Homological stability for mapping class groups of surfaces

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Abstract. We give a complete and detailed proof of Harer's stability theorem for the homology of mapping class groups of surfaces, with the best stability range presently known. This theorem and its proof have seen several improvements since Harer's original proof in the mid-80's, and our purpose here is to assemble these many additions.

1. Introduction

The purpose of this paper is to give a complete and detailed proof of Harer's stability theorem for the homology of the mapping class groups of surfaces, with the best known bound. Harer's paper has been improved a number of times over the past 35 years, by various authors and the argument given here attempts to give a "best of" from these papers.

Let $S_{g,r}$ denote a surface of genus g with r boundary components. The mapping class group of $S_{g,r}$,

$$\Gamma_{g,r} := \pi_0 \operatorname{Diff}(S_{g,r} \text{ rel } \partial),$$

is the group of components of the orientation preserving diffeomorphism group of $S_{g,r}$, where the diffeomorphisms are assumed to be the identity on the boundary of $S_{g,r}$. We consider in this paper the homology of these groups. Recall that the homology of a group G (as a group) equals the homology of its classifying space BG (as a space), and that the moduli space of Riemann surfaces $\mathcal{M}_{g,r}$ is a model for the classifying space $B\Gamma_{g,r}$ when $r > 0$, and a rational model when $r = 0$, i.e. $H_*(\Gamma_{g,r}, \mathbb{Z}) \cong H_*(\mathcal{M}_{g,r}, \mathbb{Z})$ when $r > 0$ and $H_*(\Gamma_{g,0}, \mathbb{Q}) \cong H_*(\mathcal{M}_g, \mathbb{Q})$.

Gluing a pair of pants along two or one of its boundary components define inclusions $\alpha\colon S_{g,r+1} \hookrightarrow S_{g+1,r}$ and $\beta\colon S_{g,r} \hookrightarrow S_{g,r+1}$, which induce maps on the mapping class groups

$$\alpha_g\colon \Gamma(S_{g,r+1}) \to \Gamma(S_{g+1,r}) \quad \text{and} \quad \beta_g\colon \Gamma(S_{g,r}) \to \Gamma(S_{g,r+1})$$

by extending diffeomorphisms to be the identity on the added pairs of pants.

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Note that β_g is injective, with left inverse the map

$$\delta_g\colon \Gamma(S_{g,r+1}) \to \Gamma(S_{g,r})$$

induced by gluing a disc on one of the newly created boundary components. The main theorem proved in this paper is the following:

Theorem 1.1. *Let $g \geq 0$ and $r \geq 1$. The map*

$$H_*(\alpha_g)\colon H_*(\Gamma(S_{g,r+1}),\mathbb{Z}) \to H_*(\Gamma(S_{g+1,r}),\mathbb{Z})$$

is surjective for $* \leq \frac{2}{3}g + \frac{1}{3}$ and an isomorphism for $* \leq \frac{2}{3}g - \frac{2}{3}$. The map

$$H_*(\beta_g)\colon H_*(\Gamma(S_{g,r}),\mathbb{Z}) \to H_*(\Gamma(S_{g,r+1}),\mathbb{Z})$$

is always injective and is an isomorphism $* \leq \frac{2}{3}g$.

Considering $\delta_g\colon \Gamma(S_{g,1}) \to \Gamma(S_{g,0})$ now induced by gluing a disc to the only boundary component of $S_{g,1}$, we also get a stability result for closed surfaces:

Theorem 1.2. *The map $H_*(\delta_g)\colon H_*(\Gamma_{g,1},\mathbb{Z}) \to H_*(\Gamma_{g,0},\mathbb{Z})$ is surjective for $* \leq \frac{2}{3}g + 1$ and an isomorphism for $* \leq \frac{2}{3}g$.*

Theorems 1.1 and 1.2 were first proved by Harer in [14] with a stability bound of the order of $\frac{1}{3}g$. This was improved to $\frac{1}{2}g$ shortly afterwards by Ivanov [20, 21, 22]. The first complete proof of a $\frac{2}{3}g$-range is due to Boldsen, though it is based on an earlier unpublished preprint of Harer [3, 16]. We stated in the above theorems the bounds obtained by Randal-Williams in [30]. These bounds are a slight improvement of [3]. With our knowledge of the stable homology ([25, 27], see also [23]) and using recent calculations of Morita [28, Thm. 1.1], it follows that this last range is best possible for $g = 2 \bmod 3$, and at most one off the best possible bound otherwise.

Harer's stability theorem for mapping class groups of surfaces was inspired by the analogous pre-existing theorem for general linear groups which goes back to Quillen and Borel (see [4, 31]). The general line of argument, due to Quillen, is to build for each group in the sequence considered a simplicial complex with an action of the group, so that the stabilizers of simplices are previous groups in the sequence—the spectral sequence for the action of the group on the simplicial complex decomposes then the homology of the group in terms of the homology of the stabilizers of the action, making an inductive argument possible. (The mapping class groups of surfaces being a 2-parameter family, we will need here two simplicial complexes for each pair (g,n).) For this argument to work, the simplicial complexes need to be highly connected and showing this high connectivity is the hard part of the proof.

The simplicial complexes we use here are, as in [30], two ordered arc complexes (defined in Section 2) for surfaces with boundaries, and a disc complex (defined in Section 5) for closed surfaces. The connectivity arguments are a mix of arguments from the papers [14, 17, 21, 30, 32].

The stability for mapping class groups of surfaces has been generalized in several directions. When considering surfaces with punctures, there are two different generalizations. Let $\Gamma(S^r_{g,k})$ and $\Gamma(S^{(r)}_{g,k})$ denote the mapping class group of a surface of genus g, with k boundary components and r punctures, where the punctures are assumed to be fixed by the mapping classes in the first case, and to be fixed up to permutations in the second case. Theorems 1.1 and 1.2 still hold if $\Gamma(S_{g,k})$ is replaced by $\Gamma(S^r_{g,k})$ or $\Gamma(S^{(r)}_{g,k})$, that is the maps α, β and δ defined above are isomorphisms in homology in the same range. This can be deduced from the unpunctured case by a spectral sequence argument (see [13]), or by introducing punctures into the proof. An additional stabilization map can be defined by increasing the number of punctures. For $\Gamma(S^{(r)}_{g,k})$, this map induces an isomorphism in homology in a range increasing with the number of punctures [18, Prop. 1.5]. When the surface is a punctured disc, this is Arnold's classical stability theorem for braid groups [1]. Furthermore, there are homological stability theorems for spin mapping class groups [2, 15], for mapping class groups of non-orientable surfaces [32], and more generally for moduli spaces of surfaces with certain tangential structures [30]. For higher dimensional manifolds, the homology of the mapping class groups of 3-manifolds stabilizes by connected sum and boundary connected sum with another 3-manifold [18], and for simply-connected 4-dimensional manifolds, connected sums with $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ gives a stabilization [11].

The present volume contains a survey [24] about the homology of the moduli space of curves, which gives an overview of the computation of the stable homology of mapping class groups, following the work of Madsen-Weiss and Galatius-Madsen-Tillmann-Weiss [10, 23]. (The double PCMI lecture series [9, 33] gives an alternative reference to this topic.) A survey about more general stability phenomena in the topology of moduli spaces can be found in [5].

Organization of the paper: Section 2 defines the simplicial complexes used in the proof of Theorem 1.1 and gives their main properties, though the proof of high connectivity of the complexes is postponed until Section 4. Section 3 proves Theorem 1.1 via a spectral sequence argument which builds on Section 2 and 4. Section 5 takes care of the case of closed surfaces, proving Theorem 1.2. Finally, the appendix recalls some facts about simplicial complexes and piecewise linear topology, needed in particular for the connectivity arguments in Section 4.

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2. The ordered arc complex

In this section, we define the simplicial complexes used to prove homological stability. (Basic definitions and properties of simplicial complexes are given in the appendix.) The complexes admit actions of corresponding mapping class groups of surfaces, and we study the properties of these actions. Propositions 2.2,2.3,2.4 and 2.8 give four key ingredients for the proof of homological stability given in the next section.

Consider S an oriented surface with $\partial S \neq \varnothing$. By an *arc* in S, we always mean an embedded arc intersecting ∂S only at its endpoints and doing so transversally. We work with isotopy classes of arcs, where the isotopies are assumed to fix the endpoints of the arcs.

Let b_0, b_1 be two distinct points in ∂S. We will consider in this section collections of arcs with disjointly embeddable interiors, and with endpoints the pair $\{b_0, b_1\}$. Note that the orientation of the surface induces an ordering on such collections at b_0 and at b_1.

A collection of arcs with disjoint interiors $\{a_0, \dots, a_p\}$ is called *non-separating* if its complement $S \backslash (a_0 \cup \dots \cup a_p)$ is connected.

Definition 2.1. *Let $\mathcal{O}(S, b_0, b_1)$ be the simplicial complex with set of vertices the isotopy classes of non-separating arcs with boundary $\{b_0, b_1\}$. A p–simplex of $\mathcal{O}(S, b_0, b_1)$ is a collection of $p+1$ distinct isotopy classes of arcs $\langle a_0, \dots, a_p \rangle$ which can be represented by a collection of arcs with disjoint interiors which is non-separating and such that the anticlockwise ordering of $a_0, \dots, a_p$ at b_0 agrees with the clockwise ordering at b_1.*

Up to isomorphism, there are two such complexes, depending on whether b_0 and b_1 are on the same or on different boundary components. We will denote by $\mathcal{O}^1(S)$ the complex with a choice of b_0, b_1 on the same boundary component, and $\mathcal{O}^2(S)$ the complex with a choice of b_0, b_1 on two different boundary components.

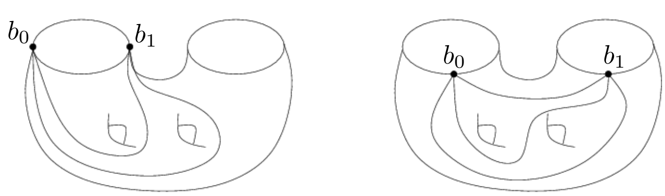

FIGURE 1. 1-simplex of $\mathcal{O}^1(S)$ and 2-simplex of $\mathcal{O}^2(S)$

The action of the mapping class group $\Gamma(S) = \pi_0 \operatorname{Diff}(S, \partial S)$ on the surface S induces an action on $\mathcal{O}(S, b_0, b_1)$. The remaining of the section gives four properties of this action.

Proposition 2.2 (Ingredient 1)**.** *For the complex $\mathcal{O}^i(S_{g,r})$, $i = 1, 2$, we have:*

(1) $\Gamma(S_{g,r})$ *acts transitively on p–simplices for each p.*

(2) *There exists isomorphisms*

$$St_{\mathcal{O}^1}(\sigma_p) \xrightarrow[\cong]{s_1} \Gamma(S_{g-p-1,r+p+1}) \quad \textit{and} \quad St_{\mathcal{O}^2}(\sigma_p) \xrightarrow[\cong]{s_2} \Gamma(S_{g-p,r+p-1}),$$

where $St_{\mathcal{O}^i}(\sigma_p)$ denotes the stabilizer of a p–simplex σ_p of $\mathcal{O}^i(S)$.

Proof. Let $i = 1, 2$ and $\sigma = \langle a_0, \ldots, a_p \rangle$ be a p–simplex of $\mathcal{O}^i(S)$ represented by arcs $a_0, \ldots, a_p$ with disjoint interiors in S. We consider the surface S "cut along σ", i.e. the surface $S \backslash \sigma = S \backslash (N_0 \cup \cdots \cup N_p)$ with N_j a small neighborhood of a_j. Its Euler characteristic satisfies $\chi(S\backslash\sigma) = \chi(S) + p + 1$ as a cellular decomposition of $S\backslash\sigma$ can be obtained from one of S by doubling the arcs a_j. We can moreover count (and describe) the boundary components of $S\backslash\sigma$: in addition to the $r - i$ components of ∂S disjoint from b_0, b_1, $\partial(S\backslash\sigma)$ has

- (when $i = 1$) $p+2$ components labeled $[\partial_0^+ S * a_0], [\bar{a}_0 * a_1], \ldots, [\bar{a}_{p-1} * a_p], [\bar{a}_p * \partial_0^- S]$,
- (when $i = 2$) $p+1$ components labeled $[\partial_0 S * a_0 * \partial_1 S * \bar{a}_p], [\bar{a}_0 * a_1], \ldots, [\bar{a}_{p-1} * a_p]$,

where $a_i, \bar{a}_i$ denote the left and right side of the arc, and $\partial_0^+ S, \partial_0^- S, \partial_0 S$ and $\partial_1 S$ are as shown in Figure 2.

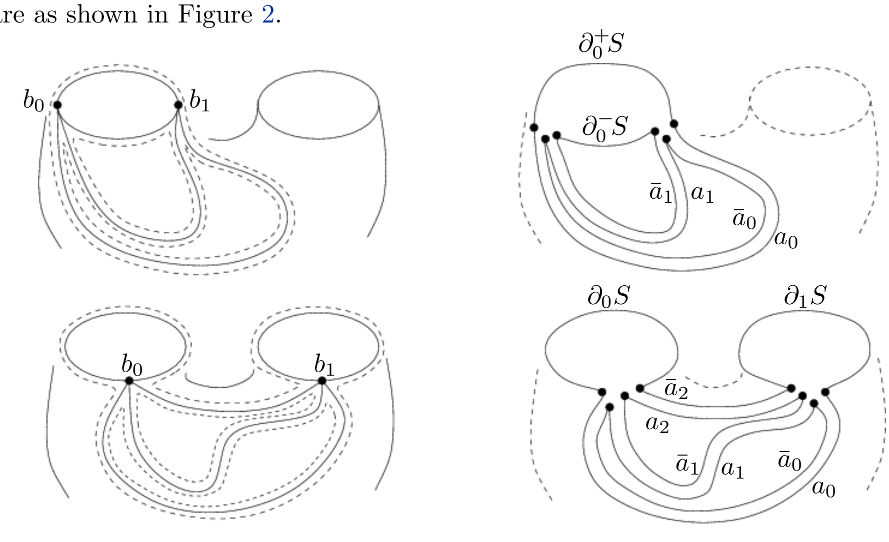

Figure 2. Cutting along the simplices of Figure 1

As $S\backslash\sigma$ is connected by assumption, we have $S\backslash\sigma \cong S_{g_\sigma, r_\sigma}$ with Euler characteristic

$$\chi(S\backslash\sigma) = \quad 2 - 2g_\sigma - r_\sigma \quad = \quad 2 - 2g - r + p + 1 \quad = \chi(S) + p + 1$$

By the above when $i=1$, $r_\sigma = r+p+1$ and thus $g_\sigma = g-p-1$, and when $i=2$, $r_\sigma = r+p-1$ and thus $g_\sigma = g-p$.

As g_σ and r_σ depend only on p, not on the simplex itself, it follows that the complement of any two p–simplices σ, σ' are diffeomorphic. Moreover, we can choose a diffeomorphism $S\backslash\sigma \cong S\backslash\sigma'$ which is compatible with the labels of the boundary by arcs of σ (resp. σ'), and hence glues to a diffeomorphism of S taking σ to σ'. Property (1) in the proposition follows.

The map $\Gamma(S\backslash\sigma) \to \Gamma(S)$ which glues $S\backslash\sigma$ back along σ has image a subgroup of $St_{\mathcal{O}^i}(\sigma)$. We want to show that this map defines an isomorphism $\Gamma(S\backslash\sigma) \cong St_{\mathcal{O}^i}(\sigma)$. The second part of the proposition will then follow as, by the above, $S\backslash\sigma$ is a surface of type $S_{g-p-1,r+p+1}$ when $i=1$ and of type $S_{g-p,r+p-1}$ when $i=2$.

