Universal coding for distributions over co-trees

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***Abstract*—For an arbitrary ordered set, we consider the model of all distributions P for which an element which precedes another element is considered the more significant one in the sense that the implication $a \leq b \Rightarrow P(a) \geq P(b)$ holds. It will be shown that if the ordered set is a finite co-tree, then the universal predictor for the indicated model or, equivalently, the corresponding universal code, can be determined exactly via an algorithm of reasonably low complexity.**

I. Introduction and background material

We study finite co-trees, i.e. finite ordered sets Λ for which every non-maximal element a has a unique immediate successor.

We think of a co-tree as an *oriented graph* and refer to the elements as *nodes*. If there is only one maximal node, the co-tree is *suspended* with the maximal node as *top node*. Some suspended co-trees are depicted in Figure 1. Note that we have named the nodes in a systematic self-explanatory manner corresponding to their *levels*, counted from the top with the maximal nodes in level 0.

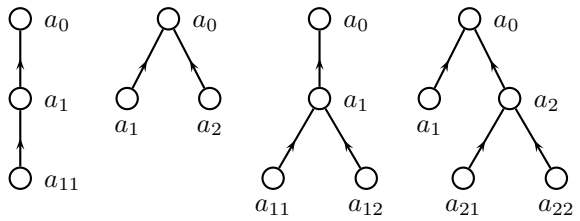

Figure 1. Some simple suspended co-trees

The co-trees shown in Figure 1 are suspended co-trees with *uniform branching* defined by a *branching pattern* $(k_1, \cdots, k_n)$ of natural numbers (for each $\nu = 1, \cdots, n$, k_ν branches emerge from each node in level $\nu - 1$).

For a general co-tree, the *left sections* $a^{\downarrow} = \{b \in \Lambda | b \leq a\}$ play a special role.

$M_+^1(\Lambda)$ is the set of distributions over Λ. The *order model* $\mathcal{P} = \mathcal{P}(\Lambda)$, which is the object we will study, is the set of all $P \in M_+^1(\Lambda)$ for which $P(a) \geq P(b)$ whenever $a \leq b$ [1].

Denoting by U_a the uniform distribution over $a^{\downarrow}$, we find that $\mathcal{P}(\Lambda)$ is a *simplex* with the U_a's as extremal elements. From the barycentric decomposition of $P \in \mathcal{P}$, $P = \sum_{a \in \Lambda} w_a U_a$, we define the *spectrum* of P by $\sigma(P) = \{a | w_a > 0\}$.

By a *code* (really, a code length function) over the *alphabet* Λ we shall here understand a function κ on Λ which satisfies *Kraft's equality* $\sum_{a \in \Lambda} e^{-\kappa(a)} = 1$. Note that we have chosen to work with natural units rather than with bits. The set of all codes over Λ is denoted $\mathrm{K}(\Lambda)$.

For $P \in M_+^1(\Lambda)$ and $\kappa \in \mathrm{K}(\Lambda)$ we consider *average code length*, $\langle \kappa, P \rangle = \sum_{a \in \Lambda} \kappa(a) P(a)$. One should aim at choosing κ so as to minimize this quantity. If P is fixed, the minimum is assumed for the code *adapted to* P, given by $\kappa(a) = \ln \frac{1}{P(a)}$ for $a \in \Lambda$ and the minimum value is the *entropy* of P, $\mathrm{H}(P) = \sum_{a \in \Lambda} P(a) \ln \frac{1}{P(a)}$. When κ is adapted to P, we also say that P is the distribution which *matches* κ.

Let $P \in M_+^1(\Lambda)$ and $\kappa^* \in \mathrm{K}(\Lambda)$. The *redundancy* associated with P and κ^* is the difference $\langle \kappa^*, P \rangle - \mathrm{H}(P)$ between the actual average code length and the minimal achievable value. This quantity is the *Kullback-Leibler divergence* between P and the distribution P^* matching κ^*, in standard notation: $\mathrm{D}(P \| P^*) = \sum_{a \in \Lambda} P(a) \ln \frac{P(a)}{P^*(a)}$. We reflect our emphasis on both the distribution and the code, by defining $\mathrm{D}(P \| \kappa^*) = \mathrm{D}(P \| P^*)$ whenever P^* matches κ^*.

