# Nash equilibrium in a game of calibration

Omar Glonti, Peter Harremoës, Zaza Khechinashvili
and Flemming Topsøe *
OG and ZK: I. Javakhishvili Tbilisi State University
Laboratory of probabilistic and statistical methods
Tbilisi, Georgia
PH and FT: Department of Mathematics, University of Copenhagen
Universitetsparken 5, 2100 Copenhagen, Denmark
Corresponding author: FT: topsoe@math.ku.dk

**Abstract**

A general game between market and investor is studied and properties which are based on the notion of Nash equilibrium are derived. The results have the potential to unify and to simplify previous research. As an illustration, a problem of calibration in a simple model of stock price development is treated. A quantitative method is suggested which makes it possible to take belief in a certain trend into account even when there is no empirical evidence available to support such a belief.

**Keywords**. Nash equilibrium, model calibration, relative entropy, reverse relative entropy.

## 1 A calibration game

In Samperi [18], cf. also [19], [20], it was observed that an information theoretical game is of significance for certain optimization problems of financial mathematics. This was based on Topsøe [21], [22] and is also related to Csiszár [4]. Certain improvements of the information theoretical game with

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emphasis on the concept of *Nash equilibrium* were given in Harremoës and Topsøe [14], in Topsøe [23], [24], and in Harremoës [13]. Of relevance is also Csiszár and Matúš [5] and Grünwald and Dawid [11].

The game we shall introduce builds on [24]. However, the considerations are not restricted to quantities defined by relative entropy (Kullback-Leibler divergence) as in [24], and also, we shall use terms from financial mathematics in order to assist the reader in applying the approach and results to problems from this field. We have included some arguments which are adaptations of arguments from [24] in order to make the paper self-contained.

We consider two sets, $\mathcal{S}_I$ and $\mathcal{S}_{II}$. These play the role of strategy sets for the two players in the game to be introduced. The first player we think of as "the market" , the second as "the investor" . Neutrally, we refer to the players as *Player I* and *Player II*. Elements of $\mathcal{S}_I$ we refer to as *states of the market* whereas elements of $\mathcal{S}_{II}$ are referred to as *investment strategies* or, more specifically, as *calibration strategies* or just *calibrations*.

We assume that

$$\mathcal{S}_I \subseteq \mathcal{S}_{II}. \tag{1}$$

Qualitatively speaking, the rationale for this assumption is that the investor should be allowed to use whatever means are available to assist him, whereas the rules and behaviour which apply to the market – including also theoretical principles we believe in such as the no-arbitrage principle – impose restrictions on our modelling of the market. One may argue that the nature of strategies for the two players are quite different, hence the assumption (1) does not make sense. However, for the applications we shall deal with here, there is a natural embedding of $\mathcal{S}_I$ in $\mathcal{S}_{II}$ which allows us to identify $\mathcal{S}_I$ with a subset of $\mathcal{S}_{II}$.

We assume that prior information is available to both players and given in terms of an element of $\mathcal{S}_{II}$, denoted $P_0$, and referred to as the *prior*. If $P_0 \in \mathcal{S}_I$, the model is already *calibrated*. In cases of interest, $P_0 \notin \mathcal{S}_I$, and we shall use game theoretical considerations to discuss how an appropriate calibration can then be achieved.

The objective function which we shall suggest is derived from an extended real valued function $\Phi$ on the product set $\mathcal{S}_{II} \times \mathcal{S}_{II}$. We assume that the inequalities

$$0 \leq \Phi(P\|R) \leq \infty \tag{2}$$

and the bi-implication

$$\Phi(P\|R) = 0 \Longleftrightarrow P = R \tag{3}$$

hold for every pair $(P, R) \in \mathcal{S}_{II} \times \mathcal{S}_{II}$. We use the double bar notation $\Phi(\cdot\|\cdot)$ merely to signal that $\Phi$ need not be symmetric (recall the common usage of this kind of notation for the non-symmetric relative entropy). We call $\Phi$ the *divergence function*.

By the *scope* ($\Phi$-*scope*) of $P_0$ we understand the set

$$\mathrm{scope}_\Phi(P_0) = \{P \in \mathcal{S}_{II} | \Phi(P\|P_0) < \infty\}. \tag{4}$$

We assume that

$$\mathcal{S}_I \subseteq \mathrm{scope}_\Phi(P_0). \tag{5}$$

Formally, (5) is a stronger assumption than (1).

For $(P, R) \in \mathcal{S}_I \times \mathcal{S}_{II}$, we define the *calibration gain* by

$$\Phi(P\|P_0 \rightsquigarrow R) = \Phi(P\|P_0) - \Phi(P\|R). \tag{6}$$

In view of the assumption (5), this is a well defined number in $[-\infty, \infty[$. This is the function we shall use as objective function below.