To check surjectivity, consider an element ϕ of the stabilizer of the above simplex σ. Stabilizing σ means that for each i we have an isotopy $\phi(a_i) \simeq_{h_i} a_{\theta(i)}$ for some permutation θ of $\{0,1,\dots,p\}$. We want to show that θ is the identity and ϕ is isotopic to a map that fixes the arcs pointwise.

By the isotopy extension theorem, the isotopy h_0 can be extended to an ambient isotopy, i.e. levelwise diffeomorphisms $H_0\colon S\times I \to S$ with $H_0(S,0) = id_S$ and $H_0(\phi(a_0),t) = h_0(\phi(a_0),t)$. Composing with the end diffeomorphism gives a map $\phi_1 = H_0(1,-)\circ\phi$ which satisfies $\phi_1 \simeq \phi$ and $\phi_1(a_0) = a_{\theta(0)}$.

Now suppose that we have constructed $\phi_i \simeq \phi$ with $\phi_i(a_j) = a_{\theta(j)}$ for each $j<i$. Consider $h'_i\colon I\times I \to S$, the isotopy taking $\phi_i(a_i)$ to $a_{\theta(i)}$. Approximate h'_i by a map $\tilde{h}_i$ transverse to $a_{\theta(0)},\dots,a_{\theta(i-1)}$. We will inductively make it disjoint from these arcs. Suppose that the image of $\tilde{h}_i$ is disjoint from $a_{\theta(0)},\dots,a_{\theta(k-1)}$ and consider $\tilde{h}_i$ as a map to $S\backslash(a_{\theta(0)}\cup\dots\cup a_{\theta(k-1)})$. We want to make it disjoint from $a_{\theta(k)}$. The inverse image of $a_{\theta(k)}$ is a union of circles and intervals. For each circle component (or arc component going back to the same endpoint), $\tilde{h}_i$ restricted to the disc it separates in $I\times I$ defines an element of $\pi_2(S\backslash(a_{\theta(0)}\cup\dots\cup a_{\theta(k-1)}))$ as its boundary is mapped to a subarc of $a_{\theta(k)}$. As this π_2 is trivial, $\tilde{h}_i$ is homotopic to a map g_i with no such circle intersections, i.e. such that either $g_i^{-1}(a_{\theta(k)})$ is a (possibly empty) union of intervals from $g_i^{-1}(b_0)$ to $g_i^{-1}(b_1)$. If the union in non-empty, g_i would restrict to a homotopy $a_{\theta(k)} \simeq a_{\theta(i)}$. As homotopic arcs are isotopic by [8, Thm 3.1], this would contradict the fact that $a_{\theta(k)}$ and $a_{\theta(i)}$ represent different vertices of a simplex. Hence, proceeding inductively, we can replace the isotopy h_i by a homotopy between $\phi(a_i)$ and $a_{\theta(i)}$ in $S\backslash(a_{\theta(0)}\cup\dots\cup a_{\theta(i-1)})$. Applying again [8, Thm 3.1], we get an isotopy in the same surface, which we can replace by an ambient isotopy H_i. Considering H_i as an isotopy of S fixed on $a_{\theta(0)},\dots,a_{\theta(i-1)}$, we define $\phi_{i+1} = H_i(1,-)\circ\phi_i$. Repeating the construction, we obtain $\phi_{p+1}\simeq\phi$ satisfying $\phi_{p+1}(a_i) = a_{\theta(i)}$ for each i.

We finally note that θ must be the identity as ϕ_{p+1} fixes ∂S, and hence must be isotopic to the identity in a neighborhood of ∂S.

Hence any element of the stabilizer of σ is in the image of $\Gamma(S\backslash\sigma) \to \Gamma(S)$. We are left to show injectivity of that map. This can be seen as follows: To a non-separating arc I in S is associated a fibration

$$\mathrm{Diff}(S \text{ rel } \partial S \cup I) \to \mathrm{Diff}(S \text{ rel } \partial S) \to Emb^{\partial}_{ns}(I,S)$$

where $Emb^{\partial}_{ns}(I,S)$ denotes the space of embeddings of a non-separating arc in S with ∂I mapping to chosen points $A, B \in \partial S$. The fibration is induced by restricting a diffeomorphism of S to the given arc. By [12, Thm 5], $\pi_1(Emb^{\partial}(I,S)) = 0$. Hence the long exact sequence of homotopy groups of the fibrations gives an injection $\pi_0(\mathrm{Diff}(S \text{ rel } \partial S \cup I)) \hookrightarrow \pi_0(\mathrm{Diff}(S \text{ rel } \partial S))$, i.e. $\Gamma(S\backslash I) \hookrightarrow \Gamma(S)$. This corresponds to the case where σ is a vertex. Repetitive use of this inclusion gives the desired result for any simplex σ. □

Consider the maps $\alpha\colon S_{g,r+1} \to S_{g+1,r}$ and $\beta\colon S_{g,r} \to S_{g,r+1}$ which glue a strip respectively on two and one boundary components (see Figure 3). There are induced maps on the mapping class groups

$$\alpha_g\colon \Gamma(S_{g,r+1}) \to \Gamma(S_{g+1,r}) \quad \text{and} \quad \beta_g\colon \Gamma(S_{g,r}) \to \Gamma(S_{g,r+1})$$

by extending the mapping classes to be the identity on the added strips. (These maps are isomorphic to the maps α_g and β_g described in the introduction in terms of pairs of pants.) Moreover, if b_0, b_1 are as in the Figure 3, we get induced maps

$$\alpha\colon \mathcal{O}^2(S_{g,r+1}) \to \mathcal{O}^1(S_{g+1,r}) \quad \text{and} \quad \beta\colon \mathcal{O}^1(S_{g,r}) \to \mathcal{O}^2(S_{g,r+1})$$

equivariant with respect to the group maps α_g and β_g.

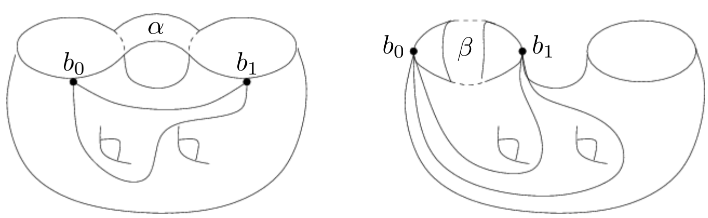

FIGURE 3. The maps α and β

The second ingredient is a new ingredient in Randal-Williams' proof. It is a symmetry property of the maps α and β which will be crucial in the spectral sequence argument. It roughly says that, on stabilizers of the action of the mapping class group $\Gamma(S)$ on the complexes $\mathcal{O}^i(S)$, α induces β and β induces α.

Proposition 2.3 (Ingredient 2)**.** *Let α, β be as in Figure 3. Given a p–simplex σ_p in $\mathcal{O}^2(S)$, we have*

$$\begin{array}{ccccc}
\Gamma(S_{g,r+1}) & \hookleftarrow & St_{\mathcal{O}^2}(\sigma_p) & \xrightarrow[\cong]{s_2} & \Gamma(S_{g-p,r+p}) \\
\downarrow{\scriptstyle \alpha_g} & & \downarrow & & \downarrow{\scriptstyle \beta_{g-p}} \\
\Gamma(S_{g+1,r}) & \hookleftarrow & St_{\mathcal{O}^1}(\alpha(\sigma_p)) & \xrightarrow[\cong]{s_1} & \Gamma(S_{g-p,r+p+1})
\end{array}$$

and given a p–simplex σ_p in $\mathcal{O}^1(S)$, we have

$$\begin{array}{ccccc}
\Gamma(S_{g,r}) & \hookleftarrow & St_{\mathcal{O}^1}(\sigma_p) & \xrightarrow[\cong]{s_1} & \Gamma(S_{g-p-1,r+p+1}) \\
\downarrow{\scriptstyle \beta_g} & & \downarrow & & \downarrow{\scriptstyle \alpha_{g-p-1}} \\
\Gamma(S_{g,r+1}) & \hookleftarrow & St_{\mathcal{O}^2}(\beta(\sigma_p)) & \xrightarrow[\cong]{s_2} & \Gamma(S_{g-p,r+p})
\end{array}$$

where s_1, s_2 are the isomorphisms of Proposition 2.2.

Proof. Consider first the map α. On S, α is defined by gluing a strip, one side glued to $\partial_0 S$ and one to $\partial_1 S$. In $S \backslash \sigma_p$, the two components are part of a single boundary component, the one denoted $[a_0 * \partial_0 S * \bar{a}_p * \partial_1 S]$ in the proof of Proposition 2.2. Hence the strip in glued to a single boundary in $S \backslash \sigma_p$. As the stabilizer of σ_p is identified with the mapping class group of $S \backslash \sigma_p$, α induces the map β on $St_{\mathcal{O}_2}(\sigma_p)$.

For β, both ends of the strip are glued to $\partial_0 S$: one end to $\partial_0 S^+$ and one to $\partial_0 S^-$ in the notation of Proposition 2.2 (see also Figure 2). Given a p–simplex σ_p of $\mathcal{O}_1(S)$, we have that $\partial_0 S^+$ and $\partial_0 S^-$ are in two different boundary components of $S \backslash \sigma_p$, namely the components denoted $[\partial_0^+ S * a_0]$ and $[\bar{a}_p * \partial_0^- S]$ in the proof of Proposition 2.2. Hence the map β induces the map α on $St_{\mathcal{O}_1}(\sigma_p)$. □

The complexes $\mathcal{O}^1$ and $\mathcal{O}^2$ are defined to "undo" the maps α and β in the sense that the inclusion of stabilizers of vertices $St_{\mathcal{O}_i}(\sigma_0) \hookrightarrow \Gamma(S)$ are the maps α and β respectively. The third ingredient makes this precise, as part of a stronger statement.

Proposition 2.4 (Ingredient 3)**.** *Let S_α and S_β denote S union a strip glued via α and β respectively as in Figure 3. The maps $\alpha\colon \Gamma(S) \to \Gamma(S_\alpha)$ and $\beta\colon \Gamma(S) \to \Gamma(S_\beta)$ are injective. Moreover, for any vertex σ_0 of $\mathcal{O}^i(S)$, there are curves c_α, c_β (given in Figure 4) in S_α and S_β such that conjugation by Dehn twists t_{c_α} and t_{c_β} along these curves fits into commutative diagrams*

$$\begin{array}{ccc}
St_{\mathcal{O}^2}(\sigma_0) & \hookrightarrow & St_{\mathcal{O}^1}(\alpha(\sigma_0)) \\
\downarrow & {\scriptstyle t_{c_\alpha}} \swarrow & \downarrow \\
\Gamma(S) & \xhookrightarrow{\alpha} & \Gamma(S_\alpha)
\end{array}
\qquad\qquad
\begin{array}{ccc}
St_{\mathcal{O}^1}(\sigma_0) & \hookrightarrow & St_{\mathcal{O}^2}(\beta(\sigma_0)) \\
\downarrow & {\scriptstyle t_{c_\beta}} \swarrow & \downarrow \\
\Gamma(S) & \xhookrightarrow{\beta} & \Gamma(S_\beta)
\end{array}$$

i.e. there are conjugations $St_{\mathcal{O}^1}(\alpha(\sigma_0)) \sim_{t_{c_\alpha}} \alpha(\Gamma(S))$ *in* $\Gamma(S_\alpha)$ *relative to* $\alpha(St_{\mathcal{O}^2}(\sigma_0))$, *and* $St_{\mathcal{O}^2}(\beta(\sigma_0)) \sim_{t_{c_\beta}} \beta(\Gamma(S))$ *in* $\Gamma(S_\beta)$ *relative to* $\beta(St_{\mathcal{O}^1}(\sigma_0))$.

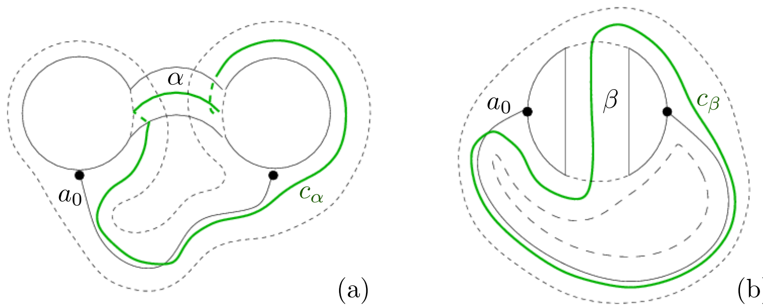

FIGURE 4. The curves c_α and c_β of Proposition 2.4 for $\sigma_0 = \langle a_0 \rangle$

Note that the stabilizer of any two vertices of $\mathcal{O}^i(S)$ are conjugate in $\Gamma(S)$ as $\Gamma(S)$ acts transitively on the vertices of $\mathcal{O}^i(S)$. So the proposition implies in particular that the stabilizer of a vertex in $\mathcal{O}^1(S_\alpha)$ is isomorphic to $\Gamma(S)$, and that so is the stabilizer of a vertex in $\mathcal{O}^2(S_\beta)$.

The existence of the particular Dehn twist used (t_{c_α} and t_{c_β}) is also used by Harer and Boldsen in their proof the 2/3 stability range. In Randal-Williams, the above property is called 1–triviality.

Proof. Let $\bar{S}$ denote either S_α or S_β. Suppose σ_0 is represented by an arc a_0 in S, and denote also by a_0 the corresponding arc in $\bar{S}$. The first step of the proof in both cases is to exhibit an arc a_1 in $\bar{S}$ disjoint from a_0 (except at the endpoints) such that $St_{\mathcal{O}^1}(a_1) = \alpha(\Gamma(S))$ (resp. $St_{\mathcal{O}^2}(a_1) = \beta(\Gamma(S))$) and $St_{\mathcal{O}^1}(\langle a_0, a_1\rangle) = \alpha(St_{\mathcal{O}^2}(a_0))$ (resp. $St_{\mathcal{O}^2}(\langle a_0, a_1\rangle) = \alpha(St_{\mathcal{O}^1}(a_0)))$ as subgroups of $\Gamma(\bar{S})$. In other words, we will show that both diagrams can be seen as being of the form

$$\begin{array}{ccc} St(\langle a_0, a_1\rangle) & \hookrightarrow & St(a_0) \\ \downarrow & & \downarrow \\ St(a_1) & \hookrightarrow & \Gamma(\bar{S}) \end{array}$$

The arc a_1 is given in Figure 5(a),(c) in both cases. To show that it satisfies the above, it is enough to produce a map $\phi \in \mathrm{Diff}(\bar{S})$ such that

(1) $\phi \simeq id$ and ϕ is constant on a_0,

(2) ϕ takes a neighborhood of $a_1 \cup \partial\bar{S}$ to a neighborhood of $\partial S \cup (\bar{S}\backslash S)$.

Indeed, an element $g \in St(a_1) \leq \Gamma(\bar{S})$ can be assumed to fix a neighborhood of a_1 and $\partial\bar{S}$. Then its conjugate $c_\phi(g) = \phi^{-1} g \phi$ fixes the strip $\bar{S}\backslash S$ and a neighborhood of ∂S, and hence is in the image of $\Gamma(S)$. And conversely, $c_{\phi^{-1}}(g)$ identifies elements in the image of $\Gamma(S)$ with elements of $St(a_1)$. But as $\phi \simeq id$,

g and $c_\phi(g)$ (resp. $c_{\phi^{-1}}(g)$) represent the same element in $\Gamma(\bar{S})$ and c_ϕ is actually the identity. As a_0 is not affected, c_ϕ restricts to an identification of $St(\langle a_0, a_1\rangle)$ with the image of $St(a_0)$ in $\Gamma(\bar{S})$.

