}))$. There is a morphism in $I/\mathrm{C}_A(X)$ between these two objects, namely the morphism (F,E,λ). These assemble to a natural transformation between the identity functor on $I/\mathrm{C}_A(X)$ and the composition $\Lambda\Pi$. □

Remark 4.6. The category $\mathbf{Exp}_A$ has a symmetric monoidal structure induced by the disjoint union of sets, and the inclusion $I : \mathbf{Exp}_A \to \mathbf{Cantor}_A$ is a symmetric monoidal functor. Hence the equivalence of classifying spaces in Proposition 4.5 is given by a map of monoids.

4.2 Level expansions

The category $\mathbf{Exp}_A$ has morphisms generated by expansions and bijections. We will now decompose the expansions into simpler types of expansions, which we call *level expansions*, and construct a category $\mathbf{Lev}_A$ where the morphisms will now be generated by level expansions and bijections.

Definition 4.7. An expansion E of X is called a *level expansion* if there exists a subset P of X such that $E = (P \times A) \cup Q$ as a subset of $\mathrm{C}_A^+(X)$, where Q is the complement of P in X.

Note that any expansion E of a set X, considered as a morphism (E, X, id) in $\mathbf{Exp}_A$, can be factored as a composition of level expansions. There is even a canonical such factorization using the height filtration of $\mathrm{C}_A^+(X)$, but we will not use it. This says in particular that the composition of two level expansions does not necessarily yield a level expansion. For that reason, we do not get a category of level expansions right away. To define a category of level expansions analogous to $\mathbf{Exp}_A$, we will start with a simplicial complex of level expansions, and then pass to its poset of simplices.

Definition 4.8. Given a set X, we let $\mathrm{L}(X)$ denote the simplicial complex whose vertices are the expansions E of X, and where a set of $p+1$ expansions $E_0, \dots, E_p$ forms a p–simplex if, after re-ordering them, all of them are level expansions of E_0, and there exists pairwise disjoint subsets $P_1, \dots, P_p \subseteq E_0$ such that E_i is also the level expansion of E_{i-1} along P_i for each $i = 1, \dots, p$.

A pair of expansions of X forms an edge in $\mathrm{L}(X)$ if and only if one is a level expansion of the other. As every expansion can be factored as a composition of level expansions, this shows that $\mathrm{L}(X)$ is connected for every finite set X. The following result shows that these complexes are in fact contractible.

Proposition 4.9. *For all finite sets X, the complex $\mathrm{L}(X)$ is contractible.*

Proof. If Y is an expansion of X, we define $\mathrm{L}(X,Y)$ to be the full subcomplex of $\mathrm{L}(X)$ whose vertices are expansions E of X admitting Y as an expansion. Compactness of the spheres implies that every homotopy class can be represented by a map into some $\mathrm{L}(X,Y) \subseteq \mathrm{L}(X)$. It is therefore sufficient to show that the complexes $\mathrm{L}(X,Y)$ are contractible.

For an expansion E of X, we define its *height* to be $h(E) = (|E| - |X|)/(n-1)$. This is the number of simple expansions (expanding a single element x once) needed to obtain E from X. Suppose that the expansion Y has height h. Let $F_i\mathrm{L}(X,Y)$ denote the full subcomplex of $\mathrm{L}(X,Y)$ on the vertices of height at least i. This defines a descending filtration of $\mathrm{L}(X,Y)$.

$$\mathrm{L}(X,Y) = F_0\mathrm{L}(X,Y) \supseteq F_1\mathrm{L}(X,Y) \supseteq F_2\mathrm{L}(X,Y) \supseteq \cdots \supseteq F_h\mathrm{L}(X,Y) = \{Y\}$$

For each $i < h$, the complex $F_i\mathrm{L}(X,Y)$ is obtained from $F_{i+1}\mathrm{L}(X,Y)$ by attaching cones on the vertices E of height i along their links, because no two such vertices are part of the same simplex. The link of E inside $F_{i+1}\mathrm{L}(X,Y)$ identifies with the poset of non-trivial level expansions of E admitting Y as an expansion. (It also identifies with a subposet of the poset of non-empty subsets of E under inclusion.) This is a contractible poset as it has a maximal element, namely the level expansion along the largest possible subsets of E which still admits Y as an expansion, proving that $\mathrm{L}(X,Y)$ is contractible. □

Let $\mathcal{L}(X)$ denote the poset of simplices of $\mathrm{L}(X)$. The elements of $\mathcal{L}(X)$ are the simplices of $\mathrm{L}(X)$, and the morphisms are the inclusions among them.

Note that a p-simplex with vertices $E_0, \dots, E_p$ as in Definition 4.8 can be identified with (E_0, P), where E_0 is an expansion of X, and $P = (P_1, \dots, P_p)$ is a p-tuple of pairwise disjoint subsets of E_0. Under this identification, the faces of σ are

$$\partial_j(E_0, P_1, \dots, P_p) = \begin{cases} (E_1, P_2, \dots, P_p) & j = 0 \\ (E_0, P_1, \dots, P_j \cup P_{j+1}, \dots, P_p) & 0 < j < p \\ (E_0, P_2, \dots, P_{p-1}) & j = p. \end{cases}$$

In other words, the poset $\mathcal{L}(X)$ can be identified with the poset of the (E, P), with E an expansion of X and $P = (P_1, \dots, P_p)$ a p-tuple of pairwise disjoint subsets of E_0, for some $p \geqslant 0$. We have $(E, P) \leqslant (F, Q)$ if there is an i such that F is the expansion of E along the union $P_1 \cup \dots \cup P_i$, and

$$Q_j = P_{i+\kappa(1)+\dots+\kappa(j-1)+1} \cup \dots \cup P_{i+\kappa(1)+\dots+\kappa(j)}$$

for some function $\kappa\colon [q] \to \mathbb{N}$ such that $i + \kappa(1) + \dots + \kappa(q) \leqslant p$.

Note that a bijection $\lambda : X \to Y$ induces a map of simplicial complexes $\mathrm{L}(\lambda)\colon \mathrm{L}(X) \to \mathrm{L}(Y)$, and hence of poset of simplices $\mathcal{L}(\lambda)\colon \mathcal{L}(X) \to \mathcal{L}(Y)$. More than that, if (E, P) is mapped to (F, Q) by $\mathcal{L}(\lambda)$, then P and Q must be tuples of the same size and λ also induces a bijection $\lambda : E \to F$ with the property that $\lambda(P_i) = Q_i$ for each i, which we also write as $\lambda(P) = Q$.