Clearly, if the investor chooses $R = P_0$, the calibration gain will be 0. The investor should attempt to select a calibration which is closer to the "true" state of the market. In order to discuss this more closely, we introduce the two-person zero-sum *calibration game*, denoted $\gamma = \gamma(\mathcal{S}_I, \mathcal{S}_{II}, P_0, \Phi)$, which has $(P, R) \curvearrowright \Phi(P\|P_0 \rightsquigarrow R)$ as pay-off function for Player II (and as cost function for Player I). The usual minimax/maximin thinking of game theory then applies and leads us to consider the minimax value $\Phi_{\min}$ and the maximin value $\Gamma_{\max}$ given by

$$\Phi_{\min} = \inf_{P \in \mathcal{S}_I} \sup_{R \in \mathcal{S}_{II}} \Phi(P\|P_0 \rightsquigarrow R), \tag{7}$$

$$\Gamma_{\max} = \sup_{R \in \mathcal{S}_{II}} \inf_{P \in \mathcal{S}_I} \Phi(P\|P_0 \rightsquigarrow R). \tag{8}$$

Note that the supremum in (7) can be identified as $\Phi(P\|P_0)$, hence

$$\Phi_{\min} = \inf_{P \in \mathcal{S}_I} \Phi(P\|P_0). \tag{9}$$

The corresponding infimum in (8) cannot readily be identified. We denote it by $\Gamma(R\|P_0)$ and call it the *calibration risk* associated with the strategy $R$:

$$\Gamma(R\|P_0) = \inf_{P \in \mathcal{S}_I} \Phi(P\|P_0 \rightsquigarrow R). \tag{10}$$

Clearly,

$$0 \leq \Gamma_{\max} \leq \Phi_{\min} < \infty. \tag{11}$$

If $\Gamma_{\max} = \Phi_{\min}$, this is the *value* of the game and the game is said to be in *equilibrium*. If $R \in \mathcal{S}_{II}$ and $\Gamma(R\|P_0) = \Gamma_{\max}$, the calibration $R$ is an *optimal* *calibration.* If $P \in \mathcal{S}_I$ and $\Phi(P\|P_0) = \Phi_{\min}$, the market state $P$ is an *optimal state.* Using terminology from game theory, a pair $(P^*, R^*) \in \mathcal{S}_I \times \mathcal{S}_{II}$ is a *Nash equilibrium pair* if the two saddle value inequalities:

$$\Phi(P^*\|P_0 \rightsquigarrow R) \leq \Phi(P^*\|P_0 \rightsquigarrow R^*) \leq \Phi(P\|P_0 \rightsquigarrow R^*) \tag{12}$$

hold for $(P, R) \in \mathcal{S}_I \times \mathcal{S}_{II}$.

Let us analyze what can be said when (12) holds. First note that from the first inequality (applied to $R = P^*$) and from (2), (3) and (5) it follows that $R^* = P^*$. Then, the first inequality of (12) is automatic and the second inequality tells us that $\Gamma(P^*\|P_0) \geq \Phi(P^*\|P_0)$. As the reverse inequality is a trivial consequence of the minimax inequality (11), we conclude that $\Gamma(P^*\|P_0) = \Phi(P^*\|P_0) = \Gamma_{\max} = \Phi_{\min}$. Thus, the game is in equilibrium and $P^*$, viewed as a state of the market, is an optimal strategy for Player I and $P^*$, viewed as a calibration, is an optimal calibration strategy for Player II.

We call $P^*$ the *bi-optimal strategy.* We shall see below that it is unique. We noted that the essential demand on $P^*$ $(= R^*)$ is that

$$\Phi(P\|P_0 \rightsquigarrow P^*) \geq \Phi(P^*\|P_0) \text{ for } P \in \mathcal{S}_I \tag{13}$$

holds. This inequality we call the *Nash inequality* (associated with the game $\gamma$). In view of the equilibrium property established, (13) may be written in the form

$$\Phi(P\|P_0) \geq \Phi(P\|P^*) + \Phi_{\min} \text{ for } P \in \mathcal{S}_I. \tag{14}$$

From (14) it follows immediately that $P^*$ is the unique optimal strategy for Player I. Similarly, from $\Gamma(R\|P_0) \leq \Phi(P^*\|P_0 \rightsquigarrow R)$ we see that

$$\Gamma(R\|P_0) \leq \Gamma_{\max} - \Phi(P^*\|R) \tag{15}$$

and it follows that $P^*$ is the unique optimal calibration for Player II. The inequality (14) is the *Pythagorean inequality* associated with the game. This inequality is widely used (in the setting when relative entropy is taken for the divergence $\Phi$) and goes back to Čencov [2] and to Csiszár [4]. Its "dual" (15), can be found in Topsøe [21] (for relative entropy).