The map ϕ is obtained by thickening the neighborhood of $\partial\bar{S}\cup a_1$ to include the shaded areas of $\bar{S}\backslash S$ shown in Figure 5(b),(d). This is possible as the shaded areas are discs intersecting $\partial\bar{S}\cup a_1$ in an arc in their boundary.

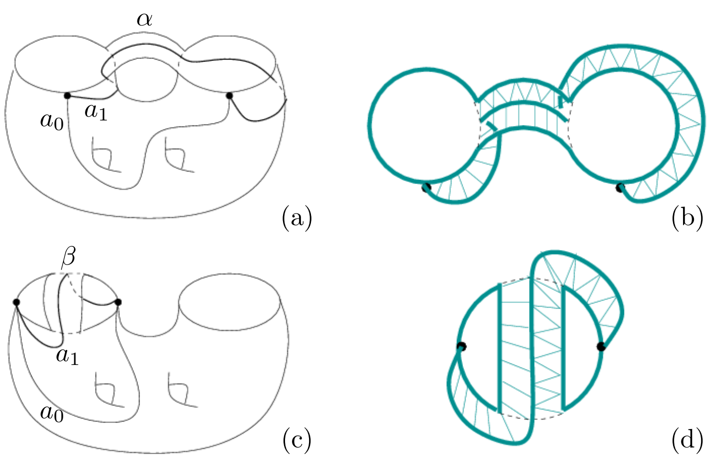

FIGURE 5. The arc a_1 and the retraction ϕ

Now in both cases, the arcs a_0, a_1 are ordered in the same way at b_0 and b_1 (that is they do *not* form a 1-simplex of $\mathcal{O}^i$). It was noted by Harer ([16], see also [3, Prop. 3.2]) that in this case, the Dehn twist ψ along the closed curve $[a_0 * a_1]$ takes a_1 to a_0. Conjugation by ψ^{-1} takes $St(a_0)$ to $St(a_1)$. As it lives in a neighborhood of a_0, a_1, it commutes with the elements of $St(a_0, a_1)$ and hence conjugation by ψ is the identity on that subgroup. (The curve $[a_0 * a_1]$ is the curve drawn in Figure 4 in each case.)

The above gives conjugations between the image of $\Gamma(S)$ in $\Gamma(\bar{S})$ and the subgroups $St_{\mathcal{O}^i}(a_0)$. Injectivity of the maps α and β then follows from the fact, proved in Proposition 2.2, that $\Gamma(S)$ is isomorphic to $St_{\mathcal{O}^i}(a_0)$. □

To be able to use Ingredient 3, we need to study the commutative square occuring in Proposition 2.4 from the point of view of group homology, which we do now.

Given a group G, we use the bar construction to compute the homology of G. Hence a k-chain $c \in C_k(G)$ is of the form $c = \sum_i z_i(g_1^i, \dots, g_k^i)$ with $g_j^i \in G$

and $z_i \in \mathbb{Z}$. By a *commuting diagram*, we will mean a diagram of the form

$$\begin{array}{ccccccc} \star & \xrightarrow{a_1} & \star & \longrightarrow \cdots & \star & \xrightarrow{a_k} & \star \\ {\scriptstyle b_0}\downarrow & & \downarrow{\scriptstyle b_1} & & {\scriptstyle b_{k-1}}\downarrow & & \downarrow{\scriptstyle b_k} \\ \star & \xrightarrow{c_1} & \star & \longrightarrow \cdots & \star & \xrightarrow{c_k} & \star \end{array}$$

with $a_i, b_i, c_i \in G$ and $a_i b_i = b_{i-1} c_i$ for each i. Such a commuting diagram defines a copy of $\Delta^k \times I$ in the classifying space of G and hence an element of $C_{k+1}(G)$. Explicitly (and with a choice of orientation), this chain is

$$(a_1, \dots, a_k, b_k) - (a_1, \dots, a_{k-1}, b_{k-1}, c_k) + \cdots + (-1)^k (b_0, c_1, \dots, c_k).$$

Lemma 2.5. *Let H, G_1, G_2 be subgroups of a group G fitting into a diagram*

$$\begin{array}{ccc} H & \hookrightarrow & G_1 \\ \downarrow & \overset{t}{\swarrow} & \downarrow \\ G_2 & \hookrightarrow & G \end{array}$$

i.e. such that G_1 and G_2 are conjugated by $t \in G$, with a common subgroup H fixed by the conjugation. Then the map $H_k(G_1, H) \longrightarrow H_k(G, G_2)$ induced by the inclusion factors as

$$\begin{array}{ccc} H_k(G_1, H) & \longrightarrow & H_k(G, G_2) \\ \downarrow{\scriptstyle \partial} & \nearrow{\scriptstyle (-1)^k(\_ \times t)} & \\ H_{k-1}(H) & & \end{array}$$

where $(h_1, \dots, h_{k-1}) \times t$ is the class given by the commuting diagram

$$\begin{array}{ccccccc} \star & \xrightarrow{h_1} & \star & \longrightarrow \cdots & \star & \xrightarrow{h_{k-1}} & \star \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow & & \downarrow{\scriptstyle t} \\ \star & \xrightarrow{h_1} & \star & \longrightarrow \cdots & \star & \xrightarrow{h_{k-1}} & \star \end{array}$$

A more general version of this lemma can be found in [29, Lem 6.2].

Proof. A class $[c] \in H_k(G_1, H)$ is of the form $c = \sum_i z_i (g_1^i, \dots, g_k^i)$ with each $g_j^i \in G_1$ and $z_i \in \mathbb{Z}$, and with boundary dc a chain in H. Extending the above notation, denote by $c \times t$ the $(k+1)$-chain given by the linear combination of commuting diagrams

$$\begin{array}{ccccccc} \star & \xrightarrow{g_1^i} & \star & \longrightarrow \cdots & \star & \xrightarrow{g_k^i} & \star \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow & & \downarrow{\scriptstyle t} \\ \star & \xrightarrow[t^{-1}g_1^i t]{} & \star & \longrightarrow \cdots & \star & \xrightarrow[t^{-1}g_k^i t]{} & \star \end{array}$$

One computes that $d(c \times t) = (-1)^{k+1} c + dc \times t + (-1)^k t^{-1} ct$. (In particular, the map $\_ \times t : C_k(H) \to C_{k+1}(G)$ is a chain map as $t^{-1}ct = c$ for any $c \in C_k(H)$.) As t conjugates G_1 into G_2, we have that $[t^{-1}ct] = 0$ in $H_*(G, G_2)$, and hence the image of $[c]$ in $H_*(G, G_2)$ is equal to $(-1)^k [dc \times t]$. □

We will use Proposition 2.4 via the following two corollaries.

Corollary 2.6. *Let σ_0 be a vertex of $\mathcal{O}^2(S)$ and $\alpha(\sigma_0)$ its image in $\mathcal{O}^1(S_\alpha)$. Then the map induced on relative homology by including the stabilizers*

$$H_*(St_{\mathcal{O}^1}(\alpha(\sigma_0)), St_{\mathcal{O}^2}(\sigma_0)) \longrightarrow H_*(\Gamma(S_\alpha), \Gamma(S))$$

is the zero map.

Proof. Applying Lemma 2.5 to Proposition 2.4, it is enough to show that the map

$$\_ \times t_{c_\alpha} : H_{*-1}(St_{\mathcal{O}^2}(\sigma_0)) \longrightarrow H_*(\Gamma(S_\alpha), \Gamma(S))$$

is the zero map. Recall that t_{c_α} is a Dehn twist along the curve c_α of Figure 4 (a). Let $\sigma_0 = \langle a_0 \rangle$. As can be seen in the figure, the curve is non-separating in a neighborhood of $S_\alpha \backslash S \cup \partial S \cup a_0$. Let c'_α be one of the components of ∂S appearing in Figure 4 (a) pushed to the interior of S_α. Note that c'_α is also non-separating in the neighborhood. Hence the complements of the two curves are diffeomorphic and there exists a diffeomorphism g of the neighborhood fixing its boundaries taking c_α to c'_α. Let $\bar{g} \in \Gamma(S_\alpha)$ be the class of g extended by the identity to the whole of S_α. Then $\bar{g}$ commutes with the image of $St_{\mathcal{O}^2}(\sigma_0)$ in $\Gamma(S_\alpha)$.

Given $[c] \in H_{k-1}(St_{\mathcal{O}^2}(\sigma_0))$, the chain $c \times t_{c_\alpha} \times \bar{g} \in C_{k+1}(\Gamma(S_\alpha))$ has boundary $d(c \times t_{c_\alpha} \times \bar{g})$

$$= (-1)^{k+1} c \times t_{c_\alpha} + d(c \times t_{c_\alpha}) \times \bar{g} + (-1)^k \, \bar{g}^{-1}(c \times t_{c_\alpha}) \, \bar{g}$$
$$= (-1)^{k+1} c \times t_{c_\alpha} + (-1)^k c \times \bar{g} + 0 + (-1)^{k+1} \, t_{c_\alpha}^{-1} c \, t_{c_\alpha} \times \bar{g} + (-1)^k c \times t_{c'_\alpha}.$$

As $t_{c_\alpha}^{-1} c \, t_{c_\alpha} = c$ the middle terms cancel and the result follows from the fact that $c \times t_{c'_\alpha} \in \Gamma(S)$. □

For the map β, we have a similar situation with c_β is a curve in a neighborhood N of $S_\beta \backslash S \cup \partial S \cup a_0$ (see Figures 4 (b) and 6), but now c_β is separating. Consider an arc a in S_β as in Figure 6, joining the two dashed boundary components of N and otherwise disjoint from it. (Such an arc exists as a_0 is non-separating.) Then c_β is non-separating in $N \cup a$.

Corollary 2.7. *Let σ_0 be a vertex of $\mathcal{O}^1(S)$. Then the composition*

$$H_{*-1}(\Gamma(S')) \longrightarrow H_{*-1}(St_{\mathcal{O}^1}(\sigma_0)) \xrightarrow{\_ \times t_{c_\beta}} H_*(\Gamma(S_\beta), \Gamma(S))$$

is the zero map, where S' is the complement in S_β of a neighborhood of $S_\beta \backslash S \cup \partial S \cup a_0 \cup a$, with a as above, and $\Gamma(S') \to St_{\mathcal{O}^1}(\sigma_0)$ is the inclusion.

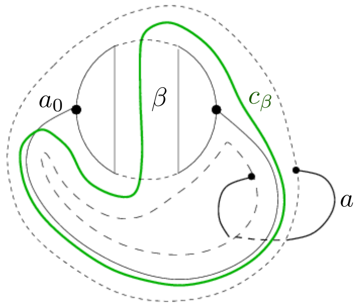

FIGURE 6. The arc a

The proof is the same as for Corollary 2.6 using the fact that c_β and the component of ∂S in the figure are non-separating in $S_\beta \backslash S'$.

The last (but not least) ingredient is the connectivity of the complex. The proof is deferred to Section 4 as it does require some work...

Proposition 2.8 (Ingredient 4)**.** *The complex $\mathcal{O}^i(S_{g,r})$ is $(g-2)$–connected.*

3. Spectral sequence argument

This section gives the proof of the main theorem, Theorem 1.1. The proof is a double spectral sequence argument build on the action of the mapping class group of a surface S on the complexes $\mathcal{O}^1(S)$ and $\mathcal{O}^2(S)$. It relies on the geometric results proved in Section 2, and on the connectivity results of Section 4. We follow the line of argument of [30].

To a simplicial complex X, we associate the augmented chain complex $(\tilde{C}_*(X), \partial)$ defined by $\tilde{C}_p(X) = \mathbb{Z}X_p$, the free module over the set of p–simplices of X (each with a chosen orientation), for $p \geq 0$, and $\tilde{C}_{-1}(X) = \mathbb{Z}$. The differential ∂ induced by the face maps and the augmentation: $\partial_p = \sum_{i=0}^{p}(-1)^i d_i$ for $p \geq 1$ and ∂_0 maps the vertices of X to the generator of $\tilde{C}_{-1}(X)$.

Let X, Y be simplicial complexes with simplicial actions of groups G and H respectively. A homomorphism $\phi\colon G \to H$ together with an equivariant map $f\colon X \to Y$ (with respect to that homomorphism) induces a map of double chain complexes

$$F\colon \tilde{C}_*(X) \otimes_G E_*G \longrightarrow \tilde{C}_*(Y) \otimes_H E_*H$$

where (E_*G, d) (resp. (E_*H, d)) is a free G– (resp. H–)resolution of $\mathbb{Z}$. Now consider the double complex (mapping cone in the q-direction)

$$C_{p,q} = (\tilde{C}_p(X) \otimes_G E_{q-1}G) \oplus (\tilde{C}_p(Y) \otimes_H E_q H)$$

with horizontal differential taking $(a \otimes b, a' \otimes b')$ to $(\partial a \otimes b, \partial a' \otimes b')$ and vertical differential taking $(a \otimes b, a' \otimes b')$ to $(a \otimes db, a' \otimes db' + F(a \otimes b))$.

The horizontal and vertical filtrations of such a double complex give two spectral sequences, both converging to the homology of the total complex. We will use the following two properties of these spectral sequences:

(SS1) If X is $(c-1)$–connected and Y is c–connected, then the E^1–term of the horizontal spectral sequence, which is the homology of $C_{p,q}$ with respect to the horizontal differential, is 0 in the range $p+q \leq c$ (noting that $\tilde{C}_p(X)$ only contributes to $C_{p,q}$ when $q > 0$). Hence the other spectral sequence converges to 0 in the range $p+q \leq c$.

(SS2) The E^1–term of the vertical spectral sequence is the relative homology group $E^1_{p,q} = H_q\big(\tilde{C}_p(Y) \otimes_H E_*H, \tilde{C}_p(X) \otimes_G E_*G\big)$ as the columns of $C_{p,q}$ are the mapping cones of the map F (with p fixed). If the actions of G and H are transitive on X and Y, a relative version of Shapiro's lemma identifies this homology group with

$$E^1_{p,q} = H_q(St_Y(\sigma_p), St_X(\sigma_p))$$

where $St_X(\sigma_p)$ and $St_Y(\sigma_p)$ are the stabilizers in X and Y of some p–simplex σ_p of X and its image in Y. Note that this formulation also includes the case $p=-1$ with $St_X(\sigma_{-1}) = G$ and $St_Y(\sigma_{-1}) = H$ as the action is trivial on the "(-1)–simplex".

Recall from the previous section the maps

$$\alpha_g \colon \Gamma(S_{g,r+1}) \to \Gamma(S_{g+1,r}) \quad \text{and} \quad \beta_g \colon \Gamma(S_{g,r}) \to \Gamma(S_{g,r+1}).$$

Denote by $H(\alpha_g)$ the relative homology group $H(\Gamma_{g+1,r}, \Gamma_{g,r+1}; \mathbb{Z})$ corresponding to the map α_g, and $H(\beta_g)$ the relative homology group $H(\Gamma_{g,r+1}, \Gamma_{g,r}; \mathbb{Z})$ corresponding to β_g. The main theorem considered in this paper (Theorem 1.1) can be restated as follows:

Theorem 3.1. (1) *$H_i(\alpha_g) = 0$ for $i \leq \frac{2g+1}{3}$ and* (2) *$H_i(\beta_g) = 0$ for $i \leq \frac{2g}{3}$.*

Proof. We prove the theorem by induction on g. To start the induction, note that statements (1) for genus 0 and (2) for genus 0,1 are trivially true as they are just concerned with H_0. Let (1_g) and (2_g) denote the truth of (1) and (2) in the theorem for genus g. The induction will go in two steps:

Step 1: For $g \geq 1$, $(2_{\leq g})$ implies (1_g).
Step 2: For $g \geq 2$, $(1_{<g})$ and (2_{g-1}) imply (2_g).