We can now construct categories $\mathbf{Lev}_A$ from the posets $\mathcal{L}(X)$ in a way similar to the one in which we constructed the categories $\mathbf{Exp}_A$ from the posets $\mathcal{E}(X)$, where we consider all objects (E, P) of the poset $\mathcal{L}(X)$ for all sets X, and where morphisms are generated by bijections and "level expansions," now interpreted as the poset structure of $\mathcal{L}(X)$ for some X.

Definition 4.10. The category $\mathbf{Lev}_A$ has objects (E, P) for E a finite set and P a p-tuple of non-empty pairwise disjoint subsets of E (for some $p \geqslant 0$). The morphisms $(E, P) \to (F, Q)$ are given by triples (F', Q', λ) where $(E, P) \leqslant (F', Q')$ in $\mathcal{L}(E)$ and $\lambda : F' \to F$ is a bijection such that $\lambda(Q') = Q$.

Note in particular that, when there is a morphism from (E, P) to (F, Q) in $\mathbf{Lev}_A$, then there exists an expansion F' of E and a bijection $F' \cong F$. This means that forgetting the tuples P of subsets induces a functor $\mathbf{Lev}_A \to \mathbf{Exp}_A$.

Proposition 4.11. *The forgetful functor*

$$J\colon \mathbf{Lev}_A \to \mathbf{Exp}_A\,, (E, P) \longmapsto E$$

induces an equivalence $|\mathbf{Lev}_A| \simeq |\mathbf{Exp}_A|$ *on classifying spaces.*

Proof. We show that the fiber of J over X is homotopy equivalent to $\mathcal{L}(X)$. The result then follows from Proposition 4.9, which says that $\mathrm{L}(X) \simeq |\mathcal{L}(X)|$ is contractible.

The objects of the fiber $J\backslash X$ can be written as tuples $(E,P,\lambda\colon E'\cong E)$ for (E,P) an object of $\mathbf{Lev}_A$, an expansion E' of X and λ a bijection, as the pair (E',λ) defines a morphism $X\to J(E,P)$ in $\mathbf{Exp}_A$. A morphism from $(E,P,\lambda\colon E'\cong E)$ to $(F,Q,\mu\colon F'\cong F)$ is given by a morphism $\varphi\colon(E,P)\to(F,Q)$ in $\mathbf{Lev}_A$ such that $(F',\mu)=J(\varphi)\circ(E',\lambda)$.

Just as in the proof of Proposition 4.5, there is then a pair $\Lambda\colon\mathcal{L}(X)\leftrightarrow X\backslash J\colon\Pi$ of functors with

$$\Pi(E,P,\lambda\colon E'\cong E)=(E',\lambda^{-1}P)$$
$$\Lambda(E,P)=(E,P,\mathrm{id}\colon E=E).$$

Clearly the composition $\Pi\Lambda$ is the identity functor. On the other hand, we have

$$\Lambda\Pi(E,P,\lambda\colon E'\cong E)=(E',\lambda^{-1}P,\mathrm{id}\colon E'=E')$$

and λ defines a natural transformation with the identity. □

Remark 4.12. The category $\mathbf{Lev}_A$ has a symmetric monoidal structure induced by the disjoint union of sets, and the functor inducing the equivalence in Proposition 4.11 is monoidal with respect to that structure. Compare Remark 4.6. Consequently, the equivalence is given by a map of monoids.

4.3 Thomason's homotopy colimit

We will now recall Thomason's explicit model for the homotopy colimit (4.2).

The symmetric monoidal category $\mathbf{Tho}_A$ has objects the tuples $m[X_1,\dots,X_m]$ with $m\geqslant 0$ and each X_i a finite set. Morphisms are given as tuples

$$(\psi,\mu,f_1,\dots,f_n)\colon m[X_1,\dots,X_m]\longrightarrow n[Y_1,\dots,Y_n]$$

with a surjection $\psi\colon[m]\to[n]$, a function $\mu\colon[m]\to\mathbb{N}$, and bijections

$$f_j\colon X(\psi,\mu)_j\longrightarrow Y_j,$$

where $[m]=\{1,\dots,m\}$ as before and where we have already used the notation

$$X(\psi,\mu)_j=\coprod_{i\in\psi^{-1}(j)}X_i\times A^{\mu(i)}.$$

The composition of such morphisms is defined by composition of the surjections, corresponding "multi-additions" of the values $\mu(i)$ and composition of the bijections f_j, appropriately multiplied with powers of A. The monoidal structure is given on objects by

$$m[X_1,\dots,X_m]\oplus n[Y_1,\dots,Y_n]=(m+n)[X_1,\dots,X_m,Y_1,\dots,Y_n].$$

Note that a morphism $1[X]\to 1[Y]$ in the category $\mathbf{Tho}_A$ is given by a bijection $X\times A^{\mu}\cong Y$ for some $\mu\in\mathbb{N}$. These are powers of very special level expansions. A bit more generally, if E is a level expansion of X that expands a subset P of X once, and leaves the complement $Q = X \setminus P$ intact, we have a zigzag

$$1[X] \longleftarrow 2[P,Q] \longrightarrow 1[E]$$

in $\mathbf{Tho}_A$: The arrow to the left is given by $X = P \sqcup Q$, and the arrow to the right by $E = (P \times A) \sqcup Q$. This suggests that $2[P,Q]$ is the better object to represent the expansion than either of the sides. For general level expansions, we can say the following.

Lemma 4.13. *There is a functor*

$$\alpha_X : \mathcal{L}(X) \to \mathbf{Tho}_A$$

that sends the object (E,P) to the object $p+1[P,E \setminus P]$, when $P = (P_1,\dots,P_p)$ and where $E \setminus P$ denotes the set $E \setminus (P_1 \cup \dots \cup P_p)$.

