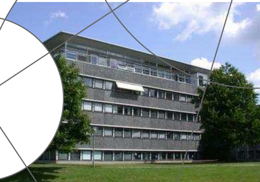
From Truth, Belief and Knowledge to Tsallis Entropy

Flemming Topsøe, topsoe@math.ku.dk
Department of Mathematical Sciences, University of Copenhagen



indications: key notions, key questions

truth
belief, tendency to act
knowledge, perception, extended experience
interaction
experiment, preparation, control
description, effort, information

what is “information”? – a potential saving of effort!
what then is “entropy”? – minimal effort, given the truth!
and, what *c a n* we know? – well, depends on your belief ...

information: cost and associated effort

What is the cost of information or, how much are you willing to pay – or *have* to pay – in order to know that an event has happened?

Or, what is the effort you are willing to/have to allocate?

Depends on the probability t, you *believe* the event has: $\kappa(t)$. κ tells us the individual effort. It is the effort-function or the descriptor.

effort $\longleftrightarrow$ description ?

Requirements: $\kappa(1) = 0$, κ is smooth (and decreasing). Further, natural with normalization via the differential cost $\iota = -\kappa'(1)$. If $\iota = 1$, we obtain natural units, nats; if $\iota = \ln 2$, we measure in binary units, bits.

the power hierarchy, the exponential hierarchy

Which kind of descriptors would you expect?

Note that any "reasonable" monotone function f defines a descriptor via linearization. Simply take

$$\kappa(t) = \frac{f(1) - f(t)}{f'(1)} .$$

Suggestions: The power hierarchy is defined from the functions $t \mapsto t^a$ $(-\infty < a < \infty)$ and gives the descriptors:

$$t \mapsto \frac{1 - t^a}{a} .$$

... and the exponential hierarchy is defined from the functions $t \mapsto b^t$ $(b > 0)$ and gives the descriptors

$$t \mapsto \frac{1 - b^{t-1}}{\ln b} .$$

But are any of these "sensible"? – and what does that mean?

accumulated effort in a probabilistic context

Consider distributions over a discrete alphabet $\mathbb{A}$:
$x = (x_i)_{i\in\mathbb{A}}$ represents truth , $y = (y_i)_{i\in\mathbb{A}}$ represents belief .
Accumulated effort (expected per observation) is

$$\Phi(x,y) = \sum_{i\in\mathbb{A}} x_i \kappa(y_i) \,.$$

Φ satisfies the perfect match principle (for short is proper) if

$$\Phi(x,y) \geq \Phi(x,x)$$

with equality only for $y = x$ (or $\Phi(x,x) = \infty$).

Theorem There is only one descriptor κ, the classical descriptor, for which Φ above is proper, viz. (nats)

$$\kappa(t) = \ln \frac{1}{t} \,.$$

questioning the basic definition

Surely, $\Phi(x, y) = \sum x_i \kappa(y_i)$ is the right expression for accumulated effort **as seen by someone, who knows the truth as well as the belief** ...

... but is this how **you** perceive accumulated effort?

What if the x_i's above are not what you perceive as truth?

... perhaps this also depends on what you believe – and Φ should rather be something like $\sum \pi(x_i, y_i)\kappa(y_i)$.

Let's leave the probabilistic setting for a while and go philosophical:

the beginnings of a philosophy of information

The whole is the world , $\mathcal{V}$

Situations from the world involve Nature and you, Observer .

Nature has no mind but holds the truth (x),

Observer has a creative mind,
– seeks the truth (x)
– is confined to belief (y)
– aims at knowledge (z).

But what is "knowledge"? Knowledge is
– the synthesis of extensive experience
– an expression of how Observer perceives situations from $\mathcal{V}$
– a manifestation of truth for Observer, for you.

interaction and effort

Proposal: Knowledge (z) depends on truth and belief via a characteristic interactor Π: $\boxed{z = \Pi(x, y)}$ $\mathcal{V} = \mathcal{V}_\Pi$.

$\Pi_1 : (x, y) \mapsto x$ defines the classical world $\mathcal{V}_1$
$\Pi_0 : (x, y) \mapsto y$ defines a black hole $\mathcal{V}_0$
$\Pi_q : (x, y) \mapsto qx + (1 - q)y$ defines mixtures, Tsallis' $\mathcal{V}_q$'s

Associated with $\mathcal{V}_\Pi$ are, possibly many, effort functions, Φ's. Extending the previous definition, an effort function is proper if it satisfies the perfect match principle (PMP):
$\Phi(x, y) \geq \Phi(x, x)$ with equality iff there is a perfect match, i.e. $y = x$ (or $\Phi(x, x) = \infty$).

Thesis Given $\mathcal{V}_\Pi$, there is at most one proper Φ-function

entropy, divergence, the fundamental inequality

Abstract modelling involves effort (Φ), entropy (H), and divergence (D). Φ is assumed proper.