Let us summarize our discussion in a form which is convenient for applications:

**Theorem 1.** *Consider the calibration game $\gamma = \gamma(\mathcal{S}_I, \mathcal{S}_{II}, P_0, \Phi)$ and assume that* (2), (3) *and* (5) *hold. If there exists $P^* \in \mathcal{S}_I$ such that the inequality* (13) *holds, then $\gamma$ is in equilibrium, $(P^*, P^*)$ is the Nash equilibrium pair for $\gamma$, and $P^*$ is the unique optimal strategy for each of the players. Furthermore, the Pythagorean and the dual Pythagorean inequalities,* (14) *and* (15)*, hold.*

It is comforting to note that the converse to this result also holds. Indeed, under the assumptions (2), (3) and (5) it is easy to see that if $\gamma$ is in equilibrium and if optimal strategies exist for both players, then the strategies coincide and the Nash inequality (13) holds for the common strategy. Therefore, if the game considered allows optimal strategies for each of the players, the only possible equilibrium type is that there exists a Nash equilibrium pair. From [24] and [13], see also [12], one will see what may happen if the players do not have optimal strategies. However, it is believed that for applications to mathematical finance, the situation we have focused on is the most important one.

Normally, the strategy set $\mathcal{S}_I$ can be extended without changing the value of the game or the bi-optimal strategy. One way to view this is to associate with each pair $(P^*, P_0)$ with $P^* \in \mathrm{scope}_\Phi(P_0)$ a *maximal model*

$$\mathcal{S}_{P^*,P_0} = \{P \in \mathrm{scope}_\Phi(P_0) | \Phi(P\|P_0 \rightsquigarrow P^*) \geq \Phi(P^*\|P_0)\}. \tag{16}$$

Taking this set as strategy set for Player I, we see that it is the largest set $\mathcal{S}$ for which the game $\gamma(\mathcal{S}, \mathcal{S}_{II}, P_0, \Phi)$ is in equilibrium with $P^*$ as bi-optimal strategy and $P_0$ as prior.

Consider again a game $\gamma(\mathcal{S}_I, \mathcal{S}_{II}, P_0, \Phi)$ satisfying (2), (3) and (5) and assume that the game is in equilibrium and that the bi-optimal strategy $P^*$ exists. Often, $P^*$ is difficult to determine exactly but it may be possible to estimate how close $P^*$ is to a suitable guess $Q \in \mathcal{S}_I$ by applying the inequality

$$\Phi(P^*\|Q) + \Phi(Q\|P^*) \leq \Phi(Q\|P_0) - \Gamma(Q\|P_0) \tag{17}$$

which follows from (14) and (15).

Finally, we introduce a concept which is particularly useful in linear models (e.g. reflecting a martingale condition) when relative entropy is taken for the divergence $\Phi$. The concept makes sense for any calibration game. So consider a general game $\gamma = \gamma(\mathcal{S}_I, \mathcal{S}_{II}, P_0, \Phi)$. A calibration $Q \in \mathcal{S}_{II}$ is said to be *robust* in case $\Phi(P\|P_0 \rightsquigarrow Q)$ is independent of $P$ for $P \in \mathcal{S}_I$. Clearly, if $Q$ is robust and if $Q \in \mathcal{S}_I$, then the Nash inequality (13) holds. Thefore, we obtain the following result directly from Theorem 1:

**Corollary 1.** *Assume that* (2), (3) *and* (5) *are satisfied for the game* $\gamma = \gamma(\mathcal{S}_I, \mathcal{S}_{II}, P_0, \Phi)$ *and that* $Q \in \mathcal{S}_I$ *is a robust calibration. Then* $\gamma$ *is in equilibrium and has* $Q$ *as the bi-optimal strategy.*

In certain cases, this result may be applied by first searching for robust calibrations and then searching among these for one in the strategy set $\mathcal{S}_I$. This approach is the one we will adopt in Section 2.1.

Possible applications of the results in this section include research as contained in Samperi [18], [19], Bellini and Frittelli [1], Goll and Rüschendorf [9], Grandits and Rheinländer [10], Rüschendorf [17], Delbaen et al [6], Kabanov et al [15], Cherny et al [3], Glonti et al [8] and others.

# 2 A model of stock price development

In this section we study a simple and well known model of stock price development. In particular, we follow-up on the study of Glonti, Jamburia, Kapanadze and Khechinashvili [7].

Consider a model of stock price development $(S_k)_{k\geq 0}$ where the price at time $k$ is given by

$$S_k = S_{k-1}(1+\rho_k)\,;k\geq 1 \tag{18}$$

with $S_0 > 0$ deterministic and $(\rho_k)_{k\geq 1}$ an iid sequence of random variables with values in $]-1,\infty[$. Previous experience or knowledge gained by other means is given in terms of the prior distribution $P_0$ of the $\rho_k$'s. This may not render $(S_k)_{k\geq 0}$ a martingale w.r.t. the filtration $(\mathcal{F}_k)_{k\geq 0}$ of $\sigma$-fields generated, for $k \geq 0$, by $S_0, S_1, \cdots, S_k$. Therefore, the model based on the prior distribution $P_0$ may not respect the no arbitrage principle and it is to be expected that market forces will eventually lead to a state, expressed in terms of a new distribution, $P$, of the $\rho_k$'s which respects the martingale condition

$$E_P(S_k|\mathcal{F}_{k-1}) = S_{k-1}\ ;k\geq 1. \tag{19}$$

In view of (18), this condition amounts to the vanishing of the mean value of the $\rho_k$'s in the new model based on $P$. By the assumption of identical distribution, this mean value does not depend on $k$.