Proof. We need to show functoriality of the indicated assignment. So we assume that we have a morphism $(E,P) \leqslant (F,Q)$ in the category $\mathcal{L}(X)$. Recall that this means that there is an i such that F is the expansion of E along the union $P_1 \cup \dots \cup P_i$, and the Q_j are given by

$$Q_j = P_{i+\kappa(1)+\dots+\kappa(j-1)+1} \cup \dots \cup P_{i+\kappa(1)+\dots+\kappa(j)}$$

for some function $\kappa\colon [q] \to \mathbb{N}$ such that $i+\kappa(1)+\dots+\kappa(q) \leqslant p$. This data precisely yields a morphism

$$(p+1)[P,E \setminus P] \to (q+1)[Q,F \setminus Q]$$

in $\mathbf{Tho}_A$: The surjection $\psi\colon [p+1] \to [q+1]$ has

$$\psi^{-1}(j) = \{i+\kappa(j-1)+1,\dots,i+\kappa(j-1)+\kappa(j)\}$$

for $j = 1,\dots,q$ (with the convention $\kappa(0) = 0$) and $\psi^{-1}(q+1) = \{1,\dots,i,p+1\}$, the function μ sends the integers $1,\dots,i$ to 1 and the other ones to 0, and the bijections are given by

$$(P_1 \times A) \sqcup \dots \sqcup (P_i \times A) \sqcup (E \setminus P) = F \setminus Q.$$

This finishes the definition of α_X on morphisms, and it is readily checked that this is compatible with composition and identities. □

Lemma 4.14. *For every bijection $\lambda : X \to Y$, there is a natural isomorphism*

$$\alpha_X \cong \alpha_Y \circ \mathcal{L}(\lambda)$$

of functors $\mathcal{L}(X) \to \mathbf{Tho}_A$, for $\mathcal{L}(\lambda)\colon \mathcal{L}(X) \to \mathcal{L}(Y)$ the poset isomorphism induced by λ.

Proof. Given an object (E,P), we have that $\mathcal{L}(\lambda)(E,P) = (\mathcal{L}(\lambda)E, \mathcal{L}(\lambda)P)$, so that

$$\alpha_X(E,P) = p+1[P,E \setminus P]$$

and

$$\alpha_Y \circ \mathcal{L}(\lambda) = p+1[\mathcal{L}(\lambda)P, \mathcal{L}(\lambda)(E \setminus P)].$$

We define the natural isomorphism between these as follows: The required surjection $[p+1] \to [p+1]$ is the identity, the function μ is constant 0, and the bijections $P_j \cong \mathcal{L}(\lambda)P_j$ and $E \setminus P \cong \mathcal{L}(\lambda)(E \setminus P)$ are the ones induced by λ. □

Let **Cat** denote the category of all small categories, and consider the functor $\mathcal{L}\colon \mathbf{Set}^{\times} \to \mathbf{Cat}$ taking a set X to the poset $\mathcal{L}(X)$. The Grothendieck construction $\mathbf{Gr}(\mathcal{L})$ is a category with objects the triples (X,E,P), with X a finite set and (E,P) an object of $\mathcal{L}(X)$, and a morphism $(X,E,P) \to (Y,F,Q)$ for each bijection $\lambda\colon X \to Y$ with $(\lambda(E),\lambda(P)) \leqslant (F,Q)$ in $\mathcal{L}(Y)$. Note that there is a forgetful functor

$$U\colon \mathbf{Gr}(\mathcal{L}) \longrightarrow \mathbf{Lev}_A$$

defined on objects by $U(X,E,P) = (E,P)$.

Proposition 4.15. *The functors α_X assemble to define a functor $H\colon \mathbf{Lev}_A \to \mathbf{Tho}_A$.*

Proof. The functors α_X of Lemma 4.13 and natural transformations of Lemma 4.14 together define a functor $\widehat{H}\colon \mathbf{Gr}(\mathcal{L}) \longrightarrow \mathbf{Tho}_A$ defined on objects by

$$\widehat{H}(X,E,P) = \alpha_X(E,P) = p+1[P,E\setminus P].$$

The result follows from noting that this functor factors through $\mathbf{Lev}_A$. □

To study the fibers of the functor $H\colon \mathbf{Lev}_A \to \mathbf{Tho}_A$, we introduce simplicial complexes $\mathrm{L}_0(X_1,\dots,X_k)$ associated to families $(X_1,\dots,X_k)$ of pairwise disjoint finite sets X_j. These will be subcomplexes of the complex $\mathrm{L}(X)$ defined in Section 4.2, where $X = X_1 \sqcup \cdots \sqcup X_k$ is their disjoint union.

The vertices of $\mathrm{L}_0(X_1,\dots,X_k)$ are the sets E of the form

$$E = (X_1 \times A^{\mu(1)}) \sqcup \cdots \sqcup (X_k \times A^{\mu(k)}) \tag{4.3}$$

for some function $\mu\colon [k] \to \mathbb{N}$. A set of $p+1$ distinct vertices $E_0,\dots,E_p$ forms a p–simplex if, possibly after reordering the sets E_i, the set is the set of vertices of a p–simplex $(E_0,P_1,\dots,P_p)$ of $\mathrm{L}(X)$ with each of the pairwise disjoint sets P_i a disjoint union of some of the $X_j \times A^{\mu(j)}$.

The set of vertices is a subset of the set of expansions of X that is canonically isomorphic to $\mathbb{N}^k$. A set of $p+1$ elements $\mu_0,\dots,\mu_p$ of $\mathbb{N}^k$ forms a p–simplex if, possibly after reordering them, $\mu_j - \mu_i$ takes values in $\{0,1\}$ for each $i<j$. In other words, from one E_i to the next, we only multiply some of the sets X_j one extra time with A, and each X_j is multiplied with A at most once between E_0 and E_p.

We let $\mathcal{L}_0(X_1,\dots,X_p)$ denote the poset of simplices of the simplicial complex $\mathrm{L}_0(X_1,\dots,X_p)$, and just as with $\mathcal{L}(X)$, we identify its objects with the tuples $(E,P_1,\dots,P_p)$ with E is as in (4.3) a vertex of $\mathrm{L}_0(X_1,\dots,X_k)$ and each P_i a disjoint union of $X_j \times A^{\mu(j)}$.

Proposition 4.16. *For each family $(X_1,\dots,X_k)$ of pairwise disjoint finite sets, the simplicial complex $\mathrm{L}_0(X_1,\dots,X_k)$ is contractible.*

Proof. The proof is essentially the same as for the complexes $\mathrm{L}(X)$. We use a filtration of the complex by the number of simple expansions needed to get from $X = X_1 \sqcup \cdots \sqcup X_k$ to the given vertex. Identifying the vertices with the elements of $\mathbb{N}^k$, we define the *height* of a given vertex μ

as $h(\mu) = \mu(1) + \cdots + \mu(k)$, and we let $F_h = F_h\mathrm{L}_0(X_1, \ldots, X_k)$ be the full subcomplex of vertices of height at most h. We have

$$\{X\} = F_0 \subset F_1 \subset \cdots \subset \mathrm{L}_0(X_1, \ldots, X_k).$$

We show by induction that each F_h is contractible. As any map from a sphere into the complex has image in a finite filtration stage by compactness, this will imply the statement.

