Non-stationary Markov chains related to continued fractions

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Outline of goal and approach

Research started differently but, with hinsight, the goal can be presented as follows:

to study stochastic processes which model demographic factors in populations consisting of individuals belonging to various age classes

Much too ambitious! Let us narrow down:

- *age classes* are consecutive integers in some interval $[i, j]$
- a *population* is made up of *families*,
- families consist of individuals, all females, either *mothers, daughters* or *free women*
- further restrictions are obtained from:
  1. *combinatorial rules* which limit the allowed structure of families in a population
  2. *significance factors* which makes it possible to assign a *weight* (*"significance"*, degree of *"importance"*) to each family.

Combinatorial rules, significance factors

States: α (mother), γ (daughter) and β (free woman). Let Ω_i^j be the population of all families over $[i,j]$ according to the following combinatorial rules:

- each family $\omega \in \Omega_i^j$ contains $j-i+1$ members, one of each of the age-classes $i, \cdots, j$
- every mother has only one daughter who is ...
- ... exactly one age-class younger than mother
- no daughter can be a mother as well
- the youngest member cannot be a mother

Significance factors, all assumed positive: $(\alpha_\nu)_{\nu\geq 1}$, $(\beta_\nu)_{\nu\geq 0}$ and $(\gamma_\nu)_{\nu\geq 0}$.

- $Z_i^j(\omega)$, *weight* of $\omega \in \Omega_i^j$ is the product of the significance factors of its members
- significance factor of a mother of age ν is α_ν
- significance factor of a daughter of age ν is γ_ν
- significance factor of a free woman of age ν is β_ν

the partition function

In view of combinatorial restrictions, assume $\gamma_\nu \equiv 1$. Thus only (α_ν) and (β_ν) are needed. Define *partition function* $Z = (Z_i^j)$ by, for each $[i, j]$:

$$Z_i^j = \text{ total weight of families in } \Omega_i^j = \sum_{\omega \in \Omega_i^j} Z_i^j(\omega)$$

(with special cases: $Z_i^{i-1} = 1, Z_i^{i-2} = 0$).

Proposition: basic identities

$$Z_i^j = \alpha_\nu Z_i^{\nu-2} Z_{\nu+1}^j + Z_i^{\nu-1} Z_\nu^j \,, \qquad (1)$$

$$Z_i^j = \alpha_j Z_i^{j-2} + \beta_j Z_i^{j-1} \,, \qquad (2)$$

$$Z_i^j = \alpha_{i+1} Z_{i+2}^j + \beta_i Z_{i+1}^j \,. \qquad (3)$$

Proof. (1): Is there a mother of age ν or is there not? (2) and (3): Take $\nu = j$ or $\nu = i + 1$. □

introducing randomness

Define probability spaces (Ω_i^j, P_i^j) by:

$$P_i^j(\omega) = \frac{Z_i^j(\omega)}{Z_i^j}\ ;\ \omega \in \Omega_i^j$$

Let X_ν $(\nu \in [i,j])$ be the natural projection maps. E.g. $\{X_\nu = \alpha\}$ means "family member of age ν is a mother". Distributions of X_ν's easy to identify, e.g.

$$P_i^j(X_\nu = \alpha) = \frac{Z_i^{\nu-2}\alpha_\nu Z_{\nu+1}^j}{Z_i^j}.$$

Define $\Omega_i = \Omega_i^\infty$ with natural Borel structure by:
$\Omega_i = \{(\omega_i, \cdots)|\forall \nu > i : \omega_\nu = \alpha \Leftrightarrow \omega_{\nu-1} = \gamma\}$.

(Ω_i, P_i) is *limit model* of (Ω_i^j, P_i^j) as $j \to \infty$ if, for every event A which depends on finitely many coordinates, $P_i(A) = \lim_{j\to\infty} P_i^j(proj_i^j(A))$. Clearly, these measures are unique. But do they exist?

Main results

Put $K_n = \beta_n + \frac{\alpha_{n+1}}{\beta_{n+1}+} \ \frac{\alpha_{n+2}}{\beta_{n+2}+} \ \frac{\alpha_{n+3}}{\beta_{n+3}+} \quad \cdots.$

Theorem, (i): Following conditions are equivalent:

- For some i, the limit model P_i exists
- For all i, the limit models P_i exist
- the continued fractions above converge
- $\lim_{n\to\infty} \frac{\alpha_1\alpha_2\cdots\alpha_{2n+1}}{Z_1^{2n} Z_1^{2n+1}} = 0$
- $\sum_{n=1}^{\infty} \beta_n \alpha_n^{-1} \alpha_{n-1} \alpha_{n-2}^{-1} \cdots \alpha_1^{(-1)^n} = \infty$.

(ii): If so, then, for $i \geq 0$, $(X_\nu)_{\nu \geq i}$ is a (non-stationary) Markov process with start distribution

$$P_i(X_i = \alpha/\beta/\gamma) = (0, \frac{\beta_i}{K_i}, \frac{\alpha_{i+1}}{K_i K_{i+1}})$$

and one step transition probabilities ($n - 1 \curvearrowright n$)

$$\begin{bmatrix} 0 & \frac{\beta_n}{K_n} & \frac{\alpha_{n+1}}{K_n K_{n+1}} \\ 0 & \frac{\beta_n}{K_n} & \frac{\alpha_{n+1}}{K_n K_{n+1}} \\ 1 & 0 & 0 \end{bmatrix}$$

A conjecture

Define the **demographic constants** λ_i in any reasonable way as expression for the expected fraction of mothers in a large family, e.g. via ergodicity (does it hold?) or as (E_i^j denoting expectation w.r.t. P_i^j)

$$\lim_{j\to\infty} \frac{E_i^j(\text{ number of mothers in } \omega)}{j-i+1}$$

or as $\lim_{\nu\to\infty} P_i(X_\nu = \alpha)$.

Conjecture (follows from known results?) All these definitions give the same value for λ_i.

Example In the *stationary case* $\alpha_\nu \equiv 1$, $\beta_\nu \equiv 1$ this is OK. For this case, $Z_i^j = F_{j-i+2}$ (*$j-i+2$'nd Fibonacci number*) and $K_n = \frac{1}{2} + \frac{1}{2}\sqrt{5}$. One finds $\lambda_i = \frac{5-\sqrt{5}}{10} = \frac{\rho}{\sqrt{5}} \approx 0.2764$. (Is example known?)