Returning to the order model $\mathcal{P} = \mathcal{P}(\Lambda)$, we define the *guaranteed redundancy* of any code by $\mathrm{R}(\kappa^*) = \sup_{P \in \mathcal{P}} \mathrm{D}(P \| \kappa^*)$ and the *minimax redundancy* by $\mathrm{R}_{\min} = \inf_{\kappa^* \in \mathrm{K}(\Lambda)} \mathrm{R}(\kappa^*)$.

It may be seen directly, see also Lemma 1 below, that there exists a unique code, κ^*, *the universal code*, such that $\mathrm{R}(\kappa^*) = \mathrm{R}_{\min}$. The distribution which matches the universal code is the *universal predictor*. It is considered the most unbiased representation of the model $\mathcal{P}$. The two universal objects identified, are those which we shall aim at characterizing by an algorithm of low complexity.

[1] this model – and not the alternative choice of all order-preserving distributions – is considered to be the natural one, a main reason being that if a precedes b ($a \leq b$), this is taken as a sign that a is more "significant" than b, hence, for sensible distributions, one should have $P(a) \geq P(b)$ rather than the other way round. In terms of coding (see below) our choice appears even more natural as it reflects the good sense of associating the shorter code words to the more significant events.

We shall use a special instance of a result from general optimization theory, which is often ascribed to Kuhn and Tucker, cf. [**?**]. In our terminology it reads:

Lemma 1: Consider the order model $\mathcal{P} = \mathcal{P}(\Lambda)$ associated with a finite co-tree Λ. Let $P^* \in \mathcal{P}$ and let κ^* be the adapted code. Then a necessary and sufficient condition that these are, respectively the universal predictor and the universal code is that, for some constant R, the following two conditions hold:

$$\mathrm{D}(U_a\|\kappa^*) = R \quad \text{for } a \in \sigma(P^*)\,, \tag{1}$$

$$\mathrm{D}(U_a\|\kappa^*) \le R \quad \text{for all } \ a \in \Lambda\,. \tag{2}$$

If (1) and (2) hold, $\mathrm{R}_{\min} = R$.

Proof: Here follows a simple intrinsic proof of the sufficiency: By convexity of redundancy in the first variable and as the U_a's are the extremal elements of $\mathcal{P}$, it follows from (2) that $\mathrm{R}(\kappa^*) \le R$. On the other hand, for every $\kappa \in \mathrm{K}(\Lambda)$, we have, using (1) and a special identity:

$$\begin{aligned}\mathrm{R}(\kappa) &= \sum_{a\in\sigma(P^*)} w_a \mathrm{R}(\kappa) \ge \sum_{a\in\sigma(P^*)} w_a \mathrm{D}(U_a\|\kappa)\\ &= \sum_{a\in\sigma(P^*)} w_a \mathrm{D}(U_a\|\kappa^*) + \mathrm{D}(P^*\|\kappa)\\ &= R + \mathrm{D}(P^*\|\kappa)\,.\end{aligned}$$

Regarding the identity used, the *compensation identity*, see [**?**]. Thus, for every κ, the in itself interesting inequality (a kind of "reverse Pythagorean inequality") $\mathrm{R}(\kappa) \ge R + \mathrm{D}(P^*\|\kappa) \ge R$ holds. The desired result follows as $\mathrm{D}(P^*\|\kappa) \ge 0$ with equality if and only if $\kappa = \kappa^*$.

Necessity may also be proved by an intrinsic argument, modeled after [**?**]. ∎

The *spectrum of* Λ is defined as $\sigma(\Lambda) = \sigma(P^*)$. It turns out that the difficulty in determining the universal objects P^* and κ^* really only lies in determining the spectrum. Table 1 shows the nature of all universal predictors and the associated spectra for the four concrete co-trees in Figure 1. The correctness of the table is easily checked by appeal to Lemma **??**. The normalizing factor, Z, which appears in the table is related to minimax redundancy as $\mathrm{R}_{\min} = \ln Z$. Considering Case 3, we realize that even for very simple examples, the spectrum may be "defect" in the sense that $\sigma(\Lambda) \ne \Lambda$. Note that the defect disappears in this example if you either remove a node (Cases 1 and 2) or add one (Case 4).