Clearly, the subcomplex F_0 is a point, hence contractible. Suppose that F_{h-1} is contractible. The complex F_h is build from F_{h-1} by attaching a cone on each vertex of height h along its link in F_{h-1}. Given a vertex μ of height h, its link in F_{h-1} is the subcomplex of vertices ν of height at most $h-1$ that satisfy that $\varepsilon = \mu - \nu$ takes values in $\{0,1\}$. In particular, every element of the link of μ in F_{h-1} is in the link of the minimal such element, namely the vertex ν with $\varepsilon(i) = 1$ whenever $\mu(i) \geqslant 1$. Hence the link of μ in F_{h-1} is contractible. Given that F_{h-1} was contractible, it follows that F_h is also contractible. □

Proposition 4.17. *The functor H induces an equivalence $|\mathbf{Lev}_A| \simeq |\mathbf{Tho}_A|$.*

Proof. We show that the fiber of the functor $H\colon \mathbf{Lev}_A \to \mathbf{Tho}_A$ over $k[X_1, \ldots, X_k]$ is equivalent to $\mathcal{L}_0(X_1, \ldots, X_k)$. The proof is analogous to that of Propositions 4.5 and 4.11, only longer because the objects and morphisms in the present case are more complicated.

The fiber has as objects the pairs consisting of an object (E,P) of the category $\mathbf{Lev}_A$ and a morphism $f\colon k[X_1, \ldots, X_k] \to p+1[P, E \setminus P]$ in the category $\mathbf{Tho}_A$. That morphism f is given by a surjective map $\pi\colon [k] \to [p+1]$, some function $\mu\colon [k] \to \mathbb{N}$, and bijections $\lambda_j\colon X(\pi,\mu)_j \to P_j$, where we set $P_{p+1} = E \setminus P$ and

$$X(\pi,\mu)_j = \bigsqcup_{i \in \pi^{-1}(j)} X_i \times A^{\mu(i)}.$$

The morphism f factors canonically as

$$k[X_1, \ldots, X_k] \longrightarrow p+1[X(\pi,\mu)_1, \ldots, X(\pi,\mu)_{p+1}] \xrightarrow{\bar{f}} p+1[P, E \setminus P], \tag{4.4}$$

where the first morphism is given by π, the function μ, and the identity maps, and where the second morphism is given by the identity, the function 0, and the λ_j. In particular, such a morphism can only exist if the expansion E has the form

$$\bigsqcup_{j=1}^{k} X_j \times A^{\mu(j)} \tag{4.5}$$

for some function $\mu\colon [k] \to \mathbb{N}$. A morphism from an object (E,P,f) to another object (F,Q,g) is given by a morphism $c\colon (E,P) \leqslant (F',Q') \cong (F,Q)$ in $\mathbf{Lev}_A$ satisfying $g = H(c) \circ f$. Let us spell out in detail what this means: First, recall that $(E,P) \leqslant (F',Q')$ means that the expansion F' is obtained from the expansion E by multiplying the subset $P_1 \sqcup \cdots \sqcup P_i$ with the set A (for some index $i \geqslant 0$), that the subsets Q'_j are each a disjoint union of consecutive subsets P_j with $j > i$, and that we have $Q'_1 \sqcup \cdots \sqcup Q'_q = P_{i+1} \sqcup \cdots \sqcup P_q$ for some index $q \leqslant p$. Second, recall also that, if g is given by a surjection $\chi\colon [k] \to [q+1]$, some function $\nu\colon [k] \to \mathbb{N}$, and bijections κ_j, then the condition $g = H(c) \circ f$

yields

$$\nu(h) = \begin{cases} \mu(h)+1 & h \leqslant i \\ \mu(h) & h > i. \end{cases}$$

We can then use $H(c)$ to define a morphism c' that makes the diagram

$$\begin{array}{ccc} p+1[X(\pi,\mu)_1,\ldots,X(\pi,\mu)_{p+1}] & \xrightarrow{c'} & q+1[X(\chi,\nu)_1,\ldots,X(\chi,\nu)_{q+1}] \\ \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle \overline{g}} \\ p+1[P,E\setminus P] & \xrightarrow[H(c)]{} & q+1[Q,F\setminus Q] \end{array}$$

in $\mathbf{Tho}_A$ commute, where $\overline{f}$ and $\overline{g}$ are the isomorphisms coming from the canonical factorization (4.4) introduced above.

We can now define the functor $\Pi\colon H\setminus k[X_1,\ldots,X_k] \to \mathcal{L}_0(X_1,\ldots,X_k)$ on objects by

$$\Pi(E,P,f) = (E, X(\pi,\mu)_1,\ldots,X(\pi,\mu)_p),$$

when E is as in (4.5) for f determined by maps π,μ and λ_j. On morphism, the functor Π is given by $c \mapsto c'$.

There is also a functor $\Lambda\colon \mathcal{L}_0(X_1,\ldots,X_k) \to H\setminus k[X_1,\ldots,X_k]$ in the other direction. It is defined on objects by

$$\Lambda(E,P) = (E,P,f),$$

where f gets its μ from (4.5), and the surjection π and the bijection λ_j from the P_k. If $(E,P) \leqslant (F,Q)$, then this determines a morphism under $k[X_1,\ldots,X_k]$.

On the one hand, the composition $\Pi\Lambda$ is the identity. On the other hand, the composition $\Lambda\Pi$ takes an object (E,P,f) with E as in (4.5) to

$$\Lambda\Pi(E,P,f) = (\bigsqcup_{j=1}^{k} X_j \times A^{\mu(j)}, X(\pi,\mu)_1,\ldots,X(\pi,\mu)_p, f'),$$

where f' relates to f as c' relates to c. Now f brings bijections λ_j that give a natural isomorphism from this to (E,P,f), finishing the proof. □

Remark 4.18. The functor inducing the equivalence in Proposition 4.17 is monoidal with respect to the symmetric monoidal structures. Compare with Remark 4.12. Consequently, the equivalence is given by a map of monoids.