Entropy is defined as minimal effort, given the truth, divergence as excess (or redundant) effort:

$$H(x) = \Phi(x,x)\,; \quad D(x,y) = \Phi(x,y) - H(x)\,.$$

The properness of Φ may be expressed in terms of D by the fundamental inequality of information theory (FI):

$$D(x,y) \geq 0 \quad \text{with equality iff } y = x\,.$$

Further notions and properties are best discussed for probabilistic modelling.

probabilistic modelling (discrete)

Truth-, belief- and knowledge instances are $x = (x_i)$, $y = (y_i)$ and $z = (z_i)$ (i ranging over an alfabet $\mathbb{A}$).
x and y are probability distributions, z just a function on $\mathbb{A}$.

Interaction, Π, acts via the local interactor π:
$\big(\Pi(x,y)\big)_{i\in\mathbb{A}} = \big(\pi(x_i,y_i)\big)_{i\in\mathbb{A}}$. π is always assumed sound , i.e. $\pi(s,t) = s$ if $t = s$ (perfect match).
π is weakly consistent if $\forall x \forall y : \sum z_i = 1$. Strong consistency requires that z is always a probability distribution.

Proposition: Only the π_q's given by $\pi_q(s,t) = qs + (1-q)t$ are weakly consistent; strong consistency requires $0 \leq q \leq 1$.

accumulated effort, the one and only

Accumulated effort always chosen among $\Phi_{\pi,\kappa}$ where κ is a descriptor and

$$\Phi_{\pi,\kappa}(x,y) = \sum_{i\in\mathbb{A}} \pi(x_i,y_i)\kappa(y_i)\,.$$

Theorem (modulo regularity conditions). Given $\pi = \pi(s,t)$, let $\pi'_2 = \frac{\partial\pi}{\partial t}$ and put $\chi(t) = \pi'_2(t,t)$.
Only one among the $\Phi_{\pi,\kappa}$'s can be proper, viz. the solution to

$$t\kappa'(t) + \chi(t)\kappa(t) = -1\,,\ \kappa(1) = 0\,. \quad (*)$$

If π is consistent, hence one of the π_q's, then a proper $\Phi_{\pi,\kappa}$ exists iff $q > 0$ ($q = 0$ OK as a singular case, though).
If so, the unique descriptor concerned is the one depending linearly on t^{q-1}, i.e. $\boxed{\kappa_q(t) = \ln_q \frac{1}{t}}$ (recall: $\ln_q u = \frac{1}{1-q}(u^{1-q}-1)$).

gross effort, pointwise fundamental inequality

Introduce gross (accumulated) effort and gross entropy by adding a term representing overhead cost (or effort):

$$\text{gross effort: } \tilde{\Phi}(x,y) = \sum_{i\in\mathbb{A}} \big(\pi(x_i,y_i)\kappa(y_i) + y_i\big) = \Phi(x,y) + 1\,,$$

$$\text{gross entropy: } \tilde{H}(x) = \sum_{i\in\mathbb{A}} \big(x_i\kappa(x_i) + x_i\big) = H(x) + 1\,.$$

Clearly, "gross divergencee"=divergence and, defining the divergence generator by
$\delta(s,t) = \big(\pi(s,t)\kappa(t) + t\big) - \big(s\kappa(s) + s\big)$, one has
$D(x,y) = \sum \delta(x_i,y_i)$. We refer to the inequality $\delta \geq 0$ as the pointwise fundamental inequality (PFI). Clearly
PFI $\Longrightarrow$ FI. **Conjecture** Converse also true
In practice, PMP and FI are always proved via PFI !

given κ, which world are you in?

Given π, we insist, when possible, to choose κ such that the resulting function Φ is proper. This gives a unique choice, the ideal descriptor. You determine κ from π, but

Warning: you cannot determine π from κ

Thus knowing the entropy function does not reveal the world.

Examples: Let $\pi = \pi_q$ $(q > 0)$ and consider π^ξ of the form

$$\pi^\xi(s,t) = \xi^{-1}\Big(\pi\big(\xi(s),\xi(t)\big)\Big)\,.$$

Then the differential equation (*) is unchanged, hence you find the same descriptor κ_q. E.g. for $\xi(u) = \ln u$, $\pi^\xi(s,t) = s^q t^{1-q}$; by PFI, the associated effort is proper.

Problem which κ's are associated with (meaningful) π's?

e.g. $\kappa(t) = \frac{1}{2}(t^{-2} - 1)$; or $\kappa(t) = 1 - \exp(t-1)$?

what *c a n* we know? (abstract modelling)

Setting: World $\mathcal{V}_\pi$ with ideal descriptor and effort fct. Φ.
I.J. Good (1952): **Belief is a tendency to act !**

To us, this is expressed via controls, w's. There is a bijection $y \leftrightarrow w$ ($w = \hat{y}$; $y = \check{w}$). In our probabilistic modelling this is given by $w_i = \kappa(y_i)$; $i \in \mathbb{A}$.
Expressed via controls, the effort function is denoted Ψ:
$\Psi(x, w) = \Phi(x, y)$ with $y \leftrightarrow w$.