The natural interpretations related to codes as well as the significance of the problem outlined as one of *general universal prediction and coding* (general, because many other models than models related to order structure may be considered) is recognized in the information theoretical literature since long. Suffice it here to refer to the survey article [**?**] (and the references therein). For the interesting connection to problems related to capacity, see [**?**].

The motivation behind our very special study is many sided. Firstly, the class of order models for co-trees appears to be the most comprehensive class of models for which an exact determination of universal objects can be provided. For the subclass of order models based on linearly ordered sets, a complete result was developed by Ryabko, cf. [**?**]. For the larger subclass of co-trees with uniform branching, an algorithm was announced in [**?**] but the details never published.

As another motivation we note, as pointed out to us by Boris Ryabko, cf. also [**?**], that for certain applications to biology, information about biological species is sometimes available only in inconclusive form resulting – not in the direct determination of their relative numbers – but only in an ordering among the species, from the more frequent to the less frequent ones. Modelling as done here based on a co-tree is one possibility, though modelling based on trees rather than co-trees appear just as interesting, or perhaps even more interesting. However, such models are without reach if you insist on expressing the universal objects in closed form.

Our results may serve as a useful reservoir of examples for future research. Regarding limitations of future related research, we remark that it is often difficult to develop exact results expressed in terms of standard functions for other desirable models, either based on order structures other than co-trees (e.g. trees) or on other constructs (such as Bernoulli models). This will also be clear from research by Harremoës and Topsøe, in preparation. Thus, it is not possible to develop exact results of the nature here discussed if single observation from a source in $\mathcal{P}$ is replaced by sequential models based on more extensive sampling (Galois theory provides a technical hindrance for this).

Table I
UNIVERSAL PREDICTORS FOR THE CO-TREES IN FIGURE 1

	Case 1	**Case 2**	**Case 3**	**Case 4**
$\sigma(\Lambda)$	Λ	Λ	$\Lambda \setminus \{a_1\}$	Λ
$P^*(a_0)$	$\frac{4}{27}\cdot\frac{1}{Z}$	$\frac{1}{27}\cdot\frac{1}{Z}$	$\frac{1}{16}\cdot\frac{1}{Z}$	$\frac{27}{3125}\cdot\frac{1}{Z}$
$P^*(a.)$	$\frac{1}{4}\cdot\frac{1}{Z}$	$2\times\frac{1}{Z}$	$\frac{1}{16}\cdot\frac{1}{Z}$	$\frac{1}{Z}, \frac{1}{27}\cdot\frac{1}{Z}$
$P^*(a..)$	$\frac{1}{Z}$		$2\times\frac{1}{Z}$	$2\times\frac{1}{Z}$
$\mathrm{R}_{\min} = \ln Z$	$\ln\frac{151}{108}$	$\ln\frac{55}{27}$	$\ln\frac{17}{8}$	$\ln\frac{256979}{84375}$

II. A PROCESS OF RELATIVIZATION

Experience tells us that for typical optimization problems of the nature we are studying, "normalization" (via a "partition function") is natural. This is for instance reflected in the division by Z in Table **??**. A natural idea then is to facilitate the search for universal objects by a prior normalization. We find it advantageous to work with codes rather than with distributions and define the *relativized universal code* for $\mathcal{P}(\Lambda)$ as the function $\tilde{\kappa}^* = \kappa^* - \mathrm{R}_{\min}$. By a process of "de-relativization", it is easy to regain κ^* from $\tilde{\kappa}^*$: Indeed,

$$\mathrm{R}_{\min} = \ln \sum_{a\in\Lambda} e^{-\tilde{\kappa}^*(a)}\,. \tag{3}$$

We may characterize $\tilde{\kappa}^*$ among all *monotone* functions $\phi : \Lambda \to \mathbb{R}$ (monotone, i.e. $a \leq b \Rightarrow \phi(a) \leq \phi(b)$). Some preparations: A node $a \in \Lambda$ is *ϕ-active* if either a is a maximal node or else $\phi(a) < \phi(a^+)$ with a^+ the immediate successor of a. We shall use the notation $\phi^\sigma(L)$ for the sum $\sum_{b\in L} \phi(b)$. Further, $N(a)$ denotes the number of elements in $a^\downarrow$ and we put $\overline{N}(a) = N(a) \ln N(a)$. With only a modest amount of extra work, one can transform Lemma 1 into the following criteriom:

Lemma 2: A real-valued function ϕ defined on Λ coincides with the relativized universal code $\tilde{\kappa}^*$ if and only if it is monotone and satisfies the two requirements:

$$\phi^\sigma(a^\downarrow) = \overline{N}(a) \text{ for every } \phi\text{-active node } a\,, \tag{4}$$

$$\phi^\sigma(a^\downarrow) \leq \overline{N}(a) \text{ for every node } a \in \Lambda\,. \tag{5}$$

The set of nodes which are active for $\tilde{\kappa}^*$ is identical to the spectrum $\sigma(\Lambda)$.

Based on Lemma **??**, one can prove the following useful result:

Proposition 2.1: Every minimal node of Λ is active. The relativized universal code is non-negative and vanishes on the minimal nodes – and nowhere else. Therefore, the universal predictor assumes its maximal value on the minimal nodes and any other node has a strictly smaller probability.

Thus, the relativized universal code measures code length relative to the shortest codeword. The basic information we are looking for can adequately be represented by $\tilde{\kappa}^*$ and $\mathrm{R}_{\min}$ (calculated from (**??**)).

III. A TOP-DOWN CONSTRUCTION

We aim at developing an algorithm which determines the key objects $\tilde{\kappa}^*$ and $\mathrm{R}_{\min}$. A preliminary construction is needed which in itself is an algorithm but of intolerably high complexity. The construction will be explained in this section whereas the algorithm will not be presented until the next section. For both sections, we focus on explanations on what is going on without providing the necessary technical justification. Suffice it here to say that the key lies in certain combinatorial identities which allow the comparison among so-called *brackets*, to be introduced below.

Key to our approach is the notion of a *blocking set*: A set $B \subseteq \Lambda$ is a *blocking set for* $a \in \Lambda$ if B is a hereditary subset of $a^\downarrow \setminus \{a\}$ which contains all minimal elements of $a^\downarrow \setminus \{a\}$. We associate two sets with such a set, the *remainder in* a, $S_B(a)$, and the *ceiling in* a, $T_B(a)$:

$$S_B(a) = a^\downarrow \setminus B\,, \tag{6}$$

$$T_B(a) = \quad \text{set of maximal elements of } B\,. \tag{7}$$

The elements in $T_B(a)$ are the first elements in B we meet on paths from a to a minimal node. The sets $S_B(a)$ are always non-empty as are the sets B and $T_B(a)$, unless a is a minimal node of Λ.

Let B be a blocking set for a. We define the *bracket of a in B*, by

$$[a, B] = \frac{\overline{N}(a) - \overline{N}^\sigma\big(T_B(a)\big)}{|S_B(a)|}\,.$$

Note that the denominator above has a similar structure as the numerator since $|S_B(a)| = N(a) - N^\sigma(T_B(a))$. Also note that in the extreme case when a is a minimal node, B has to be empty and the definition above gives $[a, \emptyset] = 0$.

One can prove that there exists a set-theoretically largest blocking set for a with maximal bracket. This set we denote $B^*(a)$ and the actual value of the bracket we denote $[a]_{\max}$. Further, the remainder in a and the ceiling in a associated with $B^*(a)$ we denote $S^*(a)$, respectively $T^*(a)$.

The *ceiling hierarchy* $(T_i^*)_{i\geq 0}$ is defined in a "top-down" manner by the following recursive recipe:

$$T_i^* = \begin{cases} \text{the set of maximal nodes} & \text{if } i = 0\,, \\ \bigcup_{a\in T_{i-1}^*} T^*(a) & \text{if } i \geq 1\,. \end{cases} \tag{8}$$

Clearly, T_i^* is eventually empty and the sets $S^*(t)$ with $t \in T^*$ constitute a decomposition of Λ. For a node $t \in T^* \setminus T_0^*$ we denote by $\mu(t)$, the "mother" of t, the first node in T^*, larger than t.

