Combinatorics of continued fractions

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Continued fractions, some basics

Finite continued fractions are expressions of the form

$$b_0 + \frac{a_1}{b_1 +} \ \frac{a_2}{b_2 +} \ \cdots \frac{a_n}{b_n}.$$

When possible, understood, e.g., for $n = 3$ as

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3}}}.$$

The a's and b's are the **elements** (**partial numerators** and **denominators**). Preferred notation:

$$K \begin{pmatrix} - & a_1 & . & . & . & a_n \\ b_0 & b_1 & . & . & . & b_n \end{pmatrix} = \frac{A_n}{B_n}.$$

Precisely, (canonical) **numerators** and **denominators**:

$$\frac{A_n}{B_n} = \frac{A_0^n}{B_0^n} = b_0 + \frac{a_1}{\frac{A_1^n}{B_1^n}} = \frac{b_0 A_1^n + a_1 B_1^n}{A_1^n}$$

$\frac{A_n}{B_n} = C_n$, the n'th **approximant**. **Convergence** for infinite continued fractions means $(C_n)_{n \geq 0}$ converges!

An example

Why bother? Many reasons (number theory, computation, ...). An example: **Lamberts expansion** (1770):

$$\ln(1+x)$$

$$= \frac{x}{1+} \; \frac{1^2x}{2+} \; \frac{1^2x}{3+} \; \frac{2^2x}{4+} \; \frac{2^2x}{5+} \; \frac{3^2x}{6+} \; \frac{3^2x}{7+} \; \cdots$$

$$= K\begin{pmatrix} - & x & 1^2x & 1^2x & 2^2x & 2^2x & 3^2x & 3^2x & \cdots \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \end{pmatrix}$$

It converges for all $x > -1$! Here are the first 7 approximants to $\ln 2 = 0.693147180\cdots$:

1 0.6667 0.7000 0.6923 0.6933 0.69312 0.69315

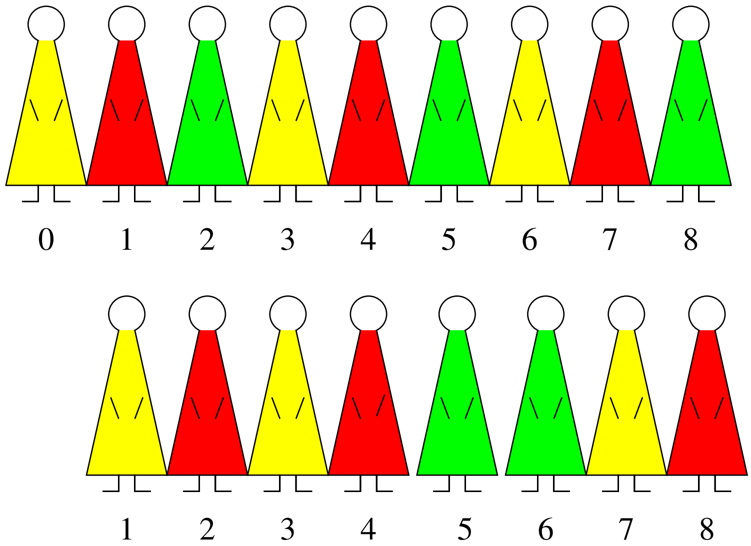
To obtain the same precision from the Taylor series expansion you need more than 100000 terms!

An observation

Problem: Formulas for A_n and B_n!
One possible solution: By recurrence relations!
A better solution, illustrated by an example:

$$A_8 = \cdots + a_1 a_4 a_7 b_2 b_5 b_8 + \cdots, \text{ (a sum of 55 terms)},$$
$$B_8 = \cdots + a_2 a_4 a_8 b_5 b_6 + \cdots, \text{ (a sum of 34 terms)}.$$

... apparently connected with **families** of **mothers**, **daughters** and **free women**!! Look here:



Later: First family, say ω, has $M(\omega) = \{1, 4, 7\}$, $D(\omega) = \{0, 3, 6\}$ and $F(\omega) = \{2, 5, 8\}$.

