Proof. Assume that

\[ P = \left( p'_1, \ldots, p'_n, p''_1, \ldots, p''_n, 0, \ldots, 0 \right) \]

with \( p'_1 \geq \cdots \geq p'_n \geq x > p''_1 \geq \cdots \geq p''_n \). Put \( s' = \sum p'_i, s'' = 1 - s' \). Fix \( p'_1, \ldots, p'_n \) and consider \( K \subseteq \mathbb{R}^{n''} \) and \( G : K \rightarrow \mathbb{R} \) defined by

\[ K = \{(q_1, \ldots, q_{n''}) \mid \sum q_j = s'', 0 \leq q_j \leq x\}, \]
\[ G(q) = F(p'_1, \ldots, p'_n, q_1, \ldots, q_{n''}, 0, \ldots, 0). \]

\( K \) is compact and convex, \( G : K \rightarrow \mathbb{R} \) concave and continuous. So \( G \) assumes its minimal value at an extremal point of \( K \), say at \( q^* = (q^*_1, \ldots, q^*_n) \). Assume that \( q^*_1 \geq \cdots \geq q^*_n \). Then \( q^* \) is of the form

\[ q^* = (x, x, \ldots, x, r, 0, \ldots, 0) \]

with \( 0 < r < x \). May now assume that:

\[ P = (p'_1, \ldots, p'_n, r, 0, \ldots, 0) \]

where \( 0 < r < x \) (or possibly \( r = 0 \)). By convexity of \( f \) in \([x, 1]\),

\[ F(p'_1, \ldots, p'_n, r, 0, \ldots, 0) \geq F(p_0, \ldots, p_0, r, 0, \ldots, 0) \]

where \( p_0 = s' / n' \). With \( P_0 = (p_0, \ldots, p_0, r, 0, \ldots, 0) \) we are done. \( \square \)