We may now forget about the structure of the model we started with (given by (18)) and translate everything to conditions involving distributions on $\mathbb{R}$. Let $M_+^1(\mathbb{R})$ denote the set of probability distributions on $\mathbb{R}$ and agree to use the bracket notation $\langle \cdot, P\rangle$ for mean values w.r.t. distributions $P \in M_+^1(\mathbb{R})$. We find it convenient to denote by $id : \mathbb{R} \to \mathbb{R}$ the identity map on $\mathbb{R}$.

The prior is then a distribution $P_0 \in M_+^1(\mathbb{R})$ and regarding the condition that it be concentrated on $]-1,\infty[$ this condition may in fact be ignored as it does not play any role for the further analysis. We do assume that $\langle id, P_0\rangle \neq 0$, either $\langle id, P_0\rangle < 0$ – when we speak of a *negative trend* – or $\langle id, P_0\rangle > 0$ – when we speak of a *positive trend*.

For the acceptable market states, now modelled by distributions $P \in M_+^1(\mathbb{R})$, the martingale condition amounts to the condition $\langle id, P\rangle = 0$. We also find it natural to demand that $P$ is equivalent to $P_0$ (i.e. that the two measures have the same null sets). We denote by $M(P_0)$ the set of *admissable market states* thus arrived at:

$$M(P_0) = \{P \in M_+^1(\mathbb{R}) | P \equiv P_0, \langle id, P\rangle = 0\}. \tag{20}$$

Regarding this set, we assume that $M(P_0) \neq \emptyset$. In fact, we may assume that $M(P_0)$ contains more than one distribution (hence $M(P_0)$ contains infinitely many distributions), i.e. that the market is *incomplete* (the situation with a complete market may be conceived as a special, singular case).

In (20) we agreed on the set of admissable strategies for Player I (the market) in the game we shall study. Regarding Player II (the investor), we conceive the available strategies – likewise given as distributions $P \in M_+^1(\mathbb{R})$ – as calibrations (chosen by the investor, not enforced by the market as was the case above when we defined $M(P_0)$). Accordingly, the investor will be allowed to choose strategies which do not respect the martingale condition. On the other hand, we only allow strategies which in some sense can be derived from the prior $P_0$. To be precise, we define the set $M_+^1(P_0)$ of *admissable calibration strategies* (for the investor) by

$$M_+^1(P_0) = \{P \in M_+^1(\mathbb{R}) | P \ll P_0\}, \tag{21}$$

i.e. as the set of distributions which are absolutely continuous w.r.t. $P_0$.

It remains to specify the divergence function $\Phi$ before a game theoretic setting as discussed in Section 1 makes sense. We shall work with *Csiszár $\phi$-divergences* for distributions on $\mathbb{R}$. They are given by the usual formula

$$\Phi(P\|Q) = \int \phi(\frac{dP}{dQ})dQ \tag{22}$$

($\infty$ in case this integral does not make sense) where $\phi$ is some convex function on $\mathbb{R}_+$ which vanishes at 1 and is strictly convex at that point. Here, we shall only pay special attention to *relative entropy* (*Kullback-Leibler divergence*) given by

$$D(P\|Q) = \int \ln \frac{dP}{dQ} dP \tag{23}$$

(corresponding to $\phi(u) = u \ln u$) and to *reverse relative entropy* given by

$$D^{inv}(P\|Q) = D(Q\|P) \tag{24}$$

(corresponding to $\phi(u) = -\ln u$).

The games we shall study only depend on $P_0 \in M_+^1(\mathbb{R})$ and on the chosen divergence $\Phi$ and we denote these games by $\gamma(P_0, \Phi)$. They are defined to be the calibration games $\gamma(M(P_0), M_+^1(P_0), P_0, \Phi)$ of Section 1.

In order to simplify and also to ensure that the condition (5) holds whatever the divergence $\Phi$, we shall assume that $P_0$ is discrete with finite support, $supp(P_0)$. We may then write $P_0$ in the form

$$P_0 = \sum_{i=0}^{n} p_{0,i}\delta_{a_i} \tag{25}$$

with the $a_i$ distinct and with $a_0 = 0$ (0 is treated as a special value where $P_0$ may or may not have positive mass). Here, $\delta_a$ denotes a unit mass at $a \in \mathbb{R}$. We assume that $p_{0,i} > 0$ for $1 \le i \le n$. The points $a_0$ (if $p_{0,0} > 0$) and $a_1, \cdots, a_n$ are referred to as the *locations* in the model. As $M(P_0)$ is assumed to contain more than one distribution, there are both positive and negative locations. Therefore, $n \ge 2$ and we may assume that $a_1 < 0$ and $a_n > 0$.