Populations

Ω_i^j is the **population** of all **families** over $[i, j]$ which only consist of females, viz.

mothers (α's)
daughters (γ's) and
free women (β's)

according to the rules:
- a family $\omega \in \Omega_i^j$ contains one member of each age-class $i, i+1, \cdots, j$,
- every mother has only one daughter,
- a daughter is one age-class younger than the mother,
- no daughter can be a mother,
- the youngest member (of age i) cannot be a mother.

Put $M(\omega)$, $D(\omega)$, $F(\omega)$ = sets of mothers, daughters and free women, respectively, identified by age.

A **cut-point** in ω allows split of the family in a "young" and an "old" part. Opposite process: **concatanation**. Special cases: $\Omega_i^{i-1} = \{\emptyset\}$, $\Omega_i^{i-2} = \emptyset$.

partition functions

Add analytical elements through three series: $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 0}$ and $(\gamma_n)_{n\geq 0}$ and define, for an interval $[i,j]$, the **pointwise partition function** by

$$Z_i^j(\omega) = \prod_{\nu\in M(\omega)} \alpha_\nu \prod_{\nu\in D(\omega)} \gamma_\nu \prod_{\nu\in F(\omega)} \beta_\nu$$

and the (accumulated) **partition function** by

$$Z_i^j = \sum_{\omega\in\Omega_i^j} Z_i^j(\omega)$$

(special cases: $Z_i^{i-1} = 1, Z_i^{i-2} = 0$).
Assume for simplicity and without loss of generality that $\gamma_\nu \equiv 1$ and think of the "continued fraction case".

Theorem 1. $A_n = Z_0^n, B_n = Z_1^n$.

Proof. Start is OK, recursion too. □

Example: If $\alpha_\nu \equiv 1$, $\beta_\nu \equiv 1$, then Z_0^n= number of families in the population $\Omega_0^n = 0, 1, 1, 2, 3, 5, \cdots$, the **Fibonacci numbers**. Clearly, also $= \sum \binom{n-\nu}{\nu}$.

reflection

An example of an identity with varying sequences:

Theorem 2.

$$Z\begin{pmatrix} \text{-} & \alpha_{i+1} & \cdot & \cdot & \cdot & \alpha_{j-1} & \alpha_j \\ \beta_i & \beta_{i+1} & \cdot & \cdot & \cdot & \beta_{j-1} & \beta_j \\ \gamma_i & \gamma_{i+1} & \cdot & \cdot & \cdot & \gamma_{j-1} & \text{-} \end{pmatrix}$$

$$= Z\begin{pmatrix} \text{-} & \alpha_j & \cdot & \cdot & \cdot & \alpha_{i+2} & \alpha_{i+1} \\ \beta_j & \beta_{j-1} & \cdot & \cdot & \cdot & \beta_{i+1} & \beta_i \\ \gamma_{j-1} & \gamma_{j-2} & \cdot & \cdot & \cdot & \gamma_i & \text{-} \end{pmatrix}.$$

Proof. Use the fact that Ω_i^j is equivalent with itself under the map $\omega \curvearrowright \tilde{\omega}$ which corresponds to the reflection determined by

$$\begin{aligned} & i \curvearrowright \text{ never mind (!)}, \\ & i+1 \curvearrowright j, \\ & \cdots, \\ & j \curvearrowright i+1. \end{aligned}$$

Now, contributions to first term from ω matches contributions to second term from $\tilde{\omega}$. □

basic identities

Special notation: $c_i^j = c_i \cdot c_{i+1} \cdots c_j$ with $c_i^{i-1} = 1$.