5 Homotopy theory and homology

In this section, we identify the spectrum $\mathbb{K}(\mathbf{Tho}_A)$ from the preceding section with the Moore spectrum for $\mathbb{Z}/(n-1)$, where n is the cardinality of A as above. We will use this and the relationship between the category $\mathbf{Tho}_A$ and the Higman–Thompson groups to give, in Section 6, concrete computations of the homology of the Higman–Thompson groups.

5.1 Entry of the Moore spectra

For any integer $n \geqslant 1$ let $\mathbb{M}_n$ denote the homotopy cofiber of the multiplication by n map on the sphere spectrum $\mathbb{S}$, so that there is a homotopy cofiber sequence as follows.

$$\mathbb{S} \xrightarrow{n} \mathbb{S} \longrightarrow \mathbb{M}_n \tag{5.1}$$

The spectrum $\mathbb{M}_n$ is the Moore spectrum for $\mathbb{Z}/n$, also known as the *mod n Moore spectrum*.

Theorem 5.1. *Let A be a finite set of cardinality $n \geqslant 2$. There is an equivalence of spectra*

$$\mathbb{K}(\mathbf{Tho}_A) \simeq \mathbb{M}_{n-1}.$$

Proof. Recall that the category $\mathbf{Tho}_A$ is defined from the diagram of categories on $\mathbb{N}$ which takes the unique object to the category $\mathbf{Set}^\times$ and the arrow k to the functor Σ_A^k that takes the product with A^k. By the Barratt–Priddy–Quillen theorem, we have that $\mathbb{K}(\mathbf{Set}^\times) \simeq \mathbb{S}$ and the functor Σ_A induces multiplication with the cardinality n of A on $\mathbb{S}$. We apply Thomason's formula (4.1) to our definition (4.2) and get

$$\mathbb{K}(\mathbf{Tho}_A) = \mathbb{K}(\operatorname*{hocolim}_{\mathbb{N}}(\mathbf{Set}^\times, \Sigma_A)) \simeq \operatorname*{hocolim}_{\mathbb{N}}(\mathbb{K}(\mathbf{Set}^\times), \mathbb{K}(\Sigma_A)) \simeq \operatorname*{hocolim}_{\mathbb{N}}(\mathbb{S}, n).$$

There is a spectral sequence

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(\mathbb{N}; (\pi_q\mathbb{S}, n)) \implies \pi_* \operatorname*{hocolim}_{\mathbb{N}}(\mathbb{S}, n)$$

that computes the homotopy groups of the homotopy colimit, see [Tho82, Sec. 3]. Here $(\pi_q\mathbb{S}, n)$ is the diagram of abelian groups on $\mathbb{N}$ that takes the object to $\pi_q\mathbb{S}$ and the morphism 1 to multiplication by n. The monoid ring $\mathbb{Z}[\mathbb{N}] \cong \mathbb{Z}[\mathrm{T}]$ is polynomial on one generator T, so that we can use the standard Koszul resolution

$$\mathbb{Z}[\mathrm{T}] \xrightarrow{\mathrm{T}-1} \mathbb{Z}[\mathrm{T}] \longrightarrow \mathbb{Z}$$

to compute the E^2 page. As the resolution has length one, the spectral sequence degenerates at E^2, and yields a long exact sequence

$$\cdots \longrightarrow \pi_*\mathbb{S} \xrightarrow{n-1} \pi_*\mathbb{S} \longrightarrow \pi_* \operatorname*{hocolim}_{\mathbb{N}}(\mathbb{S}, n) \longrightarrow \cdots. \tag{5.2}$$

Let $\varepsilon \in \pi_0 \operatorname{hocolim}_{\mathbb{N}}(\mathbb{S}, n) = [\mathbb{S}, \operatorname{hocolim}_{\mathbb{N}}(\mathbb{S}, n)]$ be the image of $1 \in \pi_0\mathbb{S} = [\mathbb{S}, \mathbb{S}]$ in the long exact sequence. From the long exact sequence we get that $(n-1)\varepsilon = 0$. Therefore, this map factors through the Moore spectrum to give a map

$$\overline{\varepsilon} \colon \mathbb{M}_{n-1} \longrightarrow \operatorname*{hocolim}_{\mathbb{N}}(\mathbb{S}, n).$$

Comparison of the long exact sequence obtained from the homotopy cofibration sequence (5.1) with the long exact sequence (5.2) shows that $\overline{\varepsilon}$ induces an isomorphism on homotopy groups, and thus that it is an equivalence. □

Together with Corollary 4.2 the proposition gives the following result.

Corollary 5.2. *For any finite set A of cardinality $n \geqslant 2$ there is an equivalence of spectra*

$$\mathbb{K}(\mathbf{Cantor}_A^{\times}) \simeq \mathbb{M}_{n-1}.$$

Remark 5.3. It is instructive to work out the implications of Corollary 5.2 on the level of components. The cofiber sequence (5.1) shows that the group $\pi_0\mathbb{M}_{n-1}$ is the cokernel of the multiplication by $n-1$ map on the group $\pi_0\mathbb{S} = \mathbb{Z}$, so that it is cyclic of order $n-1$. On the other hand the abelian group $\pi_0\Omega^\infty\mathbb{K}(\mathbf{Cantor}_A^{\times}) = \pi_0\mathrm{K}(\mathbf{Cantor}_A^{\times})$ is the group completion of the abelian monoid $\pi_0|\mathbf{Cantor}_A^{\times}|$. The latter can be identified with $\{0,1,\ldots,n-1\}$ as a set, where an integer r corresponds to the free Cantor algebra $\mathrm{C}_{n,r}$ of type n on r generators. The monoid structure is dictated by $\mathrm{C}_{n,r} \oplus \mathrm{C}_{n,s} \cong \mathrm{C}_{n,r+s}$ and $\mathrm{C}_{n,r+(n-1)} \cong \mathrm{C}_{n,r}$ if $r \geqslant 1$. Note that the neutral element 0 is the only invertible element in this monoid. The group completion of this monoid is $\mathbb{Z}/(n-1)$ by the theorem, but this can of course also be worked out by hand: Once 1 is inverted, the element $n-1$ is identified with 0 and we immediately obtain the group $\mathbb{Z}/(n-1)$.