We can now state our first main result:

Theorem 1: [top-down construction] Let a be any node in Λ and determine the unique node $t \in T^*$ for which $a \in S^*(t)$. Then $\tilde{\kappa}^*(a) = [t]_{\max}$. The spectrum consists of all maximal nodes as well as all nodes t which have a mother $\mu(t)$ with $[\mu(t)]_{\max} > [t]_{\max}$.

In the examples we know, $\sigma(\Lambda) = T^*$ but we do not know if this holds generally.

IV. THE ALGORITHM

Theorem **??** does not in itself constitute an acceptable algorithm as the search for the B^*-sets – on which the construction relies – will be far too complex if we just search for these sets in the blind among all possible blocking sets for the nodes in question. Fortunately, it turns out that a systematic search "from the bottom" can be performed in polynomial time.

The resulting algorithm consists of three parts: initialization, repeated calls of the *central subroutine*, and a final collection of results.

During initialization, basic data about the co-tree are provided. This includes access to the N- and $\overline{N}$-numbers as well as a decomposition of Λ in *minimality components* M_i obtained by successive removals of minimal nodes. Also, key data, see below, are listed for the trivial case of minimal nodes, nodes in M_0.

The central part of the algorithm consists of repeated calls to a subroutine which, for a given node a, calculates *key data*, $B^*(a), S^*(a)$ and $T^*(a)$ as well as the maximal bracket $[a]_{\max}$, that is, provided these data are already available for nodes below a. This is why these calls are made in succession from below, through the minimality components.

The final part works top-down and is quite trivial in view of Theorem **??**.

Before embarking on a clarifying example, we sketch the central subroutine in Figure 2.

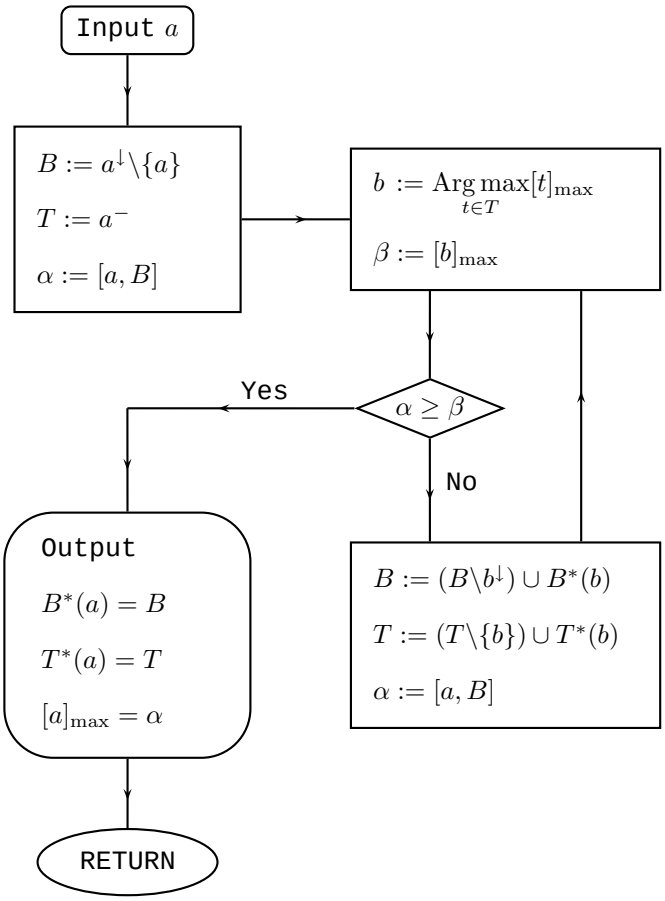

Figure 2. Flow diagram for the central subroutine

After input of a node a, the largest blocking set for a, $B = a^{\downarrow} \setminus \{a\}$ is tested: Could it be the sought set $B^*(a)$? In order to decide this, the ceiling T and the bracket α are noted. Every node in T is strictly less than a and hence, when the subroutine is called with a as input, all maximal brackets $[t]_{\max}$ with $t \in T$ will be known. The largest of these values is noted and stored in β and (one of) the corresponding nodes is noted and stored in b. It turns out, that if the test $\alpha \geq \beta$ is positive, B must be the sought set and we go to the output and return to the basic algorithm. If, however, the test is negative, we can diminish the set B as shown and try with a new triple B, T, α. The procedure is then repeated until the test is positive.