Theorem 3 (basic identities)**.**

$$Z_i^j = \alpha_\nu Z_i^{\nu-2} Z_{\nu+1}^j + Z_i^{\nu-1} Z_\nu^j\,, \tag{1}$$

$$Z_i^j = \alpha_j Z_i^{j-2} + \beta_j Z_i^{j-1}\,, \tag{2}$$

$$Z_i^j = \alpha_{i+1} Z_{i+2}^j + \beta_i Z_{i+1}^j\,, \tag{3}$$

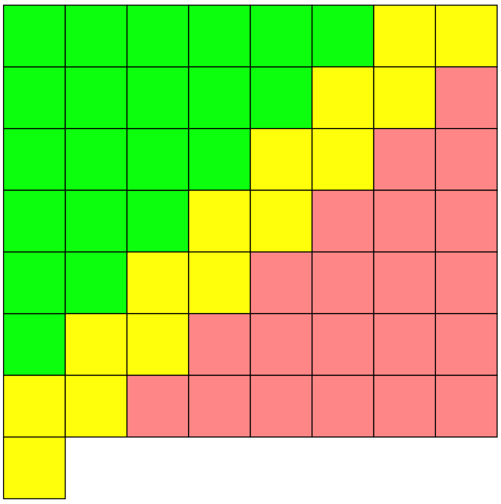
$$Z_i^j = \beta_i^j + \sum_{\nu=i+1}^{j} \beta_i^{\nu-2} \alpha_\nu Z_{\nu+1}^j\,, \tag{4}$$

$$Z_i^j = \beta_i^j + \sum_{\mu=i+1}^{j} \alpha_\mu \beta_i^{\mu+1} Z_i^{\mu-2}\,. \tag{5}$$

Proof. (1): Is there a mother of age ν or is there not?
(2) and (3): Take $\nu = j$ or $\nu = i + 1$.
(4) and (5): Consider youngest, respectively oldest mother. □

(2) is the **primary identity**, (3) the **adjoint identity**
Both are **three-term recurrence relations**.

the extended partition function diagram



Theorem 4. *Primary and adjoint identities hold with*

$$Z_i^j = \frac{(-1)^{i-j-1}}{a_{j+1}} Z_{j+2}^{i-2} \text{ for } i \geq j+2\,.$$

Columns / rows are linearly independent solutions of the primary/ adjoint recurrence relation. For the "natural" isomorphism between the two solution spaces the i'th column corresponds to the $i-2$'th row.

Quotients corresponding to a fixed pair of rows converges iff all such quotients converges iff the continued fraction converges.

the determinant formulas

Theorem 5 (the swopping identity)**.** *Given populations:*

$$\left(\Omega_{i_t}^{j_t}\right)_{t\in T}$$

and a permutation $t \curvearrowright \sigma(t)$ *of* T*. Let*

$$\mathbb{Z} = \sum_{(\omega_t)} \prod_t Z_{i_t}^{j_t}(\omega_t)\,,$$

summation being over all $(\omega_t)_{t\in T} \in \prod_t \Omega_{i_t}^{j_t}$ *which have a common cut-point. Let* $\mathbb{Z}^\sigma$ *be the corresponding quantity related to the populations* $\left(\Omega_{i_t}^{j_{\sigma(t)}}\right)_{t\in T}$*.*

Then $\mathbb{Z} = \mathbb{Z}^\sigma$*.*

Proof. Swop! □

Corollary 1 (determinant formula)**.** *Put*

$$\Delta_{i,j}^{k,l} = \begin{vmatrix} Z_i^k & Z_i^l \\ Z_j^k & Z_j^l \end{vmatrix}$$

Then

$$\Delta_{i,j}^{k,l} = (-1)^{k-j}\alpha_j^{k+1} Z_i^{j-2} Z_{k+2}^l\,.$$

introducing randomness

Assume a_i's and b_i's all > 0. With weight of $\omega \in \Omega_i^j$ proportional to $Z_i^j(\omega)$ you define probability distributions P_i^j and natural random elements X_ν; $\nu \in [i, j]$. For example,

$$P_i^j(X_\nu = \alpha) = \alpha_\nu \frac{Z_i^{\nu-2} Z_{\nu+1}^j}{Z_i^j}\,.$$

Theorem 6. *The probability distributions can be extended in a natural way to probability distributions over infinite families iff the continued fraction is convergent.*

Define the **demographic constants** λ_i as

$$\lim_{j\to\infty} \frac{E_i^j(|\omega|)}{j-i+1}$$

with $|\omega|$ denoting the number of mothers in ω.

Conjecture $\lambda_i = \lim_{\nu\to\infty} \lim_{j\to\infty} P_i^j(X_\nu = \alpha)\,.$

Example In simple examples this is OK. For the Fibonacci case, $\lambda_i = \frac{5-\sqrt{5}}{10} = \frac{\rho}{\sqrt{5}} \approx 0.2764$.