5.2 The (stable) homology of the Higman–Thompson groups

If $\mathbb{X}$ is a spectrum, its associated infinite loop space $\Omega^\infty\mathbb{X}$ is a group-like monoid, that is a monoid whose monoid of components is a group. It follows that all its components are homotopy equivalent, and in the following, we denote by $\Omega_0^\infty\mathbb{X}$ the component corresponding to the zero element $0 \in \pi_0\mathbb{X}$. We will now relate the homology of the zeroth component $\Omega_0^\infty\mathbb{K}(\mathbf{Cantor}_A)$ to that of the Higman–Thompson groups. Together with Corollary 5.2 this will yield the main identification we are after, namely the isomorphism of the homology of the Higman–Thompson groups with that of $\Omega_0^\infty\mathbb{M}_{n-1}$.

Recall from Section 3 the stabilization homomorphism $\sigma_r\colon \mathrm{V}_{n,r} \to \mathrm{V}_{n,r+1}$ and let

$$\mathrm{V}_{n,\infty} = \mathrm{colim}(\mathrm{V}_{n,1} \longrightarrow \mathrm{V}_{n,2} \longrightarrow \cdots)$$

denote the associated *stable group*. The homology of $\mathrm{V}_{n,\infty}$ is usually called the *stable homology* of the groups $\mathrm{V}_{n,r}$ for $r \geqslant 1$, but a direct consequence of our stability theorem, Theorem 3.6, is that

$$\mathrm{H}_*(\mathrm{V}_{n,r}) \cong \mathrm{H}_*(\mathrm{V}_{n,\infty}).$$

So in the case at hand, *all* the homology is stable and hence it is enough to identify the homology of $\mathrm{V}_{n,\infty}$.

Given a monoid M, one can form its bar construction $\mathrm{B}M$ and the loop space $\Omega\mathrm{B}M$ is a group-like space, known as the *group completion of M*. The group completion theorem of McDuff and Segal [McDS76] identifies in good cases the homology of $\Omega\mathrm{B}M$ with that of its "stable part." We will use this theorem to compute the homology of $\Omega^\infty\mathbb{K}(\mathbf{Cantor}_A^\times) \simeq \Omega B|\mathbf{Cantor}_A^\times|$, the group completion of the E_∞ monoid $|\mathbf{Cantor}_A^\times|$.

Theorem 5.4. *Let $A = \{1,\ldots,n\}$ with $n \geqslant 2$. There is a map*

$$\mathrm{BV}_{n,\infty} \longrightarrow \Omega_0^\infty\mathbb{K}(\mathbf{Cantor}_A^\times)$$

which induces an isomorphism in homology with all systems of local coefficients on $\Omega_0^\infty\mathbb{K}(\mathbf{Cantor}_A^\times)$.

Proof. We apply the group completion theorem to the monoid $M = |\mathbf{Cantor}_A^\times|$. More precisely, we will use Theorem 1.1 in [R-W13], which makes explicit the relevant result in [McDS76]. The monoid M is homotopy commutative, in fact E_∞, as it is the classifying space of a symmetric monoidal category. It has components indexed by $0,\ldots,n-1$, forming a monoid in the way describe in Remark 5.3. To apply Theorem 1.1 of [R-W13], we use the constant sequence of elements of M given by $\mathrm{C}_{n,1},\mathrm{C}_{n,1},\ldots$. We need to check that for every $m \in M$, the component of m in M is a right factor of the component of some finite sum $\mathrm{C}_{n,1} \oplus \cdots \oplus \mathrm{C}_{n,1}$, which is obvious as every component can be reached this way except for the zero component which is a right factor of any such sum.

Form the colimit

$$M_\infty = \operatorname{colim}\big(|\mathbf{Cantor}_A^\times| \xrightarrow{\oplus \mathrm{C}_{n,1}} |\mathbf{Cantor}_A^\times| \xrightarrow{\oplus \mathrm{C}_{n,1}} \cdots\big).$$

Theorem 1.1 in [R-W13] says that there is a homology isomorphism

$$M_\infty \longrightarrow \Omega\mathrm{B}|\mathbf{Cantor}_A^\times| \simeq \Omega^\infty\mathbb{K}(\mathbf{Cantor}_A)$$

with respect to all local coefficient systems on the target. Now the zeroth component of M_∞ can be identified with the colimit on classifying space of the maps

$$\mathrm{V}_{n,0} \longrightarrow \mathrm{V}_{n,1} \xrightarrow{\sigma_1} \cdots \longrightarrow \mathrm{V}_{n,n-1} \xrightarrow{\sigma_{n-1}} \mathrm{V}_{n,n} \xrightarrow{\sigma_n} \mathrm{V}_{n,n+1} \longrightarrow \cdots$$

of groups , and this colimit of groups is the group $\mathrm{V}_{n,\infty}$ in the stability statement. The result follows. ☐

We are now ready to prove the main result of this text.

Proof of Theorem A. As a consequence of our stability result, Theorem 3.6, for all $r \geqslant 1$, the stabilization map $\sigma_r\colon \mathrm{V}_{n,r} \to \mathrm{V}_{n,r+1}$ induces an isomorphism in homology with coefficients in any $\mathrm{H}_1(\mathrm{V}_{n,\infty})$–module. Note that, in particular, we have an isomorphism $\mathrm{H}_1(\mathrm{V}_{n,r}) \cong \mathrm{H}_1(\mathrm{V}_{n,\infty})$, so $\mathrm{H}_1(\mathrm{V}_{n,\infty})$–modules are the same as $\mathrm{H}_1(\mathrm{V}_{n,r})$–modules. It follows that the map $\mathrm{BV}_{n,r} \to \mathrm{BV}_{n,\infty}$ induces an isomorphism in homology with abelian coefficients for all $r \geqslant 1$. Theorem 5.4 gives that there is a map $\mathrm{BV}_{n,\infty} \to \Omega_0^\infty(\mathbf{Cantor}_A^\times)$ which induces an isomorphism in homology with all local coefficients on the target. (Note that $\pi_1\Omega_0^\infty(\mathbf{Cantor}_A^\times) \cong \mathrm{H}_1(\Omega_0^\infty(\mathbf{Cantor}_A^\times)) \cong \mathrm{H}_1(\mathrm{V}_{n,\infty})$, so these are again $\mathrm{H}_1(\mathrm{V}_{n,r})$–modules as above.) Corollary 5.2, we have a homotopy equivalence $\Omega_0^\infty\mathbb{K}(\mathbf{Cantor}_A^\times) \simeq \Omega_0^\infty\mathbb{M}_{n-1}$, proving the result. ☐

6 Computational consequences

We will in this section explain how Theorem A can be used to give explicit consequences about the homology of the Higman–Thompson groups. Concretely, we will compute the abelianizations and Schur multipliers of the Higman–Thompson groups directly from Theorem A, and we will also completely decide which of the groups are integrally or rationally acyclic. We recover old results with new methods, as well as prove new results. In particular, we prove the acyclicity of the Thompson group V.