For Step 1, we consider the spectral sequence described above for the actions of $G = \Gamma_{g,r+1}$ on $X = \mathcal{O}^2(S_{g,r+1})$ and of $H = \Gamma_{g+1,r}$ on $Y = \mathcal{O}^1(S_{g+1,r})$ with the homomorphism $\phi \colon G \to H$ and the map $f \colon X \to Y$ both induced by the map $\alpha \colon S_{g,r+1} \to S_{g+1,r}$ of Figure 3. As the action is transitive in both cases (Propositions 2.2), we can apply (SS2) from above which says that the vertical spectral sequence has the form $E^1_{p,q} = H_q(St_Y(\sigma_p), St_X(\sigma_p))$. When $p = -1$, we have

$$E^1_{-1,q} = H_q(\Gamma_{g+1,r}, \Gamma_{g,r+1}) = H_q(\alpha_g)$$

which are the groups we are interested in. By Propositions 2.3, the other groups are identified with

$$E^1_{p,q} = H_q(\beta_{g-p}) \quad \text{for } \; p \geq 0.$$

Hence we will be able to apply induction to these terms of the spectral sequence. We want to deduce that $E^1_{-1,q} = 0$ for $q \leq \frac{2g+1}{3}$. This will follow from the following three claims:

Claim 1: $E^\infty_{-1,q} = 0$ for $q \leq \frac{2g+1}{3}$.

Claim 2: The E^1–term is as in Figure 7, i.e. there are no possible sources of differentials to kill classes in $E^1_{-1,q}$ with $q \leq \frac{2g+1}{3}$, except possibly for $d^1 \colon E^1_{0,q} \to E^1_{-1,q}$ when $q = \frac{2g+1}{3}$ (i.e. when the fraction is an integer).

Claim 3: The map $d^1 \colon E^1_{0,q} \to E^1_{-1,q}$ is the 0-map.

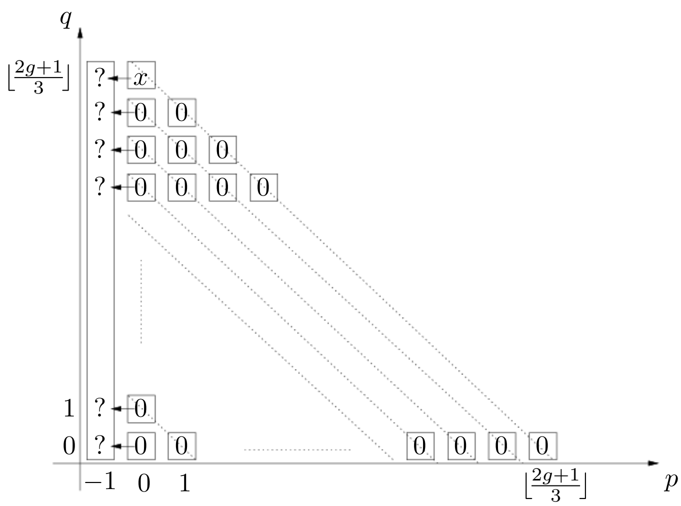

FIGURE 7. Spectral sequence for Step 1. The possible sources of differentials for the "?" are along the dotted diagonals.

Claims 1 and 2 imply immediately that $E^1_{-1,q} = 0$ for $q < \frac{2g+1}{3}$ as "it must die by E^∞" (Claim 1) and "nobody can kill it" (Claim 2). Claim 3 gives that this also holds when $q = \frac{2g+1}{3}$ as the only differential with a possibly non-trivial source is the zero map, and hence won't kill anything in the target.

By Proposition 2.8, X is $(g-2)$–connected and Y is $(g-1)$–connected. Applying (SS1) from above, we get that $E^\infty_{p,q} = 0$ for $p+q \leq g-1$. In particular, $E^\infty_{-1,q} = 0$ for $q \leq g$. As $\frac{2g+1}{3} \leq g$ when $g \geq 1$, Claim 1 follows.

The sources of differentials to $E^1_{-1,q}$ are the terms $E^{p+1}_{p,q-p}$ for $p \geq 0$. As $E^1_{p,q} = H_q(\beta_{g-p})$ when $p \geq 0$, by induction we know that $E^1_{p,q} = 0$ when $q \leq$

$\frac{2(g-p)}{3} = \frac{2g-2p}{3}$ and $p \geq 0$. Hence $E^1_{p,q-p} = 0$ for $q \leq \frac{2g+p}{3}$ and $p \geq 0$, i.e. they are all 0 for any $p \geq 0$ if $q \leq \frac{2g}{3}$ or for $p \geq 1$ if $q = \frac{2g+1}{3}$. This is Claim 2.

Claim 3 is given by Corollary 2.6: The map $d^1\colon E^1_{0,q} \to E^1_{-1,q}$ is the map $H_q(St_{\mathcal{O}^1}(\alpha(\sigma_0)), St_{\mathcal{O}^2}(\sigma_0)) \to H_q(\Gamma(S_\alpha), \Gamma(S))$ of the corollary, where $S = S_{g,r+1}$ and $S_\alpha = S_{g+1,r}$.

For Step 2, the argument is essentially the same. We consider the spectral sequence described above for the actions of $G = \Gamma_{g,r}$ on $X = \mathcal{O}^1(S_{g,r})$ and of $H = \Gamma_{g,r+1}$ on $Y = \mathcal{O}^2(S_{g,r+1})$ with the homomorphism $\phi\colon G \to H$ and the map $f\colon X \to Y$ both induced by the map $\beta\colon S_{g,r} \to S_{g,r+1}$. We can again apply (SS2) by Proposition 2.2 and get that that the vertical spectral sequence has the form $E^1_{p,q} = H_q(St_Y(\sigma_p), St_X(\sigma_p))$. When $p = -1$, we have

$$E^1_{-1,q} = H_q(\Gamma_{g,r+1}, \Gamma_{g,r}) = H_q(\beta_g)$$

which are the groups we are interested in. By Propositions 2.3, the other groups are identified with

$$E^1_{p,q} = H_q(\alpha_{g-p-1}) \quad \text{for } \ p \geq 0$$

Hence we will be able to apply induction to these terms, to deduce that $E^1_{-1,q} = 0$ for $q \leq \frac{2g}{3}$. As in the previous case, this follows from three claims:

Claim 1: $E^\infty_{-1,q} = 0$ for $q \leq \frac{2g}{3}$.

Claim 2: The E^1–term is as in Figure 7, though with $\lfloor \frac{2g+1}{3} \rfloor$ replace by $\lfloor \frac{2g}{3} \rfloor$, i.e. there are no possible sources of differentials to kill classes in $E^1_{-1,q}$ with $q \leq \frac{2g}{3}$, except possibly for $d^1\colon E^1_{0,q} \to E^1_{-1,q}$ when $q = \frac{2g}{3}$.

Claim 3: The map $d^1\colon E^1_{0,q} \to E^1_{-1,q}$ is the 0-map.

By Proposition 2.8, X and Y are $(g-2)$–connected. Applying (SS1), we get that $E^\infty_{p,q} = 0$ for $p + q \leq g - 2$. In particular, $E^\infty_{-1,q} = 0$ for $q \leq g - 1$. As $\lfloor \frac{2g}{3} \rfloor \leq g - 1$ when $g \geq 2$, Claim 1 follows.

The sources of differentials to $E^1_{-1,q}$ are the terms $E^{p+1}_{p,q-p}$ for $p \geq 0$. As $E^1_{p,q} = H_q(\alpha_{g-p-1})$ when $p \geq 0$, by induction we know that $E^1_{p,q} = 0$ when $q \leq \frac{2(g-p-1)+1}{3} = \frac{2g-2p-1}{3}$ and $p \geq 0$. Hence $E^1_{p,q-p} = 0$ for $q \leq \frac{2g+p-1}{3}$ and $p \geq 0$, i.e. they are all 0 for any $p \geq 0$ if $q \leq \frac{2g-1}{3}$ or for $p \geq 1$ if $q = \frac{2g}{3}$. This is Claim 2.

Claim 3 is a consequence of Corollary 2.7, using induction: The map $d^1\colon E^1_{0,q} \to E^1_{-1,q}$ is the map $H_q(St_{\mathcal{O}^2}(\beta(\sigma_0)), St_{\mathcal{O}^1}(\sigma_0)) \to H_q(\Gamma(S_\beta), \Gamma(S))$ mapping the top row to the bottom row of the second square in Proposition 2.4, where $S = S_{g,r}$ and $S_\beta = S_{g,r+1}$ here. Applying Lemma 2.5 to the proposition, we get that this map factors through the map

$$\_ \times t_{c_\beta}\colon H_{q-1}(St_{\mathcal{O}^1}(\sigma_0)) \longrightarrow H_q(\Gamma(S_\beta), \Gamma(S))$$

with t_{c_β} a Dehn twist along the curve c_β of Figure 4 (b). Let S' be as in Corollary 2.7. By the corollary, the composition

$$H_{q-1}(\Gamma(S')) \longrightarrow H_{q-1}(St_{\mathcal{O}^1}(\sigma_0)) \xrightarrow{\_\times t_{c_\beta}} H_q(\Gamma(S_\beta),\Gamma(S))$$

is the zero map. Note now that the first map is a β-map

$$H_{q-1}(\Gamma(S')) \cong H_{q-1}(\Gamma_{g-1,r}) \ \longrightarrow\ H_{q-1}(\Gamma_{g-1,r+1}) \cong H_{q-1}(St_{\mathcal{O}^1}(\sigma_0)).$$

By induction, it is surjective: we have $H_{q-1}(\beta_{g-1}) = H_{q-1}(\Gamma_{g-1,r+1},\Gamma_{g-1,r}) = 0$ by (1_{g-1}) as $q-1 \leq \frac{2g}{3} - 1 \leq \frac{2(g-1)}{3}$. As the composition above is 0, we can deduce that the second map is also 0 and hence that the differential is 0. □

Note that the slope $\frac{2}{3}$ in the bound of the stable range is determined by the structure of the spectral sequence: to prove that $H_q(\alpha_g) = 0$ requires that $H_{q-1}(\beta_{g-1}) = 0$ (for claim 2 of step 1), which in turn requires that $H_{q-2}(\alpha_{g-3}) = 0$ (for claim 2 of step 2). Claim 1 does not interfere with the slope because the connectivity bounds of the complexes used have a higher slope, and Claim 3 (in Step 2) only requires the slope to be at most 1. The constant coefficient is decided by the connectivity of the complexes for low genus surfaces. Recall though from the introduction that we know the slope to be best possible, and the constant coefficient to be very close to best possible.

4. Connectivity argument

In this section, we give the proof of Proposition 2.8 which gives the connectivity of the complexes $\mathcal{O}^1(S)$ and $\mathcal{O}^2(S)$ used in the previous section to prove the stability theorem. The rough line of argument is to embed $\mathcal{O}^i(S)$ in a larger complex which we can show is contractible, and work backwards from there, deducing high connectivity of smaller and smaller complexes. We start by defining the relevant complexes.

Let $\Delta \subset \partial S$ be a non-empty set of points. We consider arcs in S with boundary in Δ. We say that an arc a is *trivial* if it separates S into two components, one of which is a disc intersecting Δ only in the boundary of a. Let $\mathcal{A}(S,\Delta)$ be the simplicial complex whose vertices are isotopy classes of non-trivial arcs in S with boundary in Δ. A p–simplex of $\mathcal{A}(S,\Delta)$ is a collection of $p+1$ distinct isotopy classes of arcs $\langle a_0,\ldots,a_p\rangle$ representable by arcs with disjoint interiors.

The second complex we consider is the complex of arcs between two sets of points. Given two disjoint sets of points $\Delta_0,\Delta_1 \subset \partial S$, define $\mathcal{B}(S,\Delta_0,\Delta_1) \subset \mathcal{A}(S,\Delta_0\cup\Delta_1)$ to be the subcomplex of arcs with one boundary point in Δ_0 and one in Δ_1.

Let $\mathcal{B}_0(S,\Delta_0,\Delta_1) \subset \mathcal{B}(S,\Delta_0,\Delta_1)$ be the subcomplex of non-separating collections, i.e. simplices $\sigma = \langle a_0,\ldots,a_p\rangle$ such that the complement of the arcs

$a_0, \dots, a_p$ in S is connected. (This is a subcomplex as the non-separating property is preserved under taking faces.)

Finally, the complex $\mathcal{O}(S, b_0, b_1)$ we are interested in is the subcomplex of $\mathcal{B}_0(S, \{b_0\}, \{b_1\})$ of simplices $\langle a_0, \dots, a_p\rangle$ such that the ordering of the arcs at b_0 is opposite to that at b_1. (See Definition 2.1.)

We will prove that $\mathcal{O}(S, b_0, b_1)$ is $(g-2)$–connected by first proving that $\mathcal{A}(S,\Delta)$ is contractible (in most cases), and then slowly deducing connectivity bounds for each of the complexes in the sequence

$$\mathcal{A}(S,\Delta) \overset{i_1}{\hookleftarrow} \mathcal{B}(S,\Delta_0,\Delta_1) \overset{i_2}{\hookleftarrow} \mathcal{B}_0(S,\Delta_0,\Delta_1) \overset{i_3}{\hookleftarrow} \mathcal{O}(S,b_0,b_1).$$

In the end, we only need the case of $\Delta = \{b_0, b_1\}$, but the connectivity arguments will use an induction requiring to know the connectivity of complexes with a larger number of points—this comes from the fact that, cutting the surface along arcs between points of Δ produces several copies of the original points of Δ (see Figure 2).

The connectivity arguments we will use are of three types: (1) *direct calculation* showing contractibility, (2) exhibition of a complex as a *suspension* (or wedge of such) of a "previous" complex, and (3) *inductive deduction* from the connectivity of a larger complex. The argument for the connectivity of $\mathcal{A}(S,\Delta)$ will be a mix of type (1) and (2), the deduction along i_1 in the sequence is the most intricate argument and will be a mix of the three types of arguments, while deduction along i_2 and i_3 will be purely (and simpler) type (3) arguments.

The arguments given in this section are collected from the papers [14, 17, 21, 30, 32]. Theorem 4.1, which gives the contractibility of the full arc complexes, was originally proved by Harer using the theory of train tracks [14, Sect. 2]. We give here a much simpler proof, by surgering the arcs, due to Hatcher [17]. Theorem 4.3, giving the connectivity of the complex $\mathcal{B}(S,\Delta_0,\Delta_1)$, is also originally due to Harer [14, Thm. 1.6], though there is a gap in his proof, which is fixed in [32, Thm. 2.3]. We follow here the proof given in [32] which uses a mixture of Hatcher's surgery argument, and an careful inductive deduction which we learned from reading Ivanov [21]. Deducing the connectivity of the non-separating subcomplex $\mathcal{B}_0(S,\Delta_0,\Delta_1)$ (Theorem 4.8 below) is a more standard argument that goes back at least to Harer. Finally Theorem 4.9 gives the connectivity of the ordered complex $\mathcal{O}(S, b_0, b_1)$. This was obtained by Ivanov [21, Thm. 2.10] in the case where the two points are on the same boundary component via a different sequence of complexes and where the ordering comes naturally from a Morse-theoretic argument. The general case is given by Randal-Williams in [30, Thm. A.1], deducing it from Theorem 4.8 via a combinatorial argument, similar to that of Theorem 4.8.