## 2.1 Calibration w.r.t. relative entropy

First consider the case when $\Phi = D$, relative entropy. This case really contains the archetypical information theoretic optimization problem and has been treated in a long range of different contexts, see Kapur [16]. We base the analysis on Corollary 1. For measures $P$ and $R$ which are equivalent to $P_0$, we find that

$$D(P\|P_0 \rightsquigarrow R) = \left\langle \ln \frac{dR}{dP_0}, P \right\rangle . \tag{26}$$

Therefore, we realize that if $\ln \frac{dR}{dP_0}$ is a linear combination of the constant function 1 and the identity $id$, say $\ln \frac{dR}{dP_0} = \alpha - \beta\, id$, then $R$ is a robust calibration. This leads us to consider, for every $\beta \in \mathbb{R}$, the distribution $R_\beta \in M_+^1(\mathbb{R})$ defined by

$$\ln \frac{dR_\beta}{dP_0} = -\ln Z(\beta) - \beta\, id \tag{27}$$

or, equivalently,

$$R_\beta(A) = \frac{1}{Z(\beta)} \int_A e^{-\beta x} dP_0(x) \tag{28}$$

for measurable subsets $A \subseteq \mathbb{R}$ and with $Z(\beta)$, the *partition function* evaluated at $\beta$, given by

$$Z(\beta) = \int_{-\infty}^{\infty} e^{-\beta x} dP_0(x) = \sum_{i=0}^{n} p_{0,i} e^{-\beta a_i}. \tag{29}$$

We find that

$$\frac{d}{d\beta}\ln Z = -\langle id, R_\beta\rangle. \tag{30}$$

Therefore, $R_\beta \in M(P_0)$ if and only if $\beta$ is chosen such that

$$\sum_{i=1}^{n} p_{0,i} a_i e^{-\beta a_i} = 0. \tag{31}$$

It is clear (as $a_1 < 0$ and $a_n > 0$) that this transcendental equation has a unique solution, say $\beta^*$. By Corollary 1, it then follows that the game has a Nash equilibrium pair and that $P^* = R_{\beta^*}$ is the bi-optimal strategy. Using terminology of Csiszár [4], this contains the result that $P^*$ is the $I-$ *projection* of $P_0$ on $M(P_0)$.

We also see that if the trend is negative, then $\beta^* < 0$ and if the trend is positive, $\beta^* > 0$. Consider the maximal model associated with $P_0$ and $P^*$. As

$$\begin{aligned} D(P\|P_0 \rightsquigarrow P^*) - D(P^*\|P_0) &= \left\langle \ln\frac{dP^*}{dP_0}, P\right\rangle - \left\langle \ln\frac{dP^*}{dP_0}, P^*\right\rangle \\ &= \langle -\ln Z(\beta^*) - \beta^*\, id, P - P^*\rangle \\ &= -\beta^* \langle\, id, P\rangle, \end{aligned}$$

it follows from (16) that when the trend is negative, the maximal model consists of all $P \in M_+^1(P_0)$ with mean value $\langle id, P\rangle \geq 0$. For the original model, cf. (18), this corresponds to allowing also sub martingale measures. Similarly, if the trend is positive, the maximal model corresponds to allowing also supermartingales.

We have proved the following result:

**Theorem 2.** *For $P_0$ of the form* (25) *with $\langle id, P_0\rangle \neq 0$ and $a_1 < 0\,, a_n > 0$, the game $\gamma(P_0, D)$ has a Nash equilibrium pair and the bi-optimal distribution $P^* = R_{\beta^*}$ is determined by the equations* (28), (29) *and* (31) *(with $\beta = \beta^*$).*

*If the trend is negative, respectively positive, the associated maximal model consists of all $P \in M_+^1(P_0)$ with $\langle id, P\rangle \geq 0$, respectively $\langle id, P\rangle \leq 0$.*

## 2.2 Calibration w.r.t. reverse relative entropy

We then turn our attention to another choice of divergence, viz. $\Phi = D^{inv}$, reverse relative entropy. Again, we assume that $P_0$ is of the form (25) with $\langle id, P_0\rangle \neq 0$ and $a_1 < 0\,, a_n > 0$ and we consider the game $\gamma(P_0, D^{inv})$.

In this case we have to impose extra conditions to ensure that the game has a Nash equilibrium pair:

**Theorem 3.** *Assume that $P_0$ is of the form* (25).

*If the trend is negative, a necessary and sufficient condition that the game $\gamma(P_0, D^{inv})$ has a Nash equilibrium pair is that there is precisely one positive location. When this condition is fulfilled, the maximal model (associated with the bi-optimal distribution) contains all distributions $P$ with $supp(P) = supp(P_0)$ and $\langle id, P\rangle \geq 0$.*

*If the trend is positive, the corresponding necessary and sufficient condition is that there is precisely one negative location and when this condition is fulfilled, the maximal model contains all $P$ with $supp(P) = supp(P_0)$ and $\langle id, P\rangle \leq 0$.*

*Proof.* Throughout this proof we assume that the trend is negative. The case of a positive trend may be treated similarly or reduced to the case with a negative trend in an obvious manner.