What can Observer do? via control ! preparations which are sets of x's, typically denoted by $\mathcal{P}$.
A feasible preparation is one which Observer can realize.

more on preparations

Typical example (of genus 1): Fix a control w and a level h.
Set-up an experiment (!?) which constrains Natures possibilities to the preparation

$$\mathcal{P}(w,h) = \{x | \Psi(x,w) = h\}$$

or to the variant $\mathcal{P}_{\leq}(w,h) = \{x | \Psi(x,w) \leq h\}$.

Finite non-empty intersections of such level sets (or sub-level sets) constitute the feasible preparations and show what Observer can know !

equilibrium, MaxEnt and all that!

A closer study of a fixed preparation $\mathcal{P}$ requires game theory and exploits thinking of John Nash. We shall only outline this. The two players are Nature with truth instances $x \in \mathcal{P}$ as strategies and Observer with controls w as strategies. As objective function we take $\Psi = \Psi(x, w)$. The maxmin value is easily seen to be the MaxEnt value

$$H_{\max}(\mathcal{P}) = \sup_{x \in \mathcal{P}} H(x)\,.$$

If this equals the minmax value (required finite), the game is in equilibrium.

Another notion, often overlooked: A control ε^* is robust if, for some finite h, $\Psi(x, \varepsilon^*) = h$ for all $x \in \mathcal{P}$. Then h is the level of robustness.

By results of Nash:

robustness lemma, exponential families

Robustness lemma If $x^* \in \mathcal{P}$ and $\varepsilon^* = \hat{x^*}$ is robust with level h, then the game is in equilibrium with x^* and ε^* as optimal strategies, in particular, x^* is the MaxEnt strategy. (furthermore, the celebrated Pythagorean inequalities hold).

Let w be a control, let $\mathcal{L}^w$ be the preparation family of non-empty sets of the form $\mathcal{P}(w,h)$. The associated exponential family, denoted $\hat{\mathcal{E}}^w$ is the set of controls ε which are robust for all preparations in $\mathcal{L}^w$. From robustness lemma:

Consider a preparation family $\mathcal{L}^w$. Let x^* be a truth instance, put $\varepsilon^* = \hat{x^*}$ and assume that $\varepsilon^* \in \hat{\mathcal{E}}^w$. Put $h = \Psi(x^*, w)$. Then the game corresponding to $\mathcal{P}(w,h)$ is in equilibrium and has x^* and ε^* as optimal strategies. In particular, x^* is the MaxEnt distribution for $\mathcal{P}(w,h)$.

sketch of MaxEnt calculations in $\mathcal{V}_q$

Return to probabilistic setting and consider a Tsallis world $\mathcal{V} = \mathcal{V}_q$, cor. to π_q with $q > 0$.
Fix $y \longleftrightarrow w$. Then $\mathcal{L}^w$ consists of all preparations $\mathcal{P}$ for which $\Psi(x, w)$ is constant over $\mathcal{P}$.
But $\Psi(x, w) = \sum \big(qx_i + (1-q)y_i\big) w_i$ so condition is equivalent to $\sum x_i w_i$ being constant over $\mathcal{P}$.
For fixed constants α and β this implies that $\sum x_i(\alpha + \beta w_i)$ is constant over $\mathcal{P}$.
Now, if $\alpha + \beta w$ is a control, say w^*, $\sum x_i w_i^*$ is constant over $\mathcal{P}$, hence $\Psi(x, w^*)$ is constant over $\mathcal{P}$, i.e. $w^* \in \hat{\mathcal{E}}^w$ and the robustness lemma applies.
Then, given β, try to adjust α so that $\alpha + \beta w$ is a control.
Classically, α is the logarithm of the partition function.
Finally, adjust β ($\approx$ inverse temperature) to desired level ...

what have we achieved?

• found a reasonably transparent interpretation of Tsallis entropy
• developed a basis for an abstract theory
• clarified role of FI via PMP; focus on PFI as the natural basis for establishing FI and hence PMP
• identified the unit of entropy as an overhead
• answered the question "what *can* we know"
• found good (*the* right ?) definition of an exponential family
• indicated dual role of preparations and exponential families
• exploited games and wisdom of Nash, enabled MaxEnt calculations without introducing Lagrange multipliers
• separated Nature from Observer in key expressions

what needs being done?

- interaction, how?
- description, how?
- control, how?
- expand, quantum setting ...
- link to information geometry
- ...

thank you !