We start by proving the contractibility of the full arc complex $\mathcal{A}(S,\Delta)$:

Theorem 4.1. *$\mathcal{A}(S,\Delta)$ is contractible, unless S is a disc or an annulus with Δ included in a single component of ∂S, in which case it is $(q+2r-7)$–connected, where $q=|\Delta|$ and $r=1,2$ is the number of boundary components of S.*

Note that even though we are mostly interested in surfaces of positive genus, the bound for the connectivity of $\mathcal{A}(S,\Delta)$ in the case of discs will actually play a role in the proof of the connectivity of the next complex, $\mathcal{B}(S,\Delta_0,\Delta_1)$.

We start by a suspension lemma, i.e. type (2) argument:

Lemma 4.2. *Suppose $\mathcal{A}(S,\Delta)\neq\varnothing$ and Δ' is obtained from Δ by adding an extra point in a component of ∂S already containing a point of Δ. If $\mathcal{A}(S,\Delta)$ is d–connected, then $\mathcal{A}(S,\Delta')$ is $(d+1)$–connected.*

One can actually show that $\mathcal{A}(S,\Delta')$ is homeomorphic to the suspension of $\mathcal{A}(S,\Delta)$ (see [17], revised version) but we will not need that.

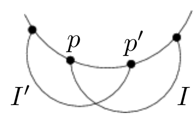

FIGURE 8.

Proof. Suppose $\Delta'=\Delta\cup\{p'\}$ and $p\in\Delta$ is a closest element to p' in ∂S. Let I and I' be the arcs drawn in Figure 8, where the points of Δ' to the left of p and right of p' may be equal to p' or p. We have a decomposition

$$\mathcal{A}(S,\Delta')=\mathrm{Star}(I)\cup_{\mathrm{Link}(I)}X$$

where X is the subcomplex of $\mathcal{A}(S,\Delta')$ of collections of arcs not containing I. We will show that X deformation retracts onto the star of I', and hence that it is contractible. The result then follows from the fact that the link of I is isomorphic to $\mathcal{A}(S,\Delta)$. (For $d\geq 0$, van Kampen's theorem implies that $\mathcal{A}(S,\Delta')$ is simply connected, so that it is enough to check connectivity in homology, which follows from the Mayer-Vietoris exact sequence.)

$\mathrm{Star}(I')$ is exactly the subcomplex of X of arcs without endpoints at p. The idea of the retraction $X\to\mathrm{Star}(I')$ is to move the arcs one by one from p to p' along I', as shown in Figure 9.

Note that the only arc that would become trivial when slided from p to p' is I, and it is not in X.

The retraction can be made explicit as follows. We want a map $f\colon I\times X\to X$ so that $f(s,\mathrm{Star}(I'))=id_{\mathrm{Star}(I')}$ and $f(1,X)\subset\mathrm{Star}(I')$. Given a simplex $\sigma=\langle a_0,\dots,a_q\rangle$ of X, we can consider the arcs of σ attached at p. Suppose that there are k 'germs of arcs' $\gamma_1,\dots,\gamma_k$ occurring in that order at p, with γ_i a germ of the arc a_{j_i}, where it is possible that $j_i=j_{i'}$ for some $i\neq i'$ if the arc a_{j_i} has

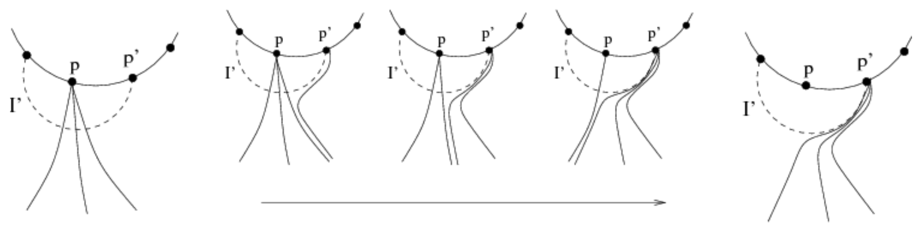

FIGURE 9. Simplices r_1, r_2, r_3 of the retraction in the case of 3 germs of arcs at p

both its endpoints at p. There is a sequence of k $(q+1)$–simplices $r_1(\sigma), \dots, r_k(\sigma)$ associated to σ, where $r_i(\sigma)$ is obtained from σ by moving the first i germs of arcs at p to p' and keeping the last $k-i+1$ germs, so that γ_i has a copy both at p and at p'. (See Figure 9 for an example when $k=3$.) If L denotes the operator on arcs that moves the first germ of the arc at p to p', we have $r_i(\sigma) = \langle b_0, \dots, b_{q+1} \rangle$ with $b_l = L^{\epsilon_i(l)}(a_l)$ for $l \le q$ and $b_{q+1} = L^{\epsilon_i(j_i)+1}(a_{j_i}) = L^{\epsilon_{i+1}(j_i)}(a_{j_i})$, where $\epsilon_i(l)$ is the number of $j < i$ such that γ_j is a germ of a_l.

A point in a simplex σ corresponds to a weighted collection of arcs via the barycentric coordinates $(t_0, \dots, t_q)$, the arc a_i having weight t_i. Assign to the ith germ γ_i the weight $w_i = t_{j_i}/2$. As $\sum_{j=0}^{p} t_j = 1$, we have $\sum_{i=1}^{k} w_i \le 1$. Now for $\sum_{j=1}^{i-1} w_j \le s \le \sum_{j=1}^{i} w_j$, define the retraction by

$$f(s, [\sigma, (t_0, \dots, t_q)]) = [r_i(\sigma), (v_0, \dots v_{q+1})]$$

where the weight $v_i = t_i$ except for the pair

$$(v_{j_i}, v_{q+1}) = (t_{j_i} - 2(s - \sum_{j=1}^{i-1} w_j), 2(s - \sum_{j=1}^{i-1} w_j))$$

i.e. the weight of (b_{j_i}, b_{q+1}) goes from $(t_{j_i}, 0)$ to $(0, t_{j_i})$ as s goes from $\sum_{j=1}^{i-1} w_j$ to $\sum_{j=1}^{i} w_j$. For $\sum_{i=1}^{k} w_i \le s \le 1$, define $f(s, (\sigma, (t_0, \dots, t_q)))$ to be constant, equal to $f(\sum_{i=1}^{k} w_i, (\sigma, (t_0, \dots, t_q)))$. Note that $f(1, (\sigma, (t_0, \dots, t_q)))$ lies in the face of r_k which is in $\mathrm{Star}(I')$.

This deformation is continuous as going to a face of σ corresponds to a t_i (and the corresponding w_j if any) going to zero. □

Proof of Theorem 4.1. We first consider the special case of the disc and cylinder with all points of Δ in one boundary component of S. Figure 10(a) shows that $\mathcal{A}(D^2, \Delta)$ is non-empty as soon as Δ has 4 points, and 10(b) that $\mathcal{A}(S^1 \times I, \Delta)$ is non-empty if $S^1 \times I$ has two points in one boundary component. As $4+2-7 = -1$ $(q = 4, r = 1)$ and $2+4-7 = -1$ $(q, r = 2)$, and (-1)–connected means non-empty, the theorem, for the special cases, is true when $r = 1$ and $q \le 4$, and when $r = 2$ and $q \le 2$. The result then follows more generally for any q by the lemma, which shows that the connectivity of these complexes goes up by one each time q goes up by one. We assume from now on that we are not in the special cases.

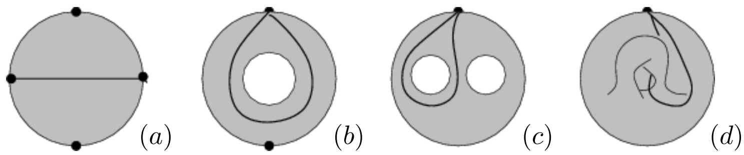

FIGURE 10. Checking non-emptiness for the disc, cylinder, pair of pants and genus 1 surface

For the general case, the lemma allows us to assume that there is at most one point of Δ in each boundary component. We claim first that $\mathcal{A}(S,\Delta)$ is non-empty: this is clear if Δ has at least 2 points, as they lie in different boundary components—any arc connecting two such points will be non-trivial. If Δ has only one point, we have that S has genus at least one or has at least three boundary components. Figure 10(c) and (d) show that there is at least one non-trivial arc in both of these cases.

Now fix a point p of Δ and an arc a of $\mathcal{A}(S,\Delta)$ with at least one of its endpoints at p. Fix also a germ of a at p. We want to define a retraction of the complex onto the star of a. The argument is similar to that of the proof of Lemma 4.2, and is summarized by Figure 11. Given a q–simplex σ of $\mathcal{A}(S,\Delta)$,

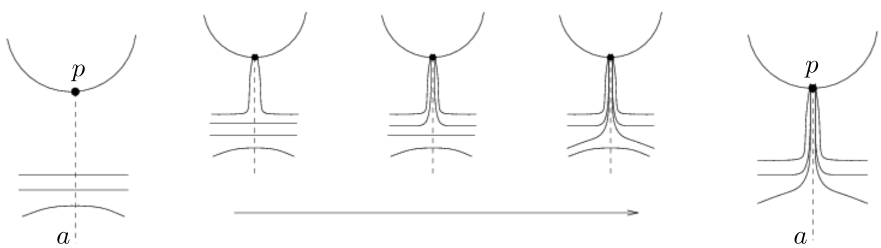

FIGURE 11. Retraction of $\mathcal{A}(S,\Delta)$ in the case of 3 germs of arcs crossing a

we represent it by a simplex with minimal and transverse intersection with a. If there are k intersections at germs of arcs $\gamma_1,\dots,\gamma_k$, we define as in the lemma k intermediate simplices $r_1(\sigma),\dots,r_k(\sigma)$ by successively cutting the arcs at the intersection points and connecting the new endpoints to p along a.

More precisely, if a_i intersects a at a point x, we can define $L(a_i)$ and $R(a_i)$ to be the arcs obtained from a_i by cutting a_i at x and joining the new endpoints to p along a, then pushing the arcs a little so they become disjoint of a, of a_i and of each other—there is then one arc on each side of a between x and p and we call $L(a_i)$ (resp. $R(a_i)$) the arc running to the left (resp. right) of a towards p. If

$L(a_i)$ or $R(a_i)$ is a trivial arc, we forget it. As there is at most one point of Δ per boundary components, and arcs intersect a minimally, the only trivial arcs that can occur in this process are of the type shown in Figure 12 where one side of the surgered arc becomes trivial. In particular both $L(a_i)$ and $R(a_i)$ cannot be trivial at the same time.

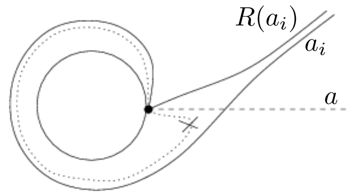

FIGURE 12. Arc where surgery creates one trivial arc (and one non-trivial one)

Define $r_i(\sigma)$ to be the $q+l_i$–simplex, with $1 \leq l_i \leq i+2$, with vertices $L^{\epsilon_i(l)}(a_l)$ and $R^{\epsilon_i(l)}(a_l)$ (if non-trivial), where $\epsilon_i(l) \in \{0,\ldots,k\}$ is the number germs γ_j included in a_l with $j<i$, and where $L^0(a_l)=R^0(a_l)=a_l$ is a single vertex. Then the retraction is as in the lemma though when a_{j_i} is replaced by two arcs $R(a_{j_i})$ and $L(a_{j_i})$, the weight of $(a_{j_i}, R(a_{j_i}), L(a_{j_i}))$ in r_i will go from $(t_{j_i},0,0)$ to $(0,t_{j_i}/2,t_{j_i}/2)$ so that the total weight stays equal to 1.

For this retraction to be well-defined, we need that the arcs obtained by cutting do not depend on the representative of σ. This follows from the fact that isotopic collections of arcs in minimal transverse intersection are isotopic through minimal transverse intersection. This can be proved by modifying a given an isotopy as in the proof of Proposition 2.2. □

Next we deduce the connectivity of the subcomplex $\mathcal{B}(S,\Delta_0,\Delta_1)$ of arcs between two subsets Δ_0 and Δ_1 of Δ. This is the most intricate of all the connectivity argument. To begin with, we need a definition to be able to state the connectivity bound: Disjoint sets $\Delta_0,\Delta_1\subset\partial S$ define a decomposition of ∂S into vertices (the points of $\Delta_0\cup\Delta_1$), edges between the vertices, and circles without vertices. We say that an edge is *pure* if both its endpoints are in the same set, Δ_0 or Δ_1. We say that an edge is *impure* otherwise. Note that the number of impure edges is always even.

Theorem 4.3. *The complex $\mathcal{B}(S,\Delta_0,\Delta_1)$ is $(4g+r+r'+l+m-6)$–connected, where g is the genus of S, r its number of boundary components, r' the number of components of ∂S containing points of $\Delta_0\cup\Delta_1$, l is half the number of impure edges and m is the number of pure edges.*

The proof of the theorem is in the same spirit as that of the previous theorem: the general argument works only in a restricted situation, so one first eliminates a number of cases, which we do by 'pushing' and surgery arguments as in the lemma and the theorem above. The argument for the general case will then be our first inductive deduction (argument of type (3)).

We say that a boundary component of S with points of $\Delta_0 \cup \Delta_1$ is *pure* if it is composed of pure edges, i.e. the points are either all in Δ_0 or all in Δ_1. The first lemma gives contractibility when S has a pure boundary, which is the case where the argument of Theorem 4.1 can be applied:

Lemma 4.4. *If S has at least one pure boundary component, then $\mathcal{B}(S, \Delta_0, \Delta_1)$ is contractible.*

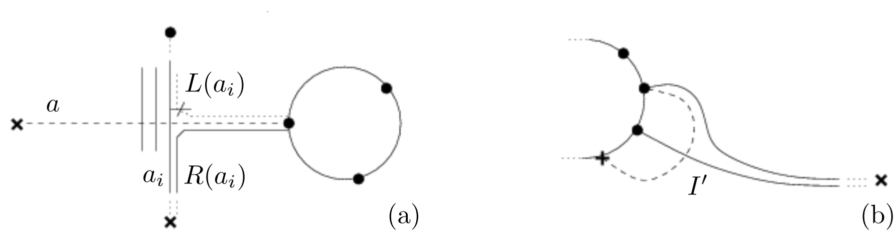

Figure 13. Lemmas 4.4 and 4.5

Proof. Choose an arc a with one boundary point on a pure boundary of S. We do a retraction onto the star of a as in the proof of Theorem 4.1, though where there is always exactly one arc which is kept after surgery as only one of $L(a_i)$ and $R(a_i)$ still has a boundary point in each of Δ_0 and Δ_1. (See Figure 13(a).) The retraction is well-defined because each newly created arc will necessarily be non-trivial, having its boundary points in two different components of ∂S. □

Lemma 4.5. *If S has at least one pure edge between a pure and an impure one in a boundary component of S, then $\mathcal{B}(S, \Delta_0, \Delta_1)$ is contractible.*

Proof. This is completely analogous to the proof of Lemma 4.2, except that the arc corresponding to I in the case of Lemma 4.2 does not exist in our complex $\mathcal{B}(S, \Delta_0, \Delta_1)$ (compare Figure 8 with Figure 13(b)). Hence the argument of Lemma 4.2 gives a contraction from $X = \mathcal{B}(S, \Delta_0, \Delta_1)$ to the star of I', showing that the complex is contractible. □

Lemma 4.6. *If the complex is non-empty, adding a pure edge between two impure edges increases the connectivity of $\mathcal{B}(S, \Delta_0, \Delta_1)$ by one.*

Proof. This is now precisely as in Lemma 4.2, as shown in Figure 14(a). □

Lemma 4.7. *When S has at least one impure edge and the complex is non-empty, adding a boundary component to S disjoint from Δ increases the connectivity of $\mathcal{B}(S, \Delta_0, \Delta_1)$ by one.*

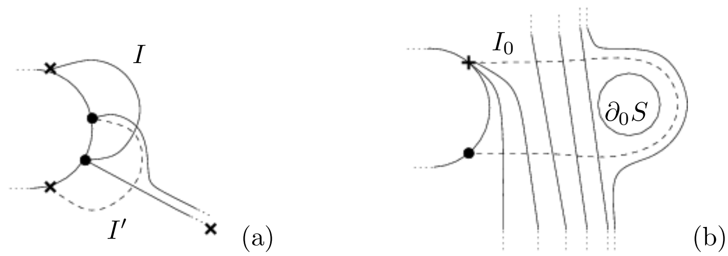

FIGURE 14. Lemmas 4.6 and 4.7

Proof. This is a variation on the proof of Lemma 4.2. Suppose $\partial_0 S$ is a boundary component of S disjoint from Δ, and let S' be $S \cup_{\partial_0 S} D^2$. We want to show that $\mathcal{F}(S,\Delta_0,\Delta_1)$ is one more connected than $\mathcal{F}(S',\Delta_0,\Delta_1)$.