The Nash inequality has the following form:

$$D^{inv}(P\|P_0 \rightsquigarrow P^*) \geq D^{inv}(P^*\|P_0). \tag{32}$$

In order to study this closer we note that for $P$ and $P^*$ in $M(P_0)$ one has:

$$\begin{aligned} &D^{inv}(P\|P_0 \rightsquigarrow P^*) - D^{inv}(P^*\|P_0) \\ &= \int \ln \frac{dP_0}{dP} dP_0 - \int \ln \frac{dP^*}{dP} dP^* - \int \ln \frac{dP_0}{dP^*} dP_0 \\ &= \int \left(1 - \frac{dP^*}{dP_0}\right) \ln \frac{dP^*}{dP} dP_0 \,, \end{aligned}$$

hence, with natural notation for the point probabilities of $P^*$ and $P$,

$$D^{inv}(P\|P_0 \rightsquigarrow P^*) - D^{inv}(P^*\|P_0) = \sum_{i=0}^{n} (p_{0,i} - p_i^*) \ln \frac{p_i^*}{p_i} \tag{33}$$

(for $i = 0$ we have to interpret the contribution to the sum as 0 in case $p_{0,0} = 0$).

Now assume that the game has a Nash equilibrium pair and let $P^* \in M(P_0)$ be the bi-optimal distribution. Then, as $P$ varies over $M(P_0)$, the right hand side in (33) assumes its minimal value, 0, for $P = P^*$. As suitable regularity conditions regarding differentiability are fulfilled, there exist Lagrange multipliers $\lambda$ and $\mu$ such that, for $i = 0, 1, \cdots, n$,

$$\frac{\partial}{\partial p_i} \left[ \sum_{j=0}^{n} (p_{0,j} - p_j^*) \ln \frac{p_j^*}{p_j} - \lambda \sum_{j=0}^{n} p_j a_j - \mu \sum_{j=0}^{n} p_j \right] = 0$$

when $P = P^*$. It follows that

$$\frac{p_i^* - p_{0,i}}{p_i^*} = \lambda a_i + \mu \tag{34}$$

for $i = 0, 1, \cdots, n$. By a multiplication with $p_i^*$ and subsequent summation, we find that $\sum_{i=0}^{n}(p_i^* - p_{0,i}) = \mu$, hence $\mu = 0$. From (34) we then see that

$$p_i^* (1 - \lambda a_i) = p_{0,i} \tag{35}$$

for $i = 0, 1, \cdots, n$. This shows that if $p_{0,0} > 0$, then $p_0^* = p_{0,0}$ (thus, this equality holds in any case) and that $\lambda \neq \frac{1}{a_i}$ for $i = 1, \cdots, n$. Further, $\lambda \neq 0$ (as $P_0 \notin M(P_0)$).

At this stage it is convenient to introduce the notation $b_i = \frac{1}{a_i}$ and to assume, as we may, that the indexing is chosen so that $b_1 < b_2 < \cdots < b_n$. From (35) we find that

$$p_i^* = -\frac{p_{0,i} b_i}{\lambda - b_i} \tag{36}$$

and hence also

$$p_{0,i} - p_i^* = \frac{\lambda p_{0,i}}{\lambda - b_i} \tag{37}$$

for $i = 1, \cdots, n$.

From (36) and from the requirement that the $p_i^*$'s be positive it follows that $\lambda$ is located in what we shall call the *central interval*, namely that interval $]b_j, b_{j+1}[$ which contains 0. It follows from (37) that

$$\sum_{i=1}^{n} \frac{p_{0,i}}{\lambda - b_i} = 0 \tag{38}$$

(recall that $p_0^* = p_{0,0}$). As is easily seen, the function $x \curvearrowright \sum_1^n p_{0,i}/(x - b_i)$ is continuous in the central interval $]b_j, b_{j+1}[$ and decreases from $+\infty$ to $-\infty$ over that interval. Therefore, $\lambda$ is uniquely determined.