The details of the algorithm may only be clear to the reader after working through the examples to follow. Before that, let us state the result obtained, our second main theorem:

Theorem 2: The algorithm described above calculates the ceiling hierarchy and thereby the universal objects associated with a co-tree Λ in polynomial time. The essential complexity is at most $6 \cdot n^3$ where n is the number of nodes in Λ.

When, above, we refer to the *essential complexity*, what we have in mind is the number of calculations and tests needed to go through the algorithm, with the understanding that quantities which are needed for the initialization of the algorithm do not contribute to this number.

We shall now demonstrate how the algorithm works in practice by investigating a particular example, the co-tree Λ of Figure 3. The work carried out during execution of the algorithm is conveniently summarized in Figure 4. We have marked in black the top-node (known to be active) as well as all nodes which occur as nodes in a ceiling constructed during the algorithm. In particular, all minimal nodes are marked in black as $\bigcup_{a\in M_1} T^*(a) = M_0$. Furthermore, we have listed the exact as well as the approximate values of all brackets which are determined during the algorithm, and we have marked with a "dagger" the one and only value, calculated for the node a_{23}, for which the $(\alpha \geq \beta)$-test of the central subroutine is not passed (in the actual case because $\overline{5} - \overline{4} \geq \overline{4}$ does *not* hold).

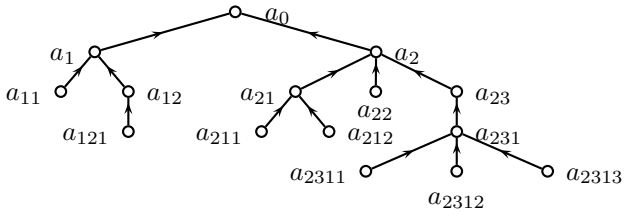

Figure 3. A "general" co-tree Λ.

The result is that all nodes except a_{231} are active. For this you have to consider the very last part of the algorithm (the "top-down" part). This can easily be done based on information listed, for example, $T^*(a_0) = \{a_1, a_2\}$ since the α-value $[a_0, B] = \overline{15} - \overline{4} - \overline{10}$ is indeed greater than the β-value $\max_{t\in T_B(a_0)}[t]_{\max} = \overline{10} - \overline{3} - \overline{5}$ when testing if the set $B = (a_0)^{\downarrow} \setminus \{a_0\}$ could be the sought set $B^*(a_0)$.

The final "top-down" part of the algorithm is necessary as not all "black" nodes need be active – some of these nodes may have been "overshadowed" by an S^*-set for a node higher up in the co-tree. But for the present example, no such overshadowing takes place.

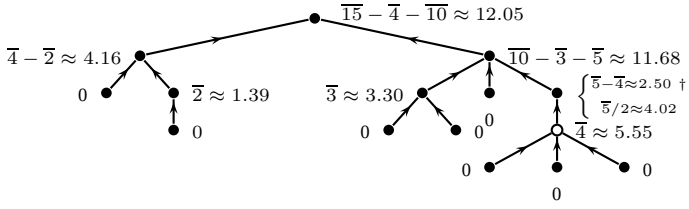

Figure 4. The algorithm for Λ.

The final result is that all values listed in Figure 3, except the one which has been "daggered away", are the correct values of the relativized universal code. For the exceptional node we

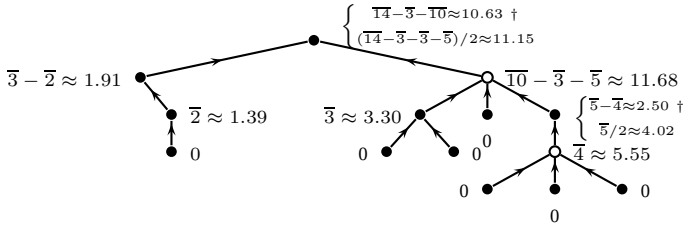

Figure 5. The algorithm for $\Lambda \setminus \{a_{11}\}$.