Call an arc I of $\mathcal{F}(S,\Delta_0,\Delta_1)$ *special* if it separates a cylinder from S, one of whose boundary components is $\partial_0 S$ and the other one intersects $\Delta_0 \cup \Delta_1$ only at the endpoints of I. As any two distinct special arcs must intersect, we have

$$\mathcal{F}(S,\Delta_0,\Delta_1) = X \bigcup_{\substack{\mathrm{Link}(I),\\ I \text{ special}}} \mathrm{Star}(I)$$

where X is the subcomplex of $\mathcal{F}(S,\Delta_0,\Delta_1)$ of simplices having no special arc among their vertices. Note that the link of any special arc I is isomorphic to $\mathcal{F}(S',\Delta_0,\Delta_1)$.

Pick a special arc I_0. We can produce a retraction of $X \cup \mathrm{Star}(I_0)$ onto $\mathrm{Star}(I_0)$. As in the proof of Lemma 4.2, we do this by producing, for any p–simplex σ intersecting I_0, a sequence of $(p+1)$–simplices $r_1,\dots,r_k$ obtained by moving the intersections of σ with I_0 one by one across $\partial_0 S$ in the way shown in Figure 14(b). Passing an arc across ∂_0 creates a trivial arc only in the case of special arcs (different from I_0), and these are assumed not to be in X. Hence the retraction is well-defined and result follows from van Kampen's theorem and the Mayer-Vietoris long exact sequence for the decomposition $\mathcal{F}(S,\Delta_0,\Delta_1) = \big(X \cup \mathrm{Star}(I_0)\big) \bigcup_{\substack{\mathrm{Link}(I),\\ I \text{ special}, I\neq I_0}} \mathrm{Star}(I)$. □

Proof of Theorem 4.3. Note first that the theorem is obviously true in the cases where we have already shown that $\mathcal{B}(S,\Delta_0,\Delta_1)$ is contractible. The proof will inductively deduce the connectivity of $\mathcal{B}(S,\Delta_0,\Delta_1)$ from that of $\mathcal{A}(S,\Delta_0\cup\Delta_1)$ in the cases not already taken care of by the lemmas.

For $g=0$ and $r'=1$, according to the theorem, non-emptiness occurs when $r+l+m \geq 4$. As $r'=1$, we have $l \geq 1$. If $l=1$, this means either $r=1$ and $m \geq 2$, $r=2$ and $m \geq 1$, or $r \geq 3$. If $l=2$, we need either $r=1$ and $m\geq 1$ or $r \geq 2$. Non-emptiness in these five cases are verified in Figure 15. The result then

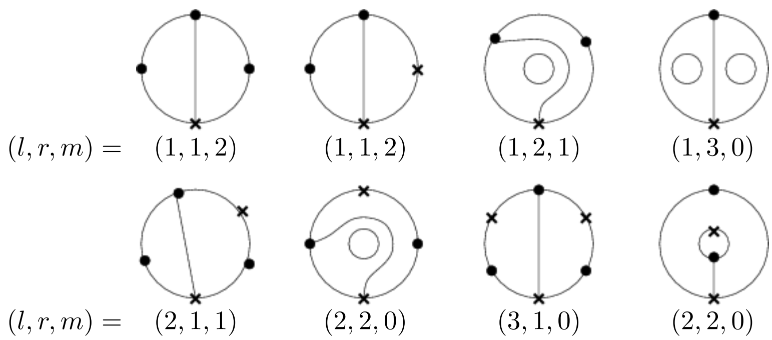

FIGURE 15. Verifying non-emptiness

follows more generally for the case $g = 0, r' = 1$ and $l = 1, 2$ by Lemmas 4.5,4.6 to change the value of m and Lemma 4.7 to change the value of r.

We will prove the result in general by induction on the lexicographically ordered triple (g, r, q), where the genus $g \geq 0$, the number of boundaries $r \geq 1$ and $q = 2l + m \geq 2$ is the cardinality of $\Delta = \Delta_0 \cup \Delta_1$. By the above, we can assume $l \geq 3$ if $g = 0$ and $r' = 1$. For each (g, q, r), it is enough to show the result in the case $r = r'$ (by Lemma 4.7) and $m = 0$ (by Lemmas 4.4,4.5,4.6), as $\mathcal{B}(S, \Delta_0, \Delta_1)$ is non-empty when $r' = 1$ and $l \geq 3$ or when $r' \geq 2$ as shown in Figure 15. The induction starts with $(g, r, q) = (0, 1, 6)$, where $q = 2l$ as $m = 0$ (which is non-empty as already checked).

So consider a surface S and a pair of sets (Δ_0, Δ_1) in ∂S with $r = r'$ and $m = 0$ (and $l \geq 3$ if $g = 0$ and $r = 1$). We need to show that $\mathcal{B}(S, \Delta_0, \Delta_1)$ is $(4g + 2r + l - 6)$–connected. Let $k \leq 4g + 2r + l - 6$ and $f \colon S^k \to \mathcal{B}(S, \Delta_0, \Delta_1)$ be a map, which we may assume to be simplicial for some PL triangulation of S^k by Theorem 6.3. By Theorem 4.1 there is an extension $\hat{f}$ of f to $\mathcal{A}(S, \Delta)$:

$$\begin{array}{ccc} S^k & \xrightarrow{f} & \mathcal{B}(S, \Delta_0, \Delta_1) \\ \big\downarrow & & \big\downarrow \\ D^{k+1} & \xrightarrow{\hat{f}} & \mathcal{A}(S, \Delta) \end{array}$$

Indeed, $\mathcal{A}(S, \Delta)$ is contractible, unless $g = 0$ and $r' = 1$ (with $r = r'$ here) in which case we need $4g + 2r + l - 6 \leq 2r + q - 7$ which is clear as $q = 2l \geq 6$ in this case. Using Theorem 6.3 again, we may moreover assume that $\hat{f}$ is simplicial with respect to some PL triangulation of D^{k+1} extending the triangulation of S^k. We are going to inductively deform $\hat{f}$ so that its image lies in $\mathcal{B}(S, \Delta_0, \Delta_1)$.

For a simplex σ of D^{k+1}, we say that σ is *bad* if its image lies in the complement of $\mathcal{B}(S, \Delta_0, \Delta_1)$ in $\mathcal{A}(S, \Delta)$, i.e. if all the vertices of σ are mapped to *pure arcs*, arcs not in $\mathcal{B}(S, \Delta_0, \Delta_1)$. Cutting S along the arcs of $\hat{f}(\sigma)$, we have a decomposition into connected components

$$(S,\Delta_0,\Delta_1)\backslash\hat{f}(\sigma) = (X_1,\Delta_0^1,\Delta_1^1)\sqcup\cdots\sqcup(X_c,\Delta_0^c,\Delta_1^c)\sqcup(Y_1,\Gamma_1)\sqcup\cdots\sqcup(Y_d,\Gamma_d)$$

where each Δ_ϵ^i is a non-empty set in ∂S inherited from Δ_ϵ, with $\epsilon = 0,1$, and Γ_j is a set containing copies of points of either Δ_0 or Δ_1, i.e. the Y_j's have only points of one type. Here by "inherited", we mean that there is a copy of a point of Δ in each X_i or Y_j neighboring it (as in Figure 2).

Let $Y_\sigma = i_1(Y_1)\cup\cdots\cup i_d(Y_d)$, where $i_j\colon Y_j\to S$ is the canonical inclusion —which is not necessarily injective on ∂Y_j. Note that each component of Y_σ has only points of one type in its boundary. We say that σ is *regular* if no arc of $\hat{f}(\sigma)$ lies inside Y_σ. When σ is regular, Y_σ is the disjoint union of the Y_j's modulo the identification of the points Γ_j's coming from the same point of Δ.

Let σ be a regular bad p–simplex of D^{k+1} maximal with respect to the ordered pair (Y_σ,p), where $(Y_{\sigma'},p') < (Y_\sigma,p)$ if $Y_{\sigma'}\subsetneq Y_\sigma$ with $\partial Y_{\sigma'}\backslash\partial Y_\sigma$ a union of non-trivial arcs in Y_σ, or if $Y_{\sigma'} = Y_\sigma$ and $p' < p$. The map $\hat{f}$ restricts on the link of such a simplex σ to a map

$$\mathrm{Link}(\sigma)\to J_\sigma = \mathcal{B}(X_1,\Delta_0^1,\Delta_1^1)*\cdots*\mathcal{B}(X_c,\Delta_0^c,\Delta_1^c)*\mathcal{A}(Y_1,\Gamma_1)*\cdots*\mathcal{A}(Y_d,\Gamma_d).$$

Indeed, if a simplex τ in the link of σ maps to a simplex of pure arcs of X_i for some i, then $Y_\sigma\subset Y_{\tau*\sigma}$ and either $Y_{\tau*\sigma} > Y_\sigma$, or $Y_{\tau*\sigma} = Y_\sigma$ and σ was not of maximal dimension. (On the other hand arcs of $\mathcal{A}(Y_j,\Gamma_j)$ could not be added to σ by regularity.)

As the triangulation of D^{k+1} is a PL triangulation, we have $\mathrm{Link}(\sigma)\cong S^{k-p}$ (see Appendix). We will show now that J_σ is at least $(k-p)$–connected.

In the case when one of the Y_j's has non-zero genus or at least two boundary components, $\mathcal{A}(Y_j,\Gamma_j)$ is contractible by Theorem 4.1 (as each boundary of Y_j has points of Γ_j) and hence so is J_σ by Proposition 6.1. So we are left to consider the case that all the Y_j's have genus 0 and one boundary, i.e. Y_j is a disc for each j.

We have $\chi(S\backslash\hat{f}(\sigma)) = \chi(S)+p'+1$, where $p'+1\le p+1$ is the number of distinct arcs in the image of σ. If X_i has genus g_i and $r_i = r_i'$ boundaries, the above equation gives $\sum_{i=1}^c(2-2g_i-r_i)+d = 2-2g-r+p'+1$ or equivalently

$$\textstyle\sum_{i=1}^c(2g_i+r_i) = 2g+r-p'+2c+d-3.$$

Moreover, we have

$$\textstyle\sum_{i=1}^c m_i+\sum_{j=1}^d q_j = 2p'+2,$$

where m_i is the number of pure edges in X_i and $q_j = |\Gamma_j|$, and $\sum_{i=1}^c l_i = l$, where l_i is half the number of impure edges in X_i.

By induction, $\mathcal{B}(X_i,\Delta_0^i,\Delta_1^i)$ is $(4g_i+2r_i+l_i+m_i-6)$–connected. Indeed for each i, we have $(g_i,r_i,q_i)<(g,r,q)$. On the other hand, $\mathcal{A}(Y_j,\Gamma_j)$ is (q_j-5)–connected by Theorem 4.1. Applying Proposition 6.1, we get

$$\begin{aligned}\mathrm{Conn}(J_\sigma) &\ge \textstyle\sum_{i=1}^c(4g_i+2r_i+l_i+m_i-4)+\sum_{j=1}^d(q_j-3)-2\\ &= 4g+2r-2p'+4c+2d-6+l+2p'+2-4c-3d-2\end{aligned}$$

$= 4g + 2r + l - d - 6 \geq 4g + 2r + l - p - 6 \geq k - p$
because $3d \leq p + 1$, and hence $d \leq p$, as the edges of Y_j are arcs of σ (as we assumed $m = 0$), minimum three edges are needed for a non-trivial Y_j and the same edge cannot be used twice in the Y_j's by regularity of σ.

Hence in all cases, there exists a PL $(k - p + 1)$–disc K with $\partial K = \mathrm{Link}(\sigma)$, and a simplicial map $F\colon K \to J_\sigma \hookrightarrow \mathcal{A}(S, \Delta)$.

Now we have $\mathrm{Star}(\sigma) = \sigma * \mathrm{Link}(\sigma)$ is a $(k+1)$–disc with boundary $\partial\sigma * \mathrm{Link}(\sigma)$ (see comments after Lemma 6.2). In the triangulation of D^{k+1}, we replace that disc with the disc $\partial\sigma * K$ which has same boundary, and we modify $\hat{f}$ in the interior of the disc using the map

$$\hat{f} * F \;:\; \partial\sigma * K \;\longrightarrow\; \mathcal{A}(S, \Delta)$$

which agrees with $\hat{f}$ on $\partial(\partial\sigma * K) = \partial\sigma * \mathrm{Link}(\sigma)$. We are left to show that we have improved the situation this way. The new simplices are of the form $\tau = \alpha * \beta$ with α a proper face of σ and β mapping to J_σ. Suppose τ is a regular bad simplex. Then each arc of β is pure and hence in $\mathcal{A}(Y_j, \Gamma_j)$ for some j. Thus $Y_\tau \subseteq Y_\sigma$. If they are equal, we must have $\tau = \alpha$ is a face of σ (by regularity of τ and σ). So $(Y_\tau, \dim(\tau)) < (Y_\sigma, p)$. Hence we have reduced the number of regular bad simplices of maximal dimension. As any bad simplex contains a regular bad subsimplex, the result follows by induction. □

We use now the connectivity of $\mathcal{B}(S, \Delta_0, \Delta_1)$ to deduce that of the subcomplex $\mathcal{B}_0(S, \Delta_0, \Delta_1)$ of non-separating simplices.

Theorem 4.8. *If Δ_0, Δ_1 are two disjoint non-empty sets of points in ∂S, then the complex $\mathcal{B}_0(S, \Delta_0, \Delta_1)$ is $(2g + r' - 3)$–connected, for g and r' as above.*

Proof. We prove the theorem as the previous one by induction on the lexicographically ordered triple (g, r, q), where $r \geq r'$ is the number of components of ∂S and $q = |\Delta_0 \cup \Delta_1| \geq 2$. To start the induction, note that the theorem is true when $g = 0$ and $r' \leq 2$ for any $r \geq r'$ and any q, and more generally that the complex is non-empty whenever $r' \geq 2$ or $g \geq 1$.

So fix (S, Δ_0, Δ_1) satisfying $(g, r, q) \geq (0, 3, 2)$. Then $2g + r' - 3 \leq 4g + r + r' + l + m - 6$. Indeed, $r \geq 1$ and $l + m \geq 1$. Moreover we assumed that either $r \geq 3$ or $g \geq 1$.