As the trend is negative, we see that $\lambda > 0$. Assume now, for the purpose of an indirect proof, that $a_{n-1}$, hence also $b_{n-1}$, is positive. Then there exists a distribution $R$ with $\langle id, R \rangle = 0$ such that $supp(R) = supp(P_0) \backslash \{a_n\}$. Let $S$ denote that distribution with $\langle id, S \rangle = 0$ for which $supp(S) = \{a_1, a_n\}$. For $0 < \varepsilon < 1$, denote by $P_\varepsilon$ the distribution

$$P_\varepsilon = (1 - \varepsilon) R + \varepsilon S.$$

Then $P_\varepsilon \in M(P_0)$ for $0 < \varepsilon < 1$. As $\lambda > 0$, it follows from (37) that $p_n^* > p_{0,n}$. Then, from (33), we see that

$$\inf_{0<\varepsilon<1} \left( D^{inv}(P_\varepsilon \| P_0 \rightsquigarrow P^*) - D^{inv}(P^* \| P_0) \right) = -\infty,$$

contradicting the Nash inequality. We conclude that $a_{n-1} < 0$, hence there is only one positive value among the locations. We have thereby proved the necessity assertion of the theorem.

In order to prove the remaining parts of the theorem, assume that there is only one positive location, $a_n$. Determine $\lambda > 0$ by (38) and consider the distribution $P^* = (p_i^*)_{i=0,1,\cdots,n}$ determined by $p_0^* = p_{0,0}$ and by (36) for $i = 1, \cdots, n$. What we have to prove is that the Nash inequality holds for all $P \in M(P_0)$, i.e., according to (33), we have to establish the validity of the inequality

$$\sum_{i=0}^{n} (p_{0,i} - p_i^*) \ln \frac{p_i^*}{p_i} \geq 0$$

for all $P = (p_i)_{i=0,1,\cdots,n} \in M(P_0)$. In order to prove at the same time also the assertion regarding the associated maximal model, we assume only that $supp(P) = supp(P_0)$ and that $\langle id, P \rangle \geq 0$. If the inequality above can be established under these conditions, the proof will be complete. This is in fact quite easy:

$$\begin{aligned}
\sum_{i=0}^{n} (p_{0,i} - p_i^*) \ln \frac{p_i^*}{p_i} &= \sum_{i=1}^{n-1} (p_{0,i} - p_i^*) \ln \frac{p_i^* p_n}{p_i p_n^*} \\
&\geq \sum_{i=1}^{n-1} (p_{0,i} - p_i^*) \left(1 - \frac{p_i p_n^*}{p_i^* p_n}\right) \\
&= \sum_{i=1}^{n-1} -\lambda a_i (p_i^* - p_i \frac{p_n^*}{p_n}) \\
&= \lambda \left[ a_n p_n^* + (\langle id, P \rangle - a_n p_n) \frac{p_n^*}{p_n} \right] \\
&\geq 0.
\end{aligned}$$

Above we used the facts $\lambda > 0$ and $p_{0,i} - p_i^* > 0$ for $i = 1, \cdots, n-1$. □

We note that it would be natural to allow in the maximal models discussed in the theorem also distributions with support strictly contained in that of $P_0$. Note however, that then (5) need not hold and one would have to extend the general theory slightly.

In the course of the proof we also determined the bi-optimal distribution:

**Theorem 4.** *Assume that the game $\gamma(P_0, D^{inv})$ has a Nash equilibrium pair. Then the bi-optimal distribution, $P^*$, is determined by $p_0^* = p_{0,0}$ and by* (36) *for* $i = 1, \cdots, n$, *where* $\lambda$ *is that number in the central interval which satisfies* (38).

In this result, we may replace the requirement that $\lambda$ belongs to the central interval, by the requirement that $\lambda$ is of the opposite sign of $\langle id, P_0 \rangle$.

# 3 The trinomial scheme

It is straight forward to apply Theorems 2, 3 and 4, combined with standard algorithms, in order to obtain efficient numeric solutions of the optimization problems connected with the games considered. It is, however, only in special cases that exact solutions can be worked out. We now consider such an instance.

What we shall study is the *trinomial scheme*, cf. Glonti et al [7], i.e. we study the case when $n = 2$. In more detail, we assume that the prior $P_0$ has positive mass at $a_1 < 0$ and at $a_2 > 0$, and also at $a_0 = 0$ (so that we are in the case of incompleteness with $M(P_0)$ containing more than one distribution). To simplify notation, we characterize distributions over the locations $a_1, a_0$ and $a_2$ by their vectors of point probabilities corresponding to these locations. The point probabilities of the prior $P_0$ are denoted $p, q$ and $r$, i.e. $P_0 = (p, q, r)$.

By Theorem 2, $\gamma(P_0, D)$ has a Nash equilibrium pair and the bi-optimal distribution, $P^*$, is determined by (28), (29) and (31). One finds the formulas:

$$P^* = \frac{1}{Z}\left(pe^{-\beta a_1}, q, re^{-\beta a_2}\right) \tag{39}$$

with $Z$ a normalization constant and

$$\beta = \frac{1}{a_2 - a_1} \ln \frac{ra_2}{-pa_1}. \tag{40}$$

By Theorems 3 and 4, $\gamma(P_0, D^{inv})$ also has a Nash equilibrium pair (independently of whether the trend is negative or positive). The bi-optimal distribution, now denoted $Q^*$, is most simply determined by noting that the point mass at 0 is the same for $Q^*$ as for $P_0$. Therefore, as also $Q^* \in M(P_0)$,

$$Q^* = \left(\frac{a_2}{a_2 - a_1}(p + r), q, \frac{-a_1}{a_2 - a_1}(p + r)\right). \tag{41}$$

We may express the formulas in a way which better allows us to compare $P^*$ and $Q^*$. We shall see that $P^*$ is related to a certain geometric average and $Q^*$, the simpler of the two distributions, to the corresponding arithmetic average.

