Sylvestor problem (prediction)

**Assume:** \( X \subseteq \text{convex}, D: X \times X \rightarrow \mathbb{R}_+ \) satisfies compensation identity (e.g. \( D = \| x - y \| \), \( D = \sum_i \ln \frac{x_i}{y_i} \)).

**Problem:** Find \( y^* \) s.t. \( R(y^*) = R_{\min} (= \inf_y R(y)), R(y) = \sup D(x, y) \).

**Introduce:** \( \hat{X} = \{ \alpha = (\alpha_x)_{x \in X} \mid \text{prob. dist. over } X \text{ w. finite supp} \} \), \( \alpha \sim \hat{\alpha} = \text{barycenter } (\sum_x \alpha_x \cdot x) \), \( \Phi(\alpha, y) = \sum_x \alpha_x D(x, y) \).

Then: \( \Phi(y, y) = H(\alpha) + D(\alpha, y) \) with \( H(\alpha) = \Phi(\alpha, \hat{\alpha}) = \sum_x \alpha_x D(x, \hat{\alpha}) \) and \( D(\alpha, y) = D(\hat{\alpha}, y) \) and,

From theorem: \( H_{\max} = R_{\min} \) (note: new \( R(y) = \sup \Phi(\alpha, y) = \sup \sum_x \alpha_x D(x, y) = \sup D(\alpha, y) = R(y) \)).

**Furthermore:** Kuhn–Tucker criterion. Given \( \alpha^*, y^* \in \mathbb{R} \) such that \( y^* = \hat{\alpha}^*, D(x, y^*) \leq R \) for all \( x \in X \), \( D(x, y^*) = R \) for every anchor (i.e. an \( x \) with \( \alpha_x > 0 \)). Then \( \alpha^* \) and \( y^* \) are optimal.

(identification, 2nd case)

**Illustration of** \( H(\alpha) \), in information theory = information transition rate. Then \( H_{\max} = \text{capacity} \).

**Proof.** For any \( y \), \( R(y) = \sum_x \alpha_x^* R(y) \)
\[ \geq \sum_x \alpha_x^* D(x, y^*) = \sum_x \alpha_x^* D(x, y^*) + D(y^*, y) \]
\[ = R + D(y^*, y) \]. Result follows as \( R(y) = R \).