find that $\overline{\kappa}^*(a_{231}) = \overline{\kappa}^*(a_{23}) = \overline{5}/2$. The relativized universal code may then be read directly from the figure when referring to Theorem **??**. As to $\mathrm{R}_{\min}$, the value is obtained by (**??**). One finds that $\mathrm{R}_{\min} = \ln A$ with

$$A = 8+2^{-2}+3^{-3}+2\cdot 5^{-5/2}+4^{-3}+3^3 2^{-10}5^{-5}+2^{18}3^{-15}5^{-5}$$

which gives $\mathrm{R}_{\min} \approx 2.12$ measured in natural units, corresponding to 3.06 bits. This may be compared with the 3 bits necessary to encode the 8 minimal nodes which are equally probable under the universal predictor.

In order to illustrate the sensitivity of the algorithm, consider also the co-tree $\Lambda^- = \Lambda \setminus \{a_{11}\}$. For this co-tree the algorithm gives a result which can conveniently be summarized in Figure 5. Again, no "overshadowing" takes place, but we note that a new inactive node pops up, the node a_2. Thus one cannot decide "locally" if a node is active or not. For this co-tree one finds $\mathrm{R}_{\min} \approx 2.01$ natural units ≈ 2.90 bits – compared to the approximately 2.81 bits needed to encode the 7 minimal nodes which have equal probability under the universal predictor.

V. Indication of further results

Theorems **??** and **??** are the main results of our research. Other results are mainly centred around co-trees with uniform branching and document the results announced earlier in [**?**]. One result gives the necessary and sufficient conditions that a co-tree with branching pattern $(k_1, \cdots, k_n)$ have full spectrum ($\sigma(\Lambda) = \Lambda$). As a corollary, this is seen to be the case if $k_1 \geq \cdots \geq k_n$, thus providing a simple and rather natural extension of Ryabko's result, [**?**].

For a specific node, say in level ν, we need only carry out tests for blocking sets which have a ceiling with all nodes in the same level, say in level μ. Denote by $[\nu, \mu]$ the associated bracket. For each ν, the central subroutine will then determine that value of μ for which $[\nu, \mu]$ is maximal. The necessary tests can be depicted in a $[\nu, \mu]$-*diagram* where $(\nu, \mu) \in \{0, \cdots, n\} \times \{1, \cdots, n+1\}$. This could look as shown in Figure 6, where we have indicated those positions which requires a test according to the central subroutine and general structural information about relationships among brackets. The starting point of the diagram (reflecting the approach "from the bottom") is at the position $(n, n+1)$ for which we have put $[n, n+1] = 0$. The final "top-down" approach of the algorithm tells us that the active nodes are the top node and then all those which emerge, popularly speaking, by "letting the sun shine from the left and considering only the visible summits". For the diagram in Figure 6, this shows that the active nodes are to be found in levels 0,2,8 and 10.

As opposed to the rough estimate in Theorem **??**, the complexity of the algorithm for co-trees with uniform branching can be evaluated precisely:

Theorem 3: The maximal number of tests that have to be carried out during the execution of the algorithm when adapted to co-trees with uniform branching is at most $2n-1$ for any branching pattern $(k_1, \cdots, k_n)$ of length n.

Finally, we mention that the phenomenon of overshadowing (cf. the discussion in Section **??**) can be illustrated by considering the co-tree with branching pattern $(1, 2, 4, 8, 16)$.

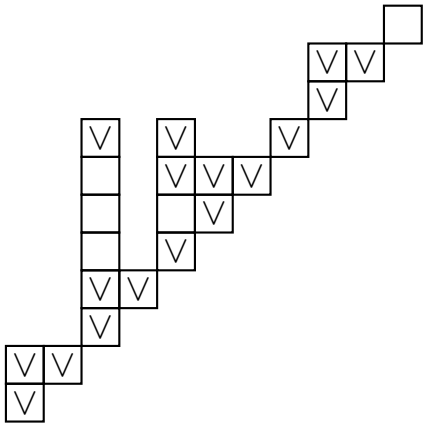

Figure 6.

Acknowledgments

Boris Ryabko introduced FT to the general problems involved and several useful discussions are acknowledged. The complex of problems have also been discussed with Peter Harremoës and a number of related results, but for other models, have been developed, though not yet published. Finally, we thank Jan Caesar for technical assistance.

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