Let $(s,t)$ be the probability vector

$$(s,t) = \left(\frac{a_2}{a_2-a_1}, \frac{-a_1}{a_2-a_1}\right) \tag{42}$$

and let

$$\tilde{p} = -pa_1\ ,\ \ \tilde{r} = ra_2 \tag{43}$$

be *weights* associated with the point masses $p$ and $r$ and their respective locations, $a_1$ and $a_2$.

The relevant averages are those of $(\tilde{p}, \tilde{r})$ w.r.t. the probability vector $(s,t)$, both the *arithmetic average*, denoted $A$, and the *geometric average*, denoted $G$, i.e.

$$A = s\tilde{p} + t\tilde{r}\ ,\quad G = \tilde{p}^s \tilde{r}^t. \tag{44}$$

We collect the key formulas and qualitative results in the following theorem:

**Theorem 5.** *Consider the trinomial scheme defined by the prior $P_0 = (p,q,r)$ corresponding to the locations $a_1 < 0, a_0 = 0$ and $a_2 > 0$. Assume that $p,q$ and $r$ are all positive and that $pa_1 + ra_2 \neq 0$. Then both games $\gamma(P_0, D)$ and $\gamma(P_0, D^{inv})$ have a Nash equilibrium pair and the corresponding bi-optimal distributions are given by*

$$P^* = \frac{1}{Z}\left(\frac{G}{-a_1}, q, \frac{G}{a_2}\right)\ ,\quad Q^* = \left(\frac{A}{-a_1}, q, \frac{A}{a_2}\right), \tag{45}$$

*where the averages $A$ and $G$ are determined by (42), (43) and (44).*

*The normalization constant $Z$ satisfies $0 < Z < 1$ and, furthermore, $G < ZA$. Accordingly, $P^*$ assigns larger weight to the location $a_0 = 0$ than $Q^*$ does and smaller weight to each of the locations $a_1$ and $a_2$.*

*Proof.* From (40) it follows that

$$e^{-\beta a_1} = \left(\frac{\tilde{r}}{\tilde{p}}\right)^t\ ,\quad e^{-\beta a_2} = \left(\frac{\tilde{r}}{\tilde{p}}\right)^{-s}$$

and then that

$$pe^{-\beta a_1} = \frac{1}{-a_1}\tilde{p}^s\tilde{r}^t\ ,\quad re^{-\beta a_2} = \frac{1}{a_2}\tilde{r}^t\tilde{p}^s\ ,$$

hence the formula for $P^*$ in (45) follows by (39). Clearly, the formula for $Q^*$ in (45) follows from (41).

By the inequality relating geometric and arithmetic means we find that

$$\begin{aligned} Z &= G\left(-\frac{1}{a_1}+\frac{1}{a_2}\right)+q \\ &< A\left(-\frac{1}{a_1}+\frac{1}{a_2}\right)+q \\ &= 1 \end{aligned}$$

(with strict inequality as $\tilde{p} \neq \tilde{r}$). It follows that $P^*(a_0) > Q^*(a_0)$. As $P^*(a_1)/Q^*(a_1) = P^*(a_2)/Q^*(a_2)$, we have $P^*(a_1) < Q^*(a_1)$, hence $G < ZA$. □

In the *symmetric case*: $a_2 = -a_1$, one finds the formulas

$$P^* = \frac{1}{Z}\left(\sqrt{pr}, q, \sqrt{pr}\right) \quad , \quad Q^* = \left(\frac{p+r}{2}, q, \frac{p+r}{2}\right) \tag{46}$$

with $Z = q + 2\sqrt{pr} = 1 - (\sqrt{p} - \sqrt{r})^2$. As it is to be expected, $P^*$ and $Q^*$ are independent of the location $a_2 = -a_1$.

We may also remark that calibration using the reverse relative entropy measure makes it possible to calibrate meaningfully a prior distribution $P_0$ which assigns probability 0 to $a_1$ or to $a_2$. Assume, say, that $r = 0$. Using the formula (41) gives

$$Q^* = \left(\frac{a_2}{a_2 - a_1}p, q, \frac{-a_1}{a_2 - a_1}p\right) \tag{47}$$

whereas the formula for calibration w.r.t. relative entropy would give the unreasonable result $P^* = (0, 1, 0)$. Calibration when $r = 0$ corresponds to a situation with negative trend when previous evidence did not show any instance of increase in stock price but, nevertheless, for one reason or another, one expects that increase in stock price could occur in the future but only corresponding to the location $a_2$.

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