Let $k \leq 2g + r' - 3$ and consider a map $f\colon S^k \to \mathcal{B}_0(S, \Delta_0, \Delta_1)$, which we may assume to be simplicial for some PL triangulation of S^k (by Theorem 6.3). This map can be extended to a simplicial map $\hat{f}\colon D^{k+1} \to \mathcal{B}(S, \Delta_0, \Delta_1)$ by Theorem 4.3 and the above calculation, for a PL triangulation of D^{k+1} extending that of S^k, using again Theorem 6.3. We call a simplex σ of D^{k+1} *regular bad* if $\hat{f}(\sigma) = \langle a_0, \dots, a_p \rangle$ and each a_j separates $S \backslash (a_0 \cup \dots \widehat{a_j} \cdots \cup a_p)$. Let σ be a regular bad simplex of maximal dimension p. Write $S \backslash \hat{f}(\sigma) = X_1 \sqcup \cdots \sqcup X_c$ with each X_i

connected. By maximality of σ, $\hat{f}$ restricts to a map

$$\mathrm{Link}(\sigma) \longrightarrow J_\sigma = \mathcal{B}_0(X_1, \Delta_0^1, \Delta_1^1) * \cdots * \mathcal{B}_0(X_c, \Delta_0^c, \Delta_1^c)$$

where each Δ_ϵ^i is inherited from Δ_ϵ and is non-empty as the arcs of $\hat{f}(\sigma)$ are impure. Each X_i has $(g_i, r_i, q_i) < (g, r, q)$, so by induction $\mathcal{B}_0(X_i, \Delta_0^i, \Delta_1^i)$ is $(2g_i + r_i' - 3)$–connected. The Euler characteristic gives $\sum_i (2 - 2g_i - r_i) = 2 - 2g - r + p' + 1$, where $p'+1 \leq p+1$ is the number of arcs in $\hat{f}(\sigma)$. We also have $\sum_i (r_i - r_i') = r - r'$, so $\sum(2g_i + r_i') = 2g + r' - p' + 2c - 3$. Now J_σ is $(\sum_i(2g_i + r_i' - 1) - 2)$–connected (using Proposition 6.1), that is $(2g + r' - p' + c - 5)$–connected. As $c \geq 2$ and $p' \leq p$, we can extend the restriction of $\hat{f}$ to $\mathrm{Link}(\sigma) \simeq S^{k-p}$ to a map $F\colon K \to J_\sigma$ with K a $(k-p+1)$–disc with boundary the link of σ. We modify $\hat{f}$ on the interior of the star of σ using $\hat{f} * F$ on $\partial\sigma * K \simeq \mathrm{Star}(\sigma)$ as in the proof of Theorem 4.3. If a simplex $\alpha * \beta$ in $\partial\sigma * K$ is regular bad, β must be trivial since β does not separate $S \backslash \hat{f}(\alpha)$, so that $\alpha * \beta = \alpha$ is a face of σ. We have thus reduced the number of regular bad simplices of maximal dimension and the result follows by induction. □

We are now (finally!) ready to prove that the ordered complex $\mathcal{O}(S, b_0, b_1)$ has the connectivity claimed:

Theorem 4.9 (Proposition 2.8)**.** *$\mathcal{O}(S, b_0, b_1)$ is $(g-2)$–connected.*

Proof. Note first that the result is true for S of genus $g = 0$ or 1. We prove the proposition by induction on g, and we may assume $g \geq 2$.

Let $k \leq g - 2$ and suppose we have a simplicial map $f\colon S^k \to \mathcal{O}(S, b_0, b_1)$. As $2g + r' - 3 \geq g - 2$, Theorem 4.8 implies that there exists an extension $\hat{f}$ of f to the disc D^{k+1} with image in $\mathcal{B}_0(S, \{b_0\}, \{b_1\})$, and we may assume that $\hat{f}$ is simplicial by Theorem 6.3. As in the last two proofs, we want to modify $\hat{f}$ so that its image lies in $\mathcal{O}(S, b_0, b_1)$.

For a simplex σ of $\mathcal{B}_0(S, \{b_0\}, \{b_1\})$, with $\sigma = \langle a_0, \ldots, a_p \rangle$ such that the arcs are ordered $a_0 < a_1 < \cdots < a_p$ in the anti-clockwise ordering at b_0, we can write uniquely $\sigma = \sigma^g * \sigma^b$ where $\sigma^g = \langle a_0, \ldots, a_i \rangle$ with i maximal such that the clockwise order at b_1 starts with $a_0 < \cdots < a_i$. Thus if $\sigma = \sigma^g$, it is a simplex of $\mathcal{O}(S, b_0, b_1)$. We say that σ is *purely bad* if σ^g is empty. (Note that vertices are always good.)

Let σ be a purely bad p–simplex. We claim that the genus of $S \backslash \sigma$ is at least $g - p$. If b_0 and b_1 lie on different boundary components, this is true regardless of the fact that the simplex is bad: Cutting along the first arc of σ reduces the number of boundary components of S without affecting the genus, and subsequent arcs can at most each reduce the genus by one (by Euler characteristic considerations). On the other hand, if b_0 and b_1 are in the same boundary component, this is true because σ is bad. Indeed, suppose that two arcs a_i, a_j of σ are ordered clockwise both at b_0 and at b_1. Then their complement $S \backslash (a_i \cup a_j)$ is a surface with the same number of boundaries as S and thus of genus $g-1$ (again because of the Euler characteristic). The remaining $p-1$ arcs of σ can each reduce the genus by at most one.

Now we want to remove purely bad simplices from the image of $\hat{f}$. Let σ be a maximal simplex of D^{k+1} such that $\hat{f}(\sigma)$ is purely bad. As σ is maximal with that property, the link of σ is mapped to $\mathcal{O}(S\backslash\hat{f}(\sigma), b_0', b_1')$, where b_0' and b_1', as shown in Figure 16, are the copies of b_0 and b_1 lying between the boundary containing b_0 and the first arc of σ at b_0 (in the anticlockwise ordering), and between the boundary containing b_1 and the first arc at b_1 (in the clockwise ordering). If σ

FIGURE 16. Purely bad 2–simplex, and vertex in its link/complement

is a p–simplex, $\hat{f}(\sigma)$ is a p'–simplex with $p' \leq p$ and, by the above, $S\backslash\hat{f}(\sigma)$ has genus at least $g-p' \geq g-p$. The link of σ is a sphere of dimension $k-p$. As $k \leq g-2$, we have $k-p \leq g-p-2$. As $\hat{f}(\sigma)$ is bad, $p' \geq 2$ and $S\backslash\hat{f}(\sigma)$ has genus $\tilde{g} < g$. By induction, the restriction of $\hat{f}$ to the link of σ extends to a map F on a $(k+1-p)$–disc K. We replace $\hat{f}$ on $\mathrm{Star}(\sigma) \simeq \partial\sigma * K$ by $\hat{f} * F$. The new purely bad simplices in the image are faces of σ and hence of smaller dimension. The result follows by induction. □

5. Closed surfaces

In this section we prove Theorem 1.2, the stability theorem for surfaces without boundary components. The idea, going back to Ivanov [22], is to build two spectral sequences computing the homology of $\Gamma_{g,1}$ and $\Gamma_{g,0}$ respectively, so that both sequences have terms involving only mapping class groups of surfaces with boundaries. Then the map $\delta_g\colon \Gamma_{g,1} \to \Gamma_{g,0}$ induced by gluing a disc, on the spectral sequences, can be identified as the left inverse of the map β already computed. We can this way verify that we have an isomorphism in a range using the stability theorem for mapping class groups of surfaces with boundaries.

Ivanov used the complex of non-separating curves in a surface. This approach requires: (1) to compute the connectivity of the complex of curves and (2) handle stabilizers of curves, which are not as well-behaved as stabilizers of arcs. The connectivity can be deduced from that of the arc complex (see Harer [14]—there is also an unpublished improved argument by Hatcher-Vogtmann, and an alternative argument by Ivanov [20]). Elements in the stabilizer of a circle may rotate the circle, permute circles in a higher simplex and even flip a circle. This can be dealt with using appropriate group extensions and comparing associated spectral sequences (see [22]).

We will instead describe here Randal-Williams' approach, which is simpler in addition to giving a slightly better range, though it will require working with the topological group of diffeomorphisms rather than with mapping class groups, and with semi-simplicial spaces instead of simplicial complexes. We start by briefly introducing the language of semi-simplicial spaces.

A *semi-simplicial space* $X_\bullet$ is a sequence of topological spaces $\{X_p\}_{p\geq 0}$ together with boundary maps $d_i\colon X_p \to X_{p-1}$ for each $0 \leq i \leq p$, satisfying the simplicial identity $d_i d_j = d_{j-1} d_i$ if $i < j$. (So a semi-simplicial space is a simplicial space without degeneracies.) The space X_p is the space of p–simplices.

We can define the realization $||X_\bullet||$ of a semi-simplicial space $X_\bullet$ like that of a simplicial set or simplicial complex by associating a topological p–simplex Δ^p to each p–simplex of X:

$$||X_\bullet|| := \coprod_{p\geq 0} X_p \times \Delta^p/_\sim$$

with the equivalence given by the face relations $(d_i x, t) \sim (x, d^i t)$, where for $t = (t_0, \dots, t_{k-1}) \in \Delta^{k-1}$, with $\sum t_i = 1$, $d^i t = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{k-1})$.

A simplicial space defines a double chain complex with set of (p,q)–chains $C_{p,q}(X_\bullet) = C_q(X_p)$, the singular chains of X_p, with vertical differential $d^V = d_{X_p}$, the differential of X_p, and horizontal differential $d^H = \sum_{i=0}^p (-1)^i d_i$, the simplicial differential. Then

$$H_*\big(C_{*,*}(X_\bullet)\,,\, d^H + (-1)^p d^V\big) = H_*(||X_\bullet||).$$

We will use below the spectral sequence associated to the vertical filtration of the double complex, which has E^1–term

$$E^1_{p,q} = H_q(X_p)$$

and converges to the homology of $||X_\bullet||$.

Let $r \geq 1$ and let $\partial_0 S, \dots, \partial_{k+r-1} S$ denote the boundary components of the surface $S_{g,k+r}$. For $0 \leq i \leq k$, define $d_i\colon \Gamma_{g,k+r} \to \Gamma_{g,k-1+r}$ to be the map that glues a disc on $\partial_i S$. These maps make

$$B\Gamma_{g,\bullet+r} = \qquad \cdots \ B\Gamma_{g,2+r} \ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \ B\Gamma_{g,1+r} \ \substack{\longrightarrow\\ \longrightarrow} \ B\Gamma_{g,r}$$

into a semi-simplicial space. (To be precise here, one needs to choose specific compatible identifications of $S_{g,k+r}$ with a disc glued on its ith boundary with $S_{g,k+r-1}$. This is can be done by choosing an appropriate decomposition of the surface into discs—by linearization, a diffeomorphism from the disc to itself is determined up to homotopy by what it does on the boundary.)

Theorem 5.1. [30] *For any $r \geq 0$, we have $||B\Gamma_{g,\bullet+r+1}|| \simeq B\,\mathrm{Diff}(S_{g,r})$.*

A direct consequence of the theorem is that $||B\Gamma_{g,\bullet+r+1}|| \simeq B\Gamma_{g,r}$ unless $r=0$ and $g=0,1$ as $\mathrm{Diff}(S_{g,r})$ has contractible components in all but these two special cases [6, 7].

We first show how to deduce Theorem 1.2 from this last result.

Proof of Theorem 1.2. We want to show that the map

$$H_*(\delta_g)\colon H_*(\Gamma_{g,1}) \to H_*(\Gamma_{g,0})$$

induced by gluing a disc, is an isomorphism for $* \le \frac{2g}{3}$ and a surjection for $* \le \frac{2g}{3}+1$. For the first two cases $g=0,1$, the non-trivial statement is that $H_*(\delta_g)$ is surjective for $*=1$.

Let $r=0$ or 1. The spectral sequence for the semi-simplicial space $B\Gamma_{g,\bullet+r+1}$ has E^1–term $E^1_{p,q}=H_q(B\Gamma_{g,p+r+1})=H_q(\Gamma_{g,p+r+1})$ and converges to $H_*(B\,\mathrm{Diff}(S_{g,r}))$ which is equal to $H_*(\Gamma_{g,r})$ unless $r=0$ and $g=0,1$. Let $\mathrm{Diff}_0(S_{g,0})$ denote the component of the identity in $\mathrm{Diff}(S_{g,0})$. The spectral sequence associated to the fibration

$$B\,\mathrm{Diff}_0(S_{g,0}) \to B\,\mathrm{Diff}(S_{g,0}) \to B\,\pi_0\mathrm{Diff}(S_{g,0}) = B\Gamma_{g,0}$$

has E^2-term $E^2_{p,q}=H_p(\Gamma_{g,0},H_q(B\,\mathrm{Diff}_0(S_{g,0})))$ converging to $H_{p+q}(B\,\mathrm{Diff}(S_{g,0}))$. As $H_1(B\,\mathrm{Diff}_0(S_{g,0}))=0$ because $\mathrm{Diff}_0(S_{g,0})$ is connected, we get $H_i(B\,\mathrm{Diff}(S_{g,0}))=H_i(\Gamma_{g,0})$ for $i=0,1$. Hence, for the purpose of proving the theorem, we can also use the semi-simplicial space $B\Gamma_{g,\bullet+r+1}$ to model the map δ_g in the cases $g=0,1$.