The results in this section are based on computations of the homology groups of the infinite loop space of the Moore spectrum $\mathbb{M}_n$ with classical methods from homotopy theory. Given a spectrum $\mathbb{X}$, the stable homotopy groups $\pi_*\mathbb{X}$ agree with the (unstable) homotopy groups $\pi_*\Omega^\infty\mathbb{X}$ of the underlying infinite loop space $\Omega^\infty\mathbb{X}$. The situation is different for homology, however. The homology of the Moore spectrum $\mathbb{M}_n$ is, up to a shift, the homology of the mod n Moore space: $\mathrm{H}_0\mathbb{M}_n \cong \mathbb{Z}/n$ and $\mathrm{H}_d\mathbb{M}_n = 0$ for $d \neq 0$. In contrast, the homology of the underlying infinite loop space $\Omega^\infty\mathbb{M}_n$ is more difficult to compute. We will here give some partial computations of these homology groups. Further computations can be obtained by working harder.

6.1 Abelianizations and Schur multipliers

In this section, we compute H_1 and H_2 as well as the first non-trivial homology group of $\mathrm{V}_{n,r}$ by computing these groups for $\Omega^\infty_0\mathbb{M}_{n-1}$. We confirm and extend the known results from the literature.

Proposition 6.1. *For all $n \geqslant 2$ and $r \geqslant 1$ there are isomorphisms*

$$\mathrm{H}_1(\mathrm{V}_{n,r}) \cong \begin{cases} 0 & n \text{ even} \\ \mathbb{Z}/2 & n \text{ odd.} \end{cases}$$

Proposition 6.2. *For all $n \geqslant 3$, we have that*

$$\mathrm{H}_d(\mathrm{V}_{n,r}) \cong \begin{cases} 0 & 0 < d < 2p-3 \\ \mathbb{Z}/p & d = 2p-3 \end{cases}$$

for p the smallest prime dividing $n-1$, and

$$\mathrm{H}_{2q-3}(\mathrm{V}_{n,r}) \neq 0$$

for q any prime dividing $n-1$.

Proposition 6.1 is essentially the case $p = 2$ of Proposition 6.2. We prove both propositions together.

Proof of Propositions 6.1 and 6.2. By Theorem A, it is equivalent to compute these homology groups for $\Omega^\infty_0\mathbb{M}_{n-1}$. The space $\Omega^\infty_0\mathbb{M}_{n-1}$ is a connected infinite loop space, and so

$$\mathrm{H}_1\Omega^\infty_0\mathbb{M}_{n-1} \cong \pi_1\Omega^\infty_0\mathbb{M}_{n-1} \cong \pi_1\mathbb{M}_{n-1}.$$

As $\pi_1\mathbb{S} \cong \mathbb{Z}/2$, the latter can easily be computed from the cofiber sequence (5.1) of spectra to be the cokernel of multiplication by $n-1$ on $\pi_1\mathbb{S} = \mathbb{Z}/2$, which proves the first proposition.

For the second proposition, we assume that $n \geqslant 3$ so that $n-1 \geqslant 2$. Let p be a prime. The p-parts of the homotopy groups of the sphere spectrum vanish below dimension $2p-3$ and $\pi_{2p-3}\mathbb{S} \otimes \mathbb{Z}/p \cong \mathbb{Z}/p$. Now multiplication by $n-1$ is an isomorphism on $\mathbb{Z}/p$ if and only if p does not divide $n-1$, and if it does, the map is zero. The result follows using the same long exact sequence now for homotopy groups with coefficients and Hurewicz's theorem. □

The following result recovers and extends the computation of Kapoudjian [Kap02], who worked out the case $r = 1$ by entirely different methods.

Proposition 6.3. *For all $n \geqslant 2$ and $r \geqslant 1$ there are group isomorphisms*

$$\mathrm{H}_2(\mathrm{V}_{n,r}) \cong \begin{cases} 0 & n \text{ even} \\ \mathbb{Z}/4 & n \equiv 3 \text{ mod } 4 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n \equiv 1 \text{ mod } 4. \end{cases}$$

Proof. Again, by Theorem A, it is equivalent to compute these homology groups for $\Omega_0^\infty\mathbb{M}_{n-1}$. Let us first tick off the case when n is even, so that $n-1$ is odd. By Proposition 6.2, when $n \geqslant 3$ with $n-1$ odd, the first possible non-trivial homology group of $\mathrm{V}_{n,r}$ is in degree $2 \cdot 3 - 3 = 3$ (if 3 divides $n-1$). In particular H_2 vanishes. If $n = 2$, the group vanishes because multiplication by $n-1$ is the identity, making $\mathbb{M}_1$ the trivial spectrum. (See also the more general Theorem 6.4.)

Let us now assume that n is odd, and write $X = \Omega_0^\infty\mathbb{M}_{n-1}$. We have that $\pi_1 X \cong \mathbb{Z}/2$. Consider the Postnikov truncation X_2, with same first and second homotopy groups as X. We have $\mathrm{H}_2 X \cong \mathrm{H}_2 X_2$. As X is an infinite loop space, its first k-invariant vanishes (see [Arl90]), so that

$$X_2 \simeq \mathrm{K}(\pi_1 X, 1) \times \mathrm{K}(\pi_2 X, 2).$$

Now Künneth theorem gives

$$\mathrm{H}_2 X_2 \cong \mathrm{H}_2\mathrm{K}(\pi_1 X, 1) \oplus \mathrm{H}_2\mathrm{K}(\pi_2 X, 2) \cong \mathrm{H}_2\mathrm{K}(\pi_2 X, 2)$$

given that $\pi_1 X \cong \mathbb{Z}/2$ has trivial H_2. As $\mathrm{H}_2\mathrm{K}(\pi_2 X, 2) \cong \pi_2 X$, we obtain that

$$\mathrm{H}_2\Omega_0^\infty\mathbb{M}_{n-1} \cong \pi_2\Omega_0^\infty\mathbb{M}_{n-1} \cong \pi_2\mathbb{M}_{n-1},$$

which reduces the question to stable homotopy theory as above. When n is odd, the homotopy cofibre sequence (5.1) only shows that this group is of order 4. We need a further computation to identify the group.

Let us assume that $n \equiv 3$ mod 4. Then $n-1$ is even but not divisible by 4, and we can use the cofibre sequence

$$\mathbb{M}_{(n-1)/2} \longrightarrow \mathbb{M}_{n-1} \longrightarrow \mathbb{M}_2$$

from the octahedral axiom to see that $\pi_2\mathbb{M}_{n-1} \cong \pi_2\mathbb{M}_2$, and this is known to be cyclic (of order 4). One way to see this is to note that the unit $\mathbb{S} \to \mathbb{KO}$ of the real topological K-theory spectrum $\mathbb{KO}$ induces an isomorphism

$$\pi_2\mathbb{M}_n \cong \mathbb{KO}_2\mathbb{M}_n \cong \mathbb{KO}^0\Sigma\mathbb{M}_n$$

for any n because $\pi_i\mathbb{S} \to \mathbb{KO}_i\mathbb{S}$ is an isomorphism for $i = 1,2$. Now $\mathbb{KO}^0\Sigma\mathbb{M}_2$ is the 0-th reduced real K-theory of the real projective plane, which is cyclic of order 4.

Lastly, if $n \equiv 1 \bmod 4$, then $n-1$ is divisible by 4. The factorization $4k = 2k\cdot 2$ gives a map $j\colon \mathbb{M}_2 \to \mathbb{M}_{4k}$ that has even degree on the bottom cell and odd degree on the top cell. Analyzing the diagram

$$\begin{array}{ccccccc}
\cdots \longleftarrow & \mathbb{KO}^0(\Sigma\mathbb{S}) \cong \mathbb{Z}/2 & \longleftarrow & \mathbb{KO}^0(\Sigma\mathbb{M}_2) \cong \mathbb{Z}/4 & \longleftarrow & \mathbb{KO}^0(\Sigma^2\mathbb{S}) \cong \mathbb{Z}/2 & \longleftarrow \cdots \\
 & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle j^*} & & \uparrow{\scriptstyle \cong} & \\
\cdots \longleftarrow & \mathbb{KO}^0(\Sigma\mathbb{S}) \cong \mathbb{Z}/2 & \longleftarrow & \mathbb{KO}^0(\Sigma\mathbb{M}_{4k}) \cong\ ? & \longleftarrow & \mathbb{KO}^0(\Sigma^2\mathbb{S}) \cong \mathbb{Z}/2 & \longleftarrow \cdots
\end{array}$$

shows that j^* cannot be zero or epi, so that $\mathbb{KO}^0\Sigma\mathbb{M}_{4k}$ must split into $\mathbb{Z}/2$ summands. □

6.2 Acyclicity results

We now deduce global results about the homology of the groups $\mathrm{V}_{n,r}$.

The following result has been suggested by Brown [Bro92, Sec. 6].

Theorem 6.4. *For all $r \geqslant 1$, the Thompson group $\mathrm{V} \cong \mathrm{V}_{2,r}$ is integrally acyclic:*

$$\mathrm{H}_d(\mathrm{V}) = \mathrm{H}_d(\mathrm{V}_{2,r}) = 0$$

for all $d \neq 0$.

Proof. For $n = 2$ multiplication by $n-1 = 1$ is homotopic to the identity, so that it is a self-equivalence of the sphere spectrum. Then the homotopy cofiber, the Moore spectrum $\mathbb{M}_1$, is contractible, and the homology of the infinite loop space vanishes. □

Theorem 6.5. *For all $n \geqslant 3$ and $r \geqslant 1$, the group $\mathrm{V}_{n,r}$ is rationally but not integrally acyclic:*

$$\mathrm{H}_d(\mathrm{V}_{n,r}) \otimes \mathbb{Q} = 0$$

for all $d \neq 0$, but

$$\mathrm{H}_{2p-3}(\mathrm{V}_{n,r}) \neq 0$$

for any prime p such that p divides $n-1$.

Proof. For $n \geqslant 2$ multiplication by $n-1$ is a rational equivalence, so that the Moore spectrum $\mathbb{M}_{n-1}$ is rationally contractible, and the rational homology groups vanish. This proves the first part of the statement. The second part of the statement is given by Proposition 6.2. □

Remark 6.6. In the case $n = 2$, rationally acyclicity of the Thompson group has earlier been shown by Brown [Bro92, Thm. 4], where the author also indicates that his proof can be adapted to prove the case $n \geqslant 3$. The case $n = 2$, still only rationally, was later reproved by Farley in [Far05].

We end by mentioning a consequence of our work for the commutator subgroups. When n is odd, Proposition 6.1 implies that the commutator subgroup $\mathrm{V}^+_{n,r}$ of $\mathrm{V}_{n,r}$ is an index-two subgroup. Let $\tilde{\Omega}^\infty_0\mathbb{K}(\mathbf{Cantor}^\times_A)$ denote the universal cover of the space $\Omega^\infty_0\mathbb{K}(\mathbf{Cantor}^\times_A)$. Shapiro's Lemma, Theorem A and Theorem 3.6 applied to the twisted coefficients $M = \mathbb{Z}\mathrm{H}_1(\mathrm{V}_{n,r}) \cong \mathbb{Z}\mathrm{H}_1(\Omega^\infty_0\mathbb{K}(\mathbf{Cantor}^\times_A))$ give the following:

Corollary 6.7. *There are homology isomorphisms*

$$\mathrm{H}_*(\mathrm{V}^+_{n,r}) \cong \mathrm{H}_*(\mathrm{V}^+_{n,\infty}) \cong \mathrm{H}_*(\tilde{\Omega}^\infty_0\mathbb{K}(\mathbf{Cantor}^\times_A)).$$

In particular, the groups $\mathrm{V}^+_{n,r}$ and $\mathrm{V}^+_{n,\infty}$ are not acyclic when n is odd.

Proof. See Sections 3.1 and 3.2 of [R-WW]. □

This answers a question of Sergiescu.

Acknowledgment

This research has been supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92) in Copenhagen. Parts of this paper were conceived while the authors were visiting the Hausdorff Research Institute for Mathematics (HIM) in Bonn. The authors would like to thank Dustin Clausen and Oscar Randal-Williams for pointing out gaps in earlier versions of this paper. The first author would also like to thank Ricardo Andrade, Ken Brown, Bjørn Dundas, Magdalena Musat, Martin Palmer, and Vlad Sergiescu for conversations related to the subject of this paper.

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