Gluing a disc on the last boundary of $S_{g,k+2}$ induces a simplicial map $B\Gamma_{g,\bullet+2}\to B\Gamma_{g,\bullet+1}$, which in turn induces a map of the corresponding spectral sequences

$$E^1_{p,q}=H_q(\Gamma_{g,p+2}) \;\longrightarrow\; \tilde{E}^1_{p,q}=H_q(\Gamma_{g,p+1}).$$

As gluing a disc is left inverse to gluing a pair of pants when both the source and target surfaces have boundaries, this map is surjective on the E^1–term, and an isomorphism for $q\le \frac{2g}{3}$ by Theorem 1.1. Hence we get an isomorphism of the targets $H_*(\Gamma_{g,1})$ and $H_*(\Gamma_{g,0})$ of the spectral sequences in all degrees $*\le\frac{2g}{3}$ and a surjection in degree $*\le\frac{2g}{3}+1$: As shown in Figure 17, surjectivity in degree

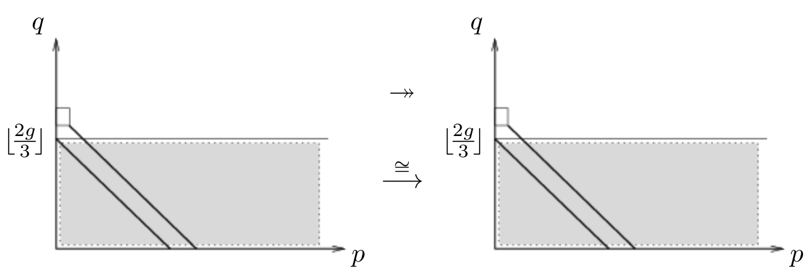

FIGURE 17. Map of spectral sequences

$q=\lfloor\frac{2g}{3}\rfloor+1$ follows from the surjection $E^\infty_{0,q}\twoheadrightarrow\tilde{E}^\infty_{0,q}$, which in turns follows from the corresponding surjection in the E^1–term by the commutativity of the diagram

$$\begin{array}{ccc} E^1_{0,q} & \twoheadrightarrow & E^\infty_{0,q} \\ \downarrow & & \downarrow \\ \tilde{E}^1_{0,q} & \twoheadrightarrow & \tilde{E}^\infty_{0,q} \end{array}$$

□

Proof of Theorem 5.1. Let $\mathrm{Conf}^{fr}(k, S_{g,r})$ denote the space of configurations of k ordered points in the interior of $S_{g,r}$, each equipped with a framing compatible with the orientation of the surface. The group $\mathrm{Diff}(S_{g,r})$ acts transitively on $\mathrm{Conf}^{fr}(k, S_{g,r})$ and the stabilizer of a point is isomorphic to $\mathrm{Diff}(S_{g,k+r})$. Let $E\,\mathrm{Diff}(S_{g,r})$ denote a contractible space with a free action of $\mathrm{Diff}(S_{g,r})$. By a continuous version of Shapiro's lemma

$$\mathrm{Conf}^{fr}(k, S_{g,r}) \times_{\mathrm{Diff}(S_{g,r})} E\,\mathrm{Diff}(S_{g,r}) \;\simeq\; B\,\mathrm{Diff}(S_{g,k+r}) \;\simeq\; B\Gamma_{g,k+r}$$

where the last equivalence holds by [6, 7] as long as $k + r > 0$. We write $\mathrm{Conf}^{fr}(k, S_{g,r})/\!\!/_{\mathrm{Diff}} := \mathrm{Conf}^{fr}(k, S_{g,r}) \times_{\mathrm{Diff}(S_{g,r})} E\,\mathrm{Diff}(S_{g,r})$. In fact, we have an equivalence of semi-simplicial spaces $B\Gamma_{g,\bullet+r+1} \simeq \mathrm{Conf}^{fr}(\bullet{+}1, S_{g,r})$ for any $r \geq 0$ with

$$\mathrm{Conf}^{fr}(\bullet{+}1, S_{g,r})/\!\!/_{\mathrm{Diff}} = \ldots \Rrightarrow \mathrm{Conf}^{fr}(2, S_{g,r})/\!\!/_{\mathrm{Diff}} \rightrightarrows \mathrm{Conf}^{fr}(1, S_{g,r})/\!\!/_{\mathrm{Diff}}$$

where, if the framed points at level k are labeled $\overrightarrow{p_0}, \ldots, \overrightarrow{p_k}$, the boundary map d_i forgets the ith point $\overrightarrow{p_i}$.

To calculate the homotopy type of the above semi-simplicial space, we first consider the semi-simplicial space

$$\mathrm{Conf}^{fr}(\bullet{+}1, S_{g,r}) = \qquad \cdots \Rrightarrow \mathrm{Conf}^{fr}(2, S_{g,r}) \rightrightarrows \mathrm{Conf}^{fr}(1, S_{g,r}).$$

For $r > 0$, we can give an explicit retraction $||\,\mathrm{Conf}^{fr}(\bullet + 1, S_{g,r})|| \xrightarrow{\sim} *$ as follows. Gluing a small collar along a boundary component of $S_{g,r}$ and retracting it shows that each $\mathrm{Conf}^{fr}(k, S_{g,r})$ is homotopy equivalent to the subspace $\mathrm{Conf}^{fr}_\epsilon(k, S_{g,r})$ of configurations at least ϵ–distant from that boundary component. Now choose a framed point $\overrightarrow{p}$ in that ϵ–neighborhood. We define a retraction from $||\,\mathrm{Conf}^{fr}_\epsilon(k, S_{g,r})|| \subset ||\,\mathrm{Conf}^{fr}(k, S_{g,r})||$ to the 0-simplex represented by $\overrightarrow{p}$ in $||\,\mathrm{Conf}^{fr}(k, S_{g,r})||$: A point $x \in ||\,\mathrm{Conf}^{fr}_\epsilon(k, S_{g,r})||$ has the form $x = [(\overrightarrow{p_0}, \ldots, \overrightarrow{p_k}), \underline{t}] \in \mathrm{Conf}^{fr}_\epsilon(k+1, S_{g,r}) \times \Delta^k$ for some $k \geq 0$. Given x, consider the $(k{+}1)$–simplex $\{(\overrightarrow{p_0}, \ldots, \overrightarrow{p_k}, \overrightarrow{p})\} \times \Delta^{k+1} \subset ||\,\mathrm{Conf}^{fr}(k, S_{g,r})||$. In that $(k{+}1)$–simplex, there is a straight line from $[(\overrightarrow{p_0}, \ldots, \overrightarrow{p_k}, \overrightarrow{p}), (\underline{t}, 0)] = [(\overrightarrow{p_0}, \ldots, \overrightarrow{p_k}), \underline{t}] = x$ to $[(\overrightarrow{p_0}, \ldots, \overrightarrow{p_k}, \overrightarrow{p}), (\underline{0}, 1)] = [\overrightarrow{p}, 1]$. Define the retraction by moving at constant speed along that line. This is continuous in x.

As crossing with a contractible space does not change the homotopy type, it follows that $\mathrm{Conf}^{fr}(\bullet+1, S_{g,r}) \times E\,\mathrm{Diff}(S_{g,r})$ is also contractible. This last semi-simplicial space admits a simplicial diagonal action of $\mathrm{Diff}(S_{g,r})$ with quotient $\mathrm{Conf}^{fr}(\bullet+1, S_{g,r})/\!\!/_{\mathrm{Diff}}$. As the action is free, we get that

$$||\,\mathrm{Conf}^{fr}(\bullet+1, S_{g,r})/\!\!/_{\mathrm{Diff}}\,|| \;\simeq\; B\,\mathrm{Diff}(S_{g,r}).$$

The result will follow in the same way for $r = 0$ if we can check that $||\,\mathrm{Conf}^{fr}(\bullet+1, S_{g,0})||$ is also contractible. Gluing a disc on the boundary component of $S_{g,1}$ induces a simplicial map $\mathrm{Conf}^{fr}(\bullet+1, S_{g,1}) \to \mathrm{Conf}^{fr}(\bullet+1, S_{g,0})$. It is a levelwise inclusion and thus a cofibration. At each simplicial level k, the cofiber $\mathrm{Conf}^{fr}(k+1, S_{g,1})/\,\mathrm{Conf}^{fr}(k+1, S_{g,0})$ can be identified with

$$\bigvee_{i=0}^{k} \big[S(D^2)\times \mathrm{Conf}^{fr}(k, S_{g,1})\big]\Big/\big[S(\partial D^2)\times \mathrm{Conf}^{fr}(k, S_{g,1})\big]$$

where the index i in the wedge records which one of the $k+1$ framed points $\overrightarrow{p_0}, \dots, \overrightarrow{p_k}$ is closest to the center of the glued disc, $S(D^2)$ denotes the sphere bundle of D^2 and records the position and framing of that point, and $\mathrm{Conf}^{fr}(k, S_{g,1})$ records the position and framing of the k other points. A configuration with no point close or closest to the center of the disc in $S_{g,0}$ is identified with the basepoint. When the point closest to the center of the disc moves away from the center, the configuration is identified with the basepoint, which is why we mod out by $S(\partial D^2) \times \mathrm{Conf}^{fr}(k, S_{g,1})$.

The cofiber at level k can be rewritten as $(S(D^2)/_{S(\partial D^2)}) \wedge \bigvee_0^k \mathrm{Conf}^{fr}(k, S_{g,1})_+$ and the cofiber of the simplicial map is the semi-simplicial space

$$(S(D^2)/_{S(\partial D^2)}) \wedge \bigvee_{i=0}^{\bullet} \mathrm{Conf}^{fr}(\bullet, S_{g,1})_+$$

with boundary maps on $\bigvee_0^\bullet \mathrm{Conf}^{fr}(\bullet, S_{g,1})_+$ induced from $\mathrm{Conf}^{fr}(\bullet+1, S_{g,0})$: Denoting the points in a configuration in the jth summand $\overrightarrow{p_0}, \dots, \overrightarrow{p_{j-1}},\ \overrightarrow{p_{j+1}}, \dots, \overrightarrow{p_k}$, the boundary map d_i on this summand forgets the ith point $\overrightarrow{p_i}$, unless $j = i$ in which case it just maps to the basepoint.

By the same argument as above, the semi-simplicial space $\bigvee_0^\bullet \mathrm{Conf}^{fr}(\bullet, S_{g,1})_+$ is equivalent to $\bigvee_0^\bullet \mathrm{Conf}_\epsilon^{fr}(\bullet, S_{g,1})_+$ and we can again define a retraction of this subspace by choosing a point $\overrightarrow{p}$ in the ϵ–neighborhood of the boundary. This time, a point in the realization has the form $x = [j, (\overrightarrow{p_1}, \dots, \overrightarrow{p_{j-1}}, \overrightarrow{p_{j+1}}, \dots, \overrightarrow{p_k}), \underline{t}]$ and we use the straight line in the simplex $\{(j, (\overrightarrow{p_1}, \dots, \overrightarrow{p_{j-1}}, \overrightarrow{p_{j+1}}, \dots, \overrightarrow{p_k}, \overrightarrow{p}))\} \times \Delta^k$, from x to the point $[j, \overrightarrow{p}, 1]$, which is identified with the basepoint for all j.

Hence the cofiber of the map $\mathrm{Conf}^{fr}(\bullet+1, S_{g,1}) \to \mathrm{Conf}^{fr}(\bullet+1, S_{g,0})$ is contractible. As the collapsed space $\mathrm{Conf}^{fr}(\bullet+1, S_{g,1})$ was contractible, we have that $\mathrm{Conf}^{fr}(\bullet+1, S_{g,0})$ is also contractible and the result follows for $r = 0$. $\square$

6. Appendix: Simplicial complexes

This short section gives the background material on simplicial complexes and piecewise linear topology needed in the rest of the paper. In particular, we consider joins of complexes and state the simplicial approximation theorem.

Combinatorially, a *simplicial complex* $X = (X_0, \mathcal{F})$ is a set of vertices X_0 together with a collection $\mathcal{F}$ of subsets of X_0 closed under taking subsets and containing all the singletons. The subsets of cardinality $p+1$ are called the *p–simplices* of X. If $\sigma = \langle x_0, \dots, x_p \rangle$ is a p–simplex of X, the subsets $\langle x_{i_0}, \dots, x_{i_k} \rangle$ of σ are called its *faces*.

To a simplicial complex X, one can associate a topological space, its *realization*, denoted $|X|$ or just X again, build as follows: $|X|$ has a 0-cell for each vertex of X, a 1-cell between any to vertices v, w such that $\langle v, w \rangle$ is a simplex of X, and more generally a p-simplex Δ^p for each p-simplex $\langle v_0, \dots, v_p \rangle$ of X with its codimension one faces identified with the simplices associated to the faces $\langle v_0, \dots, \widehat{v_j}, \dots, v_p \rangle$ of the simplex. When we talk about topological properties of a simplicial complex X, such as its connectivity, we mean the corresponding property for this associated topological space.

The *join* $X * Y$ of two simplicial complexes X and Y is the simplicial complex with vertices $X_0 \sqcup Y_0$ and a $(p+q+1)$–simplex $\sigma_X * \sigma_Y = \langle x_0, \dots, x_p, y_0, \dots, y_q \rangle$ for each p–simplex $\sigma_X = \langle x_0, \dots, x_p \rangle$ of X and q–simplex $\sigma_Y = \langle y_0, \dots, y_q \rangle$ of Y. Note that $|X * Y| = |X| * |Y|$, i.e. the realization of the join complex is the (topological) join of the realization of the two complexes. This follows from the fact that it is true for each pair of simplices.

Recall that a space (or simplicial complex) X is called *n–connected* if $\pi_i(X) = 0$ for all $i \leq n$ (where $\pi_i(X) := \pi_i(|X|)$ if X is a simplicial complex). For $n = -1$, we use the convention that (-1)*–connected* means non-empty. (For $n \leq -2$, n–connected is a void property.) Note that, by Hurewicz theorem, a simply connected space X is n-connected, $n \geq 2$, if and only if $H_*(X) = 0$ for $0 < * \leq n$.

The following proposition, which goes back at least to Milnor, tells us how to compute the connectivity of a join in terms of the connectivity of the pieces.

Proposition 6.1. [26, Lem 2.3] *Consider the join $X = X_1 * \cdots * X_k$ of k non-empty spaces. If each X_i is n_i–connected, then X is $\big(\big(\sum_{i=1}^k (n_i + 2)\big) - 2\big)$–connected.*

Note that the lemma implies that X is contractible whenever some X_i is contractible.

Given a simplex σ of a simplicial complex X, the (closed) *star of* σ, $\mathrm{Star}(\sigma)$, is the subcomplex of X of simplices containing σ, together with their faces. The *link of* σ, $\mathrm{Link}(\sigma) \subset \mathrm{Star}(\sigma)$, is the subcomplex of the star of simplices disjoint from σ. The link can also be described as the subcomplex of simplices τ disjoint from σ such that $\tau * \sigma$ is again a simplex of X, and

$$\mathrm{Star}(\sigma) = \mathrm{Link}(\sigma) * \sigma$$

where σ in the formula denotes the subcomplex of X defined by σ and its faces.

A *PL (or manifold) triangulation* K of an n–manifold M is a simplicial complex K such that $|K| \cong M$ and with the property that the link of a p–simplex σ of K is PL homeomorphic to the boundary of an $(n-p)$–simplex if σ not included in ∂K, or to an $(n-p-1)$–simplex if $\sigma \subset \partial K$.

Lemma 6.2. *(See for example* [19, Lem 1.13]*.) If D^n and S^n denote PL triangulations of the n–disc and n–sphere, then*

(i) $|D^n * D^m|$ *is an* $(n+m+1)$*–disc,*
(ii) $|D^n * S^m|$ *is an* $(n+m+1)$*–disc, and*
(iii) $|S^n * S^m|$ *is an* $(n+m+1)$*–sphere.*

Applying the Lemma to a PL triangulation K of an n–manifold, we get that, for a p–simplex σ, as $\mathrm{Star}(\sigma) = \mathrm{Link}(\sigma) * \sigma$, it is of the type $S^{n-p-1} * D^p$ (or $D^{n-p-1} * D^p$ if $\sigma \subset \partial K$), and hence the star of any simplex is an n–disc. Note moreover that, if $\sigma \not\subset \partial K$, the boundary of $\mathrm{Star}(\sigma)$ is the $(n-1)$–sphere $\partial\sigma * \mathrm{Link}(\sigma)$.

A main theorem we need for the connectivity results in Section 4 is the following:

Theorem 6.3 (Simplicial approximation)**.** [34] *Let K, L be finite simplicial complexes, and L a subcomplex of K. Let $f\colon |K| \to |X|$ be a continuous map such that the restriction $f|_L$ is a simplicial map from L to X. Then there exists a relative subdivision (K_r, L) of (K, L) and a simplicial map $g\colon K_r \to X$ such that $g|_L = f|_L$ and g is homotopic to f keeping L fixed.*

We use this theorem in Section 4 to approximate any map from a sphere into a simplicial complex by a simplicial map, and to approximate a null-homotopy of such a map, now simplicial, by a simplicial map from the disc with a triangulation extending that of the sphere. Hence we apply the theorem for the cases $(K, L) = (S^k, \varnothing)$ and $(K, L) = (D^{k+1}, S^k)$. Note that the complexes X we work with are usually not finite, but when applying the theorem, we can restrict to the (finite) subcomplex of X containing the image of the sphere or the disc. We also need the triangulations of the spheres and discs to be PL triangulations, and this can be obtained by choosing some PL triangulation of S^k (resp. D^{k+1}) and applying the theorem to it, noting that the subdivision (resp. relative subdivision) preserves the PL property.

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